Programming before Theorizing, a case study*

Ana Romero
Universidad de La Rioja
Departamento de Matemáticas y Computación
Logroño, Spain
ana.romero@unirioja.es

Francis Sergeraert
Université Joseph Fourier
Institut Fourier
Grenoble, France
Francis.Sergeraert@ujf-grenoble.fr

ABSTRACT

This paper relates how a "simple" result in combinatorial homotopy eventually led to a totally new understanding of basic theorems in Algebraic Topology, namely the Eilenberg-Zilber theorem, the twisted Eilenberg-Zilber theorem, and finally the Eilenberg-MacLane correspondance between the Classifying Space and Bar constructions. In the last case, it was an amazing lucky consequence of computations based on conjectures not yet proved. The key new tool used in this context is Robin Forman's *Discrete Vector Fields* theory.

Categories and Subject Descriptors

G [**4**]: Algorithm design and analysis—*Algebraic Topology*

General Terms

Algorithms, Design, Experimentation, Performance, Theory.

Keywords

Constructive Algebraic Topology, Fibrations, Eilenberg-Zilber theorems, Classifying Spaces, Bar Construction, Eilenberg-MacLane Spaces, Homotopy Groups

1. INTRODUCTION.

This paper is nothing but story telling, it narrates how some *new unexpected fundamental theoretical* results have been obtained in Algebraic Topology, produced by experimental evidence after numerous computations. These results are complex, not yet fully proved, but the underlying algorithms are entirely known, allowing us to implement them and to use them successfully.

Without waiting for the missing proofs, using the algorithms now available, deep modifications have been made to the Kenzo program, a program for "hard" computations in Algebraic Topology, when spectral sequences are involved; the new procedures are much shorter, simpler, more readable and, the last but not the least, much more efficient. For example, the computing time of the homotopy group $\pi_5(\Omega S^3 \cup_2 D^3)$ is now obtained in five minutes, whereas with the previous version, on the same laptop, one hour and thirty-five minutes were necessary.

With an interesting side-effect. The Kenzo program is currently too complex to be *proved*. It allows us to produce some homology and homotopy groups otherwise unreachable, and these groups remain of course *questionable*: are these results "reasonably" sure? A common criterion consists in using *two* different programs to check whether the respective results are the same. It happens our programs of *Effective Homology* remain, twenty years after the first implementations, the only programs able to process these calculations of Algebraic Topology. The *kernel* of the new version of the Kenzo program, based on this new tool called *Discrete Vector Fields*, and the previous one mainly based on the *Basic Perturbation Lemma*, are totally different. So that the check based on calculations using two different programs is now done, and the result is positive.

The paper is divided in Sections as follows.

In Section 2, *Constructive Algebraic Topology*, we recall the nature of the *computational problem* in "classical" algebraic topology, and describe the general nature of the solution designed 25 years ago, leading to a series of computer programs, the last one being the *Kenzo* program [8].

Section 3 is a brief exposition of the notion of *Discrete Vector Field*, due to Robin Forman [11]. A vector field is a possible tool to generate *constructive* homological reductions. Homology equivalences are the very heart of classical algebraic topology; *constructive* homological reductions are the heart of *constructive* algebraic topology.

The *Cradle Theorem*, Section 4, is an elementary result in combinatorial homotopy; its proof was expected a routine exercise, which surprisingly required Forman's tool, this notion of *discrete vector field*, and revealed also the power of this tool in a totally new domain.

The Cradle Theorem, so proved, made obvious how the very basic Eilenberg-Zilber theorem [10] (1950!) could also make profit of appropriate discrete vector fields, producing a new direct elegant proof, when the *constructive* classical one is rather technical, six pages of complex calculations in [9, Section 6]. The key point of the new construction, very simple, is briefly explained in Section 5.

There exists also a *twisted* Eilenberg-Zilber theorem [7], playing the same role with respect to the initial one as a

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general *fibration*, sometimes called a *twisted product*, with respect to a *trivial product.* It happens the vector field technique solves at once the non-twisted *and* the twisted Eilenberg-Zilber theorems, with *exactly* the same construction. This is sketched in Section 6.

It is possible to describe a *classifying space*, the base space of a *universal fibration*, as an *infinite* twisted product. Therefore, taking account of the previous results, it is not amazing the discrete vector field technology can also be used in this case, which gives an efficient method computing the *effective* homology of a classifying space, in particular the effective homology of the Eilenberg-MacLane spaces, the key tool to compute the *homotopy groups.* It is the subject of Section 7, where an old result conjectured by Eilenberg and MacLane in 1953, which was not proved until 1993, is now easily obtained by a totally new method. More precisely the *algorithm* producing the *reduction* wished by Eilenberg and MacLane is now available as a consequence of a simple discrete vector field, allowing us to immediately and easily implement this solution. This remarkable discrete vector field is enough to validate the calculations based on this vector field. But many expected further properties, mainly for coherence with the involved algebraic structures, remain to be proved.

