Linearization of algebraic group actions

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Abstract

This expository text presents some fundamental results on actions of linear algebraic groups on algebraic varieties: linearization of line bundles and local properties of such actions.

Contents

1 Introduction

To define algebraic varieties, one may start with affine varieties and then glue them along open affine subsets. In turn, affine algebraic varieties may be defined either as closed subvarieties of affine spaces, or intrinsically, in terms of their algebra of regular functions. A standard example is the projective space, obtained by glueing affine spaces along open subsets defined by the nonvanishing of coordinates. Projective spaces are natural ambient spaces in algebraic geometry; their locally closed subvarieties are called quasi-projective varieties. Also, recall that quasi-projectivity has an intrinsic characterization, in terms of the existence of an ample line bundle.

One may ask whether these fundamental notions and properties are still valid in the presence of group actions. More specifically, given an action of a group G on a variety X (everything being algebraic), one may ask firstly if X admits a covering by open affine G-stable subsets, and secondly, if any such subset is equivariantly isomorphic to a closed G-stable subvariety of an affine space on which G acts linearly.

The second question is easily answered in the positive. But the answer to the first question is generally negative, e.g., for the projective space $\mathbb{P}^n$ equipped with the action of its automorphism group, the projective linear group PGL_{n+1} (since this action is transitive). So it makes more sense to ask whether X admits a covering by open quasi-projective G-stable subsets, and whether any such subset is equivariantly isomorphic to a G-stable subvariety of some projective space $\mathbb{P}^n$ on which G acts linearly (i.e., via a homomorphism to GL_{n+1}).

The answer to both questions above turns out to be positive under mild restrictions on G and X:

Theorem. *Let X be a normal variety equipped with an action of a connected linear algebraic group G. Then each point of X admits an open G-stable neighborhood, equivariantly isomorphic to a G-stable subvariety of some projective space on which G acts linearly.*

This basic result is due to Sumihiro (see [Su74, Su75]); his proof is based on the notion of linearization of line bundles, introduced earlier by Mumford in his foundational work on geometric invariant theory (see [MFK94]). In loose terms, a G-linearization of a line bundle L over a G-variety X is a G-action on the variety L which lifts the given action on X, and is linear on fibres. It is not hard to show that X is equivariantly isomorphic to a subvariety of some projective space on which G acts linearly, if and only if X admits an ample G-linearized line bundle. The main point is to prove that given a line bundle L on a normal G-variety X, some positive tensor power $L^{\otimes n}$ admits a G-linearization when G is linear and connected.

As a consequence of Sumihiro's theorem, every normal variety equipped with an action of an algebraic torus T (i.e., T is a product of copies of the multiplicative group) admits a covering by open affine T-stable subsets. This is a key ingredient in the combinatorial classification of toric varieties in terms of fans (see e.g. [CLS11]) and, more generally, of the description of normal T-varieties in terms of divisorial fans (see e.g. [AHS08, La15, LS13]). The classification of equivariant embeddings of homogeneous spaces (see [LV83, Kn91, Ti11]) also relies on Sumihiro's theorem. On the other hand, examples show that this theorem is optimal, i.e., the additional assumptions on G and X cannot be suppressed.

The aim of this expository text is to present Sumihiro's theorem and some related results over an algebraically closed field, with rather modest prerequisites: familiarity with basic algebraic geometry, e.g., the contents of Chapters 1 and 2 of Hartshorne's book [Ha74]. We owe much to an earlier exposition by Knop, Kraft, Luna and Vust (see [KKV89, KKLV89]), where the ground field is assumed to be algebraically closed of characteristic zero, and to the recent article [Bri15], which deals with linearization of line bundles over possibly non-normal varieties, and an arbitrary ground field.

This text is organized as follows. In Section 2, we gather some preliminary results on algebraic groups and their actions, with special emphasis on principal bundles, associated fibre bundles, and homogeneous spaces.

Section 3 also begins with preliminary material on line bundles, invertible sheaves, and principal bundles under the multiplicative group. Then we introduce linearizations of line bundles on a G-variety, and their relation to G-quasi-projectivity, i.e., the existence of an equivariant embedding in a projective G-space (Proposition 3.2.6). We also present applications to associated fibre bundles (Corollary 3.3.3), and actions of finite groups (Proposition 3.4.8).

In Section 4, based on [KKLV89, Bri15], we obtain the main technical result of this text (Theorem 4.2.2), which provides an obstruction to the linearization of line bundles for the action of a connected algebraic group on an irreducible variety. From this, we derive an exact sequence of Picard groups for a principal bundle under a connected algebraic group (Proposition 4.3.1).

The final Section 5 presents further applications, most notably to the linearization of line bundles again (Theorem 5.2.1), Sumihiro's theorem (Theorem 5.3.3), and an equivariant version of Chow's lemma (Corollary 5.3.7). Some further developments are sketched at the end of Sections 2, 3 and 5.

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2 Algebraic groups and their actions

Throughout this text, we fix a algebraically closed base field k of arbitrary characteristic, denoted by $\mathrm{char}(k)$. By a *variety*, we mean a reduced separated scheme of finite type over k; in particular, varieties may be reducible. By a point of a variety X, we mean a closed (or equivalently, k-rational) point. We identify X with its set of points, equipped with the Zariski topology and with the structure sheaf $\mathcal{O}_X$. The algebra of global sections of $\mathcal{O}_X$ is called the algebra of regular functions on X, and denoted by $\mathcal{O}(X)$. We denote by $\mathcal{O}(X)^*$ the multiplicative group of invertible elements (also called units) of $\mathcal{O}(X)$. When X is irreducible, its field of rational functions is denoted by $k(X)$.

We will use the book [Ha74] as a general reference for algebraic geometry, and [Bo91] for linear algebraic groups.

2.1 Basic notions and examples

Definition 2.1.1. An *algebraic group* is a variety G equipped with a group structure such that the multiplication map $m : G \times G \to G$ and the inverse map $i : G \to G$ are morphisms of varieties.

