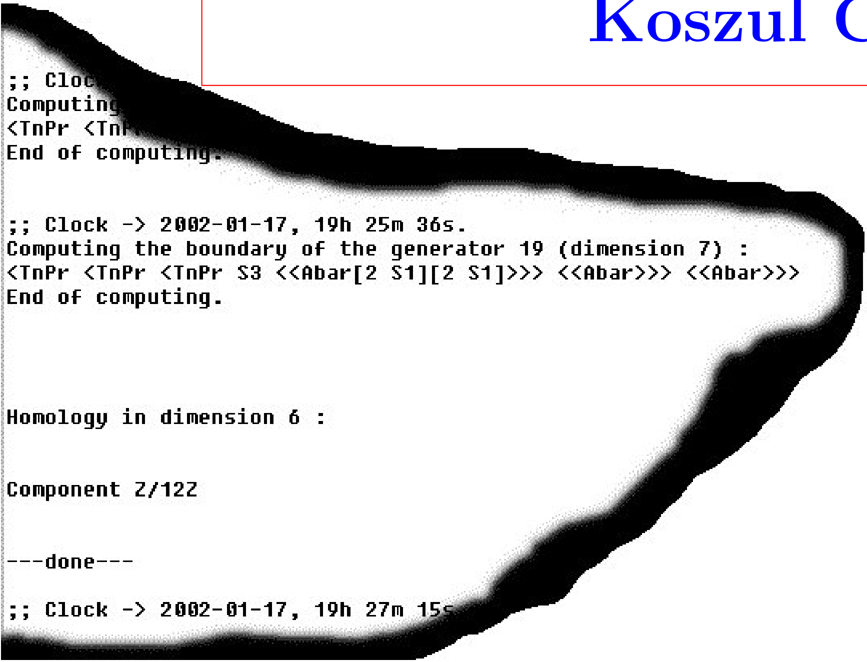
Effective Homology

of

Koszul Complexes



Francis Sergeraert, Institut Fourier, Grenoble, France

Plan

1. Tor $^{A}(M, k)$
2. A reduction solves a homological problem.
3. Basic Perturbation Lemma.
4. Effective Homology.
5. Cone construction and BPL.
6. SES_2 and SES_3 theorems.
7. Effective Tor $^{A}(A/I, k)$.
8. Aramova-Herzog bicomplex + Effective Homology

   $\mapsto$ Minimal Resolution.

k = commutative field. A = commutative k-algebra.

$x_1, \ldots, x_m \in A$. M = A-module.

<u>Definition</u>: The Koszul complex $K_A(M; x_1, \ldots, x_m)$ is a chain complex K_* of A-modules with:

$$K_n := M \otimes_k \wedge^n k^m$$

A generator of K_n is denoted by $m\ \delta x_{i_1} \cdots \delta x_{i_n}$.

Differential: $d : K_n \longrightarrow K_{n-1}$:

$$\begin{aligned} m\ \delta x_{i_1}\ \cdots\ \delta x_{i_n} \mapsto\ & + m x_{i_1}\ \delta x_{i_2} \cdots \delta x_{i_n} \\ & - m x_{i_2}\ \delta x_{i_1}\ \delta x_{i_3} \cdots \delta x_{i_n} \\ & + \cdots \\ & + (-1)^{n-1}\ m x_{i_n}\ \delta x_{i_1}\ \delta x_{i_2} \cdots \delta x_{i_{n-1}} \end{aligned}$$

"Geometrical" interpretation of Koszul complexes.

Principal case:

$$K_A(A; x_1, \ldots) = A \otimes_t \wedge k^m \ (\sim \text{total space})$$

$$\begin{aligned} A &= \text{structural algebra } (\sim \text{structural group}); \\ \wedge k^m &= \text{base coalgebra } (\sim \text{base space}); \\ t &= \text{twisting cochain } (\sim \text{twisting function}); \end{aligned}$$

General case:

$$\begin{aligned} K_A(M; x_1, \ldots) &= M \otimes_A (A \otimes_t \wedge k^m) \\ &= \text{Fibration associated to } M \otimes_A A \to M. \end{aligned}$$

Particular case: $A = k[x_1, \ldots, x_m]$.

$$K(A; x_1, \ldots, x_m) =: K(A) := A \otimes_t \wedge k^m$$
$$= \text{canonical Koszul complex of } A \text{ is } \boxed{\text{acyclic}}.$$

$$K(A) \text{ acyclic} \Leftrightarrow K(A) = \text{universal fibration of } A$$
$$\Leftrightarrow K(A) = A\text{-resolution of } k:$$

$$0 \leftarrow k \leftarrow A \leftarrow A \otimes k^m \leftarrow A \otimes \wedge^2 k^m \leftarrow \ldots$$

$\Rightarrow K(A) =$ **possible tool to compute** $\mathrm{Tor}^A(M, k)$.

Definition: M and N = A-modules $\Rightarrow$ $\mathrm{Tor}^A(M, N) =$???

Let $R_A(M)$ be an A-resolution of M,

$R_A(N)$ an A-resolution of N.

$$H_*(R_A(M) \otimes_A N) =: \mathrm{Tor}^A(M, N) := H_*(M \otimes_A R_A(N)).$$

Standard method computing $\mathrm{Tor}^A(M, k)$:

1. Compute an A-resolution $R_A(M)$ of M of A-finite type.

   (Syzygies)

2. $\Rightarrow R_A(M) \otimes_A k =$

   Chain complex of finite dimensional k-vector spaces.

3. $\Rightarrow H_*(R_A(M) \otimes_A k) = \mathrm{Tor}^A(M, k) =$

   elementary computation.

Drawbacks: 1) $R_A(M)$ = sygyzies $\Rightarrow$ not so easy.

2) It happens $\mathrm{Tor}^A(M,k) := H_*(M \otimes_A R_A(k))$

can be much more interesting !!

Theorem (Serre): $\mathcal{S}$ = PDE local system in $0 \in k^m$.

$I_{\mathcal{S}}$ = canonical ideal associated to $\mathcal{S}$.

Then $\mathcal{S}$ involutive $\Leftrightarrow \mathrm{Tor}^A(I_{\mathcal{S}},k)_+ = 0$.

But the theorem comes

from the explicit examination of $I \otimes_A R_A(k)$.

Using this theorem needs a complete solution

for the **homological problem** of $I \otimes_A R_A(k)$.

