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POSTNIKOV "INVARIANTS" IN 2004

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As yet we are ignorant
of an effective method of computing
the cohomology of a Postnikov complex
from π_n and k^{n+1} [9].

Abstract. The very nature of the so-called Postnikov *invariants* is carefully studied. Two functors, precisely defined, explain the exact nature of the connection between the category of topological spaces and the category of Postnikov towers. On one hand, these functors are in particular *effective* and lead to concrete machine computations through the general machine program Kenzo. On the other hand, the Postnikov "invariants" will be *actual* invariants only when an arithmetical decision problem – currently open – will be solved; it is even possible this problem is undecidable.

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1. Introduction

At the time of writing this paper, the so-called *Postnikov invariants* (or k-invariants) are roughly fifty years old [15]. They are a key component of standard Algebraic Topology. This notion is so important that it is a little amazing to observe some important *gaps* are still present in our working environment around this subject, still more amazing to note these gaps are seldom considered. One of these "gaps" is unfortunately an *error*, widely spread, and easy to state: the terminology "Postnikov *invariants*" is incorrect: any sensible definition of the *invariant* notion leads to the following conclusion: the Postnikov invariants are not… invariants. This is true even in the simply connected case and, to make easier the understanding, we restrict our study to this case.

First, several interesting questions of *computability* are rised by the very notion of Postnikov invariant. It is surprisingly difficult to find references related to this computability problem, as though this problem was unconsciously "hidden" (?) by the topologists. The only significant one found by the authors is the EDM title quotation[1]. In fact there are *two* distinct problems of this sort.

On the one hand, if a simply connected space is presented as a *machine object*, does there exist a general algorithm computing its Postnikov invariants? The authors have designed a general framework for *constructive* Algebraic Topology,

[1]Other possible quotations are welcome.

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giving in particular such a general algorithm [19, 16]. In the text, this process is formalized as a functor $\mathbf{SP} : \mathcal{SS}_{EH}\widetilde{\times}I \rightarrow \mathcal{P}$ where $\mathcal{SS}_{EH}$ is an appropriate category of *computable* topological spaces, and $\mathcal{P}$ is the *Postnikov category.* We will explain later the nature of the factor I, in fact the heart of our subject.

On the other hand, the *converse* problem must be considered. When a Postnikov tower is given, that is, a collection of homotopy groups and *relevant* Postnikov invariants, how to construct the corresponding topological space? The computability problem stated in the title quotation is a (small) part of this converse problem. Again, our notion of *constructive* Algebraic Topology entirely solves it. The resulting computer program Kenzo [8] allows us to give a simple concrete illustration. In fact it will be explained it is not possible to properly *state* this problem... without having its solution! Again, the strange situation, to our knowledge, has not yet been considered by topologists. Our solution for the converse problem will be formalized as a functor $\mathbf{PS} : \mathcal{P} \rightarrow \mathcal{SS}_{EH}$.

There is a lack of symmetry between the functors $\mathbf{SP} : \mathcal{SS}_{EH}\widetilde{\times}I \rightarrow \mathcal{P}$ and $\mathbf{PS} : \mathcal{P} \rightarrow \mathcal{SS}_{EH}$. Instead of our functor $\mathbf{SP} : \mathcal{SS}_{EH}\widetilde{\times}I \rightarrow \mathcal{P}$, a simpler functor $\mathbf{SP} : \mathcal{SS}_{EH} \rightarrow \mathcal{P}$, without the mysterious factor I, is expected, but in the current state of the art, such a functor *is not* available. It is a consequence of the following *open* problem: let $P_1, P_2 \in \mathcal{P}$ be two Postnikov towers; does there exist an algorithm deciding whether $\mathbf{PS}(P_1)$ and $\mathbf{PS}(P_2)$ have the same homotopy type or not? The remaining uncertainty is measured by the factor I. And because of this uncertainty, the so-called Postnikov invariants are not... invariants: the context clearly says they should be invariants of the *homotopy type*, but such a claim is equivalent to a solution of the above decision problem.

It is even possible this decision problem *does not have* any solution; in fact, our Postnikov decision problem can be translated into an arithmetical decision problem, a subproblem of the general tenth Hilbert problem to which Matiyasevich gave a negative answer [12]. If our decision problem had in turn a negative answer, it would be *definitively* impossible to transform the common Postnikov invariants to *actual* invariants.

2. The Postnikov Category and the PS Functor

Defining a functor $\mathbf{PS} : \mathcal{P} \rightarrow \mathcal{SS}_{EH}$ in principle consists in defining the *source* category, here the *Postnikov category* $\mathcal{P}$, the *target* category, the *simplicial set category* $\mathcal{SS}_{EH}$, and *then*, finally, the functor $\mathbf{PS}$ itself. It happens this is not possible in this case: the Postnikov category $\mathcal{P}$ and the functor $\mathbf{PS}$ are *mutually recursive.* More precisely, an object $P \in \mathcal{P}$ is a limit $P = \lim P_n$, every P_n being also an element of $\mathcal{P}$. Let $\mathcal{P}_n$, $n \geq 1$, be the Postnikov towers limited to dimension n. The definition of $\mathcal{P}_{n+1}$ *needs* the partial functor $\mathbf{PS}_n : \mathcal{P}_n \rightarrow \mathcal{SS}_{EH}$ where $\mathbf{PS}_n = \mathbf{PS}|\mathcal{P}_n$ and this is why the definitions of $\mathcal{P}$ and $\mathbf{PS}$ are mutually recursive.

We work only with simply connected spaces, the homotopy (or $\mathbb{Z}$-homology) groups of which being of *finite type.* It is essential, when striving to define *invariants*, to have exactly *one* object for every isomorphism class of groups of this sort, so that we adopt the following definition. No p-adic objects in our environment, which allows us to denote $\mathbb{Z}/d\mathbb{Z}$ by $\mathbb{Z}_d$; in particular $\mathbb{Z}_0 = \mathbb{Z}$.

Definition 1. A *canonical group* (abelian, of finite type) is a product $\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_k}$ where the non-negative integers d_i satisfy the divisibility condition: d_i divides d_{i+1} for $1 \leq i < k$.

