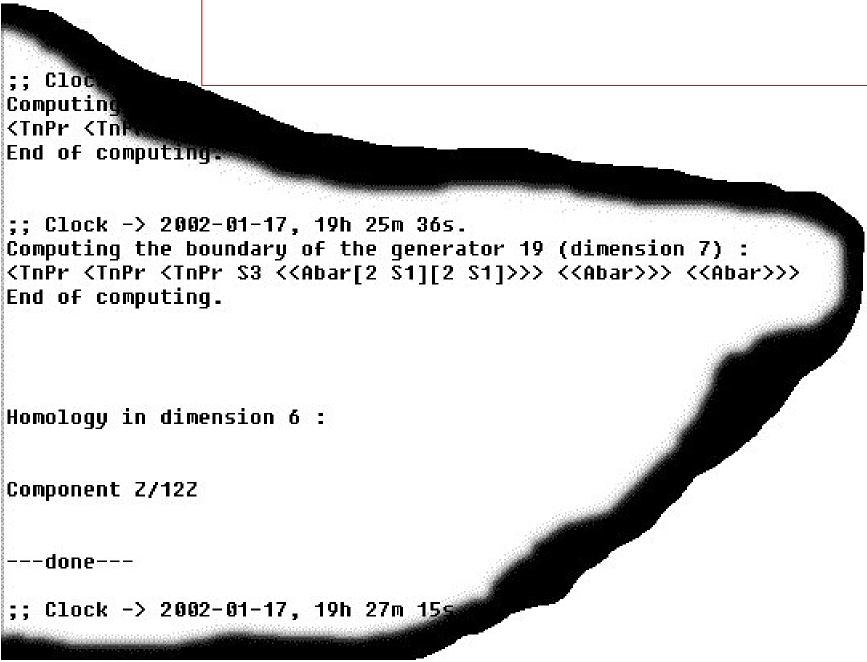
Constructive Homological Algebra I.

Homology Background



Francis Sergeraert, Institut Fourier, Grenoble, France
Genova Summer School, 2006

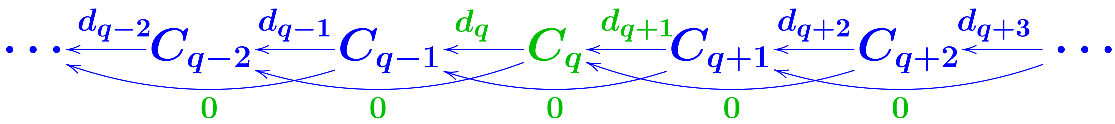
General plan:

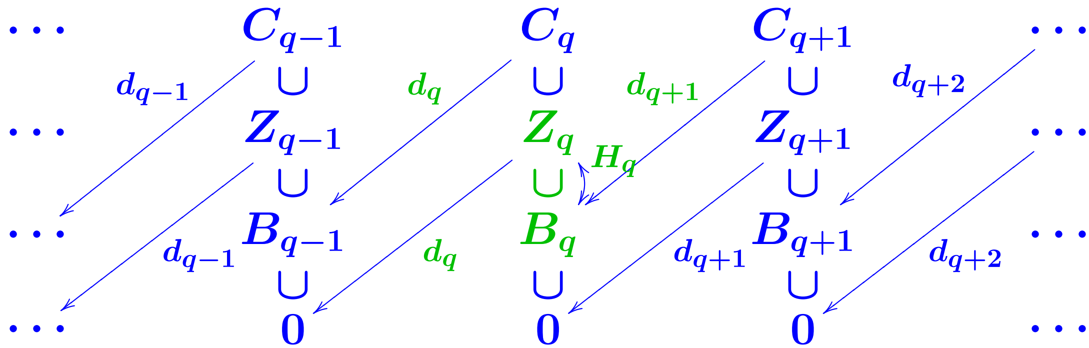
1. **Homological Algebra background.**
2. **The homological problem.**
3. **Koszul complexes.**
4. **Koszul complexes (continued).**
5. **Algebraic Topology background.**
6. **Constructive Spectral Sequences.**

Semantics of colours:

Blue	=	**“Standard” Mathematics**
Red	=	**Constructive, effective, algorithm, machine object, . . .**
Violet	=	**Problem, difficulty, obstacle, disadvantage, . . .**
Green	=	**Solution, essential point, mathematicians, . . .**

Chain complex:





$Z_q := \ker d_q$	$B_q := \operatorname{im} d_{q+1}$	$H_q := Z_q / B_q$
Cycles	Boundaries	Homology classes

