Computing all maps into a sphere

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Given topological spaces X, Y, a fundamental problem of algebraic topology is understanding the structure of all continuous maps $X \rightarrow Y$. We consider a computational version, where X, Y are given as finite simplicial complexes, and the goal is to compute $[X, Y]$, i.e., all homotopy classes of such maps.

We solve this problem in the *stable range*, where for some $d \geq 2$, we have $\dim X \leq 2d-2$ and Y is $(d-1)$-*connected*; in particular, Y can be the d-dimensional sphere S^d. The algorithm combines classical tools and ideas from homotopy theory (*obstruction theory*, *Postnikov systems*, and *simplicial sets*) with algorithmic tools from effective algebraic topology (*locally effective simplicial sets* and *objects with effective homology*).

In contrast, $[X, Y]$ is known to be uncomputable for general X, Y, since for $X = S^1$ it includes a well known undecidable problem: testing triviality of the fundamental group of Y.

In follow-up papers, the algorithm is shown to run in polynomial time for d fixed, and extended to other problems, such as the *extension problem*, where we are given a subspace $A \subset X$ and a map $A \rightarrow Y$ and ask whether it extends to a map $X \rightarrow Y$, or computing the $\mathbb{Z}_2$-*index*—everything in the stable range. Outside the stable range, the extension problem is undecidable.

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1. INTRODUCTION

Among results concerning computations in topology, probably the most famous ones are negative. For example, there is no algorithm to decide whether the fundamental group $\pi_1(Y)$ of a given space Y is trivial, i.e., whether every loop in Y can be continuously contracted to a point.[1]

Here we obtain a positive result for a closely related and fairly general problem, homotopy classification of maps;[2] namely, we describe an algorithm that works in the so-called *stable range*.

Computational topology. This paper falls into the broader area of *computational topology*, which has been a rapidly developing discipline in recent years—see, for instance, the textbooks [Edelsbrunner and Harer 2010; Zomorodian 2005; Matveev 2007].

Our focus is somewhat different from the main current trends in the field, where, on the one hand, computational questions are intensively studied in dimensions 2 and 3 (e.g., concerning *graphs on surfaces*, *knots* or 3-*manifolds*[3]), and, on the other hand, for arbitrary dimensions mainly *homology* computations are investigated.

Homology has been considered an inherently computational tool since its inception and there are many software packages that contain practical implementations, e.g., `polymake` [Gawrilow and Joswig 2000]. Thus, algorithmic solvability of homological questions is usually obvious, and the challenge may be, e.g., designing very fast algorithms to deal with large inputs. Moreover, lot of research has been devoted to developing extensions such as *persistent homology* [Edelsbrunner and Harer 2008], motivated by applications like data analysis [Carlsson 2009].

In contrast, *homotopy-theoretic problems*, as those studied here, are generally considered much less tractable than homological ones and the first question to tackle is usually the existence of any algorithm at all (indeed, many of them are algorithmically unsolvable, as the example of triviality of the fundamental group illustrates). Such problems lie at the core of algebraic topology and have been thoroughly studied from a topological perspective since the 1940s. A significant effort has also been devoted to *computer-assisted* concrete calculations, most notably of higher *homotopy groups of spheres*; see, e.g., [Kochman 1990a].

Effective algebraic topology. In the 1990s, three independent groups of researchers proposed general frameworks to make various more advanced methods of algebraic topology *effective* (algorithmic): Schön [Schön 1991], Smith [Smith 1998], and Sergeraert, Rubio, Dousson, and Romero (e.g., [Sergeraert 1994; Rubio and Sergeraert 2002; Romero et al. 2006; Rubio and Sergeraert 2005]; also see [Rubio and Sergeraert 2012] for an exposition). These frameworks yielded general *computability* results for homotopy-theoretic questions (including new algorithms for the computation of higher homotopy groups [Real 1996]), and in the case of Sergeraert and co-workers, a *practical implementation* as well.

The problems considered by us were not addressed in those papers, but we rely on the work of Sergeraert et al., and in particular on their framework of *objects with effective homology*, for implementing certain operations in our algorithm (see Sections 2 and 4).

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[1]This follows by a standard reduction, see, e.g., [Stillwell 1993], from a result of Adjan and Rabin on unsolvability of the triviality problem of a group given in terms of generators and relations; see, e.g., [Soare 2004].

[2]The definition of homotopy and other basic topological notions will be recalled later.

[3]A seminal early result in the latter direction is Haken's famous algorithm for recognizing the unknot [Haken 1961].

We should also mention that our perspective is somewhat different from the previous work in effective algebraic topology, closer to the view of theoretical computer science; although in the present paper we provide only computability results, subsequent work also addresses the computational complexity of the considered problems. We consider this research area fascinating, and one of our hopes is that our work may help to bridge the cultural gap between algebraic topology and theoretical computer science.

The problem: homotopy classification of maps. A central theme in algebraic topology is to understand, for given topological spaces X and Y, the set $[X, Y]$ of homotopy classes of maps[4] from X to Y.

Many of the celebrated results throughout the history of topology can be cast as information about $[X, Y]$ for particular spaces X and Y. An early example is a famous theorem of Hopf from the 1930s, asserting that the homotopy class of a map $f\colon S^n \to S^n$, where S^n is the n-dimensional sphere, is completely determined by an integer called the *degree* of f, thus giving a one-to-one correspondence $[S^n, S^n] \cong \mathbb{Z}$. Another great discovery of Hopf, with ramifications in modern physics and elsewhere, was a map $S^3 \to S^2$, now called by his name, that is not homotopic to a constant map.

These two early results concern *higher homotopy groups*: for our purposes, the kth *homotopy group* $\pi_k(Y)$, $k \geq 2$, of a space Y can be identified with the set $[S^k, Y]$ equipped with a suitable group operation.[5] In particular, a very important special case are the higher *homotopy groups of spheres* $\pi_k(S^n)$, whose computation has been one of the important challenges propelling research in algebraic topology, with only partial results so far despite an enormous effort (see, e.g., [Ravenel 2004; Kochman 1990b]).

The extension problem. A problem closely related to computing $[X, Y]$ is the *extension problem*: given a subspace $A \subset X$ and a map $f\colon A \to Y$, can it be extended to a map $X \to Y$? For example, the famous *Brouwer fixed-point theorem* can be re-stated as non-extendability of the identity map $S^n \to S^n$ to the ball D^{n+1}. A number of topological concepts, which may seem quite advanced and esoteric to a newcomer in algebraic topology, e.g. *Steenrod squares*, have a natural motivation in trying to solve the extension problem step by step.

Early results. Earlier developments around the extension problems are described in Steenrod's paper [Steenrod 1972] (based on a 1957 lecture series), which we can recommend, for readers with a moderate topological background, as an exceptionally clear and accessible, albeit somewhat outdated, introduction to this area. In particular, in that paper, Steenrod asks for an effective procedure for (some aspects of) the extension problem.

There has been a tremendous amount of work in homotopy theory since the 1950s, with a wealth of new concepts and results, some of them opening completely new areas. However, as far as we could find out, the *algorithmic part* of the program discussed in [Steenrod 1972] has not been explicitly carried out until now.

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[4]In this paper, all maps between topological spaces are assumed to be continuous. Two maps $f, g\colon X \to Y$ are said to be *homotopic*, denoted $f \sim g$, if there is a map $F\colon X \times [0, 1] \to Y$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$. The equivalence class of f of this relation is denoted $[f]$ and called the *homotopy class* of f.

[5]Formally, the kth homotopy group $\pi_k(Y)$ of a space Y, $k \geq 1$, is defined as the set of all homotopy classes of *pointed* maps $f\colon S^k \to Y$, i.e., maps f that send a distinguished point $s_0 \in S^k$ to a distinguished point $y_0 \in Y$ (and the homotopies F also satisfy $F(s_0, t) = y_0$ for all $t \in [0, 1]$). Strictly speaking, one should write $\pi_k(Y, y_0)$ but for a path-connected Y, the choice of y_0 does not matter. Furthermore, $\pi_k(Y)$ is trivial (has only one element) iff $[S^k, Y]$ is trivial, i.e., if every map $S^k \to Y$ is homotopic to a constant map. Moreover, if $\pi_1(Y)$ is trivial, then for $k \geq 2$, the pointedness of the maps does not matter and one can identify $\pi_k(Y)$ with $[S^k, Y]$. Each $\pi_k(Y)$ is a group, which for $k \geq 2$ is Abelian, but the definition of the group operation is not important for us at the moment.

As far as we know, the only algorithmic paper addressing the general problem of computing of $[X, Y]$ is that by Brown [Brown (jun.) 1957] from 1957. Brown showed that $[X, Y]$ is computable under the assumption that Y is 1-connected[6] and all the higher homotopy groups $\pi_k(Y)$, $2 \leq k \leq \dim X$, are *finite*. The latter assumption is rather strong[7]; in particular, Brown's algorithm is not applicable for $Y = S^d$ since $\pi_d(S^d) \cong \mathbb{Z}$.

In the same paper, Brown also gave an algorithm for computing $\pi_k(Y)$, $k \geq 2$, for every 1-connected Y. To do this, he overcame the restriction on finite homotopy groups mentioned above, and also discussed in Section 2 below, by a somewhat ad-hoc method, which does not seem to generalize to the $[X, Y]$ setting.

On the negative side, it is undecidable whether $[S^1, Y]$ is trivial (since this is equivalent to the triviality of $\pi_1(Y)$). By an equally classical result of Boone and of Novikov [Boone 1954a; 1954b; 1955; Novikov 1955] it is undecidable whether a given map $S^1 \to Y$ can be extended to a map $D^2 \to Y$, even if Y is a finite 2-dimensional simplicial complex. Thus, both the computation $[X, Y]$ and the extension problem are algorithmically unsolvable without additional assumptions on Y. These are the only previous undecidability results in this context known to us; more recent results, obtained as a follow-up of the present paper, will be mentioned later. For a number of more loosely related undecidability results we refer to [Soare 2004; Nabutovsky and Weinberger 1999; 1996] and the references therein.

New results. In this paper we prove the computability of $[X, Y]$ under a fairly general condition on X and Y. Namely, we assume that, for some integer $d \geq 2$, we have $\dim X \leq 2d - 2$, while Y is $(d-1)$-connected. A particularly important example of a $(d-1)$-connected space, often encountered in applications, is the sphere S^d. We also assume that X and Y are given as finite simplicial complexes or, more generally, as finite *simplicial sets* (a more flexible generalization of simplicial complexes; see Section 4).

An immediate problem with computing the set $[X, Y]$ of all homotopy classes of continuous maps is that it may be infinite. However, it is known that under the just mentioned conditions on X and Y, $[X, Y]$ can be endowed with a structure of a finitely generated Abelian group.[8] Our algorithm computes the isomorphism type of this Abelian group.

THEOREM 1.1. *Let $d \geq 2$. There is an algorithm that, given finite simplicial complexes (or finite simplicial sets) X, Y, where $\dim X \leq 2d - 2$ and Y is $(d-1)$-connected, computes the isomorphism type of the Abelian group $[X, Y]$, i.e., expresses it as a direct product of cyclic groups.*

Moreover, given a simplicial map $f : X \to Y$, the element of the computed direct product corresponding to $[f]$ can also be computed. Consequently, it is possible to test homotopy of simplicial maps $X \to Y$.

We remark that the algorithm does not need any certificate of the 1-connectedness of Y, but if Y is not 1-connected, the result may be wrong.

In the remainder of the introduction, we discuss related results, applications, general motivation for our work, and directions for future research. In Section 2, we will present an outline of the methods and of the algorithm. In Sections 3–5, we will intro-

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[6] A space Y is said to be *k-connected* if every map $S^i \to Y$ can be extended to D^{i+1}, the ball bounded by the spheres S^i, for $i = 0, 1, \ldots, k$. Equivalently, Y is path-connected and the first k homotopy groups $\pi_i(Y)$, $i \leq k$, are trivial.

[7] Steenrod [Steenrod 1972] calls this restriction "most severe," and conjectures that it "should ultimately be unnecessary."

[8] In particular, the groups $[X, S^d]$ are known as the *cohomotopy groups* of X; see [Hu 1959].

duce and discuss the necessary preliminaries, and then we present the algorithm in detail in Section 6.

Follow-up work. We briefly summarize a number of strengthenings and extensions of Theorem 1.1, as well as complementary hardness results, obtained since the original submission of this paper. They will appear in a series of follow-up papers.

Running time. In the papers [Čadek et al. 2012; Krčál et al. 2013] it is shown that, for every fixed d, the algorithm as in Theorem 1.1 can be implemented so that its running time is bounded by a *polynomial* in the size of X and Y.[9] The nontrivial part of this polynomiality result is a subroutine for computing *Postnikov systems*, which we use as a black box here—see Section 2. For the rest of the algorithm, verifying polynomiality is straightforward, see [Krčál 2013]; except for some brief remarks, we will not consider this issue here, in order to avoid distraction from the main topic.

The extension problem. In [Čadek et al. 2012, Theorem 1.4], it is shown that the methods of the present paper also yield an algorithm for the extension problem as defined above. The extension problem can actually be solved even for $\dim X \leq 2d - 1$, as opposed to $2d - 2$ in Theorem 1.1 (still asumming that Y is $(d-1)$-connected). Again, the running time is polynomial for d fixed.

Hardness outside the stable range. The dimension and connectivity assumptions in Theorem 1.1 turned out to be essential and almost sharp, in the following sense: In [Čadek et al. 2013], it is shown that, for every $d \geq 2$, the extension problem is undecidable for $\dim X = 2d$ and $(d-1)$-connected Y. Similar arguments show that for $\dim X = 2d$ and $(d-1)$-connected Y, deciding whether every map $X \to Y$ is homotopic to a constant map (i.e., $|[X,Y]| = 1$) is NP-hard and no algorithm is known for it [Krčál 2013, Theorem 2.1.2].

Dependence on d. The running-time of the algorithm in Theorem 1.1 can be made polynomial for every fixed d, as was mentioned above, but it depends on d at least exponentially. We consider it unlikely that the problem can be solved by an algorithm whose running time also depends polynomially on d. One heuristic reason supporting this belief is that Theorem 1.1 includes the computation of the *stable* homotopy groups $\pi_{d+k}(S^d)$, $k \leq d - 2$. These are considered mathematically very difficult objects, and a polynomial-time algorithm for computing them would be quite surprising. Another reason is that the related problem of computing the higher homotopy groups $\pi_k(Y)$ of a 1-connected simplicial complex Y was shown to be #P-hard if k, encoded in unary, is a part of input [Anick 1989; Čadek et al. 2013], and it is W[1]-hard w.r.t. the parameter k [Matoušek 2014], even for Y of dimension 4. Still, it would be interesting to have more concrete hardness results for the setting of Theorem 1.1 with variable d.

Lifting-extension and the equivariant setting. In [Čadek et al. 2013; Vokřínek 2013], the ideas and methods of the present paper are further developed and generalized to more general *lifting-extension problems* and to the *equivariant* setting, where a fixed finite group G acts freely on both X and Y, and the considered continuous maps are also required to be *equivariant*, i.e., to commute with the actions of G. The basic and important special case with $G = \mathbb{Z}_2$ will be discussed in more detail below.

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[9]Here, for simplicity, we can define the size of a finite simplicial complex X as the number of its simplices; for a simplicial set, we count only nondegenerate simplices. It is not hard to see that if the dimension of X is bounded by a constant, then X can be encoded by a string of bits of length polynomial in the number of (nondegenerate) simplices; also see the discussion in [Čadek et al. 2012].

Homotopy testing. By Theorem 1.1, it is possible to test homotopy of two simplicial maps $X \to Y$ in the stable range. It turns out that for this task, unlike for the extension problem, the restriction to the stable range is unnecessary: it suffices to assume that Y is 1-connected [Filakovský and Vokřínek 2013].

Applications, motivation, and future work. We consider the fundamental nature of the algorithmic problem of computing $[X, Y]$ a sufficient motivation of our research. However, we also hope that work in this area will bring various connections and applications, also in other fields, possibly including practically usable software, e.g., for aiding research in topology. Here we mention two applications that have already been worked out in detail.

Robust roots. A nice concrete application comes from the so-called ROB-SAT problem—robust satisfiability of systems of equations The problem is given by a rational value $\alpha > 0$ and a piecewise linear function $f \colon K \to \mathbb{R}^d$ defined by rational values on the vertices of a simplicial complex K. The question is whether an arbitrary continuous $g \colon K \to \mathbb{R}^d$ that is at most α-far from f (i.e., $\|f - g\|_\infty \leq \alpha$) has a root. In a slightly different and more special form, this problem was investigated by Franek et al. [Franek et al. 2011], and later Franek and Krčál [Franek and Krčál 2014] exhibited a computational equivalence of ROB-SAT and the extension problem for maps into the sphere S^{d-1}. The algorithm for the extendability problem based on the present paper then yields an algorithmic solution when $\dim K \leq 2d - 3$.

$\mathbb{Z}_2$-index and embeddability. An important motivation for the research leading to the present paper was the computation of the $\mathbb{Z}_2$-*index* (or *genus*) $\mathrm{ind}(X)$ of a $\mathbb{Z}_2$-space X,[10] i.e., the smallest d such that X can be equivariantly mapped into S^d. For example, the classical Borsuk–Ulam theorem can be stated in the form $\mathrm{ind}(S^d) \geq d$. Generalizing the results in the present paper, [Čadek et al. 2013] provided an algorithm that decides whether $\mathrm{ind}(X) \leq d$, provided that $d \geq 2$ and $\dim(X) \leq 2d - 1$; for fixed d the running time is polynomial in the size of X.

The computation of $\mathrm{ind}(X)$ arises, among others, in the problem of *embeddability* of topological spaces, which is a classical and much studied area; see, e.g., the survey by Skopenkov [Skopenkov 2008]. One of the basic questions here is, given a k-dimensional finite simplicial complex K, can it be (topologically) embedded in $\mathbb{R}^d$? The famous *Haefliger–Weber theorem* from the 1960s asserts that, in the *metastable range of dimensions*, i.e., for $k \leq \frac{2}{3}d - 1$, embeddability of K in $\mathbb{R}^d$ is equivalent to $\mathrm{ind}(K_\Delta^2) \leq d - 1$, where K_Δ^2, the *deleted product* of K, is a certain $\mathbb{Z}_2$-space constructed from K in a simple manner. Thus, in this range, the embedding problem is, computationally, a special case of $\mathbb{Z}_2$-index computation. A systematic study of algorithmic aspects of the embedding problem was initiated in [Matoušek et al. 2011], and the metastable range was left as one of the main open problems there (now resolved as a consequence of [Čadek et al. 2013]).

The $\mathbb{Z}_2$-index also appears as a fundamental quantity in combinatorial applications of topology. For example, the celebrated result of Lovász on Kneser's conjecture can be re-stated as $\chi(G) \geq \mathrm{ind}(B(G)) + 2$, where $\chi(G)$ is the chromatic number of a graph G, and $B(G)$ is a certain simplicial complex constructed from G (see, e.g., [Matoušek 2007]). We find it striking that prior to [Čadek et al. 2013], *nothing* seems to have been known about the computability of such an interesting quantity as $\mathrm{ind}(B(G))$.

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[10]A $\mathbb{Z}_2$-*space* is a topological space X with an action of the group $\mathbb{Z}_2$; the action is described by a homeomorphism $\nu \colon X \to X$ with $\nu \circ \nu = \mathrm{id}_X$. A primary example is a sphere S^d with the antipodal action $x \mapsto -x$. An *equivariant map* between $\mathbb{Z}_2$-spaces is a continuous map that commutes with the $\mathbb{Z}_2$ actions.

Explicit maps? Our algorithm for Theorem 1.1 works with certain implicit representations of the elements of $[X, Y]$; it can output a set of generators of the group in this representation, and it contains a subroutine implementing the group operation.

It would be interesting to know whether these implicit representations can be converted into actual maps $X \to Y$ (given, say, as simplicial maps from a sufficiently fine subdivision of X into Y) in an effective way. Given an implicit representation of a homotopy class $\kappa \in [X, Y]$, we can compute an explicit map $X \to Y$ in κ by a brute force search: go through finer and finer subdivisions X' of X and through all possible simplicial maps $X' \to Y$ until a simplicial map in κ is found. Membership in κ can be tested using Theorem 1.1; this may not be entirely obvious, but we do not give the details here, since this is only a side-remark. However, currently we have no upper bound on how fine subdivision may be required.

This would also be of interest in certain applications such as the embeddability problem—whenever we want to construct an embedding explicitly, instead of just deciding embeddability.

Various measures of complexity of embeddings have been studied in the literature, and very recently, Freedman and Krushkal [Freedman and Krushkal 2013] obtained bounds for the *subdivision complexity* of an embedding $K \to \mathbb{R}^d$. Here d and $k = \dim K$ are considered fixed, and the question is, what is the smallest $f(n)$ such that every k-dimensional complex K with n simplices that is embeddable in $\mathbb{R}^d$ has a subdivision L with at most $f(n)$ simplices that admits a linear embedding in $\mathbb{R}^d$ (i.e., an embedding that is an affine map on each simplex of L)? Freedman and Krushkal essentially solved the case with $d = 2k$ (here the embeddability can be decided in polynomial time—this is covered by [Čadek et al. 2013] but this particular case goes back to a classical work of Van Kampen from the 1930s; see [Matoušek et al. 2011]). The subdivision complexity for the other cases in the metastable range, i.e., for $k \leq \frac{2}{3}d - 1$, is wide open at present, and obtaining explicit maps $X \to Y$ in the setting of Theorem 1.1 might be a key step in its resolution.

