About Ana Romero's thesis

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Ana Romero's thesis is devoted to *constructive* (effective) versions of the Serre, Eilenberg-Moore and Bousfield-Kan spectral sequences.

Commutative Algebra, Algebraic Geometry, Arithmetic, and many other mathematical domains have now an important component consisting in implementing on computers various tools of these domains. To achieve such an ambition, it is sometimes necessary to slightly or even strongly modify the general organisation of the relevant theory. From a *mathematical* point of view, the matter at issue is the following: how to make *constructive* the corresponding theory?

It so happens the main tools of Algebraic Topology, and more generally of Homological Algebra, namely the numerous exact and spectral sequences produced in various situations, *are not* constructive. These exact and spectral sequences are frequently presented as *computing* tools; yet, except in particular cases, in fact they cannot be elementarily transformed into *algorithms* allowing a computer to determine for example some unknown homology or homotopy group.

The so-called *Effective Homology* [3] is a *constructive* presentation of Homological Algebra. Following the constructive requirement, the usual *existence* statements of homological algebra must be transformed into *algorithms* producing (computing) an *explicit* expression of the object the existence of which is claimed. If some homology group H is claimed "to be" $\mathbb{Z}/2\mathbb{Z}$, an *explicit* isomorphism between H and $\mathbb{Z}/2\mathbb{Z}$ must be produced. An *explicit* homology class is the homology class of some *explicit* cycle, and claiming that two cycles z and z' are homologous requires an *explicit* boundary preimage, that is, an *explicit* chain c satisfying $dc = z - z'$, and so on.

This point of view quickly leads to install as the main tool of Homological Algebra the *Homological Perturbation Lemma*, curiously rarely used in standard Homological Algebra. It is not then difficult to transform the simplest exact and spectral sequences into algorithms computing some groups or others; not only computing but giving also constructive versions of these groups, with explicit representatives for homology classes, with an explicit boundary preimage when a cycle is claimed homologous to 0, etc.

This is interesting but, as usual, solving a research problem produces other new ones. For example, the algorithms which are procuced in Effective Homology work as "black boxes": some homology group for example is obtained, the corresponding effective homology algorithm has used some input and has produced the result as its output; but this algorithm did not give any hint about the "reasons" which could explain such a result. The only resulting "reasons" are then a long and meaningless chain of elementary computation steps, which in fact do not give any interesting information about some underlying structure able to describe the nature of the result.

Searching for such "reasons" arouses an interesting *feedback* phenomenon. For example, unfortunately, some spectral sequence is not enough to produce an algorithm computing some desired homology group. On the contrary, effective homology is able to compute this group; but effective homology can also be used to compute *as a by-product* the corresponding spectral sequence. In fact an extra-algorithm can be added to the effective homology computation to produce *all the components* of this spectral sequence: the groups $E^r_{p,q}$ for arbitrary triples (p, q, r), the differentials $d^r_{p,q}$, arbitrary *higher* differentials included, the limit groups $E^\infty_{p,q}$, the structure of the filtration of the total homology groups H_{p+q} by the $E^\infty_{p,q}$'s. Ana Romero not only designed the appropriate theoretical algorithms to achieve this goal, but she wrote down herself an important extension[1] to the Kenzo program implementing these algorithms. It is the first time some *general* spectral sequences can actually, theoretically and concretely, be *computed*.

Designing algorithms computing these spectral sequences is the first half of Ana Romero's thesis. The process works in general for spectral sequences coming from filtered chain complexes. This is the reason why the main applications given in the memoir are the computation of Serre and Eilenberg-Moore spectral sequences, which come from appropriate filtrations of chain complexes.

Let us consider for example the computation of the homology groups of the loop space $\Omega(\Omega S^3 \cup_2 e^3)$. No *algorithm* was known for the homology groups of this loop space before effective homology. The (ordinary) Serre spectral sequence (1950) immediately gives $H_*(\Omega S^3 \cup_2 e^3)$. One of the first applications of effective homology in 1990 was the computation of the homology groups $H_*(\Omega(\Omega S^3 \cup_2 e^3))$. But Eilenberg and Moore proved a spectral sequence connects all these groups. Thanks to Ana Romero's work, this Eilenberg-Moore spectral sequence was entirely computed fifteen years later (2005), of course taking account of the obvious limitations in memory space and computing time.

These results were so satisfactory that it was tempting to extend them to other situations. The next "obvious" spectral sequence to be considered is the Adams spectral sequence. This spectral sequence connects the homology groups and the homotopy groups of a space and remains with all its other satellite spectral sequences the main tool to "compute" the homotopy groups of an arbitrary simply connected space and to study the rich underlying structure. The verb *compute* is enclosed between quotes because there is no known elementary method transforming this spectral sequence into an algorithm.

Jean-Pierre Serre designed his famous spectral sequence to obtain a number of homotopy groups of spheres. The Steenrod algebra already had an important

[1] About 2500 Lisp lines.

role in these computations, and the description of the exact role of the Steenrod algebra in this matter was given by Frank Adams thanks to his spectral sequence: its E_2 terms can be expressed as Ext-groups of cohomology groups with respect to the Steenrod algebra. To be complete, note also Edgar Brown in 1956 proved the theoretical computability of homotopy groups in the simply connected case, but through a method which does not seem concretely usable, even with the most powerful modern computers. On the contrary, effective homology allowed us to very simply reprove the computability of homotopy groups, this time through elementary methods, Postnikov and Whitehead towers, methods which have been concretely implemented and produced some homotopy groups so far unreachable.