Spectral sequence = movie: Page 0

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
E^0_{0,3} & E^0_{1,3} & E^0_{2,3} & E^0_{3,3} & E^0_{4,3} & \cdots \\
d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & \\
E^0_{0,2} & E^0_{1,2} & E^0_{2,2} & E^0_{3,2} & E^0_{4,2} & \cdots \\
d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & \\
E^0_{0,1} & E^0_{1,1} & E^0_{2,1} & E^0_{3,1} & E^0_{4,1} & \cdots \\
d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & E^0_{3,0} & E^0_{4,0} & \cdots
\end{array}
$$

Every column is a chain complex ($dd = 0$)

Spectral sequence = movie: **Page 0**

$$
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
E^0_{0,3} & E^0_{1,3} & E^0_{2,3} & E^0_{3,3} & E^0_{4,3} & \cdots \\
d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & \\
E^0_{0,2} & E^0_{1,2} & E^0_{2,2} & E^0_{3,2} & E^0_{4,2} & \cdots \\
d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & \\
E^0_{0,1} & E^0_{1,1} & E^0_{2,1} & E^0_{3,1} & E^0_{4,1} & \cdots \\
d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & d^0 \downarrow & \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & E^0_{3,0} & E^0_{4,0} & \cdots
\end{array}
$$

$$E^1_{p,q} := \ker d^0_{p,q} / \operatorname{im} d^0_{p,q+1} \qquad \Longrightarrow$$

Spectral sequence = movie: Page 1

$$
\begin{array}{ccccc}
E^1_{0,3} & E^1_{1,3} & E^1_{2,3} & E^1_{3,3} & E^1_{4,3} \\
E^1_{0,2} & E^1_{1,2} & E^1_{2,2} & E^1_{3,2} & E^1_{4,2} \\
E^1_{0,1} & E^1_{1,1} & E^1_{2,1} & E^1_{3,1} & E^1_{4,1} \\
E^1_{0,0} & E^1_{1,0} & E^1_{2,0} & E^1_{3,0} & E^1_{4,0}
\end{array}
$$

$$E^1_{p,q} := \ker d^0_{p,q} / \operatorname{im} d^0_{p,q+1}$$

Spectral sequence = movie: **Page 1**

$$\begin{array}{cccccccccc}
E^1_{0,3} & \xleftarrow{d^1} & E^1_{1,3} & \xleftarrow{d^1} & E^1_{2,3} & \xleftarrow{d^1} & E^1_{3,3} & \xleftarrow{d^1} & E^1_{4,3} & \xleftarrow{d^1} \cdots \\
E^1_{0,2} & \xleftarrow{d^1} & E^1_{1,2} & \xleftarrow{d^1} & E^1_{2,2} & \xleftarrow{d^1} & E^1_{3,2} & \xleftarrow{d^1} & E^1_{4,2} & \xleftarrow{d^1} \cdots \\
E^1_{0,1} & \xleftarrow{d^1} & E^1_{1,1} & \xleftarrow{d^1} & E^1_{2,1} & \xleftarrow{d^1} & E^1_{3,1} & \xleftarrow{d^1} & E^1_{4,1} & \xleftarrow{d^1} \cdots \\
E^1_{0,0} & \xleftarrow{d^1} & E^1_{1,0} & \xleftarrow{d^1} & E^1_{2,0} & \xleftarrow{d^1} & E^1_{3,0} & \xleftarrow{d^1} & E^1_{4,0} & \xleftarrow{d^1} \cdots
\end{array}$$

Warning: arrows d^1 entirely new. d^1 **= Chain-complex**

Spectral sequence = movie: Page 1

$$\begin{array}{ccccccccccc}
E^1_{0,3} & \xleftarrow{d^1} & E^1_{1,3} & \xleftarrow{d^1} & E^1_{2,3} & \xleftarrow{d^1} & E^1_{3,3} & \xleftarrow{d^1} & E^1_{4,3} & \xleftarrow{d^1} & \cdots \\
E^1_{0,2} & \xleftarrow{d^1} & E^1_{1,2} & \xleftarrow{d^1} & E^1_{2,2} & \xleftarrow{d^1} & E^1_{3,2} & \xleftarrow{d^1} & E^1_{4,2} & \xleftarrow{d^1} & \cdots \\
E^1_{0,1} & \xleftarrow{d^1} & E^1_{1,1} & \xleftarrow{d^1} & E^1_{2,1} & \xleftarrow{d^1} & E^1_{3,1} & \xleftarrow{d^1} & E^1_{4,1} & \xleftarrow{d^1} & \cdots \\
E^1_{0,0} & \xleftarrow{d^1} & E^1_{1,0} & \xleftarrow{d^1} & E^1_{2,0} & \xleftarrow{d^1} & E^1_{3,0} & \xleftarrow{d^1} & E^1_{4,0} & \xleftarrow{d^1} & \cdots
\end{array}$$