Every abelian group of finite type is isomorphic to *exactly one* canonical group, for example the group $\mathbb{Z}^2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{15}$ is isomorphic to the unique canonical group $\mathbb{Z}_{30} \times \mathbb{Z}_{30} \times \mathbb{Z}_0 \times \mathbb{Z}_0$; but such an isomorphism is *not*... canonical; for example, for the previous example, there exists an infinite number of such isomorphisms, and we will see this is the key point preventing us from qualifying the Postnikov invariants as invariants.

Definition 2. The category $\mathcal{SS}_{EH}$ is the category of *simply connected simplicial sets with effective homology* described in [16].

The framework of the present paper does not allow us to give a relatively complex definition of this category. Roughly speaking, an object of this category is a *machine object* coding a (possibly infinite) simply connected simplicial set with *known* homology groups; furthermore a *complete* knowledge of the homology is required: mainly every homology class has a canonical representant cycle, an algorithm computes the homology class of every cycle, and if two cycles c_0 and c_1 are homologous, an algorithm computes a chain C with $\partial C = c_1 - c_0$. For example, it is explained in [17] that $X = \Omega(\Omega(P^\infty(\mathbb{R})/P^3(\mathbb{R})) \cup_4 D^4) \cup_2 D^3$ is an object of $\mathcal{SS}_{EH}$ and the Kenzo program does compute its first homology groups, in the detailed form just briefly sketched. More generally, every "sensible" simply connected space with homology groups of finite type has the homotopy type of an object of $\mathcal{SS}_{EH}$; this statement is precisely stated in [16], the proof is not hard, it is only a repeated application of the so-called *homological perturbation lemma* [4] and the *most detailed* proof is the Kenzo computer program itself [8], a Common Lisp text of about 16,000 lines.

The definitions of the category $\mathcal{P}$ and the functor $\mathbf{PS}$ are *mutually recursive* so that we need a starting point.

Definition 3. The category $\mathcal{P}_1$ has a unique object, the void sequence $()_{2 \leq n \leq 1}$, the trivial Postnikov tower, and the functor $\mathbf{PS}_1$ associates to this unique object the trivial element $* \in \mathcal{SS}_{EH}$ with only a base point.

The next definitions of the category $\mathcal{P}_n$ and the functor $\mathbf{PS}_n$ assume the category $\mathcal{P}_{n-1}$ *and* the functor $\mathbf{PS}_{n-1} : \mathcal{P}_{n-1} \to \mathcal{SS}_{EH}$ are already available.

Definition 4. An object $P_n \in \mathcal{P}_n$ is a sequence $((\pi_m, k_m))_{2 \leq m \leq n}$, where:

- $((\pi_m, k_m))_{2 \leq m \leq n-1}$ is an element $P_{n-1} \in \mathcal{P}_{n-1}$;
- The component π_n is a canonical group;
- The component k_n is a cohomology class $k_n \in H^{n+1}(\mathbf{PS}_{n-1}(P_{n-1}), \pi_n)$;

Let us denote $X_{n-1} = \mathbf{PS}_{n-1}(P_{n-1})$. The cohomology class k_n classifies a fibration:

$$K(\pi_n, n) \hookrightarrow K(\pi_n, n) \times_{k_n} X_{n-1} \twoheadrightarrow X_{n-1} \xrightarrow{k_n} K(\pi_n, n+1) = BK(\pi_n, n).$$

- Then the functor $\mathbf{PS}_n$ associates to $P_n = ((\pi_m, k_m))_{2\leq m\leq n} \in \mathcal{P}_n$ a version *with effective homology* $X_n = \mathbf{PS}_n(P_n)$ of the total space $K(\pi_n, n) \times_{k_n} X_{n-1}$.

In particular, our version *with effective homology* of the Serre spectral sequence and our versions *with effective homology* of the Eilenberg–MacLane spaces $K(\pi, n)$ allow us to construct a version also with effective homology of the total space $K(\pi_n, n) \times_{k_n} X_{n-1}$, here denoted by X_n. We will give a typical small Kenzo demonstration at the end of this section.

A canonical forgetful functor $\mathcal{P}_n \to \mathcal{P}_{n-1}$ is defined by forgetting the last component of $((\pi_m, k_m))_{2\leq m\leq n}$, which allows us to define $\mathcal{P}$ as the projective limit $\mathcal{P} = \varprojlim \mathcal{P}_n$. If X_{n-1} is a simplicial set, then the $(n-1)$-skeletons of X_{n-1} and $K(\pi_n, n) \times_{k_n} X_{n-1}$ are the same (for the standard model of $K(\pi_n, n)$), so that if $P = \varprojlim P_n$, then the limit $\mathbf{PS}(P) = \varprojlim \mathbf{PS}_n(P_n)$ is defined also as an object of $\mathcal{SS}_{EH}$. The category $\mathcal{P}$ and the functor $\mathbf{PS} : \mathcal{P} \to \mathcal{SS}_{EH}$ are now *properly* defined.

The homotopy groups π_m's of a Postnikov tower $((\pi_m, k_m))_{2\leq m}$ can be defined *firstly* independently on the k_m's, but k_n can be *properly defined* only when $((\pi_m, k_m))_{2\leq m<n}$ is given *and only if* the functor $\mathbf{PS}_{n-1}$ is available in the environment. In other words, if the problem stated in the epigraph of the present paper cited from Encyclopedic Dictionary of Mathematics (EDM) [9] is not solved, the very notion of a Postnikov tower cannot be made *effective*.

