Minimal Resolutions

Julio Rubio Garcia
Eduardo Sáenz de Cabezon
Francis Sergeraert

June 14, 2006

> *After extensive preparations in the previous section, we are ready to harvest the fruits of our labour and compute minimal graded free resolutions.* [4, p.147]

Announcement.

The methods of Effective Homology [5] give a simple algorithm computing the minimal resolution of an A_0-module of finite type M_0, when A_0 is an ordinary polynomial ring $A_0 = k[x_1, \ldots, x_m]_0$ localized at $0 \in k^m$. Standard arguments allow us to study instead the global case of $A = k[x_1, \ldots, x_m]$, of an A-module M, and we are looking for an A_0-resolution of $M \otimes_A A_0$.

With respect to which seems the previously known methods [4, Section 4.8], the situation is the following. Our method is conceptually remarkably simple, once the very nature of effective homology is understood. On the contrary, the technicalities of the other methods are rather laborious, which of course does not mean useless. The style of our algorithm is quite different; effective homology can be seen as an automatic program writing process, deducing machine programs from simple notions of homological algebra, mainly the homological perturbation lemma. Experience in Algebraic Topology shows programs obtained in this way are simple, readable and efficient, the same in Commutative Algebra where other programs computing the effective homology of Koszul complexes have already shown the interest of the point of view and the efficiency of the programs that are so obtained.

So that it will be interesting to compare the concrete algorithms obtained with our method to the others. The situation is a little amusing: because our programs are "automatically" written down, it seems sensible to guess the resulting program shoud be close to one of the other programs. Yes or no? If yes, close to which one? Interesting questions.

The algorithm has four steps.

1. Compute a Groebner basis of the module M for an arbitrary monomial order. Replacing the generators of the Groebner basis by the leading terms produces another A-module M', a monomial module, canonically isomorphic to M as a k-vector space.

2. The *effective* homology of the Koszul complex $\mathrm{Ksz}(M')$ of M', because of the monomial generators, is easily and elementarily computed.

3. The so-called *homological perturbation lemma*[1] is applied between $\mathrm{Ksz}(M)$ and $\mathrm{Ksz}(M')$, producing the effective homology of the Koszul complex $\mathrm{Ksz}(M)$.

4. The Aramova-Herzog bicomplex [1] of M is constructed. Two further applications of the homological perturbation lemma produce the looked-for minimal resolution, with a simple explicit formula for the differentials.

Observe this organization is essentially *opposite* to the usual one: a resolution of M is most often firstly computed to obtain the (ordinary) homology of the Koszul complex. But the *effective* homology is much richer and in this case it happens the effective homology of $\mathrm{Ksz}(M)$, directly obtained without any resolutions, "contains" in particular the minimal resolution of M. The Aramova-Herzog bicomplex is to be considered as a *reading process* of this property.

For the last point, let $\rho = (f, g, h)$ be the reduction (see [5]) describing the effective homology of $\mathrm{Ksz}(M)$, the Koszul complex of M:

$$\rho = \boxed{h \circlearrowright \mathrm{Ksz}(M) \underset{f}{\overset{g}{\rightleftarrows}} H}$$

where H is the chain complex with null differentials made of k-vector spaces of dimensions the Betti numbers of $\mathrm{Ksz}(M)$. The Aramova-Herzog (ArHr) bicomplex is $\mathrm{ArHr}(M) = M \otimes_k \wedge V \otimes_k A$, where $V = \mathfrak{m}/\mathfrak{m}^2$, with an appropriate bigrading taking account of the exterior degree in the second component of the tensor product and of the polynomial degree in the third one. The horizontal differential consists in considering $\mathrm{ArHr}(M) = M \otimes \mathrm{Ksz}(A)$ and $\partial' = 1_M \otimes d_{\mathrm{Ksz}(A)}$. The vertical differential sees $\mathrm{ArHr}(M) = \mathrm{Ksz}(M) \otimes A$ and is $\partial'' = d_{\mathrm{Ksz}(M)} \otimes 1_A$.

Now the minimal resolution $R(M)$ is $R(M) = H \otimes_k A$ with the differential:

$$(\Sigma) \qquad d = (f \otimes \mathrm{id}_A) \left(\sum_{i=0}^{\infty} (-\partial'(h \otimes \mathrm{id}_A))^i \right) \partial'(g \otimes \mathrm{id}_A)$$

The series which looks infinite is in fact finite for any particular evaluation, because of a nilpotency property necessarily satisfied.