2. AN OUTLINE OF THE METHODS AND OF THE ALGORITHM

Here we present an overview of the algorithm and sketch the main ideas and tools. Everything from this section will be presented again in the rest of the paper. Some topological notions are left undefined here and will be introduced in later sections.

The geometric intuition: obstruction theory. Conceptually, the basis of the algorithm is classical *obstruction theory* [Eilenberg 1940]. For a first encounter, it is probably easier to consider a version of obstruction theory which proceeds by constructing maps $X \to Y$ inductively on the i-dimensional *skeleta*[11] of X, extending them one dimension at a time. (For the actual algorithm, we use a different, "dual" version of obstruction theory, where we lift maps from X through stages of a so-called *Postnikov system* of Y.)

In a nutshell, at each stage, the extendability of a map from the $(i-1)$-skeleton to the i-skeleton is characterized by vanishing of a certain *obstruction*, which can, more or less by known techniques, be evaluated algorithmically.

Textbook expositions may give the impression that obstruction theory is a general algorithmic tool for testing the extendability of maps (this is actually what some of the topologists we consulted seemed to assume). However, the extension at each step is generally not unique, and extendability at subsequent steps may depend, in a nontrivial way, on the choices made earlier. Thus, in principle, one needs to search an infinitely branching tree of extensions. Brown's result mentioned earlier, on comput-

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[11] The i-skeleton of a simplicial complex X consists of all simplices of X of dimension at most i.

ing $[X, Y]$ with the $\pi_k(Y)$'s finite, is based on a complete search of this tree, where the assumptions on Y guarantee the branching to be finite.

In our setting, we make essential use of the group structure on the set $[X, Y]$ (mentioned in Theorem 1.1), as well as on some related ones, to produce a finite encoding of the set of all possible extensions at a given stage.

Semi-effective and fully effective Abelian groups. The description of our algorithm has several levels. On the top level, we work with Abelian groups whose elements are homotopy classes of maps. On a lower level, the group operations and other primitives are implemented by computations with *concrete representatives* of the homotopy classes; interestingly, on the level of representatives, the operations are generally non-associative.

We need to be careful in distinguishing "how explicitly" the relevant groups are available to us. Specifically, we distinguish between *semi-effective* and *fully effective* Abelian groups: For the former, we have a suitable way of representing the elements on a computer and we can compute the various group operations (addition, inverse) on the level of representatives. For the latter, we additionally have a list of generators and relations and we can express a given element in terms of the generators (see Section 3 for a detailed discussion). A homomorphism f between two semi-effective Abelian groups is called *locally effective* if there is an algorithm that, given a representative of an element a, computes a representative of $f(a)$.

Simplicial sets and objects with effective homology. All topological spaces in the algorithm are represented as *simplicial sets*, which will be discussed in more detail in Section 4.1. Suffice it here to say that a simplicial set is a purely combinatorial description of how to build a space from simple building blocks (*simplices*), similar to a simplicial complex, but allowing more general ways of gluing simplices together along their faces, which makes many constructions much simpler and more conceptual.

For the purposes of our exposition we will occasionally talk about topological spaces specified in other ways, most notably, as *CW-complexes*—e.g., in Sections 4.3 and 5.1. However, we stress that in the algorithm, all spaces are represented as simplicial sets.

A finite simplicial set can be encoded explicitly on a computer by a finite bit string, which describes a list of all (nondegenerate) simplices and the way of gluing them together. However, the algorithm also uses a number of *infinite* simplicial sets in its computation, such as simplicial *Eilenberg–MacLane spaces* discussed below. For these, it is not possible to store the list of all nondegenerate simplices.

Instead, we use a general framework developed by Sergeraert et al. (as surveyed, e.g., in [Rubio and Sergeraert 2012]), in which a possibly infinite simplicial set is represented by a *black box* or *oracle* (we speak of a *locally effective* simplicial set). This means that we have a specified encoding of the simplices of the simplicial set and a collection of algorithms for performing certain operations, such as computing a specific face of a given simplex. Similarly, a simplicial map between locally effective simplicial sets is *locally effective* if there is an algorithm that evaluates it on any given simplex of the domain; i.e., given the encoding of an input simplex, it produces the encoding of the image simplex.

To perform global computations with a given locally effective simplicial set, e.g., compute its homology and cohomology groups of any given dimension, the black box representation of these locally effective simplicial sets is augmented with additional data structures and one speaks about *simplicial sets with effective homology*. Sergeraert et al. then provide algorithms that construct basic topological spaces, such as finite simplicial sets or Eilenberg–MacLane spaces, as simplicial sets with effective homology. More crucially, the auxiliary data structures of a simplicial set with effective homology are designed so that if we perform various topological operations, such

as the Cartesian product, the *bar construction*, the *total space of a fibration*, etc., the result is again a simplicial set with effective homology.

Postnikov systems. The target space Y in Theorem 1.1 enters the computation in the form of a *Postnikov system*. Roughly speaking, a Postnikov system of a space Y is a way of building Y from "canonical pieces", called *Eilenberg–MacLane spaces*, whose homotopy structure is the simplest possible, namely, they have a single non-trivial homotopy group. The Eilenberg–MacLane spaces occurring in the algorithm will be denoted by K_i and L_i, and they depend only on the homotopy groups of Y.

A Postnikov system has *stages* $P_0, P_1, \ldots$, where P_i reflects the homotopy properties of Y up to dimension i; in particular, $\pi_j(P_i) \cong \pi_j(Y)$ for all $j \leq i$, while $\pi_j(P_i) = 0$ for $j > i$. The isomorphisms of the homotopy groups for $j \leq i$ are induced by maps $\varphi_i \colon Y \to P_i$, which are also a part of the Postnikov system. Crucially, these maps also induce bijections $[X, Y] \to [X, P_i]$ whenever $\dim X \leq i$; in words, homotopy classes of maps $X \to Y$ from any space X of dimension at most i are in bijective correspondence with homotopy classes of maps $X \to P_i$.

The last component of a Postnikov system are mappings $k_0, k_1, \ldots$, where $k_{i-1} \colon P_{i-1} \to K_{i+1}$ is called the $(i-1)$st *Postnikov class*. Together with the group $\pi_i(Y)$, it describes how P_i is obtained from P_{i-1}.

If Y is $(d-1)$-connected, then for $i \leq 2d-2$, the Postnikov stage P_i can be equipped with an H-group structure, which is, roughly speaking, an Abelian group structure "up to homotopy" (this is where the connectivity assumption enters the picture). This H-group structure on P_i induces, in a canonical way, an Abelian group structure on $[X, P_i]$, for every space X, with no restriction on $\dim X$.

Now assuming $\dim X \leq 2d-2$, we have the bijection $[X, Y] \to [X, P_{2d-2}]$ as mentioned above, and this can serve as the definition of the Abelian group structure on $[X, Y]$ used in Theorem 1.1. Therefore, instead of computing $[X, Y]$ directly, we actually compute $[X, P_{2d-2}]$, which yields an isomorphic Abelian group. (However, the elements of $[X, P_{2d-2}]$ are not so easily related to continuous maps $X \to Y$; this is the cause of the open problem, mentioned in the introduction, of effectively finding actual maps $X \to Y$ as representatives of the generators.)

Thus, to prove Theorem 1.1, we first compute the stages $P_0, \ldots, P_{2d-2}$ of a Postnikov system of Y, and then, by induction on i, we determine $[X, P_i]$, $i \leq 2d-2$. We return the description of $[X, P_{2d-2}]$ as an Abelian group.

For the inductive computation of $[X, P_i]$ we do not need any dimension restriction on X anymore, which is important, because the induction will also involve computing, e.g., $[SX, P_{i-1}]$, where SX is another simplicial set, the *suspension* of X, with dimension one larger than that of X.

The stages P_i of the Postnikov system are built as simplicial sets with a particular property (they are *Kan* simplicial sets[12]), which ensures that every *continuous* map $X \to P_i$ is homotopic to a *simplicial* map. In this way, instead of the continuous maps $X \to Y$, which are problematic to represent, we deal only with simplicial maps $X \to P_i$ in the algorithm, which are discrete, and even finitely representable, objects.

Outline of the algorithm.

(1) As a preprocessing step, we compute, using the algorithm from [Čadek et al. 2012], a suitable representation of the first $2d-2$ stages of a Postnikov system for Y. We refer to Section 4.3 for the full specification of the output provided by this computation; in particular, we thus obtain the isomorphism types of the first $2d-2$ homotopy groups $\pi_i = \pi_i(Y)$ of Y, the Postnikov stages P_i and the Eilenberg–

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[12]The term *Kan complex* is also commonly used in the literature.

MacLane spaces L_i and K_{i+1}, $i \leq 2d - 2$, as locally effective simplicial sets, and various maps between these spaces, e.g., the Postnikov classes $k_{i-1} \colon P_{i-1} \to K_{i+1}$, as locally effective simplicial maps.

(2) Given a finite simplicial set X, the main algorithm computes $[X, P_i]$ as a fully effective Abelian group by induction on i, $i \leq 2d - 2$, and $[X, P_{2d-2}]$ is the desired output.

The principal steps are as follows:

— We construct locally effective simplicial maps $\boxplus_i \colon P_i \times P_i \to P_i$ and $\boxminus_i \colon P_i \to P_i$, $i \leq 2d - 2$ (Section 5). These induce a binary operation $\boxplus_{i*}$ and a unary operation $\boxminus_{i*}$ on $\mathrm{SMap}(X, P_i)$ that correspond to the the group operations in $[X, P_i]$ on the level of representatives. This yields, in the terminology of Section 3, a semi-effective representation for $[X, P_i]$.

— It remains to convert this semi-effective representation into a fully effective one; this is carried out in detail in Section 6. For this step, we use that $[X, L_i]$ and $[X, K_{i+1}]$ are straightforward to compute as fully effective Abelian groups since, by basic properties of Eilenberg–MacLane spaces, they are canonically isomorphic to certain cohomology groups of X. Moreover, we assume that, inductively, we have already computed $[SX, P_{i-1}]$ and $[X, P_{i-1}]$ as fully effective Abelian groups, where SX is the suspension of X mentioned above.

These four Abelian groups, together with $[X, P_i]$, fit into an *exact sequence* of Abelian groups (see Equation (8) in Section 6.1), and this is then used to compute the desired fully effective representation of $[X, P_i]$—see Section 6. Roughly speaking, what happens here is that, among the maps $X \to P_{i-1}$, we "filter out" those that can be lifted to maps $X \to P_i$ (this corresponds to evaluating an appropriate obstruction, as was mentioned at the beginning of this section), for each map that can be lifted we determine all possible liftings, and finally, we test which of the lifted maps are homotopic. Since there are infinitely many homotopy classes of maps involved in these operations, we have to work globally, with generators and relations in the appropriate Abelian groups of homotopy classes.

Remarks.

Evaluating Postnikov classes. For Y fixed, the subroutines for evaluating the Postnikov classes k_i, $i \leq 2d - 2$, could be hard-wired once and for all. In some particular cases, they are given by known explicit formulas. In particular, for $Y = S^d$, k_d corresponds to the famous *Steenrod square* [Steenrod 1947; 1972] (more precisely, to the reduction from integral cohomology to mod 2 cohomology followed by the Steenrod square Sq^2), and k_{d+1} to *Adem's secondary cohomology operation*. However, in the general case, the only way of evaluating the k_i we are aware of is using *simplicial sets with effective homology* mentioned earlier. In this context, our result can also be regarded as an algorithmization of certain *higher cohomology operations* (see, e.g., [Mosher and Tangora 1968]), although our development of the required topological underpinning is somewhat different and, in a way, simpler.[13]

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[13]Let us also mention the paper by Gonzáles-Díaz and Real [González-Díaz and Real 2003], which provides algorithms for calculating certain primary and secondary cohomology operations on a finite simplicial complex (including the Steenrod square Sq^2 and Adem's secondary cohomology operation). But both their goal and approach are different from ours. The algorithms in [González-Díaz and Real 2003] are based on explicit combinatorial formulas for these operations on the cochain level. The goal is to speed up the "obvious" way of computing the image of a given cohomology class under the considered operation. In our setting, we have no general explicit formulas available, and we can work only with the cohomology classes "locally," since they

Avoiding iterated suspensions. In order to compute $[X, P_i]$, our algorithm recursively computes all suspensions $[SX, P_j]$, $d \leq j \leq i-1$. In a straightforward implementation of the algorithm, for computing $[SX, P_{i-1}]$ we should also recursively compute $[SSX, P_{i-2}]$ etc., forming essentially a complete binary tree of recursive calls. We remark that by a slightly more complicated implementation of the algorithm, this tree of recursive calls can be truncated, since we do not really need the complete information about $[SX, P_{i-1}]$ to compute $[X, P_i]$. Essentially, we need only a system of generators of $[SX, P_{i-1}]$ and not the relations; see Remark 3.4. We stress, however, that this is merely a way to speed up the algorithm, and only by a constant factor if d is fixed.

A remark on methods. From a topological point of view, the tools and ideas that we use and combine to establish Theorem 1.1 have been essentially known.

On the one hand, there is an enormous topological literature with many beautiful ideas; indeed, in our experience, a problem with algorithmization may sometimes be an *abundance* of topological results, and the need to sort them out. On the other hand, the classical computational tools have been mostly designed for the "paper-and-pencil" model of calculation, where a calculating mathematician can, e.g., easily switch between different representations of an object or fill in some missing information by clever ad-hoc reasoning. Adapting the various methods to machine calculation sometimes needs a different approach; for instance, a recursive formulation may be preferable to an explicit, but cumbersome, formula (see, for example, [Rubio and Sergeraert 2002; Sergeraert] for an explanation of algorithmic difficulties with *spectral sequences*, a basic and powerful computational tool in topology).

We see our main contribution as that of synthesis: identifying suitable methods, putting them all together, and organizing the result in a hopefully accessible way, so that it can be built on in the future.

Some technical steps are apparently new; in this direction, our main technical contribution is probably a suitable implementation of the group operation on P_i (Section 5) and recursive testing of nullhomotopy (Section 6.4). The former was generalized and, in a sense, simplified in [Čadek et al. 2013], and the latter was extended to a more general situation in [Filakovský and Vokřínek 2013].

3. OPERATIONS WITH ABELIAN GROUPS

On the top level, our algorithm works with finitely generated Abelian groups. The structure of such groups is simple (they are direct sums of cyclic groups) and well known, but we will need to deal with certain subtleties in their algorithmic representations.

In our setting, an Abelian group A is represented by a set $\mathcal{A}$, whose elements are called *representatives*; we also assume that the representatives can be stored in a computer. For $\alpha \in \mathcal{A}$, let $[\alpha]$ denote the element of A represented by α. The representation is generally non-unique; we may have $[\alpha] = [\beta]$ for $\alpha \neq \beta$.

We call A represented in this way *semi-effective* if algorithms for the following three tasks are available:

(SE1) Provide an element $o \in \mathcal{A}$ representing the neutral element $0 \in A$.
(SE2) Given $\alpha, \beta \in \mathcal{A}$, compute an element $\alpha \boxplus \beta \in \mathcal{A}$ with $[\alpha \boxplus \beta] = [\alpha] + [\beta]$ (where $+$ is the group operation in A).
(SE3) Given $\alpha \in \mathcal{A}$, compute an element $\boxminus \alpha \in \mathcal{A}$ with $[\boxminus \alpha] = -[\alpha]$.

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are usually defined on infinite simplicial sets. That is, a cohomology class is represented by a cocycle, and that cocycle is given as an algorithm that can compute the value of the cocycle on any given simplex.

We stress that as a binary operation on $\mathcal{A}$, $\boxplus$ is not necessarily a group operation; e.g., we may have $\alpha \boxplus (\beta \boxplus \gamma) \neq (\alpha \boxplus \beta) \boxplus \gamma$, although of course, $[\alpha \boxplus (\beta \boxplus \gamma)] = [(\alpha \boxplus \beta) \boxplus \gamma]$.

For a semi-effective Abelian group, we are generally unable to decide, for $\alpha, \beta \in \mathcal{A}$, whether $[\alpha] = [\beta]$ (and, in particular, to certify that some element is nonzero).

Even if such an *equality test* is available, we still cannot infer much global information about the structure of A. For example, without additional information we cannot certify that A it is infinite cyclic—it could always be large but finite cyclic, no matter how many operations and tests we perform.

We now introduce a much stronger notion, with all the structural information explicitly available. We call a semi-effective Abelian group A *fully effective* if it is finitely generated and we have an explicit expression of A as a direct sum of cyclic groups. More precisely, we assume that the following are explicitly available:

(FE1) A list of generators $a_1, \ldots, a_k$ of A (given by representatives $\alpha_1, \ldots, \alpha_k \in \mathcal{A}$) and a list $(q_1, \ldots, q_k)$, $q_i \in \{2, 3, 4, \ldots\} \cup \{\infty\}$, such that each a_i generates a cyclic subgroup of A of order q_i, $i = 1, 2, \ldots, k$, and A is the direct sum of these subgroups.

(FE2) An algorithm that, given $\alpha \in \mathcal{A}$, computes a representation of $[\alpha]$ in terms of the generators; that is, it returns $(z_1, \ldots, z_k) \in \mathbb{Z}^k$ such that $[\alpha] = \sum_{i=1}^{k} z_i a_i$.

First we observe that, for full effectivity, it is enough to have A given by arbitrary generators and relations. That is, we consider a semi-effective A together with a list $b_1, \ldots, b_n$ of generators of A (again explicitly given by representatives) and an $m \times n$ integer matrix U specifying a complete set of relations for the b_i; i.e., $\sum_{i=1}^{n} z_i b_i = 0$ holds iff $(z_1, \ldots, z_n)$ is an integer linear combination of the rows of U. Moreover, we have an algorithm as in (FE2) that allows us to express a given element a as a linear combination of $b_1, \ldots, b_n$ (here the expression may not be unique).

LEMMA 3.1. *A semi-effective A with a list of generators and relations as above can be converted to a fully effective Abelian group.*

PROOF. This amounts to a computation of a Smith normal form, a standard step in computing integral homology groups, for example (see [Storjohann 1996] for an efficient algorithm and references).

Concretely, the Smith normal form algorithm applied on U yields an expression $D = SUT$ with D diagonal and S, T square and invertible (everything over $\mathbb{Z}$). Letting $\mathbf{b} = (b_1, \ldots, b_n)$ be the (column) vector of the given generators, we define another vector $\mathbf{a} = (a_1, \ldots, a_n)$ of generators by $\mathbf{a} := T^{-1}\mathbf{b}$. Then $D\mathbf{a} = 0$ gives a complete set of relations for the a_i (since $DT^{-1} = SU$ and the row spaces of SU and of U are the same). Omitting the generators a_i such that $|d_{ii}| = 1$ yields a list of generators as in (FE1). □

In the remainder of this section, the special form of the generators as in (FE1) will bring no advantage—on the contrary, it would make the notation more cumbersome. We thus assume that, for the considered fully effective Abelian groups, we have a list of generators and an arbitrary integer matrix specifying a complete set of relations among the generators.

Locally effective mappings. Let X, Y be sets. We call a mapping $\varphi\colon X \to Y$ *locally effective* if there is an algorithm that, given an arbitrary $x \in X$, computes $\varphi(x)$.

Next, for semi-effective Abelian groups A, B, with sets $\mathcal{A}, \mathcal{B}$ of representatives, respectively, we call a mapping $f\colon A \to B$ *locally effective* if there is a locally effective mapping $\varphi\colon \mathcal{A} \to \mathcal{B}$ such that $[\varphi(\alpha)] = f([\alpha])$ for all $\alpha \in \mathcal{A}$. In particular, we speak of a *locally effective homomorphism* if f is a group homomorphism.

LEMMA 3.2 (KERNEL). *Let $f\colon A \to B$ be a locally effective homomorphism of fully effective Abelian groups. Then $\ker(f) = \{a \in A : f(a) = 0\}$ can be represented as fully effective.*

PROOF. This essentially amounts to solving a homogeneous system of linear equations over the integers.

Let $a_1, \dots, a_m$ be a list of generators of A and U a matrix specifying a complete set of relations among them, and similarly for B, $b_1, \dots, b_n$, and V. For every $i = 1, 2, \dots, m$, we express $f(a_i) = \sum_{j=1}^n z_{ij} b_j$; then the $m \times n$ matrix $Z = (z_{ji})$ represents f in the sense that, for $a = \sum_{i=1}^m x_i a_i$, we have $f(a) = \sum_{j=1}^n y_j b_j$ with $\mathbf{y} = \mathbf{x}Z$, where $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_n)$ are regarded as *row* vectors.

Since V is the matrix of relations in B, $\sum_{j=1}^n y_j b_j$ equals 0 in B iff $\mathbf{y} = \mathbf{w}V$ for an integer (row) vector $\mathbf{w}$. So $\ker f = \{\sum_i x_i a_i : \mathbf{x} \in \mathbb{Z}^m, \mathbf{x}Z = \mathbf{w}V \text{ for some } \mathbf{w} \in \mathbb{Z}^n\}$.

Given a system of homogeneous linear equations over $\mathbb{Z}$, we can use the Smith normal form to find a system of generators for the set of all solutions (see, e.g., [Schrijver 1986, Chapter 5]). In our case, dealing with the system $\mathbf{x}Z = \mathbf{w}V$, we can thus compute integer vectors $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(\ell)}$ such that the elements $a'_k := \sum_{i=1}^m x_i^{(k)} a_i$, $k = 1, 2, \dots, \ell$, generate $\ker f$. By similar (and routine) considerations, which we omit, we can then compute a complete set of relations for the generators a'_k, and finally we apply Lemma 3.1. □

The next operation is the dual of taking a kernel, namely, factoring a given Abelian group by the image of a locally effective homomorphism. For technical reasons, when applying this lemma later on, we will need the resulting factor group to be equipped with an additional algorithm that returns a "witness" for an element being zero.