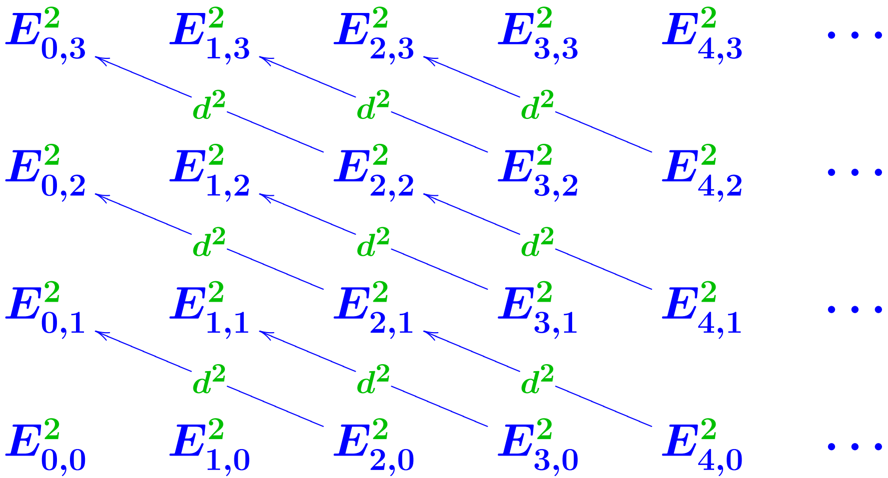
$$E^2_{p,q} := \ker d^1_{p,q} / \operatorname{im} d^1_{p+1,q} \qquad \Longrightarrow$$

Spectral sequence = movie: **Page 2**

$$
\begin{array}{cccccc}
E^2_{0,3} & E^2_{1,3} & E^2_{2,3} & E^2_{3,3} & E^2_{4,3} & \cdots \\
E^2_{0,2} & E^2_{1,2} & E^2_{2,2} & E^2_{3,2} & E^2_{4,2} & \cdots \\
E^2_{0,1} & E^2_{1,1} & E^2_{2,1} & E^2_{3,1} & E^2_{4,1} & \cdots \\
E^2_{0,0} & E^2_{1,0} & E^2_{2,0} & E^2_{3,0} & E^2_{4,0} & \cdots
\end{array}
$$

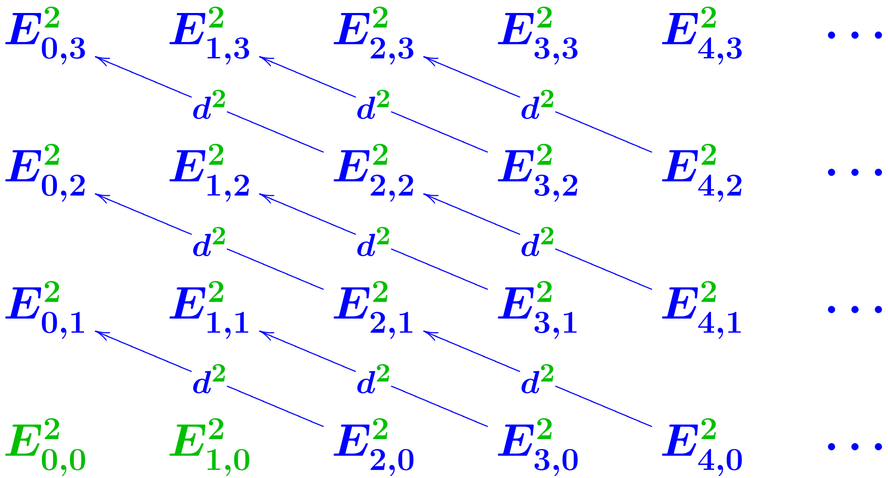
$$E^2_{p,q} := \ker d^1_{p,q} / \operatorname{im} d^1_{p+1,q}$$

