Models of algebraic group actions

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Basic definitions

This talk is about (rational) actions of algebraic groups.
For simplicity, we work over an algebraically closed ground field k.
We identify every algebraic variety with its set of k-rational points.

Recall that an *algebraic group* G is a variety equipped with a group structure such that the multiplication and inverse maps are morphisms of varieties.

A *regular action* of G on a variety X is a morphism of varieties $a : G \times X \longrightarrow X$, $(g, x) \longmapsto g \cdot x$ such that:

(i) $e \cdot x = x$ for all $x \in X$, and

(ii) $gh \cdot x = g \cdot (h \cdot x)$ for all $(g, h, x) \in G \times G \times X$.

A *rational action* of G on X is a rational map $a : G \times X \dashrightarrow X$ (i.e., a is defined on a dense open subset of $G \times X$) such that (i) and (ii) hold as equalities of rational maps.

For example, the projective linear group PGL_3 acts regularly on the projective plane $\mathbb{P}^2$. Thus, PGL_3 acts rationally on the affine plane $\mathbb{A}^2$, and on $\mathbb{P}^1 \times \mathbb{P}^1$. Likewise, $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts rationally on $\mathbb{P}^2$.

Weil's regularization theorem

The following basic result was proved by Weil (1955), with modern proofs by Zaitsev (1995) and Kraft (2018):

Theorem

For any rational action of an algebraic group G on a variety X, there exists a regular G-action on a variety Y and a G-equivariant birational map $f : X \dashrightarrow Y$.

That is, f is an isomorphism from a dense open subset $U \subset X$ to a dense open subset $V \subset Y$, and we have $f(g \cdot x) = g \cdot f(x)$ whenever both sides are defined.

We then say that Y is an *equivariant model* of X.

Question

Does every rational G-action admit a smooth projective equivariant model?

This is motivated by the classification of algebraic subgroups of the group of birational transformations $\mathrm{Bir}(X) \simeq \mathrm{Aut}_k\, k(X)$, where $k(X)$ denotes the field of rational functions. Starting with a smooth projective equivariant model, one can use the Minimal Model Program to obtain a "better" one.

Examples in low dimensions

The above question has an affirmative answer if X is a curve: then G acts regularly on its smooth projective model Y. If in addition G is connected and nontrivial, then either G is a subgroup of PGL_2 (and $Y = \mathbb{P}^1$), or G is an elliptic curve (and $Y = G$ on which G acts by translations).

The answer is also affirmative if X is a surface, by work of Blanc (2009), Fong (2021), Zimmermann (2021). If in addition G is connected, then the equivariant model Y may be taken minimal.

For example, if X is a rational surface, then Y is either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, where $n \geq 0$ (for $n = 0$ we get $\mathbb{P}^1 \times \mathbb{P}^1$). This gives back the classification of the maximal connected algebraic subgroups of the Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ due to Enriques (1893): these are exactly the conjugates of

- $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3$,
- $\mathrm{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1) = \mathrm{PGL}_2 \times \mathrm{PGL}_2$, where Aut^0 is the neutral component,
- $\mathrm{Aut}(\mathbb{F}_n) = V_n \rtimes \mathrm{GL}_2/\mu_n$, where V_n is the space of homogeneous polynomials of degree n in two variables.

Further examples

Let $X = E \times \mathbb{P}^1$, where E is an elliptic curve. Then the minimal models of X are ruled surfaces over E.

We may take for example $Y = \mathbb{P}(E)$, where E is a vector bundle of rank 2 over E, extension of two trivial line bundles. Then $G = \mathrm{Aut}^0(Y)$ lies in an exact sequence of commutative algebraic groups

$$0 \longrightarrow \mathbb{G}_a \longrightarrow G \longrightarrow E \longrightarrow 0,$$

where $\mathbb{G}_a$ denotes the additive group.

As another example, we may take $Y = \mathbb{P}(\mathcal{O}_E \oplus L)$, where L is a line bundles of degree 0 on E. Then one obtains a similar exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow G \longrightarrow E \longrightarrow 0,$$

where $\mathbb{G}_m$ denotes the additive group.

In both cases, G is a maximal connected algebraic subgroup of $\mathrm{Bir}(X)$. The maximal algebraic subgroups have been classified by Fong (2021).

Main result

Theorem

Every rational action of an algebraic group admits a normal projective equivariant model.

