HOMOGENEOUS VARIETIES UNDER SPLIT SOLVABLE ALGEBRAIC GROUPS

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Abstract. We present a modern proof of a theorem of Rosenlicht, asserting that every variety as in the title is isomorphic to a product of affine lines and punctured affine lines.

1. Introduction

Throughout this note, we consider algebraic groups and varieties over a field k. An algebraic group G is *split solvable* if it admits a chain of closed subgroups

$$\{e\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$$

such that each G_i is normal in G_{i+1} and G_{i+1}/G_i is isomorphic to the additive group $\mathbb{G}_a$ or the multiplicative group $\mathbb{G}_m$. This class features prominently in a series of articles by Rosenlicht on the structure of algebraic groups, see [Ro56, Ro57, Ro63]. The final result of this series may be stated as follows (see [Ro63, Thm. 5]):

Theorem 1. *Let X be a homogeneous variety under a split solvable algebraic group G. Then there is an isomorphism of varieties $X \simeq \mathbb{A}^m \times (\mathbb{A}^\times)^n$ for unique nonnegative integers m, n.*

Here $\mathbb{A}^m \simeq (\mathbb{A}^1)^m$ denotes the affine m-space, and $\mathbb{A}^\times = \mathbb{A}^1 \setminus \{0\}$ the punctured affine line.

Rosenlicht's articles use the terminology and methods of algebraic geometry à la Weil, and therefore have become hard to read. In view of their fundamental interest, many of their results have been rewritten in more modern language, e.g. in the book [DG70] by Demazure & Gabriel and in the second editions of the books on linear algebraic groups by Borel and Springer, which incorporate developments on "questions of rationality" (see [Bo91, Sp98]). The above theorem is a notable exception: the case of the group G acting on itself by multiplication is handled in [DG70, Cor. IV.4.3.8] (see also [Sp98, Cor. 14.2.7]), but the general case is substantially more complicated. [1]

[1] The case where k is algebraically closed and $X = G/H$ for some smooth connected subgroup $H \subset G$ is proposed as an exercise in [Sp98, §14.2].

The aim of this note is to fill this gap by providing a proof of Theorem 1 in the language of modern algebraic geometry. As it turns out, this theorem is self-improving: combined with Rosenlicht's theorem on rational quotients (see [Ro56, Thm. 2], and [BGR17, Sec. 2] for a modern proof) and some "spreading out" arguments, it yields the following stronger version:

Theorem 2. *Let X be a variety equipped with an action of a split solvable algebraic group G. Then there exist a dense open G-stable subvariety $X_0 \subset X$ and an isomorphism of varieties $X_0 \simeq \mathbb{A}^m \times (\mathbb{A}^\times)^n \times Y$ (where m, n are uniquely determined nonnegative integers and Y is a variety, unique up to birational isomorphism) such that the resulting projection $f : X_0 \to Y$ is the rational quotient by G.*

By this, we mean that f yields an isomorphism $k(Y) \stackrel{\sim}{\to} k(X)^G$, where the left-hand side denotes the function field of Y and the right-hand side stands for the field of G-invariant rational functions on X; in addition, the fibers of f are exactly the G-orbits.

As a direct but noteworthy application of Theorem 2, we obtain:

Corollary 3. *Let X be a variety equipped with an action of a split solvable algebraic group G. Then $k(X)$ is a purely transcendental extension of $k(X)^G$.*

When k is algebraically closed, this gives back the main result of [Po16]; see [CZ17] for applications to the rationality of certain homogeneous spaces.

The proof of Theorem 2 also yields a version of [Sp98, Prop. 14.2.2]:

Corollary 4. *Let X be a variety equipped with a nontrivial action of $\mathbb{G}_a$. Then there exist a variety Y, an open immersion $\varphi : \mathbb{A}^1 \times Y \to X$ and a monic additive polynomial $P \in \mathcal{O}(Y)[t]$ such that*

$$g \cdot \varphi(x, y) = \varphi(x + P(y, g), y)$$

for all $g \in \mathbb{G}_a$, $x \in \mathbb{A}^1$ and $y \in Y$.

Here P is said to be additive if it satisfies $P(y, t + u) = P(y, t) + P(y, u)$ identically; then $\mathbb{G}_a$ acts on $\mathbb{A}^1 \times Y$ via $g \cdot (x, y) = (x + P(y, g), y)$, and φ is equivariant for this action. If $\mathrm{char}(k) = 0$, then we have $P = t$ and hence $\mathbb{G}_a$ acts on $\mathbb{A}^1 \times Y$ by translation on $\mathbb{A}^1$. So Corollary 4 just means that every nontrivial $\mathbb{G}_a$-action becomes a trivial $\mathbb{G}_a$-torsor on some dense open invariant subset. On the other hand, if $\mathrm{char}(k) = p > 0$, then P is a p-polynomial, i.e.,

$$P = a_0 t + a_1 t^p + \cdots + a_n t^{p^n}$$

for some integer $n \geq 1$ and $a_0, \ldots, a_n \in \mathcal{O}(Y)$. Thus, the map

$$(P, \mathrm{id}) : \mathbb{G}_a \times Y \longrightarrow \mathbb{G}_a \times Y, \quad (g, y) \longmapsto (P(y, g), y)$$

is an endomorphism of the Y-group scheme $\mathbb{G}_{a,Y} = \mathrm{pr}_Y : \mathbb{G}_a \times Y \to Y$; conversely, every such endomorphism arises from an additive polynomial P, see [DG70, II.3.4.4]. Thus, Corollary 4 asserts that for any nontrivial $\mathbb{G}_a$-action, there is a dense open invariant subset on which $\mathbb{G}_a$ acts by a trivial torsor twisted by such an endomorphism. These twists occur implicitly in the original proof of Theorem 1, see [Ro63, Lem. 3]. [2]