Solving the homological problem for a chain complex C_*

$\Leftrightarrow$ You must be able to:

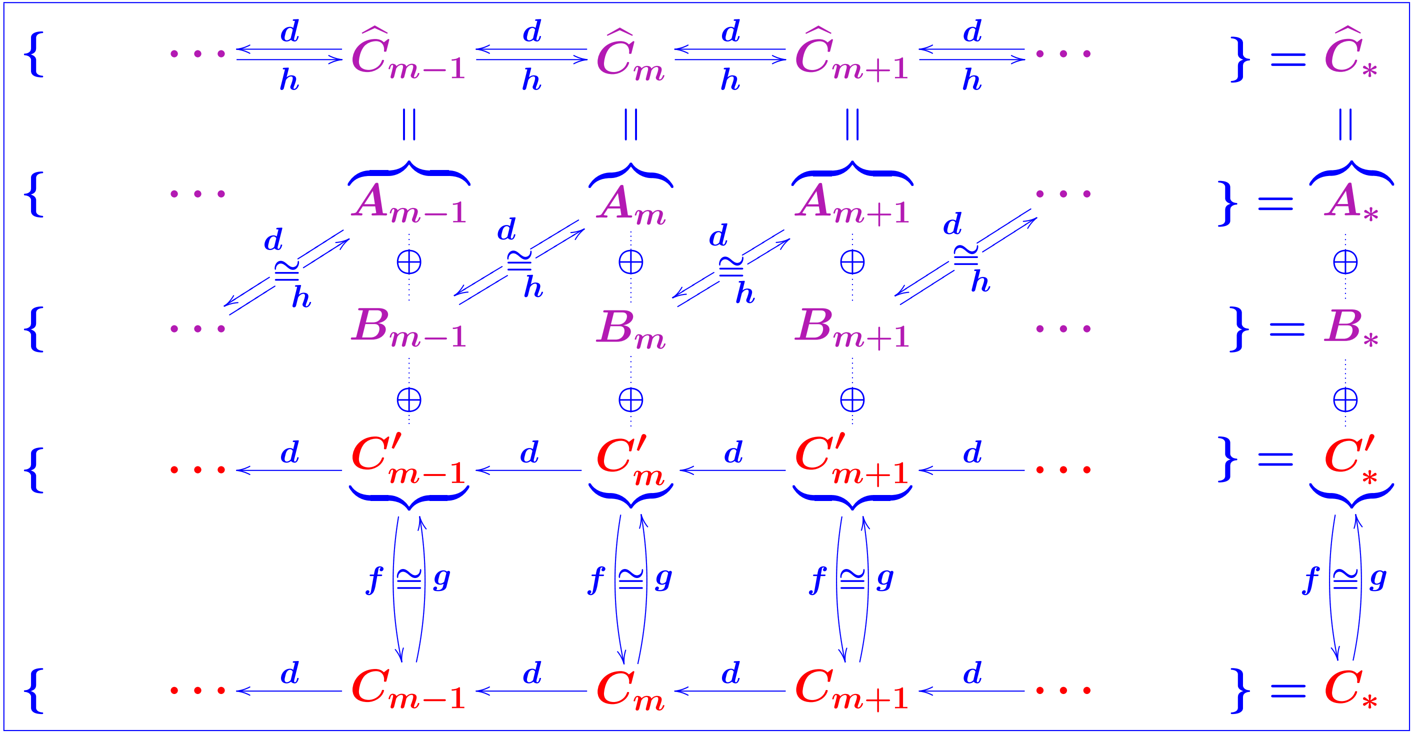
1. Determine the isomorphism class of $H_i(C_*)$ for arbitrary $i \in \mathbb{Z}$.
2. Produce a map $\rho : H_i(C_*) \to C_i$ giving a representant for every homology class.
3. Determine whether an arbitrary chain $c \in C_i$ is a cycle.
4. Compute, given an arbitrary cycle $z \in Z_i = \ker(d_i : C_i \to C_{i-1})$, its homology class $\overline{z} \in H_i(C_*)$.
5. Compute, given a cycle $z \in Z_i$ known as a boundary ($\overline{z} = 0$), a boundary-premimage $c \in C_{i+1}$ ($d_{i+1}(c) = z$).

Definition: **A (homological) reduction is a diagram:**

$$\rho : \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$$

with:

1. $\widehat{C}_*$ **and** C_* = **chain complexes.**
2. f **and** g = **chain complex morphisms.**
3. h = **homotopy operator (degree +1).**
4. $fg = \mathrm{id}_{C_*}$ **and** $d_{\widehat{C}}h + hd_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$.
5. $fh = 0$, $hg = 0$ **and** $hh = 0$.



$A_* = \ker f \cap \ker h$ $\quad$ $B_* = \ker f \cap \ker d$ $\quad$ $C'_* = \operatorname{im} g$

$$\widehat{C}_* = A_* \oplus B_* \text{ exact} \oplus C'_* \cong C_*$$

Let ρ : $h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*$ be a reduction.

Frequently:

1. **$\widehat{C}_*$ is a locally effective chain complex:**
   its homology groups are unreachable.

2. **C is an effective chain complex:**
   its homology groups are computable.

3. **The reduction ρ is an entire description of**
   the homological nature of $\widehat{C}_*$.

4. **Any homological problem in $\widehat{C}_*$ is solvable**
   thanks to the information provided by ρ.

$$\rho : \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*}$$

1. **What is $H_n(\widehat{C}_*)$?** **Solution: Compute $H_n(C_*)$.**

2. **Let $x \in \widehat{C}_n$. Is x a cycle?** **Solution: Compute $d_{\widehat{C}_*}(x)$.**

3. **Let $x, x' \in \widehat{C}_n$ be cycles. Are they homologous?**

   Solution: Look whether $f(x)$ and $f(x')$ are homologous.

4. **Let $x, x' \in \widehat{C}_n$ be homologous cycles.**

   Find $y \in \widehat{C}_{n+1}$ satisfying $dy = x - x'$?

   Solution:

   (a) Find $z \in C_{n+1}$ satisfying $dz = f(x) - f(x')$.

   (b) $y = g(z) + h(x - x')$.

Definition: (C_*, d) = given chain complex.

A perturbation δ: $C_* \to C_{*-1}$ is an operator of degree -1

satisfying $(d+\delta)^2 = 0$ ($\Leftrightarrow$ $(d\delta + \delta d + \delta^2) = 0$):

$$(C_*, d) + (\delta) \mapsto (C_*, d+\delta).$$

Problem: Let ρ : $\boxed{h \circlearrowright (\widehat{C}_*, \widehat{d}) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d)}$ be a given reduction and $\widehat{\delta}$ a perturbation of $\widehat{d}$.