Spectral sequence = movie: **Page 2**



Warning: arrows d^2 entirely new. **d^2 = Chain-complex**

Spectral sequence = movie: Page 2



$$E^3_{p,q} := \ker d^2_{p,q} / \operatorname{im} d^2_{p+1,q}$$

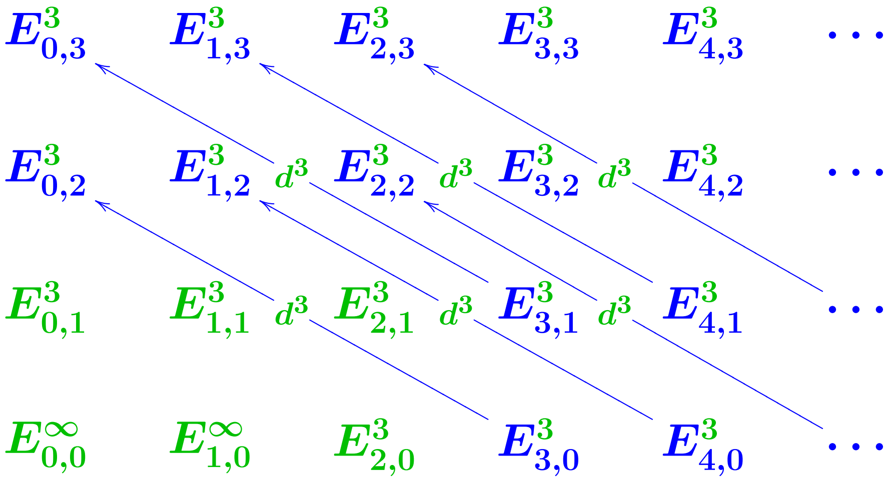
Remark: $E^n_{0,0}$ and $E^n_{0,1}$ now independent of $n \geq 2$

Spectral sequence = movie: **Page 3**

$$
\begin{array}{cccccc}
E^{3}_{0,3} & E^{3}_{1,3} & E^{3}_{2,3} & E^{3}_{3,3} & E^{3}_{4,3} & \cdots \\
E^{3}_{0,2} & E^{3}_{1,2} & E^{3}_{2,2} & E^{3}_{3,2} & E^{3}_{4,2} & \cdots \\
E^{3}_{0,1} & E^{3}_{1,1} & E^{3}_{2,1} & E^{3}_{3,1} & E^{3}_{4,1} & \cdots \\
E^{\infty}_{0,0} & E^{\infty}_{1,0} & E^{3}_{2,0} & E^{3}_{3,0} & E^{3}_{4,0} & \cdots
\end{array}
$$

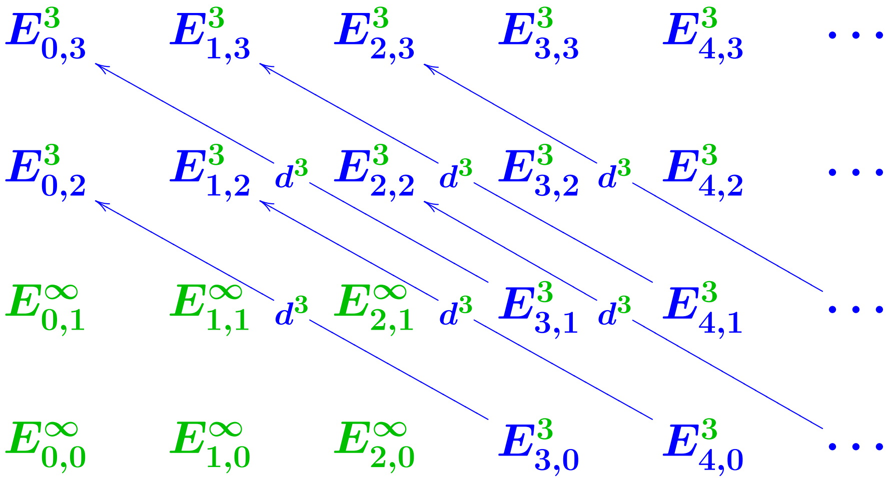
$$E^{3}_{p,q} := \ker {d^2}_{p,q} / \operatorname{im} {d^2}_{p+1,q}$$

Spectral sequence = movie: **Page 3**



Warning: arrows d^3 entirely new. **d^3 = Chain-complex**

Spectral sequence = movie: **Page 3**



$$E^4_{p,q} := \ker d^3_{p,q} / \operatorname{im} d^3_{p+1,q}$$

And so on...

Spectral sequence = movie: **Page ∞**

$$
\begin{array}{cccccc}
E^{\infty}_{0,3} & E^{\infty}_{1,3} & E^{\infty}_{2,3} & E^{\infty}_{3,3} & E^{\infty}_{4,3} & \cdots \\
E^{\infty}_{0,2} & E^{\infty}_{1,2} & E^{\infty}_{2,2} & E^{\infty}_{3,2} & E^{\infty}_{4,2} & \cdots \\
E^{\infty}_{0,1} & E^{\infty}_{1,1} & E^{\infty}_{2,1} & E^{\infty}_{3,1} & E^{\infty}_{4,1} & \cdots \\
E^{\infty}_{0,0} & E^{\infty}_{1,0} & E^{\infty}_{2,0} & E^{\infty}_{3,0} & E^{\infty}_{4,0} & \cdots
\end{array}
$$

Final state (???)