LEMMA 3.3 (COKERNEL). *Let A, B be fully effective Abelian groups with sets of representatives $\mathcal{A}, \mathcal{B}$, respectively, and let $f\colon A \to B$ be a locally effective homomorphism. Then we can obtain a fully effective representation of the factor group $C := \operatorname{coker}(f) = B/\operatorname{im}(f)$, again with the set $\mathcal{B}$ of representatives. Moreover, there is an algorithm that, given a representative $\beta \in \mathcal{B}$, tests whether β represents 0 in C, and if yes, returns a representative $\alpha \in \mathcal{A}$ such that $[f(\alpha)] = [\beta]$ in B.*

REMARK 3.4. *As will become apparent from the proof, the assumption that A is fully effective is not really necessary. Indeed, all that is needed is that A be semi-effective and that we have an explicit list of (representatives of) generators for A. In order to avoid burdening the reader with yet another piece of of terminology, however, we refrain from defining a special name for such representations.*

PROOF OF LEMMA 3.3. As a semi-effective representation for C, we we simply reuse the one we already have for B. That is, we reuse $\mathcal{B}$ (and the same algorithms for (SE1–3)) to represent the elements of C as well. To distinguish clearly between elements in B and in C, for $\beta \in \mathcal{B}$, we use the notation $b = [\beta]$ in B and $\bar{b} = \overline{[\beta]}$ for the corresponding element $b + \operatorname{im}(f)$ in C.

For a fully effective representation of C, we need the following, by Lemma 3.1: first, a complete set of generators for C (given by representatives); second, an algorithm as in (FE2) that expresses an arbitrary element of C (given as $\beta \in B$) as a linear combination of the generators; and, third, a complete set of relations among the generators.

For the first two tasks, we again reuse the solutions provided by the representation for B. Suppose $b_1, \dots, b_n$ (represented by $\beta_1, \dots, \beta_n$) generate B. Then $\bar{b}_1, \dots, \bar{b}_n$ (with the same representatives) generate C. Moreover, by assumption, we have an algorithm that, given $\beta \in \mathcal{B}$, computes integers z_i such that $[\beta] = z_1 b_1 + \dots z_n b_n$ in B; then $\overline{[\beta]} = z_1 \bar{b}_1 + \dots + z_n \bar{b}_n$ in C.

A complete set of relations among the the generators of C is obtained as follows. Let the matrix V specify a complete set of relations among the generators b_j of B, let $a_1, \ldots, a_m$ be a complete list of generators for A, and let Z be an integer matrix representing the homomorphism f with respect to the generators $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ as in the proof of Lemma 3.2. Then

$$U := \begin{pmatrix} Z \\ V \end{pmatrix}$$

specifies a complete set of relations among the $\bar{b}_j$ in C. To see that this is the case, consider an integer (row) vector $\mathbf{y} = (y_1, \ldots, y_n)$ and $\bar{b} := \sum_{j=1}^{n} y_j \bar{b}_j$. Then $\bar{b} = 0$ in C iff $b := \sum_{j=1}^{n} y_j b_j \in \mathrm{im}(f)$, i.e., iff there exists an element $a = \sum_{i=1}^{m} x_i a_i \in A$ such that $b - f(a) = 0$ in B. By definition of Z and by assumption on V, this is the case iff there are integer vectors $\mathbf{x}$ and $\mathbf{x}'$ such that $\mathbf{y} = \mathbf{x}Z + \mathbf{x}'V$, an integer combination of rows of U.

It remains to prove the second part of Lemma 3.3, i.e., to provide an algorithm that, given $\beta \in \mathcal{B}$, tests whether $\overline{[\beta]} = 0$ in C, or equivalently, whether $[\beta] \in \mathrm{im}(f)$, and if so, computes a preimage. For this, we express $[\beta] = \sum_{j=1}^{n} y_j \bar{b}_j$ as an integer linear combination of generators of B and then solve the system $\mathbf{y} = \mathbf{x}Z + \mathbf{x}'V$ of integer linear equations as above (where we rely again on Smith normal form computations). $\square$

The last operation is conveniently described using a *short exact sequence* of Abelian groups:

$$0 \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow 0 \tag{1}$$

(in other words, we assume that $f \colon A \to B$ is an injective homomorphism, $g \colon B \to C$ is a surjective homomorphism, and $\mathrm{im}\, f = \ker g$). It is well known that the middle group B is determined, up to isomorphism, by A, C, f, and g. For computational purposes, though, we also need to assume that the injectivity of f is "effective", i.e., witnessed by a locally effective inverse mapping r, and similarly for the surjectivity of g. This is formalized in the next lemma.

LEMMA 3.5 (SHORT EXACT SEQUENCE). *Let (1) be a short exact sequence of Abelian groups, where A and C are fully effective, B is semi-effective, $f \colon A \to B$ and $g \colon B \to C$ are locally effective homomorphisms, and suppose that, moreover, the following locally effective maps (typically not homomorphisms) are given:*

(i) *$r \colon \mathrm{im}\, f = \ker g \to A$ such that $f(r(b)) = b$ for every $b \in B$ with $g(b) = 0$.*[14]
(ii) *A map of representatives*[15] *$\xi \colon \mathcal{C} \to \mathcal{B}$ (where $\mathcal{B}, \mathcal{C}$ are the sets of representatives for B, C, respectively) that behaves as a section for g, i.e., such that $g([\xi(\gamma)]) = [\gamma]$ for all $\gamma \in \mathcal{C}$.*

Then we can obtain a fully effective representation of B.

PROOF. Let $a_1, \ldots, a_m$ be generators of A and $c_1, \ldots, c_n$ be generators of C, with *fixed* representative $\gamma_j \in \mathcal{C}$ for each c_j. We define $b_j := [\xi(\gamma_j)]$ for $1 \leq j \leq n$.

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[14]The equality $f(r(b)) = b$ is required on the level of group elements, and not necessarily on the level of representatives; that is, it may happen that $\varphi(\rho(\beta)) \neq \beta$, although necessarily $[\varphi(\rho(\beta))] = [\beta]$, where φ represents f and ρ represents r.

[15]For technical reasons, in the setting where we apply this lemma later, we do not get a well-defined map $s \colon C \to B$ on the level of group elements, that is, we cannot guarantee that $[\gamma_1] = [\gamma_2]$ implies $[\xi(\gamma_1)] = [\xi(\gamma_2)]$. Because of the injectivity of f, this problem does not occur for the map r.

Given an arbitrary element $b \in B$, we set $c := g(b)$, express $c = \sum_{j=1}^{n} z_j c_j$, and let $b^* := b - \sum_{j=1}^{n} z_j b_j$. Since $g(b^*) = g(b) - \sum_{j=1}^{n} z_j g(b_j) = 0$, we have $b^* \in \ker g$, and so $a := r(b^*)$ is well defined. Then we can express $a = \sum_{i=1}^{m} y_i a_i$, and we finally get $b = \sum_{i=1}^{m} y_i f(a_i) + \sum_{j=1}^{n} z_j b_j$.

Therefore, $(f(a_1), \ldots, f(a_m), b_1, \ldots, b_n)$ is a list of generators of B, computable in terms of representatives, and the above way of expressing b in terms of generators is algorithmic. Moreover, we have $b = 0$ iff $g(b) = 0$ and $r(b) = 0$, which yields equality test in B.

It remains to determine a complete set of relations for the described generators (and then apply Lemma 3.1). Let U be a matrix specifying a complete set of relations among the generators $a_1, \ldots, a_m$ in A, and V is an appropriate matrix for $c_1, \ldots, c_n$.

Let $(v_{k1}, \ldots, v_{kn})$ be the kth row of V. Since $\sum_{j=1}^{n} v_{kj} c_j = 0$, we have $b_k^* := \sum_{j=1}^{n} v_{kj} b_j \in \ker g$, and so, as above, we can express $b_k^* = \sum_{i=1}^{m} y_{ik} f(a_i)$. Thus, we have the relation $-\sum_{i=1}^{m} y_{ik} f(a_i) + \sum_{j=1}^{n} v_{kj} b_j = 0$ for our generators of B.

Let $Y = (y_{ik})$ be the matrix of the coefficients y_{ik} constructed above. We claim that the matrix

$$\begin{pmatrix} -Y & V \\ U & 0 \end{pmatrix}$$

specifies a complete set of relations among the generators $f(a_1), \ldots, f(a_m), b_1, \ldots, b_n$ of B. Indeed, we have just seen that the rows in the upper part of this matrix correspond to valid relations, and the relations given by the rows in the bottom part are valid because U specifies relations among the a_i in A and f is a homomorphism.

Finally, let

$$x_1 f(a_1) + \cdots + x_m f(a_m) + z_1 b_1 + \cdots + z_n b_n = 0 \tag{2}$$

be an arbitrary valid relation among the generators. Applying g and using $g \circ f = 0$, we get that $\sum_{j=1}^{n} z_j c_j = 0$ is a relation in C, and so $(z_1, \ldots, z_n)$ is a linear combination of the rows of V.

Let $(w_1, \ldots, w_m)$ be the corresponding linear combination of the rows of $-Y$. Then we have $\sum_{i=1}^{m} w_i f(a_i) + \sum_{j=1}^{n} z_j b_j = 0$, and subtracting this from (2), we arrive at $\sum_{i=1}^{m} (x_i - w_i) f(a_i) = 0$. Since f is an injective homomorphism, we have $\sum_{i=1}^{m} (x_i - w_i) a_i = 0$ in A, and so $(x_1 - w_1, \ldots, x_m - w_m)$ is a linear combination of the rows of U. This concludes the proof. $\square$

4. TOPOLOGICAL PRELIMINARIES

In this part we summarize notions and results from the literature. They are mostly standard in homotopy theory and can be found in textbooks—see, e.g., Hatcher [Hatcher 2001] for topological notions and May [May 1967] for simplicial notions (we also refer to Steenrod [Steenrod 1972] as an excellent background text, although its terminology differs somewhat from the more modern usage). However, they are perhaps not widely known to non-topologists, and they are somewhat scattered in the literature. We also aim at conveying some simple intuition behind the various notions and concepts, which is not always easy to get from the literature.

On the other hand, in order to follow the arguments in this paper, for some of the notions it is sufficient to know some properties, and the actual definition is never used directly. Such definitions are usually omitted; instead, we illustrate the notions with simple examples or with an informal explanation.

Even readers with a strong topological background may want to skim this part because of the notation. Moreover, in Section 4.3 we discuss an algorithmic result on the construction of Postnikov systems, which may not be well known.

CW-complexes. Below we will state various topological results. Usually they hold for fairly general topological spaces, but not for all topological spaces. The appropriate level of generality for such results is the class of *CW-complexes* (or sometimes spaces homotopy equivalent to CW-complexes).

A reader not familiar with CW-complexes may either look up the definition (e.g., in [Hatcher 2001]), or take this just to mean "topological spaces of a fairly general kind, including all simplicial complexes and simplicial sets". It is also good to know that, similar to simplicial complexes, CW-complexes are made of pieces (*cells*) of various dimensions, where the 0-dimensional cells are also called *vertices*. There is only one place, in Section 5.1, where a difference between CW-complexes and simplicial sets becomes somewhat important, and there we will stress this.

4.1. Simplicial sets

Simplicial sets are our basic device for representing topological spaces and their maps in our algorithm. Here we introduce them briefly, with emphasis on the ideas and intuition, referring to Friedman [Friedman 2012] for a very friendly thorough introduction, to [Curtis 1971; May 1967] for older compact sources, and to [Goerss and Jardine 1999] for a more modern and comprehensive treatment.

A *simplicial set* can be thought of as a generalization of simplicial complexes. Similar to a simplicial complex, a simplicial set is a space built of vertices, edges, triangles, and higher-dimensional simplices, but simplices are allowed to be glued to each other and to themselves in more general ways. For example, one may have several 1-dimensional simplices connecting the same pair of vertices, a 1-simplex forming a loop, two edges of a 2-simplex identified to create a cone, or the boundary of a 2-simplex all contracted to a single vertex, forming an S^2.
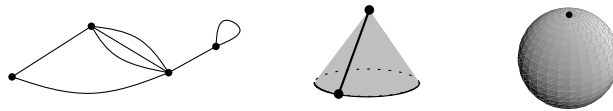
However, unlike for the still more general CW-complexes, a simplicial set can be described purely combinatorially.

Another new feature of a simplicial set, in comparison with a simplicial complex, is the presence of *degenerate simplices*. For example, the edges of the triangle with a contracted boundary (in the last example above) do not disappear—formally, each of them keeps a phantom-like existence of a degenerate 1-simplex.

Simplices, face and degeneracy operators. A simplicial set X is represented as a sequence $(X_0, X_1, X_2, \ldots)$ of mutually disjoint sets, where the elements of X_m are called the *m-simplices of* X. For every $m \geq 1$, there are $m+1$ mappings $\partial_0, \ldots, \partial_m \colon X_m \to X_{m-1}$ called *face operators*; the meaning is that for a simplex $\sigma \in X_m$, $\partial_i \sigma$ is the face of σ obtained by deleting the ith vertex. Moreover, there are $m+1$ mappings $s_0, \ldots, s_m \colon X_m \to X_{m+1}$ (opposite direction) called the *degeneracy operators*; the meaning of $s_i \sigma$ is the degenerate simplex obtained from σ by duplicating the ith vertex. A simplex is called *degenerate* if it lies in the image of some s_i; otherwise, it is *nondegenerate*. There are natural axioms that the ∂_i and the s_i have to satisfy, but we will not list them here, since we won't really use them (and the usual definition of a simplicial set is formally different anyway, expressed in the language of category theory).

We call X *finite* if it has finitely many nondegenerate simplices (every nonempty simplicial set has infinitely many degenerate simplices).

Examples. Here we sketch some basic examples of simplicial sets; again, we won't provide all details, referring to [Friedman 2012]. Let Δ^n denote the standard n-dimensional simplex regarded as a simplicial set. For $n = 0$, $(\Delta^0)_m$ consists of a single simplex, denoted by 0^m, for every $m = 0, 1, \ldots$; 0^0 is the only nondegenerate simplex. The face and degeneracy operators are defined in the only possible way.

For $n = 1$, Δ^1 has two 0-simplices (vertices), say 0 and 1, and in general there are $m+2$ simplices in $(\Delta^1)_m$; we can think of the ith one as containing i copies of the vertex 0 and $m+1-i$ copies of the vertex 1, $i = 0, 1, \ldots, m+1$. For n arbitrary, the m-simplices of Δ^n can be thought of as all nondecreasing $(m + 1)$-term sequences with entries in $\{0, 1, \ldots, n\}$; the ones with all terms distinct are nondegenerate.

In a similar fashion, every simplicial complex K can be converted into a simplicial set X in a canonical way; however, first we need to fix a linear ordering of the vertices. The nondegenerate m-simplices of X are in one-to-one correspondence with the m-simplices of K, but many degenerate simplices show up as well.

Finally we mention a "very infinite" but extremely instructive example, the *singular set*, which contributed significantly to the invention of simplicial sets—as Steenrod [Steenrod 1972] puts it, the definition of a simplicial set is obtained by writing down fairly obvious properties of the singular set. For a topological space Y, the singular set $S(Y)$ is the simplicial set whose m-simplices are all continuous maps of the standard m-simplex into Y. The ith face operator $\partial_i\colon S(Y)_m \to S(Y)_{m-1}$ is given by the composition with a canonical mapping that sends the standard $(m-1)$-simplex to the ith face of the standard m-simplex. Similarly, the ith degeneracy operator is induced by the canonical mapping that collapses the standard $(m+1)$-simplex to its ith m-dimensional face and then identifies this face with the standard m-simplex, preserving the order of the vertices.

Geometric realization. Similar to a simplicial complex, each simplicial set X defines a topological space $|X|$ (the *geometric realization of X*), uniquely up to homeomorphism. Intuitively, one takes disjoint geometric simplices corresponding to the nondegenerate simplices of X, and glues them together according to the identifications implied by the face and degeneracy operators (we again refer to the literature, especially to [Friedman 2012], for a formal definition).

k-reduced simplicial sets. A simplicial set X is called k-*reduced* if it has a single vertex and no nondegenerate simplices in dimensions 1 through k. Such an X is necessarily k-connected.

A similar terminology can also be used for CW-complexes; k-reduced means a single vertex (0-cell) and no cells in dimensions 1 through k.

Products. The *product* $X \times Y$ of two simplicial sets is formally defined in an incredibly simple way: we have $(X \times Y)_m := X_m \times Y_m$ for every m, and the face and degeneracy operators work componentwise; e.g., $\partial_i(\sigma, \tau) := (\partial_i\sigma, \partial_i\tau)$. As expected, the product of simplicial sets corresponds to the Cartesian product of the geometric realizations, i.e., $|X \times Y| \cong |X| \times |Y|$.[16] The simple definition hides some intricacies, though, as one can guess after observing that, for example, the product of two 1-simplices is not a simplex—so the above definition has to imply some canonical way of triangulating the product. It indeed does, and here the degenerate simplices deserve their bread.

Cone and suspension. Given a simplicial set X, the *cone* CX is a simplicial set obtained by adding a new vertex $*$ to X, taking all simplices of X, and, for every m-simplex $\sigma \in X_m$ and every $i \geq 1$, adding to CX the $(m + i)$-simplex obtained from

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[16]To be precise, the product of topological spaces on the right-hand side should be taken in the category of k-spaces; but for the spaces we encounter, it is the same as the usual product of topological spaces.

σ by adding i copies of $*$. In particular, the nondegenerate simplices of CX are the nondegenerate simplices of X plus the cones over these (obtained by adding a single copy of $*$). We skip the definition of face and degeneracy operators for CX as usual. The definitions are discussed, e.g., in [Goerss and Jardine 1999, Chapter III.5], although there they are given in a more abstract language, and later (in Section 6.3 below) we will state the concrete properties of CX that we will need.

We will also need the *suspension SX*; this is the simplicial set CX/X obtained from CX by contracting all simplices of X into a single vertex. The following picture illustrates both of the constructions for a 1-dimensional X:

$$X \qquad\qquad CX \qquad\qquad SX$$
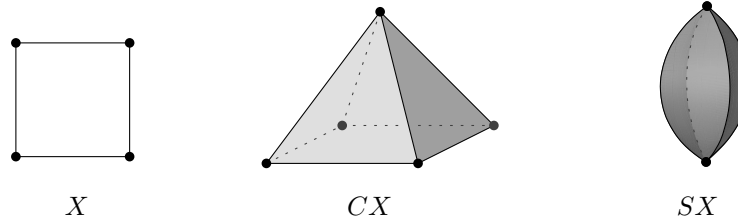
Topologically, SX is the usual (unreduced) suspension of X, which is often presented as erecting a double cone over X (or a join with an S^0). This would also be the "natural" way of defining the suspension for a simplicial complex, but the above definition for simplicial sets is combinatorially different, although topologically equivalent. Even if X is a simplicial complex, SX is not. For us, the main advantage is that the simplicial structure of SX is particularly simple; namely, for $m > 0$, the m-simplices of SX are in one-to-one correspondence with the $(m - 1)$-simplices of X.[17]

Simplicial maps and homotopies. Simplicial sets serve as a combinatorial way of describing a topological space; in a similar way, simplicial maps provide a combinatorial description of continuous maps.

A *simplicial map* $f\colon X \to Y$ of simplicial sets X, Y consists of maps $f_m\colon X_m \to Y_m$, $m = 0, 1, \ldots$, that commute with the face and degeneracy operators. We denote the set of all simplicial maps $X \to Y$ by $\mathrm{SMap}(X, Y)$.[18]

It is useful to observe that it suffices to specify a simplicial map $f\colon X \to Y$ on the *nondegenerate* simplices of X; the values on the degenerate simplices are then determined uniquely. In particular, if X is finite, then such an f can be specified as a finite object.

A simplicial map $f\colon X \to Y$ induces a continuous map $|f|\colon |X| \to |Y|$ of the geometric realizations in a natural way (we again omit the precise definition). Often we will take

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[17]Let us also remark that in homotopy-theoretic literature, one often works with *reduced* cone and suspension, which are appropriate for the category of pointed spaces and maps. For example, the *reduced suspension* ΣX is obtained from SX by collapsing the segment that connects the apex of CX to the basepoint of X. For CW-complexes, ΣX and SX are homotopy equivalent, so the difference is insignificant for our purposes.

[18]There is a technical issue to be clarified here, concerning *pointed maps*. We recall that a *pointed space* (X, x_0) is a topological space X with a choice of a distinguished point $x_0 \in X$ (the *basepoint*). In a CW-complex or simplicial set, we will always assume the basepoint to be a vertex. A *pointed map* $(X, x_0) \to (Y, y_0)$ of pointed spaces is a continuous map sending x_0 to y_0. Homotopies of pointed maps are also meant to be pointed; i.e., they must keep the image of the basepoint fixed. The reader may recall that, for example, the homotopy groups $\pi_k(Y)$ are really defined as homotopy classes of pointed maps.

If X, Y are simplicial sets, X is arbitrary, and Y is a 1-reduced (thus, it has a single vertex, which is the basepoint), as will be the case for the targets of simplicial maps in our algorithm, then every simplicial map is automatically pointed. Thus, in this case, we need not worry about pointedness.

