The Homological Hexagonal Lemma.*

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Abstract

We propose in this article a global understanding of, on the one hand, the *Homological Perturbation Theorem* (HPT) and, on the other hand, of Robin Forman's theorems about the discrete vector fields (DVFs). Forman's theorems become a simple and clear consequence of the HPT. Above both subjects the *Homological Hexagonal Lemma*, quite elementary.

1 Introduction.

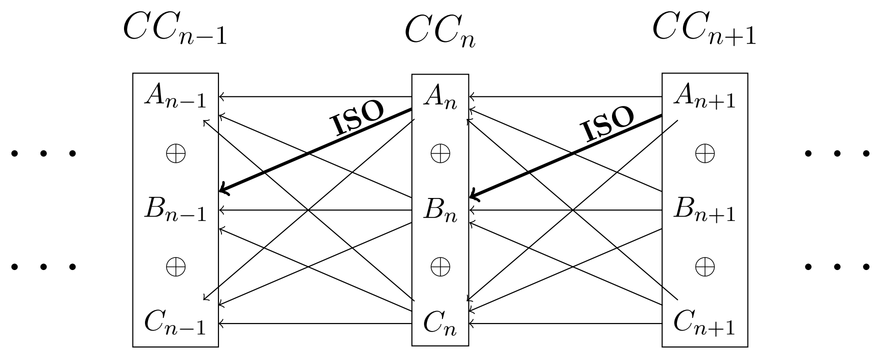
The *Homological Hexagonal Lemma* is an elementary process consisting in using the ordinary linear *Gauss reduction* in a homological context when more or less *large* chain complexes are used to *define* homology groups. Defining an object in general is not sufficient for having a *computing process* of this object, in modern language for having an *algorithm* computing this object. It happens the so-called *Homological Perturbation Theorem*[1] (HPT) has become an essential tool to obtain such algorithms, in particular when the standard exact and spectral sequences in fact do not produce such an algorithm. And the goal of this text is the following: the Homological Perturbation Theorem is a direct consequence of our Homological Hexagonal Lemma, combined with the invertibility of $1 + x$ when x is sufficiently small in an appropriate context.

The Homological Hexagonal Lemma is roughly as follows. Let (CC_*, d_*) be a chain complex where the differential admits a decomposition as clearly

*This paper is written in honour of Tornike Kadeishvily, for his 70th birthday. An opportunity to mention how we like Tornike's mathematical style, a style unfortunately now so rarely observed. To be compared with Eilenberg-MacLane's one: precision, simplicity, completeness. Devoted only to cardinal points of our vast domain of Algebraic Topology. Working together was really a happy time.

[1]Previously often called the *Basic Perturbation Lemma*; become so important that now the designation *Homological Perturbation Theorem* is preferred.

explained in this diagram:



(1)

Every chain group CC_n is a direct sum $CC_n = A_n \oplus B_n \oplus C_n$, and we assume the component $d_n^{ba} : A_n \to B_{n-1}$ of $d_n : CC_n \to CC_{n-1}$ is an isomorphism for every n. Then a homology equivalence is easily constructed $(CC_*, d_*) \Leftrightarrow (C_*, d_*^{cc} + \delta_*)$. In general, (C_*, d_*^{cc}) is not a chain complex; the homological hexagonal lemma says it is possible to modify d_*^{cc} to obtain a chain complex $(C_*, d_*^{cc} + \delta_*)$ homologically equivalent to the original one (CC_*, d_*).

Sometimes, the initial chain complex CC_* is not of finite type, while $(C_*, d_*^{cc} + \delta_*)$ is of finite type. The non-finiteness of the initial complex does not allow the topologist to have an algorithm computing its homology groups. On the contrary, the finite type of C_* automatically gives most often an elementary algorithm. The computing problem of the homology groups of (CC_*, d_*) is so solved.

The reader maybe wonders why the qualifier "hexagonal" for our lemma. It is possible to reorganize the numerous arrows of the diagram (1) to see hidden *hexagons*, which hexagons will allow us to apply an *Elementary Hexagonal Lemma*, with a diagram having this time the form of a hexagon. And the last lemma is a simple and direct consequence of the ordinary Gauss reduction process for the linear systems, the reduction process taught in every secondary school.

The homological hexagonal lemma so becomes the main component of the organization of homological algebra called *Constructive Homological Algebra* [9]. Two predominant tools are then used:

- The *Homological Perturbation Theorem*;
- The *Discrete Vector Fields* technology;

and these tools are direct consequences of our hexagonal lemma.

The *statement* of the Homological Perturbation Theorem has clearly the form of an *Implicit Function Theorem*, but the mysterious series used in the proofs, see for example the series (I) in [11, p. 27], are not presented as coming from some sort of implicit function theorem. We will see the homological hexagonal lemma fills this gap, giving a so simple understanding of the Perturbation Theorem that it is a little amazing this description does not seem to have yet been presented. Understanding the Perturbation Theorem so amounts to knowing that the inverse of $1 + x$ is $1 - x + x^2 - \cdots$ as soon as x is nilpotent or, in a topological context, when $|x| < 1$.

2 Homological Reductions.

2.1 Definitions.

Definition 1 — A *chain complex* $C_* = (C_*, d_*) = (C_n, d_n)_{n\in\mathbb{Z}}$ is a collection of abelian groups C_n, often called *chain groups*, indexed by the integers $n \in \mathbb{Z}$, and of *differentials* $d_n : C_n \to C_{n-1}$ satisfying $d_{n-1}d_n = 0$ for $n \in \mathbb{Z}$, or more simply $d^2 = 0$. [1]♣

The groups of n-cycles $Z_n(C_*)$, n-boundaries $B_n(C_*)$ and n-homology classes $H_n(C_*)$ are defined as:

$$Z_n(C_*) = \ker d_n \tag{2}$$

$$B_n(C_*) = \operatorname{im} d_{n+1} \tag{3}$$

$$H_n(C_*) = Z_n(C_*)/B_n(C_*) \tag{4}$$

Definition 2 — If $(\widehat{C}_*, \widehat{d}_*)$ and (C_*, d_*) are chain complexes, a *reduction* ρ:

$$\rho = (f, g, h) = \boxed{\begin{tikzcd} (\widehat{C}_*, \widehat{d}_*) \arrow[loop left, "h"] \arrow[r, shift right, "f"'] & (C_*, d_*) \arrow[l, shift right, "g"'] \end{tikzcd}} \tag{5}$$

is a collection of maps $f = (f_n : \widehat{C}_n \to C_n)_n$, $g = (g_n : C_n \to \widehat{C}_n)_n$ and $h = (h_n : \widehat{C}_n \to \widehat{C}_{n+1})$ satisfying the relations:

$$d_n f_n = f_{n-1}\widehat{d}_n \tag{6}$$

$$\widehat{d}_n g_n = g_{n-1}d_n \tag{7}$$

$$f_n g_n = \mathrm{id}_{C_n} \tag{8}$$

$$d_{n+1}h_n + h_{n-1}\widehat{d}_n + g_n f_n = \mathrm{id}_{\widehat{C}_n} \tag{9}$$

$$f_n h_{n-1} = 0 \tag{10}$$

$$h_n g_n = 0 \tag{11}$$

$$h_{n+1}h_n = 0 \tag{12}$$

[2]♣

A reduction ρ as in (5) can be denoted $\rho = (f, g, h) : \widehat{C}_* \Rrightarrow C_*$ or even simply $\rho : \widehat{C}_* \Rrightarrow C_*$.

The relations (6) and (7) express the maps f and g are compatible with the differentials, they are *chain complex morphisms.* The equation (8) asserts the chain complex C_* is isomorphic to the subcomplex $g(C_*) \subset \widehat{C}_*$. The next equation expresses h is a homotopy operator between gf and $\mathrm{id}_{\widehat{C}_*}$. Finally the three last equations imply $\widehat{C}_* = (\operatorname{im} g) \oplus (\ker f \cap \ker h) \oplus (\ker f \cap \ker d)$. Also $d|(\ker f \cap \ker h)$ and $h|(\ker f \cap \ker d)$ are inverse isomorphisms between their respective domains, $\ker f \cap \ker h = \operatorname{im} h$ on one hand, $\ker f \cap \ker d = \ker f \cap \operatorname{im} d$ on the other hand, as clearly displayed by the diagram of Figure 1.

In other words, the "big" complex $\widehat{C}_*$ is the direct sum of the subcomplex C'_*, isomorphic to the "small" one C_*, and the subcomplex $A_* \oplus B_*$. This chain complex is acyclic, provided with the so-called *Hodge decomposition* defined by d and h restricted to this subcomplex: $A_* = h(A_* \oplus B_*)$,

$$
\begin{tikzcd}
\{\cdots \arrow[r, <-, shift left, "d"] \arrow[r, shift right, "h"'] & \widehat{C}_{p-1} \arrow[r, <-, shift left, "d"] \arrow[r, shift right, "h"'] \arrow[d, equal] & \widehat{C}_{p} \arrow[r, <-, shift left, "d"] \arrow[r, shift right, "h"'] \arrow[d, equal] & \widehat{C}_{p+1} \arrow[r, <-, shift left, "d"] \arrow[r, shift right, "h"'] \arrow[d, equal] & \cdots\} = & \widehat{C}_* \arrow[d, equal] \\
\{\cdots & A_{p-1} & A_p & A_{p+1} & \cdots\} = & A_* \\
& \oplus & \oplus & \oplus & & \oplus \\
\{\cdots \arrow[uur, shift left, "d"] \arrow[uur, phantom, "\cong"] & B_{p-1} \arrow[uur, shift left, "d"] \arrow[uur, phantom, "\cong"] & B_p \arrow[uur, shift left, "d"] \arrow[uur, phantom, "\cong"] & B_{p+1} \arrow[uur, shift left, "d"] \arrow[uur, phantom, "\cong"] & \cdots\} = & B_* \\
& \oplus & \oplus & \oplus & & \oplus \\
\{\cdots & C'_{p-1} \arrow[l, "d"'] \arrow[d, shift right, "f"'] \arrow[d, phantom, "\cong"] & C'_p \arrow[l, "d"'] \arrow[d, shift right, "f"'] \arrow[d, phantom, "\cong"] & C'_{p+1} \arrow[l, "d"'] \arrow[d, shift right, "f"'] \arrow[d, phantom, "\cong"] & \cdots\} = \arrow[l, "d"'] & C'_* \arrow[d, shift right, "f"'] \arrow[d, phantom, "\cong"] \\
\{\cdots & C_{p-1} \arrow[l, "d"'] \arrow[u, shift right, "g"'] & C_p \arrow[l, "d"'] \arrow[u, shift right, "g"'] & C_{p+1} \arrow[l, "d"'] \arrow[u, shift right, "g"'] & \cdots\} = \arrow[l, "d"'] & C_* \arrow[u, shift right, "g"']
\end{tikzcd}
\tag{13}
$$

$\boxed{A_* = \ker f \cap \ker h}$ $\boxed{C'_* = \operatorname{im} g}$ $\boxed{B_* = \ker f \cap \ker d}$

Figure 1: Reduction Diagram

$B_* = d(A_* \oplus B_*)$, and $(dh + hd)|_{A_* \oplus B_*} = \mathrm{id}_{A_* \oplus B_*}$. Of course, A_* and B_* *are not* subcomplexes of $\widehat{C}_*$.