2.1. **Kenzo example.** Let us play the game consisting in constructing the beginning of a Postnikow tower with a $\pi_i = \mathbb{Z}_2$ at each stage and the "simplest" *non-trivial* Postnikov invariant. First $P_1 = ()$ and $X_1 = \mathbf{PS}_1(P_1) = *$. As planned, we choose $\pi_2 = \mathbb{Z}_2$ and $k_2 \in H^3(X_1, \mathbb{Z}_2) = 0$ is necessarily null, no choice. So that we define $P_2 = ((\mathbb{Z}_2, 0))$ and $X_2 = K(\mathbb{Z}_2, 2)$. The Kenzo function `k-z2` can construct this space. We show a copy of the dialog between a Kenzo user and the Lisp machine.

```
> (setf X2 (k-z2 2)) ✠
[K13 Abelian-Simplicial-Group]
```

This dialog goes on as follows. The Lisp *prompt* is the sign '`>`'. The Lisp user enters a Lisp *statement*, here "`(setf X2 (k-z2 2))`". The Maltese cross '✠' signals the end of the statement to be executed, it is added here to help the reader, but it is not visible on the user's screen. When the Lisp statement is finished, Lisp *evaluates* it, the computation time can be a microsecond or a few days or more, depending on the statement to be evaluated, and when the evaluation *terminates*, a Lisp object is *returned*, most often it is the "result" of the computation. Here the `K13` object (Kenzo object #13) is constructed and returned, it is an abelian simplicial group. A Lisp statement "`(setf some-symbol (some-function some-arguments))`" orders Lisp to make the function `some-function` work, using the arguments `some-arguments`; this function creates some object which is *returned* (displayed) *and* assigned to the symbol `some-symbol`; in this way, the created object remains reachable through the symbol locating it.

The $\mathbb{Z}$-homology in dimensions 3 and 4 of X_2 (the arguments 3 and 5 must be understood as defining $3 \leq i < 5$):

```
> (homology X2 3 5) ✠
Homology in dimension 3 :
---done---
Homology in dimension 4 :
Component Z/4Z
---done---
```

to be read $H_3 = 0$ and $H_4 = \mathbb{Z}_4$. The universal coefficient theorem implies $H^4(X_2, \mathbb{Z}_2) = \mathbb{Z}_2$, there is only one non-trivial possible $k_3 \in H^4(X_2, \mathbb{Z}_2)$ and the Kenzo function `chml-clss` (cohomology class) constructs it.

```
> (setf k3 (chml-clss X2 4)) ✠
[K125 Cohomology-Class on K30 of degree 4]
```

The attentive reader can be amazed to see this cohomology class defined on `K30` and not `K13` $= X_2$. The explanation is as follows. Let us consider the *effective homology* of X_2:

```
> (efhm X2) ✠
[K122 Equivalence K13 <= K112 => K30]
```

This is a chain equivalence between the chain complex of the considered space and some *small* chain complex, here the chain complex `K30`. In fact it is a *strong* chain equivalence, made of two *reductions* through the intermediate chain complex `K112` (see [16] for details). So that defining a cohomology class of X_2 is equivalent to defining such a class for `K30`. A *small* chain complex is a free $\mathbb{Z}$-chain complex of finite type in every dimension. The chain complex `K13` of the standard model of $X_2 = K(\mathbb{Z}_2, 2)$ is already of finite type, but the complex `K30` is much smaller. For example, in dimension 6, `K13` has 27,449 generators and `K30` has only 5.

The k_3 class allows us to define the fibration canonically associated:

$$F_3 = \left\{ K(\mathbb{Z}_2, 3) \hookrightarrow K(\mathbb{Z}_2, 3) \times_{k_3} X_2 \twoheadrightarrow X_2 \xrightarrow{k_3} K(\mathbb{Z}_2, 4) \right\}.$$

We have now the Postnikov tower $P_3 = ((\mathbb{Z}_2, 0), (\mathbb{Z}_2, k_3))$ with $X_3 = \mathbf{PS}(P_3) = K(\mathbb{Z}_2, 3) \times_{k_3} X_2$. The Kenzo program can construct our fibration F_3 and its total space X_3.

```
> (setf F3 (z2-whitehead X2 k3)) ✠
[K140 Fibration K13 -> K126]
> (setf X3 (fibration-total F3)) ✠
[K146 Kan-Simplicial-Set]
```

The fibration is modelled as a *twisting operator* $\tau_3 : X_2 \to K(\mathbb{Z}_2, 3)$ which is nothing but an avatar of k_3, and we can verify the target of τ_3 is really $K(\mathbb{Z}_2, 3)$.

```
> (k-z2 3) ✠
[K126 Abelian-Simplicial-Group]
```

We continue to the next stage of our Postnikov tower. We "choose" again $\pi_4 = \mathbb{Z}_2$, but what about the next Postnikov invariant k_4? We must choose some $k_4 \in H^5(X_3, \mathbb{Z}_2)$ so that we face the problem stated in the epigraph from EDM [9]. Fortunately, the Kenzo program *knows* how to compute the *necessary* H^5, the Kenzo program knows a (simple) solution for the EDM problem. In fact it knows the *effective* homology of the fiber space $K(\mathbb{Z}_2, 3)$:

```
> (efhm (k-z2 3)) ✠
[K268 Equivalence K126 <= K258 => K254]
```

In the same way, it knows the *effective* homology of $X_2 = K(\mathbb{Z}_2, 2)$, and the implicitly used *effective homology version* of the Serre spectral sequence, available in Kenzo, determines the effective homology of the twisted product X_3:

```
> (efhm X3) ✠
[K358 Equivalence K146 <= K348 => K344]
```

The chain-complex `K344` is of finite type, its homology groups are computable, and in this way Kenzo can compute the $\mathbb{Z}$-homology groups of X_3.

```
> (homology X3 2 6) ✠
Homology in dimension 2 :
Component Z/2Z
---done---
Homology in dimension 3 :
---done---
Homology in dimension 4 :
Component Z/2Z
---done---
Homology in dimension 5 :
Component Z/4Z
---done---
```

Finally the universal coefficient theorem implies $H^5(X^3, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and there are exactly four ways to add a new stage to our Postnikov tower with $\pi_4 = \mathbb{Z}_2$. Four possible Postnikov *invariants* k_4. In this simple case, rather misleading, it is true such a k_4 is an *invariant* of the homotopy type of the resulting space, but in the general case, we will see the situation is much more complicated; this will be explained in Section 4.2.2.

In such a case the `chml-clss` Kenzo function constructs the cohomology-class "dual" to the generator of $H_5(X_3, \mathbb{Z}) = \mathbb{Z}_4$.

```
> (setf k4 (chml-clss X3 5)) ✠
[K359 Cohomology-Class on K344 of degree 5]
```

and the process can be iterated as before, giving the fibration F_4 associated to k_4, and the total space $X_4 = \mathbf{PS}_4(P_4) = K(\mathbb{Z}_2, 4) \times_{k_4} X_3$ with $P_4 = ((\mathbb{Z}_2, 0), (\mathbb{Z}_2, k_3), (\mathbb{Z}_2, k_4))$.

