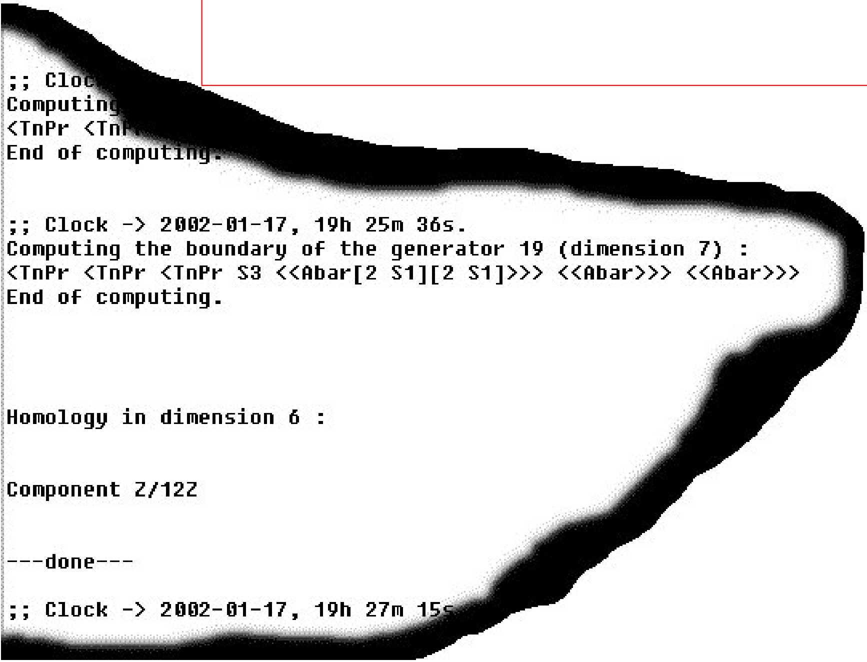
Constructive Homological Algebra V.

Algebraic Topology background



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Genova Summer School, 2006

"General" topological spaces

cannot be directly installed in a computer.

A combinatorial translation is necessary.

Main methods:

1. **Simplicial complexes.**

2. **Simplicial sets.**

Warning: Simplicial sets more complex (!)

but more powerful than simplicial complexes.

Simplicial complex $K = (V, S)$ where:

1. V = set = set of vertices of K;

2. $S \in \mathcal{P}(\mathcal{P}_*^f(V))$ (= set of simplices) satisfying:

   (a) $\sigma \in S \Rightarrow \sigma$ = non-empty finite set of vertices;

   (b) $\{v\} \in S$ for all $v \in V$;

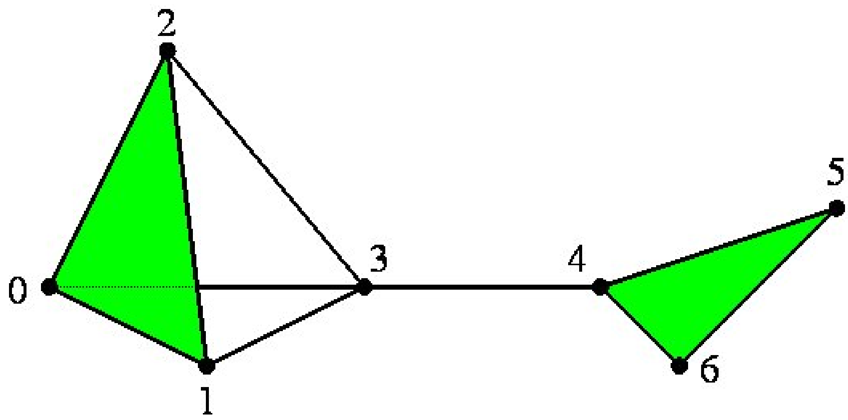
   (c) $\{(\sigma \in S)$ and $(\emptyset \neq \sigma' \subset \sigma)\} \Rightarrow (\sigma' \in S)$.

Notes:

1. V may be infinite ($\Rightarrow$ S infinite).

2. $\forall \sigma \in S$, σ is finite.