The main interest of such a reduction is the following: the "big" chain complex $\widehat{C}_*$ could be the chain complex naturally *defining* the homology groups of some object. The "small" one C_* can then be *sufficiently* small to make easy the calculation of its homology groups. The chain complex $A_* \oplus B_*$ being acyclic, the homology groups of $\widehat{C}_*$ and C_* are canonically isomorphic, so that the calculation of the homology groups of C_* produces also the homology groups of $\widehat{C}_*$. In situations where the chain complex $\widehat{C}_*$ is "too big", in particular when it is not of finite type, this can be the only available solution to reach the homology groups of $\widehat{C}_*$.

Two typical examples. The Eilenberg-MacLane space $K(\mathbb{Z}/2, 4)$ is of finite type, but not of "small type". The standard model of this space is a simplicial set with for example the next numbers $\hat{n}_i$ of simplices in dimensions 7-9:

$$\hat{n}_7 = 34359509614 \tag{14}$$

$$\hat{n}_8 = 1180591620442534312297 \tag{15}$$

$$\hat{n}_9 = 85070591730234605240519066638188154620 \tag{16}$$

The chain complex $\widehat{C}_*(K(\mathbb{Z}/2, 4))$ *defining* the homology groups of $K(\mathbb{Z}/2, 4)$ is a free $\mathbb{Z}$-complex with the same numbers of generators in degrees 7, 8 and 9. Using the incidence relations between these simplices, no computer can compute $H_8 K(\mathbb{Z}/2, 4)$, even if you use the now standard implementations of *sparse* matrices.

Yet, the method designed by Eilenberg and MacLane in [3] produces a free $\mathbb{Z}$-complex C_* with n_i generators in degree i:

$$n_7 = 4 \tag{17}$$

$$n_8 = 8 \tag{18}$$

$$n_9 = 15 \tag{19}$$

with the same homology groups, groups which can be calculated in a small fraction of second. The "standard" simplicial model of $K(\mathbb{Z}/2, 4)$ is *minimal* in the sense of Kan [6, §9] and cannot be replaced by a smaller one in *Constructive Algebraic Topology.* It happens the methods described here around the Homological Perturbation Theorem produce a *reduction* between $\widehat{C}_*(K(\mathbb{Z}/2, 4))$ and C_*, the key point to produce a version of $K(\mathbb{Z}/2, 4)$ *with effective homology.*

Another example is still more striking, the case of $K(\mathbb{Z}, 1)$. The "minimal" model of this space has an infinite number of simplices in any positive dimension. Again, Eilenberg and MacLane exhibited in [3] a reduction (called *contraction* in [3]) between the chain complex $\widehat{C}_*(K(\mathbb{Z}, 1))$, not at all of finite type, and a very small chain complex C_* with only one generator in dimensions 0 and 1 and no generator at all in higher dimensions. Also such a reduction is essential in *Constructive Algebraic Topology.*

2.2 The Homological Perturbation Theorem.

The *Homological Perturbation Theorem* is the heart of our subject. Let:

$$\rho = (f, g, h) : (\widehat{C}_*, \widehat{d}) \Rrightarrow (C_*, d) \tag{20}$$

be a (homological) reduction. It can happen a *new* differential $\widehat{d} + \widehat{\delta}$ is to be considered on the graded module $\widehat{C}_*$: a *perturbation* $\widehat{\delta}$ is added to the initial differential $\widehat{d}$; this makes sense only if the differential condition $(\widehat{d} + \widehat{\delta})^2 = 0$ is again satisfied. Then the naive reduction:

$$\xcancel{\rho = (f, g, h)} : (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \xcancel{\Rrightarrow} (C_*, d) \tag{21}$$

is in general no longer valid, no reason the components f, g and h remain compatible with the new differential $\widehat{d} + \widehat{\delta}$.

But sometimes, when a *nilpotency condition* is satisfied, a relatively simple process produces a new valid reduction. The initial homotopy h has degree $+1$, the perturbation $\widehat{\delta}$ has degree -1, the composition $h\widehat{\delta}$ has degree 0; the *nilpotency condition* is satisfied if for every $x \in \widehat{C}_*$, the iterated image $(h\widehat{\delta})^{n_x}(x)$ is null for some n_x large enough, which n_x may depend on x.

Theorem 3 (Homological Perturbation Theorem, HPT) — *Let:*

$$\rho = (f, g, h) : (\widehat{C}_*, \widehat{d}) \Rrightarrow (C_*, d) \tag{22}$$

be a reduction, and $\widehat{\delta}$ a coherent perturbation *of the differential $\widehat{d}$ satisfying the* nilpotency condition *as explained above. Then a simple process produces a new reduction:*

$$\rho' = (f', g', h') : (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \Rrightarrow (C_*, d + \delta). \tag{23}$$

The Homological Perturbation Theorem is known for a long time. A particular case was implicitly used by Eilenberg and MacLane in their seminal works on the... Eilenberg-MacLane spaces, in particular when they handle their *contractions*; for example the series used in the proof of Theorem 12.1 of [3] is to be compared to the series (56) of the Perturbation Theorem as described here.

It is generally explained the HPT is due to Michael Barratt, unpublished. The first detailed and explicit construction of the new reduction is in the thesis memoir of Shih Weishu [11] prepared under the direction of Henri Cartan. Later, Ronnie Brown [1] gave to Shih's result the "abstract" form stated above. The presentation given here reduces (!) this theorem to the elementary *Gauss reduction* process in linear algebra and the invertibility of $1+x$ when $|x| < 1$, allowing in particular easy extensions to various contexts involving topological vector spaces.

As explained in the previous section, a reduction $\widehat{C}_* \Rrightarrow C_*$ is often used to describe the homological nature of the big complex $\widehat{C}_*$ thanks to the small complex C_*. Therefore, in the framework of the HPT's statement above, if the homological nature of $(\widehat{C}_*, \widehat{d})$ is known thanks to some reduction $\widehat{C}_* \Rrightarrow C_*$, an analogous description for the *different* object $(\widehat{C}_*, \widehat{d} + \widehat{\delta})$ can often be obtained with the HPT which produces a new reduction with the *same* small graded module C_* but provided with a different differential $d+\delta$. The size of C_* being unchanged, the computation of the homology groups of $(C_*, d + \delta)$ is roughly the same as for (C_*, d).

In Shih's paper [11], the initial big complex was the chain complex of a *trivial* product of simplicial sets, and the final big complex was the chain complex of a *twisted* product: in a simplicial framework, this implies only the *differential* of the (big) chain complex is changed. It was then proved that, under quite general hypotheses, the involved perturbation satisfies the nilpotency condition; starting from the homology of the trivial product known thanks to the Künneth theorem, the HPT produces a description of the homology of the twisted product. It is Cartan-Shih's version of the Serre spectral sequence.

3 The Homological Hexagonal Lemma.

3.1 Elementary Hexagonal Lemma.

Given rational numbers $a, b, \varepsilon, \varphi, \psi, \beta$, we could study the linear system:

$$\varepsilon x + \varphi y = a \tag{24}$$

$$\psi x + \beta y = b \tag{25}$$

If ε is invertible, that is, $\neq 0$ in the rational case, we could subtract from the equation (25) the product of the equation (24) by $\psi\varepsilon^{-1}$, obtaining:

$$(\beta - \psi\varepsilon^{-1}\varphi)y = (b - \psi\varepsilon^{-1}a) \tag{26}$$

The discussion of the linear system is then easy, depending only on the nature of the coefficient $(\beta - \psi\varepsilon^{-1}\varphi)$.

This is taught in every secondary school, but rarely presented there as a *homological reduction*. Our linear system can be encapsulated in this chain complex:

$$\begin{tikzcd}
& \mathbb{Q} \arrow[dl, "0"'] & \mathbb{Q} \arrow[l, "\times\varepsilon"'] \arrow[ddl, "\times\psi"] & \\
\widehat{C}_0 = 0 & \widehat{C}_1 & \widehat{C}_2 & 0 = \widehat{C}_3 \arrow[ul, "0"'] \arrow[dl, "0"] \\
& \mathbb{Q} \arrow[ul, "0"] & \mathbb{Q} \arrow[l, "\times\beta"] \arrow[uul, "\times\varphi"'] &
\end{tikzcd} \tag{27}$$

Except $\widehat{C}_1 = \mathbb{Q}^2 = \widehat{C}_2$, the other chain groups are null. An arrow labelled for example $\times\varepsilon$ is the multiplication of the argument by ε: $(\times\varepsilon)(x) = \varepsilon x$. Discussing our linear system is nothing but studying the homological nature of our chain complex between $\widehat{C}_2$ and $\widehat{C}_1$: the differential between these chain groups is the square matrix with coefficients $\varepsilon, \varphi, \psi, \beta$ acting on a vector (x, y) with a value:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{28}$$

The Euler characteristic is 0, so that only three possibilities: $H_1 = H_2 = 0$ or $\mathbb{Q}$ or $\mathbb{Q}^2$. If ε is invertible, the value $\mathbb{Q}^2$ for the homology groups is excluded, and there remain the possibilities 0 if $(\beta - \psi\varepsilon^{-1}\varphi)$ is invertible and $\mathbb{Q}$ otherwise.