Constructing the next stage of the Postnikov tower *needs* the knowledge of $H^6(X_4, \mathbb{Z}_2)$, again a particular case of the EDM problem, and Kenzo computes in a few seconds $H^6(X_4, \mathbb{Z}_2) = \mathbb{Z}_2^4$: 16 different choices for the next Postnikov invariant k_5; again Kenzo knows how to directly construct the "simplest" non-trivial invariant k_5, in a sense which cannot be detailed here[2]; other cohomology classes could be constructed and used as well, but computations would be more complicated. Then F_5 and X_5 are constructed, but this time a few hours of computation are necessary to obtain $H^7(X_5, \mathbb{Z}_2) = \mathbb{Z}_2^5$: there are 32 different choices for the next invariant k_6 and again, in this "simple" case, such k_6 is actually an invariant of the homotopy type of the resulting space, see Section 4.2.2.

And so on.

3. Morphisms between Postnikov Towers

3.1. **The definition.** We have presented the Postnikov towers as being the objects of the Postnikov category $\mathcal{P}$, so that we must also describe the $\mathcal{P}$-morphisms. The standard considerations around homotopy groups and Kan minimal models, see for example [13], lead to the following definition.

Definition 5. Let $P = ((\pi_n, k_n))_{n \geq 2}$ and $P' = ((\pi'_n, k'_n))_{n \geq 2}$ be two Postnikov towers. A *morphism* $f : P \to P'$ is a collection of group morphisms $f = (f_n : \pi_n \to \pi'_n)_{n \geq 2}$ satisfying the following *recursive* coherence property for every n. The sub-collection $(f_i)_{2 \leq i \leq n-1}$, if coherent, defines a continuous map $\phi_{n-1} : X_{n-1}(= \mathbf{PS}(P_{n-1})) \to X'_{n-1}(= \mathbf{PS}(P'_{n-1}))$ between the $(n-1)$-th stages of the respective Postnikov towers. So that two canonical maps are defined:

- The map ϕ_{n-1} induces in a contravariant way a map $\phi^*_{n-1} : H^{n+1}(X'_{n-1}, \pi'_n) \to H^{n+1}(X_{n-1}, \pi'_n)$ between the cohomology groups;
- The map f_n induces in a covariant way a map $f_{n*} : H^{n+1}(X_{n-1}, \pi_n) \to H^{n+1}(X_{n-1}, \pi'_n)$.

Then the equality $\phi^*_{n-1}(k'_n) = f_{n*}(k_n)$ is required.

If so, a continuous map $\phi_n : X_n \to X'_n$ is defined, which allows one to continue the recursive process. The projective limit $\phi = \varprojlim \phi_n$ is then a continuous map $\phi : X = \mathbf{PS}(P) \to X' = \mathbf{PS}(P')$.

3.2. **First example.** This definition implies some *isomorphisms* between *different* Postnikov towers can exist. Let us examine when a collection $f = (f_n : \pi_n \to \pi'_n)_{n \geq 2} : ((\pi_n, k_n))_{n \geq 2} \to ((\pi'_n, k'_n))_{n \geq 2}$ is an isomorphism. On the one

[2]Depending on the Smith reduction of the boundary matrices of the small chain complex which is the main component of the effective homology of X_4.

hand, the coherence condition stated above must be satisfied; on the other hand, every f_n must be a group isomorphism; if this is the case, the obvious inverse $g = (f_n^{-1})_{n \geq 2}$ also satisfies the coherence condition and is actually an inverse of f.

The simplest example, where a non-trivial isomorphism happens, is the following. Let us consider the small Postnikov tower $P = ((\mathbb{Z}, 0), (\mathbb{Z}, k_3))$, where $k_3 \in H^4(K(\mathbb{Z}, 2))$ is $k_3 = c_1^2$, the square of the canonical generator $c_1 \in H^2(K(\mathbb{Z}, 2), \mathbb{Z})$, the first universal Chern class. The corresponding space $X = \mathbf{PS}(P)$ is the total space of a well defined fibration

$$K(\mathbb{Z}, 3) \hookrightarrow X \twoheadrightarrow K(\mathbb{Z}, 2) \xrightarrow{c_1^2} K(\mathbb{Z}, 4)$$

The same construction is valid if k_3 is replaced by $k_3' = -k_3$; the Postnikov tower $P' = ((\mathbb{Z}, 0), (\mathbb{Z}, k_3'))$ produces a *different* fibration

$$K(\mathbb{Z}, 3) \hookrightarrow X' \twoheadrightarrow K(\mathbb{Z}, 2) \xrightarrow{-c_1^2} K(\mathbb{Z}, 4).$$

It is important to understand the fibrations are not only different but they are even non-isomorphic: their classifying maps are not *homotopic*. Yet, the spaces $X = \mathbf{PS}(P)$ and $X' = \mathbf{SP}(P')$ are the same, that is, they have the same homotopy type; the following diagram is induced by the group morphism $\varepsilon_4 : K(\mathbb{Z}, 4) \overset{K(-1,4)}{\longrightarrow} K(\mathbb{Z}, 4)$ associated to the symmetry $-1 : n \mapsto -n$ in $\mathbb{Z}$, and the same for ε_3:

$$\begin{array}{ccccccc}
K(\mathbb{Z},3) & \longrightarrow & X & \longrightarrow & K(\mathbb{Z},2) & \xrightarrow{c_1^2} & K(\mathbb{Z},4) \\
{\scriptstyle \varepsilon_3}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \varepsilon_3 \widetilde{\times} =}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle =}\big\downarrow & & {\scriptstyle \varepsilon_4}\big\downarrow{\scriptstyle \cong} \\
K(\mathbb{Z},3) & \longrightarrow & X' & \longrightarrow & K(\mathbb{Z},2) & \xrightarrow{-c_1^2} & K(\mathbb{Z},4)
\end{array}$$

The $\cong$ sign between X and X' is particularly misleading. It is correct from the topological point of view: both spaces X and X' are actually homeomorphic and $\varepsilon_3 \widetilde{\times} =$ is such a homeomorphism. The $\cong$ sign is incorrect with respect to the principal $K(\mathbb{Z}, 3)$-structures: the actions of $K(\mathbb{Z}, 3)$ on the fibers of X and X' are *not compatible*; the satisfied relation is only $(\varepsilon_3 \widetilde{\times} =)(a \cdot x) = \varepsilon_3(a) \cdot (\varepsilon_3 \widetilde{\times} =)(x)$ and the principal structures would be compatible if $(\varepsilon_3 \widetilde{\times} =)(a \cdot x) = a \cdot (\varepsilon_3 \widetilde{\times} =)(x)$ were satisfied, this is why the classifying maps are opposite.