Spectral sequence = movie: **Page ∞**

$$
\begin{array}{cccccc}
E^{\infty}_{0,3} & E^{\infty}_{1,3} & E^{\infty}_{2,3} & E^{\infty}_{3,3} & E^{\infty}_{4,3} & \cdots \\
E^{\infty}_{0,2} & E^{\infty}_{1,2} & E^{\infty}_{2,2} & E^{\infty}_{3,2} & E^{\infty}_{4,2} & \cdots \\
E^{\infty}_{0,1} & E^{\infty}_{1,1} & E^{\infty}_{2,1} & E^{\infty}_{3,1} & E^{\infty}_{4,1} & \cdots \\
E^{\infty}_{0,0} & E^{\infty}_{1,0} & E^{\infty}_{2,0} & E^{\infty}_{3,0} & E^{\infty}_{4,0} & \cdots
\end{array}
$$

$$\Downarrow$$

$$H_3 = ???$$

Final state (???)

Exact result: $\exists$ filtration of H_3:

$$
\begin{array}{ccccccccccc}
& & E^{\infty}_{0,3} & & E^{\infty}_{1,2} & & E^{\infty}_{2,1} & & E^{\infty}_{3,0} & & \\
0 & \subset & H_{3,0,3} & \subset & H_{3,1,2} & \subset & H_{3,2,1} & \subset & H_{3,3,0} & = & H_3
\end{array}
$$

$$E^{\infty}_{p,q} = H_{p+q,p,q} / H_{p+q,p-1,q+1}$$

Guessing the H_n is an "extended" extension problem.

Example: $E^{\infty}_{0,3} = E^{\infty}_{1,2} = E^{\infty}_{2,1} = E^{\infty}_{3,0} = \mathbb{Z}_2 \Rightarrow$

$$H_3 = \mathbb{Z}_{16} \text{ or } \mathbb{Z}_8 \oplus \mathbb{Z}_2 \text{ or } \mathbb{Z}_4^2 \text{ or } \mathbb{Z}_4 \oplus \mathbb{Z}_2^2 \text{ or } \mathbb{Z}_2^4 \quad ???$$

John McCleary, 1985:

"User's Guide to Spectral Sequences"

"Theorem". There is a spectral sequence with $E_2^{*,*} \cong$ "something computable" and converging to H^*, something desirable.

The important observation to make about the statement of the theorem is that it gives an E_2-term of the spectral sequence but says nothing about the successive differentials d_r. Though $E_r^{*,*}$ may be known, without d_r or some further structure, it may be impossible to proceed.

John McCleary, 1985:

"User's Guide to Spectral Sequences"

It is worth repeating the caveat about differentials mentioned in chapter 1: knowledge of $E_r^{*,*}$ and d_r determines $E_{r+1}^{*,*}$ but not d_{r+1}.

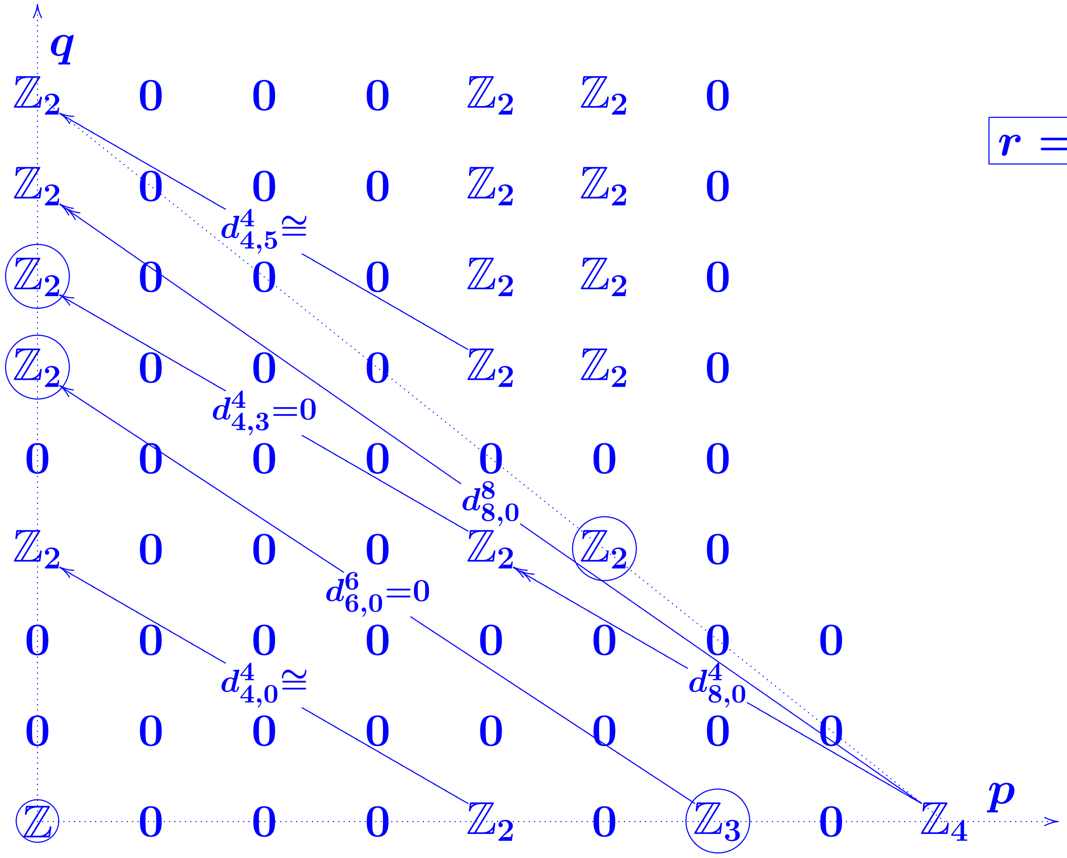
[. . .]

