(suite de la feuille 3)

Exo 9 1) on part de la base canonique $(1, X, X^2)$ de $\mathbb{R}_2[X]$.

$$\tilde{v}_1 = 1 \qquad \langle \tilde{v}_1, \tilde{v}_1 \rangle = \int_{-1}^{1} 1^2 \, dx = 2$$

$$\tilde{v}_2 = X - \frac{\langle X, \tilde{v}_1 \rangle}{\langle \tilde{v}_1, \tilde{v}_1 \rangle} \tilde{v}_1 \qquad \langle X, \tilde{v}_1 \rangle = \int_{-1}^{1} x \cdot 1 \, dx = 0$$

$$= X \qquad \langle \tilde{v}_2, \tilde{v}_2 \rangle = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$$

$$\tilde{v}_3 = X^2 - \frac{\langle X^2, \tilde{v}_1 \rangle}{\langle \tilde{v}_1, \tilde{v}_1 \rangle} \tilde{v}_1 - \frac{\langle X^2, \tilde{v}_2 \rangle}{\langle \tilde{v}_2, \tilde{v}_2 \rangle} \tilde{v}_2$$

$$= X^2 - \frac{\frac{2}{3}}{2} \cdot 1 - 0 \cdot X$$

$$= X^2 - \frac{1}{3}$$

$$\langle X^2, \tilde{v}_1 \rangle = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$$

$$\langle \tilde{v}_3, \tilde{v}_3 \rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx$$

$$= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}$$

$(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ est une base orthogonale pour $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx$

$$v_1 = \frac{1}{\sqrt{\langle \tilde{v}_1, \tilde{v}_1 \rangle}} \tilde{v}_1 = \frac{1}{\sqrt{2}}$$

$$v_2 = \frac{1}{\sqrt{\langle \tilde{v}_2, \tilde{v}_2 \rangle}} \tilde{v}_2 = \sqrt{\frac{3}{2}}\, X$$

$$v_3 = \frac{1}{\sqrt{\langle \tilde{v}_3, \tilde{v}_3 \rangle}} \tilde{v}_3 = \sqrt{\frac{45}{8}} \left(X^2 - \frac{1}{3}\right)$$

2) On note $V = C([-1,1], \mathbb{R})$, $W = \mathbb{R}_2[X]$, la projection orthogonale (pour le même prod. scal. $\langle f, g \rangle = \int_{-1}^{1} fg$) est donnée pour toute $f \in V$ par

$$\pi_W(f) = \langle f, v_1 \rangle v_1 + \langle f, v_2 \rangle v_2 + \langle f, v_3 \rangle v_3$$

$$= \frac{\langle f, \tilde{v}_1 \rangle}{\langle \tilde{v}_1, \tilde{v}_1 \rangle} \tilde{v}_1 + \frac{\langle f, \tilde{v}_2 \rangle}{\langle \tilde{v}_2, \tilde{v}_2 \rangle} \tilde{v}_2 + \frac{\langle f, \tilde{v}_3 \rangle}{\langle \tilde{v}_3, \tilde{v}_3 \rangle} \tilde{v}_3$$

(la deuxième formule évite d'introduire des racines carrées)

$\langle f, \tilde{v}_1 \rangle = \int_{-1}^{1} e^x dx = [e^x]_{-1}^{1} = e - e^{-1} = e - \frac{1}{e} = \frac{e^2-1}{e}$

$\langle f, \tilde{v}_2 \rangle = \int_{-1}^{1} \underset{u}{x} \underset{v'}{e^x} dx = -\int_{-1}^{1} e^x dx + [x e^x]_{-1}^{1}$

intégration par parties

$\int_a^b uv' = -\int_a^b u'v + [uv]_a^b$

$= -(\cancel{e} - \frac{1}{e}) + (\cancel{e} - (-e^{-1})) = \frac{2}{e}$

$\langle f, \tilde{v}_3 \rangle = \int_{-1}^{1} \underbrace{(x^2 - \frac{1}{3})}_{u} \underbrace{e^x}_{v'} dx = -\int_{-1}^{1} 2x e^x dx + [(x^2 - \frac{1}{3}) e^x]_{-1}^{1}$

