A Bousfield-Kan algorithm for computing the *effective* homotopy of a space[1]

A. ROMERO
F. SERGERAERT

This paper is devoted to a *constructive* version of the Bousfield-Kan spectral sequence (*BKSS*). The *BKSS* provides a combinatorial basis for the famous Adams spectral sequence and its descendants. Its systematic description in [1] remains a relatively difficult text, often cited by its sweet nickname, the "Yellow Monster". The modern *constructive* point of view gives an opportunity to reread this essential text and to use it to produce a new *algorithm* computing homotopy groups, more precisely computing the *effective* homotopy of a given space, a new concept much richer than the ordinary homotopy groups. Without changing the general philosophy of the *BKSS*, the *constructive* constraint leads to a significant reorganization of this rich material and, as it is most often the case, finally to a simpler and more explicit description. Combined with our own basic tools, *effective homology* and *effective homotopy*, the description of the *BKSS* given here is finally not so complicated and could also help the topologists interested by this nice subject.

68W30, 55T99, 55Q05, 18G30

1 Introduction

The origin of the Adams spectral sequence is the famous *Hurewicz theorem*: the first non-zero homology and homotopy groups of a simply connected space are the same; well, but what about the next groups? Frank Adams constructed a spectral sequence [2] starting with homological objects and converging to homotopical objects; the Hurewicz theorem is the simplest "application". The Adams spectral sequence and the related ones have been intensively used to obtain numerous homotopy groups of spaces where the homology is simple, typically the spheres, see [3, 4, 5].

Other methods are available to compute homotopy groups. The first *computational* method presented by Edgar Brown in his famous paper [6] was a constructive use of

[1]Partially supported by Ministerio de Economía y Competitividad, Spain, project MTM2013-41775-P.

the Postnikov tower; it was only a theoretical result: because of the terrible underlying inductive process, it is not yet implemented sixty years later and its concrete feasability remains hypothetical. An analogous work can be done with the Whitehead tower. Using the new concept of *effective homology* [7], a process fundamentally different from Edgar Brown's, the Postnikov and Whitehead towers have on the contrary easily been implemented, allowing us to access a few homotopy groups[2] so far unreachable, using only the effective homology versions of the Serre and Eilenberg-Moore spectral sequences.

Observing the well known power of the Adams spectral sequence, mainly due to its rich algebraic structure, it is natural to also try applying the methods of effective homology to the Adams spectral sequence. A computer can work only combinatorially and the combinatorial basis of the Adams spectral sequence is the Bousfield-Kan spectral sequence (BKSS) [1]. This spectral sequence starts with the homology groups of a simplicial set and converges to its homotopy groups; the classical Adams spectral sequence can be derived of the BKSS.

Combinatorial does not imply *constructive*[3]. In the case of the *BKSS*, its combinatorial nature does not imply it is constructive. For example one may naively hope to have an "Adams" algorithm allowing a topologist to obtain $\pi_*(X)$ from $H_*(X)$, but simple examples show such a goal is impossible: two spaces can have isomorphic homology groups and different homotopy groups. Some extra information is necessary to obtain the homotopy groups from the homology groups.

A Spectral Sequence is a family of "pages" $\{E^r_{p,q}, d^r\}_r$ of differential bigraded modules, each page being made of the homology groups of the preceding one. As expressed by John McCleary after Definition 2.2 in [8] (or Definition p.28 in the first edition by Publish or Perish), "*knowledge of $E^r_{*,*}$ and d^r determines $E^{r+1}_{*,*}$ but not d^{r+1}. If we think of a spectral sequence as a black box, then the input is a differential bigraded module, usually $E^1_{*,*}$, and, with each turn of the handle, the machine computes a successive homology according to a sequence of differentials. If some differential is unknown, then some other (any other) principle is needed to proceed.*" In most cases, it is in fact a matter of computability: the higher differentials of the spectral sequence

[2] The main successes of effective homology have been obtained for the homology groups of the loop spaces, where the Adams and Baues methods cannot be iterated beyond the second loop space [7].

[3] The qualifier *constructive* often raises difficulties for the "classical" mathematicians. The comments later in this text, mainly at the end of this introduction, also in Section 5 around the notion of *effective* homotopy, should help the reader to clarify what *constructiveness* means.

are mathematically defined, but their definition is not constructive, i.e., the differentials are not computable with the usually provided information.

In the case of the Adams spectral sequence, the module structure of the cohomology with respect to the Steenrod algebra is such an extra information which can be useful, but it is *in general* insufficient to determine the differential maps. The Adams spectral sequence is therefore not constructive.

Let X be a simply connected space. Our version of the *BKSS* is a general *algorithm*:

- Input: The *effective* homology of X.
- Output: The *effective* homotopy of X.

Effective homology [9, 7] has been designed to transform the Serre spectral sequence into a genuine algorithm computing the homology groups of a total space from the homology groups of the base space and the fibre space, in fact the *effective* homology of the total space from the *effective* homologies of the base space and the fibre space. The same process can be applied to the Eilenberg-Moore spectral sequences. The reference [7] gives examples showing this is not only a theoretical result: the proved algorithm can be written down as a computer program, and used to compute groups so far unreachable. The *effective* homology method allows one to prevent the "disconnection" of the spectral sequences from the background process, retaining the connection with the initial spaces.

The subject of the present paper is analogous. We present, organize and prove here the *BKSS* as a general algorithm $EH_*(X) \mapsto E\pi_*(X)$, the prefix E meaning *effective* (homology or homotopy). The effective homology of a space X is the ordinary one combined with extra functional objects allowing a user to solve the ambiguities often observed in "ordinary" homology, typically, the extension problems when exact or spectral sequences must be used to determine some unknown groups. The extra functional objects just mentioned cannot reasonably be used by a topologist working with pen and paper; on the contrary they can be ordinarily used with the help of a computer, thanks to *functional programming*, now standard in most scientific programming languages. Same comments for effective homotopy. This paper so opens a fascinating challenge for the topologists interested by concrete programming.

A reader of a previous version of this paper questioned the authors about the difference between the qualifiers "constructive" and "algorithmic". A *constructive* existence theorem must define a *construction* process producing a copy of the claimed existing object from the given data. In a computational framework, the required construction process is nothing but an algorithm *input* $\mapsto$ *output*, but more general situations can be considered where the "construction" process has not necessarily the form of an algorithm, see for example the book *Constructive Analysis* by Bishop and Bridges [10] for typical examples of this sort. In the present text, the computational environment is always implicitly understood, so that no difference here between "constructive" and "algorithmic".

The paper is organized as follows. After this introduction, Section 2 includes some elementary ideas about spectral sequences and simplicial sets. Then a brief description of the construction of the Bousfield-Kan spectral sequence is given in Section 3, including some necessary definitions and results. In Sections 4 and 5 we make constructive the different ingredients appearing in the definition of the Bousfield-Kan spectral sequence. This makes it possible to develop an algorithm computing all components of the spectral sequence, which is explained in Section 6. Then Section 7 presents our general algorithm for computing homotopy groups of spaces, which are provided with the natural filtration induced by the spectral sequence. The paper ends with a section of conclusions and further work.

2 Preliminaries

2.1 Spectral sequences

In this section we particularize the usual definitions of spectral sequences ([11], [8]) for the case of the Bousfield-Kan spectral sequence.

Definition 1 A spectral sequence $E = (E^r, d^r)_{r \geq 1}$ is a *second quadrant spectral sequence* if for all $r \geq 1$ one has $E^r_{p,q} = 0$ when $p > 0$ or $q < 0$.

As it is usual in the literature, in this paper we will represent second quadrant spectral sequences in the first quadrant by changing the sign of the first degree p. In other words, we put the module $E^r_{p,q}$ with $p \leq 0$ and $q \geq 0$ at the point $(-p,q)$ (which is in the first quadrant), and denote it by $E^r_{-p,q}$. The differential maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p+r,q+r-1}$ have then the bidegree $(r, r-1)$. The convergence of the spectral sequence to a graded module $H_* = \{H_n\}_{n \in \mathbb{N}}$ is therefore given by a decreasing filtration F and isomorphisms $E^\infty_{p,q} \cong F_{p-1}H_{q-p}/F_pH_{q-p}$.

It is worth emphasizing here the following ideas summarizing the main problems of spectral sequences.

Remark 2 Spectral sequences naturally arise as a very structured *shadow* of a more complicated homological background process. While the objects of the successive pages are uniquely determined (up to isomorphisms) by the previous page, there is no general algorithm to compute differentials of one page from those on the previous pages. In other words, if spectral sequences are *disconnected* from the background process, then only in special cases (typically when more information is available) can one study their convergence, and in cases which are even more special can one solve the extension problem.

The mathematician comparing the ordinary status of a spectral sequence with which is provided by effective homology and homotopy observes the *shadow* mentioned above is which remains when the functional objects, out of scope without a computer, are removed.

2.2 Simplicial sets

In this section we introduce some elementary ideas about simplicial sets, which can be considered a useful combinatorial model for topological spaces. More concretely, given a Kan simplicial set K with a base point $\star \in K_0$, an algebraic definition of the homotopy groups of K can be given such that they are isomorphic to the homotopy groups of the corresponding topological space by means of the *realization functor*. All the definitions and results of this section (and details about the connection of simplicial sets and topological spaces) can be found in [12].

Definition 3 A *simplicial set* K is a simplicial object over the category of sets, that is to say, K consists of:

- a set K_q for each integer $q \geq 0$;
- for every pair of integers (i,q) such that $0 \leq i \leq q$, *face* and *degeneracy* maps $\partial_i : K_q \to K_{q-1}$ and $\eta_i : K_q \to K_{q+1}$ satisfying the *simplicial identities:*

$$\begin{aligned}
\partial_i \partial_j &= \partial_{j-1} \partial_i && \text{if } i < j \\
\eta_i \eta_j &= \eta_{j+1} \eta_i && \text{if } i \leq j \\
\partial_i \eta_j &= \eta_{j-1} \partial_i && \text{if } i < j \\
\partial_i \eta_j &= \mathrm{Id} && \text{if } i = j, j+1 \\
\partial_i \eta_j &= \eta_j \partial_{i-1} && \text{if } i > j+1
\end{aligned}$$

The category of simplicial sets is denoted by SS.

Definition 4 A simplicial set K is called a *Kan simplicial set* if it satisfies the following *extension condition*: for every collection of q $(q-1)$-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_q$ which satisfy the compatibility condition $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j, i \neq k$, and $j \neq k$, there exists a q-simplex $x \in K_q$ such that $\partial_i x = x_i$ for every $i \neq k$.

Let us observe that the existence of the q-simplex x for each collection of $(q-1)$-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_q$ satisfying the compatibility condition does not imply it is always possible to determine it. We say that the Kan simplicial set K is a *constructive Kan simplicial set* if the desired x is given explicitly by an algorithm. It is not difficult to prove (see [12, § 17]) that every simplicial group is a constructive Kan simplicial set. The constructive Kan property of a simplicial set will be needed later in this paper.

Definition 5 Let K be a simplicial set. Two q-simplices x and y of K are said to be *homotopic*, written $x \sim y$, if $\partial_i x = \partial_i y$ for $0 \leq i \leq q$, and there exists a $(q+1)$-simplex z such that $\partial_q z = x, \partial_{q+1} z = y$, and $\partial_i z = \eta_{q-1} \partial_i x = \eta_{q-1} \partial_i y$ for $0 \leq i < q$.

If K is a Kan simplicial set, then $\sim$ is an equivalence relation on the set of q-simplices of K for every $q \geq 0$.

Let $\star \in K_0$ be a 0-simplex of K, called a *base point*; we also denote by $\star$ the degeneracies $\eta_{q-1} \ldots \eta_0 \star \in K_q$ for every q. We define $S_q(K)$ as the set of all $x \in K_q$ such that $\partial_i x = \star$ for every $0 \leq i \leq q$, which is said to be the set of q-*spheres* of K.

Definition 6 Given a Kan simplicial set K and a base point $\star \in K_0$, we define

$$\pi_q(K, \star) := \pi_q(K) := S_q(K)/(\sim)$$

The set $\pi_q(K, \star)$ admits a group structure for $q \geq 1$ and is Abelian for $q \geq 2$. It is called the q-*homotopy group* of K.