[1]Should in fact be called the *fundamental theorem of homological algebra*.

The minimal non-trivial example.

Let $A = k[x]$ (one variable) and $M = A/<x^2>$. And let us assume we do not know (!) the minimal resolution. Here the ideal is monomial and the steps 1 and 3 of our algorithm are void. The effective homology of the Koszul complex:

$$\mathrm{Ksz}(M) = [\cdots \leftarrow 0 \leftarrow M \leftarrow M.dx \leftarrow 0 \leftarrow \cdots]$$

is made of the chain complex:

$$H = [\cdots \leftarrow 0 \leftarrow k_0 \stackrel{0}{\leftarrow} k_1 \leftarrow 0 \leftarrow \cdots]$$

(where k_0 and k_1 are copies of the ground field k with respective homological degrees 0 and 1) *and* of the maps $\rho = (f, g, h)$ with:

1. $f : M \to k_0$ is defined by $f(1) = 1_0, f(x) = 0$.
2. $f : M.dx \to k_1$ is defined by $f(1.dx) = 0, f(x.dx) = 1_1$.
3. $g : k_0 \to M$ is defined by $g(1_0) = 1$.
4. $g : k_1 \to M.dx$ is defined by $g(1_1) = x.dx$.
5. $h : M \to M.dx$ is defined by $h(1) = 0, h(x) = 1.dx$.

We must guess the right differential on $H \otimes_k A$. The only non-trivial differential $d_{R(M)}(1_1 \otimes 1_A)$ comes from a unique non-null term in the series (Σ), following the path:

$$1_1 \otimes 1_A \stackrel{g\otimes \mathrm{id}_A}{\longmapsto} x \otimes dx \otimes 1_A \stackrel{\partial'}{\mapsto} x \otimes 1 \otimes x \stackrel{-h\otimes \mathrm{id}_A}{\longmapsto} -1 \otimes dx \otimes x \stackrel{\partial'}{\longmapsto} -1 \otimes 1 \otimes x^2 \stackrel{f\otimes \mathrm{id}_A}{\longmapsto} -1_0 \otimes x^2$$

and, surprise, we find the resolution $1_1 \otimes 1_A \mapsto -1_0 \otimes x^2$. You find it is a little complicated for a so trivial particular case? The point is the following: this example in a sense is *complete*, the most general case is not harder, you have here all the ingredients of the general solution, nothing more is necessary.

The key point is that the effective homology of $\mathrm{Ksz}(M)$ contains in particular the homotopy operator h, the main tool in the computation of the minimal resolution. Aramova and Herzog [1] apply the two usual bicomplex spectral sequences to their bicomplex, but the only knowledge of *cycles* representing the homology classes is not sufficient to determine the minimal resolutions. Effective homology contains in particular distinguished cycles representing the homology classes, but also much more information about the exact homological status of the chain complex with respect to these cycles.

In the crucial paper [1] — thanks to the authors for their very useful paper — the detailed examination of the computations in p.12 is instructive. In the line 13 from down, they observe "$-x_1 e_2 \wedge e_3 + x_2 e_1 \wedge e_3$ is homologous to $-x_3 e_1 \wedge e_2$", which allows them to identify both elements when computing the searched differential; in other words, a boundary can be *neglected*. But a little later, in the same situation, line 1 of p. 13 other boundaries are not at all neglected, their boundary preimages are in this case essential when constructing the differential. Why this difference? No explanation in the paper. We can suspect the authors have been a little "helped" by the right resolution known in advance? In fact the difference comes from an implicit homotopy operator h which is required to explain the difference. See the details in the next Section.

Note also how our method is simple with respect to the terrible spectral sequence computations in [1]; it is a striking illustration of the power of the methods of effective homology: they are simpler than the corresponding spectral sequences and also more efficient, in particular from an algorithmic point of view.

A few talks (Toulouse, Sevilla, Luminy, Marrakech, Karlsruhe, Logroño, Grenoble) have been recently given about these techniques and the corresponding pdf file[2] gives a more extended presentation of effective homology when used in commutative algebra. A Lisp logfile[3] gives also a typical example of computation of the *effective* homology of a Koszul complex, for the module $K[x, y, z, t]/{<}x - t^5, y - t^7, z - t^{11}{>}$.

Examples.

First Aramova-Herzog example.

