

On models of algebraic group actions

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Abstract

We show that every action of a smooth algebraic group on a variety admits a normal projective model. Along the way, we present new proofs of some basic results on algebraic transformation groups, including Weil's regularization theorem.

1 Introduction

The main motivation for this paper comes from Weil's regularization theorem (see [We55]): for any rational action of a smooth connected algebraic group G on a smooth variety X , there exists a G -action on a smooth variety Y which is G -equivariantly birationally isomorphic to X . This was extended to non-connected groups by Rosenlicht (see [Ro56, Thm. 1]), with modern proofs over an algebraically closed field by Zaitsev and Kraft (see [Za95, Kr18]).

One would like to have a model Y satisfying additional geometric properties. When working over an arbitrary field k as in [We55, Ro56], there may exist no smooth projective model (see Remark 7 for details). Is there a regular projective one? This question has an affirmative answer when X is a curve: then G acts on its regular projective model, see e.g. [De70, Prop. 5]. For surfaces, an affirmative answer is known in various degrees of generality, as a key ingredient in the classification of algebraic subgroups of groups of birational transformations (see [Bl09, §2], [Fo21, Prop. 2.4], [SZ21, Prop. 2.3]).

In this paper, we show that the model Y can be taken projective and normal. If k has characteristic 0, it follows that Y can be taken projective and smooth, by using equivariant resolution of singularities (see [Kol07, Prop. 3.9.1, Thm. 3.36]). Similarly, if X is a surface over an arbitrary field k , then Y can be taken projective and regular, by iterating the processes of blowing up the singular locus and normalizing (see [Lip78, Rem. B, p. 155]). In higher dimensions and positive characteristics, the existence of an equivariant desingularization is an open problem.

When G is connected, the existence of a normal projective model is deduced in [Br17a, Cor. 3] from two general results:

- (1) Every normal G -variety admits a covering by G -stable quasi-projective open subsets.
- (2) Every normal quasi-projective G -variety admits an equivariant completion by a normal projective G -variety.

Note that (1) fails for non-connected groups, already for the group with two elements in view of an example of Hironaka (see [Hir62] and [MFK94, Chap. IV, §3]). But it extends in a weaker form:

Theorem 1. *Let G be a smooth algebraic group, and X a G -variety. Then X contains a G -stable dense open subset which is regular and quasi-projective.*

Also, (2) extends without change:

Theorem 2. *Let G be a smooth algebraic group, and X a normal quasi-projective G -variety. Then X admits an equivariant completion by a normal projective G -variety.*

If in addition G is affine, then every normal G -variety (not necessarily quasi-projective) admits an equivariant completion by a normal proper G -variety, in view of a result of Sumihiro (see [Su74, Thm. 3], [Su75, Thm. 4.13]). We do not know whether this still holds for an arbitrary smooth algebraic group G .

This paper is organized as follows. Section 2 begins with a new proof of (1) for a connected group G ; then we derive Theorem 1 as an easy consequence. Section 3 contains auxiliary results on torsors. In Section 4, we first recall the classical construction of the associated fiber bundle, also known as contracted product; in loose terms, it constructs a G -variety from an H -variety, where H is a subgroup of G . Then we obtain an existence result for such bundles (Theorem 3) which is not used in the proofs of the main theorems, but which should have independent interest. Section 5 begins with another construction of a G -variety from an H -variety, which is somehow dual to the associated fiber bundle construction, and based on Weil restriction. This yields the main ingredient for the proof of Theorem 2. In Section 6, we present a new proof of Weil's regularization theorem, which is shorter but less self-contained than the original one. Actions of non-smooth groups are considered in Section 7; for these, we show that projective models still exist, but normal projective models do not. We also extend Weil's regularization theorem to this setting (Theorem 9).

Notation and conventions. We consider schemes over a field k with algebraic closure \bar{k} , separable closure k_s and absolute Galois group $\mathcal{G} = \text{Gal}(k_s/k)$. Morphisms and products of schemes are understood to be over k as well. Schemes are assumed to be separated and of finite type unless explicitly mentioned. A *variety* X is an integral scheme. The function field of X is denoted by $k(X)$.

An *algebraic group* G is a group scheme; we denote by $e \in G(k)$ its neutral element. By a *subgroup* H of G , we mean a subgroup scheme; then H is closed in G . A G -*scheme* is a scheme X equipped with a G -action

$$a : G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x.$$

A morphism of G -schemes $f : X \rightarrow Y$ is *equivariant* if $f(g \cdot x) = g \cdot f(x)$ identically on $G \times X$.

Given a field extension K/k and a k -scheme X , we denote by X_K the K -scheme $X \times_{\text{Spec}(k)} \text{Spec}(K)$.

2 Proof of Theorem 1

PROOF OF (1). Recall the statement: *let G be a smooth connected algebraic group, and X a normal G -variety. Then X admits a covering by G -stable quasi-projective open subsets.*

This follows from a result of Benoist asserting that every normal variety contains only finitely many maximal quasi-projective open subsets (see [Be13, Thm. 9]). An alternative proof, using methods from Raynaud's work on the quasi-projectivity of homogeneous spaces (see [Ra70]), is presented in [Br17a, Sec. 3]. We will recall the very beginning of this proof, and then use a more direct argument adapted from the proof of the quasi-projectivity of algebraic groups in [SP22, Lem. 39.8.7]. This bypasses the most technical developments of [Br17a].

Let U be an open subset of X . Since the action $a : G \times X \rightarrow X$ is a smooth morphism (see e.g. [Mi17, Lem. 1.68]), $G \cdot U = a(G \times U)$ is open in X ; also, $G \cdot U$ contains U and is clearly G -stable. Thus, it suffices to show that $G \cdot U$ is quasi-projective for any affine U .

For this, we may assume that $X = G \cdot U$. Since U is affine, its complement is the support of an effective Weil divisor D on X . We now show that D is a Cartier divisor. The product $G \times X$ is a normal variety, since G is a smooth, geometrically integral variety and X is a normal variety (use [EGA, IV.6.8.5]). Also, a is smooth and the Weil divisor $G \times D = \text{pr}_2^*(D)$ on $G \times X$ contains no fiber of a in its support, since $X = G \cdot U$. By the Ramanujam-Samuel theorem (see [EGA, IV.21.14.3]), it

follows that $G \times D$ is a Cartier divisor. Thus, $D = (e, \text{id}_X)^*(G \times D)$ is a Cartier divisor as well.

It is shown in [Br17a, Prop. 3.3] that D is ample, by using a version of the theorem of the square. We will rather construct from D an ample divisor E on X , via a norm argument taken from [SP22, Tag 0BF7].

Since G is smooth, there exists a dense open subset V of G and an étale morphism $f : V \rightarrow \mathbb{A}^n$, where $n = \dim(G)$ (see [EGA, IV.17.11.4]). Replacing V with a smaller open subset, we may assume that f is finite over its image W , a dense open subset of \mathbb{A}^n . In particular, $f : V \rightarrow W$ is finite and locally free of constant degree d .

