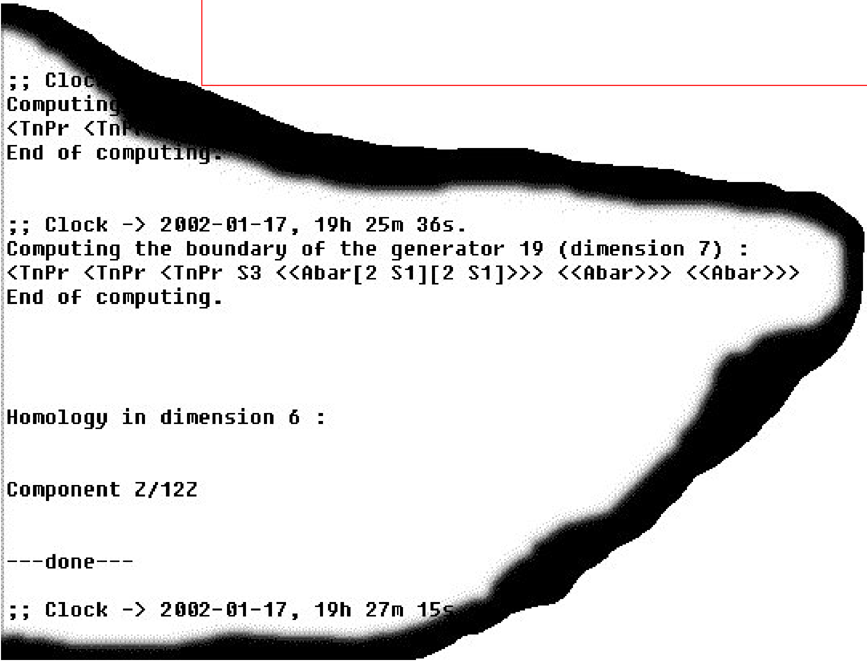
Constructive Homological Algebra II.

The Homological Problem



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Genova Summer School, 2006

“Standard” Homological Algebra is not constructive.

Examples.

1. $H_n(X) = 0$.

Translation:

$$(\forall z \in C_n(X))((dx = 0) \Rightarrow (\boxed{\exists c} \in C_{n+1}(X) \ \underline{\text{st}} \ dc = z))$$

$(\exists c)$ **constructive ??**

Generally no!!

2. $\boldsymbol{H_n(X) = \mathbb{Z}}$.

Translation:

$$\boxed{(\exists\phi)} \ \ \underline{\textbf{st}}\ \boldsymbol{\phi : \mathbb{Z} \xrightarrow{\cong} H_n(X)}$$

$(\exists\phi)$ **constructive ??**

Generally no !!

3. Making $\boldsymbol{\phi}$ **constructive:**

$\boldsymbol{\phi}$ **defined through** $\boldsymbol{\overline{\phi} : \mathbb{Z} \longrightarrow Z_n(X)}$

st the $\boxed{\textbf{induced map}}$ $\boldsymbol{\phi : \mathbb{Z} \xrightarrow{\cong} H_n(X) =}$ **isomorphism.**

Justification ??

$\phi : \mathbb{Z} \xrightarrow{\cong ??} H_n(X)$ $\overline{\phi} : \mathbb{Z} \longrightarrow Z_n(X)$

$$(\forall z \in Z_n(X))((\boxed{\exists n} \in \mathbb{Z}) \text{ \underline{st} } (\phi(n) \text{ and } z \boxed{\text{homologous}}))$$

3a. $(\exists n)$ **must be made constructive.**

3b. Justification of homologous ??

$$(\boxed{\exists c} \in C_{n+1}(X)) \text{ \underline{st} } dc = z - \phi(n)$$

$(\exists c)$ **must be made constructive.**

$\Rightarrow$ **Much work in front of us !!**

Reward: Homological algebra becomes easier !!

Solving the homological problem for a chain complex C_*

$\Leftrightarrow$ You must be able to:

1. Determine the isomorphism class of $H_i(C_*)$ for arbitrary $i \in \mathbb{Z}$.
2. Produce a map $\rho : H_i(C_*) \to C_i$
   giving a representant for every homology class.
3. Determine whether an arbitrary chain $c \in C_i$ is a cycle.
4. Compute, given an arbitrary cycle $z \in Z_i = \ker(d_i : C_i \to C_{i-1})$,
   its homology class $\overline{z} \in H_i(C_*)$.
5. Compute, given a cycle $z \in Z_i$ known as a boundary ($\overline{z} = 0$),
   a boundary-premimage $c \in C_{i+1}$ ($d_{i+1}(c) = z$).