Perhaps the same phenomenon for the Hopf fibration is easier to be understood. Usually we take S^3 as the unit sphere of $\mathbb{C}^2$ so that a *canonical* S^1-action is underlying and a *canonical* characteristic class on the quotient S^3/S^1 is deduced. But if you reverse the S^1-action, why not, the space S^3 is not modified, the quotient S^3/S^1 is not modified either, but the characteristic class is the opposite one. In other words, it is important not to forget the classifying map characterizes the isomorphism class of a principal fibration, but not the homotopy type of the total space!

3.3. **The key example.** The next example of a Postnikov tower with two stages is still rather simple, but is sufficient to understand the essential failure of the claimed Postnikov *invariants*.

Let us consider the tower $P(\ell, k) = ((\mathbb{Z}^\ell, 0), (\mathbb{Z}, k))$, the parameter ℓ being some positive integer, and k, the unique non-trivial Postnikov "invariant" being an element $k \in H^4(K(\mathbb{Z}^\ell, 2), \mathbb{Z})$. A canonical isomorphism $K(\mathbb{Z}^\ell, 2) \cong K(\mathbb{Z}, 2)^\ell$ is available. The cohomoloy ring of $K(\mathbb{Z}, 2) = P^\infty\mathbb{C}$ is the polynomial ring $\mathbb{Z}[X]$, where $X = c_1$ is the first universal Chern class, of degree 2, so that $H^*(K(\mathbb{Z}^\ell, 2), \mathbb{Z}) = \mathbb{Z}[X_i]$ with $1 \leq i \leq \ell$, every generator X_i being of degree 2. Finally, $H^4(K(\mathbb{Z}^\ell, 2), \mathbb{Z}) = \mathbb{Z}[X_i]^{[2]}$, the exponent [2] meaning we must consider only the sub-module of homogeneous polynomials of degree 2 with respect to the X_i's. Every $k \in \mathbb{Z}[X_i]^{[2]}$ thus defines a two-stage Postnikov tower $P(\ell, k) = ((\mathbb{Z}^\ell, 0), (\mathbb{Z}, k))$.

Such two *different* Postnikov towers $P(\ell, k)$ and $P(\ell', k')$ can be isomorphic. If so, the homotopy groups must me the same and $\ell = \ell'$ and it is enough to wonder whether $P(\ell, k) \overset{???}{\cong} P(\ell, k')$. A possible isomorphism $f : P(\ell, k) \to P(\ell, k')$ is made of $f_2 : \mathbb{Z}^\ell \xrightarrow{\cong} \mathbb{Z}^\ell$ and $f_3 : \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$. The component f_3 is a possible simple sign change, as in the first Example of 3.2, but the component f_2 is a $\mathbb{Z}$-linear equivalence acting on the variables $[X_i]_{1 \leq i \leq \ell}$. The coherence condition given in Definition 5 becomes $f_{3*}(k) = {f_2}^*(k')$: here f_{3*} allows one to make equivalent two classes of opposite signs, and ${f_2}^*$, much more interesting, allows one to make equivalent two classes $k, k' \in \mathbb{Z}[X_i]^{[2]}$ where k is obtained from k' by a $\mathbb{Z}$-linear change of variables. We have here identified f_2 with ϕ_2, the induced automorphism of $K(\mathbb{Z}^\ell, 2) = X_2$, the first stage of both Postnikov towers, see Definition 5.

Algebraic Topology succeeds: the topological problem of homotopy equivalence between $\mathbf{PS}(P(\ell, k))$ and $\mathbf{PS}(P(\ell, k'))$ is transformed to the algebraic problem of the $\mathbb{Z}$-linear equivalence, up to sign, between the "quadratic forms" k and k'. And this provides a complete solution because this landmark problem firstly considered by Gauss has now a complete solution, see for example [20, 21, 5].

3.4. **Higher dimensions.** But instead of working with the integer $3 = 2*2-1$, we could consider exactly the same problem with the Postnikov tower:

$$\mathcal{P}_{2d-1} \ni P(\ell, d, k) = ((\mathbb{Z}^\ell, 0), (0, 0), \ldots, (0, 0), (\mathbb{Z}, k_{2d-1} = k))$$

defined by integers $\ell \geq 1$, $d \geq 2$ and a cohomology class $k \in H^{2d}(K(\mathbb{Z}^\ell, 2), \mathbb{Z}) = \mathbb{Z}[X_i]^{[d]}$. Instead of an equivalence problem between homogeneous polynomials of degree 2, we meet the same problem but with homogeneous polynomials of degree d. And at the time of writing this paper, this problem seems entirely *open* as soon as $d \geq 3$. Now it is the right time to recall what the very notion of invariant is.

4. Invariants

4.1. **Elementary cases.** What is an *invariant*? An invariant is a process $\mathcal{I}$ which associates to every object X of some type some other object $\mathcal{I}(X)$, the relevant *invariant*; in other words, an invariant is a function. This terminology clearly says that $\mathcal{I}(X)$ *does not change* (does not *vary*) when X is replaced by X', if X and X' are equivalent in some sense: a possible relevant equivalence between X and X' should imply the *equality* – not again some other equivalence – between $\mathcal{I}(X)$ and $\mathcal{I}(X')$.