Denote by $\varphi : V \times X \rightarrow X$ the pull-back of a . Then $\mathcal{O}_{V \times X}(\varphi^*(D))$ is an invertible sheaf on $V \times X$, equipped with a section with divisor of zeroes $\varphi^*(D)$. Thus, the norm of $\mathcal{O}_{V \times X}(\varphi^*(D))$ with respect to the finite locally free morphism $f \times \text{id}_X : V \times X \rightarrow W \times X$ is an invertible sheaf \mathcal{M} on $W \times X$, equipped with a section whose divisor of zeroes E satisfies $\text{Supp}(E) = (f \times \text{id}_X)(\text{Supp}(\varphi^*(D)))$ (see [EGA, II.6.5] for the construction and properties of the norm).

Since W is open in \mathbb{A}^n , there exists an invertible sheaf \mathcal{L} on X such that $\mathcal{M} = \text{pr}_X^*(\mathcal{L})$; moreover, $\mathcal{L} \simeq \mathcal{M}_w$ for any $w \in W(k)$, where \mathcal{M}_w denotes the pull-back of \mathcal{M} under the morphism $(w, \text{id}_X) : X \rightarrow W \times X$, $x \mapsto (w, x)$. We claim that \mathcal{L} is ample.

To prove this claim, we may assume that k is algebraically closed, since \mathcal{L} is ample if and only if $\mathcal{L}_{\bar{k}}$ is ample on $X_{\bar{k}}$ (see [SGA1, VIII.5.8]). For any $w \in W(k)$, we denote by E_w the pull-back of E under (w, id_X) . Since f is finite and étale of degree d , we have $f^{-1}(w) = \{g_1, \dots, g_d\}$ where $g_i = g_i(w) \in V(k)$ for $i = 1, \dots, d$. Then $\text{Supp}(E_w) = \bigcup_{i=1}^d g_i^{-1} \text{Supp}(D)$ and hence $X \setminus \text{Supp}(E_w) = \bigcap_{i=1}^d g_i^{-1} U$ is affine. Also, note that $\text{Supp}(E_w)$ is the zero set of a section of $\mathcal{M}_w \simeq \mathcal{L}$. In view of [EGA, II.4.5.2], it suffices to show that X is the union of the $X \setminus \text{Supp}(E_w)$, where $w \in W(k)$. Since X is of finite type, it suffices in turn to check the equality $X(k) = \bigcup_{w \in W(k)} X(k) \setminus \text{Supp}(E_w)$.

Let $x \in X(k)$ and denote by V_x the set of $g \in V$ such that $x \in g^{-1}U$. Then V_x is a dense open subset of V as $X = G \cdot U$. Thus, there exists $w \in W(k)$ such that $f^{-1}(w)$ is contained in V_x . Then $x \in X \setminus \text{Supp}(E_w)$ as desired. \square

PROOF OF THEOREM 1. The regular locus X_{reg} is a dense open subset of X . We show that X_{reg} is G -stable. Since G is smooth, $G \times X_{\text{reg}}$ is regular by [EGA, IV.6.8.5]. As the action morphism $a : G \times X \rightarrow X$ is flat, it sends $G \times X_{\text{reg}}$ to X_{reg} in view of [EGA, IV.6.5.2].

Thus, we may assume that X is regular. In view of (1), there exists a dense open quasi-projective subset $U \subset X$, stable by the neutral component G^0 . Thus, U_{k_s} is a

dense open quasi-projective subset of X_{k_s} , stable by $G_{k_s}^0$. Moreover, X_{k_s} is regular by [EGA, IV.6.7.6, IV.6.8.5]. Also, there are only finitely many translates $g \cdot U$, where $g \in G(k_s)$ (since $G(k_s)/G^0(k_s)$ is finite). So $V = \bigcap_{g \in G(k_s)} g \cdot U_{k_s}$ is a dense open subset of U_{k_s} , stable by $G(k_s)$. As G is smooth, $G(k_s)$ is dense in G_{k_s} and hence V is stable by G_{k_s} ; it is also stable by the Galois group \mathcal{G} , and quasi-projective.

The assertion follows from this by a standard argument of Galois descent. More specifically, since X is of finite type, there exists a finite Galois extension K/k and an open subset W of X_K such that $V = W_{k_s}$. Then W is quasi-projective (see e.g. [GW10, Prop. 14.55]). Also, W is stable by G_K and by the Galois group \mathcal{G}_K of K/k . So the quotient $W/\mathcal{G}_K \subset X_K/\mathcal{G}_K = X$ satisfies the assertion. \square

3 Torsors

Let G be an algebraic group. A G -torsor is a faithfully flat morphism of schemes $f : X \rightarrow Y$, where X is equipped with a G -action a such that f is G -invariant and $G \times X \xrightarrow{\sim} X \times_Y X$ via (a, pr_2) . This isomorphism identifies the two projections $p_1, p_2 : X \times_Y X \rightarrow X$ with $a, \text{pr}_2 : G \times X \rightarrow X$. As a consequence, for any G -scheme Z , we have an isomorphism

$$f^* : \text{Hom}(Y, Z) \xrightarrow{\sim} \text{Hom}^G(X, Z), \quad \varphi \mapsto \varphi \circ f,$$

where the right-hand side denotes the set of G -invariant morphisms.

We now record three auxiliary results:

Lemma 1. *Let G be an algebraic group, and $f : X \rightarrow Y$ a G -torsor, where X is a normal variety. Then Y is a normal variety.*

Proof. Since X is normal and f is faithfully flat, Y is normal by [EGA, IV.6.5.4]. Also, Y is irreducible, as X is irreducible and f is surjective. \square

Lemma 2. *Let $1 \rightarrow N \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$ be an exact sequence of algebraic groups. Let X be a G -scheme, and $f : X \rightarrow Y$ an N -torsor.*

(i) *There is a unique action of Q on Y such that the diagram*

$$\begin{array}{ccc} G \times X & \xrightarrow{a} & X \\ \pi \times f \downarrow & & \downarrow f \\ Q \times Y & \xrightarrow{b} & Y \end{array}$$

commutes.

(ii) For any Q -scheme Z viewed as a G -scheme via π , the composition with f yields an isomorphism $\mathrm{Hom}^Q(Y, Z) \xrightarrow{\sim} \mathrm{Hom}^G(X, Z)$.

Proof. (i) As N is a normal subgroup of G , and f is N -invariant, the composition

$$G \times X \xrightarrow{a} X \xrightarrow{f} Y$$

is invariant under the action of N on $G \times X$ via $n \cdot (g, x) = (g, n \cdot x)$. Since the morphism $\mathrm{id}_G \times f : G \times X \rightarrow G \times Y$ is an N -torsor, it follows that there exists a unique morphism $a' : G \times Y \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{a} & X \\ \mathrm{id}_G \times f \downarrow & & \downarrow f \\ G \times Y & \xrightarrow{a'} & Y \end{array}$$

commutes. So this diagram is cartesian (e.g., since its vertical arrows are N -torsors). Using the uniqueness property for descent of morphisms, one may check that a' is a G -action. Moreover, N acts trivially on Y via a , and hence via a' by uniqueness again. Thus, a' is invariant under the N -action on $G \times Y$ via $n \cdot (g, y) = (gn^{-1}, y)$. Since $\pi \times \mathrm{id}_Y : G \times Y \rightarrow Q \times Y$ is an N -torsor, it follows that a' factors uniquely through a morphism $b : Q \times Y \rightarrow Y$. One may then check as above that b is a Q -action.