In characteristic 0, this yields the existence of a smooth projective equivariant model, by using equivariant resolution of singularities.

Likewise, surfaces in arbitrary characteristic admit an equivariant desingularization (Zariski, 1939) and hence a smooth projective equivariant model. This is unknown in higher dimensions and positive characteristics, as the existence of a canonical desingularization is an open problem.

To prove the above theorem, one starts with a variety X equipped with a regular action of an algebraic group G. One then shows:

1) X admits a dense open G-stable subset U which is normal and quasi-projective.

2) U admits an open G-equivariant immersion in a normal projective G-variety.

Proof sketch of 1)

If G is linear and connected, then the asssertions 1) and 2) follow readily from a result of Sumihiro (1974, 1975).

More specifically, the normal locus of X is open and G-stable, and hence we may assume X normal. Then X admits a covering by open G-stable subsets U_i, equivariantly isomorphic to G-stable subvarieties of the projectivization $\mathbb{P}(V_i)$ for some finite-dimensional G-module V_i.
Thus, the closure of U_i in $\mathbb{P}(V_i)$ is a projective G-variety.
Taking its normalization completes the proof.

For an arbitrary algebraic group G (possibly non-linear) with neutral component G^0, one reduces as above to X being normal.
One then shows a weaker version of Sumihiro's result:

X admits a covering by open G^0-stable subsets which are quasi-projective.

Choose such a dense open subset U. Then there are only finitely many translates $g \cdot U$, where $g \in G$ (since G/G^0 is finite). So $V = \bigcap_{g \in G} g \cdot U$ is a dense open subset of X.
Also, V is clearly G-stable and quasi-projective.

Proof sketch of 2)

Let X be a normal quasi-projective G-variety. It suffices to construct a G-equivariant immersion of X in a projective G-variety. Then we may take the closure of the image of X, and normalize it to get the result.

The idea is to reduce to the case where G is connected and linear, by using the following:

Theorem

Every algebraic group G lies in a unique exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

where H is linear and connected, and Q is a projective algebraic group.

(If G is connected, then Q is an abelian variety and the above result is known as Chevalley's structure theorem).

By the connected linear case, X admits an open H-equivariant immersion in a projective H-variety.

From this, we will derive a G-equivariant immersion in a projective G-variety via a process of "induction from H to G".

Induction of representations

Given an algebraic group G, a closed subgroup H and a finite-dimensional H-module M, the vector space

$$\mathrm{Ind}_H^G(M) = \{f \in \mathrm{Hom}(G, M) \mid f(gh^{-1}) = h \cdot f(g) \text{ for all } g \in G, h \in H\}$$

is a G-module via $(g \cdot f)(g') = f(gg')$. Here $\mathrm{Hom}(G, M) = \mathcal{O}(G) \otimes M$, where $\mathcal{O}(G)$ denotes the algebra of regular functions on G.

We have the evaluation map

$$\mathrm{ev}_e : \mathrm{Ind}_H^G(M) \longrightarrow M, \quad f \longmapsto f(e),$$

which is a morphism of H-modules and satisfies *Frobenius reciprocity*: for any G-module N, the map

$$\mathrm{Hom}^G(N, \mathrm{Ind}_H^G(M)) \longrightarrow \mathrm{Hom}^H(N, M), \quad u \longmapsto \mathrm{ev}_e \circ u$$

is an isomorphism.

If the homogeneous space G/H is a projective variety, then $\mathrm{Ind}_H^G(M)$ is a finite-dimensional vector space.

Associated fiber bundles

We still consider an algebraic group G and a closed subgroup H.
Given an H-variety Y, we consider the quotient $(G \times Y)/H$, where H acts via $h \cdot (g, y) = (gh^{-1}, h \cdot y)$. It is equipped with an action of G via $g \cdot (g', y) = (gg', y)$, and with a G-equivariant map to G/H via the projection $G \times Y \to G$. The fiber of this map at the base point $o \in G/H$ is H-equivariantly isomorphic to Y.

If the above quotient exists as a variety, we denote it by $G \times^H Y$ and call it the *associated fiber bundle*, or *contracted product*. It comes with a morphism $f : G \times^H Y \longrightarrow G/H$ and with a closed immersion $i : Y \longrightarrow G \times^H Y$ identifying Y with the fiber of f at o.