A topological counterpart of this is that, if Y is a 1-connected CW-complex, then every map $X \to Y$ is (canonically) homotopic to a map sending x_0 to y_0, and thus $[X, Y]$ is canonically isomorphic to the set of all homotopy classes of pointed maps $X \to Y$.

the usual liberty of omitting $|\cdot|$ and not distinguishing between simplicial sets and maps and their geometric realizations.

Of course, not all continuous maps are induced by simplicial maps. But the usefulness of simplicial sets for our algorithm (and many other applications) stems mainly from the fact that, if the target Y has the *Kan extension property*, then *every* continuous map $\varphi\colon |X| \to |Y|$ is homotopic to a simplicial map $f\colon X \to Y$.[19]

The Kan extension property is a certain property of a simplicial set (and the simplicial sets having it are called *Kan simplicial sets*), which need not be spelled out here—it will suffice to refer to standard results to check the property where needed. In particular, every *simplicial group* is a Kan simplicial set, where a simplicial group G is a simplicial set for which every G_m is endowed with a group structure, and the face and degeneracy operators are group homomorphisms (we will see examples in Section 4.2 below).

Homotopies of simplicial maps into a Kan simplicial set can also be represented simplicially. Concretely, a *simplicial homotopy* between two simplicial maps $f, g\colon X \to Y$ is a simplicial map $F\colon X \times \Delta^1 \to Y$ such that $F|_{X \times \{0\}} = f$ and $F|_{X \times \{1\}} = g$; here, as we recall, Δ^1 represents the geometric 1-simplex (segment) as a simplicial set, and, with some abuse of notation, $\{0\}$ and $\{1\}$ are the simplicial subsets of Δ^1 representing the two vertices. Again, if Y is a Kan simplicial set, then two simplicial maps f, g into Y are simplicially homotopic iff they are homotopic in the usual sense as continuous maps.

Locally effective simplicial sets and simplicial maps. Unsurprisingly, there is a price to pay for the convenience of representing all continuous maps and homotopies simplicially: a Kan simplicial set necessarily has infinitely many simplices in every dimension (except for some trivial cases); thus we need nontrivial techniques for representing it in a computer. Fortunately, the Kan simplicial sets relevant in our case have a sufficiently regular structure and can be handled; suitable techniques were developed and presented in [Sergeraert 1994; Rubio and Sergeraert 2002; Romero et al. 2006; Rubio and Sergeraert 2005; 2012].

For algorithmic purposes, a simplicial set X is represented in a *black box* or *oracle* manner, by a collection of various algorithms that allow us to access certain information about X. Specifically, let X be a simplicial set, and suppose that some encoding for the simplices of X by strings (finite sequences over some fixed alphabet, say $\{0, 1\}$) has been fixed.

We say that X is *locally effective* if we have algorithms for evaluating the face and degeneracy maps, i.e., i.e., given (the encoding of) a d-simplex σ of X and $i \in \{0, 1, \ldots, d\}$, we can compute the simplex $\partial_i \sigma$, and similarly for the degeneracy operators s_i.

A simplicial map $f\colon X \to Y$ is called *locally effective* if there is an algorithm that, given (an encoding of) a simplex σ of X, computes (the encoding of) the simplex $f(\sigma)$.

4.2. Eilenberg–MacLane spaces and cohomology

Cohomology. We will need some terminology from (simplicial) cohomology, such as cochains, cocycles, and cohomology groups. However, these will be mostly a convenient bookkeeping device for us, and we won't need almost any properties of cohomology.

For a simplicial complex X, an integer $n \geq 0$, and an Abelian group π, an *n-dimensional cochain* with values in π is an arbitrary mapping $c^n\colon X_n \to \pi$, i.e., a label-

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[19]The reader may be familiar with the *simplicial approximation theorem*, which states that for every continuous map $\varphi\colon |K| \to |L|$ between the polyhedra of simplicial complexes, there is a simplicial map of a *sufficiently fine subdivision* of K into L that is homotopic to φ. The crucial difference is that in the case of simplicial sets, if Y has the Kan extension property, we need not subdivide X at all!

ing of the n-dimensional simplices of X with elements of π. The set of all n-dimensional cochains is (traditionally) denoted by $C^n(X;\pi)$; with componentwise addition, it forms an Abelian group.

For a simplicial *set* X, we define $C^n(X;\pi)$ to consist only of cochains in which all *degenerate* simplices receive value 0 (these are sometimes called *normalized cochains*).

Given an n-cochain c^n, the *coboundary* of c^n is the $(n+1)$-cochain $d^{n+1} = \delta c^n$ whose value on a $\tau \in X_{n+1}$ is the sum of the values of c^n over the n-faces of τ (taking orientations into account); formally,

$$d^{n+1}(\tau) = \sum_{i=0}^{n+1} (-1)^i c^n(\partial_i \tau).$$

A cochain c^n is a *cocycle* if $\delta c^n = 0$; $Z^n(X;\pi) \subseteq C^n(X;\pi)$ is the subgroup of all cocycles (Z for *koZyklus*), i.e., the kernel of δ. The subgroup $B^n(X;\pi) \subseteq C^n(X;\pi)$ of all *coboundaries* is the image of δ; that is, c^n is a coboundary if $c^n = \delta b^{n-1}$ for some $(n-1)$-cochain b^{n-1}.

The nth (simplicial) *cohomology group* of X is the factor group

$$H^n(X;\pi) := Z^n(X;\pi)/B^n(X;\pi)$$

(for this to make sense, of course, one needs the basic fact $\delta \circ \delta = 0$).

Eilenberg–MacLane spaces as "simple ranges". The homotopy groups $\pi_k(Y)$ are among the most important invariants of a topological space Y. The group $\pi_k(Y)$ collects information about the "k-dimensional structure" of Y by probing Y with all possible maps *from* S^k. Here the sphere S^k plays a role of the "simplest nontrivial" k-dimensional space; indeed, in some respects, for example concerning homology groups, it is as simple as one can possibly get.

However, as was first revealed by the famous *Hopf map* $S^3 \to S^2$, the spheres are not at all simple concerning maps going *into* them. In particular, the groups $\pi_k(S^n)$ are complicated and far from understood, in spite of a huge body of research devoted to them. So if one wants to probe a space X with maps going *into* some "simple nontrivial" space, then spaces other than spheres are needed—and the Eilenberg–MacLane spaces can play this role successfully.

Given an Abelian group π and an integer $n \geq 1$, an *Eilenberg–MacLane space* $K(\pi, n)$ is defined as any topological space T with $\pi_n(T) \cong \pi$ and $\pi_k(T) = 0$ for all $k \neq n$. It is not difficult to show that a $K(\pi, n)$ exists (by taking a wedge of n-spheres and inductively attaching balls of dimensions $n+1, n+2, \dots$ to kill elements of the various homotopy groups), and it also turns out that $K(\pi, n)$ is unique up to homotopy equivalence.[20]

The circle S^1 is (one of the incarnations of) a $K(\mathbb{Z}, 1)$, and $K(\mathbb{Z}_2, 1)$ can be represented as the infinite-dimensional real projective space, but generally speaking, the spaces $K(\pi, n)$ do not look exactly like very simple objects.

Maps into $K(\pi, n)$. Yet the following elegant fact shows that the $K(\pi, n)$ indeed constitute "simple" targets of maps.

LEMMA 4.1. *For every $n \geq 1$ and every Abelian group π, we have*

$$[X, K(\pi, n)] \cong H^n(X;\pi),$$

where X is a simplicial complex (or a CW-complex).

This is a basic and standard result (e.g., [May 1967, Lemma 24.4] in a simplicial setting), but nevertheless we will sketch an intuitive geometric proof, since it explains

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[20]Provided that we restrict to spaces that are homotopy equivalent to CW-complexes.

why maps into $K(\pi, n)$ can be represented discretely, by cocycles, and this is a key step towards representing maps in our algorithm.

PROOF SKETCH. For simplicity, let X be a finite simplicial complex (the argument works for a CW-complex in more or less the same way), and let us consider an arbitrary continuous map $f \colon |X| \to K(\pi, n)$, $n \geq 2$.

First, let us consider the restriction of f to the $(n-1)$-skeleton $X^{(n-1)}$ of X. Since by definition, $K(\pi, n)$ is $(n-1)$-connected, $f|_{X^{(n-1)}}$ is homotopic to the constant map sending $X^{(n-1)}$ to a single point y_0 (we can imagine pulling the images of the simplices to y_0 one by one, starting with vertices, continuing with 1-simplices, etc., up to $(n-1)$-simplices). Next, the homotopy of $f|_{X^{(n-1)}}$ with this constant map can be extended to a homotopy of f with a map $\tilde{f}$ defined on all of X (this is a standard fact known as the *homotopy extension property* of X, valid for all CW-complexes, among others). Thus, $\tilde{f} \sim f$ sends $X^{(n-1)}$ to y_0.

Next, we consider an n-simplex σ of X. All of its boundary now goes to y_0, and so the restriction of $\tilde{f}$ to σ can be regarded as a map $S^n \to K(\pi, n)$ (since collapsing the boundary of an n-simplex to a point yields an S^n). Thus, up to homotopy, $\tilde{f}|_\sigma$ is described by an element of $\pi_n(K(\pi, n)) = \pi$. In this way, $\tilde{f}$ defines a cochain $c^n = c^n_{\tilde{f}} \in C^n(X; \pi)$. The following picture captures this schematically:
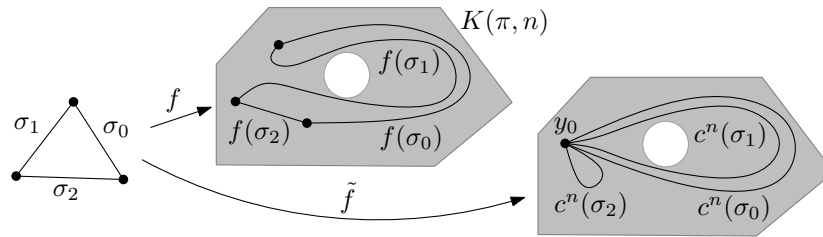
The target space $K(\pi, n)$ is illustrated as having a hole "responsible" for the nontriviality of π_n.

We note that $\tilde{f}$ is not determined uniquely by f, and $c^n_{\tilde{f}}$ may also depend on the choice of $\tilde{f}$.

Next, we observe that every cochain of the form $c^n_{\tilde{f}}$ is actually a *cocycle*. To this end, we consider an $(n+1)$-simplex $\tau \in X_{n+1}$. Since $\tilde{f}$ is defined on all of τ, the restriction $\tilde{f}|_{\partial\tau}$ to the boundary is nullhomotopic. At the same time, $\tilde{f}|_{\partial\tau}$ can be regarded as the sum of the elements of $\pi_n(K(\pi, n))$ represented by the restrictions of $\tilde{f}$ to the n-dimensional faces of τ.

Indeed, for any space Y the sum $[f]$ of two elements $[f_1], [f_2] \in \pi_n(Y)$ can be represented by contracting an $(n-1)$-dimensional "equator" of S^n to the basepoint, thus obtaining a wedge of two S^n's, and then defining f to be f_1 on one of these and f_2 on the other, as indicated in the picture below on the left (this time for $n = 2$). Similarly, in our case, the sum of the maps on the facets of τ can be represented by contracting the $(n-1)$-skeleton of τ to a point, and thus obtaining a wedge of $n+2$ n-spheres.

Therefore, we have $(\delta c^n)(\tau) = 0$, and $c^n = c^n_{\tilde{f}} \in Z^n(X; \pi)$ as claimed.

Conversely, given any $z^n \in Z^n(X; \pi)$, one can exhibit a map $\tilde{f}\colon X \to K(\pi, n)$ with $c^n_{\tilde{f}} = z^n$. Such an $\tilde{f}$ is build one simplex of X at a time. First, all simplices of dimension at most $n-1$ are sent to y_0. For every $\sigma \in X_n$, we choose a representative of the element $z^n(\sigma) \in \pi_n(K(\pi, n))$, which is a (pointed) map $S^n \to K(\pi, n)$, and use it to map σ. Then for $\tau \in X_{n+1}$, $\tilde{f}$ can be extended to τ, since $\tilde{f}|_{\partial\tau}$ is nullhomotopic by the cocycle condition for z^n. Finally, for a simplex ω of dimension larger than $n+1$, the $\tilde{f}$ constructed so far is necessarily nullhomotopic on $\partial\omega$ because $\pi_k(K(\pi, n)) = 0$ for all $k > n$, and thus an extension to ω is always possible.

We hope that this may convey some idea where the cocycle representation of maps into $K(\pi, n)$ comes from. By similar, but a little more complicated considerations, which we omit here, one can convince oneself that two maps $f, g\colon X \to K(\pi, n)$ are homotopic exactly when the corresponding cocycles $c^n_{\tilde{f}}$ and $c^n_{\tilde{g}}$ differ by a coboundary. In particular, for a given f, the cocycle $c^n_{\tilde{f}}$ may depend on the choice of $\tilde{f}$, but the cohomology class $c^n_{\tilde{f}} + B^n(X; \pi)$ does not. This finishes the proof sketch. □

A Kan simplicial model of $K(\pi, n)$. The Eilenberg–MacLane spaces $K(\pi, n)$ can be represented as Kan simplicial sets, and actually as simplicial groups, in an essentially unique way; we will keep the notation $K(\pi, n)$ for this simplicial set as well.

Namely, the set of m-simplices of $K(\pi, n)$ is given by the amazing formula

$$K(\pi, n)_m := Z^n(\Delta^m; \pi).$$

More explicitly, an m-simplex σ can be regarded as a labeling of the n-dimensional faces of the standard m-simplex by elements of the group π; moreover, the labels must add up to 0 on every $(n+1)$-face. There are $\binom{m+1}{n+1}$ nondegenerate n-faces of Δ^m, and so an m-simplex $\sigma \in K(\pi, n)_m$ is determined by an ordered $\binom{m+1}{n+1}$-tuple of elements of π.

It is not hard to define the face and degeneracy operators for $K(\pi, n)$, but we omit this since we won't use them explicitly (see, e.g., [May 1967; Rubio and Sergeraert 2012]). It suffices to say that the *degenerate* σ are precisely those labelings with two facets of Δ^m labeled identically and zero everywhere else.

In particular, for every $m \geq 0$, we have an m-simplex in $K(\pi, n)$ formed by the zero n-cochain, which is nondegenerate for $m = 0$ and degenerate for $m > 0$, and which we write simply as 0 (with the dimension understood from context). It is remarkable that the zero n-cochain on Δ^0 is the only vertex of the simplicial set $K(\pi, n)$ for $n > 0$.

We won't prove that this is indeed a simplicial model of $K(\pi, n)$. Let us just note that $K(\pi, n)$ is $(n-1)$-reduced, and its n-simplices correspond to elements of π (since an n-cocycle on Δ^n is a labeling of the single nondegenerate n-simplex of Δ^n by an element of π). Thus, each n-simplex of $K(\pi, n)$ "embodies" one of the possible ways of mapping the interior of Δ^n into $K(\pi, n)$, given that the boundary goes to the basepoint. The $(n+1)$-simplices then "serve" to get the appropriate addition relations among the just mentioned maps, so that this addition works as that in π, and the higher-dimensional simplices kill all the higher homotopy groups.

The (elementwise) addition of cochains makes $K(\pi, n)$ into a simplicial group, and consequently, $K(\pi, n)$ is a Kan simplicial set.

The simplicial sets $E(\pi, n)$. The m-simplices in the simplicial Eilenberg–MacLane spaces as above are all n-*cocycles* on Δ^m. If we take all n-*cochains*, we obtain another simplicial set called $E(\pi, n)$. Thus, explicitly,

$$E(\pi, n)_m := C^n(\Delta^m; \pi).$$

As a topological space, $E(\pi, n)$ is contractible, and thus not particularly interesting topologically in itself, but it makes a useful companion to $K(\pi, n)$. Obviously, $K(\pi, n) \subseteq E(\pi, n)$, but there are also other, less obvious relationships.

Since an m-simplex $\sigma \in E(\pi, n)$ is formally an n-cochain, we can take its coboundary $\delta\sigma$. This is an $(n + 1)$-coboundary (and thus also cocycle), which we can interpret as an m-simplex of $K(\pi, n + 1)$. It turns out that this induces a *simplicial* map $E(\pi, n) \to K(\pi, n + 1)$, which is (with the usual abuse of notation) also denoted by δ. This map is actually surjective, since the relevant cohomology groups of Δ^m are all zero and thus all cocycles are also coboundaries.

Simplicial maps into $K(\pi, n)$ and $E(\pi, n)$. We have the following "simplicial" counterpart of Lemma 4.1:

LEMMA 4.2. *For every simplicial complex (or simplicial set) X, we have*

$$\mathrm{SMap}(X, K(\pi, n)) \cong Z^n(X; \pi) \text{ and } \mathrm{SMap}(X, E(\pi, n)) \cong C^n(X; \pi).$$

We refer to [May 1967, Lemma 24.3] for a proof; here we just describe how the isomorphism[21] works, i.e., how one passes between cochains and simplicial maps. This is not hard to guess from the formal definition—there is just one way to make things match formally.

Namely, given a $c^n \in C^n(X; \pi)$, we want to construct the corresponding simplicial map $s = s(c^n) \colon X \to E(\pi, n)$. We consider an m-simplex $\sigma \in X_m$. There is exactly one way of inserting the standard m-simplex Δ^m to the "place of σ" into X; more formally, there is a unique simplicial map $i_\sigma \colon \Delta^m \to X$ that sends the m-simplex of Δ^m to σ (indeed, a simplicial map has to respect the ordering of vertices, implicit in the face and degeneracy operators). Thus, for every such σ, the cochain c^n defines a cochain $i_\sigma^*(c^n)$ on Δ^m (the labels of the n-faces of σ are pulled back to Δ^m), and that cochain is taken as the image $s(\sigma)$.

For the reverse direction, i.e., from a simplicial map s to a cochain, it suffices to look at the images of the n-simplices under s: these are n-simplices of $E(\pi, n)$ which, as we have seen, can be regarded as elements of π—thus, they define the values of the desired n-cochain.

Simplicial homotopy in $\mathrm{SMap}(X, K(\pi, n))$. Now that we have a description of simplicial maps $X \to K(\pi, n)$, we will also describe homotopies (or equivalently, simplicial homotopies) among them. It turns out that the additive structure (cocycle addition) on $\mathrm{SMap}(X, K(\pi, n)) \cong Z^n(X; \pi)$ reduces the question of whether two maps represented by cocycles c_1 and c_2 are homotopic to the question whether their difference $c_1 - c_2$ is *nullhomotopic* (homotopic to a constant map).

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[21]Both sets carry an Abelian group structure, and the bijection between them preserves these. For the set $Z^n(X; \pi)$ of cocycles, the group structure is given by the usual addition of cocycles. For the set $\mathrm{SMap}(X, K(\pi, n))$ of simplicial maps, the group structure is given by the fact that $K(\pi, n)$ is a *simplicial Abelian group*, so simplicial maps into it can be added componentwise (simplexwise).

LEMMA 4.3. *Let $c_1^n, c_2^n \in Z^n(X; \pi)$ be two cocycles. Then the simplicial maps $s_1, s_2 \in$* $\mathrm{SMap}(X, K(\pi; n))$ *represented by c_1^n, c_2^n, respectively, are simplicially homotopic iff c_1 and c_2 are* cohomologous, *i.e., $c_1 - c_2 \in B^n(X; \pi)$.*

We refer to [May 1967, Theorem 24.4] for a proof. We also remark that a simplicial version of Lemma 4.1 is actually proved using Lemmas 4.2 and 4.3.

4.3. Postnikov systems

Now that we have a combinatorial representation of maps from X into an Eilenberg–MacLane space, and of their homotopies, it would be nice to have similar descriptions for other target spaces Y. Expressing Y through its simplicial *Postnikov system* comes as close to fulfilling this plan as seems reasonably possible.

Postnikov systems are somewhat complicated objects, and so we will not discuss them in detail, referring to standard textbooks ([Hatcher 2001] in general and [May 1967] for the simplicial case) instead. First we will explain some features of a Postnikov system in the setting of topological spaces and continuous maps; this part, strictly speaking, is not necessary for the algorithm. Then we introduce a simplicial version of a Postnikov system, and summarize the properties we will actually use. Finally, we will present the subroutine used to compute Postnikov systems.

Postnikov systems on the level of spaces and continuous maps. Let Y be a CW-complex. A *Postnikov system* (also called a *Postnikov tower*) for Y is a sequence of spaces $P_0, P_1, P_2, \ldots$, where P_0 is a single point, together with maps $\varphi_i \colon Y \to P_i$ and $p_i \colon P_i \to P_{i-1}$ such that $p_i \circ \varphi_i = \varphi_{i-1}$, i.e., the following diagram commutes:

$$
\begin{array}{ccc}
 & \vdots & \\
 & P_2 & \\
 \nearrow^{\varphi_2} & \downarrow p_2 & \\
 & P_1 & \\
 \nearrow^{\varphi_1} & \downarrow p_1 & \\
 Y \xrightarrow{\ \ \varphi_0\ \ } & P_0 &
\end{array}
$$

Informally, the P_i, called the *stages* of the Postnikov system, can be thought of as successive stages in a process of building Y (or rather, a space homotopy equivalent to Y) "layer by layer" from the Eilenberg–Mac Lane spaces $K(\pi_i(Y), i)$.