(ii) This follows from the isomorphism

$$f^* : \mathrm{Hom}(Y, Z) \xrightarrow{\sim} \mathrm{Hom}^N(X, Z)$$

by taking invariants of G , or equivalently of Q . □

We also record a variant of [Br17b, Prop. 4.2.4]:

Lemma 3. *Let $f : X \rightarrow Y$ be a torsor under an abelian variety A , where X is smooth. Then there exists an A -equivariant morphism $\psi : X \rightarrow A/A[n]$ for some positive integer n , where $A[n]$ denotes the schematic kernel of the multiplication by n in A .*

Proof. We first reduce to the case where X is a variety. The irreducible components X_1, \dots, X_r of X are disjoint and stable by A . If there exist A -equivariant morphisms $\psi_i : X_i \rightarrow A/A[n_i]$ for $i = 1, \dots, r$, then we obtain the desired morphism ψ by setting $n = \prod_{i=1}^r n_i$ and $\psi|_{X_i} = \frac{n}{n_i} \psi_i$ for all i .

Therefore, we may assume that X is a variety; then so is Y . The base change $X \times_Y \operatorname{Spec} k(Y)$ is an $A_{k(Y)}$ -torsor over $k(Y)$. By [Mi17, Lem. 8.22], there exists an $A_{k(Y)}$ -equivariant morphism

$$\varphi : X \times_Y \operatorname{Spec} k(Y) \longrightarrow A_{k(Y)}/A_{k(Y)}[n] = (A/A[n])_{k(Y)}$$

for some positive integer n . Composing φ with the natural morphisms

$$\operatorname{Spec} k(X) \longrightarrow X \times_Y \operatorname{Spec} k(Y), \quad (A/A[n])_{k(Y)} \longrightarrow A/A[n]$$

yields a rational A -equivariant map $\psi : X \dashrightarrow A/A[n]$. Since X is smooth, ψ is a morphism by Weil's extension theorem (see [BLR90, Thm. 4.4/1]). \square

4 Associated fiber bundles

Consider an algebraic group G , a subgroup H , and the corresponding homogeneous space G/H with quotient morphism $q : G \rightarrow G/H$ and base point $o = q(e)$. Then q is an H -torsor (see e.g. [Mi17, 5.c]). Given an H -scheme Y , the *associated fiber bundle* is a scheme Z which fits in a cartesian square

$$\begin{array}{ccc} G \times Y & \xrightarrow{\operatorname{pr}_1} & G \\ \varphi \downarrow & & \downarrow q \\ Z & \xrightarrow{\psi} & G/H, \end{array}$$

where φ is invariant under the action of H on $G \times Y$ via

$$h \cdot (g, y) = (gh^{-1}, h \cdot y).$$

If such a scheme Z exists, then φ is an H -torsor (the pull-back of the H -torsor q under ψ) and ψ is faithfully flat (since so are φ , q and pr_1). Also, G acts on $G \times Y$ via left multiplication on itself; this action commutes with the H -action, and pr_1 is equivariant. By Lemma 2, this yields a G -action on Z such that ψ is equivariant. In particular, the fiber Z_o is an H -scheme, equivariantly isomorphic to Y . We denote Z by $G \times^H Y$.

Conversely, if Z is a G -scheme and $\psi : Z \rightarrow G/H$ a G -equivariant morphism such that $Z_o \simeq Y$ as H -schemes, then one may easily check that $Z \simeq G \times^H Y$ (see [Br17a, §2.5]).

The associated fiber bundle need not exist, as shown again by Hironaka's example (see [BB93] for details). But for any G -scheme Y , the associated fiber bundle $G \times^H Y$

exists and is isomorphic to $G/H \times Y$, where G acts diagonally; this identifies ψ with the first projection.

Another instance where $G \times^H Y$ exists is when Y is H -quasi-projective, i.e., it admits an ample H -linearized line bundle; then $G \times^H Y$ is quasi-projective (this follows from [MFK94, Prop. 7.1] or from [BLR90, 10.2/6]).

We now obtain a further existence result:

Theorem 3. *Let G be a smooth algebraic group, H a subgroup, and Y a normal quasi-projective H -variety. Then the associated fiber bundle $G \times^H Y$ exists and is a normal quasi-projective G -variety.*

Proof. By [Br17b, Thm. 2], there is a smallest normal subgroup N of H such that the quotient $Q = H/N$ is proper; moreover, N is affine and connected. Since Y is normal and quasi-projective, it is N -quasi-projective in view of [Br17a, Lem. 2.9]. Thus, $G \times^N Y$ exists and is quasi-projective. Also, $G \times Y$ is normal (as G is smooth and Y is normal), and hence $G \times^N Y$ is normal by Lemma 1.

The group $G \times H$ acts on $G \times Y$ via $(g, h) \cdot (g', y) = (gg'h^{-1}, h \cdot y)$. In view of Lemma 2, this yields an action of Q on $G \times^N Y$ which commutes with the G -action and such that the canonical morphism

$$\psi : G \times^N Y \longrightarrow G/N$$

is equivariant.

We have a unique exact sequence of algebraic groups

$$1 \longrightarrow A \longrightarrow Q \longrightarrow F \longrightarrow 1,$$

where A is an abelian variety and F is finite (see [Br17b, Lem. 3.3.7]). Denote by I the pull-back of A in H . Then we obtain two exact sequences

$$1 \longrightarrow N \longrightarrow I \longrightarrow A \longrightarrow 1, \quad 1 \longrightarrow I \longrightarrow H \longrightarrow F \longrightarrow 1.$$

We now show that the associated fiber bundle $G \times^I Y$ exists and is quasi-projective. For this, we consider the natural morphism $G/N \rightarrow G/I$ which is an A -torsor (see e.g. [Br17b, Prop. 2.8.4]); moreover, G/N is smooth. By Lemma 3, it follows that there exists an A -equivariant morphism $\varphi : G/N \rightarrow A/A[n]$ for some positive integer n . Composing φ with the A -equivariant morphism ψ yields an A -equivariant morphism $G \times^N Y \rightarrow A/A[n]$, and hence an A -equivariant isomorphism

$$G \times^N Y \simeq A \times^{A[n]} Z,$$

where Z is a closed $A[n]$ -stable subscheme of $G \times^N Y$. In particular, Z is quasi-projective and its $A[n]$ -action is free.