Let us detail which happens when ε is invertible. The homological nature of the situation can be described by the following (homological) reduction, where the only non-trivial component of the homotopy is the top arrow $\times\varepsilon^{-1}$.

$$\begin{tikzcd}
& \mathbb{Q} \arrow[dl, "0"'] \arrow[r, bend left, "\times\varepsilon^{-1}"] & \mathbb{Q} \arrow[l, "\times\varepsilon"] \arrow[ddl, "\times\psi"'] & \\
\widehat{C}_0 = 0 & \widehat{C}_1 \arrow[dd, shift right, "(-\psi\varepsilon^{-1}\ \ 1)"'] & \widehat{C}_2 \arrow[dd, shift right, "(0\ \ 1)"'] & 0 = \widehat{C}_3 \arrow[ul, "0"'] \arrow[dl, "0"] \\
& \mathbb{Q} \arrow[ul, "0"] & \mathbb{Q} \arrow[l, "\times\beta"] \arrow[uul, "\times\varphi"] & \\
C_0 = 0 & \mathbb{Q} \arrow[l, "0"] \arrow[uu, shift right, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"'] & \mathbb{Q} \arrow[l, "\beta - \psi\varepsilon^{-1}\varphi"] \arrow[uu, shift right, "\begin{pmatrix} -\varepsilon^{-1}\varphi \\ 1 \end{pmatrix}"'] & 0 = C_3 \arrow[l, "0"]
\end{tikzcd} \tag{29}$$

In fact, this reduction is better understood as the composition of two reductions, one being only a change of basis, the second one being an obvious reduction provided by a diagonal matrix. An opportunity to mention the composition of two reductions is a reduction.

Proposition 4 — *Let $\rho = (f, g, h) : A_* \Rrightarrow B_*$ and $\rho' = (f', g', h') : B_* \Rrightarrow C_*$ be two reductions. Then:*

$$\rho'\rho := (f'f, gg', h + gh'f) : A_* \Rrightarrow C_*$$

is a reduction.

♣[4] Exercise. [4]♣

The linear equations (28), taking account of the invertibility of ε, have been in fact solved by a change of basis in $\mathbb{Q}^2$. In other words, the matrix equation:

$$\begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{30}$$

can be rewritten:

$$\underbrace{\begin{pmatrix} 1 & 0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} \begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix}} \underbrace{\begin{pmatrix} 1 & \varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}} = \underbrace{\begin{pmatrix} 1 & 0 \\ -\psi\varepsilon^{-1} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}} \tag{31}$$

that is:

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi \end{pmatrix} \begin{pmatrix} x + \varepsilon^{-1}\varphi y \\ y \end{pmatrix} = \begin{pmatrix} a \\ -\psi\varepsilon^{-1}a + b \end{pmatrix} \tag{32}$$

This matrix computation can also be viewed as a particular sort of "reduction", namely an isomorphism between chain complexes described by the diagram:

$$\begin{tikzcd}
 & \mathbb{Q}^2 \arrow[dl] \arrow[dd, shift right, "{\begin{pmatrix} 1 & 0 \\ \psi\varepsilon^{-1} & 1 \end{pmatrix}}"'] & & \mathbb{Q}^2 \arrow[ll, "{\begin{pmatrix} \varepsilon & \varphi \\ \psi & \beta \end{pmatrix}}"'] \arrow[dd, shift right, "{\begin{pmatrix} 1 & \varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix}}"'] & \\
0 & & & & 0 \arrow[ul] \arrow[dl] \\
 & \mathbb{Q}^2 \arrow[ul] \arrow[uu, shift right, "{\begin{pmatrix} 1 & 0 \\ \psi\varepsilon^{-1} & 1 \end{pmatrix}}"'] & & \mathbb{Q}^2 \arrow[ll, "{\begin{pmatrix} \varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi \end{pmatrix}}"] \arrow[uu, shift right, "{\begin{pmatrix} 1 & -\varepsilon^{-1}\varphi \\ 0 & 1 \end{pmatrix}}"'] &
\end{tikzcd} \tag{33}$$

with a null homotopy, not displayed.

Now, taking advantage of the diagonal form of the lower matrix, we may define the simple reduction:

$$\begin{tikzcd}
 & \mathbb{Q}^2 \arrow[dl] \arrow[dd, shift right, "{(0\ \ 1)}"'] \arrow[rr, bend left, "{\begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 0 \end{pmatrix}}"] & & \mathbb{Q}^2 \arrow[ll, "{\begin{pmatrix} \varepsilon & 0 \\ 0 & \beta - \psi\varepsilon^{-1}\varphi \end{pmatrix}}"'] \arrow[dd, shift right, "{(0\ \ 1)}"'] & \\
0 & & & & 0 \arrow[ul] \arrow[dl] \\
 & \mathbb{Q} \arrow[ul] \arrow[uu, shift right, "{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}"'] & & \mathbb{Q} \arrow[ll, "{(\beta - \psi\varepsilon^{-1}\varphi)}"'] \arrow[uu, shift right, "{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}"'] &
\end{tikzcd} \tag{34}$$

where the only non-trivial homotopy is at the top of the diagram, allowed thanks to the invertibility of ε.

Composing the two last reductions as explained in Proposition 4 gives the reduction:

$$
\begin{tikzcd}[ampersand replacement=\&]
 \& \mathbb{Q}^2 \arrow[dl] \arrow[r, bend left, "\begin{pmatrix}\varepsilon^{-1} & 0\\ 0 & 0\end{pmatrix}"] \arrow[dd, shift right, "(-\psi\varepsilon^{-1}\ \ 1)"'] \& \mathbb{Q}^2 \arrow[l, "\begin{pmatrix}\varepsilon & \varphi\\ \psi & \beta\end{pmatrix}"] \arrow[dd, shift right, "(0\ \ 1)"'] \& \\
0 \& \& \& 0 \arrow[ul] \arrow[dl] \\
 \& \mathbb{Q} \arrow[ul] \arrow[uu, shift right, "\begin{pmatrix}0\\ 1\end{pmatrix}"'] \& \mathbb{Q} \arrow[l, "(\beta-\psi\varepsilon^{-1}\varphi)"'] \arrow[uu, shift right, "\begin{pmatrix}-\varepsilon^{-1}\varphi\\ 1\end{pmatrix}"'] \&
\end{tikzcd}
\tag{35}
$$

which is nothing but a slightly different rewriting of the reduction (29).

Instead of this simple context with the groups $\mathbb{Q}$ and $\mathbb{Q}^2$, exactly the same work: change of basis combined with an elementary reduction of a diagonal matrix, leads to the elementary "hexagonal lemma". In the figure above, you may replace the lefthand (resp. righthand) 0 by a chain group C_{n-2} (resp. C_{n+1}) of some chain complex C_*, the upper $\mathbb{Q}^2$'s by $A_{n-1} \oplus B_{n-1}$ and $A_n \oplus B_n$, and the lower $\mathbb{Q}$'s by B_{n-1} and B_n, all these objects being modules, the upper ones being decomposed in direct sums $C_{n-1} = A_{n-1} \oplus B_{n-1}$ and $C_n = A_n \oplus B_n$. The matrices then become maps between chain groups "decomposed by blocks". Producing the diagram:

$$
\begin{tikzcd}[ampersand replacement=\&]
 \& \& A_{n-1} \arrow[dl, "\delta"'] \arrow[r, bend left, "\varepsilon^{-1}"] \& A_n \arrow[l, "\varepsilon"'] \arrow[ddl, "\psi"] \& \& \\
{} \& C_{n-2} \arrow[l, dashed] \arrow[dd, shift right, "(\mathrm{id})"'] \& \oplus \& \oplus \& C_{n+1} \arrow[ul, "\eta"'] \arrow[dl, "\gamma"] \arrow[dd, shift right, "(\mathrm{id})"'] \& {} \arrow[l, dashed] \\
 \& \& B_{n-1} \arrow[ul, "\alpha"] \arrow[d, shift right, "(-\psi\varepsilon^{-1}\ \ \mathrm{id})"'] \& B_n \arrow[l, "\beta"] \arrow[uul, "\varphi"'] \arrow[d, shift right, "(0\ \ \mathrm{id})"'] \& \& \\
{} \& C_{n-2} \arrow[l, dashed] \arrow[uu, shift right, "(\mathrm{id})"'] \& B_{n-1} \arrow[l, "\alpha"'] \arrow[u, shift right, "\begin{pmatrix}0\\ \mathrm{id}\end{pmatrix}"'] \& B_n \arrow[l, "\beta-\psi\varepsilon^{-1}\varphi"'] \arrow[u, shift right, "\begin{pmatrix}-\varepsilon^{-1}\varphi\\ \mathrm{id}\end{pmatrix}"'] \& C_{n+1} \arrow[l, "\gamma"'] \arrow[uu, shift right, "(\mathrm{id})"'] \& {} \arrow[l, dashed]
\end{tikzcd}
\tag{36}
$$

Let us notice in particular that for example $\alpha(\beta - \psi\varepsilon^{-1}\varphi) = \alpha\beta - \alpha\psi\varepsilon^{-1}\varphi = \alpha\beta + \delta\varepsilon\varepsilon^{-1}\varphi$ (for $\alpha\psi + \delta\varepsilon = 0$, think of the differential starting from A_n) $= \alpha\beta + \delta\varphi = 0$, so that the lower object is again a chain complex. We have proved:

Proposition 5 (Elementary Hexagonal Lemma) — *Let (C_*, d_*)be a chain complex where the chain groups of index $(n-1)$ and n are decomposed $C_i = A_i \oplus B_i$ for $i = n-1, n$, the differentials d_i for $i = n-1, n, n+1$, being decomposed as described in the upper part of the diagram (36). We assume the component $\varepsilon : A_n \to A_{n-1}$ is invertible. Then the chain complex C_* can be canonically reduced over the same chain complex, except A_{n-1} and A_n are removed, and the differential $d'_n : B_n \to B_{n-1}$ is $d'_n = \beta - \psi\varepsilon^{-1}\varphi$.* [5]♣

Important: notice the maps α and γ are not modified, this will be essential to recursively apply this proposition.

