SPHERICAL VARIETIES

Notes by R. Devyatov, D. Fratila, V. Tsanov
of a course taught by M. Brion

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1. Introduction

The present constitutes the lecture notes from a mini course at the Summer School "Structures in Lie Representation Theory" from Bremen in August 2009.

The aim of these lectures is to describe algebraic varieties on which an algebraic group acts and the orbit structure is simple. The methods that will be used come from algebraic geometry, and representation theory of Lie algebras and algebraic groups.

We begin by presenting fundamental results on homogeneous varieties under (possibly non-linear) algebraic groups. Then we turn to the class of log homogeneous varieties, recently introduced in [7] and studied further in [8]; here the orbits are the strata defined by a divisor with normal crossings. In particular, we discuss the close relationship between log homogeneous varieties and spherical varieties, and we survey classical examples of spherical homogeneous spaces and their equivariant completions.

2. Homogeneous spaces

Let G be a connected algebraic group over $\mathbb{C}$ and $\mathfrak{g} = T_eG$ the Lie algebra of G.

Definition. A *G-variety* is an algebraic variety X with a G-action $G \times X \to X$, $(g, x) \mapsto g \cdot x$ which is a morphism of algebraic varieties.

If X is a G-variety then the Lie algebra of G acts as vector fields on X. If X is smooth we denote by $\mathcal{T}_X$ the tangent sheaf and we have a homomorphism of Lie algebras

$$\mathrm{op}_X : \mathfrak{g} \longrightarrow \Gamma(X, \mathcal{T}_X),$$

and at the level of sheaves

$$\underline{\mathrm{op}}_X : \mathcal{O}_X \otimes \mathfrak{g} \longrightarrow \mathcal{T}_X.$$

Examples: 1) *Linear algebraic groups*: $G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$ closed.

2) *Abelian varieties*, that is, complete connected algebraic groups. E.g. elliptic curves. Such groups are always commutative as will be shown below.

3) *Adjoint action*: consider the action of G on itself by conjugation. The identity $e \in G$ is a fixed point, and so G acts on $T_eG = \mathfrak{g}$. We obtain the adjoint representation $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ whose image is called the adjoint group; its kernel is the center $Z(G)$. The differential of Ad is $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$, given by $\mathrm{ad}(x)(y) = [x, y]$.

Definition. A G-variety X is called *homogeneous* if G acts transitively.

Let X be a homogeneous G-variety. Choose a point $x \in X$ and consider $G_x = \mathrm{Stab}_G(x)$ the stabilizer of x in G. Then G_x is a closed subgroup of G, since this is the fiber at x of the orbit map $G \to X$, $g \mapsto g \cdot x$. Moreover, this map factors through an isomorphism of G-varieties $X \cong G/G_x$. We actually have more than that, since the coset space has a distinguished point, namely eG_x. We have an isomorphism of G-varieties with a base point $(G/G_x, eG_x) \to (X, x)$. Note that every homogeneous variety X is smooth, and the morphism $\underline{\mathrm{op}}_X$ is surjective.

Lemma 2.1.

(i) Let X be a G-variety, where G acts faithfully, and $x \in X$. Then G_x is linear.

(ii) Let $Z(G)$ denote the center of G. Then $G/Z(G)$ is linear.

(iii) Abelian varieties are commutative groups.

Proof. (i) Let $\mathcal{O}_{X,x}$ denote the local ring of all rational functions on X defined at x, and $\mathfrak{m}_x$ denote its maximal ideal consisting of all elements vanishing at x. We will use the following two facts from commutative algebra:

$\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is a finite dimensional $\mathbb{C}$-vector space for all $n \geq 1$.

Krull's Intersection Theorem: $\bigcap_n \mathfrak{m}_x^n = \{0\}$.

Now, G_x acts faithfully on the local ring $\mathcal{O}_{X,x}$ and preserves $\mathfrak{m}_x$. This induces an action of G_x on each of the quotients $\mathcal{O}_{X,x}/\mathfrak{m}_x^n, n \geq 1$. Denote by K_n the kernel of the morphism $G_x \to \mathrm{GL}(\mathcal{O}_{X,x}/\mathfrak{m}_x^n)$. Then $\{K_n\}_n$ is a decreasing sequence of closed subgroups of G_x and $\bigcap_n K_n = \{e\}$ from Krull's Intersection Theorem. Since we are dealing with Noetherian spaces, the sequence K_n must stabilize, i.e. $K_n = \{e\}, \forall n \gg 1$. Thus we have obtained a faithful action of G_x on a finite dimensional vector space, which represents G_x as a linear algebraic group.

(ii) The image of the adjoint representation is $G/Z(G)$, a closed subgroup of $\mathrm{GL}(\mathfrak{g})$. Therefore $G/Z(G)$ is linear algebraic.

(iii) If G is an abelian variety, then $G/Z(G)$ is both complete and affine, hence a point.

□

2.1. **Homogeneous bundles.**

Definition. Let X be a G-variety and $p : E \to X$ a vector bundle. We say that E is *G-linearized* if G acts on E, the projection p is equivariant, and G acts "linearly on fibers", i.e. if $x \in X, g \in G$, then $E_x \xrightarrow{g} E_{gx}$ is linear. We will work only with vector bundles of finite rank.

If X is homogeneous, a G-linearized vector bundle E is also called *homogeneous*. If we write $(X, x) = (G/H, eH)$, then H acts linearly on the fiber E_x. In fact, there is an equivalence of categories :

$$\left(\begin{array}{c} \text{homogeneous vector} \\ \text{bundles on } G/H \end{array} \right) \simeq \left(\begin{array}{c} \text{linear representations} \\ \text{of} \quad H \end{array} \right).$$

More precisely, if E is a homogeneous vector bundle, then E_x is a linear representation of H. Conversely, if V is a linear representation of H, then

$$E = G \overset{H}{\times} V := \{(g, v) \in G \times V\}/(g, v) \sim (gh^{-1}, hv)$$

is a homogeneous vector bundle on X, with $E_x \cong V$.

Examples: 1) The tangent bundle $T_{G/H}$ corresponds to the quotient of the H-module $\mathfrak{g}$ (where H acts via the restriction of the Ad representation) by the submodule $\mathfrak{h}$.

2) The cotangent bundle $T^*_{G/H}$, with its sheaf of differential 1-forms $\Omega^1_{G/H}$, is associated with the module $(\mathfrak{g}/\mathfrak{h})^* = \mathfrak{h}^\perp \subseteq \mathfrak{g}$.

More generally, if Y is an H-variety then we can form, in a similar way, a bundle $X := G \overset{H}{\times} Y$ with projection $X \to G/H$, which is G-equivariant. The fiber over eH is Y. The bundle $G \overset{H}{\times} Y$ is called a *homogeneous fiber bundle.*

Remark. A fiber bundle X, as above, is always a complex space but it is not true in general that it is an algebraic variety. However, if Y is a locally closed H-stable subvariety of the projectivization $\mathbb{P}(V)$, where V is an H-module, then X is a variety (as follows from [18, Prop. 7.1]). This holds, for instance, if Y is affine; in particular, for homogeneous vector bundles X is always a variety.

Our next aim is to classify complete homogeneous varieties. It is well known that the automorphism group $\mathrm{Aut}(X)$ of a compact complex space is a complex Lie group (see [1, Sec. 2.3]). For any topological group G, we denote by G° the connected component of the identity element. In particular, $\mathrm{Aut}^\circ(X)$ is a complex Lie group.

Theorem 2.1. (C.P. Ramanujam [21]) If X is a complete complex algebraic variety, then $\mathrm{Aut}^\circ(X)$ is a connected algebraic group with Lie algebra $\Gamma(X, \mathcal{T}_X)$.