As $A[n]$ is finite, it follows that there is an $A[n]$ -torsor $f : Z \rightarrow W$, where W is a scheme (see [DG70, III.2.2.3, III.2.6.1]). We claim that W is quasi-projective. Indeed, Z admits an ample $A[n]$ -linearized line bundle L (as follows e.g. from [Br17a, Prop. 2.12]). By descent, we have $L \simeq f^*(M)$ for a line bundle M on W , unique up to isomorphism. Since f is finite and locally free, M is ample in view of [EGA, II.6.6.1]; this proves the claim.

Next, the projection $\text{pr}_2 : A \times Z \rightarrow Z$ lies in a commutative square

$$\begin{array}{ccc} A \times Z & \xrightarrow{\text{pr}_2} & Z \\ \downarrow & & \downarrow f \\ A \times^{A[n]} Z & \xrightarrow{\pi} & W \end{array}$$

where the vertical arrows are $A[n]$ -torsors. Thus, this square is cartesian and the morphism $\pi : G \times^N Y = A \times^{A[n]} Z \rightarrow W$ is an A -torsor. By Lemma 2, it follows that W is equipped with an action of $G \times Q/A = G \times F$. Since the composition

$$G \times^N Y \longrightarrow G/N \longrightarrow G/I$$

is G -equivariant and A -invariant, this yields a commutative diagram

$$\begin{array}{ccc} G \times^N Y & \longrightarrow & W \\ \downarrow & & \downarrow \\ G/N & \longrightarrow & G/I \end{array}$$

where all arrows are G -equivariant, and the horizontal arrows are A -torsors. As a consequence, this diagram is cartesian, so that $W = G \times^I Y$; also, the F -action on W is free. Arguing as for the construction of W , this yields a cartesian square

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ G/I & \longrightarrow & G/H \end{array}$$

where the horizontal arrows are F -torsors and V is a G -scheme. Thus, V satisfies the assertions. \square

5 Weil restriction and proof of Theorem 2

Given a morphism of schemes $\psi : Z \rightarrow W$, recall that the Weil restriction functor $\mathbf{R}_{W/k}(Z/W) = \mathbf{R}_W(Z)$ is the contravariant functor from k -schemes (not necessarily of finite type) to sets, which sends every such scheme S to the set of sections of the morphism $\psi \times \text{id}_S : Z \times S \rightarrow W \times S$. These sections may be identified with the morphisms $\sigma : W \times S \rightarrow Z$ such that $\psi \circ \sigma = \text{pr}_1$, and this identifies $\mathbf{R}_W(Z)$ with the functor considered in [BLR90, 7.6].

Next, assume that Z is quasi-projective, and W is projective. Then in view of [Gr61, p. 267], the functor $\mathbf{R}_W(Z)$ is represented by an open subscheme $\mathbf{R}_W(Z)$ of the Hilbert scheme $\text{Hilb}(Z)$. This is obtained by sending every section to its schematic image; recall that the S -points of $\text{Hilb}(Z)$ are the closed subschemes V of $Z \times S$ which are flat over S , and the schematic images of sections are those V such that $V \xrightarrow{\sim} W \times S$ via $\psi \times \text{id}_S$.

Since $\text{Hilb}(Z)$ is a disjoint union of open and closed subschemes of finite type (for which the Hilbert polynomial is specified), $\mathbf{R}_W(Z)$ satisfies the same property. Thus, $\text{Hilb}(Z)$ and $\mathbf{R}_W(Z)$ are locally of finite type.

As mentioned in [BLR90, §7.6], Weil restriction is compatible with base change. In particular, for any $w \in W(k)$ with schematic fiber Z_w , we have a morphism of functors $\mathbf{R}_W(Z) \rightarrow Z_w$ (where Z_w is identified with its functor of points), and hence a morphism of schemes

$$\text{ev}_w : \mathbf{R}_W(Z) \longrightarrow Z_w, \quad \sigma \longmapsto \sigma(w, -)$$

under the above assumptions on Z and W .

Also, if Z and W are equipped with actions of an algebraic group G such that ψ is equivariant, then we obtain compatible G -actions on $\mathbf{R}_W(Z)$ and $\text{Hilb}(Z)$. If in addition w is fixed by a subgroup H of G , then ev_w is H -equivariant.

We now consider an algebraic group G , a subgroup H of G and an H -scheme Y . We assume that the associated fiber bundle

$$\psi : Z = G \times^H Y \longrightarrow G/H = W$$

exists, and Z is quasi-projective (by Theorem 3, this holds if G is smooth and Y is normal and quasi-projective). We assume in addition that the homogeneous space G/H is proper, or equivalently projective. Then the preceding discussion yields:

Lemma 4. *With the above assumptions, the functor $\mathbf{R}_W(Z)$ is represented by an open G -stable subscheme $\mathbf{R}_W(Z)$ of $\text{Hilb}(Z)$. Moreover, the evaluation morphism $\text{ev}_o : \mathbf{R}_W(Z) \rightarrow Y$ is H -equivariant.*

Next, let X be a G -scheme, and $f : X \rightarrow Y$ an H -equivariant morphism. Then

$$\mathrm{id}_G \times f : G \times X \rightarrow G \times Y$$

is a $G \times H$ -equivariant morphism of schemes over G , where $G \times H$ acts on $G \times X$ and $G \times Y$ via $(g, h) \cdot (g', z) = (gg'h^{-1}, h \cdot z)$. By Lemma 2, this defines a G -equivariant morphism

$$G \times^H f : G \times^H X \rightarrow G \times^H Y = Z$$

of schemes over W , which pulls back to $f : X \rightarrow Y$ on fibers at o . Also, since X is a G -scheme, the morphism

$$G \times X \rightarrow W \times X, \quad (g, x) \mapsto (g \cdot o, g \cdot x)$$

factors through a G -equivariant isomorphism $G \times^H X \xrightarrow{\sim} W \times X$ of schemes over W , which pulls back to $o \times \mathrm{id}_X$ on fibers at o . This defines a G -equivariant morphism $W \times X \rightarrow Z$ of schemes over W , which pulls back to f on fibers at o . In turn, this yields a G -equivariant morphism

$$F : X \rightarrow \mathbf{R}_W(Z)$$

such that $\mathrm{ev}_o \circ F = f$.

Lemma 5. *With the above notation, assume that f is an immersion. Then F is an immersion as well.*

Proof. For any $x \in X$, the composition of homomorphisms of local rings

$$\mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{\mathbf{R}_W(Z), F(x)} \rightarrow \mathcal{O}_{X, x}$$

is an isomorphism, and hence the homomorphism $\mathcal{O}_{\mathbf{R}_W(Z), F(x)} \rightarrow \mathcal{O}_{X, x}$ is surjective. By [DG70, I.3.4.4], it follows that F is a local immersion at x . Also, F is injective, since so is f . But every injective local immersion is an immersion, since X is irreducible (see [DG70, I.3.4.5]). \square

Remark 6. We sketch an alternative approach to the above Weil restriction functor $\mathbf{R}_W(Z)$. Consider the functor $\mathbf{Hom}^H(G, Y)$ sending every scheme S to the set of H -equivariant morphisms $f : G \times S \rightarrow Y$, where H acts on $G \times S$ via right multiplication on G . Then G acts on $\mathbf{Hom}^H(G, Y)$ via left multiplication on itself. Moreover, evaluating at e yields a morphism of functors

$$\varepsilon : \mathbf{Hom}^H(G, Y) \rightarrow Y, \quad f \mapsto f(e, -),$$

which is H -equivariant (here Y is identified with its functor of points). With this notation, $\mathbf{Hom}^H(G, Y)$ is G -equivariantly isomorphic to $\mathbf{R}_W(Z)$, and this isomorphism identifies ε with ev_e .