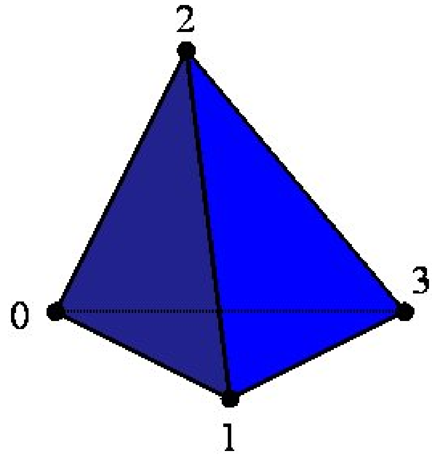
Example:



$$V = \{0, 1, 2, 3, 4, 5, 6\}$$

$$S = \left\{ \begin{array}{l} \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \\ \{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \\ \{4,5\}, \{4,6\}, \{5,6\}, \{0,1,2\}, \{4,5,6\} \end{array} \right\}$$

Disadvantages of simplicial complexes.



Example: 2-sphere :

Needs 4 vertices, 6 edges, 4 triangles.

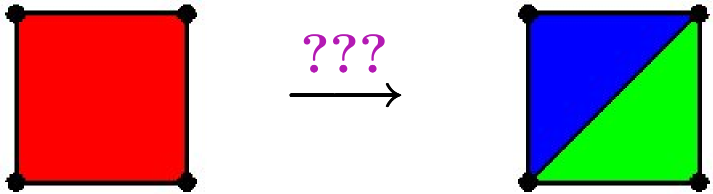
The simplicial set model needs only

1 vertex + 1 "triangle"

but an infinite number of degenerate simplices...

Product?

$\boldsymbol{\Delta^1 \times \Delta^1 = I \times I}$?



In general, constructions are difficult

with simplicial complexes.

Main differences between:

$$\textbf{Simplicial complexes} \overset{???}{\longleftrightarrow} \textbf{Simplicial Sets}$$

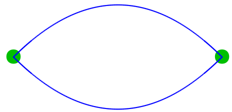
In a simplicial set:

1. **A simplex is not defined by its vertices:**

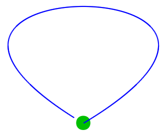
   Own existence + relations with smaller simplices.

$$X = \{X_0 \leftrightarrow X_1 \leftrightarrow X_2 \leftrightarrow X_3 \leftrightarrow \cdots\}$$

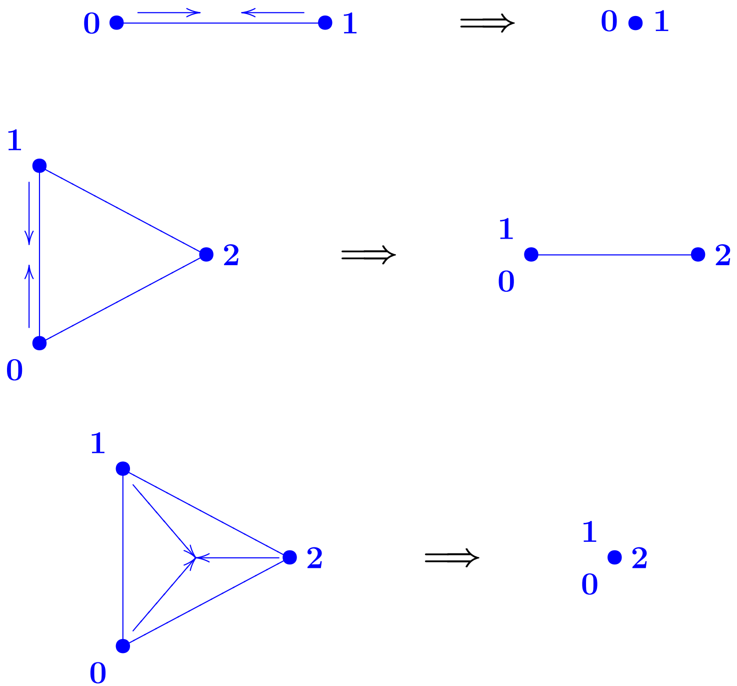
Examples: two different edges can have same vertices:



Several faces (ends) can be the same:



2. Simplices can be degenerate = more or less "collapsed".

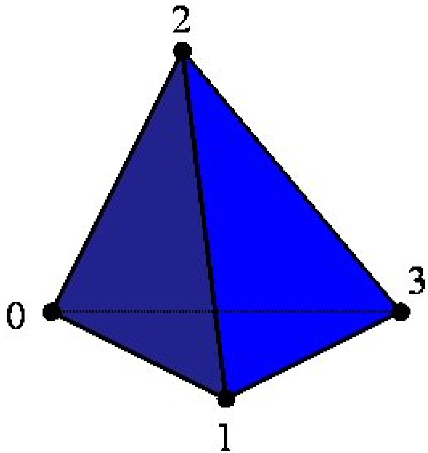


Example:

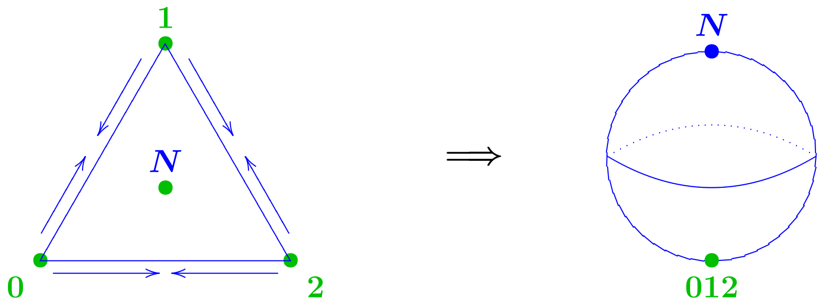
The minimal description as simplicial complex

of the two-sphere S^2

needs: 4 vertices + 6 edges + 4 triangles:



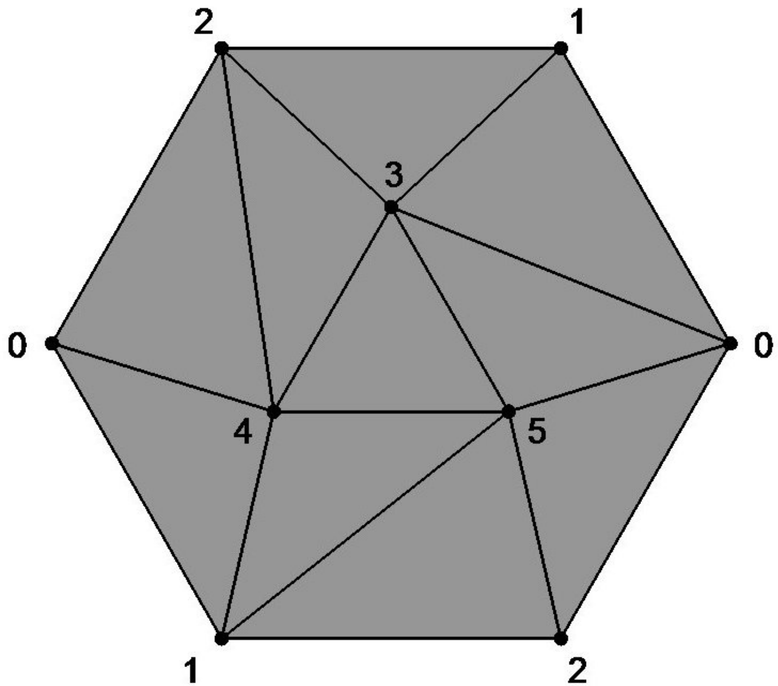
But the minimal description of the two-sphere as a simplicial set needs: 1 vertex + 1 triangle.



Note *N* is not a vertex.

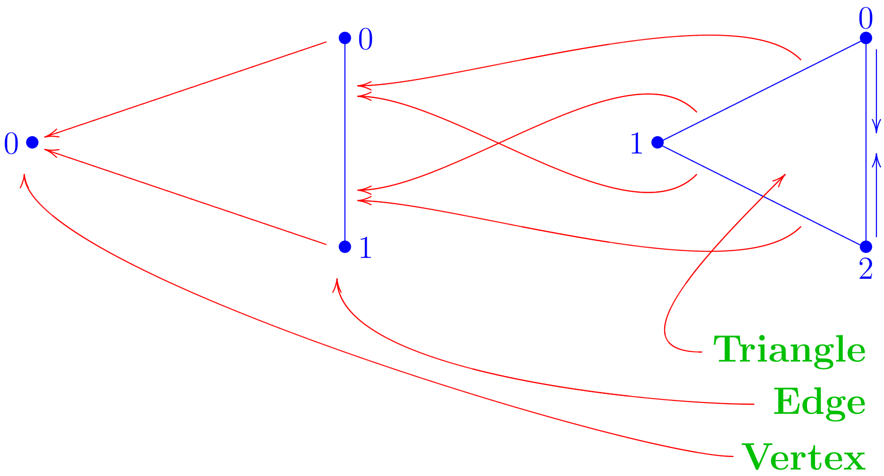
Other example = Real Projective Plane = $P^2(\mathbb{R})$.

Minimal triangulation = 6 vertices + 15 edges + 10 triangles.



Minimal presentation of $P^2\mathbb{R}$ as a simplicial set $=$

1 vertex + 1 edge + 1 triangle.



Support for the notion of simplicial set: The Δ category.