Corollary 2.2. Let X be a complete variety. Then X is homogeneous if and only if $\mathcal{T}_X$ is generated by its global sections, i.e. if and only if $\underline{\mathrm{op}}_X : \mathcal{O}_X \otimes \Gamma(X, \mathcal{T}_X) \to \mathcal{T}_X$ is surjective.

Proof. The fact that homogeneity of X implies surjectivity of $\underline{\mathrm{op}}_X$ was already noted above.

For the converse, denote $G = \mathrm{Aut}^\circ(X)$. We know, from Ramanujam's theorem, that the Lie algebra $\mathfrak{g}$ of G is identified with $\Gamma(X, \mathcal{T}_X)$. For $x \in X$ denote by $\varphi_x : G \to X$ the orbit map: $g \mapsto g \cdot x$. We observe that the surjectivity of the differential at the origin $(d\varphi_x)_e : \mathfrak{g} \to T_x X$ is equivalent to the surjectivity of the stalk map $(\underline{\mathrm{op}}_X)_x : \Gamma(X, \mathcal{T}_X) = \mathfrak{g} \to T_x X$, which is assumed to hold. Since φ_x is equivariant with respect to G, and G is homogeneous as a G-variety (considered with the left multiplication action), it follows that $d\varphi_x$ is surjective at every point. Hence φ_x is a submersion, and therefore $\mathrm{Im}(\varphi_x) = G \cdot x$ is open in X.

We proved that for every x the orbit $G \cdot x$ is open in X, but since X is a variety, it follows that $G \cdot x = X$, i.e. X is homogeneous. $\square$

Corollary 2.3. Let X be a complete variety. Then X is an abelian variety if and only if $\mathcal{T}_X$ is a trivial bundle, i.e. if and only if $\underline{\mathrm{op}}_X$ is an isomorphism.

Proof. The fact that abelian varieties have trivial tangent bundle is clear, since algebraic groups are parallelizable.

Let us show the converse implication. From Corollary 2.2 we know that X is homogeneous and hence can be written as $X = G/H$ where $G = \mathrm{Aut}^\circ(X)$ and H is the stabilizer of a given point. Now, since the tangent bundle of X is trivial we have $\dim(X) = \dim(\Gamma(X, \mathcal{T}_X)) = \dim(\mathfrak{g}) = \dim(G)$, and hence H is finite. Therefore G is complete, i.e. an abelian variety. Now, since H fixes a point and is a normal subgroup of G, it follows (from the homogeneity) that H acts trivially on X. Hence $H = \{e\}$. $\square$

In what follows we will make extensive use of the following theorem of Chevalley regarding the structure of algebraic groups (see [9] for a modern proof):

Theorem 2.4. If G is a connected algebraic group, then there exists an exact sequence of algebraic groups

$$1 \longrightarrow G_{\mathrm{aff}} \longrightarrow G \xrightarrow{p} A \longrightarrow 1 ,$$

where $G_{\mathrm{aff}} \trianglelefteq G$ is an affine, closed, connected, normal subgroup, and A is an abelian variety. Moreover, G_{aff} and A are unique.

As an easy consequence we obtain the following lemma:

Lemma 2.2. Any connected algebraic group G can be written as $G = G_{\mathrm{aff}} Z(G)^\circ$.

Proof. We have

$$G/G_{\mathrm{aff}}Z(G) = \underbrace{A/p(Z(G))}_{\text{complete}} = \frac{G/Z(G)}{G_{\mathrm{aff}}Z(G)/Z(G)} ,$$

and since $G/Z(G)$ is affine (see Lemma 2.1), it follows that $G/G_{\mathrm{aff}}Z(G)$ is complete and affine. Hence $G = G_{\mathrm{aff}}Z(G) = G_{\mathrm{aff}}Z(G)^\circ$. $\square$

We also need another important result (see [13] ch. VIII):

Theorem 2.5. (Borel's fixed point theorem)

Any connected solvable linear algebraic group that acts on a complete variety has a fixed point.

Theorem 2.6. Let X be a complete homogeneous variety. Then $X = A \times Y$, where A is an abelian variety and $Y = S/P$, with S semisimple and P parabolic in S.

Proof. Let $G := \mathrm{Aut}^\circ(X)$. Borel's theorem implies that $Z(G)^\circ_{\mathrm{aff}}$ acting on X has a fixed point. This group is normal in G and, since X is homogeneous, it follows that $Z(G)^\circ_{\mathrm{aff}}$ is trivial. Therefore, according to Chevalley's theorem, $Z(G)^\circ =: A$ is an abelian variety and $G_{\mathrm{aff}} \cap A$ is finite (since it is affine and complete).

By Lemma 2.2, the map $G_{\mathrm{aff}} \times A \to G$, defined by $(g, a) \mapsto ga^{-1}$, is a surjective morphism of algebraic groups. Its kernel is isomorphic to $G_{\mathrm{aff}} \cap A$. Thus, $G \simeq (G_{\mathrm{aff}} \times A)/K$ where K is a finite central subgroup.

The radical $R(G_{\mathrm{aff}})$ has a fixed point in X by Borel's theorem. Hence it acts trivially, and we can suppose G_{aff} semisimple. In particular, we have $Z(G_{\mathrm{aff}}) = \{e\}$, i.e. G_{aff} is adjoint. Therefore, $G_{\mathrm{aff}} \cap A = \{e\}$, i.e. $K = \{e\}$. We can conclude that $G = G_{\mathrm{aff}} \times A$.

Let $x \in X$ and consider $G_x = \mathrm{Stab}_G(x)$. From Lemma 2.1 it follows that G_x is affine and therefore $G_x^\circ \subseteq G_{\mathrm{aff}}$. Since G/G_x is complete, G/G_x° and $G_{\mathrm{aff}}/G_x^\circ$ are also complete. This implies that $G_x^\circ =: P$ is a parabolic subgroup in G_{aff}.

Now, consider the projection $G = G_{\mathrm{aff}} \times A \to G_{\mathrm{aff}}$ and its restriction $p_1 : G_x \to G_{\mathrm{aff}}$, with kernel A_x. Since $A_x = A \cap G_x$, it follows that A_x has a fixed point, and therefore acts trivially. Hence $A_x = \{e\}$. Since $[p_1(G_x) : P] < \infty$ and P is parabolic, hence connected and equal to its normalizer, we find that $p_1(G_x) = P$. We have proved that $G_x = P$. Putting all together, we get $X = G_{\mathrm{aff}}/P \times A$. □

3. Log homogeneous varieties

Definition. Let X be a smooth variety over $\mathbb{C}$ and D an effective, reduced divisor (i.e. a union of distinct subvarieties of codimension 1). We say that D has *normal crossings* if for each point $x \in X$ there exist local coordinates $t_1, ..., t_n$ at x such that, locally, D is given by the equation $t_1 \cdots t_r = 0$ for some $r \leq n$. More specifically, the completed local ring $\widehat{\mathcal{O}_{X,x}}$ is isomorphic to the power series ring $\mathbb{C}[[t_1, \ldots, t_n]]$, and the ideal of D is generated by $t_1 \cdots t_r$.

Definition. For a pair (X, D) consisting of a smooth variety and a divisor with normal crossings, we define the *sheaf of logarithmic vector fields*

$$\mathcal{T}_X(-\log D) = \left\{ \begin{array}{l} \text{derivations of } \mathcal{O}_X \text{ which} \\ \text{preserve the ideal sheaf of } D \end{array} \right\} \subset \mathcal{T}_X\ .$$

Example: Take $X = \mathbb{C}^n$ and $D = (t_1 \cdots t_r = 0), r \leq n$ the union of some of the coordinate hyperplanes. Here $\mathcal{T}_X(-\log D)$ is generated, at

$x = (0, ..., 0)$, by $t_1 \frac{\partial}{\partial t_1}, \ldots, t_r \frac{\partial}{\partial t_r}, \frac{\partial}{\partial t_{r+1}}, \ldots, \frac{\partial}{\partial t_n}$.