How to determine a new reduction:

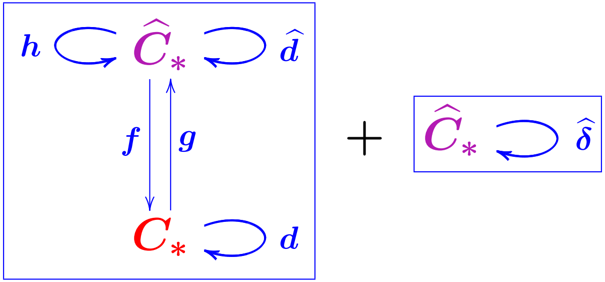
$$? : \boxed{? \circlearrowright (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \underset{?}{\overset{?}{\leftrightarrows}} (C_*, ?)}$$

describing in the same way the homology of

the chain complex with the perturbed differential?

Basic Perturbation "Lemma" (BPL):

Given:

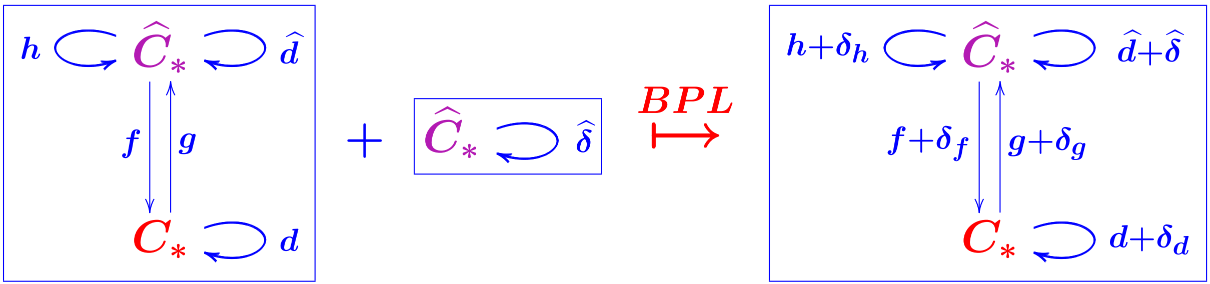


satisfying:

1. $\widehat{\delta}$ is a perturbation of the differential $\widehat{d}$;

2. The operator $h \circ \widehat{\delta}$ is pointwise nilpotent.

Then a general algorithm ***BPL*** **constructs:**



Proof:

$\phi := \sum_{i=1}^{\infty}(-1)^i(h\widehat{\delta})^i$ and $\psi := \sum_{i=1}^{\infty}(-1)^i(\widehat{\delta}h)^i$ are defined.

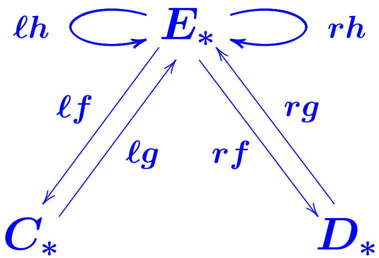
Then:

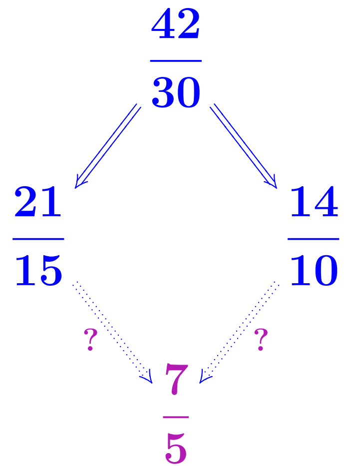
- $\delta_d := f\widehat{\delta}(\mathrm{id}_{\widehat{C}} + \phi)g = f(\mathrm{id}_{\widehat{C}} + \psi)\widehat{\delta}g$
- $\delta_f := f\psi$
- $\delta_g := \phi g$
- $\delta_h := \phi h = h\psi$

is the solution.

QED

<u>Definition</u>: A (strong chain-) equivalence $\varepsilon : C_* \Longleftarrow\!\!\!\Longrightarrow D_*$ is a pair of reductions $C_* \overset{\ell\rho}{\Longleftarrow} E_* \overset{r\rho}{\Longrightarrow} D_*$:





Normal form problem ??

More structure often necessary in C_*.

Definition: An **object with effective homology** X is a 4-tuple:

$$X = \boxed{X, C_*(X), EC_*, \varepsilon}$$

with:

1. X = an arbitrary **object** (**simplicial set**, **simplicial group**, **differential graded algebra**, ...)
2. $C_*(X)$ = the **chain complex** "traditionally" associated to X to define the **homology groups** $H_*(X)$.
3. EC_* = some **effective chain complex**.
4. ε = some **equivalence** $C_*(X) \stackrel{\varepsilon}{\Longleftarrow\!\!\!\Longrightarrow} EC_*$.

Main result of effective homology:

Meta-theorem: Let $X_{1*}, \ldots, X_{n*}$ be a collection of objects with effective homology and ϕ be a reasonable construction process:

$$\phi : (X_{1*}, \ldots, X_{n*}) \mapsto X_*.$$

Then there exists a version with effective homology ϕ_{EH}:

$$\phi_{EH}\colon \left(\boxed{X_1, C_*(X_1), EC_{1*}, \varepsilon_1}, \ldots, \boxed{X_n, C_*(X_n), EC_{n*}, \varepsilon_n}\right)$$
$$\mapsto \boxed{X, C_*(X), EC_*, \varepsilon}$$

The process is perfectly stable

and can be again used with X for further calculations.

Typical example of PBL application: the SES_2 Theorem.

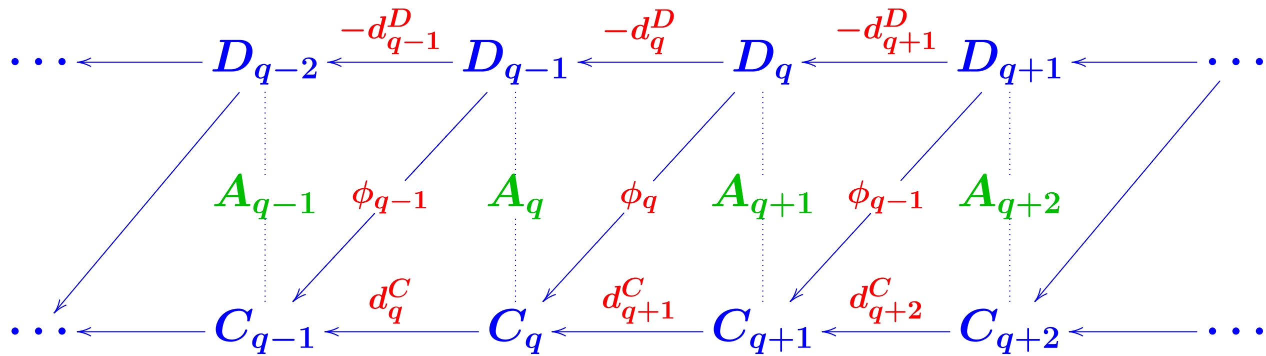
Definition: The algebraic cone construction:

Ingredients: two chain complexes C_*, D_*

and a chain-complex morphism $\phi : C_* \leftarrow D_*$.