We denote for simplicity $m(x, y)$ by xy, and $i(x)$ by x^{-1}, for any $x, y \in G$. The neutral element of G is denoted by e_G, or just by e if this yields no confusion.

Definition 2.1.2. Given two algebraic groups G, H, a *homomorphism of algebraic groups* is a group homomorphism $f : G \to H$ which is also a morphism of varieties. The *kernel* of f is its set-theoretic fibre at e_H; this is a closed normal subgroup of G.

Examples 2.1.3. (i) The *additive group* $\mathbb{G}_a$ is the affine line $\mathbb{A}^1$ equipped with the addition; the *multiplicative group* $\mathbb{G}_m$ is the punctured affine line $\mathbb{A}^1 \setminus \{0\}$ equipped with the multiplication.

(ii) For any positive integer n, the group GL_n of invertible $n \times n$ matrices is an open affine subset of the space of matrices $\mathrm{M}_n \cong \mathbb{A}^{n^2}$: the complement of the zero locus of the determinant. Since the product and inverse of matrices are polynomial in the matrix entries and the inverse of the determinant, GL_n is an algebraic group: the *general linear group*.

The determinant yields a homomorphism of algebraic groups, $\det : \mathrm{GL}_n \to \mathbb{G}_m$. Its kernel is the *special linear group* SL_n.

Likewise, for any finite-dimensional vector space V, the group $\mathrm{GL}(V)$ of linear automorphisms of V is algebraic. The choice of a basis of V yields an isomorphism of algebraic groups $\mathrm{GL}(V) \cong \mathrm{GL}_n$, where $n = \dim(V)$.

(iii) An algebraic group is called *linear* if it is isomorphic to a closed subgroup of some linear group $\mathrm{GL}(V)$.

For instance, the additive and multiplicative groups are linear, since $\mathbb{G}_m = \mathrm{GL}_1$ and $\mathbb{G}_a$ is isomorphic to the subgroup of GL_2 consisting of upper triangular matrices with diagonal coefficients 1. Further examples of linear algebraic groups include the *classical groups*, such as the orthogonal group $\mathrm{O}_n \subset \mathrm{GL}_n$ and the symplectic group $\mathrm{Sp}_{2n} \subseteq \mathrm{GL}_{2n}$.

The *projective linear group* PGL_n, the quotient of GL_n by its center (consisting of the nonzero scalar matrices, and hence isomorphic to $\mathbb{G}_m$), is a linear algebraic group as well. Indeed, PGL_n may be viewed as the automorphism group of the algebra of matrices M_n, and hence is isomorphic to a closed subgroup of GL_{n^2}. Thus, $\mathrm{PGL}(V)$ is a linear algebraic group for any finite-dimensional vector space V.

The variety $\mathrm{GL}(V)$ is affine, and hence every linear algebraic group is affine as well. Conversely, every affine algebraic group is linear, as we will see in Corollary 2.2.6.

(iv) Let C be an *elliptic curve*, i.e., C is a smooth projective curve of genus 1 equipped with a point 0. Then C has a unique structure of algebraic group with neutral element 0, and this group is commutative (see e.g. [Ha74, Prop. IV.4.8, Lem. IV.4.9]). Since the variety C is not affine, this yields examples of non-linear algebraic groups.

(v) More generally, a complete connected algebraic group is called an *abelian variety*. One can show that every abelian variety is a commutative group and a projective variety (see the book [Mum08] for these results and many further developments).

Definition 2.1.4. The *neutral component* of an algebraic group G is the connected component G^0 of G containing the neutral element.

Proposition 2.1.5. *Let G be an algebraic group.*

(i) *The variety G is smooth.*

(ii) *The (connected or irreducible) components of G are exactly the cosets gG^0, where $g \in G$.*

(iii) *The neutral component G^0 is a closed normal subgroup of G.*

(iv) *The quotient group G/G^0 is finite.*

Proof. (i) Observe that G is smooth at some point g. Since the right multiplication by any $h \in G$ is an automorphism of the variety G, this variety is smooth at gh, and hence everywhere.

(ii) Let C be a connected component of G, and choose $g \in C$. Then $g^{-1}C$ is a connected component of G by the above argument. Since $e \in g^{-1}C$, we have $g^{-1}C = G^0$, i.e., $C = gG^0$.

(iii) The inverse map i is an automorphism of the variety G that sends e to e, and hence preserves G^0. Thus, for any $g \in G^0$, the coset gG^0 contains $gg^{-1} = e$. So $gG^0 = G^0$, i.e., G^0 is a subgroup of G. Also, the conjugation by every $h \in G$ is an automorphism of the algebraic group G, and hence stabilizes G^0.

(iv) follows from (ii), since every variety has finitely many components. □

In particular, every algebraic group is *equidimensional*, i.e., its components have the same dimension.

Definition 2.1.6. An *algebraic action* of an algebraic group G on a variety X is an action

$$\alpha : G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x$$

of the abstract group G on the set X, such that α is a morphism of varieties.

For any G, X and α as above, we say that X is a *G-variety*. Also, we will just write "action" for "algebraic action" if this yields no confusion.