Section 8 is a short report about the programming work so made possible and a typical benchmark is quickly described.

Section 9 is devoted to future work.

2. CONSTRUCTIVE ALGEBRAIC TOPOLOGY.

Algebraic Topology tries to reduce *topological* problems, often difficult, to some algebraic problems, hoped more tractable. For example the Brouwer theorem states that any continuous map on the closed n-ball $f : D^n \to D^n$ has a fixed point. Algebraic topology reduces this problem to another one: the identity map $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$ cannot be factorized as a composition $\mathbb{Z} \to 0 \to \mathbb{Z}$:

$$\mathbb{Z} \xrightarrow{\ ?\ } 0 \xrightarrow{\ ?\ } \mathbb{Z} \qquad (\mathrm{id} : \mathbb{Z} \to \mathbb{Z}) \tag{1}$$

An impressive collection of Fields Medals have been attributed to algebraic topologists: Serre, Thom, Milnor, Smale, Novikov (Sergei), Quillen, Thurston, Donaldson, Friedman, Kontsevich, Voevodsky. The subject is now so important that it is sometimes unclear to decide whether someone can be qualified as an algebraic topologist.

The basic tools of algebraic topology are methods associating to topological spaces some *invariants*, such as homology groups, homotopy groups. Most textbooks of Algebraic Topology describe various methods "computing" these groups, typically the numerous *exact* and *spectral sequences.*

It so happens a careful analysis shows these methods are not *algorithms.* The *computational problem* of Algebraic Topology consists on the contrary in obtaining "general" algorithms able to compute the homology and homotopy groups.

A theoretical method was quickly obtained by Edgar Brown [6], but unfortunately concretely unapplicable, because of its complexity, even today, 55 years later. In the eighties, different methods were proposed [17, 20]; only one, due to Julio Rubio and the second author of this text [18, 15], led to concrete programs allowing us to compute some homology and homotopy groups so far unreachable. It is called *Constructive Algebraic Topology*, for "ordinary" algebraic topology often suffers from non-effective existential quantifiers, on the contrary systematically required effective in *Constructive Logic.*

Traditional methods in Algebraic Topology must often handle intermediate objects which are not of finite type, which therefore cannot be totally installed on a machine. But the *functional trick* can be used to install instead on your machine which we call *locally effective* implementations, as oracles able to answer any "local" question. Unfortunately Gödel and his friends explain such an implementation does not allow the user to compute the hoped-for invariants from such strange fuzzy objects.

To overcome this last obstacle, we finally implement our large objects as pairs connected by some "equivalences":

$$[X, C_*(X) \overset{\varepsilon}{\Longleftarrow\!\!\Longrightarrow} EC_*^X] \tag{2}$$

Here, X is some combinatorial topological object, not necessarily of finite type, therefore coded as a "locally effective" object. The chain complex $C_*(X)$ is the algebraic object *defining* its homological nature, also only locally effective. For example, the *definition* of the homology groups $H_*(X)$ is direct from $C_*(X)$, but Gödel taught us no general algorithm can *compute* these homology groups from $C_*(X)$.

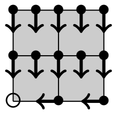
The chain complex EC_*^X on the contrary is of finite type (prefix E for *effective*) and elementary algorithms compute its homology groups. Finally, the last but not the least, the equivalence ε connects both chain complexes by a sophisticated sort of homology equivalence. This equivalence in particular ensures the homology groups of $C_*(X)$ and EC_*^X are canonically isomorphic.

Unfortunately, no general algorithm can *deduce* EC_*^X and ε from X and $C_*(X)$. When X is of finite type, such an equivalence can be chosen trivial. If X is not of finite type, sometimes, rarely, this equivalence can be deduced from some appropriate *particular* results. Then, *starting from these particular cases*, the general organization of Constructive Algebraic Topology allows the user to construct all the reasonable spaces of Algebraic Topology as objects of the type roughly described above at (2).

In this way, the computational problem of (simply connected) algebraic topology is solved. This solution, thanks to the powerful modern methods of *functional programming*, can be easily implemented. The last version of the corresponding program is the Kenzo program [8]. In particular the Kenzo program implements *constructive* versions of the famous Serre and Eilenberg-Moore spectral sequences; in fact replacing them by a simpler but more efficient process known as the Basic Perturbation Lemma [19, 5].

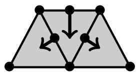
3. DISCRETE VECTOR FIELDS.

We add now a new tool to design the general organization of constructive algebraic topology, known as the theory of *Discrete Vector Fields.* Initiated by Robin Forman in his landmark paper about the *Discrete Morse Theory* [11], it is an elementary process to define and handle conveniently some *homotopy operators.* The existential quantifiers for homotopy operators in "standard" algebraic topology are most often non-constructive; *effectively* constructing these operators is on the contrary the main task in *constructive* algebraic topology, and this task is often made much easier thanks to appropriate discrete vector fields. Milnor's version of the Morse theory [12] made an intensive use of differential vector fields and Forman settled the appropriate translation of these vector fields in combinatorial topology.