Definition: A (homological) reduction is a diagram:

$$\rho : \boxed{h \circlearrowleft \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$$

with:

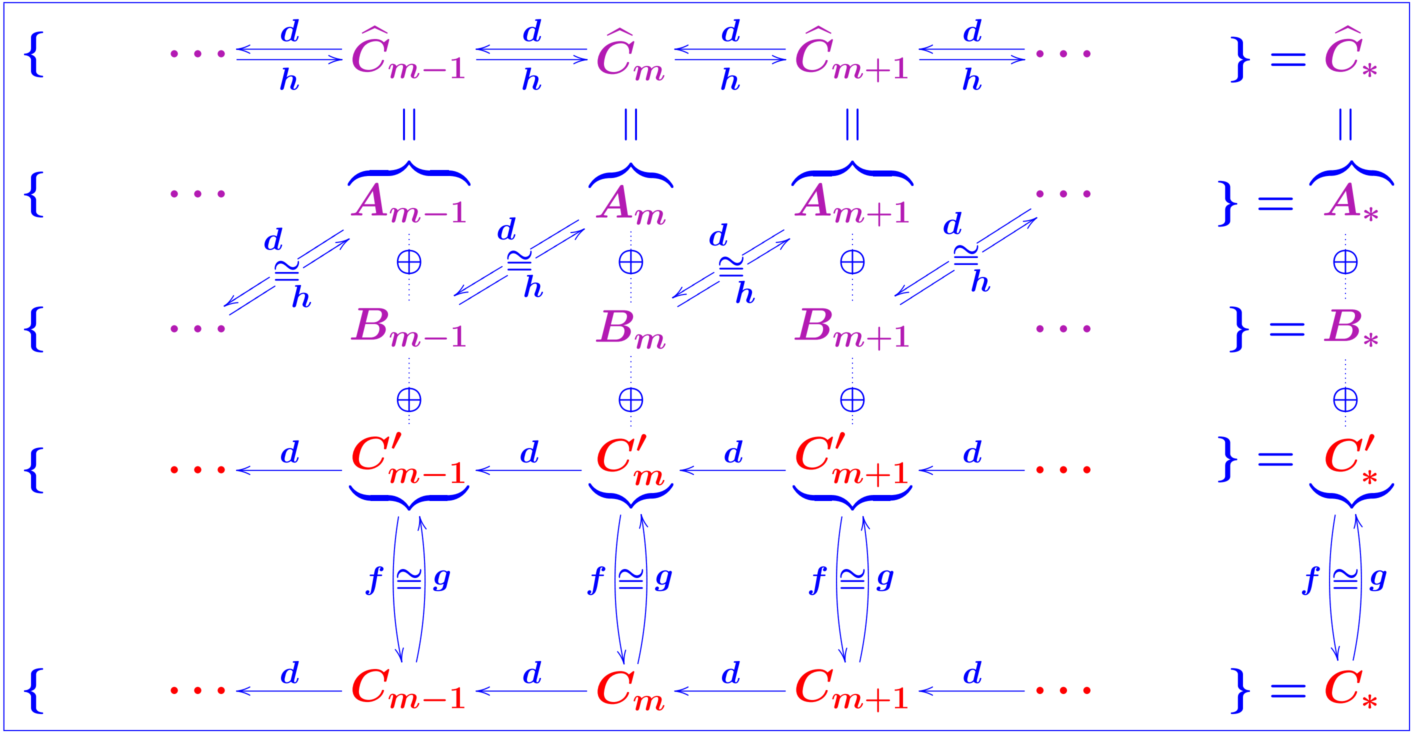
1. $\widehat{C}_*$ **and** C_* **= chain complexes.**

2. f **and** g **= chain complex morphisms.**

3. h **= homotopy operator (degree +1).**

4. $fg = \mathrm{id}_{C_*}$ **and** $d_{\widehat{C}}h + hd_{\widehat{C}} + gf = \mathrm{id}_{\widehat{C}_*}$**.**

5. $fh = 0$, $hg = 0$ **and** $hh = 0$**.**



$$\boxed{A_* = \ker f \cap \ker h} \quad \boxed{B_* = \ker f \cap \ker d} \quad \boxed{C'_* = \operatorname{im} g}$$

$$\widehat{C}_* = \boxed{A_* \oplus B_* \text{ exact}} \oplus \boxed{C'_* \cong C_*}$$

Let $\rho : \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\leftrightarrows}} C_*}$ **be a reduction.**

Frequently:

1. $\widehat{C}_*$ **is a locally effective chain complex:**

   its homology groups are unreachable.