Example 2.1.7. Let V be a finite-dimensional vector space, and $\mathbb{P}(V)$ the projective space of lines in V. For any nonzero $v \in V$, we denote by $[v] \in \mathbb{P}(V)$ the corresponding line. The projective linear group $\mathrm{PGL}(V)$ acts on $\mathbb{P}(V)$, as its full automorphism group in view of [Ha74, Ex. II.7.7.1]). We check that this action is algebraic. Choose a basis $(e_0, \ldots, e_n)$ of V; then $\mathbb{P}(V)$ is identified with $\mathbb{P}^n$ with homogeneous coordinates $x_0, \ldots, x_n$. Moreover, $\mathrm{PGL}(V)$ is identified with PGL_{n+1}, the open subset of $\mathbb{P}(M_{n+1})$ consisting of the classes $[A]$, where $A \in \mathrm{GL}_{n+1}$. Clearly, the action

$$\alpha : \mathrm{PGL}_{n+1} \times \mathbb{P}^n \longrightarrow \mathbb{P}^n, \quad ([A], [v]) \longmapsto [A \cdot v]$$

is a rational map. It suffices to show that α is defined at $([A], [e_0])$ for any $A = (a_{ij})_{0, \leq i,j \leq n} \in \mathrm{GL}_{n+1}$. For $i = 0, \ldots, n$, denote by U_i the complement of the zero locus of x_i

in $\mathbb{P}^n$; then $U_0, \ldots, U_n$ form an open affine covering of $\mathbb{P}^n$, and $[e_0] = [1 : 0 : \ldots : 0] \in U_0$. Moreover, as A is invertible, we may choose i such that $a_{i,0} \neq 0$; then $[A \cdot e_0]$ is defined and lies in U_i.

Definition 2.1.8. Given an algebraic group G and two G-varieties X, Y, we say that a morphism $f : X \to Y$ is *equivariant* if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$.

When Y is equipped with the trivial action of G (i.e., $g \cdot y = y$ for all $g \in G$ and $y \in Y$), we say that f is *G-invariant.*

Definition 2.1.9. Given an action α of an algebraic group G on a variety X, the *orbit* of a point $x \in X$ is the image of the morphism

$$\alpha_x : G \longrightarrow X, \quad g \longmapsto g \cdot x.$$

The *isotropy group* of x is the set-theoretic fibre of the *orbit map* α_x at e.

For any G, X and x as above, we denote the orbit of x by $G \cdot x$, and the isotropy group by

$$G_x = \{g \in G \mid g \cdot x = x\}.$$

Clearly, G_x is a closed subgroup of G.

Proposition 2.1.10. *Let X be a G-variety, and $x \in X$.*

(i) *The orbit $G \cdot x$ is a locally closed, smooth subvariety of X.*

(ii) *The components of $G \cdot x$ are exactly the orbits of the neutral component G^0; their dimension equals* $\dim(G) - \dim(G_x)$.

(iii) *The closure $\overline{G \cdot x}$ is the union of $G \cdot x$ and of orbits of smaller dimension.*

(iv) *Every orbit of minimal dimension is closed. In particular, X contains a closed orbit.*

Proof. (i) Since $G \cdot x$ is the image of the orbit map α_x, it is a constructible subset of X, and hence contains a nonempty open subset U of $\overline{G \cdot x}$ (see [Ha74, Ex. II.3.19]). Then $G \cdot x$ is the union of the translates $g \cdot U$, where $g \in G$. Thus, $G \cdot x$ is open in $\overline{G \cdot x}$. One may check similarly that $G \cdot x$ is smooth.

(ii) We first consider the case where G is connected. Then G is irreducible, and hence so is $G \cdot x$. For any $g \in G$, the set-theoretic fibre of α_x at $g \cdot x$ is the isotropy group $G_{g \cdot x} = gG_xg^{-1}$, which is equidimensional of dimension $\dim(G_x)$. Thus, $\dim(G \cdot x) = \dim(G) - \dim(G_x)$ by the theorem on the dimension of fibres of a morphism (see [Ha74, Ex. II.3.22]).

In the general case, $G \cdot x$ is a finite disjoint union of its G^0-orbits, which are locally closed by (i). Moreover, since $(G_y)^0 \subseteq (G^0)_y \subseteq G_y$ for any $y \in G \cdot x$, all these subgroups have the same dimension, $\dim(G_x)$. Thus, all the G^0-orbits in $G \cdot x$ have the same dimension as well. This yields the assertions.

(iii) It suffices to show that $\dim(\overline{G \cdot x} \setminus G \cdot x) < \dim(G \cdot x)$. But this holds if G is connected, since $\overline{G \cdot x}$ is irreducible in that case. In the general case, the assertion follows by using (ii).

(iv) is a direct consequence of (iii). $\square$

Corollary 2.1.11. *Let $f : G \to H$ be a homomorphism of algebraic groups, and N its kernel. Then the image $f(G)$ is a closed subgroup of H, of dimension* $\dim(G) - \dim(N)$.

Proof. Consider the action of G on H defined by $g \cdot h := f(g)h$; then $f(G)$ is the G-orbit of e_H. Clearly, the stabilizer of every $h \in H$ equals N. By Proposition 2.1.10, it follows that all orbits have the same dimension, $\dim(G) - \dim(N)$, and hence are closed. Applying this to the orbit of e_H yields the assertion. □

2.2 Representations

In this subsection, G denotes an algebraic group.

Definition 2.2.1. A (rational) *representation* of G in a finite-dimensional vector space V is a homomorphism of algebraic groups $\rho : G \to \mathrm{GL}(V)$.

Given G, V and ρ as above, we say that V is a (rational) *G-module*. Equivalently, V is equipped with an algebraic action of G, which is *linear* in the sense that the map

$$\rho(v) : V \longrightarrow V, \quad v \longmapsto g \cdot v$$

is linear for any $g \in G$.

Many notions and constructions of representation theory extend to the setting of rational representations. For example, we may define a *G-submodule* of a G-module V, as a G-stable subspace $W \subseteq V$. Also, the tensor product of any two G-modules is a G-module, and so are the symmetric powers, $\mathrm{Sym}^n(V)$, where V is a G-module and n a positive integer. The dual vector space, $V^\vee$, is a G-module as well.

Example 2.2.2. Let T be a *torus*, i.e., an algebraic group isomorphic to a product of copies of the multiplicative group $\mathbb{G}_m$. Then we may view T as the subgroup of diagonal invertible matrices in GL_r, where $r := \dim(T)$. Thus, the T-module k^r is the direct sum of the coordinate lines $\ell_1, \ldots, \ell_r$, and T acts on each ℓ_i by $t \cdot v = \chi_i(t)v$, where $\chi_1, \ldots, \chi_r$ are the diagonal coefficients.