Definition 7 Let $f : E \to B$ be a simplicial map. The map f is a *Kan fibration* if for every collection of q $(q-1)$-simplices $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_q$ of E which satisfy the compatibility condition $\partial_i x_j = \partial_{j-1} x_i, i < j, i \neq k, j \neq k$, and for every q-simplex y of B such that $\partial_i y = f(x_i), i \neq k$, there exists a q-simplex x of E such that $\partial_i x = x_i, i \neq k$, and $f(x) = y$. The simplicial set E (resp. B) is called the *total space* (resp. the *base space*) of the fibration. If Φ denotes the simplicial set generated by a vertex of B (usually the base point $\star$), then $F := f^{-1}(\Phi)$ is called the *fiber* over Φ.

Later in this paper we will need the Kan property of a fibration to be constructive; we say that f is a *constructive Kan fibration* if the desired q-simplex x of E is produced by an algorithm.

3 Description and algorithmic problems of the Bousfield-Kan spectral sequence

3.1 Definition of the spectral sequence

The Bousfield-Kan spectral sequence was introduced in [13] to establish the Adams spectral sequence [2] on a simplicial combinatorial background. Under some good conditions, the Bousfield-Kan spectral sequence associated with a simplicial set X converges to the homotopy groups of X, $\pi_*(X)$. However, as we will see later, the definition of the different levels of the spectral sequence involves complicated structures and it does not provide an algorithm computing the desired homotopy groups of a simplicial set.

The Bousfield-Kan spectral sequence is defined by means of a *tower of fibrations* which is associated with a *cosimplicial simplicial set* as explained in the following definitions. A detailed description of the construction of the spectral sequence can be found in [1].

Restriction 8 *The simplicial set X we are working with in this paper is assumed 1-reduced: only one vertex, no non-degenerate edge. It is therefore* simply connected *and all the sophisticated technicalities of [1] concerning $\pi_1(X)$ are avoided, in particular the various nilpotency conditions often required in [1]. All our results can be easily extended to the nilpotent case systematically studied in [1], but we prefer not to complicate our task with this subject, essentially disjoint from ours:* effectiveness. *Our X is therefore automatically $\mathbb{Z}$-complete and in particular $\mathbb{Z}$-good [1, Ch.III-5.4]. In short, no bad surprise about the convergence of $BKSS(X)$ toward $\pi_*(X)$ [1, Ch.V-3.7].*

Definition 9 Let X be a simplicial set with a base point $\star \in X_0$, then RX is the simplicial Abelian group defined as

$$RX = \frac{R[X]}{R[\star]}$$

where $R[X]$ denotes the simplicial $\mathbb{Z}$-module freely generated by the simplices of X, and $R[\star]$ is the simplicial submodule generated by the base point $\star$ and its degeneracies.

Note an arbitrary R-combination of q-simplices of X is only *one* q-simplex of RX.

Every simplicial group satisfies the Kan extension property [12, Theorem 17.1], so that we can consider the homotopy groups $\pi_*(RX)$. On the other hand, the group $R[X]_q$ is nothing but the standard reduced chain group $\widetilde{C}_q(X;\mathbb{Z})$. The next proposition is then a consequence of the combinatorial Kan definition of the homotopy groups [12, Theorem 22.1].

Proposition 10 *Given X a pointed simplicial set, there exists a canonical isomorphism*

$$\pi_*(RX) \cong \widetilde{H}_*(X;\mathbb{Z})$$

where $\widetilde{H}_*(X;\mathbb{Z})$ denotes the reduced homology groups of X with coefficients in $\mathbb{Z}$.

Definition 11 A *cosimplicial simplicial set* or *cosimplicial space* $\mathcal{X}$ consists of:

- for every integer $p \geq 0$, a simplicial set $\mathcal{X}^p$, called the *codimension p* component of $\mathcal{X}$;
- for every pair of integers (i,p) such that $0 \leq i \leq p$, *coface* and *codegeneracy* operators $\partial^i : \mathcal{X}^{p-1} \to \mathcal{X}^p$ (for $p \geq 1$) and $\eta^i : \mathcal{X}^{p+1} \to \mathcal{X}^p$ (both of them simplicial maps) satisfying the *cosimplicial identities:*

$$\begin{aligned}
\partial^j\partial^i &= \partial^i\partial^{j-1} && \text{if } i<j\\
\eta^j\eta^i &= \eta^i\eta^{j+1} && \text{if } i\leq j\\
\eta^j\partial^i &= \partial^i\eta^{j-1} && \text{if } i<j\\
\eta^j\partial^i &= \mathrm{Id} && \text{if } i=j,j+1\\
\eta^j\partial^i &= \partial^{i-1}\eta^j && \text{if } i>j+1
\end{aligned}$$

Cosimplicial simplicial sets form a category that we denote by CSS.

Definition 12 An augmentation of a cosimplicial space $\mathcal{X}$ consists of a simplicial set $\mathcal{X}^{-1}$ and a morphism $\partial^0 : \mathcal{X}^{-1} \to \mathcal{X}^0$ such that $\partial^1\partial^0 = \partial^0\partial^0 : \mathcal{X}^{-1} \to \mathcal{X}^1$.

In other words, a cosimplicial space $\mathcal{X}$ consists of a bigraded family $\mathcal{X} = \{\mathcal{X}^p_q\}_{p,q\in\mathbb{N}}$ with face, coface, degeneracy and codegeneracy maps $\partial_i : \mathcal{X}^p_q \to \mathcal{X}^p_{q-1}$, $\partial^j : \mathcal{X}^{p-1}_q \to \mathcal{X}^p_q$, $\eta_i : \mathcal{X}^p_q \to \mathcal{X}^p_{q+1}$, and $\eta^j : \mathcal{X}^{p+1}_q \to \mathcal{X}^p_q$, for $0 \leq i \leq q$ and $0 \leq j \leq p$. The face and degeneracy operators ∂_i and η_i must satisfy the simplicial identities (Definition 3), while for ∂^j and η^j the cosimplicial identities of Definition 11 hold. Furthermore, ∂_i and η_i commute with both coface and codegeneracy maps ∂^j and η^j.

An initial example of cosimplicial space is the *cosimplicial standard simplex* Δ, whose columns are the simplicial standard n-simplices Δ^n (see [12, Definition 5.4]).

Definition 13 The *cosimplicial standard simplex* Δ consists in codimension p of the standard p-simplex Δ^p, and the coface and codegeneracy maps are the unique morphisms $\partial^j : \Delta^{p-1} \to \Delta^p$ and $\eta^j : \Delta^{p+1} \to \Delta^p$ which map respectively the fundamental simplex $(0,\dots,p-1)$ of Δ^{p-1} to $\partial_j(0,\dots,p)$ of Δ^p and the fundamental simplex $(0,\dots,p+1)$ of Δ^{p+1} to $\eta_j(0,\dots,p)$ of Δ^p if ∂_j and η_j are the usual face and degeneracy operators of the category Δ.

Another important cosimplicial space is the cosimplicial resolution of a simplicial set, which plays an essential role in the definition of the Bousfield-Kan spectral sequence.

Definition 14 Let X be a pointed simplicial set. The *cosimplicial resolution* of X (with respect to the ring $R=\mathbb{Z}$) is the augmented cosimplicial space $\mathcal{R}X$ given by:

- for each cosimplicial degree p, the column $\mathcal{R}X^p$ is the simplicial Abelian group $R^{p+1}X$ obtained by applying $p+1$ times the functor R (Definition 9) to the simplicial set X (with the corresponding face and degeneracy maps);
- the coface and codegeneracy operators are defined as

$$\begin{array}{llll} \partial^j: & \mathcal{R}X^{p-1}=R^pX \longrightarrow & \mathcal{R}X^p=R^{p+1}X, & \partial^j=R^j\Phi R^{p-j} \\ \eta^j: & \mathcal{R}X^{p+1}=R^{p+2}X \longrightarrow & \mathcal{R}X^p=R^{p+1}X, & \eta^j=R^j\Psi R^{p-j} \end{array}$$

  where Φ and Ψ are natural transformations $\Phi:\mathrm{Id}\to R$ and $\Psi:R^2\to R$ induced by the maps $\Phi:X\to RX$ and $\Psi:R^2X\to RX$ which are given by $\Phi(x):=1*x$ for all $x\in X$ and $\Psi(1*y):=y$ for all $y\in RX$ (extended to all elements in R^2X by using the group operation in RX);
- the augmentation is given by the map $\Phi: X\to RX$.

It is worth emphasizing that each column $\mathcal{R}X^p=R^{p+1}X$ is a simplicial Abelian group, which implies that for each $q\geq 0$ the set $\mathcal{R}X^p_q$ is an Abelian group, and the face operators $\partial_i:\mathcal{R}X^p_q\to\mathcal{R}X^p_{q-1}$ and the degeneracies $\eta_i:\mathcal{R}X^p_q\to\mathcal{R}X^p_{q+1}$ are group morphisms. On the other hand, one can observe that the codegeneracy maps $\eta^j:\mathcal{R}X^{p+1}_q\to\mathcal{R}X^p_q$ are also morphisms of groups for all $j\geq 0$, but $\partial^j:\mathcal{R}X^{p-1}_q\to\mathcal{R}X^p_q$ is a group morphism only if $j\geq 1$. For $j=0$, the coface $\partial^0:\mathcal{R}X^{p-1}_q\to\mathcal{R}X^p_q$ is not a morphism of groups. For this reason, the cosimplicial space $\mathcal{R}X$ is said to be *grouplike*.

Definition 15 A cosimplicial space $\mathcal{X}$ is said to be *grouplike* if for all $p\geq 0$ the space $\mathcal{X}^p$ is a simplicial group and the operators ∂^j for $j\geq 1$ and all operators η^j are homomorphisms of simplicial groups.

Definition 16 A *tower of fibrations* if a family $(Y_n,f_n)_{n\geq 0}$ of pointed simplicial sets $\{Y_n\}_{n\geq 0}$ with fibrations $f_n:Y_n\to Y_{n-1}$:

$$\cdots \xrightarrow{f_{n+1}} Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} Y_0 \xrightarrow{f_0} Y_{-1}=\star$$

where $\star$ denotes the simplicial set with only one simplex $\star$ in each dimension. Its *inverse limit* is a simplicial set

$$\varprojlim Y_n = Y$$

with projections $p_n:Y\to Y_n$ such that $f_n\circ p_n=p_{n-1}$, satisfying the corresponding universal property for inverse limits.

Given a cosimplicial space, one can construct a tower of fibrations as follows.

Definition 17 Let $\mathcal{X}$ be a cosimplicial space and K a simplicial set. Applying the functor "$- \times K$" to every component of $\mathcal{X}$ produces the cosimplicial space $\mathcal{X} \times K$. Its p component is the simplicial set $\mathcal{X}^p \times K$. The coface ∂^j and codegeneracy η^j operators are defined as the maps:

$$\mathcal{X}^{p-1} \times K \xrightarrow{\partial^j \times \mathrm{Id}_K} \mathcal{X}^p \times K$$
$$\mathcal{X}^{p+1} \times K \xrightarrow{\eta^j \times \mathrm{Id}_K} \mathcal{X}^p \times K$$

Definition 18 Let X and Y be simplicial sets. The *function space* $\mathrm{Func}(X,Y)$ is a simplicial set whose q-simplices are the simplicial morphisms $X \times \Delta^q \to Y$ and the faces ∂_i and the degeneracies η_i are given by the compositions:

$$X \times \Delta^{q-1} \xrightarrow{\mathrm{Id}_X \times \partial^i} X \times \Delta^q \quad \longrightarrow Y$$
$$X \times \Delta^{q+1} \xrightarrow{\mathrm{Id}_X \times \eta^i} X \times \Delta^q \quad \longrightarrow Y$$

where $\partial^i : \Delta^{q-1} \to \Delta^q$ and $\eta^i : \Delta^{q+1} \to \Delta^q$ are the standard maps introduced in Definition 13.