3.2 Homological Hexagonal Lemma.

Theorem 6 (Homological Hexagonal Lemma) — *Let (CC_*, d_*) be a chain complex where every chain group is decomposed in a direct sum $CC_n = A_n \oplus B_n \oplus C_n$. Every boundary matrix $d_n : CC_n \to CC_{n-1}$ can then be decomposed by blocks:*

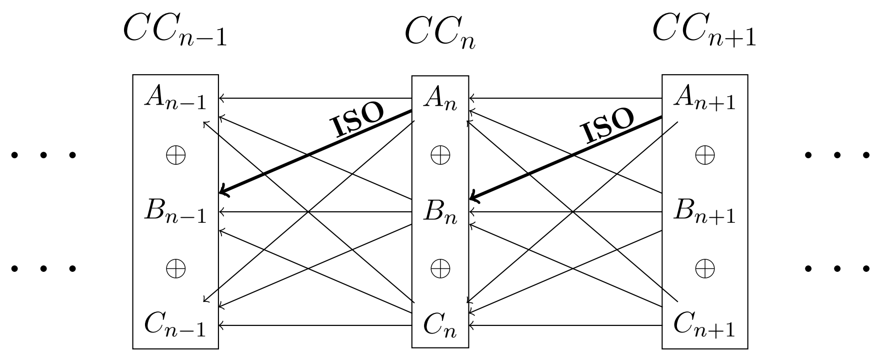
$$d_n = \begin{pmatrix} d_n^{aa} & d_n^{ab} & d_n^{ac} \\ \boxed{d_n^{ba}} & d_n^{bb} & d_n^{bc} \\ d_n^{ca} & d_n^{cb} & d_n^{cc} \end{pmatrix} \tag{37}$$

We assume every morphism $d_n^{ba} : A_n \to B_{n-1}$ is an isomorphism. Then a canonical reduction can be defined:

$$\rho = (f, g, h) : (CC_*, d_*) \Rrightarrow (C_*, d'_*) \tag{38}$$

for an appropriate differential d'_*

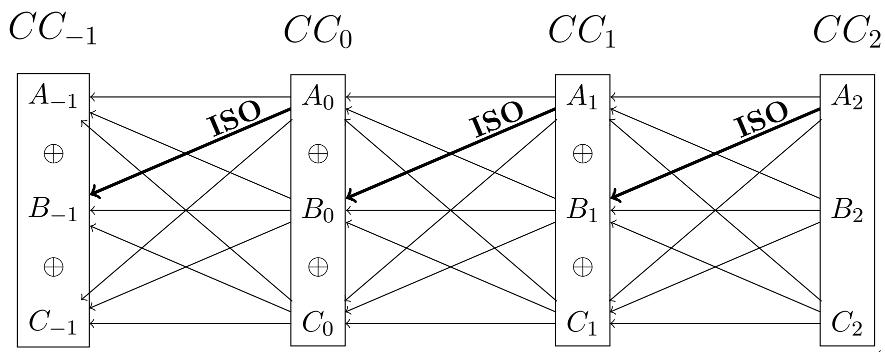
The starting situation is as follows:



(39)

where the bold arrows are assumed to be isomorphisms. Then the initial chain complex can be reduced in a smaller one where the A_n and B_n components are removed, but the maps $d_n^{cc} : C_n \to C_{n-1}$ are in general to be replaced by other ones $d'_n : C_n \to C_{n-1}$, defining a chain complex. Notice the initial d_n^{cc}'s in general *are not* a differential on C_*.

♣[6] Considering the situation between the indices -1 an 2 :

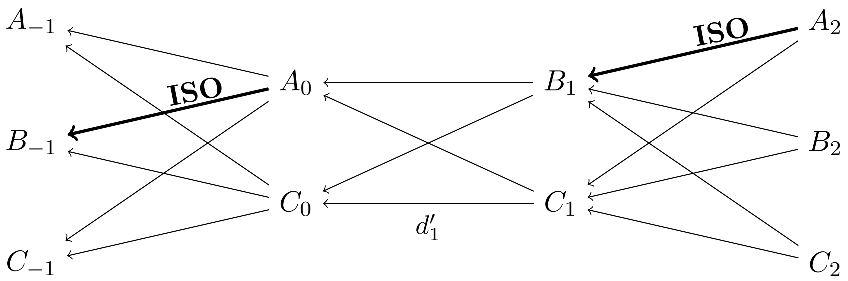


(40)

we can view this diagram as a particular case of the diagram (36) modified by these substitutions:

$$\begin{aligned} C_{n-2} &\mapsto A_{-1} \oplus B_{-1} \oplus C_{-1} \\ A_{n-1} &\mapsto B_0 \\ B_{n-1} &\mapsto A_0 \oplus C_0 \\ A_n &\mapsto A_1 \\ B_n &\mapsto B_1 \oplus C_1 \\ C_{n+1} &\mapsto A_2 \oplus B_2 \oplus C_2 \end{aligned} \tag{41}$$

In particular the arrow $\varepsilon : A_n \to A_{n-1}$ of (36) becomes the arrow $d_1^{ba} : A_1 \to B_0$ which is an isomorphism. Proposition 5 can be applied, producing a reduction of the initial chain complex on the next one, the components A_1 and B_0 being removed. Also the map $d_1^{cc} : C_1 \to C_0$ is to be replaced by $d_1' = d_1^{cc} - d_1^{ca}(d_1^{ba})^{-1}d_1^{bc}$. We obtain this diagram:

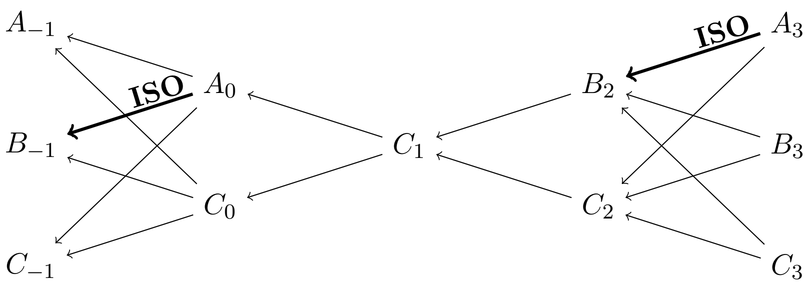


(42)

Let us recall the comment after the statement of Proposition 5: the maps outside the critical indices, 0 and 1 in this case, are *not modified* in the reduction process. In particular the maps between A_2 and B_1, also between A_0 and B_{-1} are not modified: they remain isomorphisms. So that we can again apply Proposition 5 for example between the indices 1 and 2, this time with the substitutions:

$$\begin{aligned} C_{n-2} &\mapsto A_0 \oplus C_0 \\ A_{n-1} &\mapsto B_1 \\ B_{n-1} &\mapsto C_1 \\ A_n &\mapsto A_2 \\ B_n &\mapsto B_2 \oplus C_2 \\ C_{n+1} &\mapsto A_3 \oplus B_3 \oplus C_3 \end{aligned} \tag{43}$$

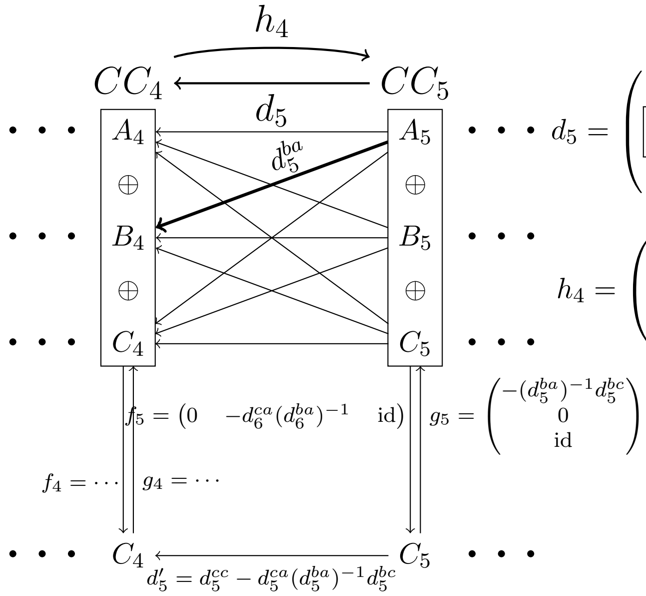
The components B_1 and A_2 are removed and we obtain a new reduced chain complex:



(44)

We can continue in the same way, removing the components A_3 and B_2, A_4 and B_3, and so on, and on the lefthand side, the components A_0 and B_{-1}, A_{-1} and B_{-2}, etc. Continuing in this way, all the A_n and B_n components are removed, letting only the C_n components. [6]♣

The details of the final reduction are analogous in any degree and given in the figure below between the degrees 5 and 4.



$$d_5 = \begin{pmatrix} d_5^{aa} & d_5^{ab} & d_5^{ac} \\ \boxed{d_5^{ba}} & d_5^{bb} & d_5^{bc} \\ d_5^{ca} & d_5^{cb} & d_5^{cc} \end{pmatrix}$$

$$h_4 = \begin{pmatrix} 0 & (d_5^{ba})^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{45}$$

Taking account of the orientation of the arrows, and of the fact that only the arrows d_n^{ba} are invertible, the non-trivial component in d'_n, (resp. f_n, g_n) is obtained by the following recipe: look for the unique path between the starting point and the arrival point, and change the sign. For example, for the non-trivial component $d_n^{ca}(d_n^{ba})^{-1}d_n^{bc}$ of d'_n, you observe the only non-trivial path between C_n and C_{n-1} consists in going first from C_n to B_{n-1} following d_n^{bc}, then in going *back* to A_n following $(d_n^{ba})^{-1}$, and finally in going to C_{n-1} following d_n^{ca}.