In the paper [1], Aramova and Herzog consider the toy example of the ideal $I = {<}x_1x_3, x_1x_4, x_2x_3, x_2x_4{>}$ in $A = k[x_1, x_2, x_3, x_4]$. The ideal is monomial and again, steps 1 and 3 of our algorithm are void. The Betti numbers of $\mathrm{Ksz}(A/I)$ are $(1, 4, 4, 1)$ and the effective homology of $\mathrm{Ksz}(A/I)$ is a diagram:

$$\rho = \boxed{h \circlearrowright \mathrm{Ksz}(A/I) \underset{f}{\overset{g}{\leftrightarrows}} H}$$

where H is the chain complex with null differentials:

$$\cdots \longleftarrow k \xleftarrow{0} k^4 \xleftarrow{0} k^4 \xleftarrow{0} k \longleftarrow \cdots$$

The arrows f and g are chain complex morphisms satisfying $fg = \mathrm{id}_H$, the self-arrow h is a homotopy between gf and $\mathrm{id}_{\mathrm{Ksz}(A/I)}$, that is, $\mathrm{id}_{\mathrm{Ksz}(A/I)} = gf + dh + hd$, and finally, the composite maps fh, hg and h^2 are null. These maps smartly express the big chain complex $\mathrm{Ksz}(A/I)$ as the direct sum of the small one H, in this case with trivial differentials, and an acyclic one ($\ker f$) *with* an *explicit* contraction h. Our Kenzo program [2] computes this effective homology in a negligible time with respect to input-output. In particular the map g defines representants for the alleged homology classes, the map f is a projection which in particular sends cycles to their homology classes, and h is the main component of a *constructive* proof of these claims.

[2] http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/Koszul.pdf
[3] http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/Koszul.txt

The minimal resolution of A/I is $R(A/I) = H \otimes A$ where a non-trivial differential must be installed. Let us apply our formula to the unique generator $\mathfrak{h}_{3,1} \otimes 1_A$ of $H_3 \otimes A$. Kenzo chooses $g(\mathfrak{h}_{3,1}) = x_2\, dx_1.dx_3.dx_4 - x_1\, dx_2.dx_3.dx_4$ and:

$$\begin{aligned}\partial'(g \otimes 1_A)(\mathfrak{h}_{3,1} \otimes 1_A) = \; & x_2\, dx_3.dx_4 \otimes x_1 \\ & -x_1\, dx_3.dx_4 \otimes x_2 \\ & +(-x_2\, dx_1.dx_4 + x_1\, dx_2.dx_4) \otimes x_3 \\ & +(x_2\, dx_1.dx_3 - x_1\, dx_2.dx_3) \otimes x_4\end{aligned}$$

Kenzo is a little luckier than Aramova and Herzog, for he had chosen:

$$\begin{aligned} g(\mathfrak{h}_{2,1}) &= -x_2\, dx_1.dx_3 + x_1\, dx_2.dx_3 \\ g(\mathfrak{h}_{2,2}) &= -x_1\, dx_3.dx_4 \\ g(\mathfrak{h}_{2,3}) &= -x_2\, dx_1.dx_4 + x_1\, dx_2.dx_4 \\ g(\mathfrak{h}_{2,4}) &= -x_2\, dx_3.dx_4 \end{aligned}$$

which is enough to imply:

$$d(\mathfrak{h}_{3,1}) = -\mathfrak{h}_{2,1} \otimes x_4 + \mathfrak{h}_{2,2} \otimes x_2 + \mathfrak{h}_{2,3} \otimes x_3 - \mathfrak{h}_{2,4} \otimes x_1$$

that is, except for legal minor differences, directly the same result as Aramova and Herzog.

Let us now *force* Kenzo to choose Aramova and Herzog's representants for the homology classes of H_2. This amounts to replacing the component g in degree 2 by another one $g' = g + d\alpha$ for α a map $\alpha : H_2 \to \mathrm{Ksz}_3(A/I)$ chosen to give the new representants. The cycle $-x_2\, dx_1.dx_i + x_1\, dx_2.dx_i$ ($i = 3$ or 4) is homologous to the cycle $-x_i\, dx_1.dx_2$ (sign error in [1]) thanks to the boundary preimage $dx_1.dx_2.dx_i$. So that we transform Kenzo's choices to Aramova and Herzog's choices by taking $\alpha(\mathfrak{h}_{2,1}) = -dx_1.dx_2.dx_3$, $\alpha(\mathfrak{h}_{2,3}) = -dx_1.dx_2.dx_4$ and $\alpha(\mathfrak{h}_{2,i}) = 0$ for $i = 2$ or 4.