More formally, it is required that for each i, the mapping φ_i induces an isomorphism $\pi_j(Y) \cong \pi_j(P_i)$ of homotopy groups for every $j \leq i$, while $\pi_j(P_i) = 0$ for all $j > i$. These properties suffice to define a Postnikov system uniquely up to homotopy equivalence, provided that Y is 0-connected and the P_i are assumed to be CW-complexes; see, e.g., Hatcher [Hatcher 2001, Section 4.3].

For the rest of this paper, we will abbreviate $\pi_i(Y)$ to π_i.

One usually works with Postnikov systems with additional favorable properties, sometimes called *standard Postnikov systems*, and for these to exist, more assumptions on Y are needed—in particular, they do exist if Y is 1-connected. In this case, the first two stages, P_0 and P_1, are trivial, i.e., just one-point spaces.

Standard Postnikov systems on the level of topological spaces are defined using the notion of *principal fibration*, which we do not need/want to define here. Let us just sketch informally how P_i is built from P_{i-1} and $K(\pi_i, i)$. *Locally*, P_i "looks like" the product $P_{i-1} \times K(\pi_i, i)$, in the sense that the *fiber* $p_i^{-1}(x)$ of every point $x \in P_{i-1}$ is (homotopy equivalent to) $K(\pi_i, i)$. However, *globally* P_i is usually *not* the product as

above; rather, it is "twisted" (technically, it is the total space of the fibration $K(\pi_i, i) \to P_i \xrightarrow{p_i} P_{i-1}$). A somewhat simple-minded analogue is the way the Möbius band is made by putting a segment "over" every point of S^1, looking locally like the product $S^1 \times [-1, 1]$ but globally, of course, very different from that product.

The way of "twisting" the $K(\pi_i, i)$ over P_{i-1} to form P_i is specified, for reasons that would need a somewhat lengthy explanation, by a mapping $k_{i-1}: P_{i-1} \to K(\pi_i, i+1)$. As we know, each such map k_{i-1} can be represented by a cocycle in $Z^{i+1}(P_{i-1}; \pi_i)$, and since it really suffices to know k_{i-1} only up to homotopy, it is enough to specify it by an element of the cohomology group $H^{i+1}(P_{i-1}; \pi_i)$. This element is also commonly denoted by k_{i-1} and called the $(i-1)$st *Postnikov class*[22] of Y.

The beauty of the thing is that P_i, which conveys, in a sense, complete information about the homotopy of Y up to dimension i, can be reconstructed from the *discrete* data given by $\pi_2, k_2, \pi_3, k_3, \ldots, k_{i-1}, \pi_i$.

For our purposes, a key fact, already mentioned in the outline section, is the following:

PROPOSITION 4.4. *If X is a CW-complex of dimension at most i, and Y is a 1-connected CW-complex, then there is a bijection between $[X, Y]$ and $[X, P_i]$ (which is induced by composition with the map φ_i).*

Simplicial Postnikov systems. To use Postnikov systems algorithmically, we represent the objects by simplicial sets and maps (this was actually the setting in which Postnikov originally defined them). Concretely, we will use the so-called *pullback representation* (as opposed to some other sources, where a *twisted product* representation can be found—but these representations can be converted into one another without much difficulty).

We let $K(\pi, n)$ and $E(\pi, n)$ stand for the particular simplicial sets as in Section 4.2. The i-th stage P_i of the Postnikov system for Y is represented as a simplicial subset of the product $P_{i-1} \times E_i \subseteq E_0 \times E_1 \times \cdots \times E_i$, where $E_j := E(\pi_j, j)$. An m-simplex of P_i can thus be written as $(\sigma^0, \ldots, \sigma^{i-1}, \sigma^i)$, where $\sigma^j \in C^j(\Delta^m, \pi_j)$ is a simplex of E_j. It will also be convenient to write $(\sigma^0, \ldots, \sigma^{i-1}) \in P_{i-1}$ as $\boldsymbol{\sigma}$ and thus write a simplex of P_i in the form $(\boldsymbol{\sigma}, \sigma^i)$.

We will introduce the following convenient abbreviations for the Eilenberg–MacLane spaces appearing in the Postnikov system (the first of them is quite standard):

$$K_{i+1} := K(\pi_i, i+1),$$
$$L_i := K(\pi_i, i).$$

The simplicial version of (a representative of) the Postnikov class k_{i-1} is a simplicial map

$$k_{i-1} \in \mathrm{SMap}(P_{i-1}, K_{i+1}).$$

Since K_{i+1} is an Eilenberg–MacLane space, we can, and will, also represent k_{i-1} as a cocycle in $Z^{i+1}(P_{i-1}, \pi_i)$.

In this version, instead of "twisting", k_{i-1} is used to "cut out" P_i from the product $P_{i-1} \times E_i$, as follows:

$$P_i := \{(\boldsymbol{\sigma}, \sigma^i) \in P_{i-1} \times E_i : k_{i-1}(\boldsymbol{\sigma}) = \delta\sigma^i\}, \tag{3}$$

where $\delta: E_i \to K_{i+1}$ is given by the coboundary operator, as was described above after the definition of $E(\pi, n)$. The map $p_i: P_i \to P_{i-1}$ in this setting is simply the projection forgetting the last coordinate, and so it need not be specified explicitly.

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[22]In the literature, *Postnikov factor* or *Postnikov invariant* are also used with the same meaning.

We remark that this describes what the simplicial Postnikov system looks like, but it does not say when it really is a Postnikov system for Y. We won't discuss the appropriate conditions here; we will just accept a guarantee of the algorithm in Theorem 4.5 below, that it computes a valid Postnikov system for Y, and in particular, such that it fulfills Proposition 4.4.

We also state another important property of the stages P_i of the simplicial Postnikov system of a simply connected Y: they are Kan simplicial sets (see, e.g. [Brown (jun.) 1957]). Thus, for any simplicial set X, there is a bijection between the set of simplicial maps $X \to P_i$ modulo simplicial homotopy and the set of homotopy classes of continuous maps between the geometric realizations. Slightly abusing notation, we will denote both sets by $[X, P_i]$.

Computing Postnikov systems. Let Y be a 1-connected locally effective simplicial set. For our purposes, we shall say that Y has a *locally effective (truncated) Postnikov system with n stages* if the following are available:

— The homotopy groups $\pi_i = \pi_i(Y)$, $2 \le i \le n$ (provided with a fully effective representation).[23]
— The stages P_i and the Eilenberg–MacLane spaces K_{i+1} and L_i, $i \le n$, as locally effective simplicial sets.
— The maps $\varphi_i \colon Y \to P_i$, $p_i \colon P_i \to P_{i-1}$, and $k_{i-1} \colon P_{i-1} \to K_{i+1}$, $i \le n$, as locally effective simplicial maps.[24]

As a preprocessing step for our main algorithm, we need the following result:

THEOREM 4.5 ([ČADEK ET AL. 2012, THEOREM 1.2]). *There is an algorithm that, given a 1-connected simplicial set Y with finitely many nondegenerate simplices (e.g., as obtained from a finite simplicial complex) and an integer n, computes a locally effective Postnikov system with n stages for Y.*

REMARKS 4.6.

(1) *In the case with π_2 through π_n all finite, each P_i, $i \le n$, has finitely many simplices in the relevant dimensions, and so a locally effective Postnikov system can be represented simply by a lookup table. Brown [Brown (jun.) 1957] gave an algorithm for computing a simplicial Postnikov system in this restricted setting.*

(2) *The algorithm for proving Theorem 4.5 combines the basic construction of Brown with the framework of* objects with effective homology *(as explained, e.g., in [Rubio and Sergeraert 2012]). We remark that the algorithm works under the weaker assumption that Y is a simplicial set with effective homology, possibly with infinitely many nondegenerate simplices.*[25]

(3) *In [Čadek et al. 2012], it is shown that for fixed n, the construction of the first n stages of a Postnikov system for Y can actually carried out in time polynomial in*

---

[23]For our algorithm, it suffices to have the π_i represented as abstract Abelian groups, with no meaning attached to the elements. However, if we ever wanted to translate the elements of $[X, P_i]$ to actual maps $X \to Y$, we would need the generators of each π_i represented as actual mappings, say simplicial, $S^i \to Y$.

[24]As explained above, the map k_{i-1} is represented by an $(i+1)$-dimensional cocycle on P_{i-1}; thus, we assume that we have an algorithm that, given an $(i+1)$-simplex $\sigma \in P_{i-1}$, returns the value $k_{i-1}(\sigma) \in \pi_i$. Let us also remark that, by unwrapping the definition, we get that the input $\sigma \in P_{i-1}$ for k_{i-1} means a labeling of the faces of Δ^{i+1} of all dimensions up to $i-1$, where j-faces are labeled by elements of π_j. Readers familiar with obstruction theory may see some formal similarity here: the $(i-1)$st obstruction determines extendability of a map defined on the i-skeleton to the $(i+1)$-skeleton, after possibly modifying the map on the interiors of the i-simplices.

[25]We also note that, for Y with only finitely many nondegenerate simplices, the maps $\varphi_i \colon Y \to P_i$ can be represented by finite lookup tables, so we do not need to require specifically that they be locally effective.

the size (number of nondegenerate simplices) of Y. The (lengthy) analysis, and even the precise formulation of this result, involve some technical subtleties and depend on the notions locally polynomial-time simplicial sets *and* objects with polynomial-time homology*, which refine the framework of locally effective simplicial sets and of objects with effective homology and were developed in [Krčál et al. 2013; Čadek et al. 2012]. We refer to [Čadek et al. 2012] for a detailed treatment.*

An example: the Steenrod square Sq^2. The Postnikov classes k_i are not at all simple to describe explicitly, even for very simple spaces. As an illustration, we present an example, essentially following [Steenrod 1947], where an explicit description is available: this is for $Y = S^d$, $d \geq 3$, and it concerns the first k_i of interest, namely, k_d. It corresponds to the *Steenrod square* Sq^2 in cohomology, which Steenrod [Steenrod 1947] invented for the purpose of classifying all maps from a $(d+1)$-dimensional complex K into S^d—a special case of the problem treated in our paper.

For concreteness, let us take $d = 3$. Then k_3 receives as the input a labeling of the 3-faces of Δ^5 by elements of $\pi_3(S^3)$, i.e., integers (the lower-dimensional faces are labeled with 0s since $\pi_j(S^3) = 0$ for $j \leq 2$), and it should return an element of $\pi_4(S^3) \cong \mathbb{Z}_2$. Combinatorially, we can thus think of the input as a function $c \colon \binom{\{0,1,\dots,5\}}{4} \to \mathbb{Z}$, and the value of k_3 turns out to be

$$\sum_{\sigma,\tau} c(\sigma)c(\tau) \; (\mathrm{mod}\ 2),$$

where the sum is over three pairs of 4-tuples σ, τ as indicated in the following picture (σ consists of the circled points and τ of the points marked by squares—there is always a two-point overlap):
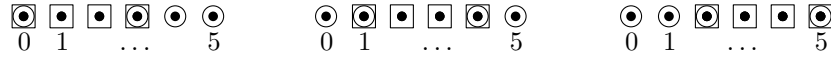
This illustrates the nonlinearity of the Postnikov classes.

5. DEFINING AND IMPLEMENTING THE GROUP OPERATION ON $[X, P_I]$

We recall that the device that allows us to handle the generally infinite set $[X, Y]$ of homotopy classes of maps, under the dimension/connectedness assumption of Theorem 1.1, is an Abelian group structure. We will actually use the group structure on the sets $[X, P_i]$, $d \leq i \leq 2d - 2$. These will be computed inductively, starting with $i = d$ (this is the first nontrivial one).

Such a group structure with good properties exists, and is determined uniquely, because P_i may have nonzero homotopy groups only in dimensions d through $2d-2$; these are standard topological considerations, which we will review in Section 5.1 below.

However, we will need to work with the underlying binary operation $\boxplus_{i*}$ on the level of representatives, i.e., simplicial maps in $\mathrm{SMap}(X, P_i)$. This operation lacks some of the pleasant properties of a group—e.g., it may fail to be associative. Here considerable care and attention to detail seem to be needed, and for an algorithmic implementation, we also need to use the *Eilenberg–Zilber reduction*, a tool related to the methods of effective homology.

5.1. An H-group structure on a space

H-groups. Let P be a CW-complex. We will consider a *binary operation* on P as a *continuous map* $\mu \colon P \times P \to P$. For now, we will stick to writing $\mu(p, q)$ for the result of

applying μ to p and q; later on, we will call the operation $\boxplus$ (with a subscript, actually) and write it in the more usual way as $p \boxplus q$.

The idea of H-groups is that the binary operation μ satisfies the usual group axioms but only *up to homotopy*. To formulate the existence of an inverse in this setting, we will also need an explicit mapping $\nu \colon P \to P$, continuous of course, representing *inverse up to homotopy*.

We thus say that

(HA) μ is *homotopy associative* if the two maps $P \times P \times P \to P$ given by $(p,q,r) \mapsto \mu(p,\mu(q,r))$ and by $(p,q,r) \mapsto \mu(\mu(p,q),r)$ are homotopic;

(HN) a distinguished element $o \in P$ (basepoint, assumed to be a vertex in the simplicial set representation) is a *homotopy neutral element* if the maps $P \to P$ given by $p \mapsto \mu(o,p)$ and $p \mapsto \mu(p,o)$ are both homotopic to the identity id_P;

(HI) ν is a *homotopy inverse* if the maps $p \mapsto \mu(\nu(p),p)$ and $p \mapsto \mu(p,\nu(p))$ are both homotopic to the constant map $p \mapsto o$;

(HC) μ is *homotopy commutative* if μ is homotopic to μ' given by $\mu'(p,q) := \mu(q,p)$.

An *Abelian H-group* thus consists of P, o, μ, ν as above satisfying (HA), (HN), (HI), and (HC).

Of course, every Abelian topological group is also an Abelian H-group. A basic example of an H-group that is typically not a group is the *loop space* ΩY of a topological space Y (see, e.g. [Hatcher 2001, Section 4.3]). For readers familiar with the definition of the fundamental group $\pi_1(Y)$, it suffices to say that ΩY is like the fundamental group but *without* factoring the loops according to homotopy.

We also define an *H-homomorphism* of an H-group (P_1, o_1, μ_1, ν_1) into an H-group (P_2, o_2, μ_2, ν_2) in a natural way, as a continuous map $h \colon P_1 \to P_2$ with $h(o_1) = o_2$ and such that the two maps $(x,y) \mapsto h(\mu_1(x,y))$ and $(x,y) \mapsto \mu_2(h(x),h(y))$ are homotopic.

A group structure on homotopy classes of maps. For us, an H-group structure on P is a device for obtaining a group structure on the set $[X,P]$ of homotopy classes of maps. In a similar vein, an H-homomorphism $P_1 \to P_2$ yields a group homomorphism $[X,P_1] \to [X,P_2]$. Here is a more explicit statement:

FACT 5.1. *Let (P,o,μ,ν) be an Abelian H-group, and let X be a space. Let μ_*, ν_* be the operations defined on continuous maps $X \to P$ by pointwise composition with μ, ν, respectively (i.e., $\mu_*(f,g)(x) := \mu(f(x),g(x))$, $\nu_*(f)(x) := \nu(f(x))$). Then μ_*, ν_* define an Abelian group structure on the set of homotopy classes $[X,P]$ by $[f]+[g] := [\mu_*(f,g)]$ and $-[f] := [\nu_*(f)]$ (with the zero element given by the homotopy class of the map sending all of X to o).*

If $h \colon P_1 \to P_2$ is an H-homomorphism of Abelian H-groups (P_1,o_1,μ_1,ν_1) and (P_2,o_2,μ_2,ν_2), then the corresponding map h_*, sending a continuous map $f \colon X \to P_1$ to $h_*(f) \colon X \to P_2$ given by $h_*(f)(x) := h(f(x))$, induces a homomorphism $[h_*] \colon [X,P_1] \to [X,P_2]$ of Abelian groups.

This fact is standard, and also entirely routine to prove. We will actually work mostly with a simplicial counterpart (which is proved in exactly the same way, replacing topological notions with simplicial ones everywhere). Namely, if X is a simplicial set, P is a Kan simplicial set, and μ, ν are simplicial maps, then by a composition as above, we obtain maps $\mu_* \colon \mathrm{SMap}(X,P) \times \mathrm{SMap}(X,P) \to \mathrm{SMap}(X,P)$ and $\nu_* \colon \mathrm{SMap}(X,P) \to \mathrm{SMap}(X,P)$, which induce an Abelian group structure on the set $[X,P]$ of simplicial homotopy classes. Similarly, if $h \colon P_1 \to P_2$ is a simplicial H-homomorphism (with everything else in sight simplicial), then $h_* \colon \mathrm{SMap}(X,P_1) \to \mathrm{SMap}(X,P_2)$ defines a homomorphism $[h_*] \colon [X,P_1] \to [X,P_2]$.

Moreover, if μ, ν are locally effective (i.e., given $\sigma, \tau \in P$, we can evaluate $\mu(\sigma, \tau)$ and $\nu(\sigma)$) and X has finitely many nondegenerate simplices, then μ_*, ν_* are locally effective as well. Indeed, as we have remarked, simplicial maps $X \to P$ are finitely representable objects, and we will have them represented by vectors of cochains.

Thus, under the above conditions, we have the Abelian group $[X, P_i]$ semi-effectively represented, where the set of representatives is $\mathrm{SMap}(X, P)$. Similarly, if $h \colon P_1 \to P_2$ is locally effective and X is has finitely many nondegenerate simplices, then $h_* \colon \mathrm{SMap}(X, P_1) \to \mathrm{SMap}(X, P_2)$ is locally effective, too.

A canonical H-group structure from connectivity. In our algorithm, the existence of a suitable H-group structure on P_i follows from the fact that P_i has nonzero homotopy groups only in the range from d to i, $i \leq 2d - 2$.

LEMMA 5.2. *Let $d \geq 2$ and let P be a $(d-1)$-reduced CW complex with distinguished vertex (basepoint) o, and with nonzero $\pi_i(P)$ possibly occurring only for $i = d, d+1, \ldots, 2d-2$. Then there are μ and ν such that (P, o, μ, ν) is an Abelian H-group, and moreover, o is a* strictly neutral *element, in the sense that $\mu(o, p) = \mu(p, o) = p$ (equalities, not only homotopy).*

Moreover, if μ' is any continuous binary operation on P with o as a strictly neutral element, then $\mu' \sim \mu$ by a homotopy stationary on the subspace $P \vee P := (P \times \{o\}) \cup (\{o\} \times P)$ (and, in particular, every such μ' automatically satisfies (HA), (HC), and (HI) with a suitable ν').

This lemma is essentially well-known, and the necessary arguments appear, e.g., in Whitehead [Whitehead 1978]. We nonetheless sketch a proof, because we are not aware of a specific reference for the lemma as stated, and also because it sheds some light on how the assumption of $(d-1)$-connectedness of Y in Theorem 1.1 is used.

The proof is based on the repeated application of the following basic fact (which is a baby version of obstruction theory and can be proved by induction of the dimension of the cells on which the maps or homotopies have to be extended).

FACT 5.3. *Suppose that X and Y are CW complexes, $A \subseteq X$ is a subcomplex, and assume that there is some integer k such that all cells in $X \setminus A$ have dimension at least k and that $\pi_i(Y) = 0$ for all $i \geq k - 1$. Then the following hold:*

 (i) *If $f \colon A \to Y$ is a continuous map, then there exists an extension $f' \colon X \to Y$ of f (i.e., $f'|_A = f$).*
(ii) *If $f \sim g \colon A \to Y$ are homotopic maps, and if $f', g' \colon X \to Y$ are arbitrary extensions of f and of g, respectively, then $f' \sim g'$ (by a homotopy extending the given one on A).*

PROOF OF LEMMA 5.2. This proof is the *only* place where it is important that we work with CW-complexes, as opposed to simplicial sets. This is because the *product* of CW-complexes is defined differently from the product of simplicial sets. In the product of CW-complexes, an i-cell times a j-cell yields an $(i+j)$-cell (and nothing else), while in products of simplicial sets, simplices of problematic intermediate dimensions appear.

Let $\varphi \colon P \vee P \to P$ be the *folding map* given by $\varphi(o, p) := p$, $\varphi(p, o) := p$, $p \in P$. Thus, the strict neutrality of o just means that μ extends φ, and we can employ Fact 5.3.

Namely, all cells in $(P \times P) \setminus (P \vee P)$ have dimension at least $2d$, and $\pi_i(P) = 0$ for $i \geq 2d - 1$. Thus, φ can be extended to some $\mu \colon P \times P \to P$, uniquely up to homotopy stationary on $P \vee P$.

From the homotopy uniqueness we get the homotopy commutativity (HC) immediately (for free). Indeed, if we define $\mu'(p, q) := \mu(q, p)$, then the homotopy uniqueness applies and yields $\mu' \sim \mu$. The homotopy associativity (HA) is also simple. Let $\psi_1, \psi_2 \colon P^3 \to P$ be given by $\psi_1(p, q, r) := \mu(\mu(p, q), r)$ and $\psi_2(p, q, r) := \mu(p, \mu(q, r))$. Then $\psi_1 = \psi_2$ on the subspace $P \vee P \vee P := (P \times \{o\} \times \{o\}) \cup (\{o\} \times P \times \{o\}) \cup (\{o\} \times \{o\} \times P)$.

Since all cells in $(P \times P \times P) \setminus (P \vee P \vee P)$ are of dimension at least $2d$, Fact 5.3 gives $\psi_1 \sim \psi_2$.