For example, one of the most popular invariants is the set of *invariant factors* of square matrices. The concerned equivalence relation is the similarity. If K is a commutative field and $A \in M_n(K)$ is an $(n \times n)$-matrix with coefficients in K representing some endomorphism of K^n, the invariant factors of A are a sequence of polynomials $\phi(A) = (\mu_1, \ldots, \mu_k)$ characterizing *in this case* the similarity class of the matrix A: two matrices A and B are similar if and only if $\phi(A) \boxed{=} \phi(B)$. Another example is the minimal polynomial $\mu_1(A)$: if two matrices are similar, they have the *same* minimal polynomial. Idem for the characteristic polynomial which is the product of the invariant factors, and so on. It is well known that, for example, the characteristic polynomial does not characterize the similarity class, yet it is an invariant: if two matrices are similar, they have the *same* characteristic polynomial. Sometimes the characteristic polynomial is sufficient to disprove the similarity between two matrices, sometimes is not. The trivial invariant consists in deciding that $\mathcal{I}(A) = *$, some fixed object, for every matrix; not very interesting but it is undoubtedly … an invariant. Symmetrically, the tentative invariant $\mathcal{I}(A) = A$ is *not* an invariant, for there exist different (!) matrices[3].

Algebraic Topology is in a sense an enormous collection of (algebraic) invariants associated to topological spaces, invariants with respect to some equivalence relation, frequently the homotopy equivalence. Typically, a homotopy group π_n is an invariant of this sort. Not frequently, with respect to some appropriate equivalence relation, a *complete* invariant maybe available. For example, H_1 is a complete invariant for the homotopy type of a finite connected graph, the genus is a complete invariant for the diffeomorphism type of a closed orientable real manifold of dimension 2.

The last two examples, quite elementary, are interesting because the difficult logical problem underlying this matter is often forgotten and easily illustrated in these cases. Let M_0 and M_1 be two closed orientable 2-manifolds that are diffeomorphic; if g denotes the genus, then $g(M_0) \boxed{=} g(M_1)$: the genus is an invariant;

[3] See `http://encyclopedia.thefreedictionary.com/invariant` for other typical examples. Another amusing bug of the standard terminology in Algebraic Topology is the expression “characteristic class” in the classical fibration theory: the usual characteristic classes are actual invariants (!) of the isomorphism class but, except in simple situations, they do not characterize (!) this isomorphism class.

furthermore it is a complete invariant because conversely $g(M_0)\boxed{=}g(M_1)$ implies both manifolds are diffeomorphic. We have framed the '=' sign because the main problem in the continuation of the story is there.

Let us consider now the case of finite graphs. In fact, it is *false* the H_1 functor is an invariant. If you take a triangle graph $G_0 = \triangle$ and a square graph $G_1 = \square$, same homotopy type, the careless topologist thinks $H_1(G_0) = H_1(G_1) = \mathbb{Z}$ so that H_1 looks like an invariant of the homotopy type, but it is important to understand this is deeply erroneous. With respect to any coherent formal definition of mathematics, in fact $H_1(\triangle) \neq H_1(\square)$, these H_1-groups are only *isomorphic*. To obtain an actual invariant of the homotopy type, you must consider the functor $\mathbf{H}_1 = \mathrm{IC} \circ H_1$, where IC is the "isomorphism class" functor, always difficult to properly define from a logical point of view, see for example [2]. But in the case of the H_1-group of a finite graph, it is a free $\mathbb{Z}$-module of finite type, it is particularly easy to determine whether two such groups are isomorphic and every topologist *implicitly* apply the IC functor without generating any error.

Such a situation is so frequent that most topologists come to confuse both notions of *functor* and *invariant*, and the case of Postnikov "invariants" is rather amazing.

4.2. **The alleged Postnikov "invariants".**

4.2.1. *Terminology.* We start with a sensible topological space, for example, a finite simply connected CW-complex E. The textbooks explain how it is possible to define or sometimes to "compute" Postnikov invariants $(k_n(E))_{n\geq 3}$. In our framework, the problem is the following:

Problem 6. How to determine a Postnikov tower $P = ((\pi_n, k_n))_{n\geq 2}$ such that E and $\boldsymbol{PS}(P)$ have the same homotopy type?

This problem, thanks to the general *Constructive* Algebraic Topology framework of the authors, now has a positive *and* constructive solution. The aforementioned textbooks also describe "solutions", which do *not* satisfy the constructive requirements which should yet be required in this context. See also [18] for another theoretical *constructive* – and interesting – solution, significantly more complex, so that it has not yet led to concrete results, that is, to machine programs.

Most topologists think that the positive solution for Problem 6 implies that k_n's of the result are "invariants" of the homotopy type of E. This is simply *false*, for any reasonable understanding of the word *invariant*, and it is rather strange such an error has been remaining for such a long time in such an important field as basic Algebraic Topology. The k_n's could be called *invariants* if they solved the next problem.

Problem 7. Construct a functor $\mathbf{SP} : \mathcal{SS}_{EH} \to \mathcal{P}$ satisfying the following properties:

(1) Some original space $E \in \mathcal{SS}_{EH}$ and $\mathbf{PS}\circ\mathbf{SP}(E)$ have the same homotopy type;

(2) If E and $E' \in \mathcal{SS}_{EH}$ have the same homotopy type, then $\mathbf{SP}(E) \boxed{=} \mathbf{SP}(E')$.

The first point is a rephrasing of Problem 6, and the second states that if E and E' have the same homotopy type, then the images $\mathbf{SP}(E)$ and $\mathbf{SP}(E')$ are $\boxed{\text{equal}}$, not only *isomorphic*. In other words the claimed "invariant" must not *change* when the source object changes in the same equivalence class; this is of course (?) the very notion of an invariant.

The non-constructive topologist easily solves the problem by replacing the category $\mathcal{P}$ by the quotient $\mathcal{P}/\text{Iso}$, and then a correct solution is obtained, but it is an artificial one. The category $\mathcal{SS}_{EH}/H$-equiv and the canonical projection $\mathcal{SS}_{EH} \to \mathcal{SS}_{EH}/H$-equiv would be much simpler, but obviously without any interest.

The right interpretation of k_n's is the following: combined with the standard homotopy groups π_n, they are to be considered as *directions for use* allowing one to reconstruct a simple object with the right homotopy type; another rephrasing of Problem 6. But it may happen two different objects E and E' with the *same* homotopy type produce *different* "directions for use", so that these "directions for use" are not invariants of the homotopy type. In fact such an accident is the most common situation, except for the topologists working only with paper and pencil.