This note is organized as follows. In Section 2, we gather background results on split solvable algebraic groups. Section 3 presents further preliminary material, on the quotient of a homogeneous space G/H by the left action of a normal subgroup scheme $N \triangleleft G$; here G is a connected algebraic group, and $H \subset G$ a subgroup scheme. In particular, we show that such a quotient is a torsor under a finite quotient of N, if either $N \simeq \mathbb{G}_m$ or $N \simeq \mathbb{G}_a$ and $\mathrm{char}(k) = 0$ (Lemma 3.4). The more involved case where $N \simeq \mathbb{G}_a$ and $\mathrm{char}(k) > 0$ is handled in Section 4; we then show that the quotient is a "torsor twisted by an endomorphism" as above (Lemma 4.3). The proofs of our main results are presented in Section 5.

Notation and conventions. We consider schemes over a field k of characteristic $p \geq 0$ unless otherwise mentioned. Morphisms and products of schemes are understood to be over k as well. A *variety* is an integral separated scheme of finite type.

An *algebraic group* G is a group scheme of finite type. By a *subgroup* $H \subset G$, we mean a (closed) subgroup scheme. A G-*variety* is a variety X equipped with a G-action

$$\alpha : G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x.$$

We say that X is G-*homogeneous* if G is smooth, X is geometrically reduced, and the morphism

$$(\mathrm{id}, \alpha) : G \times X \longrightarrow X \times X, \quad (g, x) \longmapsto (x, g \cdot x)$$

is surjective. If in addition X is equipped with a k-rational point x, then the pair (X, x) is a G-*homogeneous space*. Then $(X, x) \simeq (G/\operatorname{Stab}_G(x), x_0)$, where $\operatorname{Stab}_G(x) \subset G$ denotes the stabilizer, and x_0 the image of the neutral element $e \in G(k)$ under the quotient morphism $G \to G/\operatorname{Stab}_G(x_0)$.

Given a field extension K/k and a k-scheme X, we denote by X_K the K-scheme $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$.

We will freely use results from the theory of faithfully flat descent, for which a convenient reference is [GW10, Chap. 14, App. C].

[2] Rosenlicht was very well aware of the limitations of classical methods. He wrote in the introduction of [Ro63]: "The methods of proof we use here are refinements of those of our previous Annali paper [Ro57] and cry for improvement; there are unnatural complexities and it seems that something new that is quite general, and possibly quite subtle, must be brought to light before appreciable progress can be made."

2. Split solvable groups

We first recall some basic properties of these groups, taken from [DG70, IV.4.3] where they are called "groupes k-résolubles" (see also [Mi17, §16.g]). Every split solvable group is smooth, connected, affine and solvable. Conversely, every smooth connected affine solvable algebraic group over an algebraically closed field is split solvable (see [DG70, IV.4.3.4]).

Clearly, every extension of split solvable groups is split solvable. Also, recall that every nontrivial quotient group of $\mathbb{G}_m$ is isomorphic to $\mathbb{G}_m$, and likewise for $\mathbb{G}_a$ (see [DG70, IV.2.1.1]). As a consequence, every quotient group of a split solvable group is split solvable as well.

We now obtain a key preliminary result (a version of [Ro63, Lem. 1], see also [Sp98, Cor. 14.3.9]):

Lemma 2.1. *Let G be a split solvable group. Then there exists a chain of subgroups*

$$G_0 = \{e\} \subset G_1 \subset \cdots G_m \subset \cdots \subset G_{m+n} = G,$$

where $G_i \lhd G$ for $i = 0, \ldots, m+n$ and

$$G_{i+1}/G_i \simeq \begin{cases} \mathbb{G}_a & \text{if } i = 0, \ldots, m-1, \\ \mathbb{G}_m & \text{if } i = m, \ldots, m+n-1. \end{cases}$$

Proof. Arguing by induction on $\dim(G)$, it suffices to show that either G is a split torus, or it admits a normal subgroup N isomorphic to $\mathbb{G}_a$.

By [DG70, IV.4.3.4], G admits a normal unipotent subgroup U such that G/U is diagonalizable; moreover, U is split solvable. Since G is smooth and connected, G/U is a split torus T. Also, since every subgroup and every quotient group of a unipotent group are unipotent, U admits a chain of subgroups

$$\{e\} = U_0 \subset U_1 \subset \cdots \subset U_m = U$$

such that $U_i \lhd U_{i+1}$ and $U_{i+1}/U_i \simeq \mathbb{G}_a$ for any $i = 0, \ldots, m-1$. By [DG70, IV.4.3.14], it follows that either U is trivial or it admits a central characteristic subgroup V isomorphic to $\mathbb{G}_a^n$ for some integer $n > 0$. In the former case, $G = T$ is a split torus. In the latter case, $V \lhd G$ and the conjugation action of G on V factors through an action of T. By [Co15, Thm. 4.3], there is a T-equivariant isomorphism of algebraic groups $V \simeq V_0 \times V'$, where V_0 is fixed pointwise by T and V' is a vector group on which T acts linearly. If V' is nontrivial, then it contains a T-stable subgroup $N \simeq \mathbb{G}_a$; then $N \lhd G$. On the other hand, if V' is trivial then V is central in G; thus, every copy of $\mathbb{G}_a$ in V yields the desired subgroup N. □

