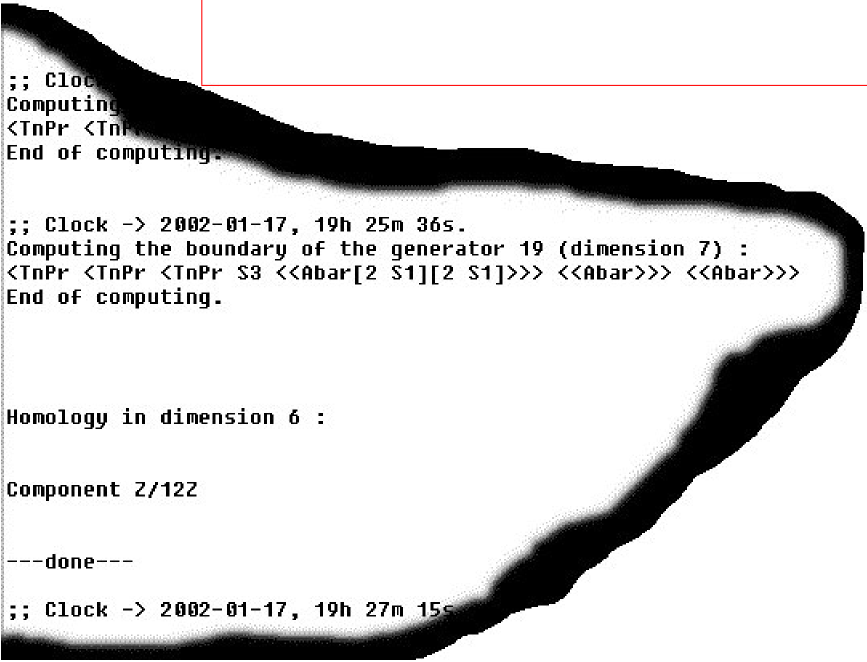
Constructive Homological Algebra IV.

Koszul complexes (continued)



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Typical simple example.

$I = < x - t^3, y - t^5 > \subset A = \mathbb{Q}[x, y, t]$.

How to compute $H_*(K(A/I)) = H_*(K(A/I\,;\,x, y, t))$?

Step 1: Compute a Groebner basis for I.

Choose a coherent monomial order,

for example DegRevLex = DRL.

$\Rightarrow$ Groebner$(I, \mathrm{DRL}) = < xt^2 - y, t^3 - x, x^2 - yt >$.

Step 2: Consider $J = < xt^2, t^3, x^2 >$

= the associated monomial ideal.

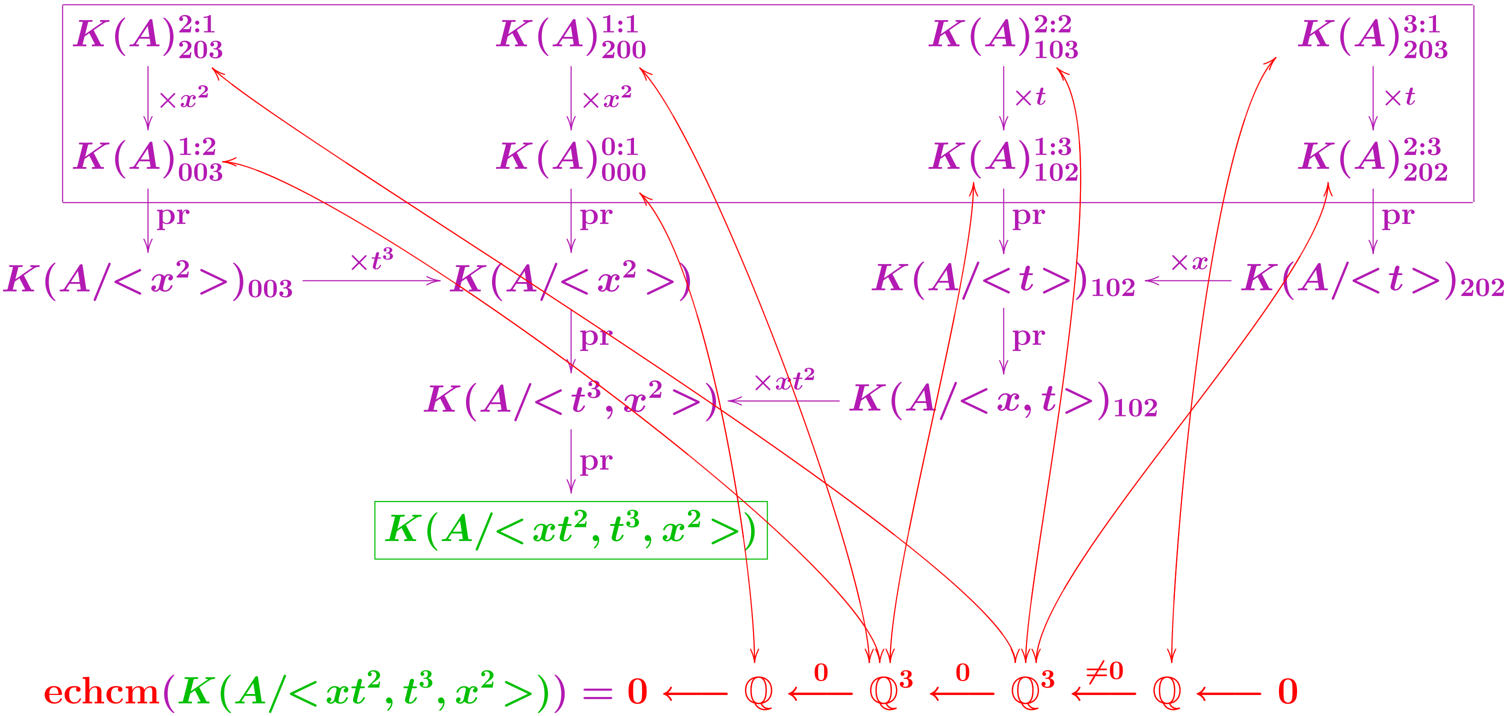
Then: 1. The $\mathbb{Q}$-vector spaces A/I and A/J are canonically isomorphic.