The component f of the reduction does not change, but the homotopy h_2 must be replaced by $h'_2 = h_2(\mathrm{id} - d\alpha f_2)$. Repeating the same computation, taking account of $g_3 = g'_3$, now the homotopy term $(h'_2 \otimes \mathrm{id}_A)\partial'(g_3 \otimes \mathrm{id}_A)(\mathfrak{h}_{3,1}) = dx_1.dx_2.dx_4 \otimes x_3 - dx_1.dx_2.dx_3 \otimes dx_4$ *is not null*, so that we must continue the expansion of the series (Σ). We find:

$$\begin{aligned}-\partial'(h'_2 \otimes \mathrm{id}_A)\partial'(g \otimes \mathrm{id}_A)(\mathfrak{h}_{3,1}) \;=\; & -dx_2.dx_4 \otimes x_1x_3 + dx_1.dx_4 \otimes x_2x_3 \\ & +dx_2.dx_3 \otimes x_1x_4 - dx_1.dx_3 \otimes x_2x_4\end{aligned}$$

but applying f or h' to the lefthand factors of the tensor products this time gives 0 and the final result is the same: Aramova-Herzog's conclusion is so justified; the *possible* pure nature of the looked-for resolution, known in advance after examining the Koszul cycles, may also be used to cancel the examination of the critical homotopy operator, but we will see our method can be applied in much more general situations, even in a non-homogeneous situation. In more complicated situations, the result could have been different: "the" minimal resolution is unique only up to chain-complex isomorphism and this set of isomorphisms is very large. In this particular case, many triangular perturbations can for example be applied to the simple expression found for $d(\mathfrak{h}_{3,1})$ without changing its intrinsic nature, and in parallel the same for "the" effective homology of the Koszul complex.

Another comment is also necessary. After all, any (correct) choice for the representants $g(\mathfrak{h}_{2,i})$ is possible, so that why it would not be possible to prefer Kenzo's choices to the initial unfortunate choices by Aramova and Herzog? The point is the following: a resolution is not only made of isomorphism classes of the boundary maps, you must make these maps fit to each other in such a way there is *equality* between appropriate kernels and images. So that when you change the cycles representing the homology classes during the computation of the component d_3 of the resolution for example, then the computation of d_2 could also be modified.

Second Aramova-Herzog example.

On one hand it is significantly simpler than the first one: the concerned module is a k-vector space of finite dimension 3, so that any computation is elementary. On another hand it is a little harder: the interesting differential to be constructed is quadratic. Note in particular it was not obvious in the previous example to obtain the effective homology: the concerned module was a k-vector space of infinite dimension, but the standard methods of effective homology know how to overcome such a problem; in fact they were invented exactly to *overcome* such a problem, see [5].

The underlying ground ring now is $A = k[x_1, x_2]$ and we consider the module $M = {<}x_1, x_2{>}/{<}x_1^2, x_2^2{>}$. The module M is a k-vector space of dimension 3. The Koszul complex is of dimension 3 in degrees 0 and 2, of dimension 6 in degree 1. The simplest form of the effective homology is well described by this figure.

	$\mathrm{Ksz}_0(M) = k^3$	$\mathrm{Ksz}_1(M) = k^6$	$\mathrm{Ksz}_2(M) = k^3$
R_1		$x_1\, dx_2$	$-x_1\, dx_1.dx_2$ $x_2\, dx_1.dx_2$
R_2	$x_1 x_2$	$x_1 x_2\, dx_1$ $x_1 x_2\, dx_2$	
R_3	x_1 x_2	$x_2\, dx_1 - x_1\, dx_2$ $x_1\, dx_1$ $x_2\, dx_2$	$x_1 x_2\, dx_1.dx_2$

Each column corresponds to a component of the Koszul complex and the (almost) canonical basis is shared in boundary preimages, cycles homologous to zero, and homology classes, each homology class being represented by a cycle not at all homologous to zero. The effective homology:

$$\rho = \boxed{h \circlearrowright \mathrm{Ksz}(M) \underset{f}{\overset{g}{\leftrightarrows}} H}$$

is read on the figure as follows. The map g consists in representing the homology classes by the cycles listed on the bottom row R_3. The map f is the inverse projection which forgets the basis vectors of the rows R_1 and R_2. The differentials and the homotopy operator h are simultaneously represented by bidirectional arrows. The chosen supplementary of the homology groups – in fact of the representing cycles – are shared in two components (R_1 and R_2) isomorphic through the differential in the decreasing direction, through the homotopy operator in the increasing direction. This diagram expresses in a very detailed way the Betti numers are $(2, 3, 1)$.

