THE GYROKINETIC LIMIT FOR THE VLASOV-POISSON SYSTEM WITH A POINT CHARGE

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ABSTRACT. We consider the asymptotics of large external magnetic field for a 2D Vlasov-Poisson system governing the evolution of a bounded density interacting with a unitary point charge. We show that the solution converges to a solution of the Euler equation with a defect measure.

1. INTRODUCTION AND MAIN RESULTS

1.1. The gyrokinetic limit for the Vlasov-Poisson system with a point charge. In this paper, we consider the asymptotical behavior of the solutions of a Vlasov-Poisson type system as ε tends to zero:

(1.1)
$$\begin{cases} \partial_t f_\varepsilon + \dfrac{v}{\varepsilon}\cdot\nabla_x f_\varepsilon + \left(\dfrac{v^\perp}{\varepsilon^2} + \dfrac{E_\varepsilon}{\varepsilon} + \dfrac{1}{\varepsilon}\dfrac{x-\xi_\varepsilon}{|x-\xi_\varepsilon|^2}\right)\cdot\nabla_v f_\varepsilon = 0, \quad (x,v)\in\mathbb{R}^2\times\mathbb{R}^2 \\ E_\varepsilon = \dfrac{x}{|x|^2} * \rho_\varepsilon, \quad \text{where } \rho_\varepsilon(t,x) = \displaystyle\int_{\mathbb{R}^2} f_\varepsilon(t,x,v)\,dv \\ \dot{\xi}_\varepsilon(t) = \dfrac{\eta_\varepsilon(t)}{\varepsilon}, \\ \dot{\eta}_\varepsilon(t) = \dfrac{\eta_\varepsilon^\perp(t)}{\varepsilon^2} + \dfrac{E_\varepsilon(t,\xi_\varepsilon(t))}{\varepsilon}, \end{cases}$$

with the initial conditions

$$f_\varepsilon(0,x,v) = f_\varepsilon^0(x,v), \quad (\xi_\varepsilon,\eta_\varepsilon)(0) = (\xi_\varepsilon^0,\eta_\varepsilon^0). \tag{1.2}$$

For each $\varepsilon > 0$, this system describes the interaction of a two-dimensional distribution of light particles (a plasma) and a heavy, unitary point charge, which are submitted to a large and constant external magnetic field, orthogonal to the plane. More precisely, the distribution of particles is represented by the positive and bounded function $f_\varepsilon = f_\varepsilon(t,x,v)$, the point charge is located at $\xi_\varepsilon(t)$, with velocity $\eta_\varepsilon(t)$. The particles are submitted to the self-consistent electric field E_ε on the one hand, and to the magnetic field represented by the terms $v^\perp/\varepsilon^2$ or $\eta_\varepsilon^\perp/\varepsilon^2$ on the other hand (here, $(x_1,x_2)^\perp = (-x_2,x_1)$).

For fixed $\varepsilon > 0$, the Cauchy theory for weak solutions of the classical Vlasov-Poisson system, namely (1.1) without charge nor magnetic field, has been settled in several works [1, 27, 18, 21]. Then, the Vlasov-Poisson system without magnetic field but with a point charge was introduced by Caprino

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Date: March 21, 2019.

2010 *Mathematics Subject Classification.* Primary 35Q83; Secondary 35A02, 35A05, 35A24.

Key words and phrases. Vlasov-Poisson, gyrokinetic limit, Euler equation.

and Marchioro [6] (with $\varepsilon = 1$). For initial data satisfying

$$
(1.3)\qquad \begin{aligned} & f_\varepsilon^0 \in L^1 \cap L^\infty(\mathbb{R}^2\times\mathbb{R}^2),\ f_\varepsilon^0 \geq 0,\ f_\varepsilon^0 \text{ is compactly supported,}\\ & \operatorname{supp}(f_\varepsilon^0) \subset \{(x,v)\in\mathbb{R}^2\times\mathbb{R}^2 \mid |x-\xi_\varepsilon^0| \geq \delta_\varepsilon\} \quad \text{for some } \delta_\varepsilon > 0, \end{aligned}
$$

global existence and uniqueness of a solution $(f_\varepsilon, \xi_\varepsilon)$ with $f_\varepsilon \in L^\infty(\mathbb{R}_+, L^1 \cap L^\infty(\mathbb{R}^2\times\mathbb{R}^2))$ compactly supported was established in [6]. We also refer to [5] for a related existence result in the case of attractive interaction between the plasma and the charge. This result can be easily extended to (1.1) with magnetic field, for each fixed $\varepsilon > 0$.

The purpose of this paper is to investigate the asymptotics of (1.1) for large external magnetic field, which corresponds to the limit ε tends to zero. We will show that under suitable bounds on the initial data, the sequence $(\rho_\varepsilon, \xi_\varepsilon)$ is relatively compact for some suitable topology on measures and we will show in Theorem 1.4 that any accumulation point (ρ, ξ) satisfies the Euler equation (E), with a defect measure. Furthermore, when the defect measure vanishes and under more regularity assumptions on ρ, (E) yields a coupled system consisting in a PDE for the evolution of ρ and an ODE for the evolution of ξ:

$$
(1.4)\qquad \begin{cases} \partial_t \rho + \left(E^\perp + \dfrac{(x-\xi)^\perp}{|x-\xi|^2}\right)\cdot\nabla\rho = 0\\ \dot{\xi}(t) = E^\perp(t,\xi(t)),\quad E = \dfrac{x}{|x|^2} * \rho. \end{cases}
$$

Before stating these theorems in subsection 1.3, we summarize the state of the art for the case without charge. The system (1.4) reduces then to the 2D incompressible Euler equation in vorticity formulation for the function ρ:

$$
(1.5)\qquad \partial_t \rho + E^\perp \cdot \nabla\rho = 0, \quad E = \frac{x}{|x|^2} * \rho.
$$

In the periodic setting without charge, Golse and Saint-Raymond [13], then Saint-Raymond [30] and also Brenier [4] established the convergence of (1.1) to the incompressible Euler equation under suitable assumptions on the initial data (see later). The same kind of result was recently obtained in [25] by different techniques. Moreover, several asymptotical regimes for linear or non linear Vlasov-like equations, leading to various nonlinear equations, were investigated in the articles [10, 11, 12, 14, 29, 15, 16], and more recently in [3, 2]. Recently, the numerical issues were studied by Filbet and Rodrigues [9], wo constructed an asymptotic-preserving scheme for the Vlasov-Poisson system in the limit of large external magnetic field.

We now turn to the system (1.4) also called vortex-wave system. It was introduced by Marchioro and Pulvirenti [24], who established global existence and uniqueness of the solution such that $\rho \in L^\infty(\mathbb{R}_+, L^1 \cap L^\infty(\mathbb{R}^2))$ and $\xi \in C^1(\mathbb{R}_+)$ never intersects the support of ρ. It was later further analyzed in, e.g., [17, 7]. We will discuss below the possibility of giving a sense to (1.4), or to (1.5), when ρ is a measure. Our definition 1.1 below, borrowed from previous works, allows to handle *vortex sheets*, namely measure-valued densities $\rho(t)$ belonging to H^{-1}.

1.2. **Some notations.** Throughout this paper,

• For $\Omega = \mathbb{R}^2$, $\Omega = \mathbb{R}^2 \times \mathbb{R}^2$ or $\Omega = \mathbb{S}^1 \times \mathbb{R}^2$, $\mathcal{M}(\Omega)$ denotes the space of bounded real Radon measures and $\mathcal{M}_+(\Omega)$ the space of bounded, positive Radon measures on Ω, $C_0(\Omega)$ the space of continous functions vanishing at infinity on Ω. We say that $\rho \in C_w(\mathbb{R}_+, \mathcal{M}_+(\Omega))$ if $\rho(t) \in \mathcal{M}_+(\Omega)$ for all $t \in \mathbb{R}_+$ and if moreover, $t \mapsto \int_\Omega \Phi(x)\, d\rho(t,x)$ is continous, for all $\Phi \in C_0(\Omega)$. The sequence $(\rho_n)_{n\in\mathbb{N}}$ is said to converge to ρ in $C_w(\mathbb{R}_+, \mathcal{M}_+(\Omega))$ if for all $T > 0$ and for all $\Phi \in C_0(\Omega)$ we have $\sup_{t\in[0,T]} \int_\Omega \Phi(x)(d\rho_n(t,x) - d\rho(t,x)) \to 0$ as $n \to +\infty$. The sequence (ρ_n) is said to converge to ρ in $L^\infty(\mathbb{R}_+, \mathcal{M}_+(\Omega))$ weak - $*$ if for all $\Phi \in L^1(\mathbb{R}_+, C_0(\Omega))$ we have $\int_{\mathbb{R}_+} \int_\Omega \Phi(t,x)(d\rho_n(t,x) - d\rho(t,x)) \to 0$ as $n \to +\infty$.

• For A, $B \in \mathcal{M}_{2,2}(\mathbb{R})$ we set $A : B = \sum_{i,j} A_{i,j} B_{i,j}$ and for $x = (x_1, x_2) \in \mathbb{R}^2$ we set $x \otimes x = x^t x = \begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix}$.

• Except in the last section, C denotes a constant changing possibly from a line to another, depending only on the uniform bounds on the initial data.

1.3. **Statements of the results.** As already mentioned, the limits of the solutions of (1.1) arising as $\varepsilon \to 0$ are measure-valued. In order to take into account such singular objects, we need to reexpress the nonlinear term $E^\perp \cdot \nabla\rho = \nabla \cdot (E^\perp \rho)$ in the sense of distributions. The formulation below, and some of its variants, was introduced by Schochet [31], Delort [8] or Poupaud [28] in the setting of weak solutions of the 2D Euler equation.

Definition 1.1 ([28], Def. 4.9)**.** Let $\rho, \mu \in \mathcal{M}_+(\mathbb{R}^2)$. For all $\Phi \in C_c^\infty(\mathbb{R}^2)$, we set

$$\mathcal{H}_\Phi[\rho, \mu] = \frac{1}{2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} H_\Phi(x,y) d\rho(x)\, d\mu(y),$$

where

$$H_\Phi(x,y) = \frac{(x-y)^\perp}{|x-y|^2} \cdot (\nabla\Phi(x) - \nabla\Phi(y)) \quad \text{if } x \neq y, \quad H_\Phi(x,x) = 0.$$

Remark 1.2. The map $(x,y) \mapsto H_\Phi(x,y)$ is defined and continuous off the diagonal $\Delta = \{(x,x) \mid x \in \mathbb{R}^2\}$. It is also bounded on $\mathbb{R}^2 \times \mathbb{R}^2$ by the mean-value theorem, hence the formulation above makes sense for ρ and μ as in Definition 1.1.