More generally, every T-module V is the direct sum of its *weight spaces*,

$$V_\chi := \{v \in V \mid t \cdot v = \chi(t)v \quad \forall\, t \in T\},$$

where χ runs over the homomorphisms of algebraic groups $\chi : T \to \mathbb{G}_m$; these are called *characters* or *weights* (see [Bo91, Prop. 8.2]).

When $T = \mathbb{G}_m$, the characters are just the power maps $t \mapsto t^n$, where $n \in \mathbb{Z}$; this yields a decomposition $V = \bigoplus_{n\in\mathbb{Z}} V_n$. For an arbitrary torus $T \cong \mathbb{G}_m^r$, the characters are exactly the Laurent monomials

$$(t_1, \ldots, t_r) \longmapsto t_1^{n_1} \cdots t_r^{n_r},$$

where $(n_1, \ldots, n_r) \in \mathbb{Z}^r$; this identifies the character group of T (relative to pointwise multiplication) with the free abelian group $\mathbb{Z}^r$, where $r = \dim(T)$.

Definition 2.2.3. A vector space V (not necessarily of finite dimension) is a *G-module*, if V is equipped with a linear action of the abstract group G such that every $v \in V$ is contained in some finite-dimensional G-stable subspace on which G acts algebraically.

For instance, given a finite-dimensional G-module V, the symmetric algebra

$$\mathrm{Sym}(V) = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(V)$$

is a G-module. Note that $\mathrm{Sym}(V) \cong \mathcal{O}(V^\vee)$, where $V^\vee$ denotes the dual vector space of V. This isomorphism is equivariant for the above action of G on $\mathrm{Sym}(V)$, and the action on $\mathcal{O}(V^\vee)$ via

$$(g \cdot f)(\ell) := f(g^{-1} \cdot \ell)$$

for any $g \in G$, $f \in \mathcal{O}(V^\vee)$ and $\ell \in V^\vee$. More generally, we have the following:

Proposition 2.2.4. *Let X be a G-variety, and consider the linear action of G on $\mathcal{O}(X)$ via $(g \cdot f)(x) := f(g^{-1} \cdot x)$ for all $g \in G$, $f \in \mathcal{O}(X)$ and $x \in X$. Then $\mathcal{O}(X)$ is a G-module.*

Proof. Let $f \in \mathcal{O}(X)$ and consider the composite map $f \circ \alpha \in \mathcal{O}(G \times X)$. By [Bo91, AG.12.4], the map

$$\mathcal{O}(G) \otimes \mathcal{O}(X) \longrightarrow \mathcal{O}(G \times X), \quad \varphi \otimes \psi \longmapsto ((g, x) \mapsto \varphi(g)\psi(x))$$

is an isomorphism. Thus, we have

$$f(g \cdot x) = \sum_{i=1}^{n} \varphi_i(g)\psi_i(x)$$

for some $\varphi_1, \ldots, \varphi_n \in \mathcal{O}(G)$ and $\psi_1, \ldots, \psi_n \in \mathcal{O}(X)$. Equivalently,

$$g \cdot f = \sum_{i=1}^{n} \varphi_i(g^{-1})\psi_i.$$

So the translates $g \cdot f$, where $g \in G$, span a finite-dimensional subspace $V = V(f) \subseteq \mathcal{O}(X)$, which is obviously G-stable. To show that G acts algebraically on V, it suffices to check that the map $g \mapsto \ell(g \cdot v)$ lies in $\mathcal{O}(G)$ for any linear form ℓ on V and any $v \in V$. We may assume that $v = h \cdot f$ for some $h \in G$. Extend ℓ to a linear form on $\mathcal{O}(X)$, also denoted by ℓ for simplicity. Then the map

$$g \longmapsto \ell(g \cdot v) = \ell(g \cdot (h \cdot f)) = \ell(gh \cdot f) = \sum_{i=1}^{n} \varphi_i(h^{-1}g^{-1})\ell(\psi_i)$$

is indeed a regular function on G. □

Proposition 2.2.5. *Let X be an affine G-variety. Then there exists a closed immersion $\iota : X \to V$, where V is a finite-dimensional G-module and ι is G-equivariant.*

Proof. We may choose a finite-dimensional subspace $V \subseteq \mathcal{O}(X)$ which generates that algebra. By Proposition 2.2.4, V is contained in some finite-dimensional G-submodule $W \subseteq \mathcal{O}(X)$. Thus, the algebra $\mathcal{O}(X)$ is G-equivariantly isomorphic to the quotient of the symmetric algebra $\mathrm{Sym}(W)$ by a G-stable ideal I. Since $\mathrm{Sym}(W) \cong \mathcal{O}(W^\vee)$, this means that X is equivariantly isomorphic to the closed G-stable subvariety of $W^\vee$ that corresponds to the ideal I. □

Corollary 2.2.6. *Every affine algebraic group is linear.*

Proof. Let G be an affine algebraic group and consider the action of G on itself by left multiplication. In view of Proposition 2.2.5, we may view G as a closed G-stable subvariety of some finite-dimensional G-module V. In other words, there exists $v \in V$ such that the orbit map $\rho_v : G \to V$ is a closed immersion, where $\rho : G \to \mathrm{GL}(V)$ denotes the representation of G in V. Then ρ is a closed immersion as well, since the algebra $\mathcal{O}(G)$ is generated by the pull-backs of regular functions on V, and hence by the maps

$$g \longmapsto \ell(g \cdot v) = \ell(\rho(g)(v)),$$

where $\ell \in V^\vee$ (these are the matrix coefficients of ρ). □

2.3 Principal bundles, homogeneous spaces

Let α be an action of an algebraic group G on a variety X. Consider the *graph* of α, i.e., the map

$$\Gamma_\alpha : G \times X \longrightarrow X \times X, \quad (g, x) \longmapsto (x, g \cdot x).$$

This is a morphism of varieties over X, where $G \times X$ is sent to X via the second projection, and $X \times X$ via the first projection; moreover, the induced morphism over any $x \in X$ is the orbit map α_x.