The chain complex H is $[0 \leftarrow k^2 \leftarrow k^3 \leftarrow k \leftarrow 0]$ with null differentials. We have to install the right differential on $H \otimes A$. With the same notations as in the previous section, the differential d_2 of the minimal resolution is obtained by a unique non-null term of the series (Σ) following the path:

$$
\begin{array}{rcl}
 & & \mathfrak{h}_{2,1} \\
(g_2 \otimes \mathrm{id}_A) & \mapsto & x_1 x_2\, dx_1.dx_2 \\
\partial' & \mapsto & x_1 x_2\, dx_2 \otimes x_1 - x_1 x_2\, dx_1 \otimes x_2 \\
-(h_1 \otimes \mathrm{id}_A) & \mapsto & -x_2\, dx_1.dx_2 \otimes x_1 - x_1\, dx_1.dx_2 \otimes x_2 \\
\partial' & \mapsto & -x_2\, dx_2 \otimes x_1^2 + (x_2\, dx_1 - x_1\, dx_2) \otimes x_1 x_2 + x_1\, dx_1 \otimes x_2^2 \\
(f_1 \otimes \mathrm{id}_A) & \mapsto & -\mathfrak{h}_{1,3} \otimes x_1^2 - \mathfrak{h}_{1,1} \otimes x_1 x_2 + \mathfrak{h}_{1,2} \otimes x_2^2,
\end{array}
$$

that is, the same result as in [1], except innocent sign changes and permutations. All the other terms produced by the series (Σ) are null.

The "path" described above makes also obvious the nilpotency argument which guarantees the convergence of the series (Σ): in $M \otimes \wedge V \otimes A$, the central term $\wedge V$ "inhales" the monomials from the lefthand factor M and partly "exhales" them to the righthand side after some processing, giving back also something on the lefthand side but with a strictly inferior degree. After a finite number of steps, certainly nothing anymore on the lefthand side. This is particularly clear in the homogeneous case, a little more dificult but interesting in the general case: the Groebner monomial orders again play an important role here.

You see in fact the nature of this example is *essentially* the same as for our initial "minimal non-trivial" example.

The favourite Kreuzer-Robbiano example.

Martin Kreuzer and Lorenzo Robbiano use a little more complicated toy example in their book [4, Chapter 4], in fact close to the first Aramova-Herzog example. Again the ring $A = k[x_1, x_2, x_3, x_4]$ but the ideal is nomore monomial: $I = <x_2^3 - x_1^2 x_3,\ x_1 x_3^2 - x_2^2 x_4, x_3^3 - x_2 x_4^2, x_2 x_3 - x_1 x4>$. It is a Groebner basis for DegRevLex, so that step 1 of the algorithm is void, but the ideal is nomore monomial and step 3 is not. Keeping the leading terms, we consider the close ideal $I' = <x_2^3, x_1 x_3^2, x_3^3, x_2 x_3>$. It is a monomial ideal and the effective homology of the Koszul complex $\mathrm{Ksz}(A/I')$ is easily computed; the Betti numbers are $(1, 4, 4, 1)$ and Kenzo gives for example as a generator of the 3-homology the cycle

$-x_3^2\,dx_1.dx_2.dx_3$. Applying the homological perturbation lemma to take account of the difference between I and I' gives the effective homology of Ksz(A/I); the new Betti numbers are certainly bounded by the previous ones, but in this simple case, they are the same. The generator of the homology in dimension 3 is now $-x_3^2\,dx_1.dx_2.dx_3 + x_2x_4\,dx_1.dx_2.dx_4 - x_1x_3\,dx_1.dx_3.dx_4 + x_2^2\,dx_2.dx_3.dx_4$. There remains to play the same game with the components f, g and h of the effective homology, and also with the differential ∂' of the Aramova-Herzog bicomplex, exactly the same game as before, nothing more, to obtain the minimal resolution:

$$0 \longleftarrow A \xleftarrow{d_1} A^4 \xleftarrow{d_2} A^4 \xleftarrow{d_3} A \longleftarrow 0$$