If some differential is unknown, then some other (any other!) principle is needed to proceed.

[. . .]

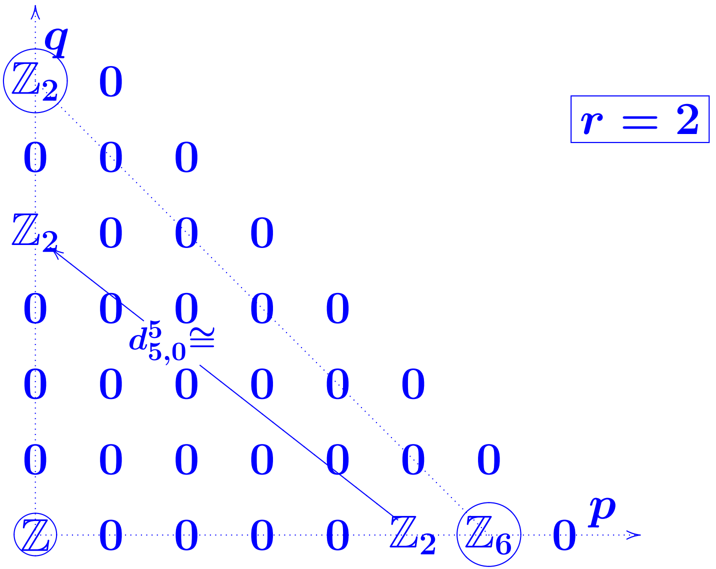
In the non-trivial cases, it is often a deep geometric idea that is caught up in the knowledge of a differential.

Example $F_3 \hookrightarrow X_5 \longrightarrow X_4$.



$H_0 = \mathbb{Z}$ $\quad H_{1-4} = 0$ $\quad H_5 = \mathbb{Z}_2$ $\quad H_6 = \mathbb{Z}_6$ $\quad H_7 = 0$ $\quad H_8 = \mathbb{Z}_2$

Example $F_4 \hookrightarrow X_6 \to X_5$.



"Result": Exact sequence: $0 \leftarrow \mathbb{Z}_6 \leftarrow H_6(X_6) \leftarrow \mathbb{Z}_2 \leftarrow 0$

$\Rightarrow \pi_6 S^3 = \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_6$???

The simplest example

to understand the nature of the problem.

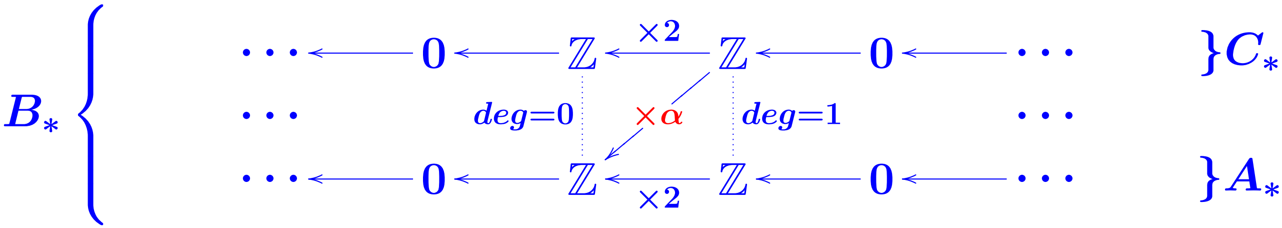
Chain complex:

$$
B_* \left\{
\begin{array}{ccccccccccc}
\cdots & \longleftarrow & \mathbf{0} & \longleftarrow & \mathbb{Z} & \xleftarrow{\times 2} & \mathbb{Z} & \longleftarrow & \mathbf{0} & \longleftarrow & \cdots \\
\cdots & & & deg{=}0 & \vdots & \swarrow {\scriptstyle \times\alpha} & \vdots & deg{=}1 & & & \cdots \\
\cdots & \longleftarrow & \mathbf{0} & \longleftarrow & \mathbb{Z} & \xleftarrow[\times 2]{} & \mathbb{Z} & \longleftarrow & \mathbf{0} & \longleftarrow & \cdots
\end{array}
\right.
\quad
\begin{array}{l}
\}C_* \\ \\ \}A_*
\end{array}
$$

$$
\Leftrightarrow \qquad \cdots \longleftarrow \mathbf{0} \longleftarrow \mathbb{Z}^2 \xleftarrow{\begin{bmatrix} 2 & 0 \\ \alpha & 2 \end{bmatrix}} \mathbb{Z}^2 \longleftarrow \mathbf{0} \longleftarrow \cdots
$$

$\Rightarrow$ **Short exact sequence of chain complexes:**

$$
\mathbf{0} \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* (= B_*/A_*) \longrightarrow \mathbf{0}
$$



Challenge: $H_*(A_*)$ **and** $H_*(C_*)$ **known** $\Rightarrow$ $H_*(B_*) = ???$

$H_0(A_*) = H_0(C_*) = \mathbb{Z}_2,\ H_m(A_*) = H_m(C_*) = 0 \quad \forall m \neq 0.$

Long exact sequence of homology $\Rightarrow$

$$\cdots \leftarrow 0 \leftarrow \mathbb{Z}_2 \leftarrow H_0(B_*) \leftarrow \mathbb{Z}_2 \leftarrow 0 \leftarrow \cdots$$

$\Rightarrow$ **Two** **possible** $H_0(B_*)$: $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ **or** $\mathbb{Z}_4$.

How to determine the right choice ???

Standard extension group theory:

$$0 \leftarrow \mathbb{Z}_2 \leftarrow E \leftarrow \mathbb{Z}_2 \leftarrow 0$$

The extension is determined by

a cohomology class $\tau \in H^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$.

$$\begin{array}{ccccc} 0 & \leftarrow\!\shortmid & 1 & \leftarrow\!\shortmid & a \\ & & 0 & \leftarrow\!\shortmid & 2a & \leftarrow\!\shortmid & b \end{array}$$

Rule: Consider $1 \in \mathbb{Z}_2$**, then an arbitrary preimage** $a \in E$;

Certainly the image of $2a$ **is** 0;

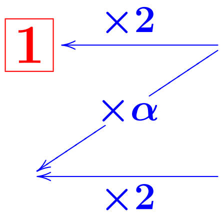
Exactness $\Rightarrow$ $2a$ **is the image of a unique** $b \in \mathbb{Z}_2$.

If $b = 0$**, then** $E = \mathbb{Z}_2 \oplus \mathbb{Z}_2$;

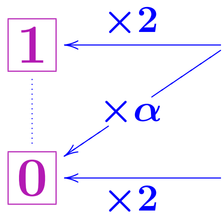
If $b = 1$**, then** $E = \mathbb{Z}_4$.

But E **is unknown!**

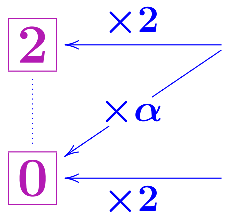
Solution: Instead of working with homology classes, work with cycles representing them.



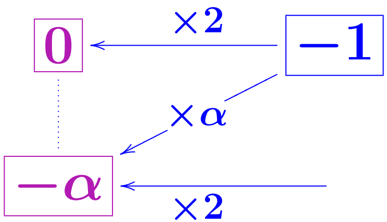
$1 \in H_0(C_*)$



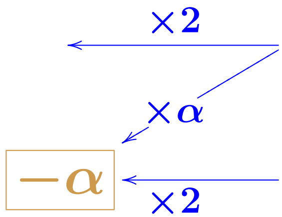
$a \in H_0(B_*)$?



$2a \in H_0(B_*)$?



$2a \in H_0(B_*)$?