2. C **is an effective chain complex:**

   its homology groups are computable.

3. **The reduction** ρ **is an entire description of**

   the homological nature of $\widehat{C}_*$**.**

4. **Any homological problem in** $\widehat{C}_*$ **is solvable**

   thanks to the information provided by ρ**.**

$$\rho : \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*}$$

1. What is $H_n(\widehat{C}_*)$? Solution: Compute $H_n(C_*)$.

2. Let $x \in \widehat{C}_n$. Is x a cycle? Solution: Compute $d_{\widehat{C}_*}(x)$.

3. Let $x, x' \in \widehat{C}_n$ be cycles. Are they homologous?

   Solution: Look whether $f(x)$ and $f(x')$ are homologous.

4. Let $x, x' \in \widehat{C}_n$ be homologous cycles.

   Find $y \in \widehat{C}_{n+1}$ satisfying $dy = x - x'$?

   Solution:

   (a) Find $z \in C_{n+1}$ satisfying $dz = f(x) - f(x')$.

   (b) $y = g(z) + h(x - x')$.

Definition: (C_*, d) = given chain complex.

A perturbation $\delta\colon C_* \to C_{*-1}$ is an operator of degree -1

satisfying $(d+\delta)^2 = 0$ ($\Leftrightarrow (d\delta + \delta d + \delta^2) = 0$):

$$(C_*, d) + (\delta) \mapsto (C_*, d+\delta).$$

Problem: Let $\rho : \boxed{h \circlearrowright (\widehat{C}_*, \widehat{d}) \underset{f}{\overset{g}{\leftrightarrows}} (C_*, d)}$ be a given reduction and $\widehat{\delta}$ a perturbation of $\widehat{d}$.

How to determine a new reduction:

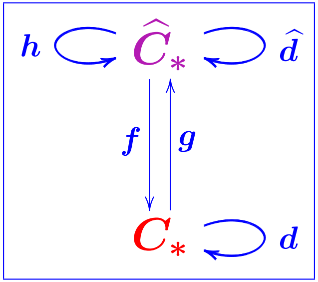
$$? : \boxed{? \circlearrowright (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \underset{?}{\overset{?}{\leftrightarrows}} (C_*, ?)}$$

describing in the same way the homology of

the chain complex with the perturbed differential?

Basic Perturbation "Lemma" (BPL):

Given:



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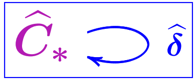
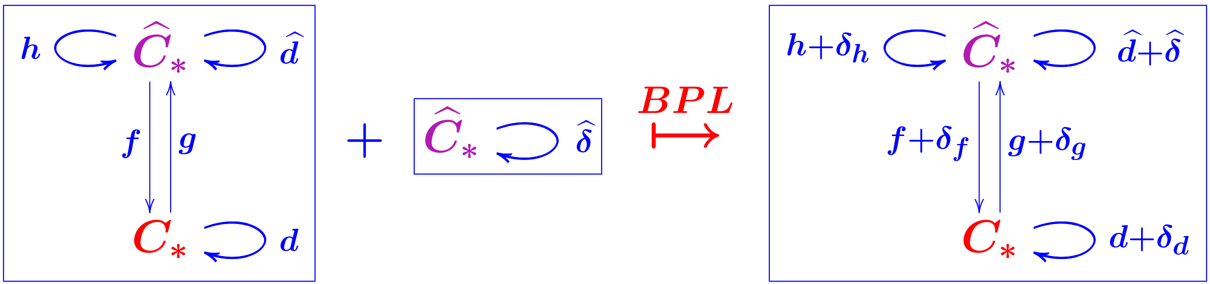


satisfying:

1. $\widehat{\delta}$ is a perturbation of the differential $\widehat{d}$;

2. The operator $h \circ \widehat{\delta}$ is pointwise nilpotent.

Then a general algorithm BPL constructs:



Proof:

$\phi := \sum_{i=1}^{\infty}(-1)^i(h\widehat{\delta})^i$ and $\psi := \sum_{i=1}^{\infty}(-1)^i(\widehat{\delta}h)^i$ are defined.