Definition 19 Let $\mathcal{X}$ and $\mathcal{Y}$ be cosimplicial spaces. The *function space* $\mathrm{Func}(\mathcal{X},\mathcal{Y})$ is a simplicial set with $\mathrm{Func}(\mathcal{X},\mathcal{Y})_q = \mathrm{CSS}(\mathcal{X} \times \Delta^q, \mathcal{Y})$ and the faces ∂_i and the degeneracies η_i being given by the compositions:

$$\mathcal{X} \times \Delta^{q-1} \xrightarrow{\mathrm{Id}_{\mathcal{X}} \times \partial^i} \mathcal{X} \times \Delta^q \quad \longrightarrow \mathcal{Y}$$
$$\mathcal{X} \times \Delta^{q+1} \xrightarrow{\mathrm{Id}_{\mathcal{X}} \times \eta^i} \mathcal{X} \times \Delta^q \quad \longrightarrow \mathcal{Y}$$

Definition 20 Let $\mathcal{X}$ be a cosimplicial space. The *total space* $\mathrm{Tot}\,\mathcal{X}$ is the simplicial set defined as the function space:

$$\mathrm{Tot}\,\mathcal{X} := \mathit{Func}(\Delta, \mathcal{X})$$

The total space of a cosimplicial space $\mathcal{X}$ can be seen as an inverse limit

$$\mathrm{Tot}\,\mathcal{X} = \varprojlim \mathrm{Tot}_n\,\mathcal{X}$$

of the simplicial sets

$$\mathrm{Tot}_n\,\mathcal{X} = \mathrm{Func}(\mathrm{sk}_n\,\Delta, \mathcal{X}) \quad \text{for } n \geq -1$$

where $\mathrm{sk}_n\,\Delta \subset \Delta$ is the *simplicial n-skeleton* of the cosimplicial simplicial set Δ; in other words, $\mathrm{sk}_n\,\Delta$ consists in codimension p of the n-skeleton of the simplicial set Δ^p.

The map $f_n : \mathrm{Tot}_n \mathcal{X} \to \mathrm{Tot}_{n-1} \mathcal{X}$ is induced by the inclusion $\mathrm{sk}_{n-1} \Delta \subset \mathrm{sk}_n \Delta$. In particular, $\mathrm{Tot}_{-1} \mathcal{X} = \star$ and $\mathrm{Tot}_0 \mathcal{X} \cong \mathcal{X}^0$. More generally, $\mathrm{Tot}_n \mathcal{X}$ depends only on $\mathcal{X}^i$ for $i \leq n$.

If $\mathcal{X}$ is grouplike [1, Ch.X-4.9 and Ch.X-6.1], the map f_n is a Kan fibration, which organizes the $\mathcal{X}^n$'s as a tower of fibrations. Moreover one can prove (see [1, Ch.X-6.3]) that if $\mathcal{X}$ is grouplike the fiber F_n of the fibration f_n is the simplicial pointed function space:

$$F_n := \mathrm{Func}_*(S^n, \mathcal{X}^n \cap \mathrm{Ker}\, \eta^0 \cap \cdots \cap \mathrm{Ker}\, \eta^{n-1})$$

where S^n is the simplicial model for the sphere of dimension n, with only two non-degenerate simplices $\star$ in dimension 0 and s^n in dimension n.

If $\mathcal{X}$ is augmented (that is, a simplicial set $\mathcal{X}^{-1}$ and a simplicial morphism $\partial^0 : \mathcal{X}^{-1} \to \mathcal{X}^0$ are given such that $\partial^1 \partial^0 = \partial^0 \partial^0 : \mathcal{X}^{-1} \to \mathcal{X}^1$), then ∂^0 induces morphisms

$$p_n : \mathcal{X}^{-1} \longrightarrow \mathrm{Tot}_n \mathcal{X}$$

which are compatible with the maps $f_n : \mathrm{Tot}_n \mathcal{X} \to \mathrm{Tot}_{n-1} \mathcal{X}$. The morphisms p_n are induced by the canonical inclusions $\mathcal{X}^{-1} \hookrightarrow \mathcal{X}^p$ obtained from the augmented cosimplicial structure of $\mathcal{X}$.

Finally, every tower of fibrations produces a spectral sequence as follows.

Let $(Y_n, f_n)_{n \geq 0}$ be a tower of fibrations, with inverse limit Y. Projections $p_n : Y \to Y_n$ are canonically defined satisfying $f_n \circ p_n = p_{n-1}$. Let F_n be the fiber of $f_n : Y_n \to Y_{n-1}$ for each $n \geq 0$. Applying homotopy groups to each fibration gives the exact couple

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{f} & \pi_*(Y_n) & \xrightarrow{f} & \pi_*(Y_{n-1}) & \xrightarrow{f} & \cdots & \xrightarrow{f} & \pi_*(Y_1) & \xrightarrow{f} & \pi_*(Y_0) \\
 & & \uparrow i & \swarrow \partial & \uparrow i & \swarrow \partial & & & \uparrow i & \swarrow \partial & \Vert \\
 & & \pi_*(F_n) & & \pi_*(F_{n-1}) & & & & \pi_*(F_1) & & \pi_*(F_0)
\end{array} \tag{1}$$

where $\partial : \pi_*(Y_{n-1}) \to \pi_{*-1}(F_n)$ is the connecting morphism and $i : \pi_*(F_n) \to \pi_*(Y_n)$ is induced by the inclusion $\mathrm{inc} : F_n \hookrightarrow Y_n$.

For each pair (p, q) such that $q \geq p$ one has the following diagram:

$$
\begin{array}{ccccc}
\downarrow f & & & & \downarrow f \\
\pi_{q-p+1}(Y_p) & & & & \pi_{q-p}(Y_{p+1}) \\
\downarrow f & & & & \downarrow f \\
\pi_{q-p+1}(Y_{p-1}) & \xrightarrow{\partial} & \pi_{q-p}(F_p) & \xrightarrow{i} & \pi_{q-p}(Y_p) \\
\downarrow f & & & & \downarrow f \\
\pi_{q-p+1}(Y_{p-2}) & & & & \pi_{q-p}(Y_{p-1}) \\
\downarrow f & & & & \downarrow f
\end{array}
\tag{2}
$$

We denote by f^r the composition $f \circ \overset{r}{\cdots} \circ f$ and we consider $i^{-1}(\mathrm{Im} f^{r-1})$ and $\mathrm{Ker} f^{r-1})$, which are subgroups of $\pi_{q-p}(F_p)$. One has the following spectral sequence.

Theorem 21 *[1, Ch.IX-4] Given a tower of fibrations $(Y_n, f_n)_{n \geq 0}$, there exists a second quadrant spectral sequence $E = (E^r, d^r)_{r \geq 1}$ given by*

$$
\begin{aligned}
E^r_{p,q} &= \frac{i^{-1}(\mathrm{Im} f^{r-1})}{\partial(\mathrm{Ker} f^{r-1})} \quad && \text{for } q \geq p \\
E^r_{p,q} &= 0 && \text{otherwise}
\end{aligned}
$$

with differential maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p+r,q+r-1}$ induced by the composition:

$$
\pi_{q-p}(F_p) \xrightarrow{i} \mathrm{Im} f^{r-1} \subseteq \pi_{q-p}(Y_p) \xrightarrow{(f^{r-1})^{-1}} \pi_{q-p}(Y_{p+r-1}) \xrightarrow{\partial} \pi_{q-p-1}(F_{p+r})
$$

This spectral sequence induces a decreasing filtration on the homotopy groups of the inverse limit Y, $\pi_*(Y)$, given by $F_n(\pi_m(Y)) = \mathrm{Ker}(p_n : \pi_m(Y) \to \pi_m(Y_n))$. Under some good *conditions (see [1, Ch.IX-5.3 and Ch.IX-5.4] for details), this spectral sequence converges to the homotopy groups of the inverse limit Y, with isomorphisms $E^\infty_{p,q} \cong F_{p-1}(\pi_{q-p}(Y))/F_p(\pi_{q-p}(Y))$.*

Let us observe that for dimension $q-p = 1$ the group $\pi_1(F_p)$ could be non-commutative and then the corresponding $E^r_{p,q}$'s could also be non-Abelian. For $q = p$, it may happen $\pi_0(F_p)$ is not a group but only a (pointed) set and in that case $E^r_{p,q}$ is defined as the set of orbits of the action of $\mathrm{Ker} f^{r-1}$. On the other hand, we have preferred not to detail the *good* conditions which ensure the convergence of the spectral sequence; as it is explained in [1], these conditions are satisfied by the Bousfield-Kan tower of fibrations when the initial simplicial set X is 1-reduced.

Theorem 21 provides a formal definition of the different groups $E^r_{p,q}$ and the differential maps $d^r_{p,q}$ of the spectral sequence associated with a tower of fibrations. If we want to *compute* the groups $E^r_{p,q}$, we need to compute first the groups $\pi_*(Y_n)$ and $\pi_*(F_n)$ appearing in the diagram (2); but it is necessary to remark here that a formal description of these groups is not sufficient; we need explicit descriptions of the maps i, f and ∂, which is possible only if we have explicit representatives for the elements of homotopy groups and conversely, given a sphere in Y_n or F_n, an algorithm must produce its homotopy class; this is covered by the notion of *effective homotopy*, see Section 5.1.

If the homotopy groups $\pi_*(Y_n)$ and $\pi_*(F_n)$ are finitely generated Abelian groups and they are explicitly known (with the corresponding generators) for all n, then it is clear that the groups $E^r_{p,q}$ and the differentials $d^r_{p,q}$ are computable because in that case the involved maps f, i and ∂ can be expressed as finite integer matrices. In this way, if we want to develop an algorithm computing the spectral sequence associated with a tower of fibrations, we will try to construct first algorithms which determine (in a constructive way) the homotopy groups of the simplicial sets Y_n and of the fiber spaces F_n.

Let us consider now a 1-reduced simplicial set X, and the associated augmented cosimplicial space $\mathcal{R}X$. As already observed, $\mathcal{R}X$ is grouplike and therefore the spaces

$$(\mathrm{Tot}_n\,\mathcal{R}X = \mathrm{Func}(\mathrm{sk}_n\,\Delta, \mathcal{R}X), f_n)_{n\geq 0}$$

define a tower of fibrations with fibers $F_n = \mathrm{Func}_*(S^n, R^{n+1}X \cap \mathrm{Ker}\,\eta^0 \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1})$. Applying Theorem 21, one obtains a spectral sequence which is called the *Bousfield-Kan spectral sequence* of X.

Theorem 22 (Bousfield-Kan spectral sequence) *[1, Ch.X-6] Let X be a simplicial set with base point $\star \in X_0$. There exists a canonical second quadrant spectral sequence $E = (E^r, d^r)_{r\geq 1}$, whose term E^1 is given by*

$$E^1_{p,q} = \pi_q(R^{p+1}X) \cap \mathrm{Ker}\,\eta^0 \cap \cdots \cap \mathrm{Ker}\,\eta^{p-1}$$

and with differential map d^1 induced by the coboundary map $\delta = \sum(-1)^j\partial^j$. Under suitable hypotheses (for instance, if X is 1-reduced) this spectral sequence converges to the homotopy groups $\pi_*(\mathrm{Tot}\,\mathcal{R}X) \cong \pi_*(X)$.

3.2 Didactic example

For a better understanding of the definition of the Bousfield-Kan spectral sequence, let us consider as a didactic example the case where the realization of the simplicial set X is the 2-sphere S^2.

q						
0	0	0	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^5$	0	
0	$\mathbb{Z}$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2^6 \oplus \mathbb{Z}_3^2$	$\mathbb{Z}^7 \oplus \mathbb{Z}_2^{15} \oplus \mathbb{Z}_3^2$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2^{11}$	0	E^1
0	0	0	$\mathbb{Z}_2$	0	0	
0	$\mathbb{Z}$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^3$	0	0	
0	0	0	0	0	0	
0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	0	0	0	
0	0	0	0	0	0	
0	$\mathbb{Z}$	0	0	0	0	
0	0	0	0	0	0	
$\mathbb{Z}$	0	0	0	0	0	
0	0	0	0	0	0	
0	0	0	0	0	0	p

Figure 1: Level 1 of the Bousfield-Kan spectral sequence of S^2.

The first level of the spectral sequence can be obtained in terms of the homology groups of different Eilenberg-MacLane spaces $K(\pi, n)$'s (see [14]) and is given by the groups in Figure 1.