Result: a chain complex $A_* = \mathrm{Cone}(\phi)$ defined by:

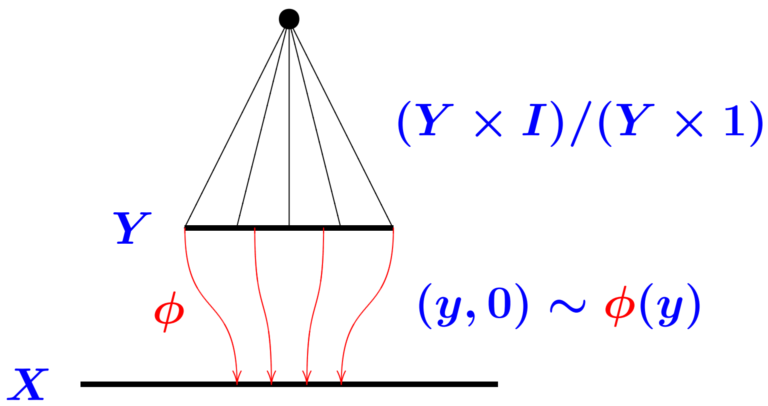
$$A_q = C_q \oplus D_{q-1} \qquad d_q^A = \begin{bmatrix} d_q^C & \phi_q \\ 0 & -d_{q-1}^D \end{bmatrix}$$



Geometrical interpretation.

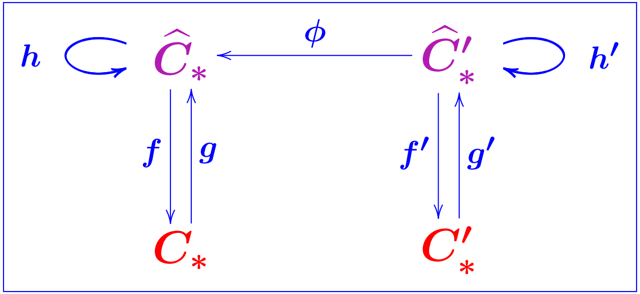
$\phi : X \leftarrow Y$ = continuous map.

$\text{Cone}(\phi) := (X \coprod (Y \times I)) \;/\; ((Y \times 1) \;\&\; (y,0) \sim \phi(y))$

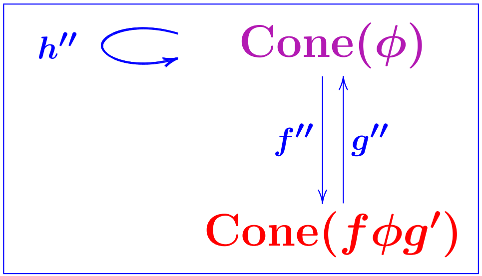


SES$_2$ Theorem: A general algorithm CR can be produced:

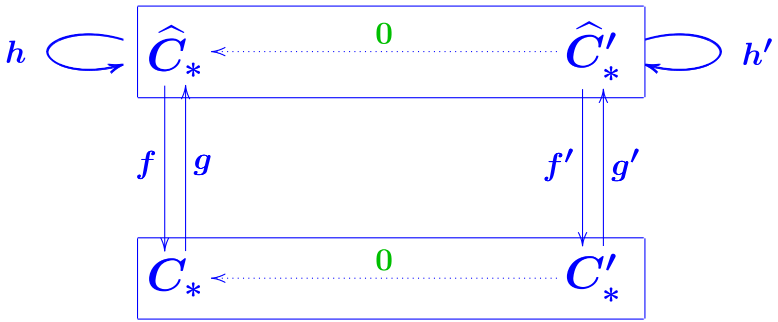
Input:



Output:



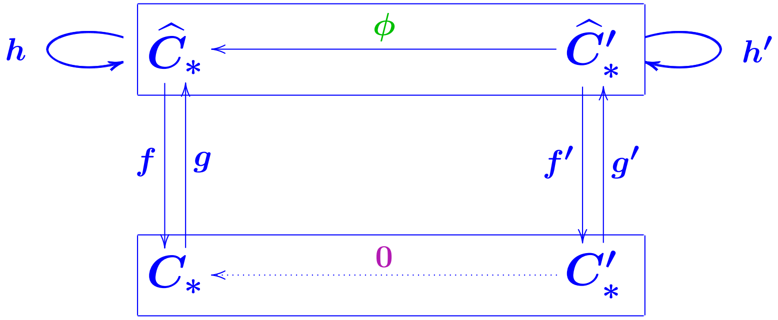
Proof: 1. Particular case $\phi = 0$: trivial (direct sums).



$$\begin{bmatrix} \widehat{d} & 0 \\ 0 & -\widehat{d}' \end{bmatrix} \quad \begin{bmatrix} d & 0 \\ 0 & -d' \end{bmatrix} \quad \begin{bmatrix} f & 0 \\ 0 & f' \end{bmatrix} \quad \begin{bmatrix} g & 0 \\ 0 & g' \end{bmatrix} \quad \begin{bmatrix} h & 0 \\ 0 & -h' \end{bmatrix}$$

$$\widehat{D} \qquad D \qquad F \qquad G \qquad H$$

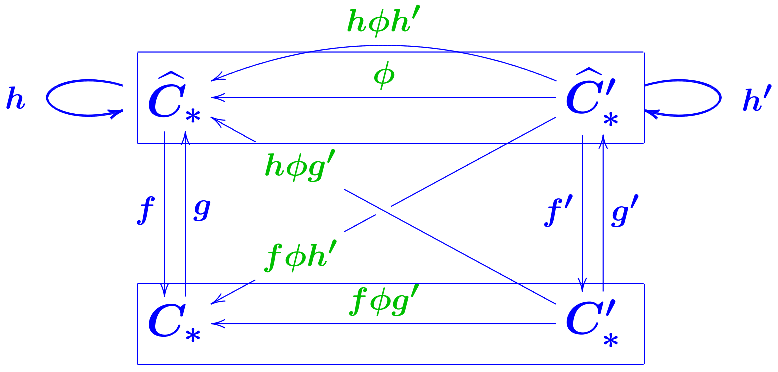
Proof: 2. Install the actual ϕ. The reduction is nomore valid.