2. $\Rightarrow$ $K(A/I)$ and $K(A/J)$ are graded $\mathbb{Q}$-vector spaces canonically isomorphic, but with non-compatible differentials:

$$d_{K(A/J)}(t^2\delta x) = 0 \quad ; \quad d_{K(A/I)}(t^2\delta x) = y.$$

Plan: 1. Compute $H_*(K(A/J))$.

2. Apply BPL to deduce $H_*(K(A/I))$.



$K(A)^{2:1}_{203}$
$K(A)^{1:1}_{200}$
$K(A)^{2:2}_{103}$
$K(A)^{3:1}_{203}$
$\times x^2$
$\times x^2$
$\times t$
$\times t$
$K(A)^{1:2}_{003}$
$K(A)^{0:1}_{000}$
$K(A)^{1:3}_{102}$
$K(A)^{2:3}_{202}$
pr
$K(A/<x^2>)_{003}$
$\times t^3$
$K(A/<x^2>)$
$K(A/<t>)_{102}$
$\times x$
$K(A/<t>)_{202}$
$K(A/<t^3, x^2>)$
$\times xt^2$
$K(A/<x,t>)_{102}$
$K(A/<xt^2, t^3, x^2>)$
$\mathrm{echcm}(K(A/<xt^2, t^3, x^2>)) = 0 \longleftarrow \mathbb{Q} \xleftarrow{0} \mathbb{Q}^3 \xleftarrow{0} \mathbb{Q}^3 \xleftarrow{\neq 0} \mathbb{Q} \longleftarrow 0$

$[K(A/I), d_I] = [K(A/J), d_J + \delta_{I,J}]$:

"Same" graded module, only the differentials are different.

But J = monomial ideal Groebner-associated to I.

$\Rightarrow$ The perturbation $\delta_{I,J}$ strictly reduces the multigrading.

Example:

$$\begin{aligned} d_J(t^2\,\delta x) &= 0 \\ d_I(t^2\,\delta x) &= y \text{ with } y \text{ "<" } t^2\,\delta x. \end{aligned}$$

The perturbation recursively replaces

leading monomials by trailing terms.

Easy-BPL $\Rightarrow$

$$[K(A/J), d_J + \delta_{I,J}] \overset{\rho'_\ell}{\Lleftarrow} [\widehat{C}_*, d_t + \widehat{\delta}_{IJ}] \;\Big|\; \overset{\rho_r}{\Rrightarrow} [\mathbb{Q}^8, d_{br}]$$

- Righthand homotopy operator h_r is multi-homogeneous.
- Perturbation $\widehat{\delta}_{IJ}$ strictly reduces the multigrading.

$\Rightarrow$ Composition $h_r \circ \delta_{IJ}$ is $\boxed{\text{pointwise nilpotent}}$.

$\Rightarrow$ BPL can be applied.

$\Rightarrow$

$$[K(A/J), d_J + \delta_{I,J}] \overset{\rho'_\ell}{\Lleftarrow} [\widehat{C}_*, d_t + \widehat{\delta}_{IJ}] \overset{\rho''_r}{\Rrightarrow} [\mathbb{Q}^8, d_{br} + \delta'_{IJ}]$$

$\Rightarrow$ Effective homology of $K(A/I)$ is obtained. QED.