For $q \leq 11$ the non-zero differential maps at level $r = 1$ are $d^1_{1,6}$, $d^1_{1,8}$, $d^1_{1,10}$, $d^1_{2,8}$, $d^1_{2,10}$, $d^1_{3,10}$ and $d^1_{3,11}$. In this particular case one can also determine the groups $E^2_{p,q}$ for $q - p \leq 6$, represented in Figure 2.

Here all the differentials d^2 must be equal to zero, which implies that these groups are already the final groups of the spectral sequence. For each dimension $q - p = 2, 3, 4$ and 5 one has only one non-zero group $E^\infty_{p,q}$, which corresponds to the homotopy group $\pi_{q-p}(S^2)$. One can observe the well-known results $\pi_2(S^2) = \pi_3(S^2) = \mathbb{Z}$ and $\pi_4(S^2) = \pi_5(S^2) = \mathbb{Z}_2$. However, for dimension $q - p = 6$ we have three non-zero groups $E^\infty_{2,8} = \mathbb{Z}_3$ and $E^\infty_{3,9} = E^\infty_{4,10} = \mathbb{Z}_2$ and two extensions are possible, $\pi_6(S^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_6$ or $\mathbb{Z}_{12}$. The BKSS apparatus says nothing about the extension problem at abutment: the BKSS is not *constructive*. On the contrary our version of the BKSS leads to Theorem 42 solving such an extension problem.

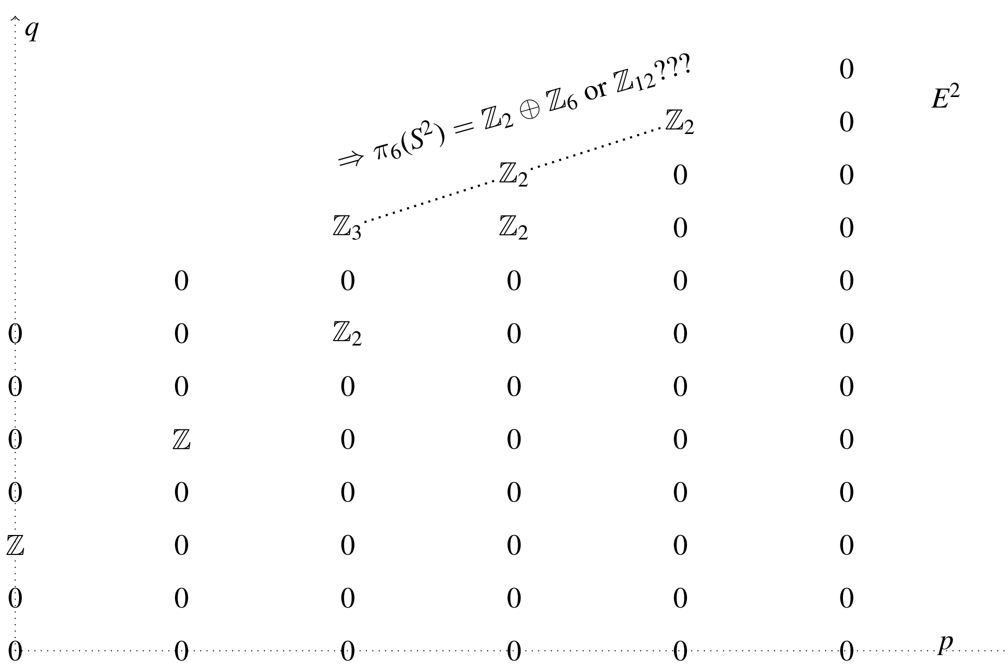


Figure 2: Level 2 of the Bousfield-Kan spectral sequence of S^2.

3.3 Algorithmic remarks

Theorem 22 provides the description of the first level of the Bousfield-Kan spectral sequence associated with a simplicial set X, but we find it convenient to remark here that this description does not allow us in general to *compute* directly the groups $E^r_{p,q}$ and the differential maps $d^r_{p,q}$ (this is a general problem of spectral sequences, as expressed in Remark 2). The columns $R^{p+1}X$ have the homotopy type of products of Eilenberg-MacLane spaces, and Cartan's algorithm [15] computes the corresponding $\pi_q(R^{p+1}X) \cong \widetilde{H}_q(R^pX)$. But to our knowledge, in our very general framework, so far no *effective* method (algorithm!) is known allowing one to determine the codegeneracy and coface operators between these homology groups, so that this definition is not sufficient to determine the groups $E^r_{p,q}$. For the computation of the different levels of the spectral sequence, the tower of fibrations associated with the cosimplicial space $\mathcal{R}X$ must be considered. One needs to determine first in a constructive way (with generators) the homotopy groups of the spaces $Y_n = \mathrm{Tot}_n\,\mathcal{R}X$ and $F_n = \mathrm{Func}_*(S^n, R^{n+1}X \cap \mathrm{Ker}\,\eta^0 \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1})$, but they are complicated (infinite) spaces and their homotopy groups are rather problematic. This makes the computation of the higher groups of the spectral sequence $E^r_{p,q}$ significantly difficult.

In a previous work [14], we used the *effective homology* method (introduced in [9] and explained in depth in [7] and [16]) to develop algorithms computing the first two levels E^1 and E^2 of the Bousfield-Kan spectral sequence of a simplicial set X, but determining the higher levels E^r for $r \geq 3$ remained open in [14]. The effective homology of a space X consists in four algorithms which provide in particular its homology groups (with the corresponding generators) and give some additional information retaining the connection with the background process which can be necessary if we want to use the space inside other topological constructions. The effective homology of chain complexes of finite type can be constructed in an elementary way, and there are also some theoretical results which provide the effective homology of some particular spaces. From this starting point, we can obtain the effective homology of more complicated spaces by applying different constructors of Algebraic Topology (see [16]).

In [14] we produced an algorithm computing the *effective* homology of RX provided that X is a 1-reduced simplicial set with effective homology. Our algorithm can be iterated producing the effective homology of R^pX for $p \geq 1$, and thanks to the canonical isomorphism $\pi_*(RX) \cong \widetilde{H}_*(X)$ one can obtain the homotopy groups $\pi_q(R^{p+1}X)$ appearing in the first level E^1 of the spectral sequence. They are finite type groups, and our algorithm provides their generators, so that the kernels $\mathrm{Ker}[\eta^i : \pi_q(R^{p+1}X) \to \pi_q(R^pX)]$ can be determined by means of some elementary matrix operations. We obtain in this way an algorithm computing the first level E^1 of the Bousfield-Kan spectral sequence of a simplicial set X. Then the differential maps $d^1_{p,q} : E^1_{p,q} \to E^1_{p+1,q}$ can also be expressed as finite integer matrices and therefore it is possible to determine their kernel and their image, and using the Smith Normal Form technique [17] we can easily compute the quotient groups $E^2_{p,q} = \operatorname{Ker} d^1_{p,q} / \operatorname{Im} d^1_{p-1,q}$. But then the second differential $d^2 : E^2_{p,q} \to E^1_{p+2,q-1}$ is in principle not known and some extra information is needed in order to determine the higher levels of the spectral sequence (as explained in Remark 2).

In the following two sections, a deep study of the fibrations in the Bousfield-Kan tower is done. Making use of a new *effective homotopy* theory and a combinatorial result named the *Cradle Theorem*, we will determine (in a constructive way) the homotopy groups of the total spaces $Y_n = \operatorname{Tot}_n \mathcal{R}X$ and the fibers $F_n = \operatorname{Func}_*(S^n, R^{n+1}X \cap \operatorname{Ker} \eta^0 \cap \cdots \cap \operatorname{Ker} \eta^{n-1})$. As explained before, once the groups $\pi_*(Y_n)$ and $\pi_*(F_n)$ are known with the corresponding generators (and when they are finitely generated Abelian groups), some elementary operations on matrices will make it possible to determine all the groups $E^r_{p,q}$ and the differential maps $d^r_{p,q}$ in the spectral sequence associated with the tower of fibrations, producing in this way the desired algorithm computing all levels of the Bousfield-Kan spectral sequence.

4 Constructive Kan property of Bousfield-Kan fibrations

In order to develop an algorithm computing the different levels of the Bousfield-Kan spectral sequence, it is necessary to prove first that the fibrations in the Bousfield-Kan tower are *constructive* Kan fibrations (cf. Definition 7).

We recall from the previous section that the Bousfield-Kan fibrations are morphisms $f_p : \operatorname{Tot}_p \mathcal{R}X \to \operatorname{Tot}_{p-1} \mathcal{R}X$, where $\operatorname{Tot}_p \mathcal{R}X$ is the simplicial set

$$\operatorname{Tot}_p \mathcal{R}X = \operatorname{Func}(\operatorname{sk}_p \Delta, \mathcal{R}X) \quad \text{for } p \geq -1,$$

which are induced by the inclusions $\operatorname{sk}_{p-1} \Delta \subset \operatorname{sk}_p \Delta$. Although the morphisms f_p are known to be Kan fibrations [1, Ch.X-6.1], this property has not been proved in a constructive way.

Theorem 23 *Let X be a 1-reduced simplicial set. The fibrations $f_p : \operatorname{Tot}_p \mathcal{R}X \to \operatorname{Tot}_{p-1} \mathcal{R}X$ in the Bousfield-Kan tower are constructive Kan fibrations.*

The proof of this theorem is not trivial. It will be based on two results of very different nature, named the *Epimorphism Theorem* and the *Cradle Theorem*, explained in the following subsections.

4.1 The Epimorphism Theorem

Let us consider a general cosimplicial space G where all the homogeneous parts G^p are Abelian simplicial groups. These groups are in particular connected by coface and codegeneracy operators ∂^i and η^i as explained before. We assume all these operators are group morphisms, except maybe ∂^0, so that G is *grouplike*, see Definition 15. For each $p \geq 0$, the *matching space* $M^pG \subset (G^p)^{p+1}$ consists of the $(p+1)$-tuples $a = (a_0, \ldots, a_p) \in G^p \times \cdots \times G^p$ satisfying the relation $\eta^i a_j = \eta^j a_{i+1}$ if $j \leq i$. Think a $(p+1)$-tuple $a \in M^pG$ is a hypothetical image of some element $b \in G^{p+1}$ by the multiple map $\eta = (\eta^0, \ldots, \eta^p)$; if so, the compatibility condition between the components a_i and the codegeneracies must be satisfied.

The following theorem can be deduced from the proof of Proposition 4.9 in [1, Ch. X]. We follow the same scheme but here more details are included trying to help the reader.

Theorem 24 *(Epimorphism theorem) The map:*

$$\eta = (\eta^0, \ldots, \eta^p) : G^{p+1} \to M^pG$$

is surjective and admits a group morphism section.

Proof If $a = (0, ..., a_i, ..., a_p) \in M^pG$ has its first i components null, it is elementary to see, using the commutation relations between cofaces and codegeneracies, that $a - \eta\partial^{i+1}a_i$ has its first $i+1$ components null. A candidate $\sigma : M^pG \to G^{p+1}$ is said having grade i if for every $a \in M^pG$, $a - \eta\sigma a$ has its first i components null. Starting from $\sigma_0 = 0$, grade 0, we can recursively define $\sigma_{i+1}a = \sigma_i a + \partial^{i+1}(a - \eta\sigma_i a)_i$, grade $i+1$, up to σ_{p+1}, grade $p+1$, that is, which is the looked-for section. □

In particular the dangerous coface ∂_0 is never used, so that σ_{p+1} is a group morphism. Finally, $G^{p+1} = \operatorname{Ker}\eta \oplus \operatorname{Im}\sigma_{p+1}$.

It is a variant of the normalization theorem.

4.2 The Cradle Theorem

The second result we are going to use in the proof of Theorem 23 is the *Cradle Theorem*, a combinatorial result based on the notion of *Discrete Vector Field*, which is an essential component of Forman's Discrete Morse Theory [18]. In order to introduce the Cradle Theorem, we need to present first some preliminary definitions.

Definition 25 An *elementary W-contraction* (also known as *elementary collapse*) is a pair (X,A) of simplicial sets, satisfying the following conditions:

(1) The component A is a *simplicial subset* of the simplicial set X.

(2) The difference $X - A$ is made of exactly *two* non-degenerate simplices $\tau \in X_q$ and $\sigma \in X_{q-1}$, the second one σ being a *face* of the first one τ.

(3) The incidence relation $\sigma = \partial_k \tau$ holds for a *unique* index $k \in 0 \ldots q$.