The sheaf $\mathcal{T}_X(-\log D)$ is locally free, and hence corresponds to a vector bundle. However, it does not correspond to a subbundle of the tangent bundle, since the quotient has support D, and hence is torsion. Observe that $\mathcal{T}_X(-\log D)$ restricted to $X \setminus D$ is nothing but $\mathcal{T}_{X\setminus D}$.

If we take the dual of the sheaf of logarithmic vector fields, we obtain the sheaf of rational differential 1-forms $\Omega^1_X(\log D)$ with poles of order at most 1 along D, called the *sheaf of differential forms with logarithmic poles*. From the previous example we see that $\Omega^1_X(\log D)$ is generated, at $x = (0, ..., 0)$, by $\frac{dt_1}{t_1}, ..., \frac{dt_r}{t_r}, dt_{r+1}, \ldots, dt_n$.

Now, suppose a connected algebraic group G, with Lie algebra $\mathfrak{g}$, acts on X and preserves D. We get the map

$$\mathrm{op}_{X,D} : \mathfrak{g} \longrightarrow \Gamma(X, \mathcal{T}_X(-\log D)) ,$$

and its sheaf version

$$\underline{\mathrm{op}}_{X,D} : \mathcal{O}_X \otimes \mathfrak{g} \longrightarrow \mathcal{T}_X(-\log D) .$$

Definition. We call a pair (X, D) as above *log homogeneous under* G, if $\underline{\mathrm{op}}_{X,D}$ is surjective. We call it *log parallelizable*, if $\underline{\mathrm{op}}_{X,D}$ is an isomorphism.

Examples: 1) Let $X = \mathbb{C}^n$, $D = (t_1 \cdots t_n = 0)$, and let $G = (\mathbb{C}^*)^n$ act on X by coordinate-wise multiplication. Then $\mathfrak{g} = \mathbb{C}^n$ acts via $(t_1 \frac{\partial}{\partial t_1}, \ldots, t_n \frac{\partial}{\partial t_n})$. Actually, in this case $\underline{\mathrm{op}}_{X,D}$ is an isomorphism, so that (X, D) is log parallelizable.

2) Let $X = \mathbb{P}^1$. Its automorphism group $G = \mathrm{PGL}(2)$ acts transitively, so X is homogeneous. Let B be the subgroup of G consisting of the images of the matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, and let $U \subset B$ consist of the images of the matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Then B acts on X with two orbits: the fixed point ∞ and its complement. Moreover, $(\mathbb{P}^1, \infty)$ is log homogeneous for B.

On the other hand, U acts on $\mathbb{P}^1$, with the same orbits, but the action on $(\mathbb{P}^1, \infty)$ is not log homogeneous.

The 1-torus $\mathbb{C}^* = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ acts on $(\mathbb{P}^1, \{0, \infty\})$, which is log parallelizable under this action.

3) Smooth toric varieties: let X be a smooth algebraic variety on which a torus $T = (\mathbb{C}^*)^n$ acts with a dense open orbit, and trivial stabilizer for points in that orbit. Thus, T can be identified with its open orbit in X. Put $D = X \setminus T$. It can be shown that D has normal crossings, and the pair (X, D) is log parallelizable for the T-action. More precisely, a smooth and complete toric variety admits a covering by open T-stable subsets, each isomorphic to $\mathbb{C}^n$ where T acts by coordinate-wise multiplication. Noncomplete smooth toric varieties admit a smooth equivariant completion satisfying the above (for these facts, see [19, Sec. 1.4]).

Remark. If (X, D) is log homogeneous under a group G, then $X_0 := X \setminus D$ consists of one G-orbit. Indeed, the map $\underline{\mathrm{op}}_{X_0} : \mathcal{O}_{X_0} \otimes \mathfrak{g} \longrightarrow \mathcal{T}_{X_0}$ is surjective, and the assertion follows by arguing as in the proof of Corollary 2.2. If (X, D) is log parallelizable, then the stabilizer of any point in $X \setminus D$ is finite.

3.1. **Criteria for log homogeneity. Criterium 1.** Let $X = G \overset{H}{\times} Y$ be a homogeneous fibre bundle. Then every G-stable divisor in X is of the form $D = G \overset{H}{\times} E$, with $E = D \cap Y$ an H-invariant divisor in Y. Moreover, (X, D) is log homogeneous (resp. log parallelizable) for G, if and only if (Y, E) is log homogeneous (resp. log parallelizable) for H°.
(The proof is easy, see [7, Prop. 2.2.1] for details).

The second criterium formulated below uses a stratification of the divisor. Let X be a G-variety, where G is a connected algebraic group, and let D be an invariant divisor with normal crossings. A stratification of D is obtained as follows. Let

$$X_1 = D, \quad X_2 = \mathrm{Sing}(D), \ldots, \quad X_m = \mathrm{Sing}(X_{m-1}), \ldots$$

Now, the strata are taken to be the connected components of $X_{m-1} \setminus X_m$. Each stratum is a smooth locally closed subvariety, and is preserved by the G-action. Let S be a stratum and $x \in S$ be a point. Let $t_1, ..., t_n$ be local coordinates for X around x such that D is defined by the equation $t_1 \cdots t_r = 0$. Then $X_{m-1} \setminus X_m$ is the set of points where precisely $m - 1$ of the coordinates are 0; in particular, $S = (t_1 = \cdots = t_r = 0)$ has codimension r in X.

The normal space to S at x is defined by

$$N = N_{S/X,x} = T_xX/T_xS.$$

It contains the normal spaces to S at x of the various strata of codimension $r-1$; these spaces are precisely the lines

$$L_i = N_{S/(t_1=\cdots=\widehat{t_i}=\cdots=t_r=0),x}$$

and we have the decomposition

$$N = L_1 \oplus \cdots \oplus L_r.$$

The stabilizer G_x acts on T_xX, T_xS and N. Since the divisor D is G-invariant, the connected component G_x° preserves each of the lines L_i, while the full stabilizer G_x is allowed, in addition, to permute them. Thus, we obtain a map

$$\rho_x : G_x^\circ \longrightarrow (\mathbb{C}^*)^r \ ,$$

with differential

$$d\rho_x : \mathfrak{g}_x \longrightarrow \mathbb{C}^r \ .$$

We can now formulate

Criterium 2. The pair (X, D) is log homogeneous (resp. log parallelizable) for G, if and only if each stratum S consists of a single G-orbit and for any $x \in S$ the map $d\rho_x$ is surjective (resp. bijective).

Furthermore, if these conditions hold, then there is an exact sequence

$$0 \longrightarrow \mathfrak{g}_{(x)} \longrightarrow \mathfrak{g} \longrightarrow \mathcal{T}_X(-\log D)_x \longrightarrow 0 \ ,$$

where $\mathfrak{g}_{(x)} = \ker(d\rho_x)$ is the stabilizer of the point x and all normal to S directions at that point.