The motivation of this definition is based on the following proposition, which is obtained by symmetrization of the variables x and y.

Proposition 1.3 ([28, 8, 31])**.** *In Definition 1.1 assume moreover that the measure ρ belongs to $L^p(\mathbb{R}^2)$ for some $p > 2$. Then we have, recalling $E = \frac{x}{|x|^2}$,*

$$\langle \nabla \cdot (E^\perp \rho), \Phi \rangle_{\mathcal{D}'(\mathbb{R}^2), \mathcal{D}(\mathbb{R}^2)} = -\mathcal{H}_\Phi[\rho, \rho].$$

We clarify now our assumptions on the initial data. To $f \in L^1$, $\rho = \int f\, dv$, ξ and $\eta \in \mathbb{R}^2$ we associate the energy

$$\mathcal{H}(f, \xi, \eta) = \frac{1}{2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^2 f(x,v)\, dx\, dv + \frac{1}{2}|\eta|^2$$
$$- \frac{1}{2} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \ln|x-y| \rho(x)\rho(y)\, dx\, dy - \int_{\mathbb{R}^2} \ln|x-\xi| \rho(x)\, dx,$$

and the momentum

$$\mathcal{I}(f,\xi,\eta)=\int_{\mathbb{R}^2}\left(|x+\varepsilon v^\perp|^2-\varepsilon^2|v|^2\right)f(x,v)\,dx\,dv+|\xi+\varepsilon\eta^\perp|^2-\varepsilon^2|\eta|^2.$$

As we shall later see, the energy and the momentum are preserved by the solutions of (1.1) that are considered in this paper.

Here we restrict our attention to initial data satisfying (1.3) for each $\varepsilon>0$. Moreover we assume the following behavior of the norms as $\varepsilon\to 0$:

$$\begin{aligned}&\sup_{0<\varepsilon<1}\left(\|f_\varepsilon^0\|_{L^1}+\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon^0(x)\,dx+|\xi_\varepsilon^0|\right)<+\infty,\\&\sup_{0<\varepsilon<1}\mathcal{H}(f_\varepsilon^0,\xi_\varepsilon^0,\eta_\varepsilon^0)<+\infty,\end{aligned}\tag{1.6}$$

and

$$\varepsilon^2\|f_\varepsilon^0\|_{L^\infty}\to 0,\quad\text{as }\varepsilon\to 0.\tag{1.7}$$

Finally, we add the condition[1] :

$$\sup_{0<\varepsilon<1}\|f_\varepsilon^0\|_{L^1}<1.\tag{1.8}$$

Our main result can now be stated as follows.

Theorem 1.4. *Let $(f_\varepsilon^0,\xi_\varepsilon^0,\eta_\varepsilon^0)$ satisfy (1.3), (1.6), (1.7) and (1.8). Let $(f_\varepsilon,\xi_\varepsilon)$ denote the corresponding global weak solution of (1.1). There exists a subsequence $\varepsilon_n\to 0$ as $n\to+\infty$ such that*
- *(ρ_{ε_n}) converges to ρ in $C_w(\mathbb{R}_+,\mathcal{M}_+(\mathbb{R}^2))$ and (ξ_{ε_n}) converges to ξ in $C^{1/2}([0,T],\mathbb{R}^2)$ for all $T>0$;*
- *$\rho\in L^\infty(\mathbb{R}_+,H^{-1}(\mathbb{R}^2))$;*
- *There exists a defect measure $\nu\in[L^\infty(\mathbb{R}_+,\mathcal{M}(\mathbb{R}^2)]^4$ such that (ρ,ξ) satisfies: for all $\Phi\in C_c^\infty(\mathbb{R}_+\times\mathbb{R}^2)$,*

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}\Phi(t,x)d(\rho(t)+\delta_{\xi(t)})(x)&=\int_{\mathbb{R}^2}\partial_t\Phi(t,x)d(\rho(t)+\delta_{\xi(t)})(x)\\&+\mathcal{H}_{\Phi(t)}[\rho+\delta_\xi,\rho+\delta_\xi]+\int_{\mathbb{R}^2}D\nabla^\perp\Phi(t,x):d\nu(t,x)\end{aligned}\tag{E}$$

in the sense of distributions on $\mathbb{R}_+$.

The next theorem specifies the structure of the defect measure:

Theorem 1.5. *Under the same assumptions as in Theorem 1.4,*
- *There exists $\nu_0=\nu_0(t,x,\theta)\in L^\infty\left(\mathbb{R}_+,\mathcal{M}_+(\mathbb{R}^2\times\mathbb{S}^1)\right)$ and there exists $(\alpha,\beta)\in L^\infty(\mathbb{R}_+,\mathbb{R})^2$ such that*

$$\nu=\int_{\mathbb{S}^1}\theta\otimes\theta\,d\nu_0(\theta)+\begin{pmatrix}-\beta\delta_\xi & \alpha\delta_\xi\\ \alpha\delta_\xi & \beta\delta_\xi\end{pmatrix}.$$

In particular, ν is symmetric.

[1] More generally, the condition is $\sup_{0<\varepsilon<1}\|f_\varepsilon^0\|_{L^1}<|\sigma|$, where σ is such that $E=\sigma x/|x|^2*\rho$.

• *The sequence* (f_{ε_n}) *converges to* $f = f(t,x,|v|)$ *in* $L^\infty(\mathbb{R}_+, \mathcal{M}_+(\mathbb{R}^2 \times \mathbb{R}^2))$ *weak -* $*$ *and* $\rho = \int f\, dv$. *Moreover, for all* Φ *continuous on* $\mathbb{S}^1$, *the sequence*

$$\int_{\mathbb{R}^2} (f_{\varepsilon_n}(t,x,v) - f(t,x,|v|))\, \Phi\left(\frac{v}{|v|}\right) |v|^2\, dv$$

converges to

$$\int_{\mathbb{S}^1} \Phi(\theta)\, d\nu_0(\theta)$$

in the sense of distributions on $\mathbb{R}_+ \times \mathbb{R}^2$.

For the asymptotics without charge, Theorems 1.4 and 1.5 were obtained by Golse and Saint-Raymond [13, Theorem A], in which case ν reduces to ν_0. The authors also derived some conditions ensuring that the defect measure is rotation invariant, so that the terms $\int \theta_1\theta_2 d\nu_0(\theta)$ and $\int(\theta_1^2 - \theta_2^2)d\nu_0(\theta)$ eventually vanish. It would be interesting to study analog criteria for this so-called phenomenom of concentration-cancellation in the present case.

It was later proved by Saint-Raymond [29] that the defect measure vanishes, so that any accumulation point is a vortex-sheet solution of the Euler equation (1.5). The global existence of such solutions had been previously obtained by Delort [8].

The equation (E) can be seen as a generalized formulation of the Euler equation (1.5) for the total measure-valued vorticity $\omega = \rho + \delta_\xi$, according to the definition given by Poupaud [28]. We stress that such solutions however do not enter the framework of vortex-sheet solutions since Dirac masses do not belong to H^{-1}.

Our last result shows that if there is no defect measure, assuming additional regularity on ρ enables to decouple the equation (E) to obtain the vortex-wave system

Theorem 1.6. *Let* (ρ, ξ) *be an accumulation point given by Theorem 1.4 and such that* ν *vanishes. If moreover* $\rho \in L^\infty_{loc}(\mathbb{R}_+, L^p(\mathbb{R}^2))$ *for some* $p > 2$ *and* $\xi \in C^1(\mathbb{R}_+, \mathbb{R}^2)$ *then* (ρ, ξ) *satisfies the system*

$$\begin{cases} \partial_t \rho + \left(E^\perp + \dfrac{(x-\xi)^\perp}{|x-\xi|^2}\right) \cdot \nabla\rho = 0 \\ \dot{\xi}(t) = E^\perp(t, \xi(t)), \end{cases}$$

where $E = \frac{x}{|x|^2} * \rho$.

Remark 1.7. It is classical (see e.g. [22]) that if $\rho \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^1 \cap L^p(\mathbb{R}^2))$ for $p > 2$ then $E = \frac{x}{|x|^2} * \rho$ belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, C^{0,1-\frac{2}{p}}) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty(\mathbb{R}^2))$.

It would be interesting to study the case of t

The plan of the paper is the following. In the next section we establish Theorems 1.4 and 1.5, decomposing the proof into several steps. We first look for a priori estimates with respect to ε. Then, the main argument is the weak formulation satisfied by $(f_\varepsilon, \xi_\varepsilon)$, derived in Proposition 2.8, in which we eventually pass to the limit by using the a priori estimates. Then we prove Theorem 1.6. Finally, we complete this paper with a section devoted to a first study of (1.4). In particular, we show that the solution is unique if the support of the density does not contain the charge.

2. Proofs of Theorem 1.4 and Theorem 1.5

Throughout this section, we consider a sequence of initial data $(f_\varepsilon^0, \xi_\varepsilon^0, \eta_\varepsilon^0)$ satisfying the assumptions (1.3) (1.6), (1.7) and (1.8). Let $(f_\varepsilon, \xi_\varepsilon)$ denote any corresponding sequence of global weak solutions to (1.1).

We will sometimes denote by

$$L_\varepsilon(t,x) = \frac{x-\xi_\varepsilon(t)}{|x-\xi_\varepsilon(t)|^2} \tag{2.1}$$

the singular electic field generated by the point charge.