This follows from the classical identification of morphisms with their graph: more specifically, for any scheme S , the data of a morphism $f : G \times S \rightarrow Y$ is equivalent to that of its graph, a closed subscheme Γ_f of $G \times S \times Y$ such that the projection $\text{pr}_{12} : G \times S \times Y \rightarrow G \times S$ is an isomorphism. Moreover, f is H -equivariant if and only if Γ_f is stable under the H -action on $G \times S \times Y$ via $h \cdot (g, s, y) = (gh^{-1}, s, h \cdot y)$. By descent for the H -torsor $\varphi_S : G \times S \times Y \rightarrow Z \times S$, the closed subschemes of $G \times S \times Y$ obtained in this way are exactly the pull-backs under φ_S of the closed subschemes $\Gamma \subset Z \times S$ such that $\Gamma \xrightarrow{\sim} W \times S$ via $\psi \times \text{id}_S$. This yields the desired isomorphism of functors. Its G -equivariance and compatibility with ε and ev_e are easily verified.

For any G -scheme X , one may show that the map

$$\text{Hom}^G(X, \mathbf{Hom}^H(G, Y)) \longrightarrow \text{Hom}^H(X, Y), \quad f \longmapsto \varepsilon \circ f$$

is an isomorphism, by using the canonical isomorphism

$$\text{Hom}(X, \mathbf{Hom}(G, Y)) \xrightarrow{\sim} \text{Hom}(G \times X, Y)$$

(see [SGA3, I.1.7.1]) and taking $G \times H$ -invariants. Equivalently, $\mathbf{Hom}^H(G, Y)$ is right adjoint to the restriction functor from G -schemes to H -schemes. A left adjoint to this restriction functor is provided by the associated fiber bundle: denoting by $i : Y \rightarrow G \times^H Y$ the canonical (H -equivariant) closed immersion, the map

$$\text{Hom}^G(G \times^H Y, X) \longrightarrow \text{Hom}^H(Y, X), \quad f \longmapsto f \circ i$$

is an isomorphism as well.

PROOF OF THEOREM 2. We first argue as in the beginning of the proof of Theorem 3. By [Br17b, Thm. 2], G has a smallest normal subgroup H such that G/H is proper; moreover, H is affine and connected. Since X is normal and quasi-projective, it is H -quasi-projective in view of [Br17a, Lem. 2.9]. Thus, there exists an H -equivariant open immersion $f : X \rightarrow Y$, where Y is an H -projective scheme; then the associated fiber bundle $Z = G \times^H Y$ exists and is quasi-projective. Since Y is proper, the projection $\text{pr}_1 : G \times Y \rightarrow G$ is proper as well. By descent (see [EGA, IV.2.7.1]), it follows that the morphism $\psi : Z \rightarrow G/H = W$ is proper as well. As W is proper, we see that Z is proper, and hence projective. By Lemma 5, the H -equivariant immersion of X in Y yields a G -equivariant immersion of X in $\mathbf{R}_W(Z)$, and hence

in $\text{Hilb}(Z)$. Since X is a variety, its schematic image in $\text{Hilb}(Z)$ is a projective G -variety. Taking its normalization yields the desired normal projective equivariant completion. \square

Remark 7. Assume that k is imperfect. Then there exist (many) k -forms G of the additive group \mathbb{G}_a such that the regular completion X of G is not smooth (see [Ru70]; here G is viewed as a smooth affine curve). The G -action on itself by translation extends uniquely to a G -action on X , which is the unique normal projective model of G . In particular, G admits no smooth projective model.

6 Proof of Weil's regularization theorem

Throughout this section, we consider a smooth algebraic group G and a smooth variety X . We first recall some notions and results from [De70, §3].

A *rational action* of G on X is a rational map

$$a : G \times X \dashrightarrow X, \quad (g, x) \longmapsto g \cdot x$$

which satisfies the following two properties:

(i) The rational map

$$(\text{pr}_1, a) : G \times X \dashrightarrow G \times X, \quad (g, x) \longmapsto (g, g \cdot x)$$

is dominant.

(ii) The rational maps

$$a \circ (\text{id}_G \times a), a \circ (m \times \text{id}_X) : G \times G \times X \dashrightarrow X, \quad (g, h, x) \longmapsto g \cdot (h \cdot x), gh \cdot x$$

are equal, where $m : G \times G \rightarrow G$ denotes the multiplication.

We denote by V the domain of definition of a ; this is an open dense subset of $G \times X$. Since G and X are smooth, V is smooth as well. For any scheme S and any $g \in G(S)$, $x \in X(S)$, we say that $g \cdot x$ is *defined* if $(g, x) \in V(S)$. Then (ii) is equivalent to

(iii) For any scheme S and any $g, h \in G(S)$, $x \in X(S)$, if $h \cdot x$ and $g \cdot (h \cdot x)$ are defined, then $gh \cdot x$ is defined and equals $g \cdot (h \cdot x)$.

For any $g \in G$ with residue field $\kappa(g)$, the fiber $\mathrm{pr}_1^{-1}(g) \subset G \times X$ is identified with $X_{\kappa(g)}$ and this identifies $V \cap \mathrm{pr}_1^{-1}(g)$ with an open subset V_g of $X_{\kappa(g)}$. In view of [De70, Lem. 1], V_g is dense in $X_{\kappa(g)}$ and the morphism

$$a_g : V_g \longrightarrow X_{\kappa(g)}, \quad x \longmapsto g \cdot x$$

is dominant. In particular, V_e is a dense open subset of X . As a consequence, the morphism $a_e : V_e \rightarrow X$ is the inclusion (indeed, we have $e \cdot x = e \cdot (e \cdot x)$ for any $x \in X$ such that $e \cdot x$ and $e \cdot (e \cdot x)$ are defined. Thus, $y = e \cdot y$ for any y in a dense open subset of X). Also, the rational map (pr_1, a) is birational (see [De70, p. 515]).

With this notation, Weil's regularization theorem asserts that *there exists a G -variety Y and a birational G -equivariant map $f : X \dashrightarrow Y$* , i.e., $f(g \cdot x) = g \cdot f(x)$ whenever $(g, x) \in V$ and f is defined at $g \cdot x$. (This result was originally stated for a geometrically integral variety X , but there is no harm in assuming X smooth).