Proof. [1] Since $\mathcal{T}_X(-\log D)$ preserves the ideal sheaf of S, we have a morphism

$$\mathcal{T}_X(-\log D)_{|S} \longrightarrow \mathcal{T}_S \ .$$

Hence, at the point x there is a linear map

$$p : \mathcal{T}_X(-\log D)_x \longrightarrow T_xS \ .$$

In suitable local coordinates $t_1, ..., t_n$ for X around x, the map p is given by the projection

$$\mathrm{span}\{t_1\frac{\partial}{\partial t_1}, ..., t_r\frac{\partial}{\partial t_r}, \frac{\partial}{\partial t_{r+1}}, ..., \frac{\partial}{\partial t_n}\} \longrightarrow \mathrm{span}\{\frac{\partial}{\partial t_{r+1}}, ..., \frac{\partial}{\partial t_n}\} \ .$$

[1] This proof was not presented in the lectures, and is taken from [7, Prop. 2.1.2].

So, we have an exact sequence

$$0 \longrightarrow \mathrm{span}\{t_1\frac{\partial}{\partial t_1}, ..., t_r\frac{\partial}{\partial t_r}\} \longrightarrow \mathcal{T}_X(-\log D)_x \xrightarrow{p} T_xS \longrightarrow 0\ .$$

Observe that the composition $p \circ \mathrm{op}_{X,D} : \mathfrak{g} \longrightarrow T_xS$ equals op_S, and hence yields an injective map $i_x : \mathfrak{g}/\mathfrak{g}_x \longrightarrow T_xS$. Thus we have a commutative diagram, where the rows are exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{g}_x & \to & \mathfrak{g} & \to & \mathfrak{g}/\mathfrak{g}_x & \to & 0 \\ & & \downarrow d\rho_x & & \downarrow \mathrm{op}_{X,D} & & \downarrow i_x & & \\ 0 & \to & \mathrm{span}\{t_1\frac{\partial}{\partial t_1}, ..., t_r\frac{\partial}{\partial t_r}\} & \to & \mathcal{T}_X(-\log D)_x & \to & T_xS & \to & 0 \end{array}$$

Since i_x is injective, the snake lemma implies that $\mathrm{op}_{X,D}$ is surjective if and only if $d\rho_x$ and i_x are surjective. Notice that i_x is onto exactly when the orbit $G \cdot x$ is open in S. This proves the first statement of the criterium.

The second statement follows: if the conditions hold, then we have an isomorphism $\ker(d\rho_x) \xrightarrow{\sim} \ker(\mathrm{op}_{X,D})$. This yields the desired exact sequence. □

3.2. **The Albanese morphism.** [2]

Using the preceding criteria, we will classify all complete log parallelizable varieties. For this, we also need some results about the Albanese morphism that we now survey (see [22] for details).

Given a variety X, there exists a *universal morphism from X to an abelian variety*, i.e., a morphism

$$f : X \to A$$

where A is an abelian variety, such that any morphism $g : X \to B$ where B is an abelian variety, admits a factorization as $g = \varphi \circ f$ for a unique morphism (of varieties) $\varphi : A \to B$. Then, by rigidity of abelian varieties, φ is the composition of a group homomorphism and a translation of A. We say that $f =: \alpha_X$ is the *Albanese morphism* of X, and $A =: A(X)$ is the *Albanese variety.*

Next, consider a pointed variety (X, x), that is, a pair consisting of a variety X together with a base point $x \in X$. We may assume that $\alpha_X(x) = 0$ (the origin of the Albanese variety). Then

$$\alpha_X : X \to A(X), \quad x \mapsto 0$$

is universal among morphisms to abelian varieties that send x to the origin.

[2]This material was not presented in the lectures and was added by M. Brion to prepare for the proof of Theorem 3.1.

For the pointed variety (G, e) where G denotes a connected algebraic group, the Albanese morphism is nothing but the quotient homomorphism $p : G \to A$ given by Chevalley's theorem; in particular, $A(G) = A$. Indeed, given an abelian variety B, every morphism $G \to B$, $e \mapsto 0$ is a group homomorphism and sends G_{aff} to 0, in view of [17, Cor. 2.2, Cor. 3.9].

More generally, consider a homogeneous space $X = G/H$ with base point $x = eH/H$. Then the product $G_{\mathrm{aff}}H \subset G$ is a closed normal subgroup, independent of the choice of x, since the quotient $G/G_{\mathrm{aff}} = A$ is commutative and hence we have $G_{\mathrm{aff}}gHg^{-1} = G_{\mathrm{aff}}H$ for all $g \in G$. Moreover, the quotient $G/G_{\mathrm{aff}}H = A/p(H)$ is an abelian variety. It follows easily that the quotient map $G/H \to G/G_{\mathrm{aff}}H$ is the Albanese morphism.

Suppose now that X is a smooth G-variety containing an open G-orbit $X_0 \cong G/H$. By Weil's extension theorem (see e.g. [17, Thm. 3.1]) the morphism $\alpha_{X_0} : G/H \to G/G_{\mathrm{aff}}H$ extends to a unique morphism

$$\alpha_X : X \to G/G_{\mathrm{aff}}H,$$

the Albanese morphism of (X, x). Since α_{X_0} is G-equivariant, then so is α_X. This defines a fibre bundle

$$X = G \overset{G_{\mathrm{aff}}H}{\times} Y$$

where the fibre $Y = \alpha_X^{-1}\alpha_X(x)$ is smooth; we say that α_X is the *Albanese fibration* of X. If G acts faithfully on X, then H is affine by Lemma 2.1, and hence $H^\circ \subset G_{\mathrm{aff}}$. In particular, $(G_{\mathrm{aff}}H)^\circ = G_{\mathrm{aff}}$.

Having these results at hand, we can now obtain the following characterization of log parallelizable varieties, due to Winkelmann (see [26]).

Theorem 3.1. Let X be a smooth, complete variety, and D be a divisor with normal crossings. Let $G = \mathrm{Aut}^\circ(X, D)$. Then (X, D) is log parallelizable for G if and only if G_{aff} is a torus and X is a fibre bundle of the form $X = G \overset{G_{\mathrm{aff}}}{\times} Y$, where Y is a smooth complete toric variety under G_{aff}. In this case, the map $X \to G/G_{\mathrm{aff}}$ is the Albanese morphism.

In particular, if (X, D) is log parallelizable, then its connected automorphism group is an extension of an abelian variety by a torus.

Proof. If we suppose that X has the described fibration properties, then log parallelizability follows directly from Criterium 1.

Conversely, suppose that (X, D) is log parallelizable. Then X contains an open G-orbit, and hence is a fibre bundle $G \overset{G_{\mathrm{aff}}H}{\times} Y$ as above. By Criterium 1 again, it follows that $(Y, D \cap Y)$ is log parallelizable under G_{aff}. Since Y is complete, there exists $y \in Y$ such that the orbit $G_{\mathrm{aff}} \cdot y$ is closed in Y (y must necessarily belong to the divisor $D \cap Y$). Then the stabilizer $(G_{\mathrm{aff}})_y$ is a parabolic subgroup of G_{aff}, in particular connected. From Criterium 2, it follows that $(G_{\mathrm{aff}})_y^\circ$ is a torus. But this implies that G_{aff} itself must be a torus. The variety Y is then a toric variety under G_{aff}.

Furthermore, since $G = G_{\mathrm{aff}} Z(G)^\circ$ (Lemma 2.2), it follows that the group G itself is commutative. Thus we must have $H = \{e\}$, and finally, $X = G \overset{G_{\mathrm{aff}}}{\times} Y$. □

Example: Let E be an elliptic curve. Let L be a line bundle on E of degree zero. Thus L is of the form $\mathcal{O}_E(p - q)$ for some $p, q \in E$. Then $G := L \setminus$ (zero section) is a principal $\mathbb{C}^*$-bundle on E. In fact, G is a connected algebraic group and we have an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G \longrightarrow E \longrightarrow 0$$

(as follows e.g. from [17, Prop. 11.2]). Take $X = \mathbb{P}(L \oplus \mathcal{O}_E)$. Then the projection $X \longrightarrow E$ is a G-equivariant $\mathbb{P}^1$-bundle, that is, X can be written as $X = G \overset{\mathbb{C}^*}{\times} \mathbb{P}^1$. The divisor is $D = G \overset{\mathbb{C}^*}{\times} \{0, \infty\}$.