2.1. **Lagrangian trajectories.**

The same arguments as in [6] imply that the unique solution f_ε to the system (1.1) is constant along the Lagrangian trajectories associated to the field $E_\varepsilon + L_\varepsilon$. More precisely, we have the representation

$$f_\varepsilon(t) = (X_\varepsilon(t), V_\varepsilon(t))_\# f_\varepsilon^0, \tag{2.2}$$

where for all $(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$, the map $t \mapsto (X_\varepsilon(t,x,v), V_\varepsilon(t,x,v))$ belongs to $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^2 \times \mathbb{R}^2)$ and is the unique absolutely continuous solution to the ODE

$$\begin{cases} \dot{X}_\varepsilon(t,x,v) = \dfrac{1}{\varepsilon} V_\varepsilon(t,x,v) \\ \dot{V}_\varepsilon(t,x,v) = \dfrac{1}{\varepsilon^2}\Big(V_\varepsilon^\perp(t,x,v) + \varepsilon(E_\varepsilon + L_\varepsilon)(t, X_\varepsilon(t,x,v))\Big), \end{cases} \tag{2.3}$$

with $(X_\varepsilon, V_\varepsilon)(0,x,v) = (x,v)$. Moreover, the repulsive interaction between the plasma and the charge ensures that if $x \neq \xi_\varepsilon^0$ then $X_\varepsilon(t,x,v) \neq \xi_\varepsilon(t)$ for all $t>0$ (see the proof of [6, Corollary 2.4]), so that $t \mapsto L_\varepsilon(t, X_\varepsilon(t,x,v))$ - and therefore also the flow map $(X_\varepsilon, V_\varepsilon)$ - is globally defined in time.

Note that since f_ε has compact velocity support, then ρ_ε belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty(\mathbb{R}^2))$ for all $\varepsilon > 0$. It follows in particular that E_ε belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty(\mathbb{R}^2))$ as well (note that its norm may blow up as ε tends to zero) and that it is almost-Lipschitz: $|E_\varepsilon(t,x) - E_\varepsilon(t,y)| \leq C_\varepsilon |x-y|(1+|\ln|x-y||)$ (see e.g. [18, (46)]). Thus it turns out that for all $(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$ the map $t \mapsto (X_\varepsilon(t,x,v), V_\varepsilon(t,x,v))$ belongs to $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^2 \times \mathbb{R}^2)$. Finally, noticing that f_ε is in $C_w(\mathbb{R}_+, L^p)$ for all $1 < p < +\infty$ because of (2.2), it is not difficult to infer that $t \mapsto E_\varepsilon(t,x)$ is continuous in time, uniformly with respect to x, so that finally, $t \mapsto (X_\varepsilon(t,x,v), V_\varepsilon(t,x,v))$ is the unique C^1 solution to the ODE (2.3).

2.2. **First a priori estimates.**

As a starting point we gather some useful facts, most of them are standard and we only sketch the proofs.

Proposition 2.1. *We have for all $t > 0$:*

$$\mathcal{H}(f_\varepsilon(t), \xi_\varepsilon(t), \eta_\varepsilon(t)) = \mathcal{H}(f_\varepsilon^0, \xi_\varepsilon^0, \eta_\varepsilon^0); \quad \|f_\varepsilon(t)\|_{L^p} = \|f_\varepsilon^0\|_{L^p}, \quad \forall 1 \leq p \leq +\infty;$$
$$\mathcal{I}(f_\varepsilon(t), \xi_\varepsilon(t), \eta_\varepsilon(t)) = \mathcal{I}(f_\varepsilon(0), \xi_\varepsilon(0), \eta_\varepsilon(0)).$$

Proof. By straightforward computations we show that $\dot{\mathcal{H}}(f_\varepsilon(t), \xi_\varepsilon(t), \eta_\varepsilon(t)) = 0$. The conservation of the norms is a consequence of (2.2). Finally, adapting the proof of [25, Proposition 2.3] to the present case with point charge, we show that $\dot{\mathcal{I}}(f_\varepsilon(t), \xi_\varepsilon(t), \eta_\varepsilon(t)) = 0$. □

Corollary 2.2. *We have*

$$\sup_{t\in\mathbb{R}_+}\sup_{\varepsilon>0}\left(\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^2 f_\varepsilon(t,x,v)\,dx\,dv+\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(t,x)\,dx\right)<+\infty,$$
$$\sup_{t\in\mathbb{R}_+}\sup_{\varepsilon>0}\left(|\xi_\varepsilon(t)|+\eta_\varepsilon(t)|\right)<+\infty,$$

and

$$\sup_{t\in\mathbb{R}_+}\|\rho_\varepsilon(t)\|_{L^2}\le C\|f_\varepsilon^0\|_{L^\infty}^{1/2}.$$

Finally,

$$\sup_{t\in\mathbb{R}_+}\sup_{\varepsilon>0}\left|\iint_{\mathbb{R}^2\times\mathbb{R}^2}\ln|x-y|\rho_\varepsilon(t,x)\rho_\varepsilon(t,y)\,dx\,dy\right|<+\infty.$$

Proof. The first estimate is established in [6] for finite interval of times. For a global in time estimate, we adapt easily the case without charge, which was handled in [25, Proposition 2.4], in the following way. We omit below the dependence upon t when not misleading. Setting

$$K_\varepsilon=\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^2 f_\varepsilon(x,v)\,dx\,dv+|\eta_\varepsilon|^2$$

we have by Proposition 2.1, by Cauchy-Schwarz inequality and by the uniform bound (1.6),

$$\begin{aligned}K_\varepsilon&\le 2\mathcal{H}(f_\varepsilon,\eta_\varepsilon,\xi_\varepsilon)+\iint_{\mathbb{R}^2\times\mathbb{R}^2}\ln_+|x-y|\rho_\varepsilon(x)\rho_\varepsilon(y)\,dx\,dy+2\int_{\mathbb{R}^2}\ln_+|x-\xi_\varepsilon|\rho_\varepsilon(x)\,dx\\&\le 2\mathcal{H}(f_\varepsilon,\eta_\varepsilon,\xi_\varepsilon)(0)+C\iint_{\mathbb{R}^2\times\mathbb{R}^2}\left(|x|+|y|\right)\rho_\varepsilon(x)\rho_\varepsilon(y)\,dx\,dy+C|\xi_\varepsilon|\|\rho_\varepsilon\|_{L^1}.\end{aligned}$$

We have used that $\ln_+ r\le r$ in the second inequality. Thus by (1.6),

$$K_\varepsilon\le C+C\left(|\xi_\varepsilon|^2+\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(x)\,dx\right)^{1/2}. \tag{2.4}$$

Applying again Proposition 2.1, and using that $2|a||b|\le\nu a^2+\nu^{-1}b^2$ for all $\nu>0$, $a,b\in\mathbb{R}$,

$$\begin{aligned}|\xi_\varepsilon|^2&+\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(x)\,dx=\mathcal{I}(f_\varepsilon,\xi_\varepsilon,\eta_\varepsilon)+2\varepsilon\int_{\mathbb{R}^2}x\cdot v^\perp f_\varepsilon(x,v)\,dx\,dv+2\varepsilon\xi_\varepsilon\cdot\eta_\varepsilon^\perp\\&\le\mathcal{I}(f_\varepsilon^0,\xi_\varepsilon^0,\eta_\varepsilon^0)+\frac12\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(x)\,dx+2\varepsilon^2\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^2 f_\varepsilon(x,v)\,dx\,dv+\frac{\gamma}{2}|\xi_\varepsilon|^2+\frac{2}{\gamma}\varepsilon^2|\eta_\varepsilon|^2\\&\le C\left(|\xi_\varepsilon^0|^2+\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon^0(x)\,dx\right)+C\varepsilon^2K_\varepsilon^0++C\varepsilon^2K_\varepsilon+\frac12\left(\gamma|\xi_\varepsilon|^2+\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(x)\,dx\right).\end{aligned}$$

By (2.4) and by (1.6) we get

$$|\xi_\varepsilon|^2+\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(x)\,dx\leq C+\frac{1}{2}\left(|\xi_\varepsilon|^2+\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(x)\,dx\right)$$
$$+C\varepsilon^2\left(|\xi_\varepsilon|^2+\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(x)\,dx\right)^{1/2},$$

therefore

$$|\xi_\varepsilon|^2+\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(x)\,dx\leq C,$$

so that also $K_\varepsilon\leq C$.

The second estimate is classical, see e.g. [13, Lemma 3.1] or [30, Lemma 2.4]: one has the interpolation inequality

$$\|\rho_\varepsilon\|_{L^2}\leq C\|f_\varepsilon\|_{L^\infty}^{1/2}\left(\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^2f_\varepsilon(t,x,v)\,dx\,dv\right)^{1/2}$$

and the estimate $K_\varepsilon\leq C$ yields the result.

Finally, we obtain the last estimate by noticing that the left-hand-side can be estimated in terms of $\|\rho_\varepsilon\|_{L^2}$, $\int|x|^2\rho_\varepsilon(x)\,dx$ and $\|\rho_\varepsilon\|_{L^1}$. The conclusion follows. $\square$

As in [25], we introduce a smooth, positive function $\overline{\rho}_\varepsilon$, compactly supported in $B(0,1)$, such that $\int\overline{\rho}_\varepsilon=\int\rho_\varepsilon$ and $\sup_{0<\varepsilon<1}\|\overline{\rho}_\varepsilon\|_{L^\infty}<+\infty$. Setting $\overline{E}_\varepsilon=(x/|x|^2)*\overline{\rho}_\varepsilon$ it is well-known that $\sup_{0<\varepsilon<1}\|\overline{E}_\varepsilon\|_{L^\infty}<+\infty$ and that $E_\varepsilon(t)-\overline{E}_\varepsilon$ belongs to $L^2(\mathbb{R}^2)$, see e.g. [20, Proposition 3.3]. In addition, by combining the previous estimates in Proposition 2.1 and Corollary 2.2 we get

$$\sup_{t\in\mathbb{R}_+}\sup_{0<\varepsilon<1}\|E_\varepsilon(t)-\overline{E}_\varepsilon\|_{L^2}<+\infty \tag{2.5}$$

(see the proof of [25, Proposition 2.5]). We remark that (ρ_ε) is therefore uniformly bounded in $L^\infty(\mathbb{R}_+,H^{-1}(\mathbb{R}^2))$ since $E_\varepsilon(t)-\overline{E}_\varepsilon=2\pi\nabla\Delta^{-1}(\rho_\varepsilon(t)-\overline{\rho}_\varepsilon)$.

We conclude this paragraph with a non concentration property that will be useful later. The following lemma was proved by Majda [19] (page 932). Other variants, using L^2 norm of the field, were established in [8, 31].