(3)

The figure above displays a discrete vector field installed on a simple cellular complex, a square divided in four squares. This cellular complex needs 9 vertices, 12 edges and 4 squares. A discrete vector field is a collection of *vectors*, each vector being simply a *pair* made of an n-cell, the *target* of the vector, and an adjacent $(n-1)$-cell, the *source* of the vector. Here, 8 vectors are made of an edge and a vertex, and 4 vectors are made of a square and an edge.

A differential vector field defines a flow, a continuous collection of paths; a discrete vector field defines in the same way (discrete) paths with an important difference: in general, many different paths start from some origin; the picture below should explain why this can happen.


(4)

A discrete vector field is *admissible* if, for every cell σ, the lengths of all the paths issued from this cell in the positive direction are bounded by a fixed number λ_σ: infinite or circular paths are forbidden.

If a discrete vector field is admissible, it defines a *contraction* of the initial cellular complex onto another one, made of the *critical cells*, that is, those cells which are not involved in the vector field. In the example of figure (3), only one cell is missing, the initial vertex $\circ$ of the square, so that, not very amazing, the whole square can be *contracted* on this unique critical vertex.

We are mainly interested by the *homological* version of this result.

Theorem 1. — *Let V be an admissible discrete vector field on a cellular complex X. Then V defines a* homological reduction *ρ:*

$$\rho = \boxed{h \circlearrowleft C_*(X) \underset{f}{\overset{g}{\leftrightarrows}} C_*(X')} \qquad (5)$$

The chain complex $C_*(X)$ defines the homology groups of the cellular complex X; the vector field defines a contracted complex X' and the associated chain complex $C_*(X')$. The homological reduction $\rho = (C_*(X), C_*(X'), f, g, h)$ is made of both involved chain complexes and three algebraic maps f, g and h. The components f and g are chain complex morphisms, the composition fg is the identity, the component h is an "ideal" homotopy between gf and $\mathrm{id}_{C_*(X)}$, that is, satisfying $\mathrm{id}_{C_*X} = gf + dh + hd$, $fh = 0$, $hg = 0$ and $hh = 0$. This is a convenient and efficient way to describe $C_*(X) \cong C_*(X') \oplus \ker f$, the complementary subcomplex $\ker f$ being *constructively* proved acyclic. Both chain complexes $C_*(X)$ and $C_*(X')$ are *homologically equivalent*, but most often the big one $C_*(X)$ carries additional structures (simplicial, algebraic, ...) which could not be installed on the small one $C_*(X')$.

Also, even if the chain complex $C_*(X)$ is not of finite type, the small chain complex $C_*(X')$ can be of finite type, so that its homology groups are elementarily computable. This will be the case in our main subject in Section 7.

4. THE CRADLE THEOREM.

Our organization of *constructive* algebraic topology produces *effective* versions of the Serre and Eilenberg-Moore spectral sequences, see for example [4]. The next spectral sequence to be processed is the Adams spectral sequence, in fact a consequence of the more basic Bousfield-Kan spectral sequence. Work about the last one is in progress, and required in particular an elementary result of combinatorial topology.

The first figure below describes how the ordinary square $[0,1] \times [0,1]$ can be continually contracted onto three of its faces, $(\{0,1\} \times [0,1]) \cup ([0,1] \times \{0\})$:


(6)

If it is required to *triangulate* the square, the same process can be *combinatorially* described as in the next figure above. You must think the highest vector contracts the above triangle onto two of its edges, then the same for the other triangle. In other words we have replaced the *differential* vector field defining the continuous lefthand contraction by a *discrete* vector field defining the combinatorial righthand contraction.

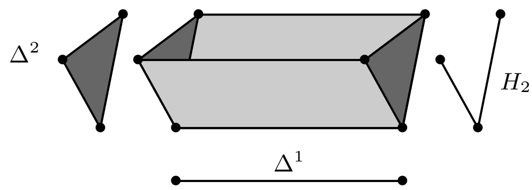
Theorem 2. **(Cradle Theorem)** — *Let p, q be two non-negative integers. Then a combinatorial contraction can be defined $P_{p,q} \Rightarrow C_{p,q}$ where:*

$$\begin{array}{rcl} \Delta^m & & \text{is the standard } m\text{-simplex} \\ P_{p,q} & = & \Delta^p \times \Delta^q \\ H_q & = & \partial\Delta^q - \mathrm{int}(\partial_0\Delta^q) \\ C_{p,q} & = & (\Delta^p \times H_q) \cup ((\partial\Delta^p) \times \Delta^q) \end{array}$$

$P_{p,q}$ is the (p,q)-prism, the ordinary triangular prism being the $(2,1)$-prism. The q-hat H_q is obtained by removing from the boundary $\partial\Delta^q$ of the standard q-simplex, the interior of the face opposite to the vertex 0. Think this "hat" is upside-down. Finally $C_{p,q}$ is the (p,q)-cradle.