Weil's regularization theorem is proved in [De70, Prop. 9] under the additional assumption that a is *pseudo-transitive*, i.e., the morphism

$$(a, \mathrm{pr}_2) : V \longrightarrow X \times X, \quad (g, x) \longmapsto (g \cdot x, x)$$

is dominant. Then Y may be taken *homogeneous* in the sense that the analogously defined morphism $G \times X \rightarrow X \times X$ is faithfully flat. We will deduce the general case from this special one, which is arguably much easier.

We start with a reduction to the case where k is *algebraically closed*. Assume that there exists a \bar{k} -variety Y which is $G_{\bar{k}}$ -birationally isomorphic to $X_{\bar{k}}$. By the principle of the finite extension (see [EGA, IV.9.1]), there exists a finite subextension K/k of \bar{k}/k , a K -variety Z equipped with an action of G_K , and a G_K -equivariant birational map

$$\psi : X_K \dashrightarrow Z.$$

By Theorems 1 and 2, we may further assume that Z is projective. Then the Weil restriction functor $\mathbf{R}_{K/k}(Z/K) = \mathbf{R}_K(Z)$ is represented by an open subscheme $\mathbf{R}_K(Z)$ of the Hilbert scheme parameterizing finite subschemes of length n of Z , where $n = [K : k]$. In particular, the scheme $\mathbf{R}_K(Z)$ is quasi-projective (see [CGP15, A.5.8] for a direct proof of this fact).

Choose an open subset $U \subset X_K$ on which ψ is defined, and let $F = X_K \setminus U$. Also, denote by $\pi : X_K \rightarrow X$ the projection. Since π is finite and surjective, $\pi(F)$ is a closed subset of X and $X \setminus \pi^{-1}\pi(F) = (X \setminus \pi(F))_K$ is a dense open subset of U . Replacing X with $X \setminus \pi(F)$, we may thus assume that ψ is a morphism. Similarly, we may further assume that ψ is an open immersion.

Denoting by $i : V \rightarrow G \times X$ the inclusion of the domain of definition of a , the G_K -equivariance of ψ yields a commutative diagram

$$\begin{array}{ccccc} V_K & \xrightarrow{i_K} & G_K \times_K X_K & \xrightarrow{\text{id} \times \psi} & G_K \times_K Z \\ & \searrow a_K & & & \downarrow b \\ & & X_K & \xrightarrow{\psi} & Z, \end{array}$$

where b denotes the action. Since Weil restriction commutes with fibered products (see [CGP15, A.5.2]), this yields in turn a commutative diagram

$$\begin{array}{ccccc} \mathbf{R}_K(V_K) & \xrightarrow{\mathbf{R}_K(i_K)} & \mathbf{R}_K(G_K) \times \mathbf{R}_K(X_K) & \xrightarrow{\text{id} \times \mathbf{R}_K(\psi)} & \mathbf{R}_K(G_K) \times \mathbf{R}_K(Z) \\ & \searrow \mathbf{R}_K(a_K) & & & \downarrow \mathbf{R}_K(b) \\ & & \mathbf{R}_K(X_K) & \xrightarrow{\mathbf{R}_K(\psi)} & \mathbf{R}_K(Z), \end{array}$$

where $\mathbf{R}_K(i_K)$, $\mathbf{R}_K(\psi)$ and $\text{id} \times \mathbf{R}_K(\psi)$ are open immersions (see [BLR90, 7.6/2]). We also have a commutative square

$$\begin{array}{ccc} V & \xrightarrow{i} & G \times X \\ j_V \downarrow & & \downarrow j_G \times j_X \\ \mathbf{R}_K(V_K) & \xrightarrow{\mathbf{R}_K(i_K)} & \mathbf{R}_K(G_K) \times \mathbf{R}_K(X_K), \end{array}$$

where the adjunction morphism j_V is a closed immersion, and likewise for j_G and j_X (see [CGP15, A.5.7]). Thus, $\gamma = \mathbf{R}_K(\psi) \circ j_X$ is a (locally closed) immersion. Also, we have $\mathbf{R}_K(a_K) \circ j_V = j_X \circ a$ by functoriality (see [CGP15, A.5.7] again). By concatenating the two diagrams above, this yields a commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{i} & G \times X & \xrightarrow{\text{id} \times \gamma} & G \times \mathbf{R}_K(Z) \\ & \searrow a & & & \downarrow c \\ & & X & \xrightarrow{\gamma} & \mathbf{R}_K(Z), \end{array}$$

where c denotes the pull-back to G of the action of $\mathbf{R}_K(G_K)$. Denote by W the schematic image of γ ; then W is G -stable, since the formation of the schematic image commutes with flat base extension. Thus, γ factors through an open immersion

$\delta : X \rightarrow W$ which lies in a commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{i} & G \times X & \xrightarrow{\text{id} \times \delta} & G \times W \\ & \searrow a & & & \downarrow d \\ & & X & \xrightarrow{\delta} & W, \end{array}$$

where d denotes the G -action on W . Thus, W is the desired model.

So we may assume that k is algebraically closed. At this stage we may conclude by using the main result of [Kr18], but we prefer to give a fully independent proof.

We now construct the “variety of orbits” for the rational action of G on X , by adapting arguments from [Sp89, IV.1, IV.2]. The group $G(k)$ acts on the field of rational functions $k(X)$ by automorphisms. Denote by K the fixed subfield; then K is finitely generated over k , and hence equals $k(Y)$ for some variety Y equipped with a dominant rational map $f : X \dashrightarrow Y$. Replacing X and Y with suitable dense open subsets, we may assume that they are both affine, and f is a morphism. Since the extension $k(X)/K$ is separable (see [Sp89, Lem. IV.1.5]), we may further assume that f is smooth. Finally, using generic freeness (see [EGA, IV.6.9.2]), we may assume that the $\mathcal{O}(Y)$ -module $\mathcal{O}(X)$ is free.

By density of k -rational points, we have $f(g \cdot x) = f(x)$ for all $g \in G$ and $x \in X$ such that $g \cdot x$ is defined. Thus, $(a, \text{pr}_2) : V \rightarrow X \times X$ yields a morphism

$$\gamma : V \longrightarrow X \times_Y X.$$

We claim that γ is *schematically dominant*. As $X \times_Y X = \text{Spec } \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(X)$, this is equivalent to the injectivity of $\gamma^\# : \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) \rightarrow \mathcal{O}(V)$. To show this, let $u_1, \dots, u_r, v_1, \dots, v_r \in \mathcal{O}(X)$ such that $\gamma^\#(\sum_{i=1}^r u_i \otimes v_i) = 0$. We may assume that u_1, \dots, u_r are part of a basis of the $\mathcal{O}(Y)$ -module $\mathcal{O}(X)$; then they are linearly independent over K . We then have $\sum_{i=1}^r u_i(g \cdot x)v_i(x) = 0$ for all $x \in X$ and $g \in G(k)$ such that $g \cdot x$ is defined. Thus, we have $\sum_{i=1}^r (g \cdot u_i)v_i = 0$ in $k(X)$ for any $g \in G(k)$. By [Sp89, Lem. IV.1.5] again, it follows that $v_1 = \dots = v_r = 0$. This proves the claim.