Proposition 2.3. *Let $\rho\in\mathcal{M}_+(\mathbb{R}^2)$ such that $I=\int_{\mathbb{R}^2}|x|^2\rho(x)\,dx<+\infty$. Assume that*

$$H(\rho)=\iint_{\mathbb{R}^2\times\mathbb{R}^2}|\ln|x-y||\rho(x)\rho(y)\,dx\,dy<+\infty.$$

Then there exists $C>0$ depending only on $\int\rho, I$ and H, such that for all $0<r<1/2$ we have

$$\sup_{x_0\in\mathbb{R}^2}\int_{B(x_0,r)}\rho(x)\,dx\leq C|\ln r|^{-1/2}.$$

In particular, it follows directly from Corollary 2.2 that

Proposition 2.4. *There exists $C > 0$ such that*

$$\sup_{t\in\mathbb{R}_+}\ \sup_{0<\varepsilon<1}\ \sup_{x_0\in\mathbb{R}^2}\ \sup_{0<r<1/2} |\ln r|^{1/2}\int_{B(x_0,r)}\rho_\varepsilon(t,x)\,dx < +\infty.$$

2.3. **Some estimates for the charge.** In this paragraph we focus on the dynamics of the charge by looking for estimates on the time integral $\int E_\varepsilon(\xi_\varepsilon)\,dt$.

Proposition 2.5. *Let $(x,v) \in \mathbb{R}^2\times\mathbb{R}^2$. Then for all $t\in\mathbb{R}_+$,*

$$\frac{1}{|X_\varepsilon(t,x,v)-\xi_\varepsilon(t)|} \le \varepsilon^2\frac{d^2}{dt^2}|X_\varepsilon(t,x,v)-\xi_\varepsilon(t)| + \frac{|V_\varepsilon(t,x,v)-\eta_\varepsilon(t)|}{\varepsilon} + |E_\varepsilon(t,X_\varepsilon(t,x,v))| + |E_\varepsilon(t,\xi_\varepsilon(t))|.$$

Proof. We compute, writing $(X_\varepsilon,V_\varepsilon) = (X_\varepsilon(t,x,v),V_\varepsilon(t,x,v))$ for simplicity,

$$\frac{d}{dt}|X_\varepsilon-\xi_\varepsilon| = \left(\frac{X_\varepsilon-\xi_\varepsilon}{|X_\varepsilon-\xi_\varepsilon|},\frac{V_\varepsilon-\eta_\varepsilon}{\varepsilon}\right),$$

so

$$\begin{aligned}\frac{d^2}{dt^2}|X_\varepsilon-\xi_\varepsilon| &= \frac{|V_\varepsilon-\eta_\varepsilon|^2}{\varepsilon^2|X_\varepsilon-\xi_\varepsilon|} + \frac{1}{\varepsilon^3}\left(\frac{X_\varepsilon-\xi_\varepsilon}{|X_\varepsilon-\xi_\varepsilon|},V_\varepsilon^\perp-\eta_\varepsilon^\perp\right)\\ &\quad+\frac{1}{\varepsilon^2}\left(\frac{X_\varepsilon-\xi_\varepsilon}{|X_\varepsilon-\xi_\varepsilon|},E_\varepsilon(X_\varepsilon)-E_\varepsilon(\xi_\varepsilon)\right)+\frac{1}{\varepsilon^2}\frac{1}{|X_\varepsilon-\xi_\varepsilon|}\\ &\quad-\frac{1}{\varepsilon}\frac{(X_\varepsilon-\xi_\varepsilon,V_\varepsilon-\eta_\varepsilon)}{|X_\varepsilon-\xi_\varepsilon|^2}\left(\frac{X_\varepsilon-\xi_\varepsilon}{|X_\varepsilon-\xi_\varepsilon|},\frac{V_\varepsilon-\eta_\varepsilon}{\varepsilon}\right),\end{aligned}$$

therefore

$$\begin{aligned}\frac{d^2}{dt^2}|X_\varepsilon-\xi_\varepsilon| &\ge \frac{|V_\varepsilon-\eta_\varepsilon|^2}{\varepsilon^2|X_\varepsilon-\xi_\varepsilon|} - \frac{1}{\varepsilon^2}\frac{(X_\varepsilon-\xi_\varepsilon,V_\varepsilon-\eta_\varepsilon)^2}{|X_\varepsilon-\xi_\varepsilon|^3}\\ &\quad-\frac{1}{\varepsilon^3}|V_\varepsilon-\eta_\varepsilon|-\frac{1}{\varepsilon^2}|E_\varepsilon(X_\varepsilon)|-\frac{1}{\varepsilon^2}|E_\varepsilon(\xi_\varepsilon)|+\frac{1}{\varepsilon^2}\frac{1}{|X_\varepsilon-\xi_\varepsilon|}\end{aligned}$$

and the conclusion follows. □

Corollary 2.6. *We have*

$$|E_\varepsilon(t,\xi_\varepsilon(t))| \le C\varepsilon^2\frac{d^2}{dt^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}|x-\xi_\varepsilon(t)|f_\varepsilon(t,x,v)\,dx\,dv + \frac{C}{\varepsilon}.$$

Proof. Integrating the inequality given by Proposition 2.5 with respect to the measure $f_\varepsilon^0(x,v)\,dx\,dv$ we get after changing variable

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{\rho_\varepsilon(t,x)}{|x-\xi_\varepsilon(t)|}\,dx &\le \varepsilon^2\frac{d^2}{dt^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}|x-\xi_\varepsilon(t)|f_\varepsilon(t,x,v)\,dx\,dv\\ &\quad+\frac{1}{\varepsilon}\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v-\eta_\varepsilon|f_\varepsilon(t,x,v)\,dx\,dv\\ &\quad+\int_{\mathbb{R}^2}|E_\varepsilon(t,x)|\rho_\varepsilon(t,x)\,dx + \|f_\varepsilon^0\|_{L^1}|E_\varepsilon(t,\xi_\varepsilon)|.\end{aligned}$$

On the one hand, we have by (2.5)

$$\int_{\mathbb{R}^2} |E_\varepsilon(t,x)|\rho_\varepsilon(t,x)\,dx \leq \|E_\varepsilon(t)-\overline{E}_\varepsilon\|_{L^2}\|\rho_\varepsilon(t)\|_{L^2} + \|\overline{E}_\varepsilon\|_{L^\infty}\|f_\varepsilon^0\|_{L^1}$$
$$\leq C(\|f_\varepsilon^0\|_{L^\infty}^{1/2} + \|f_\varepsilon^0\|_{L^1}) \leq \frac{C}{\varepsilon},$$

where we have used the bounds (1.6) and (1.7) in the last inequality.

On the other hand, Corollary 2.2 yields by Cauchy-Schwarz inequality

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v-\eta_\varepsilon(t)|f_\varepsilon(t,x,v)\,dx\,dv$$
$$\leq C\left(\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^2 f_\varepsilon(t,x,v)\,dx\,dv + |\eta_\varepsilon(t)|^2\|f_\varepsilon^0\|_{L^1}\right)^{1/2}\|f_\varepsilon(t)\|_{L^1}^{1/2} \leq C.$$

Finally,

$$\|f_\varepsilon^0\|_{L^1}|E_\varepsilon(t,\xi_\varepsilon)| \leq \sup_{0<\varepsilon<1}\|f_\varepsilon^0\|_{L^1}\int_{\mathbb{R}^2}\frac{\rho_\varepsilon(t,x)}{|x-\xi_\varepsilon(t)|}\,dx,$$

thus we obtain the desired estimate in view of the assumption (1.8). □

Corollary 2.7. *We have*

$$\int_s^t |E_\varepsilon(\tau,\xi_\varepsilon(\tau))|\,d\tau \leq C\varepsilon + \frac{C}{\varepsilon}(t-s).$$

Proof. We set

$$I_\varepsilon(t) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} |x-\xi_\varepsilon(t)|f_\varepsilon(t,x,v)\,dx\,dv = \iint_{\mathbb{R}^2\times\mathbb{R}^2} |X_\varepsilon(t,x,v)-\xi_\varepsilon(t)|f_\varepsilon^0(x,v)\,dx\,dv,$$

so that by the system (2.3), using again the estimates of Corollary 2.2 we get

$$\left|\frac{d}{dt}I_\varepsilon(t)\right| \leq \frac{1}{\varepsilon}\iint_{\mathbb{R}^2\times\mathbb{R}^2} |V_\varepsilon(t,x,v)-\eta_\varepsilon(t)|f_\varepsilon^0(x,v)\,dx\,dv \leq \frac{C}{\varepsilon}.$$

Then, integrating in time the inequality in Corollary 2.6 yields

$$\int_s^t |E_\varepsilon(\tau,\xi_\varepsilon(\tau))| \leq C\varepsilon^2\left(\left|\frac{dI_\varepsilon}{dt}(t)\right| + \left|\frac{dI_\varepsilon}{dt}(s)\right|\right) + \frac{C(t-s)}{\varepsilon},$$

which implies the claim of the corollary. □

2.4. **Weak formulation.** In order to study the asymptotical equation for (1.1), we reexpress the system (1.1), using a weak formulation that was derived in [13] as a starting point in the case without charge.

Proposition 2.8. *We have*

$$\partial_t\rho_\varepsilon + \nabla_x\cdot\Big((E_\varepsilon^\perp + L_\varepsilon^\perp)\rho_\varepsilon\Big)$$
$$= \nabla_x\cdot\left(\left[\nabla_x\cdot\int_{\mathbb{R}^2} v\otimes v f_\varepsilon\,dv\right]^\perp\right) + \varepsilon\nabla_x\cdot\partial_t\int_{\mathbb{R}^2} v^\perp f_\varepsilon\,dv,$$

where we recall the definition (2.1) *for* L_ε.

Proof. See the equations (3.8) and (3.9) in the proof of Lemma 3.2 in [13], substituting E_ε by $E_\varepsilon + L_\varepsilon$. □

In order to deal with the singular term $L_\varepsilon^\perp \rho_\varepsilon$ as ε tends to zero, for ρ_ε converging to a Radon measure, we shall actually symmetrize the nonlinear term as in Definition 1.1 with respect to the total measure $\rho_\varepsilon + \delta_{\xi_\varepsilon}$. This can be done by taking into account the dynamics of the charge.