3. Quotients of homogeneous spaces by normal subgroups

Let G be an algebraic group, $H \subset G$ a subgroup, and $N \triangleleft G$ a smooth normal subgroup. Then H acts on N by conjugation. The semi-direct product $N \rtimes H$ defined by this action (as in [Mi17, Sec. 2.f]) is equipped with a homomorphism to G, with schematic image the subgroup $NH \subset G$. Recall that $H \triangleleft NH \subset G$ and $NH/H \simeq N/N \cap H$. Denote by

$$q : G \longrightarrow G/H, \quad r : G \longrightarrow G/NH$$

the quotient morphisms. Then q is an H-torsor, and hence a categorical quotient by H. Since r is invariant under the H-action on G by right multiplication, there exists a unique morphism $f : G/H \longrightarrow G/NH$ such that the triangle

$$\begin{array}{ccccc} & & G & & \\ & \swarrow^{q} & & \searrow^{r} & \\ G/H & & \xrightarrow{\quad f \quad} & & G/NH \end{array}$$

commutes.

We will also need the following observation (see [Mi17, Prop. 7.15]):

Lemma 3.1. *With the above notation, the square*

$$\begin{array}{ccc} G \times NH/H & \xrightarrow{a} & G/H \\ \downarrow{\scriptstyle \mathrm{pr}_G} & & \downarrow{\scriptstyle f} \\ G & \xrightarrow{r} & G/NH \end{array}$$

is cartesian, where a denotes the restriction of the action $G \times G/H \to G/H$ and pr_G denotes the projection.

Proof. Since r is an NH-torsor, we have a cartesian square

$$\begin{array}{ccc} G \times NH & \xrightarrow{m} & G \\ \downarrow{\scriptstyle \mathrm{pr}_G} & & \downarrow{\scriptstyle r} \\ G & \xrightarrow{r} & G/NH, \end{array}$$

where m denotes the restriction of the multiplication $G \times G \to G$. Also, the square

$$\begin{array}{ccc} G \times NH & \xrightarrow{m} & G \\ \downarrow{\scriptstyle (\mathrm{id},q)} & & \downarrow{\scriptstyle q} \\ G \times NH/H & \xrightarrow{a} & G/H \end{array}$$

is commutative, and hence cartesian since the vertical arrows are H-torsors. As q is faithfully flat, this yields the assertion by descent. $\square$

For simplicity, we set $X = G/H$ and $Y = G/NH$. These homogeneous spaces come with base points x_0, y_0 such that $f(x_0) = y_0$.

Lemma 3.2. (i) *With the above notation, f is G-equivariant and N-invariant, where G (and hence N) acts on X, Y by left multiplication.*

(ii) *f is smooth, surjective, and its fibers are exactly the N-orbits.*

(iii) *The morphism*

$$\gamma : N \times X \longrightarrow X \times_Y X, \quad (n, x) \longmapsto (x, n \cdot x)$$

is faithfully flat.

(iv) *The map $f^\# : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ yields an isomorphism $\mathcal{O}_Y \xrightarrow{\sim} f_*(\mathcal{O}_X)^N$, where the right-hand side denotes the subsheaf of N-invariants.*

(v) *If $N \cap H$ is central in G, then f is a $N/N \cap H$-torsor.*

Proof. (i) Let R be an algebra, $g \in G(R)$ and $x \in X(R)$. As q is faithfully flat, there exist a faithfully flat R-algebra R' and $g' \in G(R')$ such that $x = g' \cdot x_0$. Then $f(g \cdot x) = f(gg' \cdot x_0) = gg' \cdot y_0 = g \cdot (g' \cdot y_0) = g \cdot f(x)$ in $Y(R')$, and hence in $Y(R)$. This yields the G-equivariance of f.

If $g \in N(R)$ then $gg' = g'n$ for some $n \in N(R')$. Thus, $f(gg' \cdot x_0) = f(g' \cdot x_0)$, i.e., $f(g \cdot x) = f(x)$, proving the N-invariance.

(ii) Observe that NH/H is homogeneous under the smooth algebraic group N, and hence is smooth. Thus, $\mathrm{pr}_G : G \times NH/H \to G$ is smooth as well. It follows that f is smooth by using Lemma 3.1 and the faithful flatness of r. Also, f is surjective since so are pr_G and r.

Let K/k be a field extension, $x \in X(K)$, and $y = f(x)$. There exist a field extension L/K and $g \in G(L)$ such that $x = g \cdot x_0$. Thus, $y = g \cdot y_0$ and the fiber X_y satisfies $(X_y)_L = g(X_{y_0})_L$. Also, $X_{y_0} = N \cdot x_0$ in view of Lemma 3.1 together with the isomorphisms $N \cdot x_0 \simeq N/N \cap H \simeq NH/H$. Thus, $(X_y)_L = g \cdot (Nx_0)_L = (Ng \cdot x_0)_L = (N \cdot x)_L$, and therefore $X_y = N_K \cdot x$ by descent.