The usual Serre and Eilenberg-Moore spectral sequences are not constructive and the Adams spectral sequence is not constructive either, which did not prevent from using computers for auxiliary computations, a striking application of this sort being for example the computations of Λ-algebras by Martin Tangora [4]. Nevertheless, it is a little amazing that forty-five years after its invention, the question of making constructive an object as important as the Adams spectral sequence has not yet been considered. Please note our challenge is a constructive version for the *unstable* Adams spectral sequence for an *arbitrary* simply connected space with effective homology.

It is true this problem is terribly more sophisticated than the same problem for the Serre and Eilenberg-Moore spectral sequences. These difficulties are mainly concentrated around two points. It is clear the modern presentation of the Adams spectral sequence through the more general Bousfield-Kan spectral sequence, which has also a more algebraic flavour, is to be used.

On the one hand the Bousfield-Kan spectral sequence does not come from a filtered chain complex, but from an exact couple produced by a tower of fibrations. This difficulty is real but not so complex. After all the relevant tower of fibrations is nothing but a cofiltration of its limit space, and the standard convergence properties of the Bousfield-Kan spectral sequence in the simply connected case allows the constructive topologist to easily overcome this obstacle. Easy in theory: programming the solution is postponed to future work and is by itself an interesting programming challenge in a totally new domain.

On the other hand, and still more new and interesting, the Bousfield-Kan spectral sequence is a spectral sequence of *homotopy* groups, not of homology groups as in Serre and Eilenberg-Moore spectral sequences. Examining carefully the subject with the usual *constructive* requirements leads to the following conclusion: effective homology is not enough, effective *homotopy* is now required. It is classical to present homotopy groups as homology groups thanks to John Moore's trick, but this is possible only for simplicial *groups*. Unfortunately, the stages of the relevant tower of fibrations are not simplicial groups, they are only Kan spaces, this is unavoidable, so that an *effective* version of the usual Kan theory is required to make constructive the Bousfield-Kan spectral sequence.

In a sense the results which are obtained in Ana Romero's thesis are the most advanced results which can be obtained toward a constructive Bousfield-Kan spectral sequence *without effective homotopy.* The Bousfield-Kan spectral sequence is based upon a simplicial-cosimplical set $R_\infty X$ repetitively using the magic R-construction, the object R being some ring, for example $\mathbb{Z}$: given some simplicial set X, a simplicial *group* RX is obtained by replacing every simplex σ of X by the "line of simplices" $R\sigma$, and there is an obvious way to assemble these lines as a simplicial group. John Moore taught us that $\pi_*(RX) = H_*(X)$ and Bousfield and Kan systematically used this observation to connect the homology and the homotopy groups of X by their famous spectral sequence.

So that one of the unavoidable requirements to make constructive the Bousfield-Kan spectral sequence is the following: if the simplicial set X is given with effective homology, is it possible to present RX also with effective *homology*? The answer is positive, and is certainly the main contribution of Ana Romero's work toward a constructive version of the Bousfield-Kan spectral sequence. Ana Romero proves this problem can be reduced to the computability of the effective homology of Eilenberg-MacLane spaces $K(R, n)$'s ($\mathbb{Z}$ only if $R = \mathbb{Z}$!). And this computability is a consequence of the effective version of the Eilenberg-Moore spectral sequence, in fact already underlying in the fantastic work by Eilenberg, Maclane and Cartan about the $H_*(K(\pi, n))$'s in the fifties. Iterating this result, Ana Romero proves all the "vertical" components of the simplicial-cosimplicial Bousfield-Kan space $R_\infty X$ are objects with effective homology, of course when the initial object X is so too.

Once this result is obtained, Ana Romero shows how this gives an *algorithm* computing the $E^1_{p,q}$ and $E^2_{p,q}$ groups of the Bousfield-Kan spectral sequence of an *arbitrary* simply connected space X with effective homology. For example the standard simplicial presentation of $\Omega S^3 \cup_2 e^3$ is infinite, but it is a space *with effective homology*, so that the $E^1_{p,q}$ and $E^2_{p,q}$ groups of the corresponding Bousfield-Kan spectral sequence are now computable. Ana Romero gives also a few indications in her memoir to explain how the method of the so-called *unique spectral relations* of Bousfield and Kan could be used to go further toward the next $E^r_{p,q}$'s and the corresponding differentials, but it is clear an available general effective homotopy theory would produce more comfortable and synthetic results.

Effective Homotopy is by itself at least another thesis subject; the general style of effective homology must be followed, but the framework of Kan's theory, much less *algebraic*, makes the work significantly more technical. Typically the exact Serre homotopy sequence connects the homotopy groups of the ingredients of a fibration, so that the effective homotopy version must produce an *algorithm* computing for example the effective homotopy of the total space when the effective homotopy of the base and fiber spaces is given. The Bousfield-Kan spectral sequence with respect to this exact homotopy sequence is then "nothing but" the same as an ordinary homological spectral sequence with respect to collections of exact sequences.

The professional topologists will be a little amazed not to see the various Steenrod algebras in Ana Romero's thesis. In fact the role of these algebras is here postponed to the rich structure connecting the groups $H_*K(\mathbb{Z}, n)$, which gives also other ideas. The main other *theoretical* solution for effective homology is *operadic*, cf. for example [2]. It happens the most algebraic versions of the unavoidable

E_∞-operad are also made of $K(\mathbb{Z}, n)$'s [1] and it is clear all these points of view are connected to each other in a rich way where most is still to be discovered. As usual, studying some right research subject opens other new research domains.

References

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