Proposition 2.9. *Let $\Phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$. We have for all $t \geq 0$*

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Phi(t,x)\rho_\varepsilon(t,x)\,dx + \Phi(t,\xi_\varepsilon(t)) - \int_{\mathbb{R}^2} \Phi(0,x)\rho_\varepsilon(0,x)\,dx - \Phi(0,\xi_\varepsilon(0))\\
&= \int_0^t \int_{\mathbb{R}^2} \partial_t\Phi(s,x)\rho_\varepsilon(s,x)\,ds\,dx + \int_0^t \partial_t\Phi(s,\xi_\varepsilon(s))\,ds\\
&+ \int_0^t \mathcal{H}_{\Phi(s,\cdot)}[\rho_\varepsilon(s,\cdot)+\delta_{\xi_\varepsilon(s)}, \rho_\varepsilon(s,\cdot)+\delta_{\xi_\varepsilon(s)}]\,ds\\
&- \int_0^t \int_{\mathbb{R}^2} \left(D\nabla^\perp\Phi(s,x) : \int_{\mathbb{R}^2} v\otimes v f_\varepsilon(s,x,v)\,dv \right) dx\,ds\\
&- \int_0^t \eta_\varepsilon(s)\cdot \left(D\nabla^\perp\Phi(s,\xi_\varepsilon(s))\,\eta_\varepsilon(s)\right) ds\\
&+ R_\varepsilon(t),
\end{aligned}$$

where

$$|R_\varepsilon(t)| \leq C\|\Phi\|_{W^{3,\infty}(\mathbb{R}_+\times\mathbb{R}^2)}(1+t)\varepsilon.$$

We recall that $\mathcal{H}_\Phi[\cdot,\cdot]$ is defined in Definition 1.1.

Proof. We apply Proposition 2.8 with the test function Φ. After symmetrizing the term $E_\varepsilon^\perp \rho_\varepsilon$ as in Definition 1.1, we obtain

(2.6)

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Phi(t,x)\rho_\varepsilon(t,x)\,dx - \int_{\mathbb{R}^2} \Phi(0,x)\rho_\varepsilon(0,x)\,dx = \int_0^t \int_{\mathbb{R}^2} \partial_t\Phi(s,x)\rho_\varepsilon(s,x)\,ds\,dx\\
&+ \int_0^t \mathcal{H}_{\Phi(s,\cdot)}[\rho_\varepsilon(s),\rho_\varepsilon(s)]\,ds\\
&+ \int_0^t \int_{\mathbb{R}^2} L_\varepsilon^\perp(s,x)\cdot\nabla\Phi(s,x)\rho_\varepsilon(s,x)\,dx\,ds\\
&- \int_0^t \int_{\mathbb{R}^2} \left(D\nabla^\perp\Phi(s,x) : \int_{\mathbb{R}^2} v\otimes v f_\varepsilon(s,x,v)\,dv \right) dx\,ds + R_\varepsilon^1,
\end{aligned}$$

where

$$\begin{aligned}
R_\varepsilon^1 &= \varepsilon \int_0^t \iint_{\mathbb{R}^2} f_\varepsilon(s,x,v)v^\perp\cdot\partial_t\nabla\Phi(s,x)\,dv\,dx\,ds\\
&- \varepsilon \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon(t,x,v)v^\perp\cdot\nabla\Phi(t,x)\,dx\,dv + \varepsilon \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)v^\perp\cdot\nabla\Phi(0,x)\,dx\,dv,
\end{aligned}$$

so that by Corollary 2.2,

$$|R_\varepsilon^1| \leq C\varepsilon\left(t\|\partial_t\nabla\Phi\|_{L^\infty} + \|\nabla\Phi\|_{L^\infty}\right).$$

Next, we insert the motion of the point charge. We introduce the combination

$$h_\varepsilon(t) = \xi_\varepsilon(t) + \varepsilon\eta_\varepsilon(t)^\perp,$$

so that by the mean-value theorem and by Corollary 2.2 again,

$$\Phi(t,\xi_\varepsilon(t))-\Phi(0,\xi_\varepsilon(0))=\Phi(t,h_\varepsilon(t))-\Phi(0,h_\varepsilon(0))+R_\varepsilon^2,$$

with

$$|R_\varepsilon^2|\leq C\varepsilon\|\nabla\Phi\|_{L^\infty}.$$

On the other hand, we observe that

$$\dot{h}_\varepsilon(t)=E_\varepsilon^\perp(t,\xi_\varepsilon(t)),$$

therefore

$$\begin{aligned}\Phi(t,h_\varepsilon(t))-\Phi(0,h_\varepsilon(0))&=\int_0^t\partial_t\Phi(s,h_\varepsilon(s))\,ds+\int_0^t E_\varepsilon^\perp(s,\xi_\varepsilon(s))\cdot\nabla\Phi(s,h_\varepsilon(s))\,ds\\&=\int_0^t\partial_t\Phi(s,\xi_\varepsilon(s))\,ds+\int_0^t E_\varepsilon^\perp(s,\xi_\varepsilon(s))\cdot\nabla\Phi(s,\xi_\varepsilon(s))\,ds\\&+\int_0^t E_\varepsilon^\perp(s,\xi_\varepsilon(s))\cdot[\nabla\Phi(s,h_\varepsilon(s))-\nabla\Phi(s,\xi_\varepsilon(s))]\,ds\\&+R_\varepsilon^3,\end{aligned}$$

where

$$|R_\varepsilon^3|\leq C\varepsilon t(\|\partial_t\nabla\Phi\|_{L^\infty}+\|\partial_t\Phi\|_{L^\infty}).$$

Moreover, we observe that

$$\begin{aligned}&\int_{\mathbb{R}^2}L_\varepsilon^\perp(s,x)\cdot\nabla\Phi(s,x)\rho_\varepsilon(s,x)\,dx+E_\varepsilon^\perp(s,\xi_\varepsilon(s))\cdot\nabla\Phi(s,\xi_\varepsilon(s))\\&\qquad=\int_{\mathbb{R}^2}L_\varepsilon^\perp(s,x)\cdot\big[\nabla\Phi(s,x)-\nabla\Phi(s,\xi_\varepsilon(s))\big]\rho_\varepsilon(s,x)\,dx\\&\qquad=2\mathcal{H}_{\Phi(s,\cdot)}[\rho_\varepsilon(s),\delta_{\xi_\varepsilon(s)}].\end{aligned}$$

Noticing that

$$\mathcal{H}_\Phi[\rho_\varepsilon+\delta_{\xi_\varepsilon},\rho_\varepsilon+\delta_{\xi_\varepsilon}]=\mathcal{H}_\Phi[\rho_\varepsilon,\rho_\varepsilon]+2\mathcal{H}_\Phi[\rho_\varepsilon,\delta_{\xi_\varepsilon}]$$

and inserting this latter in (2.6) we obtain

(2.7)
$$\begin{aligned}&\int_{\mathbb{R}^2}\Phi(t,x)\rho_\varepsilon(t,x)\,dx-\int_{\mathbb{R}^2}\Phi(0,x)\rho_\varepsilon(0,x)\,dx+\Phi(t,\xi_\varepsilon(t))-\Phi(0,\xi_\varepsilon(0))\\&=\int_0^t\int_{\mathbb{R}^2}\partial_t\Phi(s,x)\rho_\varepsilon(s,x)\,ds\,dx+\int_0^t\partial_t\Phi(s,\xi_\varepsilon(s))\,ds\\&+\int_0^t\mathcal{H}_{\Phi(s,\cdot)}[\rho_\varepsilon(s)+\delta_{\xi_\varepsilon(s)},\rho_\varepsilon(s)+\delta_{\xi_\varepsilon(s)}]\,ds\\&-\int_0^t\int_{\mathbb{R}^2}\left(D\nabla^\perp\Phi(s,x):\int_{\mathbb{R}^2}v\otimes vf_\varepsilon(s,x,v)\,dv\right)dx\,ds\\&+\int_0^t E_\varepsilon^\perp(s,\xi_\varepsilon(s))\cdot[\nabla\Phi(s,h_\varepsilon(s))-\nabla\Phi(s,\xi_\varepsilon(s))]\,ds\\&+R_\varepsilon^1+R_\varepsilon^2+R_\varepsilon^3.\end{aligned}$$

We next estimate the last (non remainder) term in (2.7). We have

$$\int_0^t E_\varepsilon^\perp(s,\xi_\varepsilon(s))\cdot[\nabla\Phi(s,h_\varepsilon(s))-\nabla\Phi(s,\xi_\varepsilon(s))]\,ds$$
$$=\int_0^t E_\varepsilon^\perp(s,\xi_\varepsilon(s))\cdot[D\nabla\Phi(s,\xi_\varepsilon(s))\,\varepsilon\eta_\varepsilon^\perp(s)]\,ds+R_\varepsilon^4,$$

with

$$|R_\varepsilon^4|\le C\varepsilon^2\sup_{t\in\mathbb{R}_+}|\eta_\varepsilon(t)|^2\|D^3\Phi\|_{L^\infty}\int_0^t|E_\varepsilon(s,\xi_\varepsilon(s))|\,ds\le C\varepsilon\|D^3\Phi\|_{L^\infty}(\varepsilon^2+t),$$

where we used Corollary 2.7.

We now claim that

$$\int_0^t E_\varepsilon^\perp(s,\xi_\varepsilon(s))\cdot[D\nabla\Phi(s,\xi_\varepsilon(s))\,\varepsilon\eta_\varepsilon^\perp(s)]\,ds$$
$$=-\int_0^t\eta_\varepsilon(s)\cdot[D\nabla^\perp\Phi(s,\xi_\varepsilon(s))\,\eta_\varepsilon(s)]\,ds+R_\varepsilon^5, \tag{2.8}$$

where

$$|R_\varepsilon^5|\le C\varepsilon(1+t)\|\Phi\|_{W^{3,\infty}},$$

which together with (2.7) and the estimates on the remainders will yield the conclusion of the proposition.