Next, let $f : X \to Y$ be a G-invariant morphism, where Y is a variety. Then the image of Γ_α is contained in the fibred product $X \times_Y X$; moreover, equality holds (as sets) if and only if the (set-theoretic) fibres of f are exactly the G-orbits in X. Also, note that Γ_α induces a bijection $G \times X \to X \times_Y X$ if and only if the abstract group G acts freely on X with quotient map f. This motivates the following:

Definition 2.3.1. Let X be a G-variety, and $f : X \to Y$ a G-invariant morphism. We say that f is a *(principal) G-bundle* (or *G-torsor*) *over* Y if it satisfies the following conditions:

(i) f is faithfully flat.

(ii) The map

$$\Gamma : G \times X \longrightarrow X \times_Y X, \quad (g, x) \longmapsto (x, g \cdot x)$$

is an isomorphism.

Remark 2.3.2. Condition (i) just means that f is flat and surjective; it implies that the morphism f is open.

Condition (ii) is equivalent to the square

(1)
$$\begin{array}{ccc} G \times X & \xrightarrow{p_2} & X \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

being cartesian, where p_2 denotes the second projection. Note that all the maps in this square are faithfully flat: this clearly holds for p_2, and hence for α, since it is identified with p_2 via the automorphism of $G \times X$ given by $(g, x) \mapsto (g, g \cdot x)$.

We now obtain some basic properties of principal bundles:

Proposition 2.3.3. *Let $f : X \to Y$ be a G-bundle.*

(i) *The morphism f is smooth.*

(ii) *The map $f^{\#} : \mathcal{O}_Y \to f_*(\mathcal{O}_X)^G$ is an isomorphism, where the right-hand side denotes the subsheaf of G-invariants in $f_*(\mathcal{O}_X)$.*

(iii) *The morphism f is affine if and only if G is linear.*

Proof. (i) Since f is assumed to be flat, it suffices to show that its scheme-theoretic fibres are equidimensional and smooth (see [Ha74, Thm. III.10.2]). But in view of the cartesian square (1), these fibres are isomorphic to those of the projection $p_2 : G \times X \to X$. This yields the assertion, since G is smooth and equidimensional

(ii) As f is faithfully flat, it suffices to show that the induced map

$$u : \mathcal{O}_X = f^*(\mathcal{O}_Y) \to f^* f_*(\mathcal{O}_X)^G$$

is an isomorphism. By [Ha74, Prop. III.9.3], we have a natural isomorphism

$$f^* f_*(\mathcal{F}) \xrightarrow{\cong} p_{2*}\alpha^*(\mathcal{F}) \tag{2}$$

for any quasi-coherent sheaf $\mathcal{F}$ on X. This yields a natural isomorphism $f^*(f_*(\mathcal{O}_X)) \cong p_{2*}(\mathcal{O}_{G\times X})$. Moreover, we have for any open subset U of X:

$$\Gamma(U, p_{2*}(\mathcal{O}_{G\times X})) = \Gamma(G \times U, \mathcal{O}_{G\times X}) = \mathcal{O}(G) \otimes \mathcal{O}(U)$$

in view of [Bo91, AG.12.4]. Thus, $p_{2*}(\mathcal{O}_{G\times X}) = \mathcal{O}(G) \otimes \mathcal{O}_X$. So we obtain an isomorphism

$$f^* f_*(\mathcal{O}_X) \xrightarrow{\cong} \mathcal{O}(G) \otimes \mathcal{O}_X,$$

equivariant for the G-action on $\mathcal{O}(G) \otimes \mathcal{O}_X$ via left multiplication on $\mathcal{O}(G)$. Thus, taking G-invariants yields an isomorphism

$$v : f^* f_*(\mathcal{O}_X)^G \xrightarrow{\cong} \mathcal{O}_X,$$

and one may check that v is the inverse of u.

(iii) If the morphism f is affine, then its fibres are affine as well. Thus, G is affine, and hence linear in view of Corollary 2.2.6.

Conversely, assume that G is linear and Y is affine; we then show that X is affine. By (the proof of) [Ha74, Thm. III.3.7], it suffices to check that the functor of global sections $\Gamma(X, -)$ is exact on the category of quasi-coherent sheaves on X. Note that $\Gamma(X, \mathcal{F}) = \Gamma(Y, f_*(\mathcal{F}))$ for any such sheaf $\mathcal{F}$; also, $\Gamma(Y, -)$ is exact, since Y is affine. Thus, it suffices to show that f_* is exact. For this, we use again the isomorphism (2); note that α^* is exact (as α is flat), and p_{2*} is exact (as G is affine). Thus, $f^* f_*$ is exact. Since f is faithfully flat, this yields the exactness of f_*. □

Remarks 2.3.4. (i) The three statements of Proposition 2.3.3 are samples of permanence properties of morphisms under faithfully flat descent. Such properties are systematically investigated in [EGAIV, §2]; we will freely use some of its results in the sequel.

(ii) By Proposition 2.3.3, every G-bundle $f : X \to Y$ is a quotient morphism in the sense of [Bo91, 6.1]; equivalently, f is a geometric quotient in the sense of [MFK94, Def. 0.6]. It follows that f is a *categorical quotient*, i.e., for any G-invariant morphism $\varphi : X \to Z$, where Z is a variety, there exists a unique morphism $\psi : Y \to Z$ such that $\varphi = \psi \circ f$ (see [Bo91, Lem. 6.2] or [MFK94, Prop. 0.1]).

Definition 2.3.5. A *morphism* from a G-bundle $f' : X' \to Y'$ to a G-bundle $f : X \to Y$ consists of a G-equivariant morphism $\varphi : X' \to X$ and a morphism $\psi : Y' \to Y$ such that the square

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\psi} & Y \end{array}$$

is cartesian.