$$\begin{bmatrix} \widehat{d} & \phi \\ 0 & -\widehat{d}' \end{bmatrix} \quad \begin{bmatrix} d & 0 \\ 0 & -d' \end{bmatrix} \quad \begin{bmatrix} f & 0 \\ 0 & f' \end{bmatrix} \quad \begin{bmatrix} g & 0 \\ 0 & g' \end{bmatrix} \quad \begin{bmatrix} h & 0 \\ 0 & -h' \end{bmatrix}$$

$$\widehat{D} \qquad D \qquad F \qquad G \qquad H$$

<u>Proof</u>: 3. Apply the Basic Perturbation Lemma:



$$\begin{bmatrix} \widehat{d} & \phi \\ 0 & -\widehat{d}' \end{bmatrix} \quad \begin{bmatrix} d & f\phi g' \\ 0 & -d' \end{bmatrix} \quad \begin{bmatrix} f & f\phi h' \\ 0 & f' \end{bmatrix} \quad \begin{bmatrix} g & -h\phi g' \\ 0 & g' \end{bmatrix} \quad \begin{bmatrix} h & h\phi h' \\ 0 & -h' \end{bmatrix}$$

$$\widehat{D} \qquad D \qquad F \qquad G \qquad H$$

QED.

Why the terminology SES$_2$ theorem?

A morphism $\phi : A_* \leftarrow B_*$ produces

an effective Short Exact Sequence of chain complexes:

$$0 \longrightarrow A_* \underset{i}{\overset{\rho}{\rightleftarrows}} \mathrm{Cone}(\phi) \underset{j}{\overset{\sigma}{\rightleftarrows}} B_* \longrightarrow 0$$

and the SES$_2$ theorem is an algorithm:

$$[\mathrm{Reduction}(A_*) + \mathrm{Reduction}(B_*)] \;\mapsto\; \mathrm{Reduction}(\mathrm{Cone}(\phi))$$

Notation: $\rho : \widehat{C}_* \Rrightarrow C_* \quad \Leftrightarrow \quad \rho : \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*}$.

Theorem (**Easy Basic Perturbation Lemma**):

$\boxed{\rho : (\widehat{C}_*, \widehat{d}) \Rrightarrow (C_*, d)} + \boxed{\delta : C_* \to C_{*-1} = \text{perturbation of } d}$

$\mapsto \boxed{\rho' : (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \Rrightarrow (C_*, d + \delta)}$.

Proof: $(\widehat{C}_*, \widehat{d}) = (A_*, \widehat{d}) \oplus (C'_*, d')$ **with** $(C'_*, d') \cong (C, d)$.

Copy into (C'_*, d') **the perturbation** $\delta \quad \mapsto (C'_*, d' + \delta')$.

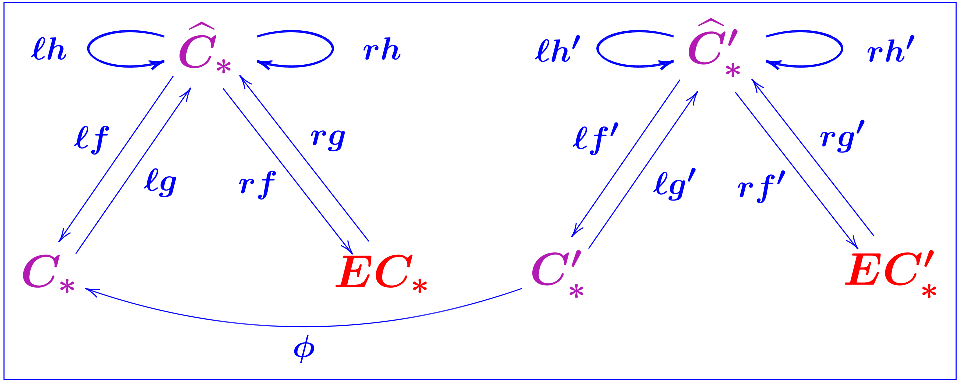
Solution $= \rho : ((A_*, \widehat{d}) \oplus (C'_*, d' + \delta')) \Rrightarrow (C_*, d + \delta)$.

QED

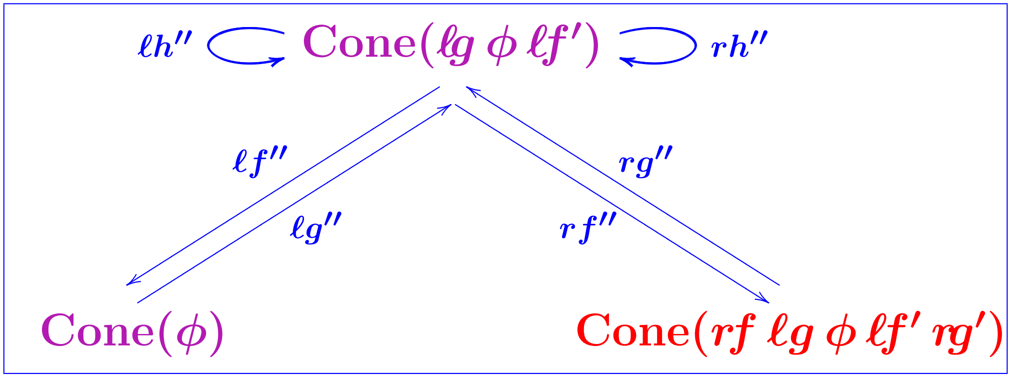
Cone-Equivalence Theorem:

A general algorithm CE can be produced:

Input:



Output:



SES$_3$ Theorem:

Let $(A, i, \rho, B, j, \sigma, C)$ be

an effective short exact sequence of chain-complexes:

$$0 \longrightarrow A_* \underset{i}{\overset{\rho}{\leftrightarrows}} B_* \underset{j}{\overset{\sigma}{\leftrightarrows}} C_* \longrightarrow 0$$

where:

1. **The i and j arrows are chain complex morphisms.**
2. **The ρ and σ arrows are graded module morphisms.**
3. $\mathrm{id}_{A_*} = \rho \circ i \;\; ; \;\; \mathrm{id}_{B_*} = i \circ \rho + \sigma \circ j \;\; ; \;\; \mathrm{id}_{C_*} = j \circ \sigma$.