The square example before is the particular case $p = q = 1$. In this case, the 0-face $\partial_0\Delta^1$ is $\{1\}$ with itself as interior (!), so that the 1-hat H_1 is only the other end $\{0\}$ of the interval Δ^1. The contraction $P_{1,1} \Rightarrow C_{1,1}$ was sketched before stating Theorem 2.

The (1,2)-cradle could be viewed as an actual rudimentary cradle.



(7)

Once the notion of discrete vector field is understood, the particular cases of the reductions $P_{1,1} \Rightarrow C_{1,1}$ and $P_{1,2} \Rightarrow C_{1,2}$ are not difficult, but the general case is not so easy. You must design an appropriate sorting method to organize by ordered pairs all the simplices not inside the cradle, describing the combinatorial contraction onto the cradle. Finally this requires an amusing game roughly described in the next section for the close but more basic result known as the Eilenberg-Zilber theorem.

For example, for $p = q = 8$, you have to sort in the appropriate way 265,728 simplices of various dimensions in the canonical triangulation of $\Delta^8 \times \Delta^8$, requiring 4 seconds with an efficient program. But the same program is out of memory to process the case $p = q = 10$, because of an unavoidable exponential complexity.

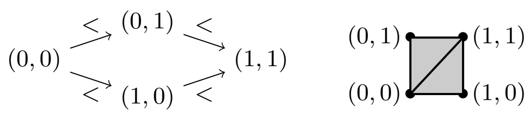
In fact studying this Cradle Theorem led us to a new combinatorial understanding of the essential Eilenberg-Zilber theorem, to be sketched now.

5. A NEW PROOF FOR THE EILENBERG-ZILBER THEOREM.

The Eilenberg-Zilber theorem [10] solves the following problem. The general *simplicial* framework in combinatorial topology requires *triangulated* models for the topological spaces to be processed. What about products? The product of two 1-simplices is a square, which can be triangulated as the union of two triangles. The exact correspondence between a square and this collection of two triangles is the particular case of the Eilenberg-Zilber theorem for the bidimension $(1, 1)$. If two simplicial sets X and Y are in your environment, the product $X \times Y$ has a natural decomposition in prisms $P_{p,q} = \Delta^p \times \Delta^q$ and there remains to triangulate these prisms; such a prism is naturally divided in $\binom{p+q}{p,q}$ $(p+q)$-simplices, and Eilenberg-Zilber describes the exact connection between the chain complex generated by the prisms and the one generated by the simplices after the decomposition.

We will not give detailed explanations in this paper, but we can explain the key idea, very simple, leading to our new understanding of the Eilenberg-Zilber theorem.

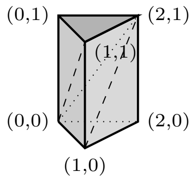
A 1-simplex $[0, 1] = I = \Delta^1$ has two vertices with a canonical order $0 < 1$. The square $\Delta^1 \times \Delta^1$ has four vertices, and it is natural to consider the *product order*, which is not a total order, the vertices $(0, 1)$ and $(1, 0)$ cannot be compared:



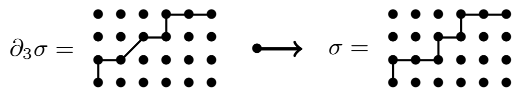
(8)

Observe the *maximal chains* correspond to the canonical decomposition of the square in two triangles. This idea is valid for any prism $P_{p,q}$. The $(p+1)\times(q+1)$ vertices of $P_{p,q}$ inherit a natural product order, and a $(p+q)$-simplex of the canonical triangulation of the prism is exactly a maximal chain for this product order. The picture below illustrates the case $(p, q) = (2, 1)$, the last one which can be illustrated in our poor world with only three dimensions:

- $(0, 0) < (0, 1) < (1, 1) < (2, 1)$
- $(0, 0) < (1, 0) < (1, 1) < (2, 1)$
- $(0, 0) < (1, 0) < (2, 0) < (2, 1)$



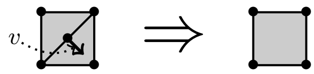
In fact it is possible to illustrate by a simple "picture" this decomposition in *every* bidimension (p, q); for example, with obvious conventions, the righthand picture below is a "drawing" of the 8-simplex $\sigma = (0,0)\text{-}(0,1)\text{-}(1,1)\text{-}(2,1)\text{-}(2,2)\text{-}(3,2)\text{-}(3,3)\text{-}(4,3)\text{-}(5,3)$:



(9)

There are $\binom{8}{5,3} = 56$ 8-simplices of this sort. Their subsimplices must not be forgotten; for example the 3-face $\partial_3\sigma$ is obtained in skipping the vertex #3, that is, the fourth one, giving the other drawn 7-simplex; a (discrete) *vector* is also sketched from $\partial_3\sigma$ to σ, to be used later. This convenient *plane* representation of high-dimensional simplices is possible in general thanks to the *product poset* $\{0 \ldots p\} \times \{0 \ldots q\}$.