Proof of (2.8)**.** Recalling that $\varepsilon E_\varepsilon(\xi_\varepsilon)^\perp=\eta_\varepsilon+\varepsilon^2\dot\eta_\varepsilon^\perp$, we have

$$\int_0^t\varepsilon E_\varepsilon^\perp(s,\xi_\varepsilon(s))\cdot[D\nabla\Phi(s,\xi_\varepsilon(s))\,\eta_\varepsilon^\perp(s)]\,ds$$
$$=\int_0^t\eta_\varepsilon(s)\cdot[D\nabla\Phi(s,\xi_\varepsilon(s))\,\eta_\varepsilon^\perp(s)]\,ds+\varepsilon^2\int_0^t\dot\eta_\varepsilon^\perp(s)\cdot[D\nabla\Phi(s,\xi_\varepsilon(s))\,\eta_\varepsilon^\perp(s)]\,ds$$
$$=I+J.$$

On the one hand, for $F=\nabla\Phi(s,\cdot)$, a simple computation shows that

$$a\cdot(DF\,a^\perp)=-a\cdot(D(F^\perp)\,a),\quad\forall a\in\mathbb{R}^2,$$

hence

$$I=-\int_0^t\eta_\varepsilon(s)\cdot[D\nabla^\perp\Phi(s,\xi_\varepsilon(s))\,\eta_\varepsilon(s)]\,ds.$$

On the other hand, integrating by parts in J we get

$$J=-\varepsilon^2\int_0^t\eta_\varepsilon^\perp(s)\cdot\frac{d}{ds}[D\nabla\Phi(s,\xi_\varepsilon(s))\,\eta_\varepsilon^\perp(s)]\,ds+R_\varepsilon^6, \tag{2.9}$$

where

$$R_\varepsilon^6=\varepsilon^2\eta_\varepsilon^\perp(t)\cdot[D\nabla\Phi(t,\xi_\varepsilon(t))\,\eta_\varepsilon^\perp(t)]-\varepsilon^2\eta_\varepsilon^\perp(0)\cdot[D\nabla\Phi(0,\xi_\varepsilon(0))\,\eta_\varepsilon^\perp(0)]$$

so that

$$|R_\varepsilon^6|\le C\varepsilon^2\|D^2\Phi\|_{L^\infty}.$$

Next, we compute

$$-\varepsilon^2\int_0^t\eta_\varepsilon^\perp(s)\cdot\frac{d}{ds}[D\nabla\Phi(s,\xi_\varepsilon(s))\,\eta_\varepsilon^\perp(s)]\,ds$$
$$=-\varepsilon^2\int_0^t\eta_\varepsilon^\perp(s)\cdot[\frac{d}{ds}(D\nabla\Phi(s,\xi_\varepsilon(s)))\,\eta_\varepsilon^\perp(s)]\,ds-\varepsilon^2\int_0^t\eta_\varepsilon^\perp(s)\cdot[D\nabla\Phi(s,\xi_\varepsilon(s))\,\dot\eta_\varepsilon^\perp(s)]\,ds$$

and using that $|\dot{\xi}_\varepsilon| = |\eta_\varepsilon|/\varepsilon \leq C/\varepsilon$ in the first term of the RHS we obtain

$$\begin{aligned}
&-\varepsilon^2 \int_0^t \eta_\varepsilon^\perp(s) \cdot \frac{d}{ds}[D\nabla\Phi(s,\xi_\varepsilon(s))\, \eta_\varepsilon^\perp(s)]\, ds \\
&= -\varepsilon^2 \int_0^t \eta_\varepsilon^\perp(s) \cdot [D\nabla\Phi(s,\xi_\varepsilon(s))\, \dot{\eta}_\varepsilon^\perp(s)]\, ds + R_\varepsilon^7,
\end{aligned}$$

with

$$|R_\varepsilon^7| \leq Ct\,\varepsilon\, \|D^3\Phi\|_{L^\infty}.$$

Coming back to (2.9), we obtain therefore

$$J = -\varepsilon^2 \int_0^t \eta_\varepsilon^\perp(s) \cdot [D\nabla\Phi(s,\xi_\varepsilon(s))\, \dot{\eta}_\varepsilon^\perp(s)]\, ds + R_\varepsilon^8$$

with

$$|R_\varepsilon^8| \leq C\varepsilon(1+t)\|\Phi\|_{W^{3,\infty}}.$$

Now, we observe that for all $F \in C^1(\mathbb{R}^2,\mathbb{R}^2)$, we have

$$a \cdot (DF\, b) = b \cdot (DF\, a) + \mathrm{curl}(F) a^\perp \cdot b, \quad \forall (a,b) \in \mathbb{R}^2 \times \mathbb{R}^2,$$

where we have defined $\mathrm{curl}(F) = \partial_2 F_1 - \partial_1 F_2$. Applying this to $F = \nabla\Phi(s,\cdot)$, so that $\mathrm{curl}(F) = 0$, to $a = \eta_\varepsilon^\perp$ and $b = \dot{\eta}_\varepsilon^\perp$, we get

$$J = -J + R_\varepsilon^8,$$

hence (2.8) follows.

□

2.5. **Estimate on the trajectory of the charge.**

Corollary 2.10. *Let $T > 0$. There exists $K_0 > 1$ and $\varepsilon_0 > 0$, depending only on T, such that for all $0 < \varepsilon < \varepsilon_0$ and for all $0 \leq s < t \leq T$,*

$$|\xi_\varepsilon(t) - \xi_\varepsilon(s)| \leq K_0 \left((t-s)^{1/2} + \varepsilon^{1/3} \right). \tag{2.10}$$

Proof. By Proposition 2.9 and by Remark 1.2, we have for all $\Phi \in C_c^\infty(\mathbb{R}^2)$ and for all $0 \leq s < t \leq T$

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \Phi(x)\rho_\varepsilon(t,x)\, dx + \Phi(\xi_\varepsilon(t)) - \int_{\mathbb{R}^2} \Phi(x)\rho_\varepsilon(s,x)\, dx - \Phi(\xi_\varepsilon(s)) \right| \\
&\leq C\|\Phi\|_{W^{2,\infty}}(t-s) \sup_{\tau\in[0,T]} \left(\|\rho_\varepsilon(\tau)\|_{L^1}^2 + \|\rho_\varepsilon(\tau)\|_{L^1} + \iint_{\mathbb{R}^2} |v|^2 f_\varepsilon(\tau,x,v)\, dx\, dv + |\eta_\varepsilon(\tau)|^2 \right) \\
&+ |R_\varepsilon(t)| + |R_\varepsilon(s)|,
\end{aligned}$$

and by Corollary 2.2 it follows that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \Phi(x)\rho_\varepsilon(t,x)\, dx + \Phi(\xi_\varepsilon(t)) - \int_{\mathbb{R}^2} \Phi(x)\rho_\varepsilon(s,x)\, dx - \Phi(\xi_\varepsilon(s)) \right| \\
&\leq C\|\Phi\|_{W^{2,\infty}}(t-s) + C(1+T)\,\varepsilon\,\|\Phi\|_{W^{3,\infty}}.
\end{aligned} \tag{2.11}$$

Let $K_1 > 1$ be a sufficiently large number to be determined later, depending only on T. Let $\varepsilon_0 = K_1^{-6}$. We first claim that

$$\begin{aligned}
&\forall 0 < \varepsilon < \varepsilon_0, \quad \forall 0 \leq s < t \leq T \quad \text{with } t-s \leq K_1^{-4}, \\
&|\xi_\varepsilon(t) - \xi_\varepsilon(s)| \leq 2K_1 \left((t-s)^{1/2} + \varepsilon^{1/3} \right).
\end{aligned} \tag{2.12}$$

Otherwise, there exist $0<\varepsilon<\varepsilon_0$ and $0\le s<t\le T$ with $t-s\le K_1^{-4}$ but $|\xi_\varepsilon(t)-\xi_\varepsilon(s)|>2K_1\left((t-s)^{1/2}+\varepsilon^{1/3}\right)$. We set

$$\Phi(x)=\chi\left(\frac{x-\xi_\varepsilon(s)}{K_1\left((t-s)^{1/2}+\varepsilon^{1/3}\right)}\right),$$

where χ is a cut-off function such that $\chi=1$ on $B(0,1)$ and χ vanishes on $B(0,2)^c$. In particular, we have $\Phi(\xi_\varepsilon(t))=0$ and $\Phi(\xi_\varepsilon(s))=1$. Moreover, since $K_1(t-s)^{1/2}<1$ and $K_1\varepsilon^{1/3}\le K_1^{-1}<1$, we have

$$\|\Phi\|_{W^{2,\infty}}\le CK_1^{-2}(t-s)^{-1},\quad \|\Phi\|_{W^{3,\infty}}\le CK_1^{-3}\varepsilon^{-1}.$$

In view of (2.11) and using Proposition 2.4, we get

$$\begin{aligned}1&\le 2\sup_{\tau\in[0,T]}\int_{B(\xi_\varepsilon(\tau),2K_1\left((t-s)^{1/2}+\varepsilon^{1/3}\right)}\rho_\varepsilon(\tau,x)\,dx+C(1+T)\,K_1^{-3}\\&\le 2\sup_{\tau\in[0,T]}\int_{B(\xi_\varepsilon(\tau),2K_1^{-3})}\rho_\varepsilon(\tau,x)\,dx+C(1+T)\,K_1^{-3}\\&\le C\left(|\ln(2K_1^{-3})|^{-1/2}+(1+T)\,K_1^{-3}\right)\le\frac12\end{aligned}$$

if we choose K_1 sufficiently large (note that this choice may be done explicit). This yields a contradiction, and (2.12) follows.