4.2.2. *The SP functor, first try.* Let us briefly describe the standard solution of Problem 6, a solution which can be easily made constructive thanks to [19, 16, 18]. Let E be some reasonable[4] simply connected space. There are many ways to determine the[5] Postnikov tower $P = \mathbf{SP}(E)$ and one of them is illustrated here with the beginning of the simplest case, the 2-sphere S^2. Hurewicz indicates $\pi_2 = H_2 = \mathbb{Z}$; the invariant k_2 is necessarily null. The next step invokes the Whitehead fibration:

$$K(\mathbb{Z}, 1) \hookrightarrow E^3 \twoheadrightarrow S^2 \xrightarrow{c_1} K(\mathbb{Z}, 2).$$

where c_1 is the canonical cohomology class, in this case the first Chern class of the complex structure of S^2. The first stage of the Postnikov tower is $X_2 = K(\mathbb{Z}, 2) = P^\infty\mathbb{C}$ and the first stage of the complementary Whitehead tower is the total space $E^3 = S^3$: our fibration is nothing but the Hopf fibration. Then $\pi_3(S^2) = \pi_3(S^3) = H_3(S^3, \mathbb{Z}) = \mathbb{Z}$, so that the next Postnikov invariant is some $k_3 \in H^4(X_2, \mathbb{Z}) = H^4(K(\mathbb{Z}, 2), \mathbb{Z}) = \mathbb{Z}$. How to determine this cohomology class?

In general we obtain a fibration

$$E^n \hookrightarrow E \to X_{n-1}$$

where X_{n-1} is the $(n-1)$-stage of the Postnikov tower containing homotopy groups $(\pi_i)_{2 \le i \le n-1}$, and E^n is the complementary n-stage of the Whitehead tower [7, Proposition 8.2.5] containing homotopy groups $(\pi_i)_{i \ge n}$; in the Kan

[4]That is, an $\mathcal{SS}_{EH}$-object, see [16].

[5]In fact *some* Postnikov tower…

context of [13, § 8], E^n is the n-th Eilenberg subcomplex of E. How to deduce a cohomology class $k_n \in H^{n+1}(X_{n-1}, \pi_n)$? The $(n-1)$-connectivity of E^n produces a transgression morphism $H^n(E^n, \pi_n) \rightarrow H^{n+1}(X_{n-1}, \pi_n)$; the group $H^n(E^n, \pi_n)$ contains a fundamental Hurewicz class and the image of this class in $H^{n+1}(X_{n-1}, \pi_n)$ is the wished k_n. In the particular case of S^2 this process leads to an isomorphism $H^3(S^3, \mathbb{Z}) \xrightarrow{\cong} H^4(K(\mathbb{Z}, 2), \mathbb{Z})$ so that k_3 is the image of the fundamental cohomology class of S^3, that is, the (?) generator c_1^2 of $H^4(K(\mathbb{Z}, 2), \mathbb{Z})$. Sure?

As usual we have light-heartedly mixed intrinsic objects and isomorphism classes of these objects. The *correct* isomorphism to be considered for our example is $H^3(E^3, \pi_3\underline{(E^3)}) \cong H^4(K(\pi_2\underline{(S^2)}, 2), \pi_3\underline{(E^3)})$ where E^3 is now the total space of the *canonical* fibration $K(\pi_2\underline{(S^2)}, 1) \hookrightarrow E^3 \twoheadrightarrow S^2$; this isomorphism is actually canonical. But no canonical ring structure for $\pi_3(E^3)$ so that speaking of c_1^2 does not make sense. There is actually a canonical element $k_3 \in H^4(K(\pi_2\underline{(S^2)}, 2), \pi_3\underline{(E^3)})$, but such an element deeply depends on S^2 itself and cannot be qualified as an *invariant* of the *homotopy type* of S^2. An actual invariant should be taken in the "absolute" (independent of S^2) group $H^4(K(\mathbb{Z}, 2), \mathbb{Z})$, but such a choice depends on an isomorphism $\pi_3(E^3) \cong \mathbb{Z}$; two such isomorphisms are possible so that in this case, k_3 is defined *up to sign*: it is well known the Hopf fibration and the "opposite" one produce the "same" total space.

This is the reason why in the definition of a Postnikov tower, see Definition 1, we have decided to have only *one* group for each isomorphism class; this is easy and can be done in a *constructive* way. The goal being to obtain *invariants*, we had to design our Postnikov towers as a catalogue of possible Postnikov towers, in such a way that there are no redundant copies up to isomorphism in this collection; bearing this point in mind, it was mandatory to have only one copy for every isomorphism class of group. But this was not enough, for it is today impossible to take the same precaution for the second components, k_n's, the so-called Postnikov invariants.

For example, if the concerned homotopy groups are finite, then the number of possible k-invariants is finite, so that the related equivalence problem is theoretically solved; this was already noted by Edgar Brown [3], which conversely implies (!) he did not know how to solve the general case. On the contrary, when the homotopy groups have infinite automorphism groups, there is no known way of transforming pseudo-invariants to actual invariants.

We understand now the reason of the repetitive remark in Section 2.1: "In this particular case, k_n is actually an invariant of the homotopy type"; we decided to systematically choose $\pi_n = \mathbb{Z}_2$, but the automorphism group of $\mathbb{Z}_2$ is *trivial*; no non-trivial automorphism of the constructed tower can exist and *then* k_n's are actual *invariants*.

But if some user intends to use Postnikov invariants to try to prove the spaces E and E' have different homotopy types, then the following can happen. A calculation could respectively produce the Postnikov towers $((\mathbb{Z}^\ell, 0), (0, 0), \ldots,$

$(\mathbb{Z}, k_{2d-1}))$ and $((\mathbb{Z}^\ell, 0), \ldots, (\mathbb{Z}, k'_{2d-1}))$ (see Section 3.4). If fortunately $k_{2d-1} = k'_{2d-1}$, our user can be sure the homotopy types are the same but if on the contrary $k_{2d-1} \neq k'_{2d-1}$, then he has to decide whether two homogeneous polynomials of degree d are linearly equivalent or not and for $d \geq 3$ no general solution is known. *Maybe* they are equivalent, *maybe* not; because the alleged invariants may… vary, in general our user cannot conclude: the claimed invariants cannot play the role ordinarily expected for invariants. Qualifying them as invariants is therefore a deep *error*.