In the case $(p, q) = (1, 1)$ the reader could kindly admit the following scheme describes the desired correspondence between a square divided in two triangles and the brute square:



(10)

The lefthand square, divided in two triangles, is provided with a discrete vector field, made of a unique (!) vector v pairing the edge $\sigma = (0,0)\text{-}(1,1)$ with the triangle $\tau = (0,0)\text{-}(1,0)\text{-}(1,1)$. Think the lower triangle is collapsed on two edges while the upper one is inflated. This is a vague interpretation of the Eilenberg-Zilber theorem in the simple case $(p, q) = (1, 1)$.

The relation $\sigma = \partial_1\tau$ is satisfied. In the framework of product posets, this pairing is also described as follows:

(11)

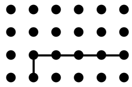
We see on this picture the *source* of the vector v is the *edge* (two vertices) $(0,0)\text{-}(1,1)$ and the *target* is the *triangle* (three vertices) $(0,0)\text{-}(1,0)\text{-}(1,1)$.

Going back now to the figure (9) in the case of the bidimension $(5, 3)$, the pair made of $\partial_3\sigma$ and σ is also a vector of the discrete vector field to be constructed in this case. More precisely such a pair is constructed as follows. You run the path describing a simplex from down-left to up-right; you look for the *first* "event", only the first one:

- If ever you run a diagonal  , then you decide this simplex is the *source* of a vector of the discrete vector field to be defined, the corresponding *target* being obtained by replacing this diagonal by a bend  . The dimension of the last simplex is one more.

- Conversely, if ever you run a bend  , (not a bend ) then you decide this simplex is the *target* of a vector of the discrete vector field to be defined, the corresponding *source* being obtained by replacing this bend by a diagonal  . The dimension of the last simplex is one less.

In this way, almost all the simplices of $P_{p,q}$ are divided in disjoint pairs. Except those "without" any event, which remain "alone", which necessarily have a simple form, such as:



(12)

which corresponds to a "remaining" $(4,1)$-prism in this case, obtained by *collapsing* all the simplices inside it onto its boundary, except one which on the contrary is *inflated* to fill in the prism. Please admit this complex collapsing-inflating process is totally described by the pairing so roughly described. This very briefly sketched process is made rigorous by the marvelous technique of the discrete vector fields, which finally at once reproves the famous Eilenberg-Zilber theorem [10]. In a way which, when implemented in our computer programs, strikingly improved readability *and* efficiency.

THEOREM 3. — *Let X and Y be two simplicial sets. Then a vector field $V_{X,Y}$ can be defined on the simplicial product $S(X \times Y)$ which defines the Eilenberg-Zilber homological reduction:*

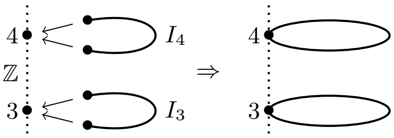
$$\rho_{EZ} = \boxed{h \circlearrowright C_*(S(X \times Y)) \underset{f}{\overset{g}{\leftrightarrows}} C_*(P(X \times Y))} \quad (13)$$

$S(X \times Y)$ is the presentation of the product $X \times Y$ as a simplicial set, $P(X \times Y)$ is the presentation of the same product as a union of prisms, and both $C_*(-)$'s are the respective associated chain complexes. The lefthand chain complex carries a *simplicial* structure: the righthand one, much smaller, cannot carries such a structure.

Every pair made of a p-simplex α of X and a q-simplex β of Y produces *one* prism $P_{\alpha,\beta}$ in $P(X \times Y)$; this prism can be triangulated and divided in $\binom{p+q}{p,q}$ $(p+q)$-simplices in $S(X \times Y)$, as explained above. The reduction $\rho_{EZ} = (f, g, h)$, thanks to the properties required for its components, expresses the large chain complex as the direct sum of a subcomplex isomorphic to the small one and another one *constructively* acyclic. In this way the preferred presentation of the product can be freely used according to the environment, each presentation having advantages and disadvantages.

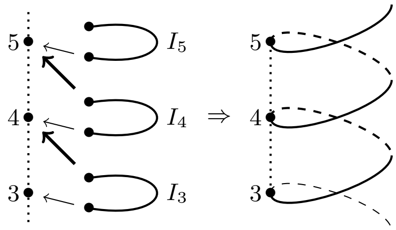
6. THE TWISTED EILENBERG-ZILBER THEOREM.

Jean-Pierre Serre was rewarded in 1954 by a Fields Medal. His work about spectral sequences and homotopy groups of spheres really was a revolution in Algebraic Topology. An essential ingredient was the notion of *fibration*, *initiated* by Hurewicz in 1935. In particular Serre proved most constructions in Algebraic Topology can me modelled as generalized fibrations. A fibration is a sort of *twisted* product, as it is illustrated by the simplest fibration. The usual product $\mathbb{Z} \times S^1$ is an infinite stack of circles. The circle $\#n$ in this stack can be viewed as a 1-simplex, an interval I_n, where both ends are attached to the vertex $n \in \mathbb{Z}$.