Only one non-null component in h_{n-1}, deduced from the invertible component of d_n.

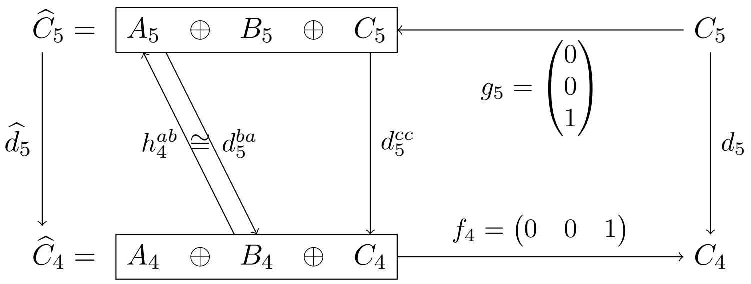
4 Homological Perturbation Theorem.

♣[3] This section is devoted to *the* proof of Theorem 3.

We start with a reduction:

$$\rho = (f, g, h) : \widehat{C}_* \Rrightarrow C_* \tag{46}$$

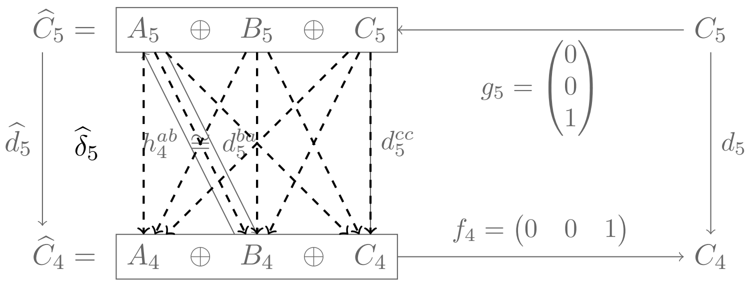
Figure 1 also known as “equation” (13) gives in detail the organization of such a reduction. As it was already done before, we detail which happens between the degrees 4 and 5, only which is relevant. The orientation of the diagram is changed to save some space.



(47)

$$\widehat{d}_5 = \begin{pmatrix} 0 & 0 & 0 \\ d_5^{ba} & 0 & 0 \\ 0 & 0 & d_5^{cc} \end{pmatrix} \qquad h_4 = \begin{pmatrix} 0 & h_4^{ab} = (d_5^{ba})^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is the initial situation. Now we introduce a perturbation $\widehat{\delta}_*$ of the differential $\widehat{d}_*$ of $\widehat{C}_*$.



(48)

This perturbation $\widehat{\delta}$ has arbitrary components:

$$\widehat{\delta}_n = \begin{pmatrix} \widehat{\delta}_n^{aa} & \widehat{\delta}_n^{ab} & \widehat{\delta}_n^{ac} \\ \widehat{\delta}_n^{ba} & \widehat{\delta}_n^{bb} & \widehat{\delta}_n^{bc} \\ \widehat{\delta}_n^{ca} & \widehat{\delta}_n^{cb} & \widehat{\delta}_n^{cc} \end{pmatrix} \tag{49}$$

except the differential condition $(\widehat{d}_{n-1}+\widehat{\delta}_{n-1})(\widehat{d}_n+\widehat{\delta}_n) = 0$ must be satisfied.

Question: could we apply the *homological hexagonal lemma*, that is, Theorem 6, to obtain a reduction for the new "big" chain complex $(\widehat{C}_*, \widehat{d}_* + \widehat{\delta}_*)$?

The answer is simple: if the *new* components $(\widehat{d}+\widehat{\delta})_n^{ba} = d_n^{ba} + \widehat{\delta}_n^{ba}$ are invertible, then the homological hexagonal lemma may be applied. But $d_n^{ba} + \widehat{\delta}_n^{ba} = d_n^{ba}(\mathrm{id} + (d_n^{ba})^{-1}\widehat{\delta}_n^{ba}) = d_n^{ba}(\mathrm{id} + h_{n-1}^{ab}\widehat{\delta}_n^{ba})$ and the last expression is invertible if and only if $\mathrm{id} + h_{n-1}^{ab}\widehat{\delta}_n^{ba}$ is invertible.

It is exactly here that we are in a situation where an *implicit function theorem* can be applied, under the most elementary form: in an appropriate context, if a is invertible, and if a' is sufficiently small, then $a + a'$ is also invertible. Here id is invertible and we have to work with the "size" of $h_{n-1}^{ab}\widehat{\delta}_n^{ba}$ to obtain the invertibility of $\mathrm{id} + h_{n-1}^{ab}\widehat{\delta}_n^{ba}$.

In a purely algebraic context, it is enough to ask for the (pointwise) nilpotency of $h_{n-1}^{ab}\widehat{\delta}_n^{ba}$. That is, if for every $x \in A_n$ there exists a ν_x such that $(h_{n-1}^{ab}\widehat{\delta}_n^{ba})^{\nu_x}(x) = 0$ then we obtain:

$$(\mathrm{id} + h_{n-1}^{ab}\widehat{\delta}_n^{ba})^{-1} = \sum_{i=0}^{\infty}(-1)^i \left(h_{n-1}^{ab}\widehat{\delta}_n^{ba}\right)^i \tag{50}$$

Finally:

$$\begin{aligned}(d_n^{ba} + \widehat{\delta}_n^{ba})^{-1} &= \left(\sum_{i=0}^{\infty}\left(h_{n-1}^{ab}\widehat{\delta}_n^{ba}\right)^i\right) \\ &= h_{n-1}^{ab} - h_{n-1}^{ab}\widehat{\delta}_n^{ba}h_{n-1}^{ab} + h_{n-1}^{ab}\widehat{\delta}_n^{ba}h_{n-1}^{ab}\widehat{\delta}_n^{ba}h_{n-1}^{ab} + \cdots\end{aligned} \tag{51}$$

The form obtained here for the (pointwise) nilpotency condition is useful, even often easier than the one which is given in the common statement of the HPT as in Theorem 3. How to obtain the last one?

The map h_{n-1} is very particular:

$$h_{n-1} = \begin{pmatrix} 0 & h_{n-1}^{ab} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{52}$$

while $\widehat{\delta}_n$ is almost arbitrary, see formula (49). So that:

$$h_{n-1}\widehat{\delta}_n = \begin{pmatrix} h_{n-1}^{ab}\widehat{\delta}_n^{ba} & h_{n-1}^{ab}\widehat{\delta}_n^{bb} & h_{n-1}^{ab}\widehat{\delta}_n^{bc} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{53}$$

and the nilpotency condition is satisfied for $h_{n-1}^{ab}\widehat{\delta}_n^{ba}$ if and only if it is satisfied for $h_{n-1}\widehat{\delta}_n$.

We mentioned in the comments of the diagram (45) the essential role played in the final formulas for the obtained reduction by the inverse term $(d_5^{ba})^{-1}$. Let us compute:

$$\begin{aligned}(d_n^{ba} + \widehat{\delta}_n^{ba})^{-1} &= (\mathrm{id} + (d_n^{ba})^{-1}\widehat{\delta}_n^{ba})^{-1}(d_n^{ba})^{-1} = (\mathrm{id} + h_{n-1}^{ab}\widehat{\delta}_n^{ba})^{-1}h_{n-1}^{ab} \\ &= h_{n-1}^{ab}\sum_{i=0}^{\infty}(-1)^i\left(\widehat{\delta}_n^{ba}h_{n-1}^{ab}\right)^i\end{aligned} \tag{54}$$

Using the formula:

$$\begin{pmatrix} 0 & h & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \zeta \\ \eta & \theta & \iota \end{pmatrix}\begin{pmatrix} 0 & h & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & h\delta h & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{55}$$

we see the critical term obtained at (54) is nothing but the unique non-null term at position 2-1 in the 3-3 matrix:

$$h'_{n-1} = h_{n-1}\sum_{i=0}^{\infty}(-1)^i(\widehat{\delta}_n h_{n-1})^i \tag{56}$$

We so obtain the usual formula for the homotopy operator h' of the new reduction obtained after perturbation:

$$h' = h\sum_{i=0}^{\infty}(-1)^i(\widehat{\delta}h)^i \tag{57}$$

It is then convenient to introduce:

$$\varphi = \sum_{i=0}^{\infty}(-1)^i(\widehat{\delta}h)^i \qquad\qquad \psi = \sum_{i=0}^{\infty}(-1)^i(h\widehat{\delta})^i \tag{58}$$

and continuing the same sort of analysis, using the detailed formulas given at (45), we obtain the new reduction:

$$\rho' = (f', g', h') : (\widehat{C}_*, \widehat{d}_* + \widehat{\delta}_*) \Rrightarrow (C_*, d_* + \delta_*) \tag{59}$$

where:

$$h' = h\varphi = \psi h \tag{60}$$
$$f' = f\varphi \tag{61}$$
$$g' = \psi g \tag{62}$$
$$\delta = f\widehat{\delta}\psi g = f\varphi\widehat{\delta}g \tag{63}$$

formulas which are so simple and so "canonical" they are not difficult to remember by heart. In particular $f\psi = \varphi g = 0$, while φf, ψf, $g\varphi$ and $g\psi$ do not make sense, and the formulas (61) and (62) are the only possible. Also, in the formula (60) (resp. (63)), it is natural to "privilege" h (resp. $\widehat{\delta}$) and again no other choice. [3]♣

The last formulas are elegant, but we must mention the understanding of the homological perturbation theorem given by the diagrams (45), (47) and (48) is much better. We will see a striking illustration when studying Forman's theorems about this notion of *Discrete Vector Field* (DVF). In fact it is when the author studied the DVFs and Forman's point of view that the presentation given here of the HPT became obvious.