The existence of a homotopy inverse is not that simple, and actually, we won't need it (since we will construct an inverse explicitly). For a proof, we thus refer to the literature: every 0-connected CW-complex with an operation satisfying (HA) and (HN) also satisfies (HI); see, e.g., [Whitehead 1978, Theorem X.2.2, p. 461]. □

5.2. A locally effective H-group structure on the Postnikov stages

Now we are in the setting of Theorem 1.1; in particular, Y is a $(d-1)$-connected simplicial set. Let P_i, $i \geq 0$, denote the ith stage of a locally effective simplicial Postnikov system for Y, as in Section 4; we will consider only the first $2d-2$ stages. Since Y is $(d-1)$-connected, P_0 through P_{d-1} are trivial (one-point), and each P_i is $(d-1)$-reduced. We will occasionally refer to the $P_d, P_{d+1}, \ldots, P_{2d-2}$ as the *stable stages* of the Postnikov system.

By Lemma 5.2, we know that the stable stages possess a (canonical) H-group structure. But we need to define the underlying operations on P_i concretely as simplicial maps and, mainly, make them effective. Since P_i is typically an infinite object, we will have just *local effectivity*, i.e., the operations can be evaluated algorithmically on any given pair of simplices.

From now on, we will denote the "addition" operation on P_i by $\boxplus_i$, and use the infix notation $\boldsymbol{\sigma} \boxplus_i \boldsymbol{\tau}$. Similarly we write $\boxminus_i \boldsymbol{\sigma}$ for the "inverse" of $\boldsymbol{\sigma}$. For a more convenient notation, we also introduce a *binary* version of $\boxminus_i$ by setting $\boldsymbol{\sigma} \boxminus_i \boldsymbol{\tau} := \boldsymbol{\sigma} \boxplus_i (\boxminus_i \boldsymbol{\tau})$.

Preliminary considerations. We recall that an m-simplex of P_i is written as $(\sigma^0, \sigma^1, \ldots, \sigma^i)$, with $\sigma^i \in C^i(\Delta^m; \pi_i(Y))$. Thus, its components are cochains. One potential source of confusion is that we already *have* a natural addition of such cochains defined; they can simply be added componentwise, as effectively as one might ever wish.

However, this *cannot* be used as the desired addition $\boxplus_i$. The reason is that the Postnikov classes k_{i-1} are generally nonlinear, and thus k_{i-1} is typically not a homomorphism with respect to cochain addition. In particular, we recall that P_i was defined as the subset of $P_{i-1} \times E_i$ "cut out" by k_{i-1}, i.e., via $k_{i-1}(\boldsymbol{\sigma}) = \delta\sigma^i$, where $\boldsymbol{\sigma} = (\sigma^0, \ldots, \sigma^{i-1})$. Therefore, P_i is usually not even closed under the cochain addition.

Our approach to define a suitable operation $\boxplus_i$ is inductive. Suppose that we have already defined $\boxplus_{i-1}$ on P_{i-1}. Then we will first define $\boxplus_i$ on special elements of P_i of the form $(\boldsymbol{\sigma}, 0)$, by just adding the $\boldsymbol{\sigma}$'s according to $\boxplus_{i-1}$ and leaving 0 in the last component.

Another important special case of $\boxplus_i$ is on elements of the form $(\boldsymbol{\sigma}, \sigma^i) \boxplus_i (\mathbf{0}, \tau^i)$. In this case, in spite of the general warning above against the cochain addition, the last components *are* added as cochains: $(\boldsymbol{\sigma}, \sigma^i) \boxplus_i (\mathbf{0}, \tau^i) = (\boldsymbol{\sigma}, \sigma^i + \tau^i)$. The main result of this section constructs a locally effective $\boxplus_i$ that extends the two special cases just discussed.

Let us remark that by definition, $\boxplus_i$ and $\boxminus_i$, as simplicial maps, operate on simplices of every dimension m. However, in the algorithm, we will be using them only up to $m \leq 2d-2$, and so in the sequel we always implicitly assume that the considered simplices satisfy this dimensional restriction.

The main result on $\boxplus_i, \boxminus_i$. The following proposition summarizes everything about $\boxplus_i, \boxminus_i$ we will need.

PROPOSITION 5.4. *Let Y be a $(d-1)$-connected simplicial set, $d \geq 2$, and let $P_d, P_{d+1}, \ldots, P_{2d-2}$ be the stable stages of a locally effective Postnikov system with $2d-2$ stages for Y. Then each P_i has an Abelian H-group structure, given by locally effec-*

tive simplicial maps $\boxplus_i\colon P_i \times P_i \to P_i$ *and* $\boxminus_i\colon P_i \to P_i$*with the following additional properties:*

(a) $(\boldsymbol{\sigma},\sigma^i)\boxplus_i(\mathbf{0},\tau^i) = (\boldsymbol{\sigma},\sigma^i+\tau^i)$ *for all* $(\boldsymbol{\sigma},\sigma^i) \in P_i$ *and* $\tau^i \in L_i$ *(we recall that* $L_i = K(\pi_i,i)$*.)*
(b) $\boxminus_i(\mathbf{0},\sigma^i) = (\mathbf{0},-\sigma^i)$ *for all* $\sigma^i \in L_i$.
(c) *The projection* $p_i\colon P_i \to P_{i-1}$ *is a strict homomorphism, i.e.,* $p_i(\boldsymbol{\sigma}\boxplus_i\boldsymbol{\tau}) = p_i(\boldsymbol{\sigma})\boxplus_{i-1} p_i(\boldsymbol{\tau})$ *and* $p_i(\boxminus_i\boldsymbol{\sigma}) = \boxminus_{i-1}p_i(\boldsymbol{\sigma})$ *for all* $\boldsymbol{\sigma},\boldsymbol{\tau} \in P_i$.
(d) *If, moreover,* $i < 2d-2$*, then the Postnikov class* $k_i\colon P_i \to K_{i+2}$ *is an* H-*homomorphism (with respect to* $\boxplus_i$ *on* P_i *and the simplicial group operation* $+$*, addition of cocycles, on* K_{i+2}*).*

As was announced above, the proof of this proposition proceeds by induction on i. The heart is an explicit and effective version of (d), which we state and prove as a separate lemma.

LEMMA 5.5. *Let* P_i *be a* $(d-1)$-*connected simplicial set, and let* $\mathbf{0},\boxplus_i,\boxminus_i$ *be an Abelian* H-*group structure on* P_i*, with* $\boxplus_i,\boxminus_i$ *locally effective. Let* $k_i\colon P_i \to K_{i+2}$ *be a simplicial map, where* $i < 2d-2$*. Then there is a locally effective simplicial map* $A_i\colon P_i \to E_{i+1}$ *such that, for all simplices* $\boldsymbol{\sigma},\boldsymbol{\tau}$ *of equal dimension,* $A_i(\boldsymbol{\sigma},\mathbf{0}) = A_i(\mathbf{0},\boldsymbol{\tau}) = 0$*, and*

$$k_i(\boldsymbol{\sigma}\boxplus_i\boldsymbol{\tau}) = k_i(\boldsymbol{\sigma}) + k_i(\boldsymbol{\tau}) + \delta A_i(\boldsymbol{\sigma},\boldsymbol{\tau}).$$

We recall that $\delta\colon E_{i+1} \to K_{i+2}$ is the simplicial map induced by the coboundary operator, and that a simplicial map $f\colon P_i \to K_{i+2}$ is nullhomotopic iff it is of the form $\delta \circ F$ for some $F\colon P_i \to E_{i+1}$ (see Lemma 4.3). Therefore, the map A_i is an "effective witness" for the nullhomotopy of the map $(\boldsymbol{\sigma},\boldsymbol{\tau}) \mapsto k_i(\boldsymbol{\sigma}\boxplus_i\boldsymbol{\tau}) - k_i(\boldsymbol{\sigma}) - k_i(\boldsymbol{\tau})$, and so it shows that k_i is an H-homomorphism.

We postpone the proof of the lemma, and prove the proposition first.

PROOF OF PROPOSITION 5.4. As was announced above, we proceed by induction on i. As an inductive hypothesis, we assume that, for some $i < 2d-2$, locally effective simplicial maps $\boxplus_i,\boxminus_i$ providing an H-group structure on P_i have been defined satisfying (a)–(c) in the proposition.

This inductive hypothesis is satisfied in the base case $i = d$: in this case we have $P_d = L_d$, and $\boxplus_d$ and $\boxminus_d$ are the addition and additive inverse of cocycles (under which L_d is even a simplicial Abelian group). Then (a),(b) obviously hold and (c) is void.

In order to carry out the inductive step from i to $i+1$, we first apply Lemma 5.5 for P_i, $\boxplus_i$, and k_i, which yields a locally effective simplicial map $A_i\colon P_i \times P_i \to E_{i+1}$ with $A_i(\boldsymbol{\sigma},\mathbf{0}) = A_i(\mathbf{0},\boldsymbol{\tau}) = 0$ and $k_i(\boldsymbol{\sigma}\boxplus_i\boldsymbol{\tau}) = k_i(\boldsymbol{\sigma}) + k_i(\boldsymbol{\tau}) + \delta A_i(\boldsymbol{\sigma},\boldsymbol{\tau})$, for all $\boldsymbol{\sigma},\boldsymbol{\tau}$. As was remarked after the lemma, this implies that k_i is an H-homomorphism with respect to $\boxplus_i$.

Next, using A_i, we define the operations $\boxplus_{i+1},\boxminus_{i+1}$ on P_{i+1}. We set

$$(\boldsymbol{\sigma},\sigma^{i+1})\boxplus_{i+1}(\boldsymbol{\tau},\tau^{i+1}) := (\boldsymbol{\sigma}\boxplus_i\boldsymbol{\tau},\omega^{i+1}), \quad \text{where } \omega^{i+1} := \sigma^{i+1} + \tau^{i+1} + A_i(\boldsymbol{\sigma},\boldsymbol{\tau}). \quad (4)$$

Why is $\boxplus_{i+1}$ simplicial? Since $\boxplus_i$ is simplicial, it suffices to consider the last component, and this is a composition of simplicial maps, namely, of projections, A_i, and the operation $+$ in the simplicial group E_{i+1}. Clearly, $\boxplus_{i+1}$ is also locally effective.

We also need to check that P_{i+1} is closed under this $\boxplus_{i+1}$. We recall that, for $\boldsymbol{\sigma} \in P_i$, the condition for $(\boldsymbol{\sigma},\sigma^{i+1}) \in P_{i+1}$ is $k_i(\boldsymbol{\sigma}) = \delta\sigma^{i+1}$. Using this condition for $(\boldsymbol{\sigma},\sigma^{i+1}),(\boldsymbol{\tau},\tau^{i+1}) \in P_{i+1}$, together with $\boldsymbol{\sigma}\boxplus_i\boldsymbol{\tau} \in P_i$ (inductive assumption), and the property of k_i above, we calculate $k_i(\boldsymbol{\sigma}\boxplus_i\boldsymbol{\tau}) = k_i(\boldsymbol{\sigma}) + k_i(\boldsymbol{\tau}) + \delta A_i(\boldsymbol{\sigma},\boldsymbol{\tau}) = \delta\sigma^{i+1} + \delta\tau^{i+1} + \delta A_i(\boldsymbol{\sigma},\boldsymbol{\tau}) = \delta\omega^{i+1}$, and thus $(\boldsymbol{\sigma},\sigma^{i+1})\boxplus_{i+1}(\boldsymbol{\tau},\tau^{i+1}) \in P_{i+1}$ as needed.

Part (a) of the proposition for $\boxplus_{i+1}$ follows from (4) and the property $A_i(\mathbf{0}, \boldsymbol{\tau}) = 0 = A_i(\boldsymbol{\sigma}, \mathbf{0})$. In particular, $(\mathbf{0}, 0)$ is a strictly neutral element for $\boxplus_{i+1}$.

Moreover, as a continuous map, $\boxplus_{i+1}$ fulfills the assumptions on μ' in Lemma 5.2, and thus it satisfies the axioms of an Abelian H-group operation.

Next, we define the inverse operation $\boxminus_{i+1}$ by

$$\boxminus_{i+1}(\boldsymbol{\sigma}, \sigma^{i+1}) := (\boxminus_i \boldsymbol{\sigma}, -\sigma^{i+1} - A_i(\boldsymbol{\sigma}, \boxminus_i \boldsymbol{\sigma})).$$

It is simplicial for the same reason as that for $\boxplus_{i+1}$, and by a computation similar to the one for $\boxplus_{i+1}$ above, we verify that P_{i+1} is closed under $\boxminus_{i+1}$.

To verify that this $\boxminus_{i+1}$ indeed defines a homotopy inverse to $\boxplus_{i+1}$, we check that it actually is a *strict* inverse. Inductively, we assume $\boldsymbol{\sigma} \boxminus_i \boldsymbol{\sigma} = \mathbf{0}$ for all $\boldsymbol{\sigma} \in P_i$, and from the formulas defining $\boxplus_{i+1}$ and $\boxminus_{i+1}$, we check that $(\boldsymbol{\sigma}, \sigma^{i+1}) \boxminus_{i+1} (\boldsymbol{\sigma}, \sigma^{i+1}) = (\mathbf{0}, 0)$. Another simple calculation yields (b) for $\boxminus_{i+1}$.

Part (c) for $\boxplus_{i+1}$ and $\boxminus_{i+1}$ follows from the definitions and from $A_i(\mathbf{0}, \mathbf{0}) = 0$. This finishes the induction step and proves the proposition. $\square$

PROOF OF LEMMA 5.5. Here we will use ("locally") some terminology concerning chain complexes (e.g., chain homotopy, homomorphism of chain complexes), for which we refer to the literature (standard textbooks, say [Hatcher 2001]).

First we define the *nonadditivity map* $a_i \colon P_i \times P_i \to K_{i+2}$ by

$$a_i(\boldsymbol{\sigma}, \boldsymbol{\tau}) := k_i(\boldsymbol{\sigma} \boxplus_i \boldsymbol{\tau}) - k_i(\boldsymbol{\sigma}) - k_i(\boldsymbol{\tau}).$$

(Thus, the map a_i measures the failure of k_i to be strictly additive with respect to $\boxplus_i$.) We want to show that $a_i = \delta A_i$ for a locally effective A_i.

Let us remark that the *existence* of A_i can be proved by an argument similar to the one in Lemma 5.2. That argument works for CW-complexes, and as was remarked in the proof of that lemma, it is essential that the product of an i-cell and a j-cell is an $(i+j)$-cell and *nothing else*. For simplicial sets the product is defined differently, and if we consider $P_i \times P_i$ as a simplicial set, we do get simplices of "unpleasant" intermediate dimensions there.

We will get around this using the *Eilenberg–Zilber reduction* (which is also one of the basic tools in effective homology—but we won't need effective homology directly); here, we follow the exposition in [Gonzalez-Diaz and Real 2005] (see also [Rubio and Sergeraert 2012, Sections 7.8 and 8.2]). Loosely speaking, it will allow us to convert the setting of the simplicial set $P_i \times P_i$ to a setting of a tensor product of chain complexes, where only terms of the "right" dimensions appear.

We note that A_i is defined on an infinite object, so we cannot compute it globally—we need a local algorithm for evaluating it, yet its answers have to be globally consistent over the whole computation.

First we present the Eilenberg–Zilber reduction for an arbitrary simplicial set P with basepoint (and single vertex) o. The reduction consists of three locally effective maps[26] AW, EML and SHI that fit into the following diagram:

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[26]The acronyms stand for the mathematicians *Alexander* and *Whitney*, *Eilenberg* and *Mac Lane*, and *Shih*, respectively.

Here $C_*(\cdot)$ denotes the (normalized) chain complex of a simplicial set, with integer coefficients (so we omit the coefficient group in the notation). For brevity, chains of all dimensions are collected into a single structure (whence the star subscript), and $\otimes$ is the tensor product. Thus, $(C_*(P) \otimes C_*(P))_n = \bigoplus_{i+j=n} C_i(P) \otimes C_j(P)$. The operators AW and EML are homomorphisms of chain complexes, while SHI is a *chain homotopy* operator raising the degree by $+1$. Thus, for each n, we have $\mathrm{AW}_n \colon C_n(P \times P) \to (C_*(P) \otimes C_*(P))_n$, $\mathrm{EML}_n \colon (C_*(P) \otimes C_*(P))_n \to C_n(P \times P)$, and $\mathrm{SHI}_n \colon C_n(P \times P) \to C_{n+1}(P \times P)$.

We refer to [Gonzalez-Diaz and Real 2005, pp. 1212–1213] for explicit formulas for AW and EML in terms of the face and degeneracy operators. We give only the formula for SHI, since A_i will be defined using SHI_{i+1}, and we summarize the properties of AW, EML, SHI relevant for our purposes.

The operator SHI_n operates on n-chains on $P \times P$. The formula given below specifies its values on the "basic" chains of the form (σ^n, τ^n); here σ^n, τ^n are n-simplices of P, but (σ^n, τ^n) is interpreted as the chain with coefficient 1 on (σ^n, τ^n) and 0 elsewhere. The definition then extends to arbitrary chains by linearity.

Let p and q be non-negative integers. A (p,q)-*shuffle* (α, β) is a partition

$$\{\alpha_1 < \cdots < \alpha_p\} \cup \{\beta_1 < \cdots < \beta_q\}$$

of the set $\{0, 1, \ldots, p+q-1\}$. Put

$$\mathrm{sig}(\alpha, \beta) = \sum_{i=1}^{p} (\alpha_i - i + 1).$$

Let $\gamma = \{\gamma_i, \ldots, \gamma_r\}$ be a set of integers. Then s_γ denotes the compositions of the degeneracy operators $s_{\gamma_1} \ldots s_{\gamma_r}$ (the s_m are the degeneracy operators of P, and ∂_m are its face operators). The operator SHI is defined by

$$\mathrm{SHI}(\sigma^0, \tau^0) = 0,$$

$$\mathrm{SHI}(\sigma^m, \tau^m) = \sum_{T(m)} (-1)^{\epsilon(\alpha,\beta)} (s_{\bar{\beta}+\bar{m}} \partial_{m-q+1} \cdots \partial_m \sigma^m, s_{\alpha+\bar{m}} \partial_{\bar{m}} \cdots \partial_{m-q-1} \tau^m),$$

where $T(m)$ is the set of all $(p+1, q)$-shuffles such that $0 \le p + q \le m - 1$,

$$\bar{m} = m - p - q, \quad \epsilon(\alpha, \beta) = \bar{m} - 1 + \mathrm{sig}(\alpha, \beta),$$

$$\alpha + \bar{m} = \{\alpha_1 + \bar{m}, \ldots, \alpha_{p+1} + \bar{m}\}, \quad \bar{\beta} + \bar{m} = \{\bar{m} - 1, \beta_1 + \bar{m}, \ldots, \beta_q + \bar{m}\}.$$

The above formula shows that SHI_n is locally effective, in the sense that, if a chain $c_n \in C_n(P \times P)$ is given in a locally effective way, by an algorithm that can evaluate the coefficient for each given n-simplex of $P \times P$, then a similar algorithm is available for the $(n+1)$-chain $\mathrm{SHI}_n(c_n)$ as well.

The first fact we will need is that for every n, the maps satisfy the following identity (where ∂ denotes the boundary operator in $C_*(P \times P)$):

$$\mathrm{id}_{C_n(P \times P)} - \mathrm{EML}_n \circ \mathrm{AW}_n = \mathrm{SHI}_{n-1} \circ \partial + \partial \circ \mathrm{SHI}_n. \tag{5}$$

This identity says that SHI_n is a chain homotopy between $\mathrm{EML}_n \circ \mathrm{AW}_n$ and the identity on $C_n(P \times P)$.

The second fact, which follows directly from the formulas in [Gonzalez-Diaz and Real 2005], is that the operators EML and SHI behave well with respect to the basepoint o and its degeneracies, in the following sense: For every n and for every (nondegenerate) n-dimensional simplex τ^n of P (regarded as a chain),

$$\mathrm{EML}_n(o \otimes \tau^n) = \pm(o^n, \tau^n), \qquad \mathrm{EML}_n(\tau^n \otimes o) = \pm(\tau^n, o^n), \tag{6}$$

where o^n is the (unique) n-dimensional degenerate simplex obtained from o. The images in (6) lie in the subgroup $C_n(P \vee P) \subseteq C_n(P \times P)$. Moreover, the operator SHI_n maps $C_n(P \vee P)$ into $C_{n+1}(P \vee P)$, i.e., the chains $\mathrm{SHI}(o^n, \tau^n)$ and $\mathrm{SHI}(\tau^n, o^n)$ are linear combinations of simplices of the form (o^{n+1}, σ^{n+1}) and (σ^{n+1}, o^{n+1}), respectively, where σ^{n+1} ranges over certain $(n+1)$-dimensional simplices of P.