(14)

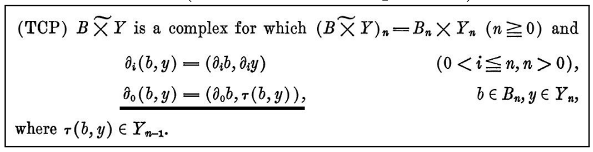
In short, we have decided the identifications $\partial_0 I_n = n$ and $\partial_1 I_n = n$. Now we suddenly decide, instead of the first identification, a different one: $\partial_0 I_n = n + 1$, see the bold arrows ↖ below:



(15)

You understand in the first case, we have constructed a trivial product $\mathbb{Z} \times S^1$, while in the second case, the "twisted" method produced a so-called *twisted product*, which in this case is isomorphic to the real line. In the first case, the "vertical projection" is the standard projection $\mathbb{Z} \times S^1 \to S^1$ while in the second case, the obtained map $\mathbb{R} \to S^1$ is isomorphic to the exponential map $t \mapsto e^{2\pi i t}$. For us, the only important point here is the following: *only the* 0*-face operator has been modified* to transform the trivial product into a twisted one.

This technique has been systematically generalized to arbitrary dimensions many years ago in [2], a landmark article from which we extract this "image", p.651 which gives the axioms for a TCP (twisted cartesian product) $B \tilde{\times} Y$:



(TCP) $B \tilde{\times} Y$ is a complex for which $(B \tilde{\times} Y)_n = B_n \times Y_n$ $(n \geqq 0)$ and

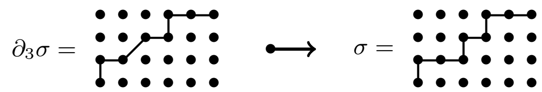
$\partial_i(b, y) = (\partial_i b, \partial_i y)$ $\quad (0 < i \leqq n, n > 0)$,

$\partial_0(b, y) = (\partial_0 b, \tau(b, y))$, $\quad b \in B_n, y \in Y_n$,

where $\tau(b, y) \in Y_{n-1}$.

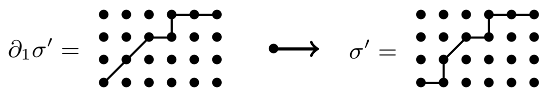
This describes the simplicial structure of the TCP when a *twisting function* τ is defined, a function satisfying a few appropriate properties. The reader understands we just want to highlight that, in the general situation, the same remarkable property is observed: for the *most general* combinatorial definition of a twisted product, it is enough to *perturb* the ordinary 0-face of the trivial product, not the other i-faces for $i \geq 1$, a property not at all obvious a priori.

Now an extraordinary *miracle*. We repeat the illustration about the lovely vector field solving the Eilenberg-Zilber problem:



(16)

The vector sketched above starts from the 3-face of an 8-simplex, going to this simplex. The key point is the face index, 3 in this case. It is possible such a vector invokes a 1-face :



(17)

But it is definitively impossible such a "vector" *comes from a* 0*-face*. The game based on diagonals and bends is such that the i-incidence relation between both involved simplices is possible only with $i > 0$. This is the miracle, because on the contrary, in a twisted product, the *twisted* property is concerned only by the 0-face operator.

We cannot give the details in the setting of this article, see the preprint [14], but because of this miracle, *exactly* *the same* vector field can be used in the general case of a twisted product, to obtain which is usually called the *twisted* Eilenberg-Zilber theorem. Compare the efficiency of this discrete vector field technique with the 9 years between the references [10] and [7]; and also with the complexity of the last article.

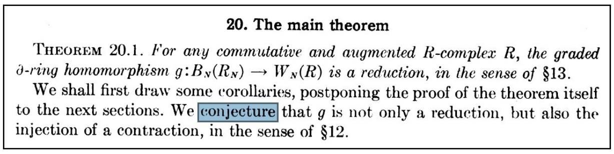
THEOREM 4. *Let $X \times_\tau Y$ be some twisted product of two simplicial sets X and Y. Then a vector field $V_{X,Y}$ can be defined on the simplicial product $S(X \times_\tau Y)$ which defines the twisted Eilenberg-Zilber reduction:*

$$\rho_{\tau EZ} = \boxed{h \circlearrowright C_*(S(X \times_\tau Y)) \underset{f}{\overset{g}{\leftrightarrows}} C_*(P(X \times_\tau Y))}$$

The components $S(X \times_\tau Y)$ and $P(X \times_\tau Y)$ are the respective *simplicial* and *prismatic* presentations of the twisted product $X \times_\tau Y$.

7. AN OLD RESULT CONJECTURED BY EILENBERG AND MACLANE.

Another image, the statement of the main theorem of the paper [9], where we have highlighted the word “conjecture”:



20. The main theorem

THEOREM 20.1. *For any commutative and augmented R-complex R, the graded ∂-ring homomorphism $g: B_N(R_N) \rightarrow W_N(R)$ is a reduction, in the sense of* §13.