It is then said A is obtained from X by an elementary W-contraction, and X is obtained from A by an elementary W-extension.

If the condition 3 is not satisfied, the homotopy types of A and X could be different.

Definition 26 A *W-contraction* (or *collapse*) is a pair (X,A) of simplicial sets satisfying the following conditions:

(1) The component A is a *simplicial subset* of the simplicial set X.

(2) There exists a sequence $(A_i)_{0 \leq i \leq m}$ with:

    (a) $A_0 = A$ and $A_m = X$.

    (b) For every $0 < i \leq m$, the pair (A_i, A_{i-1}) is an elementary W-contraction.

In other words, a W-contraction is a finite sequence of elementary W-contractions. If (X,A) is a W-contraction, then a topological contraction $X \to A$ can be defined.

'W' stands for J.H.C. Whitehead, who undertook [19] a systematic study of the notion of *simple homotopy type*, defining two simplicial objects X and Y as having the same simple homotopy type if they are equivalent modulo the equivalence relation generated by the elementary W-contractions and W-extensions.

A W-contraction can be seen as a finitary version of *anodyne extension* (see [20]).

Definition 27 Let (X,A) be a W-contraction. A *description by a filling sequence* of this property is an ordering $\phi = (\sigma_1, \sigma_2, \ldots, \sigma_{2r-1}, \sigma_{2r})$ of all non-degenerate simplices of the difference $X - A$ satisfying the following properties. Let $A_0 = A$ and $A_i = A_{i-1} \cup \sigma_i$ for $1 \leq i \leq 2r$. Then:

(1) Every face of σ_i is in A_{i-1}.

(2) The simplex σ_{2i-1} is a face of the simplex σ_{2i}, so that the pair (A_{2i}, A_{2i-2}) is an elementary W-contraction.

(3) $A_{2r} = X$.

The list $(\sigma_1, \sigma_2, \ldots, \sigma_{2r-1}, \sigma_{2r})$ is called a *W-list*.

Such a description is a particular case of Forman's *Discrete Vector Field* [18]. In our case the vector field is $V = \{(\sigma_{2i-1}, \sigma_{2i})_{0<i\leq r}\}$.

Proposition 28 *Let ϕ be a filling sequence describing the W-contraction (X,A). Then, if Y is a constructive Kan simplicial set and $f : A \to Y$ is a simplicial morphism, the filling sequence ϕ canonically defines a simplicial extension of f to $f' : X \to Y$.*

Proof The W-list $\phi = (\sigma_1, \sigma_2, \ldots, \sigma_{2r-1}, \sigma_{2r})$ produces a sequence of elementary contractions (A_{2i}, A_{2i-2}) for $0 < i \leq r$, the difference $A_{2i} - A_{2i-2}$ being made of the simplices σ_{2i} and σ_{2i-1} which satisfy $\partial_k \sigma_{2i} = \sigma_{2i-1}$ for a unique k. Supposing that f' is defined on A_{2i-2}, then we consider the compatible simplices $f'(\partial_j \sigma_{2i}) \in Y$ for $j \neq k$ and we define $f'(\sigma_{2i})$ by applying the constructive Kan property of Y and $f'(\sigma_{2i-1}) := \partial_k f'(\sigma_{2i})$. Starting with $f'|_{A_0} = f$, we define recursively f' over all elementary contractions (A_{2i}, A_{2i-2}); for the last one $A_{2r} = X$, so that we obtain the extension $f' : X \to Y$. □

Definition 29 Let p, q be two natural numbers. The prism $\Delta^{p,q}$ is the simplicial set $\Delta^{p,q} := \Delta^p \times \Delta^q$.

We have to define the cradle $C^{p,q}$, a simplicial subcomplex of the prism $\Delta^{p,q}$.

Definition 30 The q-hat Λ^q is the subcomplex $\Lambda^q \subset \Delta^q$ made of all the faces of Δ^q except the 0-face.

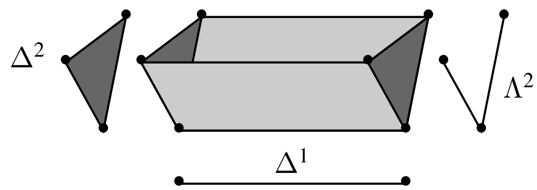
Let us observe that the hat Λ^q is the union of q simplices of dimension $q-1$; adding the missing face $\partial_0 \Delta^q$ would produce the boundary $\partial \Delta^q$ of the q-simplex.

Generalizing the previous definition, one can define $\Lambda^q_j \subset \Delta^q$ for $0 \leq j \leq q$ as the subcomplex made of all the faces of Δ^q except the j-face. In particular, $\Lambda^q_0 = \Lambda^q$.

Definition 31 The (p,q)-cradle is the simplicial subcomplex $C^{p,q} \subset \Delta^{p,q}$ defined by:

$$C^{p,q} = (\Delta^p \times \Lambda^q) \cup (\partial \Delta^p \times \Delta^q)$$

This designation *cradle*, due to Julio Rubio, is inspired by the particular case $p = 1$ and $q = 2$.



Theorem 32 *(Cradle Theorem [21]) Let $p, q \in \mathbb{N}$ and $0 \leq j \leq q$. The pair $(X, A) = (\Delta^{p,q}, C_j^{p,q})$ is a W-contraction.*

Similar W-contractions $(\Delta^{p,q}, C_j^{p,q})$ can be obtained for $C_j^{p,q} := (\Delta^p \times \Lambda_j^q) \cup (\partial\Delta^p \times \Delta^q)$, for $0 \leq j \leq q$.

It is obvious the cradle $C_j^{p,q}$ is topologically a strong deformation retract of the prism $\Delta^{p,q}$. The combinatorial version of this observation required here, not amazing, is more difficult than expected; the proof given in [21] obtains the desired contraction thanks to an appropriate discrete vector field. It happens this discrete vector field gives a new interesting understanding of the Eilenberg-Zilber theorem, finally leading to the "right" proof of the old Eilenberg-MacLane conjecture about the correspondence between Classifying Space and Bar constructions (see [22]). In [20], a different constructive proof of the Cradle Theorem is stated for the particular case of $p = 1$. Following the same ideas, a different constructive proof could maybe be given also for the Cradle Theorem.

4.3 Proof of the constructive Kan property of the Bousfield-Kan fibrations

We can finally present the proof of Theorem 23, which claims that the maps $f_p : \operatorname{Tot}_p \mathcal{R}X \to \operatorname{Tot}_{p-1} \mathcal{R}X$ involved in the definition of the Bousfield-Kan spectral sequence are constructive Kan fibrations.

Proof Let us recall that a map $f : E \to B$ is said to be a *constructive Kan fibration* if an algorithm σ_f is provided such that given a dimension q, and index k, a list of q $(q-1)$-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_q$ of E which satisfy the compatibility condition $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j, i \neq k$ and $j \neq k$, and a q-simplex y of B such that $\partial_i y = f(x_i)$ for $i \neq k$, then σ_f *returns* a q-simplex x of E such that $\partial_i x = x_i$ for $i \neq k$ and $f(x) = y$.

In our case, given a dimension q and an index k, a list of q $(q-1)$-simplices of $E = \mathrm{Tot}_p\,\mathcal{R}X$ $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_q$ consists of a list of morphisms of cosimplicial spaces $x_i = \alpha_i : \mathrm{sk}_p\,\Delta \times \Delta^{q-1} \to \mathcal{R}X$, and a q-simplex y of $B = \mathrm{Tot}_{p-1}\,\mathcal{R}X$ is a cosimplicial morphism $y = \beta : \mathrm{sk}_{p-1}\,\Delta \times \Delta^q \to \mathcal{R}X$. We must provide a q-simplex x of $E = \mathrm{Tot}_p\,\mathcal{R}X$, that is, a map $x = \gamma : \mathrm{sk}_p\,\Delta \times \Delta^q \to \mathcal{R}X$, such that $\partial_i \gamma = \alpha_i$ for $i \neq k$ and $f_p(\gamma) = \beta$.

The last condition $f_p(\gamma) = \beta$ implies that the definition of γ on the columns $\mathcal{X}^0, \ldots, \mathcal{X}^{p-1}$ of the cosimplicial space $\mathcal{X} := \mathrm{sk}_p\,\Delta \times \Delta^q$ coincides with β. For the columns $\mathcal{X}^s$ for $s > p$, the definition of γ is deduced from the column $\mathcal{X}^p$ and the cofaces ∂^i, and then it only remains to define γ over the column $\mathcal{X}^p = \mathrm{sk}_p\,\Delta^p \times \Delta^q = \Delta^p \times \Delta^q$. In other words, we have to define $\gamma^p : \Delta^{p,q} \to \mathcal{R}X^p = R^{p+1}X$.

Taking account of the compatibility conditions between the α_i's, this amounts to giving a cosimplicial map $\alpha : \mathrm{sk}_p\,\Delta \times \Lambda^q_k \to \mathcal{R}X$. In particular, for the p-column we have $\alpha^p : \Delta^p \times \Lambda^q_k \to R^{p+1}X$. Similarly, the q-simplex y of $B = \mathrm{Tot}_{p-1}\,\mathcal{R}X$ is a cosimplicial morphism $y = \beta : \mathrm{sk}_{p-1}\,\Delta \times \Delta^q \to \mathcal{R}X$, which for codimension p provides $\beta^p : \mathrm{sk}_{p-1}\,\Delta^p \times \Delta^q = \partial\Delta^p \times \Delta^q \to R^{p+1}X$. The union of the domains of α^p and β^p is the cradle $C^{p,q}_k$, and both maps are compatible on the common domain due to the condition $\partial_i y = f(x_i)$. This defines a unique map $\alpha^p\beta^p : C^{p,q}_k \to R^{p+1}X$; we have to extend this map into a map $\gamma^p : \Delta^{p,q} \to R^{p+1}X$ which will provide the lifting y looked for compatible with the x_i's. But let us remark that not every γ^p extending $\alpha^p\beta^p$ is correct: the desired γ must be compatible with the codegeneracy maps η^i. The compatibility with η^i's is ensured by applying Proposition 28 and Theorem 24 as follows.

We consider the map $\eta : H' \to H$ introduced in Subsection 4.1 and the codimension $p-1$, that is, for each integer q one has $H \subset (\mathcal{R}X^{p-1}_q)^p = (R^p X_q)^p$ is the set of p-tuples $a = (a_0, \ldots, a_{p-1}) \in \mathcal{R}X^{p-1}_q \times \cdots \times \mathcal{R}X^{p-1}_q$ satisfying the relation $\eta^i a_j = \eta^j a_{i+1}$ if $j \leq i$, and $H' = \mathcal{R}X^p_q = R^{p+1}X_q$. The composition

$$C^{p,q}_k \xrightarrow{\alpha^p\beta^p} R^{p+1}X \xrightarrow{\eta} H$$

is a simplicial morphism and H is a simplicial group and therefore satisfies the Kan property, so that applying Proposition 28 we can obtain an extension of this morphism to a map $f' : \Delta^{p,q} \to H$. Then we apply the section $\sigma : H \to H'$ of Theorem 24 to construct a map $\gamma^p : \Delta^{p,q} \to R^{p+1}X$, which is an extension of $\alpha^p\beta^p : C^{p,q}_k \to R^{p+1}X$ and is compatible with the codegeneracy maps η^i (because the tuples $a \in H$ satisfy the relation $\eta^i a_j = \eta^j a_{i+1}$ if $j \leq i$). This proves the constructive Kan property of the fibrations in the Bousfield-Kan tower. □

5 Effective homotopy in the Bousfield-Kan tower of fibrations

Let us recall that our goal consists in developing an algorithm computing the different stages of the Bousfield-Kan spectral sequence, and for this task we need to determine (in a constructive way, that is, with the corresponding generators) the different homotopy groups of the spaces $Y_n = \mathrm{Tot}_n \mathcal{R}X$ and $F_n = \mathrm{Func}_*(S^n, R^{n+1}X \cap \mathrm{Ker}\,\eta^0 \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1})$ appearing in the Bousfield-Kan tower of fibrations.

5.1 Effective homotopy theory

The computation of homotopy groups is one of the most challenging problems in Algebraic Topology. Although several theoretical methods have been designed trying to determine homotopy groups of spaces, most of them are not constructive and cannot be directly implemented in a computer and the only available computer programs cannot be applied in all situations.