Possible or even likely that one or several of our predecessors, Eilenberg, MacLane, Barratt, Cartan, Shih, ..., had such a presentation in mind, but it is a pity it was not made available.

Let us notice also that in a topological context, slighter hypotheses could be considered. For example, if our chain complexes are made of Banach vector spaces, then $||h^{ab}_{n-1}\widehat{\delta}^{ba}_n|| < 1$ is enough to ensure the convergence of the series (50): we so obtain the topological versions of the Perturbation Homological Theorem given in [2]. In case of Fréchet spaces provided with collections of norms, the smoothing process à la Nash-Moser-Schwartz as explained in [10] may sometimes produce the same result.

5 Discrete Vector Fields.

This section gives the main definitions around the notion of *Discrete vector Field* (DVF), due to Robin Forman, see [5].

From now on, all the chain groups of our chain complexes are R-free modules with respect to a commutative unitary ground ring R.

Definition 7 — A *cellular complex* is a chain complex $C_* = (C_*, d_*, \beta_*)$ where every β_n is a *distinguished R-basis* of the corresponding C_n.

The most standard examples come from combinatorial topology, where the objects of β_n are made of n-dimensional (geometrical) *cells* put together according to some or other process. Think for example of the simplicial complexes, the simplicial sets, the cubical sets, the CW-complexes, ... The chain complex defining the homology of such a combinatorial object is made of free R-modules provided with the distinguished bases made of the constituent cells. Also, in commutative algebra, many chain complexes are made of vector spaces based for example on monomials, elements of Gröbner bases, ... But the "abstract" notion of *cellular chain complex* as given in the above definition is sufficient and convenient.

Important: We do not assume the chain complex C_* is of finite type, in other words, the basis β_n can be non-finite, frequent in *constructive* homological algebra, in particular in *constructive* algebraic topology.

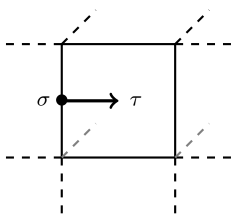
Definition 8 — Let $C_* = (C_*, d_*, \beta_*)$ be a cellular complex. An element $c \in \beta_n$ is called an *n-cell*, or a *cell of dimension n*. We always assume $\beta_m \cap \beta_n = \emptyset$ if $m \neq n$, so that the *dimension* $\dim(c)$ of a *cell* c is an integer unambiguously defined.

Definition 9 Let $C_* = (C_*, d_*, \beta_*)$ be a cellular complex. The cell c' is a *face* of the cell c if $\dim(c') = \dim(c) - 1$ and the coefficient of c' in dc is non-null. It is a *regular face* of c it this coefficient is *invertible* in the ring R.

In particular, if $R = \mathbb{Z}$, then this coefficient for a regular cell must be ± 1. If the ground ring R is a field, any face is a regular face.

Definition 10 — Let C_* be a cellular chain complex. A (discrete) *vector* v is a pair $v = (\sigma, \tau)$ made of a cell τ and of a *regular* face σ of τ. Then σ is called the *source* of the vector v, and τ is its *target*.

In a geometrical context, you could think of a vector $v = (\sigma, \tau)$ as a usual "secondary school vector" drawn from the "center" of the cell σ, the source cell, up to the "center" of the cell τ, the target cell. For example, in a square of a cubical complex:



(64)

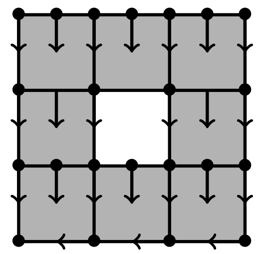
but in fact our discrete vector is the "abstract" pair of cells (σ, τ), here of dimensions 1 and 2, nothing else; in particular the cell σ is a regular face of the cell τ: with the most tempting orientations of these cells, $-\sigma$ appears in $d\tau$ and -1 is invertible in any unitary ring.

Definition 11 — Let C_* be a cellular chain complex. Then a *Discrete Vector Field* (DVF) in C_* is a collection $V = \{v_i\}_{i \in I} = \{(\sigma_i, \tau_i)\}_{i \in I}$ of (discrete) vectors satisfying the following property: the family of sets $\{\{\sigma_i, \tau_i\}\}_{i \in I}$ is pairwise disjoint.

In other words, $\{\sigma_i, \tau_i\} \cap \{\sigma_j, \tau_j\} = \emptyset$ for $i \neq j$. The dimensions of the σ_i's and τ_i's are arbitrary, except that $\dim(\sigma_i) = \dim(\tau_i) - 1$ for every i; but for $i \neq j$, $\dim(\sigma_i) \neq \dim(\sigma_j)$ is possible (and frequent).

The common didactic example is the square annulus provided with the vector field described on this figure:

$C_* =$



(65)

It is a small cubical complex with 16 vertices, 24 edges and 8 squares. Our vector field is made of 15 vectors having a vertex as source, and 8 vectors having as source an edge. Only two cells are neither source nor target.

Definition 12 — If $V = \{(\sigma_i, \tau_i)\}$ is a discrete vector field in a cellular chain complex C_*, then a *critical cell* χ is a cell $\chi \notin \cup_i \{\sigma_i, \tau_i\}$.

Therefore a partition $\cup_n \beta_n = S \coprod T \coprod K$ is defined: S is the set of the *source* cells, T is the set of the *target* cells and K is the set of the *critical* cells. The dimensions of these cells are arbitrary, except that for every source cell $\sigma \in S$ of dimension n, there corresponds a target cell $\tau \in T$ of dimension $n+1$. The dimensions of the remaining cells, the critical cells, are arbitrary. These sets T, S and K are the disjoint unions of the homogeneous components T_n, S_n and K_n for $n \in \mathbb{Z}$.

Definition 13 — Let V be a DVF on a cellular chain complex C_*. Let T, S and K the sets of target, source and critical cells, as just defined above. Then the *source map* $s : T \to S$ (resp. the *target map* $t : S \to T$) is defined by $s(\tau) = \sigma$ (resp. $t(\sigma) = \tau$) if (σ, τ) is a vector of V. These maps have homogeneous componentss $s_n : T_n \to S_{n-1}$ and $t_{n-1} : S_{n-1} \to T_n$ which are inverse of each other.

We are not concerned in this text by the ordinary vector fields of differential analysis, which allows us to call a discrete vector field more simply as a *vector field*.

Definition 14 — Let C_* be a cellular chain complex and σ some $(n-1)$-cell and τ some n-cell. Then the *incidence number* $\varepsilon(\sigma, \tau)$ of σ with respect to τ is the coefficient of σ in $d\tau$.

The cell σ is a face of τ if $\varepsilon(\sigma, \tau) \neq 0$, a regular face if $\varepsilon(\sigma, \tau)$ is invertible in the ground ring R.

Such a vector field defines a differential and also a codifferential on the underlying complex.

Definition 15 — Let $V = \{(\sigma_i, \tau_i)\}_i$ be a vector field on the cellular chain complex C_*. Then the *differential* $d_V : C_* \to C_{*-1}$ (resp. the *codifferential* $d'_V : C_* \to C_{*+1}$) is defined as follows: if τ_i is the target cell of the vector (σ_i, τ_i), then $d_V(\tau_i) = \varepsilon(\sigma_i, \tau_i)\sigma_i$ (resp. $d'_V(\tau_i) = 0$); if σ_i is the source cell of the vector (σ_i, τ_i), then $d'_V(\sigma_i) = \varepsilon(\sigma_i, \tau_i)^{-1}\tau_i$ (resp. $d_V(\sigma_i) = 0$); finally, if χ is a critical cell, then $d_V(\chi) = d'_V(\chi) = 0$.

We recall an incidence number $\varepsilon(\sigma_i, \tau_i)$ associated to a vector $(\sigma_i, \tau_i) \in V$ is necessarily R-invertible: the cell σ_i is a *regular* face of τ_i. These differential and codifferential are some sorts of linear extensions of $s : T \to S$ and $t : S \to T$, taking account of the incidence numbers.

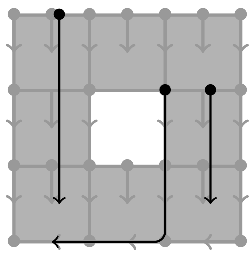
In a sense, the differential d_V consists in keeping from the original differential d of C_* only the terms corresponding to vectors, taking account of the incidence numbers. The codifferential d'_V is more or less the "same" but with the "reverse" orientation, possible because of the nature of the vector field. In particular, if σ is a source cell (resp. τ a target cell), then $d_V d'_V(\sigma) = \sigma$ (resp. $d'_V d_V \tau = \tau$).

There is an obvious notion of *target graded module* $RT = \oplus_{\tau \in T} R\tau$ and of *source graded module* $RS = \oplus_{\sigma \in S} R\sigma$; they are graded submodules of C_* but not subcomplexes. Then $d_V : RT \to RS$ and $d'_V : RS \to RT$ are inverse R-isomorphisms. An analogous *critical graded module* RK may be defined. The homogeneous components of dimension n of these graded modules are denoted by RT_n, RS_n and RK_n, so that $C_n = RT_n \oplus RS_n \oplus RK_n$.

The reader probably guesses that we arrive at a situation where the homological hexagonal lemma again could be applied. But another technicality is necessary. An extra condition is required, our vector field must be *admissible*, a property depending on the dynamical structure of the vector field.

Definition 16 — Let $V = \{(\sigma_i, \tau_i)\}_i$ be a vector field in a cellular chain complex C_*. A *V-path*, or simply a *path*, is a sequence $(v_j)_{1 \le j \le m} = ((\sigma_{i_j}, \tau_{i_j}))_{1 \le j \le m}$ of vectors satisfying the following property: for every $j < m$, $\sigma_{i_{j+1}}$ is a face of τ_{i_j} different from σ_{i_j}. The *length* of such a path is m.