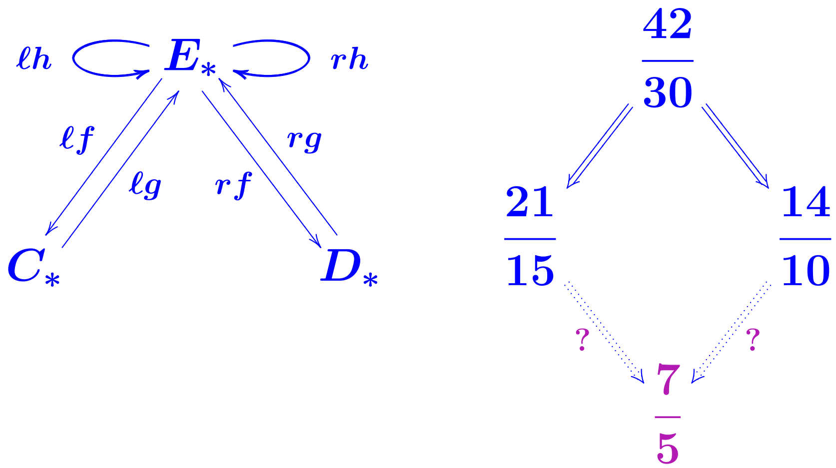
Then:

- $\delta_d := f\widehat{\delta}(\mathrm{id}_{\widehat{C}} + \phi)g = f(\mathrm{id}_{\widehat{C}} + \psi)\widehat{\delta}g$
- $\delta_f := f\psi$
- $\delta_g := \phi g$
- $\delta_h := \phi h = h\psi$

is the solution.

QED

Definition: A (strong chain-) equivalence $\varepsilon : C_* \Longleftarrow\!\!\!\Longrightarrow D_*$ is a pair of reductions $C_* \overset{\ell\rho}{\Lleftarrow} E_* \overset{r\rho}{\Rrightarrow} D_*$:



Normal form problem ??

More structure often necessary in C_*.

Definition: An object with effective homology X is a 4-tuple:

$$X = \boxed{X, C_*(X), EC_*, \varepsilon}$$

with:

1. X = an arbitrary object (simplicial set, simplicial group, differential graded algebra, ...)
2. $C_*(X)$ = the chain complex "traditionally" associated to X to define the homology groups $H_*(X)$.
3. EC_* = some effective chain complex.
4. ε = some equivalence $C_*(X) \stackrel{\varepsilon}{\Longleftarrow\!\!\Longrightarrow} EC_*$.

Main result of effective homology:

Meta-theorem: Let $X_{1*}, \ldots, X_{n*}$ be a collection of objects with effective homology and ϕ be a reasonable construction process:

$$\phi : (X_{1*}, \ldots, X_{n*}) \mapsto X_*.$$

Then there exists a version with effective homology ϕ_{EH}:

$$\phi_{EH}: (\boxed{X_1, C_*(X_1), EC_{1*}, \varepsilon_1}, \ldots, \boxed{X_n, C_*(X_n), EC_{n*}, \varepsilon_n})$$

$$\mapsto \boxed{X, C_*(X), EC_*, \varepsilon}$$

The process is **perfectly stable**

and can be again used with X for further calculations.

Typical example of PBL application: the SES_2 Theorem.

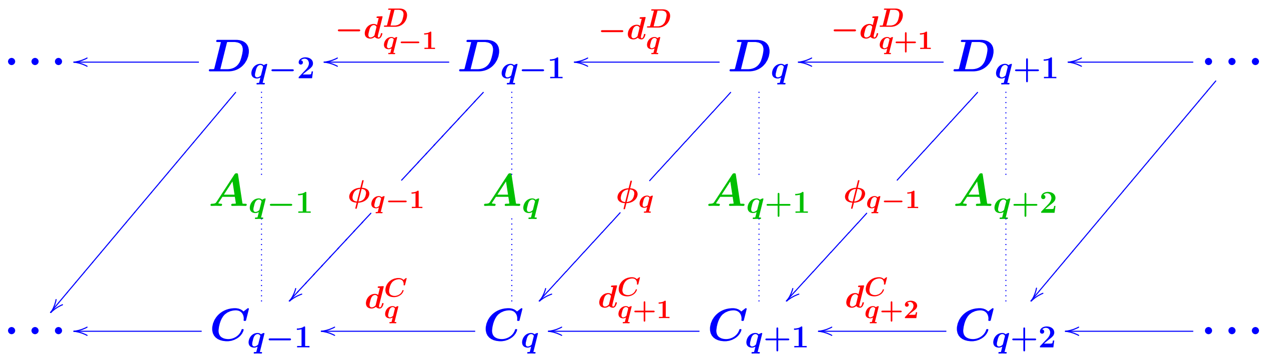
<u>Definition</u>: The algebraic cone construction:

Ingredients: two chain complexes C_*, D_*

and a chain-complex morphism $\phi : C_* \leftarrow D_*$.

Result: a chain complex $A_* = \mathrm{Cone}(\phi)$ defined by:

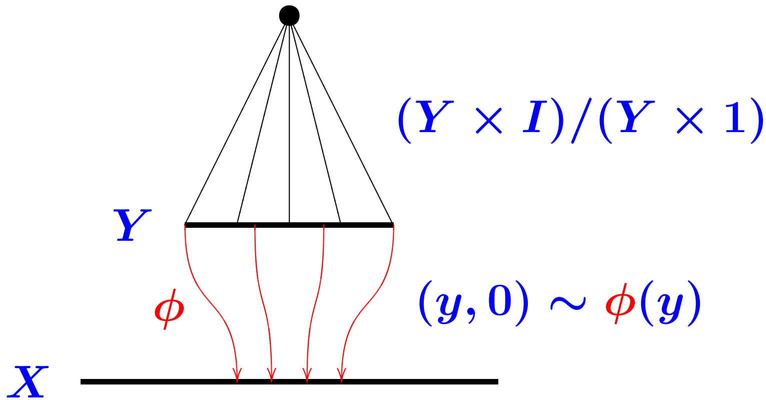
$$A_q = C_q \oplus D_{q-1} \qquad d_q^A = \begin{bmatrix} d_q^C & \phi_q \\ 0 & -d_{q-1}^D \end{bmatrix}$$



Geometrical interpretation.

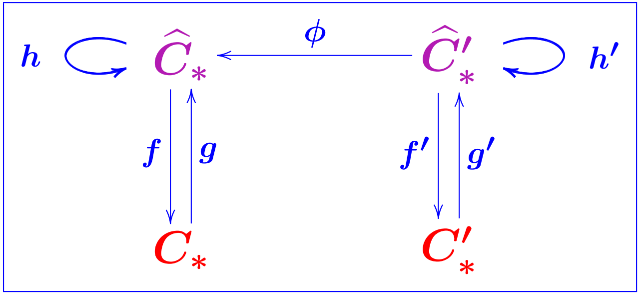
$\phi : X \leftarrow Y$ = continuous map.

$$\text{Cone}(\phi) := \left(X \coprod (Y \times I)\right) / \left((Y \times 1) \;\&\; (y,0) \sim \phi(y)\right)$$

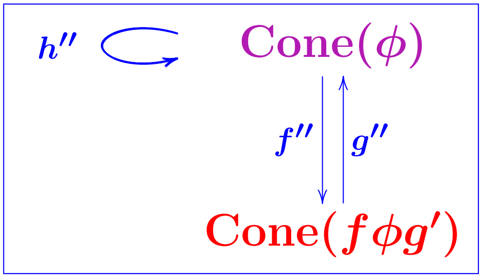


SES$_2$ Theorem: A general algorithm CR can be produced:

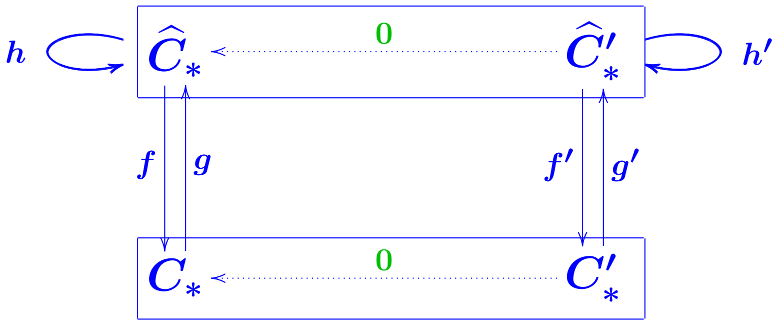
Input:



Output:

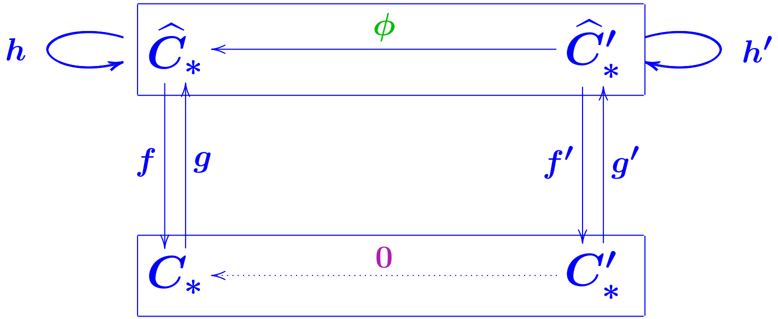


Proof: 1. Particular case $\phi = 0$: trivial (direct sums).



$$\underbrace{\begin{bmatrix} \widehat{d} & 0 \\ 0 & -\widehat{d}' \end{bmatrix}}_{\widehat{D}} \quad \underbrace{\begin{bmatrix} d & 0 \\ 0 & -d' \end{bmatrix}}_{D} \quad \underbrace{\begin{bmatrix} f & 0 \\ 0 & f' \end{bmatrix}}_{F} \quad \underbrace{\begin{bmatrix} g & 0 \\ 0 & g' \end{bmatrix}}_{G} \quad \underbrace{\begin{bmatrix} h & 0 \\ 0 & -h' \end{bmatrix}}_{H}$$

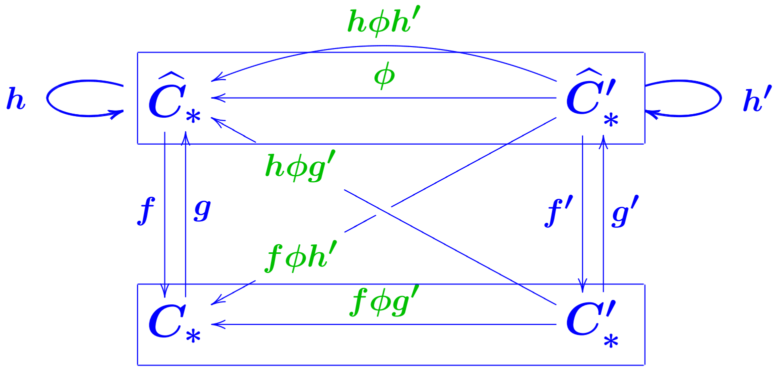
Proof: **2. Install the actual ϕ. The reduction is nomore valid.**



$$\begin{bmatrix} \hat{d} & \phi \\ 0 & -\hat{d}' \end{bmatrix} \quad \begin{bmatrix} d & 0 \\ 0 & -d' \end{bmatrix} \quad \begin{bmatrix} f & 0 \\ 0 & f' \end{bmatrix} \quad \begin{bmatrix} g & 0 \\ 0 & g' \end{bmatrix} \quad \begin{bmatrix} h & 0 \\ 0 & -h' \end{bmatrix}$$

$$\widehat{D} \qquad D \qquad F \qquad G \qquad H$$

Proof: 3. Apply the Basic Perturbation Lemma:



$$\underbrace{\begin{bmatrix} \widehat{d} & \phi \\ 0 & -\widehat{d}' \end{bmatrix}}_{\widehat{D}} \quad \underbrace{\begin{bmatrix} d & f\phi g' \\ 0 & -d' \end{bmatrix}}_{D} \quad \underbrace{\begin{bmatrix} f & f\phi h' \\ 0 & f' \end{bmatrix}}_{F} \quad \underbrace{\begin{bmatrix} g & -h\phi g' \\ 0 & g' \end{bmatrix}}_{G} \quad \underbrace{\begin{bmatrix} h & h\phi h' \\ 0 & -h' \end{bmatrix}}_{H}$$

QED.

Why the terminology SES$_2$ theorem?