Corollary:

$$H_*(K(A/I)) \text{ is a } k\text{-vector space of finite dimension.}$$

More precisely: $H_*(K(A/I)) \leq H_*(K(A/J))$.

Example:

$$\begin{aligned} I &= \langle x - t^3, y - t^5 \rangle \\ &= \langle xt^2 - y, t^3 - x, x^2 - yt \rangle \\ J &= \langle xt^2, t^3, x^2 \rangle \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} H_*(K(A/J) &= (k^1, k^3, k^2) \\ H_*(K(A/I)) &= (k^1, k^2, k^1) \end{aligned}$$

Local character of the analysis given by the Koszul process.

The "differential" terms δx in the Koszul complex are producers of generators of $\mathfrak{m}/\mathfrak{m}^2$.

With $\mathfrak{m}$ being the maximal ideal of $k[x_1, \ldots, x_m]$ $\boxed{\text{at } (0, \ldots, 0)}$.

Example.

$J = \,<x^2>\, \subset k[x] \Rightarrow H_*(k[x]/J) = (k, k)$.

$I = \,<(x-1)^2>\, \subset k[x] \Rightarrow H_*(k[x]/I) = \boxed{(0, 0)}$.

Correct natural framework:

$$A_0 = k[x_1, \ldots, x_m]_0 \qquad (\ ]_0 = \text{ localized at } 0)$$

$$A_0 = \{p/q\} \text{ with } p, q \in A = k[x_1, \ldots, x_m],\ q(0) \neq 0.$$

But for $I_0 =$ ideal in A_0 and $I = I_0 \cap A$:

$$H_*(K(A/I)) \xrightarrow{\boxed{\cong}} H_*(K(A_0/I_0))$$

Every ideal I of A should be replaced by its 0-localization:

$$I_0 = (I \otimes_A A_0) \cap A$$

Example: $<(x-1)^2>_0 = <1> = A$.

Theorem: Let M be an A-module with $A = k[x_1, \ldots, x_m]_0$.

Then there is a canonical bijection between:

1. Effective homologies of $K(M)$;
2. Effective A-resolutions of M.

The same between:

1. Minimal effective homologies of $K(M)$;
2. Minimal effective A-resolutions of M.

Key tool: Aramova-Herzog bicomplex:

$$\boldsymbol{AH(M) = M \otimes \wedge \otimes A}$$

with $\boldsymbol{\wedge = \wedge(\mathfrak{m}/\mathfrak{m}^2)}$.

Two possible readings of $\mathbf{AH}(\boldsymbol{M})$:

$$\mathrm{AH}(M) = \underbrace{M \otimes K(A)}_{1} = \underbrace{K(M) \otimes A}_{2}$$

First reading: $\boldsymbol{K(A)}$ **acyclic** $\Rightarrow \mathrm{AH}(M) \sim M$.

Second reading: $\mathrm{AH}(M) = K(M) \otimes A$.

Effective homology of $K(M) \Leftrightarrow (K(M) \sim H_*)$

with $H_* =$ **chain-complex of** k**-vector spaces of finite type.**

$$\Rightarrow K(M) \otimes A \sim H_* \otimes A$$

where $H_* \otimes A =$

chain-complex of $\boxed{A\text{-free}}$**-modules of finite type.**

Combining both readings $\Rightarrow$ **Equivalence** $M \Longleftrightarrow (H_* \otimes A)$.

Interpretation of $\Longleftrightarrow$ $\Rightarrow$ **Resolution.**

The Aramova-Herzog bicomplex $\mathrm{AH}(M)$.

$\mathrm{AH}(M) :=$

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M\otimes\wedge^3\otimes A_0 & \xrightarrow{\partial''} & M\otimes\wedge^2\otimes A_1 & \xrightarrow{\partial''} & M\otimes\wedge^1\otimes A_2 & \xrightarrow{\partial''} & M\otimes\wedge^0\otimes A_3 & \longrightarrow & 0 \\
{\scriptstyle\partial'}\downarrow & & {\scriptstyle\partial'}\downarrow & & {\scriptstyle\partial'}\downarrow & & \downarrow & & \\
M\otimes\wedge^2\otimes A_0 & \xrightarrow{\partial''} & M\otimes\wedge^1\otimes A_1 & \xrightarrow{\partial''} & M\otimes\wedge^0\otimes A_2 & \longrightarrow & 0 & & \\
{\scriptstyle\partial'}\downarrow & & {\scriptstyle\partial'}\downarrow & & \downarrow & & & & \\
M\otimes\wedge^1\otimes A_0 & \xrightarrow{\partial''} & M\otimes\wedge^0\otimes A_1 & \longrightarrow & 0 & & & & \\
{\scriptstyle\partial'}\downarrow & & \downarrow & & & & & & \\
M\otimes\wedge^0\otimes A_0 & \longrightarrow & 0 & & & & & & \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