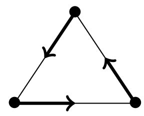
In other words, following a path consists in playing the following game. You choose a vector, and you look for a face of the target cell of this vector which is the source of *another* vector, in general several choices are possible, or maybe no choice. For example, for the vector field defined at (65), three possible paths are drawn below, with respective lengths 3, 4 and 2.

$C_* =$  (66)

The targets of all the vectors of a path have the same dimension. In the figure above, this dimension is 2 for two of them, 1 for the third one.

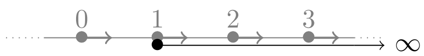
Definition 17 — Let V be a vector field in a cellular chain complex C_*. This vector field is *admissible* if for every source cell σ, the length of every path starting from σ is bounded by some integer λ_σ.

For the vector field (65), the maximal length of a path is 6, so that this vector field is admissible. If a circular path is possible, the vector field is not admissible, the simplest example being the boundary of a triangle with the circular vector field.



(67)

Most users of these notions do not consider the case of cellular complexes not of finite type, and the requirement of non-circularity is then enough. In more general complexes, there could exist a non-circular path, but with infinite length. For example the cubical complex of the intervals $[i-1, i] \subset \mathbb{R}$ for $i \in \mathbb{Z}$ can be provided with the vector field $\{([i-1], [i-1, i])\}_{i \in \mathbb{Z}}$, making possible a path starting for example from 1 with an arbitrary length. Such a vector field is not admissible.



(68)

6 Forman's theorems for the discrete vector fields.

♣[20] We show in this section how an admissible discrete vector field V on a cellular chain complex (C_*, d_*, β_*) generates a canonical reduction of (C_*, d_*) on $(K_*, d_*^{kk} + \delta_*)$ for an appropriate complement δ_*.

As explained in the previous section the vector field decomposes every chain group $C_n = RT_n \oplus RS_n \oplus RK_n$. The differential $d_n : C_n \to C_{n-1}$ may be written "by blocks":

$$d_n = \begin{pmatrix} d_n^{tt} & d_n^{ts} & d_n^{tk} \\ \boxed{d_n^{st}} & d_n^{ss} & d_n^{sk} \\ d_n^{kt} & d_n^{ks} & d_n^{kk} \end{pmatrix} \tag{69}$$

A superscript ab means the component of domain B_n and of codomain A_{n-1} is considered, with A and B being RT, RS or RK.

The keypoint is the following: the admissibility of the vector field implies the component d_n^{st} is invertible, making it possible to apply the homological hexagonal lemma.

The admissibility is necessary to ensure this invertibility. In the case of the triangle with the circular vector field, the boundary matrix in dimension 1 could be:

$$d_1 = d_1^{st} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{70}$$

not invertible. While for the example (68) of the real line and the vector field without any critical cell, the boundary matrix could be $d_1([i-1,i]) = d_1^{st}([i-1,i]) = [i]-[i-1]$. So that an inverse image of $[0]$ should be for example $-[0,1]-[1,2]-[2,3]-\cdots$ with an infinite (!) number of terms, in fact illegal in C_1.

Let us consider now the general case of a cellular chain complex (C_*, d_*, β_*) provided with an admissible vector field $V = (\sigma_i, \tau_i)$.

Definition 18 — Let τ be a target cell. The *height* $\dagger(\tau)$ of τ is the maximal length of a V-path starting from $s(\tau)$, or if you prefer, having $(s(\tau), \tau)$ as its first vector.

The cell τ being a target cell, the vector field V has a unique vector (σ, τ) with $\sigma = s(\tau)$, see Definition 13.

In general, a target cell τ has four sorts of faces. A face can be *the* source cell $\sigma = s(\tau)$ associated to the target cell τ via the map s; a face can be *another* source cell $\sigma' \neq \sigma$; a face can be *another* target cell τ' with lower dimension – do not forget the dimensions of the target cells may be non-constant, frequent; and finally a face can be a critical cell.

In the second case, let $\sigma' \neq \sigma$ be a source cell in the faces of τ which is not the source of τ. Then the definition of a path implies a path using the vector (σ, τ) may be extended by the vector $(\sigma', t(\sigma'))$. This implies $\dagger(\tau) \geq \dagger(t(\sigma')) + 1$. More precisely, there exists such a σ' with $\dagger(\tau) = \dagger(t(\sigma')) + 1$. If no σ' of this sort at all, this means $\dagger(\tau) = 1$.

The next lemma will be useful, saying in fact a finite triangular matrix with invertible terms on the diagonal is invertible; true also for an infinite matrix when the non-diagonal part is poinwise nilpotent.

Lemma 19 — *Let M and M' be R-modules provided with decompositions:*

$$M = \oplus_{i \in \mathbb{N}} M_i \qquad M' = \oplus_{i \in \mathbb{N}} M'_i \tag{71}$$

Let $f : M \to M'$ be a linear map such that the "diagonal components" $f_{i,i}$ are isomorphisms, and the subdiagonal terms $f_{j,i}$ are null for $j > i$. Then f is an isomorphism.

♣[19] Let us denote $M_{\leq i} := \oplus_{j \leq i} M_j$, and$M_{<i} := \oplus_{j<i} M_j$ and the same for $M'_{\leq i}$ and $M'_{<i}$. Some components of f are:

$$\begin{aligned} f_{\leq i, \leq i} &: M_{\leq i} \to M'_{\leq i} \\ f_{i,i} &: M_i \to M'_i \\ f_{<i,i} &: M_i \to M'_{<i} \\ f_{<i,<i} &: M_{<i} \to M_{<i} \end{aligned} \tag{72}$$

The modules M and M' are inductive limits of $M_{\leq i}$ and $M'_{\leq i}$. Also f is the inductive limit of $f_{\leq i, \leq i}$ and it is enough to prove the last one is an isomorphism. Let $x' = x'_i + x'_{<i}$ be an element of $M'_{\leq i}$; we look for a preimage $x = x_i + x_{<i}$ in $M_{\leq i}$. Then $f_{\leq i, \leq i}(x) = f_{i,i}(x_i) + f_{<i,i}(x_i) + f_{<i,<i}(x_{<i})$. The unique possible choice $x_i = f_{i,i}^{-1}(x'_i)$ leads after an elementary computation to the equation:

$$f_{<i,<i}(x_{<i}) = x'_{<i} - f_{<i,i}(f_{i,i}^{-1}(x'_i)) \tag{73}$$

But $f_{<i,<i} = f_{\leq i-1,\leq i-1}$ and we may recursively assume the last one is an isomorphism, giving the (unique) solution for $x_{<i}$. [19]♣

With the notations above, if $g = f^{-1}$, then the formula (73) implies the following formulas:

$$\begin{aligned} g_{i,i} &:= (f_{i,i})^{-1} \\ g_{<i,i} &:= -g_{<i,<i} f_{<i,i} g_{i,i} \end{aligned} \tag{74}$$

are a recursive definition of g.

Let us return to our admissible vector field V on the cellular chain complex C_*. Definition 18 allows us to divide RT_n, the target submodule of dimension n, as a direct sum:

$$RT_n = \oplus_{i \geq 1} RT_{n,i} \tag{75}$$

where $RT_{n,i}$ is generated by the target cells of T_n of height i, those cells such as the maximal paths starting from this cell have length i. The same for RS_{n-1}, after having naturally extended to a source cell $\sigma \in S_{n-1}$ the notion of height by $\dagger(\sigma) := \dagger(t(\sigma))$. Also the boundary component d_n^{st} has a diagonal component defined by:

$$(d_n^{st})_{i,i}(\tau) = \varepsilon(s(\tau), \tau) s(\tau) \tag{76}$$

if τ is an n-cell of height i. The comments after Definition 18 imply the "matrix" $d_n^{st} : \oplus_{i \geq 1} RT_{n,i} \rightarrow \oplus_{i \geq 1} RS_{n-1,i}$ is *triangular*: if σ' is a source cell face of τ different from $\sigma = s(\tau)$, then $\dagger(\sigma') < \dagger(\tau) = \dagger(s(\tau))$ is satisfied.

You may notice that $(d_n^{st})_{i,i}$ is nothing but the differential $d_{V,n}|RT_{n,i} : RT_{n,i} \rightarrow RS_{n-1,i}$ given in Definition 15, a differential which is an isomorphism, so that Lemma 19 implies d_n^{st} is an isomorphism, and the homological hexagonal theorem produces the next theorem, a rephrasing of both Forman's theorems about DVFs, a point detailed later. [20]♣

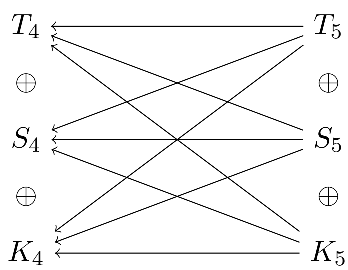
Theorem 20 (Robin Forman's Theorems) — *Let V be an admissible discrete vector field on a cellular chain complex (C_*, d_*, β_*). Then this vector field defines a canonical homological reduction:*

$$\rho_V = (f_V, g_V, h_V) : (C_*, d_*) \Rrightarrow (RK_*, d_*^{kk} + \delta_*) \tag{77}$$

The graded module RK_* is the critical component of the decomposition $C_* = RT_* \oplus RS_* \oplus RK_*$ according to the nature target, source or critical of the cells. The differential d_* is so decomposed by blocks in 9 components, d_*^{kk} being the one between RK_* and RK_{*-1}.

You remark the nature of Forman's theorems has a nature close to a result of the HPT. Would it be possible to produce a different demonstration thanks to the HPT? The answer is positive. This second proof of Theorem 20 is given now.