$= -2 \cdot \frac{2}{e} + (\frac{2}{3} e - \frac{2}{3} e^{-1})$

$= -\frac{14}{3} \frac{1}{e} + \frac{2}{3} e = \frac{2e^2 - 14}{3e}$

Meilleure approximation de e^x dans $\mathbb{R}_2[X]$ est le polynôme

$$\pi_W(e^x) = \frac{e - \frac{1}{e}}{2} + \frac{\frac{2}{e}}{\frac{2}{3}} X + \frac{\frac{2e - 14\frac{1}{e}}{3}}{\frac{8}{45}} (X^2 - \frac{1}{3}) =$$

$$= \frac{33 - 3e^2}{4e} + \frac{3}{e} X + \frac{15e^2 - 105}{4e} X^2$$

Pour $f(x) = \cos(x)$, on trouve

$$\pi_W(\cos(x)) = \left(\frac{45}{2}\cos(1) - 15\sin(1)\right) X^2 + \left(6\sin(1) - \frac{15}{2}\cos(1)\right)$$

(note que $\langle f, \tilde{v}_2 \rangle = \int_{-1}^{1} x \cos x \, dx = 0$

car on intègre une fonction impaire entre des bornes opposées).

Pour $f(x) = \sqrt{x+1}$, on trouve

$$\pi_W(\sqrt{x+1}) = \sqrt{2}\left(-\frac{1}{7} X^2 + \frac{2}{5} X + \frac{5}{7}\right)$$

Ex 12 rappel : droite des moindres carrés est donnée par $y = ax + b$ avec

$$\begin{cases} a = \dfrac{\sum_{j=1}^{n} (x_j y_j - \bar{x}\bar{y})}{\sum_{j=1}^{n} (x_j - \bar{x})^2} \\ b = \bar{y} - a\bar{x} \end{cases}$$

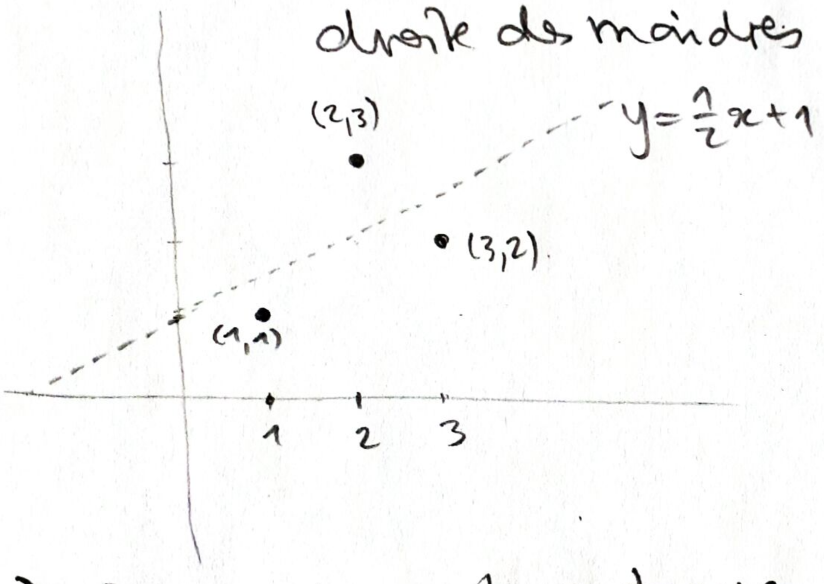
1)

x_j	y_j	$x_j y_j$	$x_j y_j - \bar{x}\bar{y}$	$x_j - \bar{x}$
1	1	1	-3	-1
2	3	6	2	0
3	2	6	2	1
$\bar{x} = 2$	$\bar{y} = 2$	$\bar{x}\bar{y} = 4$	$\sum(x_j y_j - \bar{x}\bar{y}) = 1$	$\sum(x_j - \bar{x})^2 = 2$

$$a = \frac{1}{2}$$

$$b = 2 - \frac{1}{2} \cdot 2 = 1.$$

droite des moindres carrés : $y = \frac{1}{2}x + 1.$

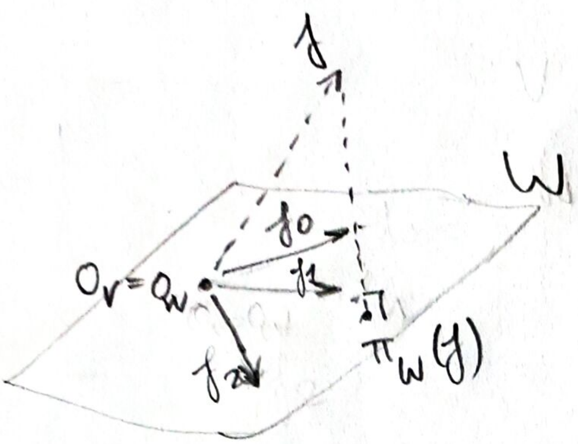


2) Pour cet exemple, on trouve $y = \frac{7}{2}x - 3$.