The *effective homotopy* method [23] was designed trying to compute homotopy groups of simplicial sets in a constructive way. It is based on the ideas of the effective homology technique [9, 7], implemented in the Kenzo system [24], which makes it possible to determine homology groups of complicated spaces and has obtained some results (for example homology groups of iterated loop spaces of a loop space modified by a cell attachment, components of complex Postnikov towers, etc.) which were not known before. Moreover, Kenzo can also compute some homotopy groups and has allowed to detect an error in a theorem published in [25] (see [26] for details on these calculations).

The Kan simplicial sets K which will be considered in this paper will be *connected* simplicial sets, that is to say, such that $\pi_0(K)$ has only one homotopy class, $\pi_0(K) = \{\star\}$. Furthermore, since we aim to work with the groups $\pi_*(K)$ in a constructive way, we only consider Kan simplicial sets whose homotopy groups $\pi_*(K)$ are Abelian groups of finite type.

The main notion of the effective homotopy method is the following definition.

Definition 33 The *effective homotopy* of a constructive Kan simplicial set K is a graded 4-tuple $(\pi_q, f_q, g_q, h_q)_{q\geq 1}$ where:

- The component π_q is a standard presentation of a finitely generated Abelian group (that is to say, each π_q is a direct sum of several copies of the infinite cyclic group $\mathbb{Z}$ and some finite primary cyclic groups $\mathbb{Z}_{p_i^q}$, $\pi_q = \mathbb{Z}^{\alpha_q} \oplus \mathbb{Z}_{p_1^q}^{\beta_1^q} \oplus \cdots \oplus \mathbb{Z}_{p_r^q}^{\beta_r^q}$. The component π_q is therefore a well defined Abelian group of finite type in some canonical form, inside which the usual computations can be done). As we will see later, this group will be isomorphic to the desired homotopy group $\pi_q(K) = S_q(K)/(\sim)$.
- The component g_q is an algorithm $g_q : \pi_q \to S_q(K)$ giving for every "abstract" homotopy class $a \in \pi_q$ a sphere $x = g_q(a) \in S_q(K)$ representing this homotopy class.
- The component f_q is an algorithm $f_q : S_q(K) \to \pi_q$ computing for every sphere $x \in S_q(K)$ "its" homotopy class $a = f_q(x) \in \pi_q$. The algorithm f_q must induce an isomorphism $f_q : S_q(K)/(\sim) \xrightarrow{\cong} \pi_q$ and the composition $f_q g_q$ must be the identity of π_q.
- The component h_q is an algorithm $h_q : \mathrm{Ker} f_q \to K_{q+1}$ satisfying $\partial_i h_q = \star$ for all $0 \leq i \leq q$ and $\partial_{q+1} h_q = \mathrm{Id}_{\mathrm{Ker} f_q}$. This algorithm produces a *certificate* for a sphere $x \in S_q(K)$ claimed having a null homotopy class by the algorithm f_q.

We can also say that the graded 4-tuple $(\pi_q, f_q, g_q, h_q)_{q \geq 1}$ provides *a solution for the homotopical problem* of K.

The problem now is how one can determine the effective homotopy of a given Kan simplicial set K. As done in the effective homology framework [16], we will start with some spaces whose effective homotopy can be directly determined (for example, Eilenberg-MacLane spaces $K(\pi, n)$'s for finitely generated Abelian groups π and $n \geq 1$, see [23] for details), and then different constructors of Algebraic Topology (for instance, fibrations) should produce new spaces with effective homotopy. The following result allows one to compute the effective homotopy of the total space of a constructive Kan fibration if the base and fiber spaces are objects with effective homotopy.

Theorem 34 *[23] An algorithm can be written down:*

- **Input:**
  - *A constructive Kan fibration $p : E \to B$ where B is a constructive Kan complex (which implies the fiber F and E are also constructive Kan simplicial sets), and F or B is simply connected.*
  - *Effective homotopies for the simplicial sets F and B.*

- **Output:** *An effective homotopy for the Kan simplicial set E.*

Let us emphasize here that in general, given a fibration $F \hookrightarrow E \xrightarrow{p} B$, it is not possible to determine the homotopy groups of the total space, $\pi_*(E)$, from the groups $\pi_*(F)$ and $\pi_*(B)$. To illustrate this problem it suffices to consider a trivial fibration $S^1 \hookrightarrow S^1 \times S^2 \to S^2$ and the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$; both fibrations have the same base and fiber spaces but the homotopy groups of the total spaces $S^1 \times S^2$ and S^3 are different. However, if the three spaces involved in the fibration are constructive simplicial sets, f is a constructive fibration and both F and B are provided with effective homotopy, our algorithm determines the groups $\pi_*(E)$.

If we want to apply our algorithm computing the effective homotopy of a fibration to the fibrations $f_n : Y_n \to Y_{n-1}$ in the Bousfield-Kan tower, we need the fibers F_n and the first space Y_0 to be objects with effective homotopy and the fibrations f_n to satisfy the constructive Kan property.

5.2 Effective homotopy of the first space

The first space in the tower of fibrations of Bousfield-Kan is $Y_0 = RX$, the simplicial Abelian group freely generated by the simplices of X; it is in particular a *constructive* Kan complex since it is a simplicial Abelian group (see [12] for the explicit construction of the Kan property in that case). Moreover, it is well-known that, given X a pointed simplicial set, there exists a canonical isomorphism $\pi_*(RX) \cong \widetilde{H}_*(X;\mathbb{Z})$ where $\widetilde{H}_*(X;\mathbb{Z})$ denotes the reduced homology groups of X with coefficients in $\mathbb{Z}$.

Let us observe that, if X is a finite simplicial set (as for instance one of the spheres S^p), its homology groups $\widetilde{H}_*(X;\mathbb{Z})$ (with generators) can be elementarily computed and therefore it is not difficult to construct the graded 4-tuple $(\pi_q, f_q, g_q, h_q)_{q\geq 1}$ defining the effective homotopy of RX. In a more general situation, if the simplicial set X has *effective homology* (a construction similar to the effective homotopy, for the determination of homology groups of chain complexes, see [9] or [16] for details), thanks to the isomorphism $\pi_*(RX) \cong \widetilde{H}_*(X;\mathbb{Z})$ it is also easy to determine the effective homotopy of the simplicial Abelian group RX. Therefore, if X is a simplicial set with effective homology (which includes the particular case of X being a simplicial set of finite type), then $Y_0 = RX$ has effective homotopy.

Proposition 35 *Let X be a simplicial set with effective homology. Then the first space in the Bousfield-Kan tower of fibrations, $Y_0 = RX$, is an object with effective homotopy.*

5.3 Effective homotopy of the fibers

Let us study now the fibers F_n in the Bousfield-Kan fibrations, which are the pointed function spaces

$$F_n = \mathrm{Func}_*(S^n, R^{n+1}X \cap \mathrm{Ker}\,\eta^0 \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1})$$

The spaces F_n are simplicial Abelian groups and therefore they are in particular constructive Kan simplicial sets. The simplicial functor $\mathrm{Func}_*(S^n, -)$ can be seen as a model for the topological iterated loop space Ω^n, and in particular it is well-known that it moves forward the homotopy groups of effective Kan simplicial sets K, that is, $\pi_q(\mathrm{Func}_*(S^n, K)) \cong \pi_{q+n}(K)$. However, the constructiveness of this isomorphism is not obvious. A previous paper [21] constructs an explicit version of this isomorphism, to be understood as an isomorphism between the effective homotopy groups of $\mathrm{Func}_*(S^n, K)$ and K: given a q-sphere of $\mathrm{Func}_*(S^n, K)$, the isomorphism constructs a $(q+n)$-sphere of K, etc. The proof is divided into several lemmas which use the effective homotopy theory and require some applications of our Cradle Theorem, showing the usefulness of this combinatorial result in a different context.

Using the (explicit) isomorphism $\pi_q(\mathrm{Func}_*(S^n, K)) \cong \pi_{q+n}(K)$ one can deduce that the homotopy groups of F_n are:

$$\begin{aligned}
\pi_q(F_n) &= \pi_q(\mathrm{Func}_*(S^n, R^{n+1}X \cap \mathrm{Ker}\,\eta^0 \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1})) \\
&\cong \pi_{q+n}(R^{n+1}X \cap \mathrm{Ker}\,\eta^0 \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1}) \\
&\cong \pi_{q+n}(R^{n+1}X) \cap \mathrm{Ker}\,\eta^0 \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1}
\end{aligned}$$

where the codegeneracy maps on the last term of the equation are the corresponding maps induced on the homotopy groups, $\eta^i : \pi_*(R^{n+1}X) \to \pi_*(R^nX)$.

Let us also observe that the second relation $\pi_{q+n}(R^{n+1}X \cap \mathrm{Ker}\,\eta^0 \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1}) \cong \pi_{q+n}(R^{n+1}X) \cap \mathrm{Ker}\,\eta^0 \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1}$ is easily made explicit thanks to the fact of $R^{n+1}X$ being a simplicial Abelian group. For each $0 \leq i \leq n-1$, one can define inverse group morphisms $\phi : \pi_{q+n}(R^{n+1}X \cap \mathrm{Ker}\,\eta^i \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1}) \to \pi_{q+n}(R^{n+1}X \cap \mathrm{Ker}\,\eta^{i+1} \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1}) \cap \mathrm{Ker}\,\eta^i$ given by $\phi([x]) = [x]$ and $\psi : \pi_{q+n}(R^{n+1}X \cap \mathrm{Ker}\,\eta^{i+1} \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1}) \cap \mathrm{Ker}\,\eta^i \to \pi_{q+n}(R^{n+1}X \cap \mathrm{Ker}\,\eta^i \cap \cdots \cap \mathrm{Ker}\,\eta^{n-1})$ given by $\psi([x]) = [x - \partial^i \eta^i x]$. In this way, the composition of the two relations in the previous equation is an explicit isomorphism.

We want to compute now the effective homotopy of F_n. We recall first that $R^{n+1}X$ satisfies $\pi_*(R^{n+1}X) \cong \widetilde{H}_*(R^nX)$. For $n \geq 1$, R^nX is an infinite simplicial set, so that in principle it is not easy to compute its (reduced) homology groups. However, in [14] we developed an algorithm which determines the effective homology of RX from the effective homology of the simplicial set X, supposing that X is 1-reduced. This algorithm can be iterated n times computing in this way the effective homology of R^nX, which provides us the desired effective homotopy of $R^{n+1}X$ whenever X is a 1-reduced simplicial set with effective homology. The groups $\pi_{q+n}(R^{n+1}X)$ (with the generators) can then be computed and are finitely generated groups, so that the kernels $\operatorname{Ker}\eta^i$ can be easily determined by means of some simple matrix operations. We obtain in this way the effective homotopy of the fibers F_n.

Proposition 36 *An algorithm can be written down:*

- **Input:**
  - *A 1-reduced simplicial set X with effective homology.*
  - *An integer $n \geq 0$.*
- **Output:** *An effective homotopy for the fiber* $F_n = \operatorname{Func}_*(S^n, R^{n+1}X \cap \operatorname{Ker}\eta^0 \cap \cdots \cap \operatorname{Ker}\eta^{n-1})$.

5.4 Effective homotopy of the total spaces

In order to determine the effective homotopy of the total spaces $Y_n = \operatorname{Tot}_n \mathcal{R}X$ in the Bousfield-Kan tower of fibrations, we apply Theorem 34 over the different fibrations in an iterative way:

(1) The first base space $Y_0 = RX$ has effective homotopy (Proposition 35).

(2) The first fiber $F_1 = \operatorname{Func}_*(S^1, R^2X \cap \operatorname{Ker}\eta^0)$ has effective homotopy (Proposition 36).

(3) The first fibration $f_1 : Y_1 \to Y_0$ is a constructive Kan fibration (Theorem 23).

(4) Applying Theorem 34, we deduce that our total space Y_1 has effective homotopy.

(5) Now Y_1 is also the base space of the second fibration $f_2 : Y_2 \to Y_1$.

(6) The second fibre F_2 has also effective homotopy (Proposition 36).

(7) The second fibration f_2 is a constructive Kan fibration (Theorem 23).