Now, we split $[0,T]$ as $[0,T]=\cup_{i=0}^{N-1}[t_i,t_{i+1}]$, with $|t_{i+1}-t_i|=K_1^{-4}$, for $i=1,\dots N-1$, and $|t_1-t_0|\le K_1^{-4}$. Let $0<\varepsilon<\varepsilon_0$. Let $0\le s<t\le T$ such that $|t-s|>K_1^{-4}$ and $i<j$ such that $t\in[t_i,t_{i+1})$ and $s\in[t_j,t_{j+1})$. We have by (2.12)

$$\begin{aligned}|\xi_\varepsilon(t)-\xi_\varepsilon(s)|&\le|\xi_\varepsilon(t)-\xi_\varepsilon(t_i)|+|\xi_\varepsilon(t_i)-\xi_\varepsilon(t_j)|+|\xi_\varepsilon(t_j)-\xi_\varepsilon(s)|\\&\le 2K_1\left(|t-t_i|^{1/2}+2\varepsilon^{1/3}+|s-t_j|^{1/2}\right)\\&+2K_1(N+1)\left(K_1^{-2}+\varepsilon^{1/3}\right)\\&\le 2K_1^{-1}(N+3)+2K_1(N+3)\varepsilon^{1/3}\\&\le 2K_1(N+3)\left(|t-s|^{1/2}+\varepsilon^{1/3}\right).\end{aligned}$$

Taking $K_0=2K_1(N+3)$, we are led to the result. □

2.6. **Time equicontinuity for the densities.**

In this paragraph we prove the following

Lemma 2.11. *Let $T>0$. There exists $K_1>0$ and $\varepsilon_0>0$, depending only on T, such that for all $0<\varepsilon<\varepsilon_0$ and for all $0\le s<t\le T$,*

$$\|\rho_\varepsilon(t)-\rho_\varepsilon(s)\|_{W^{-3,1}(\mathbb{R}^2)}\le K_1\left((t-s)^{1/2}+\varepsilon^{1/3}\right).$$

Proof. As for Corollary 2.10, the proof relies on Proposition 2.9 and Remark 1.2. By (2.11) we have for all $\Phi\in C_c^\infty(\mathbb{R}^2)$ and for all $0\le s<t\le T$

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\Phi(x)\rho_\varepsilon(t,x)\,dx-\int_{\mathbb{R}^2}\Phi(x)\rho_\varepsilon(s,x)\,dx\right|\\&\le|\Phi(\xi_\varepsilon(t))-\Phi(\xi_\varepsilon(s))|+C\|\Phi\|_{W^{2,\infty}}(t-s)+C(1+T)\,\varepsilon\,\|\Phi\|_{W^{3,\infty}}.\end{aligned}$$

In view of Corollary 2.10, this yields by the mean-value theorem

$$\Big|\int_{\mathbb{R}^2}\Phi(x)\rho_\varepsilon(t,x)\,dx-\int_{\mathbb{R}^2}\Phi(x)\rho_\varepsilon(s,x)\,dx\Big|\leq K_0\|\nabla\Phi\|_{L^\infty}\Big((t-s)^{1/2}+\varepsilon^{1/3}\Big)$$
$$+C\|\Phi\|_{W^{2,\infty}}(t-s)+C(1+T)\,\varepsilon\,\|\Phi\|_{W^{3,\infty}},$$

from which the conclusion follows.

□

Remark 2.12. In [13] (see also [25]), it is proved instead that the sequence of densities is uniformly bounded in $C^{1/2}(\mathbb{R}_+,W^{-2,1}(\mathbb{R}^2))$. Here we loose one derivative, due to the contribution of the point charge appearing in the estimate for the remainder in the proof of Proposition 2.9.

2.7. **Compactness.** In this paragraph we use the previous estimates to show that

Proposition 2.13. *There exists a subsequence such that* (ρ_{ε_n}) *converges to some* ρ *in* $C_w(\mathbb{R}_+,\mathcal{M}_+(\mathbb{R}^2))$ *as* $n\to+\infty$*, and* ρ *belongs to* $L^\infty(\mathbb{R}_+,H^{-1}(\mathbb{R}^2))$*. The sequence* (ξ_{ε_n}) *converges to some* ξ *in* $C^{1/2}([0,T],\mathbb{R}^2)$ *for all* $T>0$*.*

To show Proposition 2.13 we shall use a straightforward adaptation of Ascoli's theorem:

Lemma 2.14. *Let* $T>0$*. Let* (F,d) *be a complete metric space. Let* (g_ε) *be a family of* $C([0,T],F)$ *such that*

(1) *For all* $t\in[0,T]$*, the family* $(g_\varepsilon(t))$ *is relatively compact in* F*;*
(2) *There exists* $C>0$ *and a sequence* $r_\varepsilon\to 0$ *as* $\varepsilon\to+0$ *such that for all* $t,s\in[0,T]$*, for all* $\varepsilon>0$*,* $d(g_\varepsilon(t),g_\varepsilon(s))\leq C|t-s|^{1/2}+r_\varepsilon$*.*

Then the family (g_ε) *is relatively compact in* $C([0,T],F)$*.*

Recalling that (ρ_ε) is uniformly bounded in $L^\infty(\mathbb{R}_+,\mathcal{M}_+(\mathbb{R}^2))$ and in view of Lemma 2.11, we can apply this Lemma to $F=W^{-3,1}$ for any $T>0$. Arguing as in the proof of Lemma 3.2 in [31, Lemma 3.2] and using a diagonal argument, we can then show the existence of $\varepsilon_n\to 0$ as $n\to+\infty$ such that (ρ_{ε_n}) converges to some ρ in $C_w(\mathbb{R}_+,\mathcal{M}_+(\mathbb{R}^2))$. By (2.5), the family (ρ_ε) is bounded in $L^\infty(\mathbb{R}_+,H^{-1}(\mathbb{R}^2))$ so we infer that ρ belongs to $L^\infty(\mathbb{R}_+,H^{-1}(\mathbb{R}^2))$.

On the other hand, the sequence (ξ_{ε_n}) is uniformly bounded in $L^\infty(\mathbb{R}_+,\mathbb{R}^2)$ in view of Corollary 2.2. Recalling that Corollary 2.10 holds, applying Lemma 2.14 and a diagonal argument, we obtain a subsequence, still denoted in the same way, such that (ξ_{ε_n}) converges to some ξ in $C^{1/2}([0,T],\mathbb{R}^2)$ for all $T>0$. Therefore the proposition is proved.

2.8. **Existence of a defect measure.** The following lemma in an extension of Lemma 3.3 in [13].

Lemma 2.15 ([13], Lemma 3.3)**.** *Under the assumptions of Theorem 1.4, the sequence* (f_{ε_n}) *is relatively compact in* $L^\infty(\mathbb{R}_+,\mathcal{M}(\mathbb{R}^2\times\mathbb{R}^2))$ *weak -* $*$*. Moreover, any accumulation point* f *satisfies*

$$\nabla_v\cdot(v^\perp f)=0,\quad \textit{in } \mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^2\times\mathbb{R}^2).$$

Proof. We follow the arguments of [13]. We have

$$v^\perp \cdot \nabla_v f_\varepsilon = -\partial_t(\varepsilon^2 f_\varepsilon) - \nabla_x \cdot (\varepsilon v f_\varepsilon) - \nabla_v \cdot (\varepsilon(L_\varepsilon + E_\varepsilon) f_\varepsilon).$$

By Corollary 2.2, the first two terms of the RHS converge to zero in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R}^2$. We next focus on the last term. Let Φ be a test function with support included in $[0,R] \times B_{\mathbb{R}^2}(0,R)^2$ for some $R > 0$. We have by (2.5)

$$\begin{aligned}
&\varepsilon \left| \int_0^{+\infty} \iint_{\mathbb{R}^2\times\mathbb{R}^2} E_\varepsilon(t,x) \cdot \nabla_v \Phi(t,x,v) f_\varepsilon(t,x,v)\, dx\, dv\, dt \right| \\
&\le \varepsilon \|\nabla_v \Phi\|_{L^\infty} R \left(\|\rho_\varepsilon\|_{L^\infty(L^2)} \|E_\varepsilon - \overline{E}_\varepsilon\|_{L^\infty(L^2)} + \|\rho_\varepsilon\|_{L^\infty(L^1)} \|\overline{E}_\varepsilon\|_{L^\infty} \right) \\
&\le C \|\nabla_v \Phi\|_{L^\infty} \left(\varepsilon \|f_\varepsilon^0\|_{L^\infty}^{1/2} + \varepsilon \right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\varepsilon \left| \int_0^{+\infty} \iint_{\mathbb{R}^2\times\mathbb{R}^2} L_\varepsilon(t,x) \cdot \nabla_v \Phi(t,x,v) f_\varepsilon(t,x,v)\, dx\, dv\, dt \right| \\
&\le \varepsilon \|\nabla_v \Phi\|_{L^\infty} \pi R^2 \, \|f_\varepsilon^0\|_{L^\infty} \int_0^R \left(\int_{B(\xi_\varepsilon(t), \|f_\varepsilon^0\|_{L^\infty}^{-1/2})} \frac{dx}{|x - \xi_\varepsilon(t)|} \right) dt \\
&+ \varepsilon \|\nabla_v \Phi\|_{L^\infty} \|f_\varepsilon^0\|_{L^\infty}^{1/2} \int_0^R \int_{\mathbb{R}^2 \setminus B(\xi_\varepsilon(t), \|f_\varepsilon^0\|_{L^\infty}^{-1/2})} \int_{\mathbb{R}^2} f_\varepsilon(t,x,v)\, dx\, dv\, dt \\
&\le C \|\nabla_v \Phi\|_{L^\infty} \left(\varepsilon \|f_\varepsilon^0\|_{L^\infty}^{1/2} + \varepsilon \|f_\varepsilon^0\|_{L^\infty}^{1/2} \right) \\
&\le C \varepsilon \|f_\varepsilon^0\|_{L^\infty}^{1/2}.
\end{aligned}$$

Since $\varepsilon \|f_\varepsilon^0\|_{L^\infty}^{1/2}$ tends to zero as $\varepsilon \to 0$ by assumption (1.7), we infer that $v^\perp \cdot \nabla_v f_\varepsilon = \nabla_v \cdot (v^\perp f_\varepsilon) \to 0$ in the sense of distributions.

Now, the sequence (f_{ε_n}) is uniformly bounded in $L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^2 \times \mathbb{R}^2))$, thus it is relatively compact in $L^\infty(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^2 \times \mathbb{R}^2))$ weak - $*$. Let f be an accumulation point. In view of the previous estimates we obtain in the limit: $\nabla_v \cdot (v^\perp f) = 0$ in the sense of distributions.