Remarks 2.3.6. (i) Given a variety Y, the projection $p_2 : G \times Y \to Y$ is a G-bundle, where G acts on $G \times Y$ via its action on G by left multiplication. Such a G-bundle is called *trivial*.

One easily checks that the endomorphisms of the trivial G-bundle are exactly the maps of the form $(g, y) \mapsto (gf(y), y)$, where $f : Y \to G$ is a morphism. As a consequence, every such endomorphism is an automorphism.

(ii) For any G-bundle $f : X \to Y$ and any morphism $\psi : Y' \to Y$, the induced morphism $\varphi : X \times_Y Y' \to Y'$ is a G-bundle as well, where G acts on $X \times_Y Y'$ via its action on X. This defines the *pull-back* of a G-bundle.

(iii) Given a G-variety X, a G-invariant morphism $f : X \to Y$ is a G-bundle if and only if there exists a faithfully flat morphism $\psi : Y' \to Y$ such that the pull-back $X \times_Y Y' \to Y'$ is trivial. Indeed, this follows from the permanence property of isomorphisms under faithfully flat descent (see [EGAIV, Prop. 2.7.1]).

(iv) Every morphism of G-bundles over the same variety Y is an isomorphism. Indeed, this holds for trivial bundles by (i). As any two G-bundles can be trivialized by a common pull-back, the general case follows by using the above permanence property again.

(v) A G-bundle $f : X \to Y$ is called *locally trivial*, if Y admits an open covering V_i $(i \in I)$ such that the pull-back bundle $f^{-1}(V_i) \to V_i$ is trivial for any $i \in I$.

For example, the nth power map

$$f_n : \mathbb{G}_m \longrightarrow \mathbb{G}_m, \quad t \longmapsto t^n$$

is a μ_n-bundle, where $\mu_n \subset \mathbb{G}_m$ denotes the subgroup of nth roots of unity (here n is a positive integer, not divisible by $\mathrm{char}(k)$). Also, f_n is not locally trivial when $n \geq 2$, as every open subset of $\mathbb{G}_m$ is connected.

Definition 2.3.7. Let $f : X \to Y$ be a G-bundle, and Z a G-variety. The *associated fibre bundle* is a variety W equipped with a G-invariant morphism $\varphi : X \times Z \to W$, where G

acts on $X \times Z$ via $g \cdot (x, z) := (g \cdot x, g \cdot z)$, and with a morphism $\psi : W \to Y$, such that the square

$$\begin{array}{ccc} X \times Z & \xrightarrow{p_1} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{\psi} & Y \end{array}$$

is cartesian.

With the above notation, φ is a G-bundle: the pull-back of f by ψ. Thus, the triple (W, φ, ψ) is uniquely determined by the G-bundle $f : X \to Y$ and the G-variety Z. Also, ψ is faithfully flat and its fibres are isomorphic to Z, since these assertions hold after pull-back by the faithfully flat morphism f. We will denote W by $X \times^G Z$.

The associated fibre bundle need not exist in general, as we will see in Example 5.3.5. But it exists when f is trivial (just take $W = Y \times Z$), and hence when f is locally trivial (by a glueing argument). We will obtain other sufficient conditions for the existence of the associated fiber bundle in Corollaries 3.3.3 and 5.3.4.

Next, let G be a linear algebraic group, and $H \subseteq G$ a closed subgroup; consider the action of H on G by right multiplication. Then there exists a quotient morphism

$$f : G \longrightarrow G/H,$$

where G/H is a smooth quasi-projective variety. The *homogeneous space* G/H is equipped with a transitive G-action such that f is equivariant, and with a base point $\xi := f(e)$ having isotropy group H. When the subgroup H is normal in G, there is a unique structure of algebraic group on G/H such that f is a homomorphism with kernel H (see [Bo91, Thm. 6.8] for these results).

Returning to an arbitrary closed subgroup H of G, the differential of f at e lies in an exact sequence

$$0 \longrightarrow T_e(H) \longrightarrow T_e(G) \xrightarrow{df_e} T_\xi(G/H) \longrightarrow 0. \tag{3}$$

Moreover, f may be realized as the orbit map $G \mapsto G \cdot x$ for some point $x \in \mathbb{P}(V)$, where V is a finite-dimensional G-module (see again [Bo91, Thm. 6.8], and its proof).

Proposition 2.3.8. *With the above notation and assumptions, f is an H-bundle.*

Proof. The morphism f is surjective by construction. Also, by generic flatness, f is flat over some nonempty open subset $V \subseteq G/H$. As G acts transitively on G/H, the translates $g \cdot V$ ($g \in G$) cover G/H. Since f is equivariant, it must be flat everywhere. In view of the exact sequence (3) together with [Ha74, Prop. III.10.4], the map f is smooth at e, and hence everywhere by the above argument.

Next, consider the fibred product $G \times_{G/H} G$: it is equipped with two projections p_1, p_2 to G, which are smooth as f is smooth and the square (1) is cartesian. It follows that $G \times_{G/H} G$ is a smooth variety. Also, note that the map

$$\Gamma : G \times H \longrightarrow G \times_{G/H} G, \quad (x, y) \longmapsto (x, xy^{-1})$$

is bijective and $G \times H$-equivariant, where $G \times H$ acts on itself via $(g, h) \cdot (x, y) := (gx, yh^{-1})$, and on $G \times_{G/H} G$ via $(g, h) \cdot (x, y) := (gx, gyh^{-1})$. Moreover, the latter action is transitive, and the isotropy group of $(e,e) \in G \times_{G/H} G$ is trivial. By [Bo91, Prop. 6.7], to show that Γ is an isomorphism, it suffices to check that its differential at (e,e) is an isomorphism. But $d\Gamma_{(e,e)}$ may be identified with the map

$$T_e(G) \times T_e(H) \longrightarrow T_e(G) \times_{T_\xi(G/H)} T_e(G), \quad (x,y) \longmapsto (x, x-y),$$

which is indeed an isomorphism in view of the exact sequence (3). □

Some further developments.