with the matrices:

$$d_1 = \left[\; x_1^2x_3 - x_2^3, -x_1x_3^2 + x_2^2x_4, x_2x_4^2 - x_3^3, -x_1x_4 + x_2x_3 \;\right]$$

$$d_2 = \begin{bmatrix} 0 & -x_3 & -x_4 & 0 \\ -x_3 & -x_1 & -x_2 & x_4 \\ x_1 & 0 & 0 & -x_2 \\ x_2x_4 & -x_2^2 & -x_1x_3 & -x_3^2 \end{bmatrix} \qquad d_3 = \begin{bmatrix} -x_2 \\ -x_4 \\ x_3 \\ -x_1 \end{bmatrix}$$

Another toy example.

Let us finally consider now the non-homogeneous ideal:

$$I = <t^5 - x, t^3y - x^2, t^2y^2 - xz, t^3z - y^2, t^2x - y, tx^2 - z, x^3 - ty^2, y^3 - x^2z, xy - tz>$$

This ideal seems more complicated than the previous one, but in a sense in fact it is not. This ideal is obtained by applying the DegRevLex Groebner process to $I = <x - t^5, y - t^7, z - t^{11}>$ and the simple arithmetic nature of the toric generators allow us to expect a simple minimal resolution. But the program ignores this expression of I and it is interesting to observe the result of its study: the minimal resolution is in principle a machine to analyze the *deep* structure of an ideal or module. Macaulay2's `resolution` gives for A/I a resolution with Betti numbers $(1, 7, 11, 6, 1)$ which is not minimal[4]. On the contrary, Singulars `mres` computes the minimal resolution, necessarily equivalent to ours; but to our knowledge, Singular does not give any information about the connection between the homology of the Koszul complex and this minimal resolution, in particular between the *effective* character of the homology of the Koszul complex and the *effective* character of the obtained resolution. No indication in [3] about these subjects. See [6] for details about our point of view.

The approximate monomial module A/I' has Betti numbers $(1, 9, 15, 8, 1)$. Applying the homological perturbation lemma between Ksz(A/I') and Ksz(A/I) gives the effective homology of the last one. The Betti numbers are, surprise, $(1, 3, 3, 1)$. For example a generator for the 3-homology is $-x^2\,dt.dx.dy + tx\,dt.dx.dz\; -$

[4]But the writer of the part of this text is not at all a Macaulay2 expert; using the rich set of Macaulay2 procedures, it is certainly possible to compute the minimal resolution.

$t^4 dt.dy.dz + dx.dy.dz$. The same process as before using the Aramova-Herzog bicomplex now describes a possible minimal resolution. The differentials can be:

$$d_1 = \left[-t^2x + y, -tx^2 + z, -t^5 + x\right]$$

$$d_2 = \begin{bmatrix} 0 & t^5 - x & tx^2 - z \\ t^5 - x & 0 & -tx^2 + y \\ -tx^2 + z & -t^2x + y & 0 \end{bmatrix}$$

$$d_3 = \begin{bmatrix} -t^2x + y \\ tx^2 - z \\ -t^5 + x \end{bmatrix}$$

With respect to the series (Σ), each term of degree k in the previous matrices comes from a term of the series with $i = k - 1$. Here all the terms of the series are null for $i \geq 5$: in fact the degree corresponds to the number of applications of ∂'.

References

[1] Annetta Aramova and Jürgen Herzog. *Free resolutions and Koszul homology.* Journal of Pure and Applied Algebra, 1995, vol.105, pp 1-16.

[2] Xavier Dousson, Julio Rubio, Francis Sergeraert and Yvon Siret. *The Kenzo program.* `www-fourier.ujf-grenoble.fr/~sergerar/Kenzo/`

[3] Bert-Martin Greuel and Gerhard Pfister. *A **Singular** Introduction to Commutative Algebra.* Springer, 2002.

[4] Martin Kreuzer, Lorenzo Robbiano. *Computational Commutative Algebra 2.* Springer, 2005.

[5] Julio Rubio, Francis Sergeraert. *Constructive Algebraic Topology.* Bulletin des Sciences Mathématiques, 2002, vol. 126, pp. 389-412.

[6] Julio Rubio, Francis Sergeraert. *Genova Lecture Notes.* `www-fourier.ujf-grenoble.fr/~sergerar/Papers/Genova-Lecture-Notes.pdf`