3.3. **The Tits morphism.** Let $(X_0, x_0) = (G/H, eH)$ be a homogeneous space. For each $x \in X_0$, the isotropy Lie algebra is denoted by $\mathfrak{g}_x$. All these isotropy Lie algebras are conjugate to $\mathfrak{h}$, and in particular have the same dimension. Let

$$\mathcal{L} := \{\mathfrak{l} \subset \mathfrak{g} \text{ Lie subalgebra} \mid \dim \mathfrak{l} = \dim \mathfrak{h}\},$$

be the variety of Lie subalgebras of $\mathfrak{g}$. The group G acts on $\mathcal{L}$ via the adjoint action on $\mathfrak{g}$. We have a G-equivariant map

$$\begin{array}{rccc} \tau: & X_0 & \longrightarrow & \mathcal{L} \\ & x & \longmapsto & \mathfrak{g}_x . \end{array}$$

This map is called the *Tits morphism.* The image of τ is

$$\tau(X_0) = G \cdot \mathfrak{h} = G/N_G(\mathfrak{h}) = G/N_G(H^\circ) .$$

Thus τ is a fibration, with fibre

$$N_G(H^\circ)/H = (N_G(H^\circ)/H^\circ)/(H/H^\circ) .$$

Observe that $N_G(H^\circ)/H^\circ$ is an algebraic group, and H/H° is a finite subgroup. Since $G = G_{\mathrm{aff}} Z(G)^\circ$, and τ is clearly $Z(G)$-invariant, the image $\tau(X_0)$ is a unique orbit under G_{aff}. If the action of G on X_0 is faithful, then H is affine by Lemma 2.1. Thus, $H^\circ \subset G_{\mathrm{aff}}$, and hence

$$\tau(X_0) = G_{\mathrm{aff}}/N_{G_{\mathrm{aff}}}(H^\circ) .$$

Now, let (X, D) be a log homogeneous variety for a group G, and take $X_0 = X \setminus D$. Then the Tits morphism defined on X_0 as above, extends to X by

$$\begin{array}{rccc} \tau : & X & \longrightarrow & \mathcal{L} \\ & x & \longmapsto & \mathfrak{g}_{(x)} \ . \end{array}$$

Notice that the Tits morphism is constant if and only of (X, D) is log parallelizable for G.

Remark. If X is a complete homogeneous variety, write $X = A \times Y$ according to Theorem 2.6. Then the Albanese and Tits morphisms are given by the two projections of this Cartesian product; respectively

$$\alpha : X \longrightarrow A \quad , \quad \tau : X \longrightarrow Y \ .$$

4. Local structure of log homogeneous varieties

Let (X, D) be a complete log homogeneous variety for a connected linear algebraic group G. Then there are only finitely many orbits of G in X, and they form a stratification (Criterium 2). Let $Z = G \cdot z = G/G_z$ be a closed orbit, through a given point z. The stabilizer G_z is then a parabolic subgroup of G. Let $R_u(G)$ and G_{red} be respectively the unipotent radical and a Levi subgroup (i.e., a maximal connected reductive subgroup) of G, so that we have the Levi decomposition

$$G = R_u(G) G_{\mathrm{red}} \ .$$

Moreover, G_{red} is unique up to conjugation by an element in $R_u(G)$ (see [20, Chap. 6] for these results.)

Arguing as in the proof of Theorem 2.6, we see that $R_u(G)$ fixes Z pointwise. Thus, G_{red} acts transitively on Z and we have

$$Z = G_{\mathrm{red}} \cdot z = G_{\mathrm{red}}/(G_{\mathrm{red}} \cap G_z) \ .$$

with $G_{\mathrm{red}} \cap G_z$ a parabolic in G_{red}. We are aiming to describe the local structure of X along Z.

More generally, let G be a connected reductive group acting on a normal variety X. Suppose $Z \subset X$ is a complete orbit of this action. Fix a point $z \in Z$. The stabilizer G_z is a parabolic subgroup of G. Let P be an opposite parabolic, i.e., $L := P \cap G_z$ is a Levi subgroup of both G_z and P. Then $P \cdot z = R_u(P) \cdot z$ is an open cell in Z. In fact, the action of the unipotent radical on this orbit is simply transitive, so that $R_u(P) \cdot z \cong R_u(P)$. With this notation, we have the following

Theorem 4.1. There exists a subvariety $Y \subset X$ containing z, which is affine, L-stable, and such that the map

$$\begin{array}{rccc} \psi : & R_u(P) \times Y & \longrightarrow & X \\ & (g, y) & \longmapsto & g \cdot y \end{array}$$

is an open immersion. In particular $Y \cap Z = \{z\}$.

Proof. [3] First notice that X can be replaced with any G-stable neighborhood of Z. A result of Sumihiro (see [24]) implies that such a neighborhood can be equivariantly embedded in a projective space $\mathbb{P}(V)$, where V is a G-module. We may even assume that X is the entire projective space $\mathbb{P}(V)$.

In this case V contains an eigenvector v_λ for G_z with weight λ, such that $z = [v_\lambda]$. There exists an eigenvector $f = f_{-\lambda} \in V^*$ for P with weight $-\lambda$, such that $f(v_\lambda) \neq 0$. Let $X_f = \mathbb{P}(V)_f \cong X \setminus (f = 0)$ be the localization of X along f. Our aim is to find an L-stable closed subvariety $Y \subset X_f$, such that $\psi : R_u(P) \times Y \longrightarrow X_f$ is an isomorphism. It is sufficient to construct a P-equivariant map

$$\varphi : X_f \longrightarrow P/L \cong R_u(P) \ .$$

Then we may take $Y = \varphi^{-1}(eL)$.

Start with

$$\begin{array}{rccc} \varphi : & X_f & \longrightarrow & \mathfrak{g}^* \\ & [v] & \longmapsto & \left(\xi \mapsto \frac{(\xi f)(v)}{f(v)}\right) \end{array}$$

Note that for $[v] \in X_f$ and $\xi \in \mathfrak{p}$ we have

$$\varphi[v](\xi) = \frac{(\xi f)(v)}{f(v)} = \frac{-\lambda(\xi) f(v)}{f(v)} = -\lambda(\xi) \ .$$

Now, choose a G-invariant scalar product on $\mathfrak{g}$. This choice yields an identification $\mathfrak{g}^* \cong \mathfrak{g}$. The composition of this identifying map and φ is a P-equivariant map, still denoted by $\varphi : X_f \longrightarrow \mathfrak{g}$. Let $\zeta \in \mathfrak{g}$ be the element corresponding to $-\lambda \in \mathfrak{g}^*$. Let $\mathfrak{n}$ be the nil-radical of $\mathfrak{p}$. We have $\mathfrak{n} = \mathfrak{p}^\perp$, and hence $\varphi : X_f \longrightarrow \zeta + \mathfrak{n}$. The affine space $\zeta + \mathfrak{n}$ consists of a single P-orbit, and we have $P_\zeta = L$. Thus

$$\varphi(X_f) = \zeta + \mathfrak{n} \cong P \cdot \zeta \cong P/L \ .$$

We have obtained the desired fibre bundle structure on X_f. □

[3] This proof, due to Knop (see [14]), was not presented in the lectures, and is taken from the notes of M. Brion.