We now apply this to $P = P_i$ (with basepoint $\mathbf{0}$). We consider the nonadditivity map a_i as an $(i+2)$-cocycle on $P_i \times P_i$, which can be regarded as a homomorphism $a_i \colon C_{i+2}(P_i \times P_i) \to \pi_{i+1}$. If we compose this homomorphism a_i on the left with both sides of the identity (5), for $n = i+2$, we get

$$a_i \circ \mathrm{id}_{C_{i+2}(P \times P)} - a_i \circ \mathrm{EML}_{i+2} \circ \mathrm{AW}_{i+2} = a_i \circ \mathrm{SHI}_{i+1} \circ \partial + a_i \circ \partial \circ \mathrm{SHI}_{i+2}. \qquad (7)$$

Now $a_i \circ \partial = 0$ since a_i is a cocycle. Moreover, every basis element of $C_*(P_i) \otimes C_*(P_i)$ in degree $i + 2 < 2d$ is of the form $\mathbf{0} \otimes \tau^{i+2}$ or $\tau^{i+2} \otimes \mathbf{0}$ (since P_i has no nondegenerate simplices in dimensions $1, \ldots, d-1$). Such elements are taken by EML into $C_{i+1}(P \vee P)$, on which a_i vanishes because $\mathbf{0}$ is a strictly neutral element for $\boxplus_i$. Thus, $a_i \circ \mathrm{EML}_{i+2} = 0$ for $i + 2 < 2d$.

Therefore, (7) simplifies to $a_i = a_i \circ \mathrm{SHI}_{i+1} \circ \partial$. Thus, if we set $A_i := a_i \circ \mathrm{SHI}_{i+1}$, then $a_i = \delta A_i$, as desired (since applying δ to a cochain α corresponds to the composition $\alpha \circ \partial$ on the level of homomorphisms from chains into π_{i+1}). Finally, the property $A_i(\mathbf{0}, \cdot) = A_i(\cdot, \mathbf{0}) = 0$ follows because the corresponding property holds for a_i and SHI_{i+1} maps $C_{i+1}(P_i \vee P_i)$ to $C_{i+2}(P_i \vee P_i)$. $\quad\square$

5.3. A semi-effective representation of $[X, P_i]$

Now let X be a finite simplicial complex or, more generally, a simplicial set with finitely many nondegenerate simplices (as we will see, the greater flexibility offered by simplicial sets will be useful in our algorithm, even if we want to prove Theorem 1.1 only for simplicial *complexes* X).

Having the locally effective H-group structure on the stable Postnikov stages P_i, we obtain the desired locally effective Abelian group structure on $[X, P_i]$ immediately.

Indeed, according to the remarks following Fact 5.1, a simplicial map $s \colon P \to Q$ of arbitrary simplicial sets induces a map $s_* \colon \mathrm{SMap}(X, P) \to \mathrm{SMap}(X, Q)$ by composition, i.e., by $s_*(f)(\sigma) = (s \circ f)(\sigma)$ for each simplex $\sigma \in P$. If P and Q are Kan, we also get a well-defined map $[s_*] \colon [X, P] \to [X, Q]$. Moreover, if s is locally effective, then so is s_* (since X has only finitely many nondegenerate simplices). In particular, the group operations on $[X, P_i]$ are represented by locally effective maps $\boxplus_{i*} \colon \mathrm{SMap}(X, P_i) \times \mathrm{SMap}(X, P_i) \to \mathrm{SMap}(X, P_i)$ and $\boxminus_{i*} \colon \mathrm{SMap}(X, P_i) \to \mathrm{SMap}(X, P_i)$.

The cochain representation. However, we can make the algorithm considerably more efficient if we use the special structure of P_i and work with cochain representatives of the simplicial maps in $\mathrm{SMap}(X, P_i)$.

We recall from Section 4 that simplicial maps into $K(\pi, n)$ and $E(\pi, n)$ are canonically represented by cocycles and cochains, respectively. Simplicial maps $X \to P_i$ are, in particular, maps into the product $E_0 \times \cdots \times E_i$, and so they can be represented by $(i+1)$-tuples of cochains $\mathbf{c} = (c^0, \ldots, c^i)$, with $c^j \in C^j := C^j(X; \pi_j)$.

The "simplicial" definition of $\boxplus_{i*}, \boxminus_{i*}$ can easily be translated to a "cochain" definition, using the correspondence explained after Lemma 4.2. For simplicity, we describe the result concretely for the unary operation $\boxminus_{i*}$; the case of $\boxplus_{i*}$ is entirely analogous, it just would require more notation.

Thus, to evaluate $(d^0, \ldots, d^i) := \boxminus_{i*}\mathbf{c}$, we need to compute the value of d^j on each j-simplex ω of X, $j = 0, 1, \ldots, i$. To this end, we first identify ω with the standard j-simplex Δ^j via the unique order-preserving map of vertices. Then the restriction of $(c^0, \ldots, c^i)$ to ω (i.e., a labeling of the faces of ω by the elements of the appropriate

Abelian groups) can be regarded as a j-simplex $\boldsymbol{\sigma}$ of P_i. We compute $\boldsymbol{\tau} := \boxminus_j \boldsymbol{\sigma}$, again a j-simplex of P_i. The component τ^j of $\boldsymbol{\tau}$ is a j-cochain on Δ^j, i.e., a single element of π_j, and this value, finally, is the desired value of $d^j(\omega)$. For $\boxplus_{i*}$ everything works similarly.

We also get that $\mathbf{0} \in \mathrm{SMap}(X, P_i)$, the simplicial map represented by the zero cochains, is a strictly neutral element under $\boxplus_{i*}$.

We have made $[X, P_i]$ into a *semi-effectively represented Abelian group* in the sense of Section 3. The representatives are the $(i+1)$-tuples $(c^0, \ldots, c^i)$ of cochains as above. However, our state of knowledge of $[X, P_i]$ is rather poor at this point; for example, we have as yet no equality test.

A substantial amount of work still lies ahead to make $[X, P_i]$ fully effective.

6. THE MAIN ALGORITHM

In order to prove our main result, Theorem 1.1, on computing $[X, Y]$, we will prove the following statement by induction on i.

THEOREM 6.1. *Let X be a simplicial set with finitely many nondegenerate simplices, and let Y be a $(d-1)$-connected simplicial set, $d \geq 2$, for which a locally effective Postnikov system with $2d-2$ stages $P_0, \ldots, P_{2d-2}$ is available. Then, for every $i = d, d+1, \ldots, 2d-2$, a fully effective representation of $[X, P_i]$ can be computed, with the cochain representations of simplicial maps $X \to P_i$ as representatives.*

Two comments on this theorem are in order. First, unlike in Theorem 1.1, there is no restriction on $\dim X$ (the assumption $\dim X \leq 2d-2$ in Theorem 1.1 is needed only for the isomorphism $[X, Y] \cong [X, P_{2d-2}]$). Second, as was already mentioned in Section 5.3, even if we want Theorem 1.1 only for a simplicial *complex* X, we need Theorem 6.1 with simplicial *sets* X, because of recursion.

First we will (easily) derive Theorem 1.1 from Theorem 6.1.

PROOF OF THEOREM 1.1. Given a Y as in Theorem 1.1, we first obtain a fully effective Postnikov system for it with $2d-2$ stages using Theorem 4.5. Then we compute a fully effective representation of $[X, P_{2d-2}]$ by Theorem 6.1. Since Y is $(d-1)$-connected and $\dim X \leq 2d-2$, there is a bijection between $[X, Y]$ and $[X, P_{2d-2}]$ by Proposition 4.4.

It remains to implement the homotopy testing. Given two simplicial maps $f, g \colon X \to Y$, we use the locally effective simplicial map $\varphi_{2d-2} \colon Y \to P_{2d-2}$ (which is a part of a locally effective simplicial Postnikov system), and we compute the cochain representations $\mathbf{c}, \mathbf{d}$ of the corresponding simplicial maps $\varphi_{2d-2} \circ f, \varphi_{2d-2} \circ g \colon X \to P_{2d-2}$. Then we can check, using the fully effective representation of $[X, P_{2d-2}]$, whether $[\mathbf{c}] - [\mathbf{d}] = 0$ in $[X, P_{2d-2}]$. This yields the promised homotopy testing algorithm for $[X, Y]$ and concludes the proof of Theorem 1.1. □

6.1. The inductive step: An exact sequence for $[X, P_i]$

Theorem 6.1 is proved by induction on i. The base case is $i = d$ (since $P_0, \ldots, P_{d-1}$ are trivial for a $(d-1)$-connected Y), which presents no problem: we have $P_d = L_d = K(\pi_d, d)$, and so

$$[X, P_d] \cong H^d(X; \pi_d).$$

This group is fully effective, since it is the cohomology group of a simplicial set with finitely many nondegenerate simplices, with coefficients in a fully effective group. (Alternatively, we could start the algorithm at $i = 0$; then it would obtain $[X, P_d]$ at stage d as well.)

So now we consider $i > d$, and we assume that a fully effective representation of $[X, P_{i-1}]$ is available, where the representatives of the homotopy classes $[f] \in [X, P_{i-1}]$

are (cochain representations of) simplicial maps $f\colon X \to P_{i-1}$. We want to obtain a similar representation for $[X, P_i]$.

Let us first describe on an intuitive level what this task means and how we are going to approach it.

As we know, every map $g \in \mathrm{SMap}(X, P_i)$ yields a map $f = p_{i*}(g) = p_i \circ g \in \mathrm{SMap}(X, P_{i-1})$ by projection (forgetting the last coordinate in P_i). We first ask the question of *which* maps $f \in \mathrm{SMap}(X, P_{i-1})$ are obtained as such projections; this is traditionally called the *lifting problem* (and g is called a *lift* of f). Here the answer follows easily from the properties of the Postnikov system: liftability of a map f depends only on its homotopy class $[f] \in [X, P_{i-1}]$, and the liftable maps in $[X, P_{i-1}]$ are obtained as the kernel of the homomorphism $[k_{(i-1)*}]$ induced by the Postnikov class. This is very similar to the one-step extension in the setting of obstruction theory, as was mentioned in the introduction. This step will be discussed in Section 6.2.

Next, a single map $f \in \mathrm{SMap}(X, P_{i-1})$ may in general have many lifts g, and we need to describe their structure. This is reasonably straightforward to do on the level of *simplicial maps*. Namely, if $\mathbf{c} = (c^0, \ldots, c^{i-1})$ is the cochain representation of f and g_0 is a fixed lift of f, with cochain representation $(\mathbf{c}, c_0^i)$, then it turns out that all possible lifts g of f are of the form (again in the cochain representation) $(\mathbf{c}, c_0^i + z^i)$, $z^i \in Z^i(X, \pi_i) \cong \mathrm{SMap}(X, L_i)$. Thus, all of these lifts have a simple "coset structure".

This allows us to compute a list of generators of $[X, P_i]$. We also need to find all *relations* of these generators, and for this, we need to be able to test whether two maps $g_1, g_2 \in \mathrm{SMap}(X, P_i)$ are homotopic. This is somewhat more complicated, and we will develop a recursive algorithm for homotopy testing in Section 6.4.

Using the group structure, it suffices to test whether a given $g \in \mathrm{SMap}(X, P_i)$ is null-homotopic. An obvious necessary condition for this is nullhomotopy of the projection $f = p_i \circ g$, which we test recursively. Then, if $f \sim 0$, we $\boxplus_{i*}$-add a suitable nullhomotopic map to g, and this reduces the nullhomotopy test to the case where g has a cochain representation of the form $(\mathbf{0}, z^i)$, $z_i \in Z^i(X, \pi_i) \cong \mathrm{SMap}(X, L_i)$.

Now $(\mathbf{0}, z^i)$ can be nullhomotopic, as a map $X \to P_i$, by an "obvious" nullhomotopy, namely, one "moving" only the last coordinate, or in other words, induced by a nullhomotopy in $\mathrm{SMap}(X, L_i)$. But there may also be "less obvious" nullhomotopies, and it turns out that these correspond to maps $SX \to P_{i-1}$, where SX is the suspension of X defined in Section 4.1. Thus, in order to be able to test homotopy of maps $X \to P_i$, we also need to compute $[SX, P_{i-1}]$ recursively, using the inductive assumption, i.e., Theorem 6.1 for $i - 1$.

The exact sequence. We will organize the computation of $[X, P_i]$ using an *exact sequence*, a basic tool in algebraic topology and many other branches of mathematics. First we write the sequence down, including some as yet undefined symbols, and then we provide some explanations. It goes as follows:

$$[SX, P_{i-1}] \xrightarrow{[\mu_i]} [X, L_i] \xrightarrow{[\lambda_{i*}]} [X, P_i] \tag{8}$$
$$\downarrow {\scriptstyle [p_{i*}]}$$
$$[X, P_{i-1}] \xrightarrow{[k_{(i-1)*}]} [X, K_{i+1}]$$

This is a sequence of Abelian groups and homomorphisms of these groups, and exactness means that the image of each of the homomorphisms equals the kernel of the successive one.

We have already met most of the objects in this exact sequence, but for convenience, let us summarize them all.

— $[SX, P_{i-1}]$ is the group of homotopy classes of maps from the suspension into the one lower stage P_{i-1}; inductively, we may assume it to be fully effective.

— $[\mu_i]$ is a homomorphism appearing here for the first time, which will be discussed later.

— $[X, L_i] \cong H^i(X; \pi_i)$ consists of the homotopy classes of maps into the Eilenberg–MacLane space $L_i = K(\pi_i, i)$, and it is fully effective.

— $[\lambda_{i*}]$ is the homomorphism induced by the mapping $\lambda_i \colon L_i \to P_i$, the "insertion to the last component"; i.e., $\lambda_i(\sigma^i) = (\mathbf{0}, \sigma^i)$. In terms of cochain representatives, λ_{i*} sends z^i to $(\mathbf{0}, z^i)$.

— $[X, P_i]$ is what we want to compute, $[p_{i*}]$ is the projection (on the level of homotopy), and $[X, P_{i-1}]$ has already been computed, as a fully effective Abelian group.

— $[k_{(i-1)*}]$ is the homomorphism induced by the composition with the Postnikov class $k_{i-1} \colon P_{i-1} \to K_{i+1} = K(\pi_i, i+1)$.

— $[X, K_{i+1}] \cong H^{i+1}(X, \pi_i)$ are again maps into an Eilenberg–MacLane space.

Let us remark that the exact sequence (8), with some $[\mu_i]$, can be obtained by standard topological considerations from the so-called *fibration sequence* for the fibration $L_i \to P_i \to P_{i-1}$; see, e.g., [Mosher and Tangora 1968, Chap. 14].[27] However, in order to have all the homomorphisms locally effective and also to provide the locally effective "inverses" (as required in Lemma 3.5), we will need to analyze the sequence in some detail; then we will obtain a complete "pedestrian" proof of the exactness with only a small extra effort. Thus, the fibration sequence serves just as a background.

The algorithm for computing $[X, P_i]$ goes as follows.

(1) Compute $[X, P_{i-1}]$ fully effective, recursively.
(2) Compute $N_{i-1} := \ker [k_{(i-1)*}] \subseteq [X, P_{i-1}]$ (so N_{i-1} consists of all homotopy classes of liftable maps), fully effective, using Lemma 3.2 and Theorem 4.5.
(3) Compute $[SX, P_{i-1}]$ fully effective, recursively.
(4) Compute the factor group $M_i := \operatorname{coker} [\mu_i] = [X, L_i]/\operatorname{im} [\mu_i]$ using Lemma 3.3, fully effective and including the possibility of computing "witnesses for 0" as in the lemma.
(5) The exact sequence (8) can now be transformed to the short exact sequence

$$0 \to M_i \xrightarrow{\ell_i} [X, P_i] \xrightarrow{[p_{i*}]} N_{i-1} \to 0$$

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[27]Let us consider topological spaces E and B with basepoints and a pointed map $p \colon E \to B$. If p has the so-called *homotopy lifting property* (which is the case for our p_i) it is called a *fibration* and the preimage F of the base point in B is called the *fibre* of p. The sequence of maps $F \xhookrightarrow{i} E \xrightarrow{p} B$ can be prolonged into the *fibration sequence*

$$\cdots \to \Omega F \xrightarrow{\Omega i} \Omega E \xrightarrow{\Omega p} \Omega B \xrightarrow{\mu} F \xrightarrow{i} E \xrightarrow{p} B$$

of pointed maps, where, for a pointed space Y, ΩY is the space of loops starting at the base point. For spaces X and Y with base points, let $\operatorname{Map}(X, Y)_*$ denote the set of all continuous pointed maps, and let $[X, Y]_*$ be the set of (pointed) homotopy classes of these maps. Then the fibration sequence yields the sequence

$$\cdots \to \operatorname{Map}(X, \Omega F)_* \to \operatorname{Map}(X, \Omega E)_* \to \operatorname{Map}(X, \Omega B)_* \to \operatorname{Map}(X, F)_* \to \operatorname{Map}(X, E)_* \to \operatorname{Map}(X, B)_*.$$

As it turns out, on the level of homotopy classes we get even the long *exact* sequence

$$\cdots \to [X, \Omega F]_* \to [X, \Omega E]_* \to [X, \Omega B]_* \to [X, F]_* \to [X, E]_* \to [X, B]_*.$$

There is a natural bijection between $[\Sigma X, E]_*$ and $[X, \Omega E]_*$, where ΣX is the *reduced* suspension of X, and so we get the long exact sequence

$$\cdots \to [\Sigma X, F]_* \to [\Sigma X, E]_* \to [\Sigma X, B]_* \to [X, F]_* \to [X, E]_* \to [X, B]_*.$$

For CW-complexes, the difference between SX and ΣX does not matter, and for the sequence $P_i \to P_{i-1} \to K_{i+1}$, which can be considered as a fibration, we arrive at (8).

(where ℓ_i is induced by exactly the same mapping λ_{i*} of representatives as $[\lambda_{i*}]$ in the original exact sequence (8)). Let $\mathcal{N}_{i-1} := \{f \in \mathrm{SMap}(X, P_{i-1}) : [k_{(i-1)*}(f)] = 0\}$ be the set of representatives of elements in N_{i-1}. Implement a locally effective "section" $\xi_i : \mathcal{N}_{i-1} \to \mathrm{SMap}(X, P_i)$ with $[p_{i*} \circ \xi_i] = \mathrm{id}$ and a locally effective "inverse" $r_i : \mathrm{im}[\lambda_{i*}] \to M_i$ with $\ell_i \circ r_i = \mathrm{id}$, as in Lemma 3.5, and compute $[X, P_i]$ fully effective using that lemma.

We will now examine steps 2,4,5 in detail, and simultaneously establish the exactness of (8).

Convention. It will be notationally convenient to let maps such as p_{i*}, $k_{(i-1)*}$, λ_{i*}, which send simplicial maps to simplicial maps, operate directly on the cochain representations (and in such case, the result is also assumed to be a cochain representation). Thus, for example, we can write $p_{i*}(\mathbf{c}, c) = \mathbf{c}$, $\lambda_{i*}(z^i) = (\mathbf{0}, z^i)$, etc. We also write $[\mathbf{c}]$ for the homotopy class of the map represented by $\mathbf{c}$.

6.2. Computing the liftable maps

Here we will deal with the last part of the exact sequence (8), namely,

$$[X, P_i] \xrightarrow{[p_{i*}]} [X, P_{i-1}] \xrightarrow{[k_{(i-1)*}]} [X, K_{i+1}].$$

First we note that, since the projection map p_i is an H-homomorphism by Proposition 5.4(c), the (locally effective) map $p_{i*} : \mathrm{SMap}(X, P_i) \to \mathrm{SMap}(X, P_{i-1})$ indeed induces a well-defined group homomorphism $[X, P_i] \to [X, P_{i-1}]$ (Fact 5.1). Similarly, the H-homomorphism k_{i-1} (Proposition 5.4(d)) induces a group homomorphism $[k_{(i-1)*}] : [X, P_{i-1}] \to [X, K_{i+1}] \cong H^{i+1}(X; \pi_i)$.

LEMMA 6.2 (LIFTING LEMMA). *We have* $\mathrm{im}[p_{i*}] = \ker[k_{(i-1)*}]$. *Moreover, if we set* $\mathcal{N}_{i-1} := \{f \in \mathrm{SMap}(X, P_{i-1}) : [k_{(i-1)*}(f)] = 0\}$*, then there is a locally effective mapping* $\xi_i : \mathcal{N}_{i-1} \to \mathrm{SMap}(X, P_i)$ *such that* $p_{i*} \circ \xi_i$ *is the identity map (on the level of simplicial maps).*

PROOF. Let us consider a map $f \in \mathrm{SMap}(X, P_{i-1})$ with cochain representation $\mathbf{c}$. Every cochain $(\mathbf{c}, c^i)$ with $c^i \in C^i(X; \pi_i)$ represents a simplicial map $X \to P_{i-1} \times E_i$, and this map goes into P_i iff the condition

$$k_{(i-1)*}(\mathbf{c}) = \delta c^i \tag{9}$$

holds. Thus, f has a lift iff $k_{(i-1)*}(\mathbf{c})$ is a coboundary, or in other words, iff $[k_{(i-1)*}(\mathbf{c})] = 0$ in $[X, K_{i+1}]$. Hence $\mathrm{im}[p_{i*}] = \ker[k_{(i-1)*}]$ indeed.

Moreover, if $k_{(i-1)*}(\mathbf{c})$ is a coboundary, we can compute some c^i satisfying (9) and set $\xi_i(f) := (\mathbf{c}, c^i)$. This involves some arbitrary choice, but if we fix some (arbitrary) rule for choosing c^i, we obtain a locally effective ξ_i as needed. The lemma is proved. □

REMARK 6.3. *In the previous proof as well as in a few more situations below, we will need to make some arbitrary choice of a particular solution to a system of linear equations over the integers. We refrain from specifying any particular such rule, but typically, such a rule will be built into any particular Smith normal form algorithm that we use as a subroutine to solve the system of integer linear equations (9).*

We have thus proved exactness of the sequence (8) at $[X, P_{i-1}]$. Step 2 of the algorithm can be implemented using Lemma 3.2. We have also prepared the section ξ_i for Step 5.