(iii) Consider the commutative triangle

$$\begin{array}{ccc} N \times X & \xrightarrow{\ \gamma\ } & X \times_Y X \\ & {\scriptstyle \mathrm{pr}_X} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & X. & \end{array}$$

Clearly, the morphism pr_X is faithfully flat. Also, pr_1 is faithfully flat, since it is obtained from f by base change. Moreover, for any field extension K/k and any $x \in X(K)$, the restriction $\gamma_x : N \times x = N_K \to X_x$ is the orbit map $n \mapsto n \cdot x$, and hence is faithfully flat by (ii). So the assertion follows from the fiberwise flatness criterion (see [EGA, IV.11.3.11]).

(iv) We have

$$\mathcal{O}_Y = r_*(\mathcal{O}_G)^{NH} = f_* q_*(\mathcal{O}_G)^{NH} = f_*(q_*(\mathcal{O}_G)^H)^N = f_*(\mathcal{O}_X)^N,$$

since q (resp. r) is a torsor under H (resp. NH).

(v) The subgroup $N \cap H \subset G$ fixes x_0 and is central in G. By a lifting argument as in (i), it follows that $N \cap H$ fixes $X = G \cdot x_0$ pointwise. Thus, the N-action on X factors uniquely through an action of $N/N \cap H$. Since the square

$$\begin{array}{ccc} G \times N/N \cap H & \xrightarrow{a} & X \\ {\scriptstyle \mathrm{pr}_G}\downarrow & & \downarrow{\scriptstyle f} \\ G & \xrightarrow{r} & Y \end{array}$$

is cartesian (Lemma 3.1) and r is faithfully flat, this yields the assertion. □

In view of the assertions (i), (ii), (iii) and (iv), f is a geometric quotient by N in the sense of [MFK94, Def. 0.7].

Next, denote by $\mathrm{Stab}_N \subset N \times X$ the stabilizer, i.e., the pullback of the diagonal in $X \times_Y X$ under γ. Then Stab_N is a closed subgroup scheme of the X-group scheme $N_X = (\mathrm{pr}_X : N \times X \to X)$, stable under the G-action on $N \times X$ via $g \cdot (n, x) = (gng^{-1}, g \cdot x)$.

Lemma 3.3. (i) *The projection* $\mathrm{pr}_X : \mathrm{Stab}_N \to X$ *is faithfully flat and* G*-equivariant. Its fiber at* x_0 *is* H*-equivariantly isomorphic to* $N \cap H$ *on which* H *acts by conjugation.*

(ii) pr_X *is finite if and only if* $N \cap H$ *is finite.*

Proof. (i) Clearly, pr_X is equivariant and its fiber $\mathrm{Stab}_N(x_0)$ is as asserted. Form the cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & G \\ \downarrow & & \downarrow{\scriptstyle q} \\ \mathrm{Stab}_N & \xrightarrow{\mathrm{pr}_X} & G/H. \end{array}$$

Then Z is equipped with a G-action such that π is equivariant, with fiber at e being $N \cap H$. As a consequence, the morphism

$$G \times N \cap H \longrightarrow Z, \quad (g, z) \longmapsto g \cdot z$$

is an isomorphism with inverse being $z \mapsto (\pi(z), \pi(z)^{-1} \cdot z)$. Via this isomorphism, π is identified with the projection $G \times N \cap H \to G$. Thus, π is faithfully flat, and hence so is pr_X.

(ii) This also follows from the above cartesian square, since π is finite if and only if $N \cap H$ is finite. □

Lemma 3.4. *Assume that* $N \not\subset H$.

(i) *If* $N \simeq \mathbb{G}_m$ *and* G *is connected, then* f *is an* $N/N \cap H$*-torsor. Moreover,* $N/N \cap H \simeq \mathbb{G}_m$.

(ii) *If* $N \simeq \mathbb{G}_a$ *and* $p = 0$*, then* f *is an* N*-torsor.*

Proof. (i) In view of the rigidity of tori (see [SGA3, Exp. IX, Cor. 5.5] or [Mi17, Cor. 12.37]), N is central in G. Also, $N \cap H$ is a finite subgroup of N, and hence $N/N \cap H \simeq \mathbb{G}_m$. So we conclude by Lemma 3.2 (v).

(ii) Likewise, $N \cap H$ is a finite subgroup of $\mathbb{G}_a$, and hence is trivial since $p = 0$. So we conclude by Lemma 3.2 (v) again. □

4. Quotients by the additive group

We first record two preliminary results, certainly well-known but for which we could locate no appropriate reference.

Lemma 4.1. *Let X be a locally noetherian scheme. Let $Z \subset \mathbb{A}^1 \times X$ be a closed subscheme such that the projection $\mathrm{pr}_X : Z \to X$ is finite and flat. Then Z is the zero subscheme of a unique monic polynomial $P \in \mathcal{O}(X)[t]$.*

Proof. First consider the case where $X = \mathrm{Spec}(A)$, where A is a local algebra with maximal ideal $\mathfrak{m}$ and residue field K. Denoting by x the closed point of X, the fiber Z_x is a finite subscheme of $\mathbb{A}^1_K$. Thus, $Z_x = V(P)$ for a unique monic polynomial $P \in K[t]$. So the images of $1, t, \ldots, t^{n-1}$ in $\mathcal{O}(Z_x)$ form a basis of this K-vector space, where $n = \deg(P)$. Also, $\mathcal{O}(Z)$ is a finite flat A-module, hence free. By Nakayama's lemma, the images of $1, t, \ldots, t^{n-1}$ in $\mathcal{O}(Z)$ form a basis of this A-module. So we have $t^n + a_1 t^{n-1} + \cdots + a_n = 0$ in $\mathcal{O}(Z)$ for unique $a_1, \ldots, a_n \in A$. Thus, the natural map $A[t]/(t^n + a_1 t^{n-1} + \cdots + a_n) \to \mathcal{O}(Z)$ is an isomorphism, since it sends a basis to a basis. This proves the assertion in this case.