$A_p = k[x_1, \ldots, x_m]^{[p]}$

$\wedge^q = \wedge^q k^m = \wedge^q(\mathfrak{m}/\mathfrak{m}^2)$

$M = A$-module

$\otimes = \otimes_k$

Horizontal $= M \otimes K(A)_q$

Vertical $= K(M) \otimes A_p$

1. Horizontal reduction.

Every horizontal complex is a homogeneous component of $M \otimes_k K(A)$:

$$0 \longrightarrow M \otimes \wedge^3 \otimes A_0 \xrightarrow{\partial''} M \otimes \wedge^2 \otimes A_1 \xrightarrow{\partial''} M \otimes \wedge^1 \otimes A_2 \xrightarrow{\partial''} M \otimes \wedge^0 \otimes A_3 \longrightarrow 0$$

But $K(A)$ acyclic $\Rightarrow$ every horizontal is 0-reducible.

Except the 0-horizontal $= M \otimes_k \wedge^0 \otimes_k S_0 = M$.

BPL $\Rightarrow$ A canonical reduction is produced:

$$\mathrm{AH}(M) \Rrightarrow M$$

2. Vertical reduction.

The p-vertical is $K(M) \otimes_k A_p$.

Let $K(M) \Rrightarrow H(K(M))$ be a reduction of $K(M)$ over the complex made of the homology groups of $K(M)$ and the null differential.

Applying this reduction to the p-vertical produces:

$$\mathrm{AH}(M)_p \Rrightarrow H(K(M)) \otimes_k A_p$$

BPL $\Rightarrow$ a canonical reduction is produced:

$$\mathrm{AH}(M) \Rrightarrow H(K(M)) \otimes_k A$$

⇒ Equivalence:

$$\boldsymbol{H(K(M)) \otimes_k A \Lleftarrow \mathbf{AH}(M) \Rrightarrow M}$$

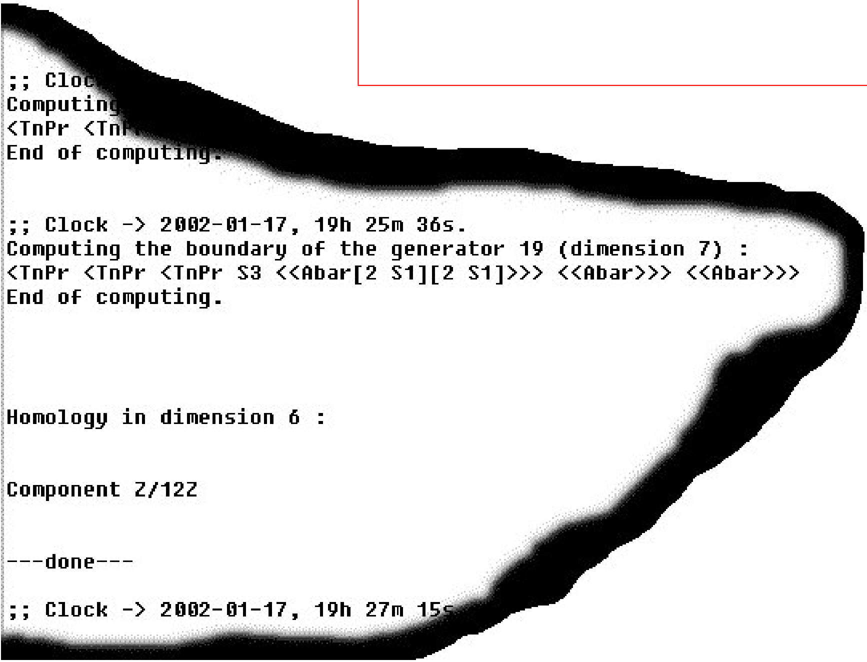
$$g_\ell : H(K(M)) \otimes_k A \to \mathbf{AH}(M), \qquad f_r : \mathbf{AH}(M) \to M$$

Then:

$$\boldsymbol{f_r \circ g_\ell : H(K(M)) \otimes_k A \longrightarrow M}$$

is the looked-for resolution.

The END



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;; Cloc
Computing
<TnPr <Tn
End of computing.

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Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.

Homology in dimension 6 :

Component Z/12Z

---done---

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