(8) We deduce from Theorem 34 that Y_2 has effective homotopy. Again this is the base of a new fibration $f_3 : Y_3 \to Y_2$, with fiber F_3.

(9) We continue the same process for all fibrations in the tower.

In this way we obtain iteratively the effective homotopy of all spaces Y_n in the tower of fibrations. In particular, the effective homotopy of Y_n gives us the homotopy groups $\pi_q(Y_n)$ (and their generators).

Theorem 37 *An algorithm can be written down:*

- **Input:**
  - { *A* 1*-reduced simplicial set* X *with effective homology.*
  - { *An integer* $n \geq 0$.
- **Output:** *An effective homotopy for the space* $Y_n = \mathrm{Tot}_n \mathcal{R}X$.

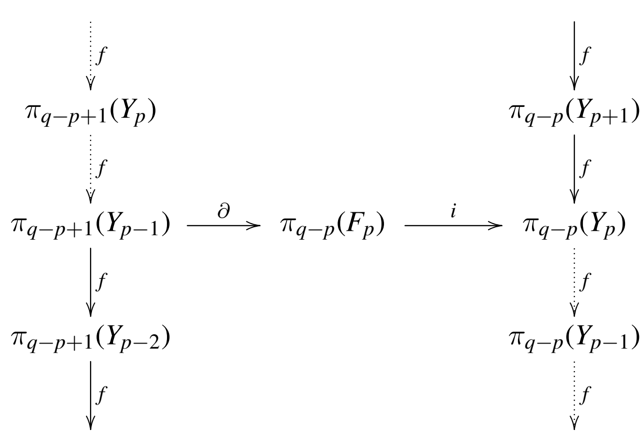
The effective homotopy of the different elements of the tower of fibrations will be used in the following sections to determine all levels of the Bousfield-Kan spectral sequence and to construct an algorithm computing the homotopy groups of a simplicial set K.

6 An algorithm computing the Bousfield-Kan spectral sequence

As seen in Section 3, the spectral sequence associated with a tower of fibrations $(Y_n, f_n)_{n \geq 0}$ is given by the formula:

$$E^r_{p,q} = \frac{i^{-1}(\mathrm{Im} f^{r-1})}{\partial(\mathrm{Ker} f^{r-1})} \quad \text{for } q \geq p$$

where $i^{-1}(\mathrm{Im} f^{r-1})$ and $\partial(\mathrm{Ker} f^{r-1})$ are subgroups of $\pi_{q-p}(F_p)$ obtained from the diagram:



(3)

The differential maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p+r,q+r-1}$ are induced by the composition:

$$\pi_{q-p}(F_p) \xrightarrow{i} \operatorname{Im} f^{r-1} \subseteq \pi_{q-p}(Y_p) \xrightarrow{(f^{r-1})^{-1}} \pi_{q-p}(Y_{p+r-1}) \xrightarrow{\partial} \pi_{q-p-1}(F_{p+r})$$

It is clear that, if all the homotopy groups $\pi_*(Y_n)$ and $\pi_*(F_n)$ are finitely generated Abelian groups and they are explicitly known through the *effective* homotopy of the spaces Y_n and F_n, then one can determine the groups $E^r_{p,q}$ and the differential maps $d^r_{p,q}$ by means of elementary operations with the integer matrices defining the maps i, f and ∂ of the diagram.

In the case of the Bousfield-Kan tower of fibrations associated with a simplicial set X, we have proved in Section 5 that the spaces $Y_n = \operatorname{Tot}_n \mathcal{R}X$ and $F_n = \operatorname{Func}_*(S^n, R^{n+1}X \cap \operatorname{Ker}\eta^0 \cap \cdots \cap \operatorname{Ker}\eta^{n-1})$ have effective homotopy, which in particular provides the homotopy groups $\pi_*(Y_n)$ and $\pi_*(F_n)$ with the generators. Therefore, we obtain the following algorithm computing the desired groups $E^r_{p,q}$ and the differential maps $d^r_{p,q}$ of the Bousfield-Kan spectral sequence of a simplicial set X.

Theorem 38 *An algorithm can be written down:*

- **Input:** *A* 1*-reduced pointed simplicial set* X *with effective homology.*
- **Output:**
  - { *The groups* $E^r_{p,q}$ *for every* $r \geq 1$ *and* $p,q \in \mathbb{N}$ *of the Bousfield-Kan spectral sequence associated with* X *(with generators).*
  - { *The differential maps* $d^r_{p,q}$ *for all* $p,q \in \mathbb{N}$ *and* $r \geq 1$*.*

This algorithm makes it possible to determine the different stages of the Bousfield-Kan spectral sequence associated with a simplicial set X (supposed to be 1-reduced and with effective homology). The implementation of the corresponding programs can be managed by means of a functional programming language as Common Lisp, even if it involves the representation of complicated (infinite!) structures. The programs will be included in a new module for the Kenzo system [24]; some functions have already been designed but our algorithm is not fully implemented yet.

As seen in Theorem 22, the Bousfield-Kan spectral sequence associated with a 1-reduced simplicial set X is known to converge to the homotopy groups of X. In this way, our algorithm makes it possible to compute the graded part of the natural filtration induced on the homotopy groups (introduced in Theorem 21). As already explained, this information does not provide a general algorithm for computing the desired homotopy groups $\pi_*(X)$ (because of extension problems), but in some particular cases the groups $E^r_{p,q}$ obtained by our algorithm in Theorem 38 could be sufficient to deduce some low dimension homotopy groups.

Let us recall from Section 3.2 that for example, in the case of the 2-sphere S^2, the level E^2 of the Bousfield-Kan spectral sequence allows us to deduce the (well-known) homotopy groups $\pi_i(S^2)$ for $i = 2, 3, 4, 5$. However, in dimension $q - p = 6$ one has three non-zero groups $E^\infty_{2,8} = \mathbb{Z}_3$ and $E^\infty_{3,9} = E^\infty_{4,10} = \mathbb{Z}_2$ and several extensions are possible, so that the spectral sequence does not determine the homotopy group $\pi_6(S^2)$.

In the following section we explain how the desired homotopy groups of a simplicial set X can be *constructively* determined directly from the tower of fibrations which produces the Bousfield-Kan spectral sequence (without determining the groups $E^r_{p,q}$ and solving the possible extension problems), obtaining in this way an algorithm computing homotopy groups of spaces. Moreover, we present an algorithm for computing the natural filtration induced on the homotopy groups by the spectral sequence (and not only the graded part which can be directly deduced from the groups $E^\infty_{p,q}$), which is a more refined invariant than the naked homotopy groups.

7 An algorithm computing the effective homotopy of a space

Let X be a simplicial set, non necessarily satisfying the Kan property.

Definition 39 A *Kan completion* KX of X is a constructive Kan simplicial set provided with an inclusion $X \hookrightarrow KX$.

Remark 40 Several methods can be considered to construct a Kan completion KX for a simplicial set X. For example, we can recursively define KX^n from $KX^0 := X$ and KX^{n+1} from from KX^n as follows. For every $0 \leq k \leq q+1$ and every collection of $q+1$ elements $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{q+1}$ of KX^n_q which satisfy the compatibility condition $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j, i \neq k$, and $j \neq k$, we add a new $(q+1)$-simplex x and a new q-simplex x' to KX^{n+1}, whose faces are deduced from the collection $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{q+1}$ (such that $\partial_i x = x_i$ for $i \neq k$and $\partial_k x = x'$). Then $KX := \cup_n KX^n$ is clearly Kan and can be called the *jigsaw model* of X.

Two Kan completions KX and KX' for a simplicial set X are homotopically equivalent in a canonical way. In particular, the jigsaw model is equivalent to $\operatorname{Tot} \mathcal{R}X$.

We can now generalize the definition of effective homotopy introduced in Section 5.1 for Kan simplicial sets as follows.

Definition 41 Let X be a simplicial set. The *effective homotopy* of X consists in a Kan completion KX, provided with a solution for its homotopical problem, that is, a graded 4-tuple $(\pi_q, f_q, g_q, h_q)_{q\geq 1}$ for KX as in Definition 33.

It is clear that, in particular, the effective homotopy of a space X provides its homotopy groups. In this section, we use the space $\operatorname{Tot}\mathcal{R}X$ as a Kan completion of X for computing its effective homotopy.

We consider the tower of fibrations $(Y_n = \operatorname{Tot}_n \mathcal{R}X, f_n)_{n\geq 0}$ appearing in the construction of the Bousfield-Kan spectral sequence of a simplicial set X. If X is 1-reduced, the homotopy groups of the inverse limit $Y = \operatorname{Tot}\mathcal{R}X = \varprojlim \operatorname{Tot}_n \mathcal{R}X$ are $\pi_m(\operatorname{Tot}\mathcal{R}X) \cong \pi_m(X) \cong \varprojlim \pi_m(\operatorname{Tot}_n \mathcal{R}X)$.

On the other hand, the first level of the Bousfield-Kan spectral sequence is defined (see Theorem 22) as $E^1_{p,q} = \pi_q(R^{p+1}X) \cap \operatorname{Ker}\eta^0 \cap \cdots \cap \operatorname{Ker}\eta^{p-1} = \pi_{q-p}(F_p)$, where $F_p := \operatorname{Func}_*(S^p, R^{p+1}X \cap \operatorname{Ker}\eta^0 \cap \cdots \cap \operatorname{Ker}\eta^{p-1})$ is the fiber of the fibration $f_p : \operatorname{Tot}_p \mathcal{R}X \to \operatorname{Tot}_{p-1}\mathcal{R}X$. In a previous work [14] we have proved that, if the simplicial set X is 1-reduced, the groups $E^1_{p,q} = \pi_{q-p}(F_p)$ satisfy $E^1_{p,q} = 0$ for $q < 2p+2$, which implies $\pi_m(F_n) = 0$ for $n > m-2$ (see [27] for a detailed proof of this result).

We observe then the long exact sequence of homotopy [12] of the fibration f_n:

$$\cdots \xrightarrow{\partial} \pi_m(F_n) \xrightarrow{\operatorname{inc}_*} \pi_m(\operatorname{Tot}_n \mathcal{R}X) \xrightarrow{f_*} \pi_m(\operatorname{Tot}_{n-1}\mathcal{R}X) \xrightarrow{\partial} \pi_{m-1}(F_n) \xrightarrow{\operatorname{inc}_*} \cdots$$

where one can easily deduce that $\pi_m(\operatorname{Tot}_n \mathcal{R}X) \cong \pi_m(\operatorname{Tot}_{n-1}\mathcal{R}X)$ for $n > m-2$. The isomorphism is explicit because f_n is a constructive Kan fibration and the base and the total spaces are objects with effective homotopy (see [23]). This implies

$$\ldots \cong \pi_m(\operatorname{Tot}_n \mathcal{R}X) \cong \pi_m(\operatorname{Tot}_{n-1}\mathcal{R}X) \cong \ldots \cong \pi_m(\operatorname{Tot}_{m-1}\mathcal{R}X) \cong \pi_m(\operatorname{Tot}_{m-2}\mathcal{R}X)$$

and then $\pi_m(X) \cong \varprojlim \pi_m(\operatorname{Tot}_n \mathcal{R}X) \cong \pi_m(\operatorname{Tot}_{m-2}\mathcal{R}X)$. Therefore, if we know the homotopy groups of the spaces $\operatorname{Tot}_n \mathcal{R}X$, the homotopy groups of X can be directly determined as $\pi_m(X) \cong \pi_m(Y_{m-2})$, without using the different components $E^r_{p,q}$ of the spectral sequence. We observe in particular $\pi_2(X) \cong \pi_2(\operatorname{Tot}_0 \mathcal{R}X) = \pi_2(RX) \cong H_2(X)$; it is the Hurewicz theorem for X is 1-reduced.

Let us remark that, thanks to Theorem 37, the spaces $Y_n = \operatorname{Tot}_n \mathcal{R}X$ have effective homotopy; in this way, the isomorphism $\pi_m(Y) \cong \pi_m(Y_{m-2})$ provides the first component

π_q of the effective homotopy of the inverse limit $Y = \mathrm{Tot}\,\mathcal{R}X = \varprojlim \mathrm{Tot}_n\,\mathcal{R}X$. Moreover, it is not difficult to construct the components g, f and h defining the effective homotopy of the inverse limit Y as follows.