♣[20]′ The initial differential of the cellular chain complex C_* looks like:



(78)

The previous proof was based on the invertibility of $d_n^{st} : RT_n \to RS_{n-1}$, allowing us to use the homological hexagonal lemma. Instead, let us consider now the chain complex:

$$
\begin{tikzcd}
RT_4 & RT_5 \arrow[ddl, "d_V"'] \\
\oplus & \oplus \\
RS_4 & RS_5 \\
\oplus & \oplus \\
RK_4 & RK_5
\end{tikzcd}
\tag{79}
$$

where $d_V : RT_n \to RS_{n-1}$ of the figure is the only non-null component of $d_V : C_n \to C_{n-1}$ of Definition 15: this differential is null, except between the components RT_n and RS_{n-1}. It is essentially nothing but the vector field V combined with the incidence numbers $\varepsilon(\sigma, \tau)$.

Compare to the Reduction Diagram, Figure 1. Here, the d_V map is invertible, the inverse being the codifferential d'_V of Definition 15. So that a very simple reduction appears:

$$\rho = (f, g, h) : (C_*, d_V) \Rrightarrow (RK_*, 0) \tag{80}$$

The map f is the canonical projection $f : C_* \to RK_*$, the map g is the canonical injection $g : RK_* \to C_*$ and finally the homotopy h is simply $(d_V)^{-1} = d'_V$, null except from RS_{n-1} to RT_n, also directly associated to the vector field.

Let us now reinstall the "right" differential d_* of C_*. This amounts to introducing a *perturbation* $\delta = d - d_V$. Applying the HPT requires the pointwise nilpotency of $h\delta = d'_V\delta$. It was observed in the HPT proof that the invertibility of $h_{n-1}^{ts}\delta^{st}$ is enough (in fact equivalent), see formula 50.

It is sufficient to examine this nilpotency property for the elements of our preferred base of RT_n, namely the target cells of dimension n. Let τ be a target n-cell and $\sigma = s(\tau)$. Identifying a component of $\delta^{st}(\tau)$ consists in looking for a face of τ which is *not* the corresponding source cell $\sigma = s(\tau)$: do not forget the perturbation is $d - \boxed{d_V}$. Let σ' be such a source cell. Applying the homotopy operator $h = d'_V$ to this source cell σ consists in considering the corresponding target cell $t(\sigma') = \tau'$, taking account of the incidence number; except this possible role of the incidence number, we are beginning to extend the "path" started from (σ, τ) by (σ', τ'). Continuing the iteration is the same game: how to extend the path $((\sigma, \tau), (\sigma', \tau'))$? The reader has already understood the nilpotency of $h_{n-1}^{ts}\delta^{st}$ at τ is nothing but the admissibility of the vector field at τ. The pointwise nilpotency $h_{n-1}^{ts}\delta^{st}$ is equivalent to the admissibility of the vector field for the vectors between dimensions $n - 1$ and n.

The HPT may therefore be applied, producing the desired reduction.

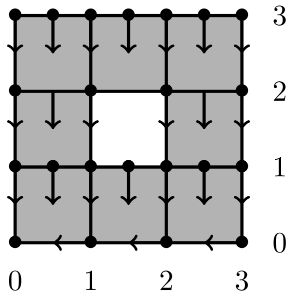
[20]′♣

Both versions of the proof of Forman's theorems are clearly equivalent, the second one is just a rewriting of the first one using the available HPT. But the second one is particularly convenient when programming. Let us examine the formulas (58-63). The formulas for φ and ψ can be recursively rewritten:

$$\varphi = \mathrm{id} - \widehat{\delta} h \varphi \quad \psi = \mathrm{id} - h \widehat{\delta} \psi \tag{81}$$

But it remains as usual to start the recursion. A *terminal* target cell τ is a cell such that no other face in $d\tau$ among the source cells than $\sigma = s(\tau)$; a path arriving at (σ, τ) cannot be extended further. A source cell σ is *terminal* if $t(\sigma)$ is terminal. Then the recursion starts with $\varphi(\sigma) = \sigma$ (resp. $\psi(\tau) = \tau$) when σ (resp. τ) is a terminal source cell (resp. terminal target cell). Using the standard recursive methods of programming, computing these maps φ and ψ is easy. From which all the components of the new reduction are deduced by the formulas (60-63).

Playing again with the square annulus, we can illustrate the use of these formulas.

$C_* =$  (82)

We use obvious coordinates allowing us to denote a vertex by two digits, for example the only critical vertex is 00, an edge by two vertices, for example the only critical edge is 12-22, and a square by two opposite vertices, for example the only square having the vertex 00 is 00□11.

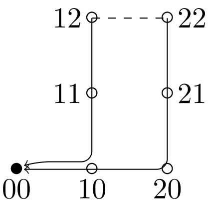
Starting from the initial "small" chain complex $(RK_*, 0)$, after perturbation, this chain complex after perturbation becomes (RK_*, d_K) with:

$$d_K = f\delta g - f\delta h\delta g + f\delta h\delta h\delta g - \cdots \tag{83}$$

for $f : C_* \to RK_*$ the canonical projection, $g : RK_* \to C_*$ the canonical injection, $h = d'_V$ the codifferential and $\delta = d - d_V$ the perturbation.

In this simple case, the differential d_K is determined by the differential of the unique critical 1-cell 12-22. Then $g(\text{12-22}) = \text{12-22}$, followed by $\delta g(\text{12-22}) = 22 - 12$ (for the orientation left to right). Applying h consists in this case in "following a vector between dimensions 0 and 1" and *then* applying δ gives the "other" vertex of the edge which is not the source cell of the same vector. But finally we must apply f which is non-null for a vertex only for the vertex 00.

Playing this game starting from the vertex 22 gives a non-null term only if you apply four times δh, giving the vertex 00. The same with the vertex 12 with three times δh. The signs will be opposite so that finally $d_K = 0$ again, but with a slightly different meaning: you must understand the 0-face 22 of 12-22, being 22 "missing" in the critical complex, is to be replaced by something else, following the indications given by the vector field: "morally" the 0-face of 12-22 is +00, while the 1-face is -00, so that finally $d_K(12 - 22) = +00 + -00 = 0$. An illustration of this state could be this one:



(84)

The hollow circles are not critical while the black one is critical. The vector field is used to supply a boundary for the edge 12-22.

It is the right moment now to explain which is the exact relation between the result obtained here by the HPT, or the homological hexagonal lemma as you prefer, the exact relation with what we called Forman's theorems. By the way, which theorems? Forman's paper [5] is a rare paper in today's mathematical landscape, simultaneously easily readable and full of simple and yet *new* ideas, in particular new results. We are concerned here only by the part of [5] devoted to the discrete vector fields, mainly the sections 6 to 8.

Let us examine with a little care the reduction provided by our Theorem 20. This reduction is:

$$\rho_V = (f_V, g_V, h_V) : (C_*, d_*) \Rrightarrow (RK_*, d_*^{kk} + \delta_*) \tag{85}$$

We are happy to obtain the new differential $d_*^{kk} + \delta_*$ on the graded module RK_*, having so the "right homology", the homology defined by (C_*, d_*). More precisely, a reduction $C_* \Rrightarrow RK_*$ is produced. This implies in particular the *image* $g_V(RK_*, d_*^{kk} + \delta_*)$ in (C_*, d_*) is a chain complex homologically equivalent to the last one. This image is called the *Morse complex* C_*^Φ in [5] and is the main subject of [5, Section 7], the homology equivalence being Theorem 7.3 of this section.

Our small chain complex $(RK_*, d_*^{kk} + \delta_*)$ is the main goal of [5, Section 8], please examine the first lines of this section when the author explains he will obtain his Morse complex *directly in terms of the critical cells.* Our chain complex $(RK_*, d_*^{kk} + \delta_*)$ is called $\mathscr{M}_*$ in [5, Section 8], and the main property of this chain complex is Theorem 8.2, where the map (8.1) in its statement is our map g_V.

By the way, what is the "Morse complex" in the case of our toy example, the square annulus? The formula (62) becomes:

$$g_V = \psi g = g - h\delta g + h\delta g\delta g - \cdots \tag{86}$$

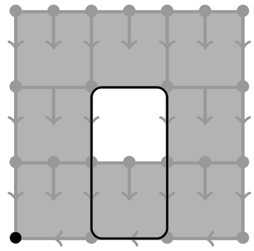
and playing the same game as above when computing the perturbation δ_* to be applied to K_*, we obtain:

$$g_V(00) = 00 \tag{87}$$

$$\begin{aligned} g_V(\text{12-22}) = {} & (\text{10-11}) + (\text{11-12}) + (\text{12-22}) \\ & - (\text{10-20}) - (\text{20-21}) - (\text{21-22}) \end{aligned} \tag{88}$$

The unique 1-cell of the Morse complex is drawned below:

$C_* =$



(89)

and it is geometrically obvious why its boundary is null. The unique 0-cell of the Morse complex is also highlighted.

In this case, the vector field (65) produces a reduction:

$$\left\{0 \leftarrow \mathbb{Z}^{16} \overset{d_1}{\leftarrow} \mathbb{Z}^{24} \overset{d_2}{\leftarrow} \mathbb{Z}^{8} \leftarrow 0\right\} \Rrightarrow \left\{0 \leftarrow \mathbb{Z} \overset{0}{\leftarrow} \mathbb{Z} \leftarrow 0 \leftarrow 0\right\} \tag{90}$$

So that — surprise! —, the homology of the square annulus is isomorphic to the homology of a circle.

7 Conclusion.

The understanding so simple given here of the HPT and of the DVF technology allowed the author and his collaborators to dramatically improve [8] their computer programs concretely implementing *Constructive Homological Algebra*. See [9] for this *algorithmic* organization of Homological Algebra.

As usual, new ideas solve pending problems, but also open new fields and raise new interesting problems. For example the systematic use of DVFs allowed us to define a totally new understanding of the Eilenberg-Zilber theorem [7], more than 60 years after the original paper [4]. In particular giving the final best implementation of the underlying homotopy operator.

The paper [8] explains how experimental evidences are produced by the computer programs implementing Constructive Homological Algebra. Opening new interesting and essential questions for the connections between natural DVFs and underlying algebraic structures, for example the numerous algebra and coalgebra structures of the chain complexes of Algebraic Topology.

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