Theorem 4.2. Let (X, D) be a complete, log homogeneous variety under a connected affine algebraic group G. Let $G = R_u(G)G_{\text{red}}$ be a Levi decomposition. Let $Z = G \cdot z$ be a closed orbit. Let P, L, Y be as in Theorem 4.1. Then $Y \cong \mathbb{C}^r$, where L acts via a surjective homomorphism to $(\mathbb{C}^*)^r$.

Proof. The tangent space T_zX is a G_z-module. The subspace T_zZ tangent to the orbit Z is a submodule. The normal space to Z at that point is

$$N = T_zX/T_zZ\ ,$$

which is in turn a G_z-module. Put $r = \dim N$. From our Criterium 2 for log homogeneity (paragraph 3.1), we deduce that $G_z = G_z^\circ$ acts on N diagonally, via a surjective homomorphism $G_z \longrightarrow (\mathbb{C}^*)^r$. So the unipotent radical $R_u(G_z)$ acts trivially, and the restriction to the Levi subgroup $L \longrightarrow (\mathbb{C}^*)^r$ is surjective as well. Let $\chi_1, ..., \chi_r$ be the corresponding characters of L.

Theorem 4.1 implies that we can decompose T_zX into a direct sum of L-modules as

$$T_zX = T_zZ \oplus T_zY\ .$$

As a consequence, there is an isomorphism of L-modules

$$T_zY \cong N\ .$$

It follows that L acts on T_zY diagonally, with weights $\chi_1, ..., \chi_r$. Now, let $\mathcal{O}(Y)$ be the coordinate ring of Y, and $\mathfrak{m}$ the maximal ideal of z. Then L acts on the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ via $-\chi_1, ..., -\chi_r$. The action on $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is given by the characters of the form $-k_1\chi_1 - \cdots - k_r\chi_r$ with $k_i \geq 0$ and $\sum k_i = k$. Since $\mathcal{O}(Y)$ is filtered by the powers $\mathfrak{m}^k$, and is a semisimple L-module, it follows that $\mathcal{O}(Y) \cong \mathbb{C}[t_1, ..., t_r]$. The coordinate t_i is taken to be an L-eigenvector in $\mathfrak{m}$ mapped to the i-th coordinate in $\mathfrak{m}/\mathfrak{m}^2$, an eigenvector with character $-\chi_i$. We can conclude that $Y \cong \mathbb{C}^r$ with a diagonal action of L. $\square$

Corollary 4.3. With the notation from the above theorem, let $B \subset G_{\text{red}}$ be any Borel subgroup. Then B has an open orbit in X.

Proof. Since all Borel subgroups of G_{red} are conjugate, it suffices to prove the statement for a particular one. So we can assume that $B \subset P$. Then we can write $B = R_u(P)(B \cap L)$, and $B \cap L$ is a Borel subgroup of L. We have $Z(L)^\circ \subset B \cap L$. By Theorem 4.2, $Z(L)^\circ$ has an open dense orbit in Y. By Theorem 4.1, we have an open immersion $R_u(P) \times Y \longrightarrow X$. This proves the corollary. $\square$

5. Spherical varieties and classical homogeneous spaces

Let G be a connected reductive group over $\mathbb{C}$. Let X be a G-variety.

Definition. X is called *spherical* if it contains an open B-orbit, where B is a Borel subgroup of G.

Definition. A closed subgroup $H \subset G$ is called spherical if the homogeneous variety G/H is spherical.

Exercise. Show that G/H is spherical if and only if there exists a Borel subgroup B such that the set BH is open in G if and only if $\mathfrak{g} = \mathfrak{b} + \mathfrak{h}$ for some Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$.

Recall that the Tits morphism for a homogeneous space $X = G/H$ is given by

$$\begin{array}{rccc} \tau : & X & \longrightarrow & \mathcal{L} \\ & x & \longmapsto & \mathfrak{g}_x \end{array}$$

where $\mathcal{L}$ is the variety of all Lie subalgebras of $\mathfrak{g}$ (one may also consider those of fixed dimension $\dim \mathfrak{h}$ as was done before). The map is G-equivariant, and its image is isomorphic to $G/N_G(\mathfrak{h})$. Thus τ defines a homogeneous fibration $\tau : G/H \longrightarrow G/N_G(\mathfrak{h})$.

Examples: 1) Every complete log homogeneous variety under a linear algebraic group G is spherical under a Levi subgroup G_{red} (see Corollary 4.3).

2) *Flag varieties*: Every homogeneous space $X = G/P$, where P is a parabolic subgroup, is spherical. This follows from the properties of the Bruhat decomposition. Since parabolic subgroups are self-normalizing, i.e. $P = N_G(\mathfrak{p})$, the Tits morphism is an isomorphism onto its image.

3) All *toric varieties* are spherical. Here $G = (\mathbb{C}^*)^n = B$ is its own Borel subgroup. The Tits morphism here is constant.

4) Let $U \subset G$ be a maximal unipotent subgroup. Let $\mathfrak{n} \subset \mathfrak{g}$ be the corresponding Lie subalgebra. Then we have the decomposition $\mathfrak{g} = \mathfrak{b}^- \oplus \mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Thus the variety G/U is spherical. It can be written as a homogeneous fibre bundle in the following form

$$G/U = G \overset{B}{\times} B/U \ .$$

The fibre B/U is isomorphic to a maximal torus T in G (we have $B = TU$). The base space is the flag variety G/B.

Now, let Y be a complete smooth toric variety under the torus T. Then G/U is embedded in $G \overset{B}{\times} Y$ which is smooth, complete and log homogeneous (see Criterium 1, paragraph 3.1). Thus $G \overset{B}{\times} Y$ is spherical. The Tits morphism is the projection map $G \overset{B}{\times} Y \longrightarrow G/B$ (recall that $N_G(\mathfrak{n}) = B$).

Remark. There are some other equivariant completions of G/U which are not log homogeneous. For example, if

$$G = SL_2 \ , \ U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$

then we have the embeddings $SL_2/U \hookrightarrow \mathbb{C}^2 = (SL_2/U) \cup \{0\} \hookrightarrow \mathbb{P}^2 = (SL_2/U) \cup \{0\} \cup \mathbb{P}^1$ (the first embedding is $gU \mapsto g \cdot e_1$), and we see that $\{0\}$ is an isolated orbit of codimension 2.

5) *Horospherical varieties*: Suppose we have a subgroup H satisfying $U \subset H \subset G$ for a maximal unipotent subgroup U. It is an exercise to show that the normalizer $P := N_G(H)$ is parabolic, $[P, P] \subset H$, and P/H is a torus. (For example, if $H = U$ then P is a Borel subgroup of G; moreover, $[P, P] = U$, and P/H is isomorphic to a maximal torus of G). Now P/H can be equivariantly embedded in a complete, smooth toric variety Y and then $X = G \overset{P}{\times} Y$ is an equivariant completion of G/H with Tits morphism $\tau \colon X \to G/P$.

6) *Reductive groups*: Let $X = G$, where $G \times G$ acts by $(x, y) \cdot z = xzy^{-1}$. Then $(G \times G)_e = \mathrm{diag}(G)$. Note that X is spherical, since $B^- \times B$ is a Borel subgroup of $G \times G$ whenever B, B^- are opposite Borel subgroups of G, and then $(B^- \times B) \cdot e = B^- B$ is open in G (since $\mathfrak{b}^- + \mathfrak{b} = \mathfrak{g}$).