4.2.3. *The SP functor, second try.* The right definition for the **SP** functor is now clear. We must add to the data some *explicit* isomorphisms between the homotopy groups $\pi_n(E)$ of the considered space E with the corresponding *canonical* groups, see Definition 1.

Definition 8. The product $\mathcal{SS}_{EH}\widetilde{\times} I$ is the set of pairs (E, α) where:

(1) The component E is a simplicial set with effective homology $E \in \mathcal{SS}_{EH}$;
(2) The component α is a collection $(\alpha_n)_{n\geq 2}$ of isomorphisms $\alpha_n : \pi_n(E) \xrightarrow{\cong} \underline{\pi_n}$ where $\underline{\pi_n}$ denotes the unique *canonical* group isomorphic to $\pi_n(E)$.

The preceding discussions can be reasonably considered as a demonstration of the next theorem.

Theorem 9. *A functor* $\mathbf{SP} : \mathcal{SS}_{EH}\widetilde{\times} I \to \mathcal{P}$ *can be defined.*

(1) *If* $(E, \alpha) \in \mathcal{SS}_{EH}\widetilde{\times} I$, *then* E *and* $\mathbf{PS}\circ\mathbf{SP}(E, \alpha)$ *have the same homotopy type.*
(2) *If* $P \in \mathcal{P}$ *is a Postnikov tower, there exists a unique* α *such that* $\mathbf{SP}(\mathbf{PS}(P), \alpha) = P$.

Thus it is tempting – and correct – to replace the **PS** functor by $\mathbf{PS} : \mathcal{P} \to \mathcal{SS}_{EH}\widetilde{\times} I$ in order to obtain a better symmetry. But ordinary topologists work with elements in $\mathcal{SS}_{EH}$, not in $\mathcal{SS}_{EH}\widetilde{\times} I$.

5. Postnikov Invariants in the Available Literature

Most textbooks speaking of Postnikov invariants (or k-invariants) use the *invariant* terminology without justifying it so that strictly speaking, there is no mathematical error in this case. For example, [7, p. 279] defines the Postnikov invariant through a transgression morphism[6] and explains "The k^i precisely constitute the stepwise obstructions…"; the statement about this obstruction is of course correct, but it seems the terminology should therefore speak of Postnikov *obstructions*? Nothing is explained about the *invariant* nature of these obstructions.

Other books speak of these invariants as objects allowing one to *reconstruct* the right homotopy type. For example, in [11, p. 412]: "The map k_n is equivalent to a class in $H^{n+2}(X_n; \pi_{n+1}(K))$ called the n-th *k-invariant* of X. These classes specify how to construct X inductively from Eilenberg-MacLane spaces".

[6] We used this method in Section 4.2.2

This should be compared with our considerations about the interpretation in terms of "directions for use" at the end of Section 4.2.1. Again, no indication in this book about the justification of the *invariant* terminology. The Section "The Postnikov Invariants" in [6, V.3.B] can be analyzed along the same lines.

In [10, VI], because of the sophisticated categorical environment, the authors prefer to define the *general* notion of a *Postnikov tower* for a space X, each one containing in particular its k_n-invariants [10, VI.5]; finally, Theorem [10, VI.5.14] proves two such Postnikov towers for the same X are *weakly* equivalent. In other words, one source object produces in general a large infinite set of (different!) k_n-invariants for every relevant n; yet, some invariant theory is interesting when different objects can produce the same invariants, not when an object produces different invariants! In fact, as explained in our text, this cannot be currently avoided, but why do not these authors make explicit the misleading status of these claimed invariants?

The book [13] systematically uses the powerful notion due to Kan of *minimal simplicial Kan-model*, often allowing a user to work in a "canonical" way, frequently allowing the same user to detect easily a non-unicity problem. In this way [13, p. 113] correctly signals that the map $B \to K(\pi, n+1)$ leading to a k_n-invariant is defined up to a π-automorphism, which is not a serious drawback: the decision problem about the possible equivalence of two k_n's under such an automorphism is easy when π is of finite type. But the author does not mention the same problem with respect to the automorphisms of the base space B, the automorphisms leading to the corresponding open problem detailed here Section 3.4.

The same author again considers the same question in the more recent textbook [14]. He defines the notion of Postnikov system in Section 22.4; the existence of *some* Postnikov system is proved, the term "k-invariant" is used only once in quotation makrs, seemingly implying that this term is not appropriate, but no explanation is given.

Hans Baues [1, p.33] on the contrary correctly respects the necessary symmetry between the source and the target of the classifying map; but the author is aware of the underlying difficulty and it is interesting to observe how he "solves" the arisen problem:

> Here $k_n(Y)$ is actually an *invariant* of the homotopy type of Y in the sense that a map $f : Y \to Z$ satisfies:
> $$(P_{n-1}f)^* k_n(Z) = (\pi_n f)_* k_n(Y)$$
> in $H^{n+1}(P_{n-1}X, \pi_n Y)$.

Clearly explained, the author says that the *invariant* is variable, but in a *functorial* way. Baues' condition is essentially the *coherence condition* of our Definition 5. If appropriate morphisms of the category $\mathcal{SS}_{EH} \widetilde{\times} I$ were defined, the functorial property of the map **SP** (Theorem 9) would be exactly Baues' relation. But it is not explained in Baues' paper why a functor can be qualified as an invariant.

Probably the reference, the most lucid one about our subject, is [22]. Chapter IX is entirely devoted to Postnikov systems. We find p. 423:

> The term 'invariant' is used somewhat loosely here. In fact k^{n+2} is a cohomology class of a space X^n, which has not been constructed in an invariant way. This difficulty, however, is not serious, for, as we shall show below, the construction of the space X^n can be made completely natural.

This text is essentially a rephrasing of Baues' explanation. Again the common confusion between the notions of invariant and functor is observed. To make "natural" its invariants, George Whitehead uses enormous singular models, so that the obtained k^{n+2} heavily depends on X itself and not only on its homotopy type. In fact Section [22, IX.4] shows Whitehead is in fact also interested in being able to reconstruct the homotopy type of X from the "natural" associated Postnikov tower, and this goal is obviously reached, but this does not provide a general machinery allowing one to detect *different homotopy types* when the associated *invariants* are *different*.

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