For an arbitrary scheme X, the assertion holds in a neighborhood of every point by the local case. In view of the uniqueness of P, this completes the proof. □

Lemma 4.2. *Let X be a locally noetherian scheme, and $H \subset \mathbb{G}_{a,X}$ a finite flat subgroup scheme. Then $H = \mathrm{Ker}(P, \mathrm{id})$ for a unique monic additive polynomial $P \in \mathcal{O}(X)[t]$, where (P, id) denotes the endomorphism*

$$\mathbb{G}_{a,X} \longrightarrow \mathbb{G}_{a,X}, \quad (g, x) \longmapsto (P(x, g), x).$$

Proof. We may assume that X is affine by the uniqueness property. Let $X = \mathrm{Spec}(A)$, then $H = V(P)$ for a unique monic polynomial $P \in A[t]$ (Lemma 4.1). We now adapt an argument from [DG70, IV.2.1.1] to show that P is an additive polynomial.

Denote by $m : \mathbb{G}_{a,X} \times_X \mathbb{G}_{a,X} \to \mathbb{G}_{a,X}$ the group law. Since H is a subgroup scheme, we have $H \times_X H \subset m^{-1}(H)$. Considering the ideals of these closed subschemes of $\mathbb{G}_{a,X} \times_X \mathbb{G}_{a,X} \simeq \mathbb{G}_a \times \mathbb{G}_a \times X = \mathrm{Spec}(A[t, u])$ yields that $P(t + u) \in (P(t), P(u))$ in $A[t, u]$. So there exist $Q, R \in A[t, u]$ such that

$$P(t + u) - P(t) - P(u) = Q(t, u)P(t) + R(t, u)P(u).$$

Since P is monic, there exist unique $Q_1, Q_2 \in A[t,u]$ such that

$$Q(t,u) = Q_1(t,u)P(u) + Q_2(t,u), \quad \deg_u(Q_2) < \deg(P) = n.$$

Thus, we have

$$P(t+u) - P(t) - P(u) - Q_2(t,u)P(t) = (Q_1(t,u)P(t) + R(t,u))P(u).$$

As the left-hand side has degree in u at most $n-1$, it follows that $Q_1(t,u)P(t)+R(t,u) = 0$ and $P(t+u)-P(t)-P(u) = Q_2(t,u)P(t)$. Considering the degree in t, we obtain $Q_2 = 0$ and $P(t+u) = P(t) + P(u)$ identically. □

Next, we return to the setting of Section 3: G is an algebraic group, $H \subset G$ a subgroup, $N \triangleleft G$ a smooth normal subgroup, and $f : X = G/H \to G/NH = Y$ the natural morphism. Since f is N-invariant (Lemma 3.2 (i)), we may view X as an Y-scheme equipped with an action of the Y-group scheme N_Y.

Lemma 4.3. *Assume in addition that $N \simeq \mathbb{G}_a$ and $N \not\subset H$. Then there exist a faithfully flat morphism of Y-group schemes $\varphi : N_Y \to \mathbb{G}_{a,Y}$ and a $\mathbb{G}_{a,Y}$-action on X such that f is a $\mathbb{G}_{a,Y}$-torsor.*

Proof. By Lemma 3.3, the stabilizer Stab_N is finite and flat over Y. Thus, $\mathrm{Stab}_N = \mathrm{Ker}(P, \mathrm{id})$ for a unique monic p-polynomial $P \in \mathcal{O}(X)[t]$ (Lemma 4.2). Also, $\mathrm{Stab}_N \subset N \times X$ is stable under the action of the abstract group $N(k)$ via $g \cdot (n,x) = (n, g \cdot x)$; as a consequence, we have $P(g \cdot x, t) = P(x,t)$ identically on X, for any $g \in N(k)$. This still holds after base change by a field extension K/k, since the formation of Stab_N commutes with such base change and hence P is invariant under any such extension. Since $N(K)$ is dense in N_K for any infinite field K, it follows that P is N-invariant. As $\mathcal{O}(X)^N = \mathcal{O}(Y)$ (Lemma 3.2 (iv)), we see that $P \in \mathcal{O}(Y)[t]$.

Choose an isomorphism $\mathbb{G}_a \xrightarrow{\sim} N$ and consider the morphism

$$\varphi = (P, \mathrm{id}) : \mathbb{G}_{a,Y} \longrightarrow \mathbb{G}_{a,Y}, \quad (t,y) \longmapsto (P(y,t), y).$$