Let G be semisimple and adjoint (i. e. $Z(G) = \{e\}$). Consider the representation $G \to \mathrm{GL}(V_\lambda)$ where V_λ is a simple G-module of highest weight λ. This defines a map $G \to \mathrm{PGL}(V_\lambda)$ that is injective for regular (dominant) λ. Let $\overline{G}$ be the closure of G in $\mathbb{P}(\mathrm{End}(V_\lambda))$. Then we have the following theorem, a reformulation in the setting of log homogeneous varieties of a result due to De Concini and Procesi. (see [12]).

Theorem 5.1. $\overline{G}$ is a smooth log homogeneous $G \times G$-variety with a unique closed orbit, and is independent of the choice of λ.

Recall that $\mathrm{End}(V_\lambda) \cong V_\lambda^* \otimes V_\lambda$ as a $G \times G$-module. Let $f_{-\lambda} \otimes v_\lambda \in V_\lambda^* \otimes V_\lambda$ be an eigenvector of $B^- \times B$ formed as the tensor product of a highest weight vector $v_\lambda \in V_\lambda$ and a corresponding functional $f_{-\lambda} \in V_\lambda^*$. Then $f_{-\lambda} \otimes v_\lambda$ is an eigenvector for $B^- \times B$, and any such eigenvector is a scalar multiple of $f_{-\lambda} \otimes v_\lambda$. Therefore, the orbit

$$(G \times G) \cdot [f_{-\lambda} \otimes v_\lambda] \subset \mathbb{P}(\mathrm{End}(V_\lambda))$$

is the unique closed orbit in $\mathbb{P}(\mathrm{End}(V_\lambda))$, and hence in $\overline{G}$.

The main step in the proof of the remaining assertions is to obtain a precise version of the local structure theorem 4.1 for $\overline{G}$, with $z := [f_{-\lambda} \otimes v_\lambda]$, $P := B \times B^-$, and $L := T \times T$. Specifically, there exists a $T \times T$-equivariant morphism

$$\varphi : \mathbb{C}^r \longrightarrow \overline{G}, \quad (0, \ldots, 0) \longmapsto z,$$

where $T \times T$ acts on $\mathbb{C}^r$ via

$$(t_1, t_2) \cdot (x_1, \ldots, x_r) := (\alpha_1(t_1 t_2^{-1}) x_1, \ldots, \alpha_r(t_1 t_2^{-1}) x_r)$$

($\alpha_1, \ldots, \alpha_r$ being the simple roots), such that the morphism

$$\psi : U \times U^- \times \mathbb{C}^r \longrightarrow \overline{G}, \quad (g, h, x) \longmapsto (g, h) \cdot \varphi(x)$$

is an open immersion; in particular, φ is an isomorphism over its image, the subvariety Y of Theorem 4.1. Since the image of ψ meets the unique closed orbit, its translates by G form an open cover of $\overline{G}$; this implies e.g. the smoothness of $\overline{G}$.

The regularity assumption for λ cannot be omitted, as shown by the following

Example: Let $G = \mathrm{PGL}_n \subset \mathbb{P}(\mathrm{Mat}_n) = X$ (so that λ is the first fundamental weight). Then $D = X \setminus G = (\det = 0)$. This is an irreducible divisor, singular along matrices of rank $\leq n-2$. Thus, (X, D) is not log homogeneous for $n \geq 3$.

We now continue with our list of examples of spherical varieties:

7) *Symmetric spaces*: Let G be a connected reductive group and let θ be an involutive automorphism of G. Let G^θ be the subgroup of elements fixed by θ. This is a reductive subgroup, and the homogeneous space G/G^θ is affine; it is called a *symmetric space* (see [23], that we will use as a general reference for symmetric spaces).

The involution θ of G yields an involution of G/G^θ that fixes the base point; one can show that this point is isolated in the fixed locus of θ. Since G/G^θ is homogeneous, it follows that each of its points is an isolated fixed point of an involutive automorphism; this is the original definition of a symmetric space, due to E. Cartan.

A symmetric space is spherical, by the *Iwasawa decomposition* that we now recall. A parabolic subgroup $P \subset G$ is called θ*-split* if P and $\theta(P)$ are opposite. Let P be a minimal θ-split parabolic subgroup. Then $L := P \cap \theta(P)$ is a θ-stable Levi subgroup of P. In fact, the derived subgroup $[L, L]$ is contained in G^θ; as a consequence, every maximal torus $T \subset L$ is θ-stable. Thus, $T = T^\theta A$, where $A := \{t \in T | \theta(t) = t^{-1}\}$, and $T^\theta \cap A$ is finite. In fact, A is a maximal θ*-split* subtorus, i.e., a θ-stable subtorus where θ acts via the inverse map.

The Iwasawa decomposition asserts that the natural map

$$R_u(P) \times A/A^\theta \longrightarrow G/G^\theta$$

is an open immersion. Since $R_u(P)A$ is contained in a Borel subgroup of G, we see that the symmetric space G/G^θ is spherical. Another consequence is the decomposition of Lie algebras

$$\mathfrak{n}(\mathfrak{p}) \oplus \mathfrak{a} \oplus \mathfrak{g}^\theta = \mathfrak{g},$$

where $\mathfrak{n}$ denotes the nilradical (see [25, Prop. 38.2.7]).

(For instance, consider the group $G \times G$ and the automorphism θ such that $\theta(x, y) = (y, x)$. Then $(G \times G)^\theta = \mathrm{diag}(G)$.)

Next, consider a G-module V_λ containing non-zero G^θ-fixed points. Let v be such a fixed point; then we have a G-equivariant map

$$G/G^\theta \longrightarrow V_\lambda, \quad gG^\theta \longmapsto g \cdot v.$$

One can show that $\dim V_\lambda^{G^\theta}$ is either 1 or 0 (see Proposition 5.1). If it is 1, the weight λ is called *spherical*. Spherical weights form a finitely-generated submonoid of the monoid of dominant weights.

Theorem 5.2. Let G be a semisimple adjoint group, θ an involution, and λ a regular spherical weight. Then the map $G/G^\theta \to \mathbb{P}(V_\lambda)$ is injective and the closure of its image is a smooth, log homogeneous G-variety, independent of λ and containing a unique closed orbit $G \cdot [v_\lambda] \cong G/\theta(P)$.

This generalization of Theorem 5.1 is again a reformulation in the setting of log homogeneous varieties of a result due to De Concini and Procesi; they have also shown that the Tits morphism

$$X := \overline{G \cdot [v_\lambda]} \longrightarrow \mathcal{L}$$

is an isomorphism over its image. This yields an alternative construction of X as the closure of $G \cdot \mathfrak{g}^\theta$ in the variety of Lie subalgebras.

Proposition 5.1. Let G be a connected reductive group, and $H \subset G$ a closed subgroup. Then H is spherical if and only if for any dominant weight λ and any character $\chi \in \mathrm{Hom}(H, \mathbb{C}^*)$ we have

$$\dim(V_\lambda)_\chi^{(H)} \leq 1 ,$$

where $(V_\lambda)_\chi^{(H)}$ denotes the subspace of all H-eigenvectors of weight χ.

Moreover, if H is reductive and $\dim V_\lambda^H \leq 1$ for any λ, then H is spherical.