We then have a dual version of Frobenius reciprocity:
for any G-variety X, the map

$$\mathrm{Hom}^G(G \times^H Y, X) \longrightarrow \mathrm{Hom}^H(Y, X), \quad u \longmapsto u \circ i$$

is an isomorphism.

Associated fiber bundles (continued)

Let M be a finite-dimensional H-module. Then the associated fiber bundle $G \times^H M$ exists, and $f : G \times^H M \to G/H$ is a G-equivariant vector bundle. Its space of global sections is identified with the induced module $\mathrm{Ind}_H^G(M)$, compatibly with the natural G-actions.

More generally, the associated fiber bundle $G \times^H Y$ exists if Y admits an H-equivariant immersion in $\mathbb{P}(M)$ for some finite-dimensional H-module M. Moreover, $G \times^H Y$ is a quasi-projective G-variety; it is normal (resp. smooth) if Y is normal (resp. smooth).

Proposition

Let Y be as above. Then the set of global sections of $f : G \times^H Y \to G/H$ may be identified with $\mathrm{Hom}^H(G, Y)$.
If G/H is projective, then $\mathrm{Hom}^H(G, Y)$ is a G-variety, and admits an open G-equivariant immersion in a projective G-variety.

The proof uses the Hilbert scheme of the projective variety $G \times^H \bar{Y}$, where $\bar{Y}$ denotes the closure of Y in $\mathbb{P}(M)$.

Induction of varieties with group action

We still consider an algebraic group G, a closed subgroup H such that G/H is projective, and an H-variety Y such that $G \times^H Y$ exists and is quasi-projective. The latter condition holds e.g. if H is linear and connected, and Y is normal and quasi-projective.

Like for induction of modules, we have an evaluation map

$$\mathrm{ev}_e : \mathrm{Hom}^H(G, Y) \longrightarrow Y, \quad f \longmapsto f(e)$$

which is an H-equivariant morphism.

Moreover, for any G-variety X, the map

$$\mathrm{Hom}^G(X, \mathrm{Hom}^H(G, Y)) \longrightarrow \mathrm{Hom}^H(X, Y), \quad u \longmapsto \mathrm{ev}_e \circ u$$

is an isomorphism.

If $u : X \to Y$ is an H-equivariant immersion, then so is the corresponding G-equivariant morphism $X \longrightarrow \mathrm{Hom}^H(G, Y)$.

Completion of the proof of 2)

Recall that it suffices to prove the following:

Let G be an algebraic group, and X a normal quasi-projective G-variety. Then X admits an equivariant immersion in a projective G-variety.

Also, recall the exact sequence

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

where H is connected and linear, and Q is projective.

By Sumihiro's theorem, we may choose an open H-equivariant immersion $i : X \to Y$, where Y is a normal projective H-variety.

Then $\mathrm{Hom}^H(G, Y)$ exists and admits an open G-equivariant immersion in a projective G-variety Z (contained in the Hilbert scheme of $G \times^H Y$).

Moreover, i corresponds to a G-equivariant immersion $X \to \mathrm{Hom}^H(G, Y)$.

We may thus view X as a G-stable subvariety of Z.

Some further developments

Weil's regularization theorem was originally proved over an arbitrary field k, using the algebro-geometric language of generic points. The modern proofs by Zaitsev and Kraft assume k algebraically closed.

An alternative proof is presented in arXiv:2202.04352. It works for any group scheme of finite type G over a field k. It is also shown that *every rational G-action on a variety admits a projective equivariant model X*.

If G is smooth, then X may be taken normal. But this does not extend to an arbitrary G. For example, let $G = \alpha_p$ (the kernel of the endomorphism $\mathbb{G}_a \longrightarrow \mathbb{G}_a$, $t \longmapsto t^p$ in characteristic $p > 0$) acting on the projective plane curve $X = (y^p z - x^{p+1} = 0)$ via $t \cdot [x : y : z] = [x : y + tz : z]$. Then X is singular and the G-action does not lift to the normalization.

The automorphism group schemes of projective curves present many open questions in positive characteristics. For example, *which Lie algebras occur as algebras of vector fields on such curves?*

It is easy to show that one gets Lie algebras of arbitrarily large dimension. By contrast, over an algebraically closed field k of characteristic 0, one only gets the subalgebras of $\mathfrak{sl}(2, k)$.