Objects: $\underline{0} = \{0\}$, $\underline{1} = \{0,1\}$, $\ldots$, $\underline{m} = \{0,1,\ldots,m\}$, $\ldots$

Morphisms:

$$\Delta(\underline{m},\underline{n}) = \{\alpha : \underline{m} \nearrow \underline{n} \ \underline{\text{st}}\ (k \leq \ell \Rightarrow \alpha(k) \leq \alpha(\ell))\}.$$

<u>Definition</u>: A simplicial set is

a contravariant functor $X : \Delta \to \text{Sets}$.

$X(m) = X_m = \{m\text{-simplices}\}$ **of** X.

$X(\alpha : \underline{m} \to \underline{n}) = X_\alpha =$

$\{$**Incidence relations of type α between X_m and X_n**$\}$.

Product construction for simplicial sets.

$X = (\{X_m\}, \{X_\alpha\})$, $Y = (\{Y_m\}, \{Y_\alpha\})$ **two simplicial sets.**

$Z = X \times Y =$???

Simple and natural definition:

$Z = X \times Y$ **defined by** $Z = (\{Z_m\}, \{Z_\alpha\})$ **with:**

$Z_m = X_m \times Y_m$

If $\Delta(\underline{n}, \underline{m}) \ni \alpha : \underline{n} \nearrow \underline{m}$:

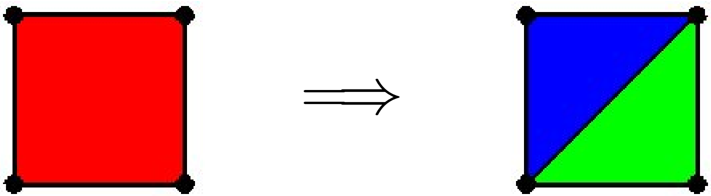
$$Z_\alpha \colon X_m \times Y_m \xrightarrow{X\alpha \times Y_\alpha} X_n \times Y_n$$

Example:

This natural product:

automatically constructs

the "right" triangulation of $I \times I$.



Twisted Products

Ingredients:

B = base space = simplical set

F = fibre space = simplicial set

G = structural group = simplicial group

$\tau : B \to G$ = twisting function

Result:

$$E = F \times_{\tau} B$$

$\Rightarrow$ **Fibration:**

$$F \hookrightarrow [E = F \times_{\tau} B] \longrightarrow B$$

Main point: twist τ = modifier of incidence relations

in $F \times B$.

<u>Example 1</u>: $B = S^1$, $G = \mathbb{Z}$, $\tau_1(s_1) = 0_0 \in G_0$

$\Rightarrow B \times_\tau G = S^1 \times \mathbb{Z} =$ trivial product.

In particular $\partial_0(s_1, \tau(s_1).k_1) = (*, k_0)$.

<u>Example 2</u>: $B = S^1$, $G = \mathbb{Z}$, ${\tau'}_1(s_1) = 1_0 \in G_0$

$\Rightarrow B \times_\tau G = S^1 \times \mathbb{Z} =$ twisted product.

Now $\partial_0(s_1, k_1) = (*, \tau'(s_1).k_0) = (*, (k+1)_0)$.

Daniel Kan's fantastic work ($\sim$ 1960 – 1980).

Every "standard" natural topological construction process has a translation in the simplicial world.

Frequently the translation is even "better".

Typical example. The loop space construction in ordinary topology gives only an H-space (= group up to homotopy).

Kan's loop space construction produces a genuine simplicial group, playing an essential role in Algebraic Topology.

Conclusion: Simplicial world = Paradise! ???

Translation process: $\boxed{\text{Topology}} \rightarrow \boxed{\text{Algebra}}$

$$X \longmapsto C_*(X)$$

$\Rightarrow C_*(X) =$ **chain complex canonically associated to** X.

$C_m(X) := \mathbb{Z}[X_m]$ **and** $d(\sigma) := \sum_{i=0}^{m} (-1)^m \partial_i^m(\sigma)$.

$H_m(X) := H_m(C_*(X))$.

"Equivalent" version: $C_*^{ND}(X)$ **with:**

$C_m^{ND}(X) := \mathbb{Z}[X_m^{ND}]$ **and** $d(\sigma) := \sum_{i=0}^{m} (-1)^m \partial_i^m(\sigma \bmod ND)$.

$H_m^{ND}(X) := H_m(C_*^{ND}(X)) \overset{\text{thr}}{=} H_m(X)$.

General work style in Algebraic Topology.

Main problem = Classification.

Main invariants = Homology groups.