Then an algorithm constructs a canonical reduction:

$$\mathrm{Cone}\,(i) \ggg C_*$$

from the data.

Proof:

1. Cancel all the differentials.

Then an obvious reduction is obtained:

$$\rho \circlearrowright \boxed{[A_*, 0] \xrightarrow{i} [B_*, 0]} \underset{j}{\overset{\sigma}{\rightleftarrows}} [C_*, 0]$$

2. Reinstall the differentials of A_* and B_*.

3. To be interpreted

as a perturbation of the differential of Cone(i).

4. Apply BPL.

$$\rho \circlearrowright \boxed{[A_*, d_A] \xrightarrow{i} [B_*, d_B]} \underset{j}{\overset{\sigma - \rho d_B \sigma}{\rightleftarrows}} [C_*, d_C]$$

QED

Corollary: Same data:

$$0 \longrightarrow A_* \underset{i}{\overset{\rho}{\rightleftarrows}} B_* \underset{j}{\overset{\sigma}{\rightleftarrows}} C_* \longrightarrow 0$$

+ $A_* \overset{\varepsilon_A}{\Longleftarrow\!\!\Longrightarrow} EA_*$ **and** $B_* \overset{\varepsilon_B}{\Longleftarrow\!\!\Longrightarrow} EB_*$ **with effective homology.**

Then an algorithm constructs $\varepsilon_C : C_* \Longleftarrow\!\!\Longrightarrow EC_*$.

Proof:

$$C_* \Lleftarrow \mathrm{Cone}\,(i) \Lleftarrow \widehat{\mathrm{Cone}\,(i)} \Rrightarrow E\mathrm{Cone}\,(i)$$

+ **Composition of reductions.**

QED.

Previous results $\Rightarrow$

A simple algorithm computes

the effective homology of $K(A/<g_1, \ldots, g_n>)$.

Typical simple example.

$I = < x - t^3, y - t^5 > \subset A = \mathbb{Q}[x, y, t].$

How to compute $H_*(K(A/I)) = H_*(K(A/I; x, y, t))$ **?**

Step 1: Compute a Groebner basis for I**.**

Choose a coherent monomial order,

for example DegRevLex = DRL.

$\Rightarrow \text{Groebner}(I, \text{DRL}) = < xt^2 - y, t^3 - x, x^2 - yt >.$

Step 2: Consider $J = < x^2, xt^2, t^3 >$ **(Lex preferred here)**

= the associated monomial ideal.

Then: 1. The $\mathbb{Q}$-vector spaces A/I and A/J are canonically isomorphic.

2. $\Rightarrow$ $K(A/I)$ and $K(A/J)$ are graded $\mathbb{Q}$-vector spaces canonically isomorphic, but with non-compatible differentials:

$$d_{K(A/J)}(t^2\delta x) = 0 \quad ; \quad d_{K(A/I)}(t^2\delta x) = y.$$

Plan: 1. Compute $H_*(K(A/J))$.

2. Apply BPL to deduce $H_*(K(A/I))$.

How to compute $H_*(A/<x^2, xt^2, t^3>)$?

Recursive process about the number of generators.

Relation between $H_*(A/<x^2, xt^2, t^3>)$ and $H_*(A/<xt^2, t^3>)$?

Exact sequence of A-modules:

$$0 \to \frac{A}{<t^2>} \xrightarrow{\times x^2} \frac{A}{<xt^2, t^3>} \xrightarrow{\text{pr}} \frac{A}{<x^2, xt^2, t^3>} \to 0$$

Remark: $<t^2> = <xt^2, t^3> : x^2 = \{a \in A \ \underline{\text{st}} \ ax^2 \in <xt^2, t^3>\}$.

$\Rightarrow$ Exact sequence of chain complexes:

$$0 \to K\left(\frac{A}{<t^2>}\right) \xrightarrow{\times x^2} K\left(\frac{A}{<xt^2, t^3>}\right) \xrightarrow{\text{pr}} K\left(\frac{A}{<x^2, xt^2, t^3>}\right) \to 0$$

$\Rightarrow$ Effective homologies of $K(A/<t^2>)$ and $K(A/<xt^2,t^3>)$
give effective homology of $K(A/<x^2,xt^2,t^3>)$

What about the first step of the recursive process?

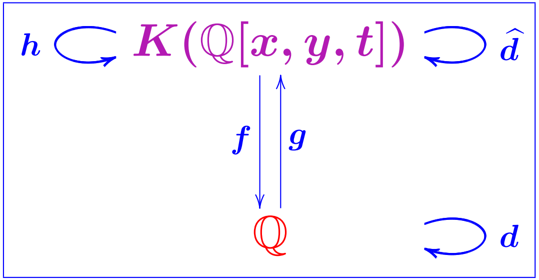
Continuing in the same way $\Rightarrow$ short exact sequence:

$$0 \to K\left(\frac{A}{<>}\right) \xrightarrow{\times t^2} K\left(\frac{A}{<>}\right) \xrightarrow{\text{pr}} K\left(\frac{A}{<t^2>}\right) \to 0$$

$\Rightarrow$

It is enough to know the effective homology of $K(A)$.

Theorem: A multi-homogeneous reduction can be produced:



with all the maps $\widehat{d}$, d, f, g and h homogeneous with respect to a $[x, y, t]$-multi-grading.

Proof.

Multi-grading of $x^\alpha y^\beta t^\gamma\ \delta x\ \delta t = [\alpha+1, \beta, \gamma+1]$

$\Rightarrow$ Koszul differential $\widehat{d}$ is multi-homogeneous.

$h(x^\alpha y^\beta t^\gamma\ \delta x\ \delta t) = 0$

$h(x^\alpha y^\beta t^3\ \delta x) = -x^\alpha y^\beta t^2\ \delta x\ \delta t$

$h(x^\alpha y^4\ \delta x) = -x^\alpha y^3\ \delta x\ \delta y$

$h(x^3\ \delta x) = 0$

$h(x^3) = x^2\ \delta x$

$\Rightarrow$ Contraction h is multi-homogeneous.

The trivial morphisms f and g

are trivially multi-homogeneous.