$b \in H_0(A_*)$

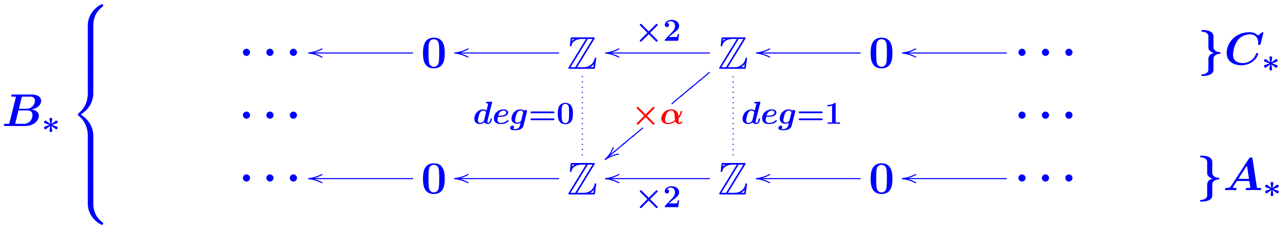
Conclusion:

α even $\Rightarrow$

$$H_0(B_*) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

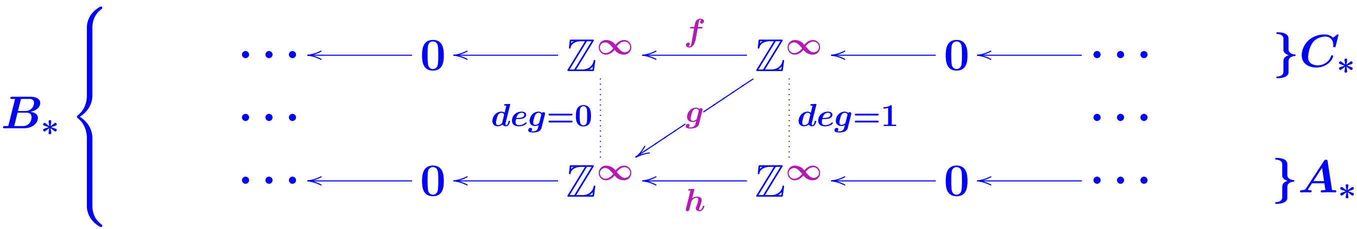
α odd $\Rightarrow$

$$H_0(B_*) = \mathbb{Z}_4$$



Lifting homology classes to explicit cycles gives a solution.

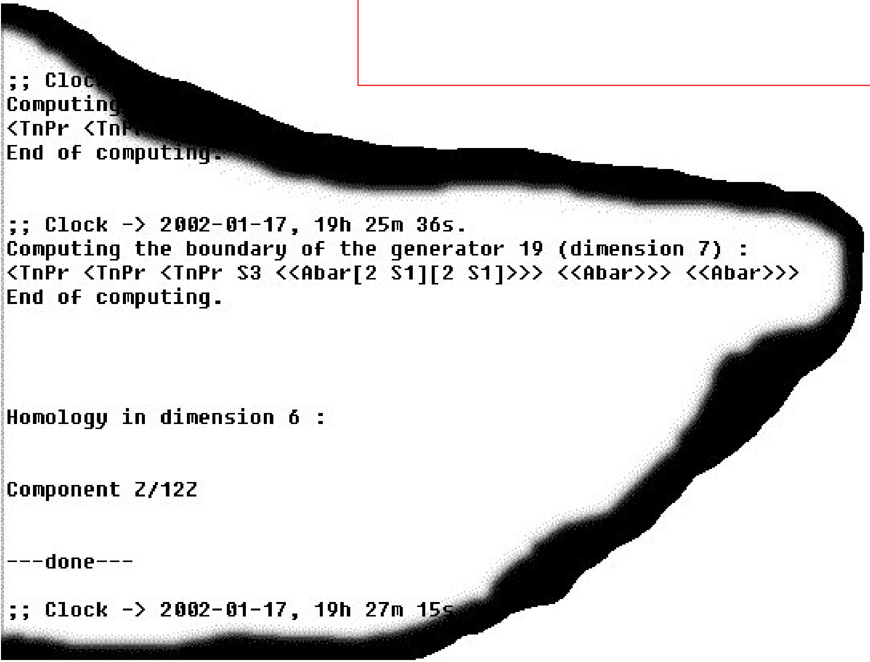
A little more general situation:



$$H_0 A_* = H_0(C_*) = \mathbb{Z}_2 \Rightarrow H_0(B_*) = ???$$

Same solution if it is possible to work in A_*, B_* and C_*.

The END


;; Cloc
Computing
<TnPr <Tn
End of computing.
;; Clock -> 2002-01-17, 19h 25m 36s.
Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.
Homology in dimension 6 :
Component Z/12Z
---done---
;; Clock -> 2002-01-17, 19h 27m 15s