The notions and results of §2.1 and §2.2 are classical. They can be found e.g. in [Bo91, §1], in the more general setting of algebraic groups over an arbitrary field. Far-reaching generalizations to group schemes are developed in [DG70] and [SGA3].

The notion of torsor presented in Definition 2.3.1 extends to the setting of schemes in several ways, depending on the choice of a Grothendieck topology (see [SGA3, IV.6.1], [BLR90, 8.1]). More specifically, the notions of torsor for the fpqc and fppf topology give back ours in the setting of varieties; here fpqc stands for faithfully flat and quasi-compact, and fppf for faithfully flat of finite presentation. Further commonly used notions are those of torsors for the étale topology, and for the finite étale topology; the latter are called *locally isotrivial*.

A G-torsor $f : X \to Y$ is locally isotrivial if and only if there exist an open covering $(V_i)_{i \in I}$ of Y and finite étale morphisms $\psi_i : V_i' \to V_i$ such that the pull-back torsors $X \times_Y V_i' \to V_i'$ are all trivial. This definition, due to Serre in [Se58], predates the introduction of the above topologies. Every fpqc torsor under a linear algebraic group G is locally isotrivial in view of [Gr59, p. 29] (see also [Ray70, Lem. XIV.1.4]). But some fppf torsors under abelian varieties are not locally isotrivial, see [Ray70, Chap. XIII].

An algebraic group G is called *special*, if every locally isotrivial G-torsor is locally trivial for the Zariski topology. This notion is also due to Serre in [Se58]; he showed in particular that every algebraic group obtained from $\mathbb{G}_a$, GL_n, SL_n and Sp_{2n} by iterated extensions is special. (For instance, $\mathbb{G}_m = \mathrm{GL}_1$ is special; this will be checked directly in Proposition 3.1.3). A full description of special groups was obtained a little later by Grothendieck in [Gr58]; in particular, the special semi-simple groups are exactly the products of special linear and symplectic groups.

The notion of special group makes sense, more generally, for algebraic groups over an arbitrary field. Describing all special groups is an open question in that setting; see [Hu16] for a characterization of special semi-simple groups.

Given an algebraic group G, possibly nonlinear, and a closed subgroup H, the homogeneous space G/H is equipped with a unique structure of variety such that the canonical map $f : G \to G/H$ is a quotient morphism. This is proved in [Ch05, 8.5 Thm. 4] when G is connected, and in [SGA3, VIA Thm. 3.2] in a much more setting (flat group schemes, locally of finite type over a local artinian ring). The approach in both of these references is more indirect than for linear groups, as G/H may not be constructed as an orbit in the projective space of a G-module (see Example 3.2.2 (iv)). One shows like in Proposition 2.3.8 that the above map f is an H-bundle. Moreover, G/H is clearly smooth, and quasi-projective in view of [Ray70, Cor. VI.2.5].

3 Line bundles over G-varieties

3.1 Line bundles, invertible sheaves, and principal $\mathbb{G}_m$-bundles

Definition 3.1.1. A *line bundle* over a variey X is a variety L equipped with a morphism $\pi : L \to X$ such that X admits an open covering $(U_i)_{i\in I}$ of X satisfying the following conditions:

(i) For any $i \in I$, there exists an isomorphism $\psi_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{A}^1$.

(ii) For any $i, j \in I$, the isomorphism

$$\theta_{ij} := \psi_j \circ \psi_i^{-1} : (U_i \cap U_j) \times \mathbb{A}^1 \xrightarrow{\cong} (U_i \cap U_j) \times \mathbb{A}^1$$

is of the form $(x, z) \mapsto (x, a_{ij}(x)z)$ for some $a_{ij} \in \mathcal{O}(U_i \cap U_j)^*$.

With the above notation, L is obtained by glueing the trivial line bundles $p_1 : U_i \times \mathbb{A}^1 \to U_i$ via the linear *transition functions* a_{ij}. Thus, each fibre L_x is a line, in the sense that it has a canonical structure of 1-dimensional vector space.

Given a line bundle $\pi : L \to X$ and a morphism $\varphi : X' \to X$, the *pull-back* $\varphi^*(L)$ is the fibred product $L \times_X X'$ equipped with its projection to X'. This is a line bundle over X', obtained by glueing the trivial line bundles $f^{-1}(U_i) \times \mathbb{A}^1 \to f^{-1}(U_i)$ via the transition functions $a_{ij} \circ f$.

A *morphism* from another line bundle $\pi' : L' \to X'$ to $\pi : L \to X$ is given by a cartesian square

$$\begin{array}{ccc} L' & \xrightarrow{\psi} & L \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{\varphi} & X, \end{array}$$

i.e., by an isomorphism $L' \xrightarrow{\cong} \varphi^*(L)$ of line bundles over X'.

The automorphisms of the trivial line bundle over X are exactly the maps of the form $(x, z) \mapsto (x, f(x)z)$, where $f \in \mathcal{O}(X)^*$. This identifies the automorphism group of the trivial bundle with the unit group $\mathcal{O}(X)^*$. In fact, the same holds for the automorphism group of any line bundle L, by using the local trivializations ψ_i.

Any two line bundles L, M over X can be trivialized over a common open covering $\mathcal{U} = (U_i)_{i\in I}$. We may thus define the *tensor product* $L \otimes M$ by glueing the trivial line bundles over U_i via the transition functions $a_{ij}\, b_{ij}$, where a_{ij} (resp. b_{ij}) denote the transition functions of L (resp. M). Likewise, the *dual line bundle*, $L^\vee$, is defined by the transition functions a_{ij}^{-1}. One may check that $L \otimes M$ and $L^\vee$ are independent of the choice of the trivializing open covering $\mathcal{U}$. Moreover, the tensor product yields an abelian group structure on the set of isomorphism classes of line bundles over X, with neutral element the trivial bundle, and inverse the dual. This abelian groups is called the *Picard group* of X, and denoted by $\mathrm{Pic}(X)$. Every morphism $f : X' \to X$ defines a pull-back homomorphism, $f^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X')$.