Easy complements of Effective Homology Theorems:

If every input is multi-homogeneous,

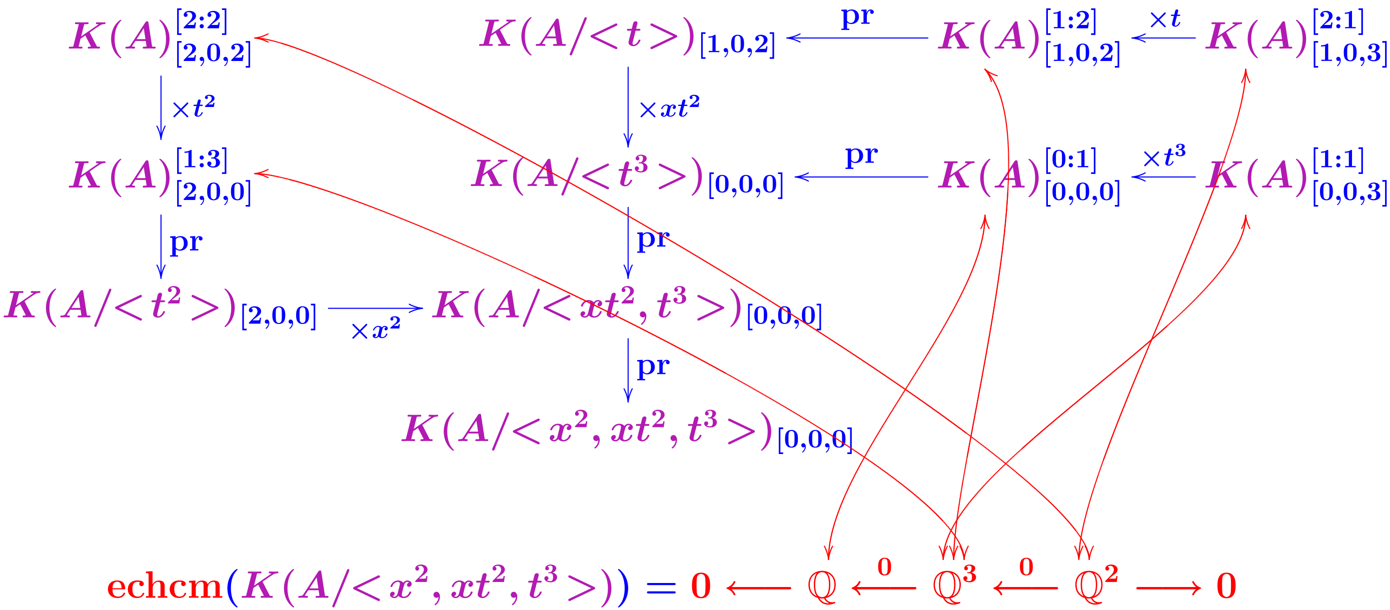
then every output is multi-homogeneous.

Applying to the SES$_3$ theorem for:

$$\mathbf{0} \longrightarrow \boldsymbol{K}(\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}]) \xrightarrow{\times t^{\boxed{2}}} \boldsymbol{K}(\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}]) \xrightarrow{\text{pr}} \boldsymbol{K}\left(\frac{\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}]}{<\boldsymbol{t}^2>}\right) \longrightarrow \mathbf{0}$$

Multiplication by $\boldsymbol{t}^{\boxed{2}}$ $\Rightarrow$ you must shift the multi-grading of the lefthand $\boldsymbol{K}(\mathbb{Q}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}])$ to get $\times \boldsymbol{t}^{\boxed{2}}$ multi-homogeneous:

$$\text{Multigrading}(\boldsymbol{x}^\alpha \boldsymbol{y}^\beta \boldsymbol{t}^\gamma\ \delta\boldsymbol{x}\ \delta\boldsymbol{t}) = [\alpha+1, \beta, \gamma+1+\boxed{2}\ ]$$



$K(A)^{[2:2]}_{[2,0,2]}$
$K(A/<t>)_{[1,0,2]}$
pr
$K(A)^{[1:2]}_{[1,0,2]}$
$\times t$
$K(A)^{[2:1]}_{[1,0,3]}$
$\times t^2$
$\times xt^2$
$K(A)^{[1:3]}_{[2,0,0]}$
$K(A/<t^3>)_{[0,0,0]}$
pr
$K(A)^{[0:1]}_{[0,0,0]}$
$\times t^3$
$K(A)^{[1:1]}_{[0,0,3]}$
pr
pr
$K(A/<t^2>)_{[2,0,0]}$
$\times x^2$
$K(A/<xt^2,t^3>)_{[0,0,0]}$
pr
$K(A/<x^2,xt^2,t^3>)_{[0,0,0]}$
$\mathbf{echcm}(K(A/<x^2,xt^2,t^3>)) = 0 \longleftarrow \mathbb{Q} \xleftarrow{0} \mathbb{Q}^3 \xleftarrow{0} \mathbb{Q}^2 \longrightarrow 0$

Intermediate result:

The SES_2 and SES_3 theorems produce an equivalence:

$$[K(A/J), d_J] \overset{\rho_\ell}{\Lleftarrow} [\widehat{C}_*, d_t] \overset{\rho_r}{\Rrightarrow} [\mathbb{Q}^6, d_{br}]$$

with:

- J = monomial ideal canonically associated to the initial ideal:

  $I = \langle xt^2 - y, t^3 - x, x^2 - yt \rangle = \langle x - t^3, y - t^5 \rangle$.

- $\widehat{C}_*$ = some chain-complex.
- $\mathbb{Q}^6$ = $\mathbb{Q}$-chain complex of finite type.
- All the objects are multigraded.
- All the morphisms are multi-homogeneous.

$[K(A/I), d_I] = [K(A/J), d_J + \delta_{I,J}]$:

"Same" graded module, only the differentials are different.

But J = monomial ideal Groebner-associated to I.

$\Rightarrow$ The perturbation $\delta_{I,J}$ strictly reduces the multigrading.

Example:

$$\begin{aligned} d_J(t^2\ \delta x) &= 0 \\ d_I(t^2\ \delta x) &= y \ \text{ with } \ y \text{ “}<\text{” } t^2\ \delta x. \end{aligned}$$

The perturbation recursively replaces

leading monomials by trailing terms.

Easy-BPL $\Rightarrow$

$$[K(A/J), d_J + \delta_{I,J}] \overset{\rho'_\ell}{\twoheadleftarrow\!\!\Leftarrow} [\widehat{C}_*, d_t + \delta'_{IJ}] \,\Big|\, \overset{\rho_r}{\Rightarrow\!\!\twoheadrightarrow} [\mathbb{Q}^6, d_{br}]$$

- Righthand homotopy operator h_r is multi-homogeneous.
- Perturbation δ'_{IJ} strictly reduces the multigrading.