Then φ is an endomorphism of the Y-group scheme $\mathbb{G}_{a,Y}$. Moreover, φ is faithfully flat, as follows from the fiberwise flatness criterion (see [EGA, IV.11.3.11]), since $\mathbb{G}_{a,Y}$ is faithfully flat over Y and for any $y \in Y$, the morphism $\varphi_y : t \mapsto P(y,t)$ is faithfully flat. Denote by K the kernel of φ. Then we have $K \times_Y X = \mathrm{Stab}_N$; thus, K is finite and flat over Y, by Lemma 3.3 and descent. Moreover, the square

$$\begin{array}{ccc} K \times_Y \mathbb{G}_{a,Y} & \xrightarrow{m} & \mathbb{G}_{a,Y} \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \varphi} \\ \mathbb{G}_{a,Y} & \xrightarrow{\varphi} & \mathbb{G}_{a,Y} \end{array}$$

is cartesian, where m denotes the group law, and pr the projection (indeed, $P(t,y) = P(u,y)$ if and only if $(u-t,y) \in K$). So φ is a K-torsor. The action

$$\alpha : \mathbb{G}_{a,Y} \times_Y X = \mathbb{G}_a \times X \longrightarrow X, \quad (t,x) \longmapsto t \cdot x$$

is a K-invariant morphism. By descent again, it follows that there is a unique morphism $\beta : \mathbb{G}_a \times X \to X$ such that the triangle

$$\begin{array}{ccc} \mathbb{G}_a \times X & \xrightarrow{\alpha} & X \\ & \searrow^{\varphi} \quad \nearrow_{\beta} & \\ & \mathbb{G}_a \times X & \end{array}$$

commutes. Thus, $\beta(t,x) = \alpha(P(f(x),t),x)$ identically on $\mathbb{G}_a \times X$. In particular, $\beta(0,x) = \alpha(0,x) = x$ identically on X. Also, β satisfies the associativity property of an action, since so does α and φ is faithfully flat. So β is an action of $\mathbb{G}_{a,Y}$ on X. Consider the associated morphism

$$\delta : \mathbb{G}_a \times X \longrightarrow X \times_Y X, \quad (t,x) \longmapsto (\beta(t,x),x)$$

as a morphism of X-schemes. For any field extension K/k and any $x \in X(K)$, we get a morphism $\delta_x : \mathbb{G}_{a,K} \to X_x$ such that $\delta_x \circ P_x = \alpha_x$. Thus, δ_x is an isomorphism by the construction of P. In view of the fiberwise isomorphism criterion (see [EGA, IV.17.9.5]), it follows that δ is an isomorphism. So f is a $\mathbb{G}_{a,Y}$-torsor relative to this action β. $\square$

5. Proofs of the main results

5.1. **Proof of Theorem 1.** We first consider the case where X is equipped with a k-rational point x_0. Then $X = G/H$ for some subgroup $H \subset G$. If G is a torus, then G/H has the structure of a split torus, and hence is isomorphic to $(\mathbb{A}^\times)^n$ for some integer $n \geq 0$. Otherwise, G admits a normal subgroup $N \simeq \mathbb{G}_a$ by Lemma 2.1. If $N \subset H$ then $X \simeq (G/N)/(H/N)$ and we conclude by induction on $\dim(G)$. So we may assume that $N \not\subset H$. Then we have a morphism

$$f : X = G/H \longrightarrow G/NH \simeq (G/N)/(NH/N).$$

Moreover, f is a $\mathbb{G}_a$-torsor by Lemma 3.2 (if $p = 0$) and Lemma 4.3 (if $p > 0$). By induction on $\dim(G)$ again, we may assume that $Y \simeq \mathbb{A}^m \times (\mathbb{A}^\times)^n$ as a variety. In particular, Y is affine, and hence the $\mathbb{G}_a$-torsor f is trivial. So $X \simeq \mathbb{A}^1 \times Y \simeq \mathbb{A}^{m+1} \times (\mathbb{A}^\times)^n$ as a variety.

To complete the proof, it suffices to show that every homogeneous G-variety has a k-rational point. This follows from a result of Rosenlicht (see [Ro56, Thm. 10]) and is reproved in [Bo91, Thm. 15.11], [Sp98, Thm. 14.3.13]. For completeness, we present a proof based on the following lemma, also due to Rosenlicht (see [Ro56, Lem., p. 425]):

Lemma 5.1. *Let X be a homogeneous variety under $G = \mathbb{G}_a$ or $\mathbb{G}_m$. Then X has a k-rational point.* [3]

Proof. Since X is a smooth curve, it admits a unique regular completion $\bar{X}$, i.e., $\bar{X}$ is a regular projective curve equipped with an open immersion $X \to \bar{X}$. Moreover, $\bar{X}$ is geometrically integral since so is X. We identify X with its image in $\bar{X}$, and denote by $Z = \bar{X} \setminus X$ the closed complement, equipped with its reduced subscheme structure. Then $Z = \coprod_{i=1}^n \mathrm{Spec}(K_i)$, where the K_i/k are finite extensions of fields.

By the smoothness of X again, we may choose a finite separable extension K/k such that X has a K-rational point x_0. Then (X_K, x_0) is a homogeneous space under G_K, and hence is isomorphic to G_K as a variety. Also, $\bar{X}_K$ is the regular completion of X_K; moreover, Z_K is reduced and $\bar{X}_K \setminus X_K = Z_K$. Since $X_K \simeq \mathbb{A}^1_K$ or $\mathbb{A}^\times_K$, it follows that $\bar{X}_K \simeq \mathbb{P}^1_K$; in particular, $\bar{X}$ is a smooth projective curve of genus 0. This identifies Z_K with $\mathrm{Spec}(K)$ (the point at infinity) if $G = \mathbb{G}_a$, resp. with $\mathrm{Spec}(K) \coprod \mathrm{Spec}(K) = \{0, \infty\}$ if $G = \mathbb{G}_m$.