Denote by η the generic point of Y ; then the generic fiber X_η is a smooth K -variety equipped with a rational action of the K -group G_K . As the morphism $\eta \rightarrow Y$ is flat and γ is schematically dominant, the base change $\gamma_\eta : V \times_Y \eta \rightarrow (X \times_Y X) \times_Y \eta$ is schematically dominant as well (see [EGA, IV.11.10.5]). So the rational map

$$(a, \text{pr}_2) : G_K \times_K X_\eta \dashrightarrow X_\eta \times_K X_\eta$$

is dominant, i.e., the rational action of G_K on X_η is almost transitive. By [De70, Prop. 9], there exists a G_K -homogeneous K -variety Z which is G_K -equivariantly birational to X_η . In particular, $K(Z) = k(X)$.

It remains to “regularize” Z . We first treat the case where G is *affine*, or equivalently linear. Then Z is G_K -quasi-projective; equivalently, there exists a homomorphism $\rho : G_K \rightarrow \mathrm{GL}_{n,K}$ and an immersion $Z \hookrightarrow \mathbb{P}_K^{n-1}$ which is equivariant relative to ρ (see e.g. [Br17a, Cor. 2.14]). Denote by $(a_{ij})_{1 \leq i, j \leq n}$ the matrix coefficients of ρ and by d their determinant. Then $a_{ij}, d^{-1} \in \mathcal{O}(G_K) = \mathcal{O}(G) \otimes_k K$. We may choose a finitely generated subalgebra R of K which contains the a_{ij} and d^{-1} , and has fraction field K . Let $Y = \mathrm{Spec}(R)$, then ρ extends to a unique homomorphism of Y -group schemes

$$\rho_Y : G_Y = G \times Y \longrightarrow \mathrm{GL}_{n,Y}.$$

Denote by W the closure of Z in $\mathbb{P}_Y^n = \mathbb{P}^n \times Y$; then W is stable under the action of G_Y via ρ_Y . The resulting morphism $G_Y \times_Y W \rightarrow W$ yields a G -action on W , and in turn the desired regularization.

For an arbitrary (smooth) algebraic group G , we first reduce to the case where G is *connected* by adapting the proof of [Ro56, Thm. 1]. Denote by G^0 the neutral component of G , and assume that X is G^0 -birationally isomorphic to a G^0 -variety. We may then assume that X is a G^0 -variety, i.e., $g \cdot x$ is defined for any $g \in G^0$ and $x \in X$. Also, we may choose $g_1, \dots, g_r \in G(k)$ such that $G = \bigcup_{i=1}^r G^0 g_i$ (since k is algebraically closed). Then $g_i \cdot x$ is defined for any $i = 1, \dots, r$ and for any $x \in \bigcap_{i=1}^r V_{g_i} = X \setminus Y$, where $Y \subsetneq X$ is closed. Thus, $h \cdot (g_i \cdot x)$ is defined for any $h \in G^0$ and i, x as above, i.e., $g \cdot x$ is defined for all $g \in G$ and $x \in X \setminus Y$.

Let $Z = \bigcap_{h \in G^0(k)} h \cdot Y$; then Z is a closed subset of X , stable by $G^0(k)$. Since G is smooth, $G^0(k)$ is schematically dense in G^0 and hence Z is G^0 -stable. We claim that $g \cdot x$ is defined for any $g \in G$ and $x \in X \setminus Z$. Indeed, $x \in X \setminus h \cdot Y$ for some $h \in G^0(k)$. Thus, $h^{-1} \cdot x$ is defined and lies in $X \setminus Y$. So $gh \cdot (h^{-1} \cdot x)$ is defined. By using (iii), this yields the claim.

Next, let W be the subset of X consisting of those x such that $g_i \cdot x$ is defined and lies in Z for some i . Then W is closed in X : indeed, W contains Z and $W \setminus Z = \bigcap_{i=1}^r g_i^{-1}(X \setminus Z)$ is closed in $X \setminus Z$. Also, W is stable by $G^0(k)$ (since the latter is normalized by the g_i) and hence by G^0 . For any $g \in G$ and $x \in X \setminus W$, we have that $g \cdot x$ is defined (as $x \notin Z$) and $g \cdot x \notin W$ (otherwise, $g_i g \cdot x \in Z$ for some i , and hence $g_j \cdot x \in Z$ for j such that $g_i g \in G^0 g_j$. So $x \in W$, a contradiction). In other terms, g is defined at any point of $X \setminus W$, and stabilizes this open subset. As a consequence, each g_i yields an automorphism of $X \setminus W$. So the G^0 -action on $X \setminus W$ extends to a G -action.

Thus, we may assume that (k is algebraically closed and) G is connected. In view of [De70, Prop. 8], we may further assume that the rational action of G on X is *faithful*, i.e., its schematic kernel is trivial. We have a unique exact sequence

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\pi} A \longrightarrow 1,$$

where N is smooth, connected and affine, and A is an abelian variety (see e.g. [Mi17, Thm. 8.28]). Also, there exists a finite Galois extension of fields L/K such that Z_L has a L -rational point. Then Z_L is a homogeneous space G_L/H , where H is affine (see [Mi17, Prop. 8.9]). Consider the exact sequence

$$1 \longrightarrow N_L \longrightarrow G_L \longrightarrow A_L \longrightarrow 1.$$

The schematic image of H in A_L is an affine subgroup, and hence is finite. Thus, the homogeneous space $G_L/N_L H$ is a quotient of A_L by a finite subgroup. Also, the natural morphism $G_L/H \rightarrow G_L/N_L H$ is the geometric quotient by the action of N_L via left multiplication on G_L . By Galois descent, it follows that the geometric quotient $Z \rightarrow Z/N_K$ exists, and Z/N_K is a torsor under the quotient of A_K by a finite subgroup. Since Z is smooth, Lemma 3 yields an A_K -equivariant morphism $Z/N_K \rightarrow A_K/A[n]_K$ for some positive integer n . Denote by I the pull-back of $A[n]$ under the homomorphism $\pi : G \rightarrow A$; then $N \subset I \subset G$ and I/N is finite. Moreover, we have a G_K -equivariant morphism $Z \rightarrow Z/N_K \rightarrow A_K/A[n]_K = G_K/I_K$ and hence $Z \simeq G_K \times^{I_K} W$ for some I_K -scheme W . Since Z is normal and quasi-projective, it is I_K -quasi-projective (see again [Br17a, Cor. 2.14]); thus, so is W . By arguing as in the affine case, one obtains an affine variety Y with function field K , and a scheme V equipped with an action of I and an I -invariant morphism $V \rightarrow Y$ having generic fiber W ; moreover, V is I -quasi-projective. Thus, $G \times^I V$ exists and yields the desired regularization.

7 Actions of non-smooth algebraic groups

Throughout this section, the ground field k has characteristic $p > 0$ (otherwise every algebraic group is smooth). We first construct actions having no normal projective model:

Example 8. We consider torsors under the “smallest” non-smooth algebraic group $G = \alpha_p$, the kernel of the Frobenius homomorphism $F : \mathbb{G}_a \rightarrow \mathbb{G}_a, z \mapsto z^p$.