$\Rightarrow$ Composition $h_r \circ \delta'_{IJ}$ is pointwise nilpotent.

$\Rightarrow$ BPL can be applied.

$\Rightarrow$

$$[K(A/J), d_J + \delta_{I,J}] \overset{\rho'_\ell}{\twoheadleftarrow\!\!\Leftarrow} [\widehat{C}_*, d_t + \delta'_{IJ}] \overset{\rho''_r}{\Rightarrow\!\!\twoheadrightarrow} [\mathbb{Q}^6, d_{br} + \delta''_{IJ}]$$

$\Rightarrow$ Effective homology of $K(A/I)$ is obtained. QED.

The Aramova-Herzog bicomplex $AH(M)$.

$AH(M) :=$

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M \otimes \wedge^3 \otimes A_0 & \xrightarrow{\partial''} & M \otimes \wedge^2 \otimes A_1 & \xrightarrow{\partial''} & M \otimes \wedge^1 \otimes A_2 & \xrightarrow{\partial''} & M \otimes \wedge^0 \otimes A_3 & \longrightarrow & 0 \\
{\scriptstyle \partial'}\downarrow & & {\scriptstyle \partial'}\downarrow & & {\scriptstyle \partial'}\downarrow & & \downarrow & & \\
M \otimes \wedge^2 \otimes A_0 & \xrightarrow{\partial''} & M \otimes \wedge^1 \otimes A_1 & \xrightarrow{\partial''} & M \otimes \wedge^0 \otimes A_2 & \longrightarrow & 0 & & \\
{\scriptstyle \partial'}\downarrow & & {\scriptstyle \partial'}\downarrow & & \downarrow & & & & \\
M \otimes \wedge^1 \otimes A_0 & \xrightarrow{\partial''} & M \otimes \wedge^0 \otimes A_1 & \longrightarrow & 0 & & & & \\
{\scriptstyle \partial'}\downarrow & & \downarrow & & & & & & \\
M \otimes \wedge^0 \otimes A_0 & \longrightarrow & 0 & & & & & & \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

$A_p = k[x_1, \ldots, x_m]^{[p]}$

$\wedge^q = \wedge^q k^m = \wedge^q(\mathfrak{m}/\mathfrak{m}^2)$

$M = A$-module

$\otimes = \otimes_k$

Horizontal $= M \otimes K(A)_q$

Vertical $= K(M) \otimes A_p$

1. Horizontal reduction.

Every horizontal complex is a homogeneous component of $M \otimes_k K(A)$:

$$0 \longrightarrow M \otimes \wedge^3 \otimes A_0 \xrightarrow{\partial''} M \otimes \wedge^2 \otimes A_1 \xrightarrow{\partial''} M \otimes \wedge^1 \otimes A_2 \xrightarrow{\partial''} M \otimes \wedge^0 \otimes A_3 \longrightarrow 0$$

But $K(A)$ acyclic $\Rightarrow$ every horizontal is 0-reducible.

Except the 0-horizontal $= M \otimes_k \wedge^0 \otimes_k S_0 = M$.

BPL $\Rightarrow$ A canonical reduction is produced:

$$AH(M) \Rrightarrow M$$

2. Vertical reduction.

The p-vertical is $K(M) \otimes_k A_p$.

Let $K(M) \Rrightarrow H(K(M))$ be a reduction of $K(M)$ over the complex made of the homology groups of $K(M)$ and the null differential.

Applying this reduction to the p-vertical produces:

$$AH(M)_p \Rrightarrow H(K(M)) \otimes_k A_p$$

BPL $\Rightarrow$ a canonical reduction is produced:

$$AH(M) \Rrightarrow H(K(M)) \otimes_k A$$

$\Rightarrow$ **Equivalence:**

$$\boldsymbol{H(K(M)) \otimes_k A \overset{\twoheadleftarrow}{\Longleftarrow} AH(M) \Rrightarrow M}$$

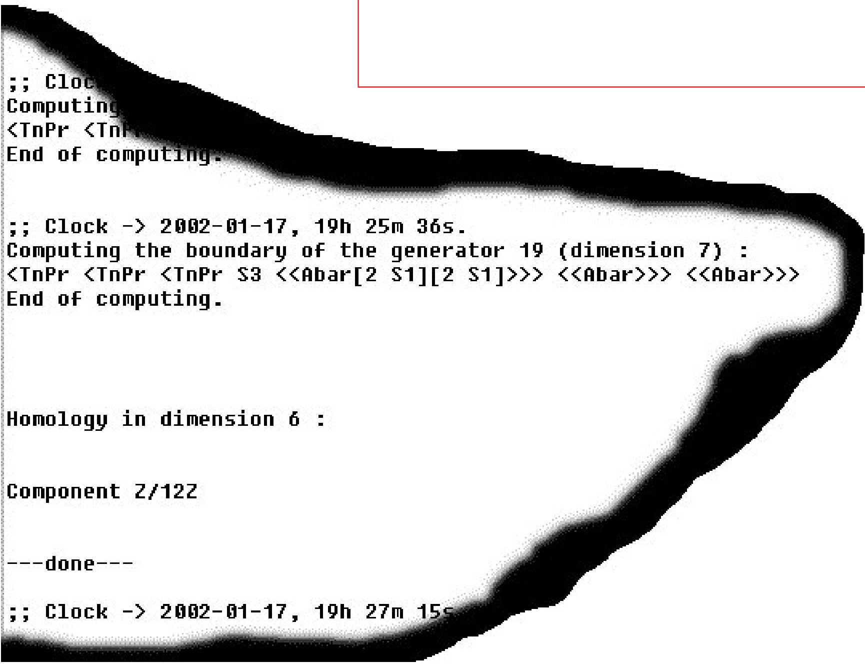
$$\boldsymbol{g_\ell : H(K(M)) \otimes_k A \to AH(M), \quad f_r : AH(M) \to M}$$

Then:

$$\boldsymbol{f_r \circ g_\ell : H(K(M)) \otimes_k A \longrightarrow M}$$

is the looked-for resolution.

The END


;; Cloc
Computing
<TnPr <Tn
End of computing.

;; Clock -> 2002-01-17, 19h 25m 36s.
Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.

Homology in dimension 6 :

Component Z/12Z

---done---

;; Clock -> 2002-01-17, 19h 27m 15s

Francis Sergeraert, Institut Fourier, Grenoble, France