In the former case, we have $Z = \mathrm{Spec}(k)$ and hence $\bar{X}$ has a k-rational point. Thus, $\bar{X} \simeq \mathbb{P}^1$, so that X has a k-rational point as well.

In the latter case, let $L = k(X)$; then L/k is separable and X_L has an L-rational point. Thus, we see as above that $\bar{X}_L \simeq \mathbb{P}^1_L$ and this identifies Z_L with $\{0, \infty\}$. In particular, $Z(L) = Z(K)$. Since K and L are linearly disjoint over k, it follows that $Z(k)$ consists of two k-rational points; we then conclude as above. □

Returning to a homogeneous variety X under a split solvable group G, we may choose $N \triangleleft G$ such that $N \simeq \mathbb{G}_a$ or $\mathbb{G}_m$ (Lemma 2.1). Also, we may choose a finite Galois extension K/k such that X has a K-rational point x_0. Let $H = \mathrm{Stab}_{G_K}(x_0)$; then (X_K, x_0) is the homogeneous space G_K/H, and hence there is a geometric quotient

$$f : X_K = G_K/H \longrightarrow G_K/N_K H$$

(Lemma 3.2). Then f is a categorical quotient, and hence is unique up to unique isomorphism. By Galois descent (which applies, since all considered varieties are affine), we obtain a G-equivariant morphism $\varphi : X \to Y$ such that $\varphi_K = f$. In particular, Y is a homogeneous variety under G/N. Arguing by induction on $\dim(G)$, we may assume that Y has a k-rational point y. Then the fiber X_y is a homogeneous N-variety, and hence has a k-rational point.

[3] This lemma is reproved in [Bo91, Prop. 15.6], but the argument there is unclear to me. In modern language, it is asserted that every smooth, geometrically rational curve is an open subvariety of a smooth complete curve of genus 0. Yet this fails for nontrivial forms of the affine line, see [Ru70, Lem. 1.1]. Also, it is asserted that the G-action on X extends to an action on its regular completion; this requires a proof.

5.2. **Proof of Theorem 2.** We may freely replace X with any dense open G-stable subvariety. In view of Rosenlicht's theorem on rational quotients mentioned in the introduction, we may thus assume that there exist a variety Y and a G-invariant morphism

$$f : X \longrightarrow Y$$

such that $k(Y) \xrightarrow{\sim} k(X)^G$ and the fiber of f at every $y \in Y$ is a homogeneous variety under $G_{\kappa(y)}$, where $\kappa(y)$ denotes the residue field at y. By generic flatness, we may further assume that f is flat.

Denoting by η the generic point of Y, the fiber X_η is a homogeneous variety under $G_\eta = G_{k(Y)}$. By Theorem 1, this yields an isomorphism

$$Z_\eta \xrightarrow{\sim} X_\eta, \tag{5.1}$$

where $Z = \mathbb{A}^m \times (\mathbb{A}^\times)^n$ for unique integers $m, n \geq 0$. This yields in turn a birational map

$$\varphi : Z \times Y \dashrightarrow X$$

such that $f \circ \varphi = \mathrm{pr}_Y$ as rational maps.

It suffices to show that there exists a dense open subvariety $Y_0 \subset Y$ such that φ is defined on $Z \times Y_0$ and yields an open immersion $Z \times Y_0 \to X$ with G-stable image. For this, we start with some reductions.

We may assume that Y is affine (by replacing X with the preimage of a dense open affine subvariety) and also that X is normal (since its normal locus is a dense open G-stable subvariety). In view of a result of Sumihiro (see [Su75, Thm. 3.9]), we may further assume that X is a locally closed G-stable subvariety of the projectivization $\mathbb{P}(V)$, where V is a finite-dimensional G-module. The closure $\bar{X}$ of X in $\mathbb{P}(V)$ and its boundary $\bar{X} \setminus X$ are G-stable. By a version of Borel's fixed point theorem (see [DG70, IV.4.3.2]), there exist a positive integer N and a nonzero $s \in H^0(\bar{X}, \mathcal{O}(N))$ which vanishes identically on $\bar{X} \setminus X$ and is a G-eigenvector. Then the dense open subvariety $\bar{X}_s$ is affine, G-stable and contained in X; thus, we may further assume that X is affine. This replaces Y with a dense open subset Y_0 (as f is flat and hence open). As Y is affine, we may choose a nonzero $t \in \mathcal{O}(Y)$ which vanishes identically on $Y \setminus Y_0$. Replacing X with X_t and Y with Y_t, we may finally assume that X, Y are affine and X is normal.

Choose a closed immersion of Y-varieties $X \to \mathbb{A}^N \times Y$; then φ yields a rational map

$$(\varphi_1, \ldots, \varphi_N, \mathrm{pr}_Y) : Z \times Y \dashrightarrow \mathbb{A}^N \times Y$$

such that the pull-back $Z_\eta \to \mathbb{A}^N_\eta$ is a closed immersion. In particular, $\varphi_1, \ldots, \varphi_N \in \mathcal{O}(Z_\eta) = \mathcal{O}(Z) \otimes_k k(Y)$. Replacing again Y with a dense open affine subvariety, we may thus assume that $\varphi_1, \ldots, \varphi_N \in \mathcal{O}(Z) \otimes_k \mathcal{O}(Y) = \mathcal{O}(Z \times Y)$. As a consequence, φ is a morphism.