Let C be an integral affine curve and consider the product $C \times \mathbb{A}^1$ with projections x, y . Choose $f \in \mathcal{O}(C)$ and denote by $X \subset C \times \mathbb{A}^1$ the zero subscheme of $y^p - f(x)$.

So we have a cartesian square

$$\begin{array}{ccc} X & \xrightarrow{y} & \mathbb{A}^1 \\ x \downarrow & & \downarrow F \\ C & \xrightarrow{f} & \mathbb{A}^1. \end{array}$$

The group G acts on $C \times \mathbb{A}^1$ via $g \cdot (x, y) = (x, y + g)$ and this action stabilizes X . The projection $x : X \rightarrow C$ is a G -torsor, the pull-back of F under f . By [DG70, III.4.6.7], every G -torsor over an affine scheme is obtained via this construction.

The affine curve X is integral if and only if f is not a p th power in the function field $k(C)$. When C is normal, this is equivalent to f not being a p th power in $\mathcal{O}(C)$, and hence to the non-triviality of the torsor x (see [DG70, III.4.6.7] again).

We now assume for simplicity that k is algebraically closed and C is smooth. Then X is non-normal if and only if the morphism $f : C \rightarrow \mathbb{A}^1$ is ramified; this holds e.g. if f has a multiple factor when expressed in local coordinates at some point.

Assuming in addition that X is integral and non-normal, we claim that *the G -action on X does not lift to an action on the normalization \tilde{X}* . Otherwise, G acts freely on \tilde{X} and hence there exists a G -torsor $q : \tilde{X} \rightarrow \tilde{X}/G$, where \tilde{X}/G is an integral affine curve (see [DG70, III.2.2.3, III.2.6.1]). Thus, the normalization morphism $\nu : \tilde{X} \rightarrow X$ descends to a morphism $\mu : \tilde{X}/G \rightarrow C$; equivalently, we have a commutative square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\nu} & X \\ q \downarrow & & \downarrow x \\ \tilde{X}/G & \xrightarrow{\mu} & C. \end{array}$$

As the vertical arrows are G -torsors, this square is cartesian. Since ν is finite, so is μ by descent. Also, since ν is birational and every open subset of X is G -stable, it follows that μ is birational as well. As C is smooth, μ is an isomorphism. So ν is an isomorphism as well, a contradiction. This proves our claim.

Next, we show that X admits no normal projective model. Indeed, X has a projective model \bar{X} , its closure in $\bar{C} \times \mathbb{P}^1$ where \bar{C} denotes the smooth projective completion of C , and G acts on $\bar{C} \times \mathbb{P}^1$ via $g \cdot (x, [y : z]) = (x, [y + gz : z])$. If X has a normal projective model X' , then the G -equivariant rational map of curves $X' \dashrightarrow X$ yields an equivariant morphism $\varphi : X' \rightarrow \bar{X}$. Thus, φ is the normalization, and hence so is its pull-back $\varphi^{-1}(X) \rightarrow X$, a contradiction to the claim.

Next, we discuss the notion of rational action in the setting of non-smooth groups, for which we could locate no convenient reference.

As in [GW10, 9.6], we define a *rational map* $f : X \dashrightarrow Y$ (where X and Y are schemes) as an equivalence class of pairs (U, g) , where U is a schematically dense open subset of X , and $g : U \rightarrow Y$ is a morphism. Here (U, g) is said to be equivalent to (V, h) if there exists a schematically dense open subset W of $U \cap V$ such that $g|_W = h|_W$. (Equivalently, f is a pseudo-morphism in the sense of [EGA, IV.20.2]). There exists a largest such subset U , the *domain of definition* of f .

Next, a rational action of an algebraic group G on a smooth variety X is defined as in Section 6, by replacing “dominant” with “schematically dominant” in (i). The arguments in the beginning of [De70, §3] adapt readily with this replacement, and yield the equivalence of (ii) and (iii). Also, for any $g \in G$ with residue field $\kappa(g)$, we may still identify $V \cap \text{pr}_1^{-1}(g)$ with a schematically dominant open subset V_g of $X_{\kappa(g)}$, and the morphism $V_g \rightarrow X$, $g \mapsto g \cdot x$ is (schematically) dominant. Here V denotes again the domain of definition of the action. Thus, we still have that $e \cdot x = x$ for all $x \in V_e$.

We may now state:

Theorem 9. *Let X be a smooth variety equipped with a rational action of an algebraic group G . Then X is G -birationally isomorphic to a projective G -variety.*

Proof. We first consider the case where G acts on X . We may choose a positive integer n such that G/G_n is smooth, where G_n denotes the n th Frobenius kernel (see e.g. [Br17b, Prop. 2.9.2]). By [Br17a, Lem. 2.8], the quotient X/G_n exists and is a G/G_n -variety. Using Theorem 1, we may further assume that X/G_n is regular and quasi-projective. We now claim that X admits a G -equivariant projective completion.

By [Br17a, Lem. 2.9], X/G_n admits an ample H -linearized line bundle, where H denotes the smallest normal subgroup of G such that G/H is proper., and H acts on X/G_n via its quotient $H/H \cap G_n = H/H_n$. Since the quotient morphism $X \rightarrow X/G_n$ is finite, X also admits an ample H -linearized line bundle. The claim follows from this by arguing as in the proof of Theorem 2 (except for the final step of normalization).

Next, we turn to the general case, where G acts rationally on X . By the above claim, it suffices to show that X is G -birationally isomorphic to a G -variety. For this, we may assume that k is algebraically closed by the first reduction step in the proof of Weil’s regularization theorem, which extends without any change. We then have $G = G_n G_{\text{red}}$, where G_{red} denotes the largest smooth subgroup of G (the closure of $G(k)$), see e.g. [Br17b, Lem. 2.8.6]. Using the regularization theorem, we may further assume that X is a G_{red} -variety. Denote by $V_n \subset G_n \times X$ the domain of definition of the rational action $a_n : G_n \times X \dashrightarrow X$. Then V_n is schematically dense

in $G_n \times X$, and stable by $G(k)$ (acting on $G_n \times X$ via $g \cdot (h, x) = (ghg^{-1}, g \cdot x)$) in view of the condition (iii) in Section 6. Thus, $V_n \cap \text{pr}_1^{-1}(e)$ is identified with a schematically dense open subset $(V_n)_e$ of X , which is stable by $G(k)$ as well. As a consequence, $(V_n)_e$ is stable by G_{red} . We may thus assume that $(V_n)_e = X$.

Since G_n is infinitesimal, the projection $\text{pr}_2 : G_n \times X \rightarrow X$ induces a homeomorphism of the underlying topological spaces, with inverse homeomorphism (e, id_X) . As V_n is open in $G_n \times X$, it follows that $V_n = G_n \times X$ as schemes, i.e., G_n acts on X . Since G_{red} also acts on X , the domain of definition of the rational G -action is the whole X by (iii) again. So X is a G -variety. \square

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