Exo 13 $V = C^0([-\pi, \pi], \mathbb{R})$

$W = \mathrm{Vect}\{f_0, f_1, f_2\}$ avec $f_n(x) = \cos(nx)$.

$f(x) = x^2$



On a vu que la famille (f_0, f_1, f_2) est orthogonale pour $\langle u, v\rangle = \int_{-1}^{1} u(x)v(x)\,dx$, donc

$$\pi_W(f) = \frac{\langle f, f_0\rangle}{\langle f_0, f_0\rangle} f_0 + \frac{\langle f, f_1\rangle}{\langle f_1, f_1\rangle} f_1 + \frac{\langle f, f_2\rangle}{\langle f_2, f_2\rangle} f_2$$

$$\langle f, f_0\rangle = \int_{-\pi}^{\pi} f(x) f_0(x)\,dx = \int_{-}^{\pi} x^2 \cdot 1\,dx = 2\int_0^{\pi} x^2\,dx$$
$$= 2\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^3}{3}$$

$$\langle f, f_1\rangle = \int_{-\pi}^{\pi} x^2 \cos(x)\,dx \underset{\uparrow}{=} 2\int_0^{\pi} \underbrace{x^2}_{u} \underbrace{\cos(x)}_{v'}\,dx$$

$x^2\cos(x)$ est paire et on intègre entre des bornes opposées

$$= 2\left(-\int_0^{\pi} 2x\sin(x)\,dx + \left[x^2\sin x\right]_0^{\pi}\right)$$
$$= -4\int_0^{\pi} \underbrace{x}_{u}\,\underbrace{\sin x}_{v'}\,dx$$
$$= -4\left(-\int_0^{\pi} 1\cdot(-\cos x)\,dx + \left[x\cdot(-\cos x)\right]_0^{\pi}\right)$$
$$= -4\int_0^{\pi} \cos x\,dx - 4\pi$$
$$= -4\left[\sin x\right]_0^{\pi} - 4\pi = -4\pi.$$

Plutôt que de calculer $\langle f_1, f_2 \rangle$, on calcule pour $n \in \mathbb{N}^*$ quelconque :

$$\langle f_1, f_n \rangle = \int_{-\pi}^{\pi} x^2 \cos(nx)\, dx$$

$$= 2 \int_0^{\pi} x^2 \cos(nx)\, dx$$

$(n \neq 0)$

$$= 2\left(-\int_0^{\pi} 2x \frac{\sin(nx)}{n}\, dx + \left[x^2 \frac{\sin(nx)}{n} \right]_0^{\pi} \right)$$

$$= -\frac{4}{n} \int_0^{\pi} \underbrace{x}_{u} \underbrace{\sin(nx)}_{v'}\, dx$$

$$= -\frac{4}{n} \left(-\int_0^{\pi} 1 \cdot \frac{-\cos(nx)}{n}\, dx + \left[x \cdot \frac{-\cos(nx)}{n} \right]_0^{\pi} \right)$$

$$= -\frac{4}{n^2} \int_0^{\pi} \cos(nx)\, dx + \frac{4}{n^2} \pi \cos(n\pi)$$

$$= -\frac{4}{n^2} \left[\frac{\sin(nx)}{n} \right]_0^{\pi} + \frac{4\pi}{n^2} \cos(n\pi)$$

$$= (-1)^n \frac{4\pi}{n^2}$$

$\begin{cases} +1 \text{ si } n \text{ est pair} \\ -1 \text{ si } n \text{ est impai[r]} \end{cases}$

donc $\cos(n\pi) = (-1)^n$.

en particulier,

$$\langle f_1, f_2 \rangle = \frac{4\pi}{2^2} = \pi.$$

On rappelle de l'exo 8 que $\langle f_0, f_0 \rangle = 2\pi$

$\langle f_1, f_1 \rangle = \langle f_2, f_2 \rangle = \pi$

donc

$$\pi_W(f) = \frac{\frac{2\pi^3}{3}}{2\pi} f_0 - \frac{4\pi}{\pi} f_1 + \frac{\pi}{\pi} f_2$$

$$= \frac{\pi^2}{3} - 4\cos x + \cos(2x)$$