A morphism $\phi : A_* \leftarrow B_*$ produces

an effective $\boxed{\text{S}}$hort $\boxed{\text{E}}$xact $\boxed{\text{S}}$equence of chain complexes:

$$0 \longrightarrow A_* \underset{i}{\overset{\rho}{\leftrightarrows}} \text{Cone}(\phi) \underset{j}{\overset{\sigma}{\leftrightarrows}} B_* \longrightarrow 0$$

and the SES$_2$ theorem is an algorithm:

$$[\text{Reduction}(A_*) + \text{Reduction}(B_*)] \;\mapsto\; \text{Reduction}(\text{Cone}(\phi))$$

Notation: $\rho : \widehat{C}_* \Rrightarrow C_* \quad \Leftrightarrow \quad \rho : \boxed{h \circlearrowright \widehat{C}_* \underset{f}{\overset{g}{\rightleftarrows}} C_*}$.

Theorem (**Easy Basic Perturbation Lemma**):

$\boxed{\rho : (\widehat{C}_*, \widehat{d}) \Rrightarrow (C_*, d)} + \boxed{\delta : C_* \rightarrow C_{*-1} = \text{perturbation of } d}$

$\mapsto \boxed{\rho' : (\widehat{C}_*, \widehat{d} + \widehat{\delta}) \Rrightarrow (C_*, d + \delta)}$.

Proof: $(\widehat{C}_*, \widehat{d}) = (A_*, \widehat{d}) \oplus (C'_*, d')$ with $(C'_*, d') \cong (C, d)$.

Copy into (C'_*, d') **the perturbation** $\delta \quad \mapsto (C'_*, d' + \delta')$.

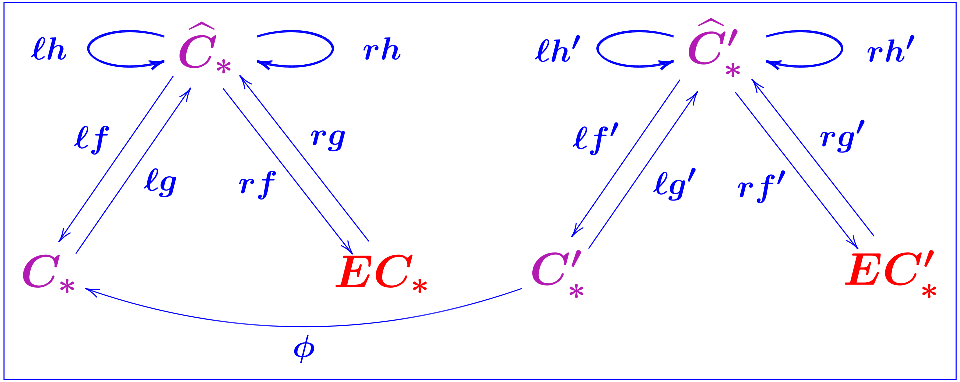
Solution $= \rho : ((A_*, \widehat{d}) \oplus (C'_*, d' + \delta')) \Rrightarrow (C_*, d + \delta)$.

QED

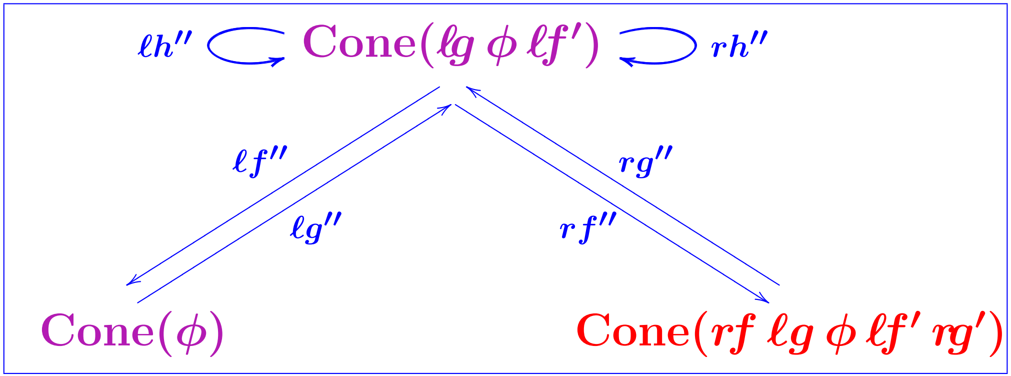
Cone-Equivalence Theorem:

A general algorithm CE can be produced:

Input:



Output:



SES$_3$ Theorem:

Let $(A, i, \rho, B, j, \sigma, C)$ be

an effective short exact sequence of chain-complexes:

$$0 \longrightarrow A_* \underset{i}{\overset{\rho}{\leftrightarrows}} B_* \underset{j}{\overset{\sigma}{\leftrightarrows}} C_* \longrightarrow 0$$

where:

1. **The i and j arrows are chain complex morphisms.**
2. **The ρ and σ arrows are graded module morphisms.**
3. $\mathrm{id}_{A_*} = \rho \circ i \quad ; \quad \mathrm{id}_{B_*} = i \circ \rho + \sigma \circ j \quad ; \quad \mathrm{id}_{C_*} = j \circ \sigma$.