A sphere $s \in S_m(Y)$ is a family of spheres $s_i \in S_m(Y_i)$ compatible with the fibrations f_i. Given an "abstract" homotopy class $a \in \pi_m \cong \pi_m(Y) \cong \pi_m(Y_{m-2})$, the effective homotopy of Y_{m-2} provides a sphere $s_{m-2} = g^{Y_{m-2}}(a) \in S_m(Y_{m-2})$. We consider $s_{m-3} = f_{m-2}(s_{m-2})$ and then in a recursive way $s_i = f_{i+1}(s_{i+1})$ for $i < m-2$. To determine the spheres $s_i \in S_m(Y_i)$ for $i > m-2$, it suffices to apply in an iterative way the constructive Kan property of the fibrations f_i. We define $g(a) := (s_i)_{i \geq 0}$.

On the other hand, let $s \equiv (s_i)_{i \geq 0} \in S_m(Y)$. We consider $s_{m-2} \in S_m(Y_{m-2})$ and "its" homotopy class $a = f^{Y_{m-2}}(s_{m-2}) \in \pi_m$ given by the effective homotopy of Y_{m-2}. We define $f(s) := a$.

Finally, let $s \in S_m(Y)$ such that $f(s) = 0 \in \pi_m$. We consider $s_{m-2} \in S_m(Y_{m-2})$. Following the previous definition of the component f, one has $f^{Y_{m-2}}(s_{m-2}) = 0$ and then the component $h^{Y_{m-2}}$ of the effective homotopy of Y_{m-2} provides an $(m+1)$-simplex $z_{m-2} \in Y_{m-2}$ such that $\partial_i z_{m-2} = \star$ for all $0 \leq i \leq m$ and $\partial_{m+1} z_{m-2} = s_{m-2}$. We consider now $z_{m-3} = f_{m-2}(z_{m-2})$ and then in a recursive way $z_i = f_{i+1}(z_{i+1})$ for $i < m-2$. On the other hand, taking into account s_{m-1}, s_{m-2} and z_{m-2}, the constructive Kan property of the fibration provides an $(m+1)$-simplex w of Y_{m-1} such that $f_{m-1}(w) = z_{m-2}$, $\partial_i w = \star$ for $1 \leq i \leq m$, $\partial_{m+1} w = s_{m-1}$ and $\partial_0 w$ is an element in $S_m(F_{m-1})$. Since $\pi_m(F_{m-1}) = 0$, algorithm $h^{F_{m-1}}$ in the effective homotopy of F_{m-1} returns an $(m+1)$-simplex $v \in F_{m-1}$ such that $\partial_i v = \star$ for all $0 \leq i \leq m$ and $\partial_{m+1} v = \partial_0 w$. Applying again the constructive Kan property of the fibration f_{m-1}, one has an $(m+2)$-simplex y of Y_{m-1} with $f_{m-1}(y) = \eta_{m+1} z_{m-2}$, $\partial_0 y = v$, $\partial_i y = \star$ for $1 \leq i \leq m$ and $\partial_{m+2} y = w$. Then we take $z_{m-1} := \partial_{m+1} y$ which satisfies $f_{m-1}(\partial_{m+1} y) = z_{m-2}$, $\partial_i \partial_{m+1} y = \star$ for $0 \leq i \leq m$ and $\partial_{m+1}\partial_{m+1} y = s_{m-1}$. Iterating the process, we build z_i for every $i > m-2$. The element $z = (z_i)_{i \geq 0} \in Y$ is the desired element providing a certificate of the sphere s claimed having a null homotopy class in π_m.

In this way, the effective homotopy of the total space $Y = \mathrm{Tot}\,\mathcal{R}X = \varprojlim \mathrm{Tot}_n\,\mathcal{R}X$ has been determined. Thanks to the canonical morphism $X \hookrightarrow \mathrm{Tot}\,\mathcal{R}X$, we obtain therefore the following algorithm computing the effective homotopy of a simplicial set X. In particular, this makes it possible to compute the homotopy groups $\pi_*(X)$.

Theorem 42 *An algorithm can be written down:*

- **Input:** *A* 1*-reduced simplicial set* X *with effective homology.*

- **Output:** *The effective homotopy of X.*

On the other hand, as stated in Theorem 21, the spectral sequence of a tower of fibrations $(Y_n, f_n)_{n \geq 0}$ induces a filtration on the homotopy groups of the inverse limit Y given by:

$$F_n(\pi_m(Y)) = \mathrm{Ker}(p_n : \pi_m(Y) \to \pi_m(Y_n))$$

In our case, the Bousfield-Kan spectral sequence produces the filtration of $\pi_m(\varprojlim \mathrm{Tot}_n \mathcal{R}X) \cong \pi_m(X)$:

$$F_n(\pi_m(X)) = \mathrm{Ker}(p_n : \pi_m(X) \to \pi_m(\mathrm{Tot}_n \mathcal{R}X))$$

Thanks to Theorem 42 and the isomorphism $\pi_m(X) \cong \pi_m(\mathrm{Tot}_{m-2} \mathcal{R}X)$, one can compute the homotopy groups $\pi_m(X)$ and determine the subgroups $\mathrm{Ker}(p_n : \pi_m(X) \to \pi_m(\mathrm{Tot}_n \mathcal{R}X))$ by means of operations on matrices, producing in this way the following algorithm.

Theorem 43 *An algorithm can be written down:*

- **Input:**
  - { *A 1-reduced simplicial set X with effective homology.*
  - { *Integers $m \geq 2$ and $n \geq 0$.*
- **Output:** *The group $F_n(\pi_m(X))$ corresponding to the natural filtration induced on $\pi_m(X)$ by the Bousfield-Kan spectral sequence.*

Theorem 42 provides a general algorithm computing homotopy groups of (1-reduced) simplicial sets with effective homology, and in particular it makes it possible to determine stable and unstable sphere homotopy groups, which is known to be an interesting problem in Algebraic Topology. Moreover, Theorem 43 makes it possible to determine the natural filtration induced by the Bousfield-Kan spectral sequence, which is a more refined invariant than the naked homotopy groups. As already said, finite type simplicial sets are objects with effective homology so that (if they are 1-reduced) one can also apply our algorithms on them. Moreover, there exist effective homology versions of many topological constructors and this makes it possible to consider a wide variety of complicated (infinite) simplicial sets which have effective homology, and then we can also compute their homotopy groups (and the corresponding filtration).

Although our algorithm computing homotopy groups of spaces is not yet implemented, there is no doubt at all about the feasibility of such a concrete implementation: these new algorithms have the same general style as numerous other algorithms already implemented in the Kenzo program [24], already producing striking results, in particular around complicated loop spaces. However, it is necessary to remark that several algorithms of exponential nature will be present in our calculations, so that one can expect complexity problems which will surely prevent us from computing homotopy groups in high dimensions, and in particular it is clear that our results could not compete with the specific procedures designed for computing homotopy groups of spheres. However, our algorithm can be applied to general spaces, which can make it possible to determine some unknown homotopy groups of complicated simplicial sets.

8 Conclusions and further work

In this paper an algorithm computing the *effective homotopy* of a simplicial set has been explained. The algorithm is based on the Bousfield-Kan spectral sequence, and its main ingredient consists in defining the effective homotopy of the different elements in the tower of fibrations which appears in the definition of the spectral sequence. The algorithm can be applied to 1-reduced simplicial sets X with effective homology, allowing in particular the computation of stable and unstable homotopy groups of spheres. We are also able to compute the natural filtration induced on the homotopy groups by the spectral sequence.

Our algorithms are not yet concretely implemented as computer programs. Although such a concrete implementation can certainly be done using the functional programming language Common Lisp, due to the exponential nature of some of the calculations one can hope the computations will not be too fast and only low dimension homotopy groups will be obtained. In order to improve the complexity of our calculations one of the main components of the pending work consists in writing down a *good* implementation of a *good* algorithm computing the *effective* homology of $K(\mathbb{Z}, n)$, an interesting subject by itself: it is easy to prove the computation of the *effective* homology of a simplicial group $R^{p+1}X$ can be reduced to the same problem for the main Eilenberg-MacLane spaces $K(\mathbb{Z}, n)$, see [14].

References

[1] A. K. Bousfield and D. M. Kan, Homotopy Limits, Completions and Localizations, Lecture Notes in Mathematics, Springer-Verlag, 1972.

[2] J. F. Adams, On the non-existence of elements of Hopf invariant one, Annals of Mathematics 72(1) (1960) 20–104.

[3] D. C. Ravenel, Complex cobordism and stable homotopy groups of spheres, Academic Press, 1986.

[4] S. O. Kochman, Stable homotopy groups of spheres, Lecture Notes in Mathematics, Springer-Verlag, 1990.

[5] S. O. Kochman and M. E. Mahowald, On the computation of stable stems, in: The Cech Centennial: A Conference on Homotopy Theory, AMS Contemporary Mathematics, vol.181, 1995.

[6] E. H. Brown, Finite computability of Postnikov complexes, Annals of Mathematics 65(1) (1957) 1–20.

[7] J. Rubio and F. Sergeraert, Constructive Algebraic Topology, *Bulletin des Sciences Mathématiques* 126(5) (2002) 389–412.

[8] J. McCleary, User's guide to spectral sequences, second edition, Cambridge University Press, 2001.

[9] F. Sergeraert, The computability problem in Algebraic Topology, Advances in Mathematics 104(1) (1994) 1–29.

[10] E. Bishop and D. Bridges, Constructive Analysis, Springer, 1985.

[11] S. MacLane, Homology, Springer, 1963.

[12] J. P. May, Simplicial objects in Algebraic Topology, Van Nostrand Mathematical Studies, University of Chicago Press, 1967.

[13] A. K. Bousfield and D. M. Kan, The homotopy spectral sequence of a space with coefficients in a ring, Topology 11 (1972) 79–106.

[14] A. Romero, Computing the first stages of the Bousfield-Kan spectral sequence, Applicable Algebra in Engineering, Communication and Computing 21 (2010) 227–248.

[15] H. Cartan, Algèbres d'Eilenberg-MacLane et homotopie, Séminaire H. Cartan, École Normal Supérieure, Paris : Exposés 2-11, 1954-55.

[16] J. Rubio and F. Sergeraert, Constructive Homological Algebra and Applications, Unpublished results (2006), http://arxiv.org/abs/1208.3816.

[17] T. Kaczynski, K. Mischaikow, and M. Mrozek, Computational Homology, Applied Mathematical Sciences, Springer, 2004.

[18] R. Forman, Morse theory for cell complexes, Advances in Mathematics 134 (1998) 90–145.

[19] J. Whitehead, Simple homotopy types, American Journal of Mathematics 82 (1960) 1–57.

[20] P. G. Goerss and J. F. Jardine, Simplicial Homotopy Theory, Birkhäuser, 1999.

[21] A. Romero and F. Sergeraert, A combinatorial tool for computing the effective homotopy of iterated loop spaces. To appear in Discrete and Computational Geometry.

[22] A. Romero and F. Sergeraert, Programming before Theorizing, a case study, in: Proceedings International Symposium on Symbolic and Algebraic Computation, ISSAC 2012, ACM, 2012, 289–296.

[23] A. Romero and F. Sergeraert, Effective homotopy of fibrations, Applicable Algebra in Engineering, Communication and Computing 23 (2012) 85–100.

[24] X. Dousson, J. Rubio, F. Sergeraert and Y. Siret, The Kenzo program, Institut Fourier, Grenoble, 1999, http://www-fourier.ujf-grenoble.fr/~sergerar/Kenzo/.

[25] R. Mikhailov and J. Wu, On homotopy groups of the suspended classifying spaces, Algebraic and Geometric Topology 10 (2010) 565–625.

[26] A. Romero and J. Rubio, Homotopy groups of suspended classifying spaces: an experimental approach, Mathematics of computation 82 (2013) 2237–2244.

[27] A. Romero, Effective Homology and Spectral Sequences, Ph.D. Thesis, Universidad de La Rioja, 2007. http://www.unirioja.es/cu/anromero/tesis.pdf

Departamento de Matemáticas y Computación, Universidad de La Rioja, Spain.

Institut Fourier, Université Joseph Fourier, France.

ana.romero@unirioja.es, Francis.Sergeraert@ujf-grenoble.fr