Denote by $\mathrm{Isol}(\varphi)$ the set of points of $Z \times Y$ which are isolated in their fiber; then $\mathrm{Isol}(\varphi)$ contains the points of Z_η. By Zariski's Main Theorem (see [EGA, III.4.4.3]), $\mathrm{Isol}(\varphi)$ is open in $Z \times Y$ and the restriction of φ to $\mathrm{Isol}(\varphi)$ factors as

$$\mathrm{Isol}(\varphi) \xrightarrow{\psi} X' \xrightarrow{\gamma} X,$$

where ψ is an open immersion and γ is finite. Replacing X' with the schematic image of ψ, we may assume that ψ is schematically dominant; then X' is a variety. Since φ is birational, so is γ; as X is normal, it follows that γ is an isomorphism. Thus, φ restricts to an open immersion $\mathrm{Isol}(\varphi) \to X$.

Consider the closed complement $F = (Z \times Y) \setminus \mathrm{Isol}(\varphi)$. Then F_η is empty, and hence the ideal $I(F) \subset \mathcal{O}(Z \times Y)$ satisfies $1 \in I(F) \otimes_{\mathcal{O}(Y)} k(Y)$. Replacing Y with a principal open subvariety, we may thus assume that $1 \in I(F)$, i.e., F is empty and $\mathrm{Isol}(\varphi) = Z \times Y$. Equivalently, $\varphi : Z \times Y \to X$ is an open immersion.

It remains to show that the image of φ is G-stable. The isomorphism (5.1) is equivariant relative to some action $\alpha : G_\eta \times_\eta Z_\eta \to Z_\eta$. We may view α as a morphism $G \times Z \times \eta \to Z$, i.e., a family $(x_1, \ldots, x_m, y_1, \ldots, y_n)$, where $x_1, \ldots, x_m \in \mathcal{O}(G \times Z \times \eta)$ and $y_1, \ldots, y_n \in \mathcal{O}(G \times Z \times \eta)^\times$ (the group of invertible elements). Shrinking Y again, we may assume that $x_1, \ldots, x_m \in \mathcal{O}(G \times Z \times Y)$ and $y_1, \ldots, y_n \in \mathcal{O}(G \times Z \times Y)^\times$. Then α is given by a morphism $G \times Z \times Y \to Z$, i.e., an action of G_Y on $Z \times Y$. Moreover, φ is G_Y-equivariant, since so is φ_η. This completes the proof of Theorem 2.

The proof of Corollary 4 is completely similar; the point is that the generic fiber X_η is a nontrivial $\mathbb{G}_{a,\eta}$-homogeneous variety, and hence is isomorphic to $\mathbb{A}^1_\eta$ on which $\mathbb{G}_{a,\eta}$ acts via a monic additive polynomial $P \in k(Y)[t]$ (Lemma 5.1). We leave the details to the reader.

Remark 5.2. (i) Theorem 1 may be reformulated as follows: every homogeneous variety X under a split solvable algebraic group G is affine and satisfies

$$\mathcal{O}(X) \simeq k[x_1, \ldots, x_m, y_1, y_1^{-1}, \ldots, y_n, y_n^{-1}],$$

where $x_1, \ldots, x_m, y_1, \ldots, y_n$ are algebraically independent. So the invertible elements of the algebra $\mathcal{O}(X)$ are exactly the Laurent monomials $c y_1^{a_1} \cdots y_n^{a_n}$, where $c \in k^\times$ and $a_1, \ldots, a_n \in \mathbb{Z}$. As a consequence, the projection

$$f : X \longrightarrow (\mathbb{A}^\times)^n$$

is uniquely determined (but the projection $X \to \mathbb{A}^m$ is not: as an example, $k[x, y, y^{-1}] \simeq k[x + P(y), y, y^{-1}]$ for any $P \in k[t]$). In fact, f is the quotient by the unipotent part U of G, as follows fom the proof of Theorem 1.

(ii) Likewise, in the setting of Theorem 2, the projection $X_0 \to (\mathbb{A}^\times)^n \times Y$ is the rational quotient by U. This theorem is known, in a more precise formulation, for a variety X equipped with an action of a connected reductive algebraic group G over an algebraically closed field of characteristic 0. Then one considers the action of a Borel subgroup of G, and uses the "local structure theorem" as in [Kn90, Satz 2.3]. The dimension of Y is the complexity of the G-action on X, and n is its rank; both are important numerical invariants of the action (see e.g. [Ti11, Chap. 2]).

These invariants still make sense in positive characteristics, and the local structure theorem still holds in a weaker form (see [Kn93, Satz 1.2]). Theorem 2 gives additional information in this setting.

(iii) Corollary 4 also holds for a variety X equipped with a nontrivial action of the multiplicative group: there exist a variety Y, a nonzero integer n and an open immersion $\varphi : \mathbb{A}^\times \times Y \to X$ such that $g \cdot \varphi(x, y) = \varphi(g^n x, y)$ identically. This follows from the fact that every nontrivial $\mathbb{G}_{m,\eta}$-homogeneous variety is isomorphic to $\mathbb{A}^\times_\eta$ on which $\mathbb{G}_{m,\eta}$ acts by the nth power map for some $n \neq 0$.

This extends to the action of a split torus T: using [Su75, Cor. 3.11], one reduces to the case where X is affine and T acts via a free action of a quotient torus T'. Then the quotient $X \to Y$ exists and is a T'-torsor, see [SGA3, Exp. IX, Thm. 5.1] for a much more general result.

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