We shall first draw some corollaries, postponing the proof of the theorem itself to the next sections. We conjecture that g is not only a reduction, but also the injection of a contraction, in the sense of §12.

In the main applications, R in fact is a *simplicial group*, a topological group with a compatible simplicial structure, $W_N(R)$ is the classifying space of this group, the base space of a *universal fibration* for the structural group R, and $B_N(R_N)$ is the *Bar construction* of the same group, a simpler process constructing *only* an “algebraic” version of the classifying space. These objects are the heart of algebraic topology, in particular the key objects to compute homotopy groups. The problem is to homologically compare both versions $W_N(R)$ and $B_N(R_N)$ of the classifying space.

Which is called a *reduction* in this extract of [9] is in the current terminology a *homology equivalence*, and which is called a *contraction* is now called a (homological) reduction.

The difference between the main theorem of [9] and the result conjectured is the following. The Theorem 20.1 states that the map g induces an isomorphism in homology. The fact of g being the injection of a contraction is much more precise. It claims a homological reduction can be constructed:

$$\rho = \boxed{h \circlearrowright W_N(R) \underset{f}{\overset{g}{\leftrightarrows}} B_N(R_N)} \quad (18)$$

Such a reduction ρ would express:

$$W_N(R) = B_N(R_N) \oplus \ker f$$

with $\ker f$ *constructively* acyclic. This is in particular the key point to be able to use this machinery to produce *algorithms* computing homotopy groups.

The only available proof of this conjecture [13] adds to [9] the use of the Homological Perturbation Lemma [5], not available in the fifties. The resulting algorithm is relatively complex and has not yet been implemented. We obtain this reduction, computationally and theoretically, via a totally different process.

It happens such a reduction ρ was *computationally* constructed a few years ago, thanks to the previous work of Julio Rubio, see [4], where the dual situation is studied in the more difficult situation of loop spaces and Cobar constructions, to obtain an efficient solution for the so-called *Adam's problem.* The same technique can be applied to classifying spaces and Bar constructions, in which case Rubio's construction produces an equivalence made of two reductions:

$$W_N(R) \overset{\rho_1}{\Lleftarrow} \mathrm{Bar}^R(W_N(R) \otimes_\tau R, \mathbb{Z}) \overset{\rho_2}{\Rrightarrow} B_N(R_N)$$

Both reductions are made of triples $\rho_1 = (f_1, g_1, h_1)$ and $\rho_2 = (f_2, g_2, h_2)$. In such a situation where the lefthand term $W_N(R)$ is enormous with respect to the righthand term $B_N(R_N)$, “often” a direct reduction $\rho_3 = (f_3, g_3, h_3) : W_N(R) \Rrightarrow B_N(R_N)$ can be constructed by the formulas $f_3 = f_2 g_1$, $g_3 = f_1 g_2$ and $h_3 = f_1 h_2 g_1$. We have not yet succeeded in understanding when such a method is correct, but after numerous computer tries, it was obvious in this case the ρ_3 so obtained is the reduction which was desired sixty years ago by Eilenberg and MacLane. The situation is so complex we have not yet proved this result.

Now the structure of the big object $W_N(R)$, a *classifying space*, is a sort of infinite twisted product:

$$W_N(R) = \cdots (\cdots ((((R \times_\tau R) \times_\tau R) \times_\tau R) \times_\tau R) \cdots \times_\tau R) \cdots \quad (19)$$

Only a “sort” of infinite product, for the product process is combined with a complex suspension process shifting the dimensions, making the adjustment of our Eilenberg-Zilber vector field rather problematic.

But finally we quickly found this vector field by a funny process, helped by another nice property of these Discrete vector fields. The hoped-for vector field should generate the reduction (18). Key point: it happens that, if this is correct, then the vector field satisfying this property is *entirely determined* by this reduction.

We explained above the reduction guessed by Eilenberg and MacLane was firstly identified as a very indirect consequence of Julio Rubio's previous work, without any proof to justify the result. Because of the reverse dependence *reduction ⇒ vector field*, it was tempting, instead of studying directly the complex infinite product (19), to use this process to obtain the vector field from the reduction not yet proved, but experimentally known.

The guess was correct. Computational experiments following this idea quickly gave a remarkably simple vector field, of the same sort as the Eilenberg-Zilber vector fields. Reversing the process, implementing the vector field so obtained as a general process, other calculations showed this simple vector field really produces the reduction conjectured by Eilenberg and MacLane, guessed by a dirty trick from Julio Rubio's equivalence.

Now the situation is much better. We clearly have the right vector field, we have the right reduction, and the process already used for proving in a different way the Eilenberg-Zilber theorems should work also to obtain along the same lines a complete proof for the reduction (18). Anyway, because the reduction is obtained through a vector field, it is enough to validate the computations obtained using it. Are missing some compatibility properties between the various involved algebraic structures, quite interesting, but currently not necessary for our computations.