$\boldsymbol{X}$ **given.** $\boldsymbol{H_*(X) = ???}$

Game rule: Please find a fibration:

$$\boldsymbol{F \hookrightarrow E \to B}$$

where $\boldsymbol{X = F}$ **or** $\boldsymbol{E}$ **or** $\boldsymbol{B}$

and where the homology of both other terms is known.

Then use the fibration!

Main tools: $(F \hookrightarrow E \to B)$

Serre spectral sequence:

$$H_*(B; H_*(F)) \Rightarrow H_*(E)$$

Eilenberg-Moore spectral sequence I:

$$\mathrm{Cobar}^{H_*(B)}(H_*(E), \mathbb{Z}) \Rightarrow H_*(F)$$

Eilenberg-Moore spectral sequence II:

$$\mathrm{Bar}^{H_*(G)}(H_*(E), H_*(F)) \Rightarrow H_*(F)$$

But these spectal sequences are not algorithms!

Example. $\pi_6 S^3 = ???$.

Solution (Serre). Consider 7 fibrations:

$$
\begin{array}{rrcccl}
(\mathrm{EM_2\text{-}SS}) & \underline{K(\mathbb{Z},1)} & \hookrightarrow & * & \to & K(\mathbb{Z},2) \\
(\mathrm{EM_2\text{-}SS}) & \underline{K(\mathbb{Z}_2,1)} & \hookrightarrow & * & \to & K(\mathbb{Z}_2,2) \\
(\mathrm{EM_2\text{-}SS}) & K(\mathbb{Z}_2,2) & \hookrightarrow & * & \to & K(\mathbb{Z}_2,3) \\
(\mathrm{EM_2\text{-}SS}) & K(\mathbb{Z}_2,3) & \hookrightarrow & * & \to & K(\mathbb{Z}_2,4) \\
(\mathrm{S\text{-}SS}) & K(\mathbb{Z},2) & \hookrightarrow & X_4 & \to & \underline{S^3} \\
(\mathrm{S\text{-}SS}) & K(\mathbb{Z}_2,3) & \hookrightarrow & X_5 & \to & X_4 \\
(\mathrm{S\text{-}SS}) & K(\mathbb{Z}_2,4) & \hookrightarrow & \boxed{X_6} & \to & X_5
\end{array}
$$

Solution: $\pi_6(S^3) = H_6(X_6)$

Effective Homology gives

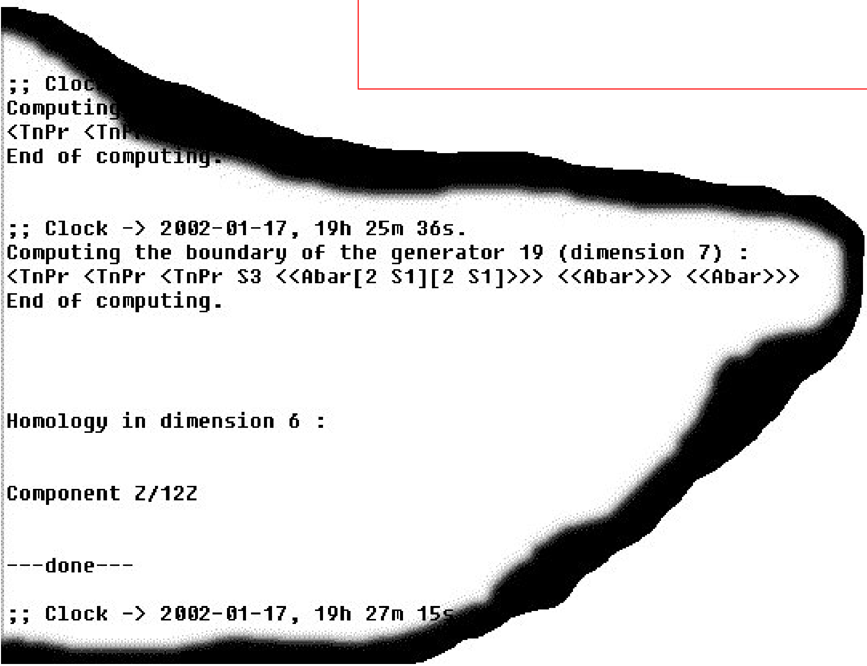
effective versions

of Serre and Eilenberg-Moore spectral sequences.

$\Rightarrow$ Basic Algebraic Topology

is within range of Symbolic Computation.

The END



```
;; Cloc
Computing
<TnPr <TnP
End of computing.

;; Clock -> 2002-01-17, 19h 25m 36s.
Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.

Homology in dimension 6 :

Component Z/12Z

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