□

Proposition 2.16. *Under the assumptions of Theorem 1.4, there exists a subsequence $(f_{\varepsilon_{n_k}})$ and there exists $f = f(t,x,|v|) \in L^\infty(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^2 \times \mathbb{R}^2))$ such that (f_{ε_n}) converges to f in $L^\infty(\mathbb{R}_+, \mathcal{M}_+(\mathbb{R}^2 \times \mathbb{R}^2))$ weak - $*$ and such that $\rho = \int f\, dv$. Moreover, there exists a measure $\nu_0 \in L^\infty(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^2 \times \mathbb{S}^1))$ such that for all Φ continuous on $\mathbb{S}^1$,*

$$\int_{\mathbb{R}^2} \left(f_{\varepsilon_{n_k}}(t,x,v) - f(t,x,|v|) \right) \Phi\left(\frac{v}{|v|} \right) |v|^2\, dv$$

converges to

$$\int_{\mathbb{S}^1} \Phi(\theta)\, d\nu_0(\theta)$$

in the sense of distributions on $\mathbb{R}_+ \times \mathbb{R}^2$. In particular, we have:

$$\int_{\mathbb{R}^2} v_1 v_2 f_{\varepsilon_{n_k}}\, dv \to \int_{\mathbb{S}^1} \theta_1 \theta_2 d\nu_0(\theta) \quad \textit{as } k \to +\infty$$

and

$$\int_{\mathbb{R}^2} (v_2^2 - v_1^2) f_{\varepsilon_{n_k}}\, dv \to \int_{\mathbb{S}^1} (\theta_2^2 - \theta_1^2) d\nu_0(\theta) \quad \textit{as } k \to +\infty$$

in the sense of distributions on $\mathbb{R}_+ \times \mathbb{R}^2$.

Proof. In view of Lemma 2.15, which extends [13, Lemma 3.3], we may argue exactly as in the beginning of the proof of [13, Theorem A] (pages 802–803) to find a measure ν_0 satisfying the previous properties. We do not provide the details here. □

2.9. **Proofs of Theorems 1.4 and Theorem 1.5 completed.** We consider the subsequence $(\rho_{\varepsilon_{n_k}})$ of (ρ_{ε_n}), which we still denote by (ρ_{ε_n}) for simplicity. In order to prove Theorem 1.4 we have to pass to the limit in the weak formulation given by Proposition 2.9. Let Φ be a test function and let $t \geq 0$. On the one hand, the compactness statements of Proposition 2.13 directly imply that

$$\int_{\mathbb{R}^2} \Phi(t,x)\rho_{\varepsilon_n}(t,x)\, dx + \Phi(t,\xi_{\varepsilon_n}(t)) - \int_{\mathbb{R}^2} \Phi(0,x)\rho_{\varepsilon_n}(0,x)\, dx - \Phi(0,\xi_{\varepsilon_n}(0))$$
$$- \int_0^t \int_{\mathbb{R}^2} \partial_t \Phi(s,x)\rho_{\varepsilon_n}(s,x)\, ds\, dx - \int_0^t \partial_t\Phi(s,\xi_{\varepsilon_n}(s))\, ds$$

converges to

$$\int_{\mathbb{R}^2} \Phi(t,x)\rho(t,x)\, dx + \Phi(t,\xi(t)) - \int_{\mathbb{R}^2} \Phi(0,x)\rho(0,x)\, dx - \Phi(0,\xi(0))$$
$$- \int_0^t \int_{\mathbb{R}^2} \partial_t \Phi(s,x)\rho(s,x)\, ds\, dx - \int_0^t \partial_t\Phi(s,\xi(s))\, ds$$

as n tends to ∞.

We turn now to the nonlinear terms in Proposition 2.9. The sequence (ρ_{ε_n}) is uniformly bounded in $L^\infty(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^2))$ and it satisfies the non-concentration property of Proposition 2.4. It was proved in [8] (see also [19, 31]) that this, together with the convergence of (ρ_{ε_n}) to ρ, implies the convergence of $\int \mathcal{H}_{\Phi(s,\cdot)}[\rho_{\varepsilon_n}(s), \rho_{\varepsilon_n}(s)]\, ds$ to $\int \mathcal{H}_{\Phi(s,\cdot)}[\rho(s),\rho(s)]\, ds$. By Proposition 2.4, this also yields the convergence of $\int \mathcal{H}_{\Phi(s,\cdot)}[\rho_{\varepsilon_n}(s), \delta_{\xi_{\varepsilon_n}}(s)]\, ds$ to $\int \mathcal{H}_{\Phi(s,\cdot)}[\rho(s), \delta_\xi(s)]\, ds$ (this is done in the proof of (25) in [26]).

We finally handle the last terms of Proposition 2.9. By virtue of Proposition 2.16 we already know that

$$\int_0^t \int_{\mathbb{R}^2} \left(D\nabla^\perp \Phi(s,x) : \int_{\mathbb{R}^2} v \otimes v f_{\varepsilon_n}(s,x,v)\, dv \right) dx\, ds$$

converges to

$$\int_0^t \int_{\mathbb{R}^2} \left(D\nabla^\perp \Phi(s,x) : \int_{\mathbb{S}^1} \theta \otimes \theta\, d\nu_0(s,x,\theta) \right) dx\, ds.$$

Moreover, since (η_{ε_n}) is uniformly bounded, there exists α and β in $L^\infty(\mathbb{R}_+, \mathbb{R})$ such that, after extracting a subsequence (still denoted in the same way),

$\eta_{\varepsilon_n,1}\eta_{\varepsilon_n,2}$ converges to α and $\eta_{\varepsilon_n,2}^2 - \eta_{\varepsilon_n,1}^2$ to 2β in $L^\infty(\mathbb{R}_+)$ weak - $*$. But $D\nabla^\perp\Phi(s,\xi_{\varepsilon_n}(s))$ converges to $D\nabla^\perp\Phi(s,\xi(s))$ locally uniformly on $\mathbb{R}_+$, so

$$\int_0^t \eta_{\varepsilon_n}(s)\cdot[D\nabla^\perp\Phi(s,\xi_{\varepsilon_n}(s))\,\eta_{\varepsilon_n}(s)]\,ds$$

converges to

$$\int_0^t (\partial_{11}-\partial_{22})\Phi(s,\xi(s))\alpha(s) + 2\partial_{12}\Phi(s,\xi(s))\beta(s)\,ds.$$

Considering the measure

$$d\nu(t,x) = \int_{\mathbb{S}^1}\theta\otimes\theta d\nu_0(t,x,\theta) + \begin{pmatrix} -\beta(t) & \alpha(t) \\ \alpha(t) & \beta(t)\end{pmatrix}\delta_{\xi(t)} \in L^\infty(\mathbb{R}_+,\mathcal{M}(\mathbb{R}^2)),$$

we conclude the proof.

3. Proof of Theorem 1.6

We begin with the derivation of the first equation for ρ. Let $\eta:\mathbb{R}_+\to[0,1]$ be smooth such that η vanishes on $[0,1]$ and $\eta=1$ on $[2,+\infty)$ and set $\eta_\delta=\eta(\cdot/\delta)$, which converges to 1 almost everywhere.

Let Φ be a test function and

$$\Phi_\delta(t,x) = \Phi(t,x)\eta_\delta\left(|x-\xi(t)|\right),$$

so that $\Phi_\delta(t,\xi(t)) = \partial_t\Phi_\delta(t,\xi(t))\equiv 0$ and $\nabla\Phi_\delta(t,\xi(t))\equiv 0$.

On the one hand, by Lebesgue's dominated convergence theorem, $\int\Phi_\delta(t,x)\rho(t,x)\,dx$ tends to $\int\Phi(t,x)\rho(t,x)\,dx$ as $\delta\to 0$.

Next, as noted in Remark 1.7, we have $E\in L^\infty_{\rm loc}(L^\infty)$. Moreover, as $\rho\in L^1_{\rm loc}(L^p)$ for $p>2$ the quantity $\frac{1}{|x-\xi|})\rho$ belongs to $L^1_{\rm loc}$. According to Proposition 1.3 we may reexpress the nonlinear term of (E) as

$$\begin{aligned}
\mathcal{H}_{\Phi_\delta(t)} &= \int_{\mathbb{R}^2}\left(E^\perp(t,x)+\frac{(x-\xi(t))^\perp}{|x-\xi(t)|^2}\right)\cdot\nabla\Phi_\delta(t,x)\rho(t,x)\,dx\\
&= \int_{\mathbb{R}^2}\left(E^\perp(t,x)+\frac{(x-\xi(t))^\perp}{|x-\xi(t)|^2}\right)\cdot\nabla\Phi(t,x)\eta_\delta\left(|x-\xi(t)|\right)\rho(t,x)\,dx\\
&+ \int_{\mathbb{R}^2}E^\perp(t,x)\cdot\frac{x-\xi(t)}{|x-\xi(t)|}\Phi(t,x)\eta_\delta'\left(|x-\xi(t)|\right)\rho(t,x)\,dx\\
&= I_\delta + J_\delta,
\end{aligned}$$

where we have used that $a^\perp\cdot a=0$. On the one hand, Lebesgue's dominated convergence theorem implies that I_δ converges to

$$\int_{\mathbb{R}^2}\left(E^\perp(t,x)+\frac{(x-\xi(t))^\perp}{|x-\xi(t)|^2}\right)\cdot\nabla\Phi(t,x)\rho(t,x)\,dx$$

as $\delta\to 0$. On the other hand, we have by Hölder's inequality

$$|J_\delta|\le\frac{C}{\delta}\int_{|x-\xi(t)|\le 2\delta}|E(t,x)||\rho(t,x)|\,dx\le\frac{C}{\delta}\|E(t)\|_{L^\infty}\|\rho(t)\|_{L^p}\delta^{2-\frac{2}{p}},$$

so J_δ vanishes in the limit $\delta\to 0$.

Finally, we compute

$$\partial_t\Phi_\delta(t,x) = \partial_t\Phi(t,x)\eta_\delta\left(|x-\xi(t)|\right) + \Phi(t,x)\frac{1}{\delta}\eta'\left(|x-\xi(t)|\right)\dot\xi(t)\cdot\frac{\xi(t)-x}{|x-\xi(t)|}$$

and using that $|\dot{\xi}(t)| \leq C$ we find as above that the integral $\int \partial_t \Phi_\delta(t,x)\rho(t,x)\,dx$ converges to $\int \partial_t \Phi(t,x)\rho(t,x)\,dx$ as $\delta \to 0$. Therefore, we have proved that ρ satisfies the first equation of (1.4) in the sense of distributions. Inserting this equation in (E) for any function Φ not necessarily vanishing near $\xi(t)$, we infer that

$$\frac{d}{dt}\Phi(t,\xi(t)) = \partial_t \Phi(t,\xi(t)) + E^\perp(t,\xi(t)) \cdot \nabla \Phi(t,\xi(t)),$$

which yields the second equation for ξ.

Acknowledgments During the preparation of this work the author has been partially supported by the French ANR projects SchEq ANR-12-JS-0005-01, GEODISP ANR-12-BS01-0015-01, and INFAMIE ANR-15-CE40-01.

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