6.3. Factoring by maps from SX

We now focus on the initial part

$$[SX, P_{i-1}] \xrightarrow{[\mu_i]} [X, L_i] \xrightarrow{[\lambda_{i*}]} [X, P_i]$$

of the exact sequence (8), and explain how the suspension comes into the picture. We remark that $[\lambda_{i*}]$ is a well-defined homomorphism, for the same reason as $[p_{i*}]$ and $[k_{(i-1)*}]$; namely, λ_i is an H-homomorphism by Proposition 5.4(a).

The kernel of $[\lambda_{i*}]$ describes all homotopy classes of maps $X \to L_i$ that are nullhomotopic as maps $X \to P_i$. To understand how they arise as images of maps $SX \to P_{i-1}$, we first need to discuss a representation of nullhomotopies as maps from the cone.

Maps from the cone. A map $X \to Y$ between two topological spaces is nullhomotopic iff it can be extended to a map $CX \to Y$ on the cone over X; this is more or less a reformulation of the definition of nullhomotopy. The same is true in the simplicial setting if the target is a Kan simplicial set, such as P_i.

We recall that the n-dimensional nondegenerate simplices of CX are of two kinds: the n-simplices of X and the cones over the $(n-1)$-simplices of X. In the language of cochains, this means that, for any coefficient group π, we have

$$C^n(CX; \pi) \cong C^{n-1}(X; \pi) \oplus C^n(X; \pi),$$

and thus a cochain $b \in C^n(CX; \pi)$ can be written as (e, c), with $e \in C^{n-1}(X; \pi)$ and $c \in C^n(X; \pi)$. We also write $c = b|_X$ for the restriction of b to X. The coboundary operator $C^n(CX; \pi) \to C^{n+1}(CX, ; \pi)$ then acts as follows:

$$\delta(e, c) = (-\delta e + c, \delta c).$$

Rephrasing Lemma 4.3 in the language of extensions to CX, we get the following:

COROLLARY 6.4. *A map* $f \in \mathrm{SMap}(X, L_i)$*, represented by a cocycle* $c^i \in Z^i(X; \pi_i)$*, is nullhomotopic iff there is a cocycle* $b \in Z^i(CX; \pi) \cong \mathrm{SMap}(CX, L_i)$ *such that* $b|_X = c$.

This describes the homotopies in $\mathrm{SMap}(X, L_i)$, which induce the "obvious" homotopies in $\mathrm{im}\, \lambda_{i*}$. Let us now consider an element in the image of λ_{i*}, i.e., a map $g \colon X \to P_i$ with a cochain representation $(\mathbf{0}, c^i)$. By the above, a nullhomotopy of g can be regarded as a simplicial map $G \colon CX \to P_i$ whose cochain representation $(\mathbf{b}, b^i)$ satisfies $(\mathbf{b}|_X, b^i|_X) = (\mathbf{0}, c^i)$ (here $\mathbf{b}|_X = (b^0|_X, \dots, b^{i-1}|_X)$ is the componentwise restriction to X). Thus, the projection $F := p_{i*} \circ G \in \mathrm{SMap}(CX, P_{i-1})$ is represented by $\mathbf{b}$ with $\mathbf{b}|_X = \mathbf{0}$, and hence it maps all of the "base" X in CX to $\mathbf{0}$.

Recalling that SX is obtained from CX by identifying X to a single vertex, we can see that such F exactly correspond to simplicial maps $SX \to P_{i-1}$ (here we use that P_{i-1} has a single vertex $\mathbf{0}$). Thus, maps in $\mathrm{SMap}(SX, P_{i-1})$ give rise to nullhomotopies of maps in $\mathrm{im}\, \lambda_{i*}$.

After this introduction, we develop the definition of μ_i and prove the exactness of our sequence (8) at $[X, L_i]$.

The homomorphism μ_i. Since the nondegenerate $(i+1)$-simplices of SX are in one-to-one correspondence with the nondegenerate i-simplices of X, we have the isomorphism of the cochain groups

$$D_i \colon C^{i+1}(SX; \pi_i) \to C^i(X; \pi_i).$$

Moreover, this is compatible with the coboundary operator (up to sign):

$$\delta D_i(c) = -D_i(\delta c).$$

Alternatively, if we identify the $(i+1)$-cochains on SX with those $(i+1)$-cochains $b = (e, c) \in C^{i+1}(CX; \pi_i)$ for which $b|_X = c = 0$, then the isomorphism is given by $D_i(e, 0) =$

e. The coboundary formula $\delta(e, c) = (-\delta e + c, \delta c)$ for CX indeed gives $D_i(\delta(e, 0)) = D_i(-\delta e, 0) = -\delta e = -\delta D_i(e, 0)$.

Because of the compatibility with δ, D_i restricts to an isomorphism $Z^{i+1}(SX; \pi_i) \to Z^i(X; \pi_i)$ (which we also denote by D_i). This induces an isomorphism $[D_i] : H^{i+1}(SX; \pi_i) \to H^i(X; \pi_i)$.

Translating from cochains to simplicial maps, we can also regard D_i as an isomorphism $\mathrm{SMap}(SX, K_{i+1}) \to \mathrm{SMap}(X, L_i)$, (where, as we recall, $K_{i+1} = K(\pi_i, i+1)$ and $L_i = K(\pi_i, i)$), and $[D_i]$ as an isomorphism $[SX, K_{i+1}] \to [X, L_i]$.

Now we define $\mu_i \colon \mathrm{SMap}(SX, P_{i-1}) \to \mathrm{SMap}(X, L_i)$ by

$$\mu_i := D_i \circ k_{(i-1)*}.$$

That is, given $F \in \mathrm{SMap}(SX, P_{i-1})$, we first compose it with k_{i-1}, which yields a map in $\mathrm{SMap}(SX, K_{i+1})$ represented by a cocycle in $Z^{i+1}(SX; \pi_i)$. Applying D_i means reinterpreting this as a cocycle in $Z^i(X; \pi_i)$ representing a map in $\mathrm{SMap}(X, L_i)$, which we declare to be $\mu_i(F)$. This, clearly, is locally effective, and $[\mu_i]$ is a well-defined homomorphism $[SX, P_{i-1}] \to [X, L_i]$ (since $[D_i]$ and $[k_{(i-1)*}]$ are well-defined homomorphisms).

The connection of this definition of μ_i to the previous considerations on nullhomotopies may not be obvious at this point, but the lemma below shows that μ_i works.

LEMMA 6.5. *The sequence (8) is exact at $[X, L_i]$, i.e.,* $\mathrm{im}\,[\mu_i] = \ker\,[\lambda_{i*}]$.

PROOF. First we want to prove the inclusion $\mathrm{im}\,[\mu_i] \subseteq \ker\,[\lambda_{i*}]$. To this end, we consider $F \in \mathrm{SMap}(SX, P_{i-1})$ arbitrary and want to show that $[\lambda_{i*}(\mu_i(F))] = 0$ in $[X, P_i]$.

As was discussed above, we can view F as a map $\overline{F} \colon CX \to P_{i-1}$ that is zero on X. Let **b** be the cochain representation of $\overline{F}$; thus, $\mathbf{b}|_X = \mathbf{0}$.

Let $z^i \in Z^i(X; \pi_i)$ be the cocycle representing $\mu_i(F)$. Then $(0, z^i) \in C^{i-1}(X; \pi_i) \oplus C^i(X; \pi_i)$ represents a map $CX \to E_i$, and $(\mathbf{b}, (0, z^i))$ represents a map $G \colon CX \to P_{i-1} \times E_i$.

We claim that G actually goes into P_i, i.e., is a lift of $\overline{F}$. For this, we just need to verify the lifting condition (9), which reads $k_{(i-1)*}(\mathbf{b}) = \delta(0, z^i)$.

By the coboundary formula for the cone, we have $\delta(0, z^i) = (z^i, 0)$, while $k_{(i-1)*}(\mathbf{b}) = (z^i, 0)$ by the definition of $\mu_i(F)$. So $G \in \mathrm{SMap}(CX, P_i)$ is indeed a lift of $\overline{F}$. At the same time, $(\mathbf{b}, (0, z^i))|_X = (\mathbf{0}, z^i)$, and so G is a nullhomotopy for the map represented by $(\mathbf{0}, z^i)$, which is just $\lambda_{i*}(\mu_i(F))$.

To prove the reverse inclusion $\mathrm{im}\,[\mu_i] \supseteq \ker\,[\lambda_{i*}]$, we proceed similarly. Suppose that $z^i \in Z^i(X; \pi_i)$ represents a map $f \in \mathrm{SMap}(X, L_i)$ with $[\lambda_{i*}(f)] = 0$ in $[X, P_i]$. Then $\lambda_{i*}(f)$ has the cochain representation $(\mathbf{0}, z^i)$, and there is a nullhomotopy $G \in \mathrm{SMap}(CX, P_i)$ for it, with a cochain representation $(\mathbf{b}, (a^{i-1}, z^i))$, where $\mathbf{b}|_X = \mathbf{0}$.

Since $\mathbf{b}|_X = \mathbf{0}$, **b** represents a map $\overline{F} \in \mathrm{SMap}(CX, P_{i-1})$ zero on X, which can also be regarded as $F \in \mathrm{SMap}(SX, P_{i-1})$. Let $\tilde{z}^i$ represent $\mu_i(F)$. Since G is a lift of $\overline{F}$, the lifting condition $k_{(i-1)*}(\mathbf{b}) = \delta(a^{i-1}, z^i)$ holds. We have $k_{(i-1)*}(\mathbf{b}) = (\tilde{z}^i, 0)$, again by the definition of μ_i, and $\delta(a^{i-1}, z^i) = (-\delta a^{i-1} + z^i, \delta z^i)$ by the coboundary formula for the cone. Hence $\tilde{z}^i - z^i = \delta a^{i-1}$, which means that $[z^i] = [\tilde{z}^i]$. Thus $[f] = [\mu_i(F)] \in \mathrm{im}\,[\mu_i]$, and the lemma is proved. □

Having $[\mu_i]$ defined as a locally effective homomorphism, we can employ Lemma 3.3 and implement Step 4 of the algorithm.

6.4. Computing nullhomotopies

The next step is to prove the exactness of the sequence (8) at $[X, P_i]$.

LEMMA 6.6. *We have* $\mathrm{im}\,[\lambda_{i*}] = \ker\,[p_{i*}]$.

PROOF. The inclusion $\operatorname{im}[\lambda_{i*}] \subseteq \ker[p_{i*}]$ holds even on the level of simplicial maps, i.e., $\operatorname{im}\lambda_{i*} \subseteq \ker p_{i*}$. Indeed, $p_{i*}(\lambda_{i*}(z^i)) = p_{i*}(\mathbf{0}, z^i) = \mathbf{0}$.

For the reverse inclusion, consider $(\mathbf{c}, c^i) \in \operatorname{SMap}(X, P_i)$ and suppose that $[p_{i*}(\mathbf{c}, c^i)] = [\mathbf{c}] = 0 \in [X, P_{i-1}]$. We need to find some $z^i \in Z^i(X; \pi_i)$ with $[(\mathbf{0}, z^i)] = [(\mathbf{c}, c^i)]$ in $[X, P_i]$.

A suitable z^i can be constructed by taking a nullhomotopy $CX \to P_{i-1}$ for $\mathbf{c}$ and lifting it. Namely, let $\mathbf{b}$ represent a nullhomotopy for $\mathbf{c}$, i.e., $\mathbf{b}|_X = \mathbf{c}$, and let $(\mathbf{b}, b^i)$ be a lift of $\mathbf{b}$ (it exists because CX is contractible and thus *every* map on it can be lifted). We set
$$z^i := c^i - (b^i|_X).$$

We need to verify that z^i is a cocycle. This follows from the lifting conditions $k_{(i-1)*}(\mathbf{c}) = \delta c^i$ and $k_{(i-1)*}(\mathbf{b}) = \delta b^i$, and from the fact that $k_{(i-1)*}(\mathbf{b})|_X = k_{(i-1)*}(\mathbf{b}|_X) = k_{(i-1)*}(\mathbf{c})$ (this is because applying $k_{(i-1)*}$ really means a composition of maps, and thus it commutes with restriction). Indeed, we have $\delta z^i = \delta c^i - \delta(b^i|_X) = k_{(i-1)*}(\mathbf{c}) - k_{(i-1)*}(\mathbf{c}) = 0$.

It remains to to check that $[(\mathbf{c}, c^i)] = [(\mathbf{0}, z^i)]$. We calculate $[(\mathbf{c}, c^i)] - [(\mathbf{0}, z^i)] = [(\mathbf{c}, c^i) \boxminus_{i*} (\mathbf{0}, z^i)] = [(\mathbf{c}, c^i - z^i)] = [(\mathbf{c}, b^i|_X)] = [(\mathbf{b}|_X, b^i|_X)] = 0$, since $(\mathbf{b}, b^i)$ is a nullhomotopy for $(\mathbf{b}|_X, b^i|_X)$. □

Defining the inverse for λ_{i*}. Now we consider the cokernel $M_i = [X, L_i]/\operatorname{im}[\mu_i]$ as in Step 4 of the algorithm, and the (injective) homomorphism $\ell_i \colon M_i \to [X, P_i]$ induced by $[\lambda_{i*}]$.

The last thing we need for applying Lemma 3.5 in Step 5 is a locally effective map $r_i \colon \operatorname{im}\ell_i \to M_i$ with $\ell_i \circ r_i = \operatorname{id}$.

Let $\mathcal{R}_i$ be the set of representatives of the elements in $\operatorname{im}\ell_i = \operatorname{im}[\lambda_{i*}]$; by the above, we can write $\mathcal{R}_i = \{(\mathbf{c}, c^i) \in \operatorname{SMap}(X, P_i) : [\mathbf{c}] = 0\}$.

For every $(\mathbf{c}, c^i) \in \mathcal{R}_i$ we set
$$\rho_i(\mathbf{c}, c^i) := z^i,$$

where z^i is as in the above proof of Lemma 6.6 (i.e., $z^i = c^i - (b^i|_X)$, where $(\mathbf{b}, b^i)$ is a lifting of some nullhomotopy $\mathbf{b}$ for $\mathbf{c}$). This definition involves a choice of a particular $\mathbf{b}$ and b^i, which we make arbitrarily (see above) for each $(\mathbf{c}, c^i)$.

LEMMA 6.7. *The map ρ_i induces a map $r_i \colon \operatorname{im}[\lambda_{i*}] \to [X, L_i]$ such that $\ell_i \circ r_i = \operatorname{id}$.*

PROOF. In the proof of Lemma 6.6 we have verified that $[\lambda_{i*}(\rho_i(\mathbf{c}, c^i))] = [(\mathbf{c}, c^i)]$, so $\lambda_{i*} \circ \rho_i$ acts as the identity on the level of homotopy classes. It follows that r_i is well-defined, since ℓ_i is injective and thus the condition $\ell_i \circ r_i = \operatorname{id}$ determines r_i uniquely. □

We note that, since we assume $[X, P_{i-1}]$ fully effective, we can algorithmically test whether $[\mathbf{c}] = 0$, i.e., whether $\mathbf{c}$ represents a nullhomotopic map—the problem is in computing a concrete nullhomotopy $\mathbf{b}$ for $\mathbf{c}$.

We describe a recursive algorithm for doing that. For more convenient notation, we will formulate it for computing nullhomotopies for maps in $\operatorname{SMap}(X, P_i)$, but we note that, when evaluating ρ_i, we actually use this algorithm with $i - 1$ instead of i. Some of the ideas in the algorithm are very similar to those in the proof of the exactness at $[X, P_i]$ (Lemma 6.6 above), so we could have started with a presentation of the algorithm instead of Lemma 6.6, but we hope that a more gradual development may be easier to follow.

The nullhomotopy algorithm. So now we formulate a recursive algorithm `NullHom(c, `c^i`)`, which takes as input a cochain representation of a nullhomotopic map in $\operatorname{SMap}(X, P_i)$ (i.e., such that $[(\mathbf{c}, c^i)] = 0$), and outputs a nullhomotopy $(\mathbf{b}, b^i)$ for $(\mathbf{c}, c^i)$.

The required nullhomotopy $(\mathbf{b}, b^i)$ will be $\boxplus_{i*}$-added together from several nullhomotopies; this decomposition is guided by the left part of our exact sequence (8). Namely, we recursively find a nullhomotopy for $\mathbf{c}$ and lift it, which reduces the original problem to finding a nullhomotopy for a map in $\operatorname{im}\lambda_{i*}$, of the form $(\mathbf{0}, z^i)$, as in the proof of Lemma 6.6. Then, using the fact that ℓ_i is an isomorphism, we find nullhomotopies witnessing that $[z^i] = 0$ in M_i. Here we need the assumption that the representation of M_i allows for computing "witnesses of zero" as in Lemma 3.3.

For this to work, we need the fact that if $\mathbf{b}_1$ is a nullhomotopy for $\mathbf{c}_1$ and $\mathbf{b}_2$ is a nullhomotopy for $\mathbf{c}_2$, then $\mathbf{b}_1 \boxplus_{i*} \mathbf{b}_2$ is a nullhomotopy for $\mathbf{c}_1 \boxplus_{i*} \mathbf{c}_2$. This is because $\boxplus_{i*}$ operates on mappings by composition, and thus it commutes with restrictions—we have already used the same observation for k_{i*}.

The base case of the algorithm is $i = d$. Here, as we recall, $P_d = L_d = K(\pi_d, d)$, and a nullhomotopic c^d means that $c^d \in Z^d(X; \pi_d)$ is a coboundary. We thus compute $e \in Z^{d-1}(X; \pi_d)$ with $c^d = \delta e$, and the desired nullhomotopy is $(e, \delta e)$ (indeed, $(e, \delta e)$ specifies a valid map $CX \to L_d$ since, by the coboundary formula for the cone, it is a cocycle).

Now we can state the algorithm formally.

Algorithm `NullHom(c, $c^i$).`

A. (Base case) If $i = d$, return $(\mathbf{b}, b^d) = (\mathbf{0}, (e, \delta e))$ as above and stop.

B. (Recursion) Now $i > d$. Set $\mathbf{b}_0 := \texttt{NullHom}(\mathbf{c})$, and let $(\mathbf{b}_0, b_0^i)$ be an arbitrary lift of $\mathbf{b}_0$.

C. (Nullhomotopy coming from SX) Set $z^i := c^i - (b_0^i|_X)$, and use the representation of M_i to find a "witness for $[z^i] = 0$ in M_i". That is, compute $F \in [SX, P_{i-1}]$ such that $[z^i] = [\tilde{z}^i]$ in $[X, L_i]$, where $\tilde{z}^i$ is the cocycle representing $\mu_i(F)$. Let a be the cochain representation of the map $\overline{F} \in \mathrm{SMap}(CX, P_{i-1})$ corresponding to F.

D. (Nullhomotopy in $[X, L_i]$) Compute $e \in Z^{i-1}(X; \pi_i)$ with $\tilde{z}^i - z^i = \delta e$. (Then, as in the base case $i = d$ above, $(e, \delta e)$ is a nullhomotopy for $\tilde{z}^i - z^i$, and thus $(\mathbf{0}, (e, \delta e))$ is a nullhomotopy for $(\mathbf{0}, \tilde{z}^i - z^i)$.)

E. Return

$$(\mathbf{b}, b^i) := (\mathbf{b}_0, b_0^i) \boxplus_{i*} \Big((\mathbf{a}, (0, \tilde{z}^i)) \boxplus_{i*} (\mathbf{0}, (e, \delta e)) \Big).$$

PROOF OF CORRECTNESS. First we need to check that z^i in Step C indeed represents 0 in M_i. This is because, as in the proof of Lemma 6.6, $[(\mathbf{0}, z^i)] = [\lambda_{i*}(z^i)] = 0$, and since ℓ_i is injective, we have $[z^i] = 0$ in M_i as claimed. So the algorithm succeeds in computing some $(\mathbf{b}, b^i)$, and we just need to check that it is a nullhomotopy for $(\mathbf{c}, c^i)$.

All three terms in the formula in Step E are valid representatives of maps $CX \to P_i$ (for $(\mathbf{b}_0, b_0^i)$ this follows from the inductive hypothesis, for $(\mathbf{a}, (0, \tilde{z}^i))$ we have checked this in the first part of the proof of Lemma 6.5, and for $(\mathbf{0}, (e, \delta e))$ we have already discussed this). So $(\mathbf{b}, b^i)$ also represents such a map, and all we need to do is to check that $(\mathbf{b}|_X, b^i|_X) = (\mathbf{c}, c^i)$:

$$\begin{aligned}
(\mathbf{b}|_X, b^i|_X) &= (\mathbf{b}_0|_X, b_0^i|_X) \boxplus_{i*} \Big((\mathbf{a}|_X, \tilde{z}^i) \boxplus_{i*} (\mathbf{0}, \delta e) \Big) \\
&= (\mathbf{c}, b_0^i|_X) \boxplus_{i*} \Big((\mathbf{0}, \tilde{z}^i) \boxplus_{i*} (\mathbf{0}, z^i - \tilde{z}^i) \Big) \\
&= (\mathbf{c}, b_0^i|_X + \tilde{z}^i + z^i - \tilde{z}^i) = (\mathbf{c}, b_0^i|_X + (c^i - (b_0^i|_X))) = (\mathbf{c}, c^i).
\end{aligned}$$

Thus, the algorithm correctly computes the desired nullhomotopy. □

As we have already explained, the algorithm makes ρ_i locally effective, and so Step 5 of the main algorithm can be implemented. This completes the proof of Theorem 6.1.

Acknowledgments

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