8. ABOUT COMPUTING TIMES.

These results are mainly *algorithms*. When the discrete vector field proof for the Eilenberg-Zilber theorems was discovered, *implementing* these theorems in our programs through discrete vector fields significantly improved time *and* space complexities.

It was therefore tempting, without waiting for the complete proof of the reduction conjectured by Eilenberg and MacLane, to modify the computation of the effective homology of the classifying spaces in the Kenzo program, so far following Julio Rubio's algorithm. Furthermore, the new implementation systematically using our discrete vector fields is much simpler.

A typical example is the computation of the homotopy group $\pi_5(\Omega S^3 \cup_2 D^3)$. It is elementary to prove the first non-null homotopy group of $\Omega S^3 = \{\text{continuous maps } S^1 \to S^3\}$ is $\pi_2(\Omega S^3) = \mathbb{Z}$, so that this makes sense to attach a 3-disk D^3 to this space by an attachment map $S^2 \to \Omega S^3$ of degree 2; let us call $X_2 = \Omega S^3 \cup_2 D^3$ the bizarre space so obtained. The simplicial model of this space is not of finite type, but thanks to the work of Julio Rubio about the Adams problem, it is a space with effective homology, and the Kenzo program can be used to apply the Whitehead tower method.

To compute the fifth homotopy group, the following sequence of fibrations is constructed:

$$\begin{array}{rlcl}
K(\mathbb{Z}/2,1) \longrightarrow X_3 \longrightarrow X_2 & & \pi_2(X_2) = \mathbb{Z}/2 \\
K(\mathbb{Z}/2,2) \longrightarrow X_4 \longrightarrow X_3 & & \pi_3(X_3) = \mathbb{Z}/2 \\
K(\mathbb{Z},3) \longrightarrow X_4' \longrightarrow X_4 & & \pi_4(X_4) = \mathbb{Z}/4 + \mathbb{Z} \\
K(\mathbb{Z}/2,3) \longrightarrow X_4'' \longrightarrow X_4' & & \pi_4(X_4') = \mathbb{Z}/4 \\
K(\mathbb{Z}/2,3) \longrightarrow X_5 \longrightarrow X_4'' & & \pi_4(X_4'') = \mathbb{Z}/2
\end{array}$$

Each fibre space $K(G, n)$ is an Eilenberg-MacLane space, which uses the first non-null homotopy group G of the base space to construct a total space, an appropriate *twisted* product, with this group cancelled or decreased. Finally the Whitehead tower produces $\pi_5(\Omega S^3 \cup_2 D^3) = \pi_5(X_2) = \pi_5(X_3) = \cdots = \pi_5(X_4'') = \pi_5(X_5) = H_5(X_5) = (\mathbb{Z}/2)^4$. The space X_5 is 4-connected, which implies $\pi_5(X_5) = H_5(X_5)$, which homology group is computable. In particular, the space of the continuous maps $S^5 \to \Omega S^3 \cup_2 D^3$ has 16 connected components.

On an ordinary laptop, with the previous version of the Kenzo program, this computation needed 1h35m, while with the new technique of discrete vector fields to obtain the effective homology of all these Eilenberg-MacLane spaces, now the computation is done in 5 minutes. The *time* complexity for this particular computation has been divided by 19. The logfile shows also the *space* complexity has been divided by 2.3.

9. TO DO.

The various structures so obtained with the help of discrete vector fields can reasonably be claimed the definitive versions of the Eilenberg-Zilber theorems, ordinary or twisted; the same for the reduction conjectured by Eilenberg and MacLane and also for the effective homology of the Eilenberg-MacLane spaces or more generally for the classifying spaces.

The underlying structures present in these structures are very reach: in particular many algebra structures, coalgebra structures, module structures, comodule structures are involved. It is the first time a *direct reduction*:

$$C_*(K(G,n)) \Rrightarrow \mathrm{Bar}^n(C_*(K(G,0)))$$

is obtained by a simple process, very simple to program, so simple that when implemented, the corresponding programs are immediately significantly improved. Studying the exact algebraic coherence properties which are satisfied by these reductions is an essential subject, where much work remains to do.

These algorithms are simple but raise new problems of recursiveness which cannot be explained in this brief report. We cannot pretend our implementation is the best one, maybe some significant progress can be again obtained with regard to this question.

Using the same strategy, we have identified the vector fields allowing us to construct a direct reduction:

$$\Omega^n(X) \Rrightarrow \widetilde{\mathrm{Cobar}}^n(X)$$

but the situation is terribly more complicated, it is no longer a matter of iterated Cobar constructions, the terrible A_∞-coalgebra structures are then involved, this is why the tilde above the Cobar operator. Taking account of Forman's connection [11] between discrete vector fields and homotopy types, this should finish in a very general and relatively simple solution for the longstanding problem of geometrical models of finite type for the homotopy types of iterated loop spaces [1, 3, 16].

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