Then an algorithm constructs a canonical reduction:

$$\mathrm{Cone}\,(i) \Rrightarrow C_*$$

from the data.

Proof:

1. **Cancel all the differentials.**

Then an obvious reduction is obtained:

$$\rho \circlearrowleft \boxed{[A_*, 0] \xrightarrow{i} [B_*, 0]} \underset{j}{\overset{\sigma}{\leftrightarrows}} [C_*, 0]$$

2. **Reinstall the differentials of A_* and B_*.**

3. **To be interpreted**

   as a perturbation of the differential of Cone(i).

4. **Apply BPL.**

$$\rho \circlearrowleft \boxed{[A_*, d_A] \xrightarrow{i} [B_*, d_B]} \underset{j}{\overset{\sigma - \rho d_B \sigma}{\leftrightarrows}} [C_*, d_C]$$

QED

<u>Corollary</u>: Same data:

$$0 \longrightarrow A_* \underset{i}{\overset{\rho}{\leftrightarrows}} B_* \underset{j}{\overset{\sigma}{\leftrightarrows}} C_* \longrightarrow 0$$

$+$ $A_* \overset{\varepsilon_A}{\Longleftarrow\!\!\Longrightarrow} EA_*$ **and** $B_* \overset{\varepsilon_B}{\Longleftarrow\!\!\Longrightarrow} EB_*$ **with effective homology.**

Then an algorithm constructs $\varepsilon_C : C_* \Longleftarrow\!\!\Longrightarrow EC_*$.

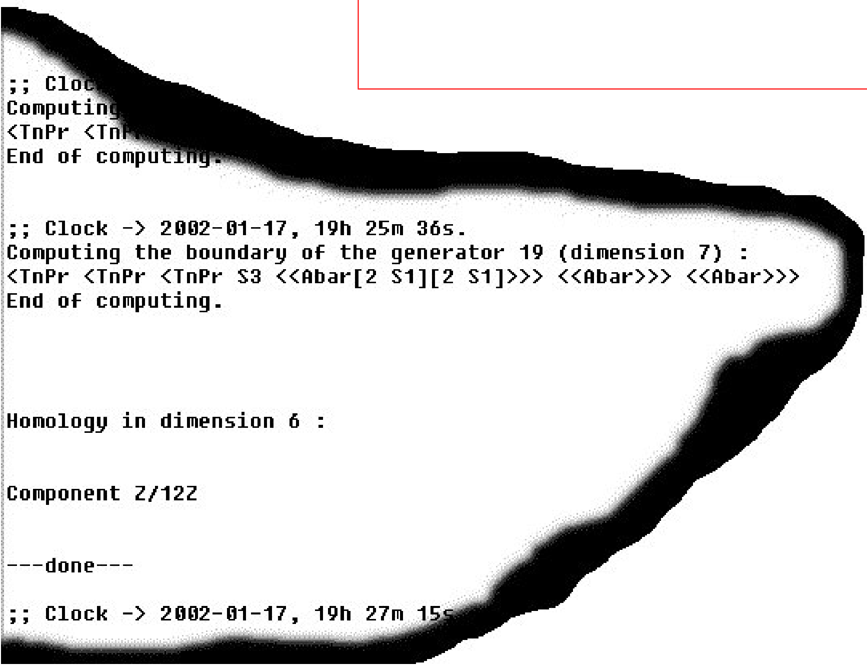
<u>Proof</u>:

$$C_* \Lleftarrow \mathrm{Cone}\,(i) \Lleftarrow \widehat{\mathrm{Cone}\,(i)} \Rrightarrow E\mathrm{Cone}\,(i)$$

$+$ **Composition of reductions.**

QED.

The END


End of computing.
;; Clock -> 2002-01-17, 19h 25m 36s.
Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.
Homology in dimension 6 :
Component Z/12Z
---done---
;; Clock -> 2002-01-17, 19h 27m