Proof. It is known that the $G \times G$-module $\mathbb{C}[G]$ can be decomposed as follows (see e.g. [25, Thm. 27.3.9])

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \text{ dominant weight}} V_\lambda^* \otimes V_\lambda.$$

The embeddings of the direct summands are given by

$$f \otimes v \longmapsto a_{f,v} = (g \mapsto f(gv)).$$

Let H be spherical and consider $v_1, v_2 \in (V_\lambda)_\chi^{(H)}$. Let B be a Borel subgroup such that BH is open in G, and choose $f \in (V_\lambda^*)^{(B)}$. Then

$$\frac{a_{f,v_2}}{a_{f,v_1}} \in \mathbb{C}(G)$$

is an invariant for the right H-action. It is also an invariant for the left B-action. Thus,

$$\frac{a_{f,v_2}}{a_{f,v_1}} \in \mathbb{C}(G)^{B \times H} = \mathbb{C}^*,$$

since $B \times H$ has an open orbit in G. Hence, there exists $t \in \mathbb{C}^*$ such that $a_{f,v_2} = t a_{f,v_1}$. Now,

$$0 = f(gv_2) - tf(gv_1) = f(gv_2 - tgv_1) = f(g(v_2 - tv_1)).$$

But V_λ is irreducible and $f \neq 0$. Hence $v_2 = tv_1$. This shows the "only if" part of the first assertion.

We now show the second assertion. Let H be reductive and such that $\dim V_\lambda^H \leq 1$ for all dominant λ. By a theorem of Rosenlicht, to show that G/H contains an open B-orbit, it suffices to show that every rational B-invariant function on G/H is constant, i.e., $\mathbb{C}(G/H)^B = \mathbb{C}^*$. Since G/H is affine, $\mathbb{C}(G/H)$ is the fraction field of $\mathbb{C}[G/H]$. Let $f \in \mathbb{C}(G/H)^B$. Then the set of all "denominators" $D \in \mathbb{C}[G/H]$ such that $fD \in \mathbb{C}[G/H]$, is a non-zero B-stable subspace of $\mathbb{C}[G/H]$. Hence this subspace contains an eigenvector of B, i.e., we may write $f = f_1/f_2$, where $f_1, f_2 \in \mathbb{C}[G/H]_\mu^{(B)} = \mathbb{C}[G]_\mu^{(B)\times H}$. Using the above decomposition of the $G \times G$-module $\mathbb{C}[G]$, it follows that

$$f_i = a_{\phi, v_i} \quad (i = 1, 2)$$

where $\phi \in (V_\lambda^*)^{(B)}$, $v_1, v_2 \in V_\lambda^H$, and $V_\lambda = V_\mu^*$. Thus, $v_2 = tv_1$, and $f = t$.

The proof in the non-reductive case relies on the same ideas; the details will not be given here. □

Proposition 5.2. Let $H \subset G$ be a spherical subgroup, and $N_G(H)$ its normalizer. Then $N_G(H)/H$ is *diagonalizable* (i. e., it is isomorphic to a subgroup of some $(\mathbb{C}^*)^N$). Moreover, $N_G(H) = N_G(\mathfrak{h}) = N_G(H^\circ)$.

Proof. For any homogeneous space G/H, the quotient $N_G(H)/H$ acts on G/H on the right as follows:

$$\gamma \cdot gH = g\gamma^{-1}H = gH\gamma^{-1}.$$

This yields an isomorphism

$$N_G(H)/H = \mathrm{Aut}^G(G/H).$$

Also, note that $N_G(H) \subset N_G(H^\circ) = N_G(\mathfrak{h})$.

We now prove the first assertion in the case that H is reductive. Then the natural action of $N_G(H)/H$ on $\mathbb{C}[G/H]$ is faithful, since $\mathbb{C}(G/H)$ is the fraction field of $\mathbb{C}[G/H]$. But we have a decomposition

$$\mathbb{C}[G/H] \cong \bigoplus_\lambda V_\lambda^* \otimes V_\lambda^H$$

as $G \times N_G(H)/H$-modules, in view of the decomposition of $\mathbb{C}[G]$ as $G \times G$-modules. Moreover, each non-zero V_λ^H is a line, by Proposition 5.1. Thus, $N_G(H)/H$ acts on V_λ^H via a character χ_λ, and this yields the desired embedding $N_G(H)/H \hookrightarrow (\mathbb{C}^*)^N$.

The argument in the case of a non-reductive subgroup H follows similar lines, by replacing invariants of H with eigenvectors.

It remains to show that $N_G(H) \supset N_G(H^\circ)$. For this, observe that H° is spherical. Hence the group $N_G(H^\circ)/H^\circ$ is diagonalizable; in particular, commutative. So $N_G(H^\circ)/H^\circ$ normalizes H/H°, i.e., $N_G(H^\circ)$ normalizes H. □

We state without proof the following important result, with contributions by several mathematicians (among which Demazure, De Concini, Procesi, Knop, Luna) and the final step by Losev (see [16]).

Theorem 5.3. Let G/H be a spherical homogeneous space. Then

(1) G/H admits a log homogeneous equivariant completion.
(2) If $H = N_G(H)$, then $\overline{G \cdot \mathfrak{h}} \subset \mathcal{L}$ is a log homogeneous equivariant completion with a unique closed orbit.

Definition. A *wonderful variety* is a complete log homogeneous G-variety X with a unique closed orbit.

The G-orbit structure of wonderful varieties is especially simple: the boundary divisor has the form $D = D_1 \cup \ldots \cup D_r$, with D_i irreducible and smooth. The closed orbit is $D_1 \cap \ldots \cap D_r$, and the orbit closures are precisely the partial intersections $D_{i_1} \cap \ldots \cap D_{i_s}$, where $1 \leq i_1 < \cdots < i_s \leq r$. In particular, r is the codimension of the closed orbit, also known as the *rank* of X.

For a wonderful variety X, the Tits morphism $\tau \colon X \to \mathcal{L}$ is finite. In particular, the identity component of the center of G acts trivially on X, and hence we may assume that G is semisimple.

Let us discuss some recent results and work in progress on the classification of wonderful varieties.

Theorem 5.4. There exist only finitely many wonderful G-varieties for a given semisimple group G.

This finiteness result, a consequence of [2, Cor. 3.2], is obtained via algebro-geometric methods (invariant Hilbert schemes) which are non-effective in nature. On the other hand, a classification program developed by Luna has been completed for many types of semi-simple groups: in type A by Luna himself (see [15]), D by Bravi and Pezzini (see [3]), E by Bravi (see [4]) and F by Bravi and Luna (see [6]).

There is a geometric approach to Luna's program, initiated by Bravi and Cupit-Foutou (see [5]) via invariant Hilbert schemes, and currently developed by Cupit-Foutou (see [10, 11]). The starting point is the following geometric realization of wonderful varieties: let X be such a variety, with open orbit G/H, and let $v \in (V_\lambda)_\chi^{(H)}$, where λ and χ are regular. Then X is the normalization of the orbit closure $\overline{G \cdot [v]} \subset \mathbb{P}(V_\lambda)$.

This orbit closure may be non-normal, as shown by the example of $\mathbb{P}^1 \times \mathbb{P}^1$ viewed as the wonderful completion of SL_2/T. If $V = V_n = \mathbb{C}[x,y]_n$ and $v = x^p y^q, p \neq q$, then $\overline{SL_2 \cdot [v]} \subset \mathbb{P}(V)$ is singular, but its normalization is $\mathbb{P}^1 \times \mathbb{P}^1$. (Here SL_2 acts on $\mathbb{C}[x,y]_n$ in the usual way.)

Finally, the structure of general complete log homogeneous varieties reduces to those of wonderful and of toric varieties, in the following sense. Let X be a log homogeneous equivariant completion of a spherical homogeneous space G/H. Let $\underline{X}$ be the wonderful completion of $G/N_G(H)$. Then the natural map $G/H \to G/N_G(H)$ extends (uniquely) to an equivariant surjective map $\tau : X \to \underline{X}$. Moreover, the general fibers of τ are finite disjoint unions of complete, smooth toric varieties. (Indeed, τ is just the Tits morphism, and its general fibers are closures of $N_G(H)/H$, a finite disjoint union of tori). We refer to [7, Sec. 3.3] for further results on that reduction.

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