Definition 3.1.2. A *section* of a line bundle L is just a section of its structure morphism π, i.e., a morphism $s : X \to L$ such that $\pi \circ s = \mathrm{id}$.

The sections of the trivial line bundle over X are exactly the maps $(x, z) \mapsto (x, f(x)z)$, where $f \in \mathcal{O}(X)$. More generally, the trivializations ψ_i identify the sections of L with the families $(f_i)_{i \in I}$, where the $f_i \in \mathcal{O}(U_i)$ satisfy $f_i = a_{ij} f_j$ on $U_i \cap U_j$ for all $i, j \in I$. In particular, the sections of L form an $\mathcal{O}(X)$-module, denoted by $\Gamma(X, L)$.

Likewise, the *local sections* of L, i.e., the sections over open subsets of X, define a sheaf of $\mathcal{O}_X$-modules that we will denote by $\mathcal{L}$. Clearly, the sheaf of local sections of the trivial line bundle is just the structure sheaf, $\mathcal{O}_X$. As a consequence, $\mathcal{L}$ is an invertible sheaf of $\mathcal{O}_X$-modules. Moreover, the assignement $L \mapsto \mathcal{L}$ yields a bijective correspondence between isomorphism classes of line bundles over X and isomorphism classes of invertible sheaves; this correspondence is compatible with pull-backs, tensor products, and duals. This identifies the Picard group of X with that defined in [Ha74, II.6] via invertible sheaves. Also, note that $\Gamma(X, L) = \Gamma(X, \mathcal{L})$ with the above notation; every morphism $f : X' \to X$ yields a pull-back morphism $f^* : \Gamma(X, L) \to \Gamma(X', f^*(L))$.

Next, observe that every line bundle $\pi : L \to X$ is equipped with an action of $\mathbb{G}_m$ by scalar multiplication on fibres. The fixed locus L_0 of this action is exactly the image of the zero section; thus, $L_0 \cong X$ and the complement,

$$L^\times := L \setminus L_0,$$

is an open $\mathbb{G}_m$-stable subset of L. Clearly, π is invariant, and restricts to a $\mathbb{G}_m$-bundle

$$\pi^\times : L^\times \longrightarrow X,$$

which pulls back to the trivial $\mathbb{G}_m$-bundle on each trivializing open subset U_i.

The action of $\mathbb{G}_m$ on L yields an action on the sheaf of algebras $\pi_*(\mathcal{O}_L)$ on X; we have an isomorphism of such sheaves

$$\pi_*(\mathcal{O}_L) \cong \bigoplus_{n=0}^{\infty} \mathcal{L}^{\otimes(-n)}, \tag{4}$$

where the algebra structure on the right-hand side arises from the isomorphisms

$$\mathcal{L}^{\otimes r} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes s} \xrightarrow{\cong} \mathcal{L}^{\otimes(r+s)}, \tag{5}$$

and each $\mathcal{L}^{\otimes r}$ is the subsheaf of weight r for the $\mathbb{G}_m$-action. In other words, the local functions on L that are homogeneous of degree n on fibres are exactly the local sections of $L^{\otimes(-n)}$. Replacing L with its dual and taking global sections, we obtain an isomorphism of graded algebras

$$\mathcal{O}(L^{-1}) \cong \bigoplus_{n=0}^{\infty} \Gamma(X, L^{\otimes n}).$$

Likewise, we obtain isomorphisms

$$\pi^\times_*(\mathcal{O}_{L^\times}) \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^{\otimes n}, \quad \mathcal{O}(L^\times) \cong \bigoplus_{n \in \mathbb{Z}} \Gamma(X, L^{\otimes n}).$$

Proposition 3.1.3. *Let $f : X \to Y$ be a $\mathbb{G}_m$-bundle and consider the linear action of $\mathbb{G}_m$ on $\mathbb{A}^1$ with weight -1. Then the associated bundle $\pi : X \times^{\mathbb{G}_m} \mathbb{A}^1 \to Y$ exists. Moreover, f is isomorphic to the $\mathbb{G}_m$-bundle $\pi^\times : L^\times \to Y$. In particular, f is locally trivial.*

Proof. The sheaf of $\mathcal{O}_Y$-algebras $\mathcal{A} := f_*(\mathcal{O}_X)$ is equipped with an action of $\mathbb{G}_m$, and hence with a grading

$$\mathcal{A} \cong \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n,$$

where $\mathcal{A}_n$ denotes the subsheaf of weight n; then each $\mathcal{A}_n$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules. For the trivial bundle $p_2 : G\times Y \to Y$, we have $\mathcal{A} = p_{2*}(\mathcal{O}_{G\times Y}) \cong \mathcal{O}(G)\otimes\mathcal{O}_Y \cong \mathcal{O}_G[t,t^{-1}]$, as follows from [Bo91, AG.12.4]; as a consequence, $\mathcal{A}_n \cong \mathcal{O}_G\, t^n$ for all n. Since f becomes trivial after some faithfully flat pull-back, it follows that each $\mathcal{A}_n$ is invertible, and the multiplication map $\mathcal{A}_r \otimes_{\mathcal{O}_Y} \mathcal{A}_s \to \mathcal{A}_{r+s}$ is an isomorphism for all r, s. Let $\mathcal{L} := \mathcal{A}_1$, then we obtain an isomorphism of sheaves of graded $\mathcal{O}_Y$-algebras

$$\mathcal{A} \cong \bigoplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n},$$

where the multiplication in the right-hand side is given by the maps (5). Since the morphism f is affine (Proposition 2.3.3 (i)), it follows that the $\mathbb{G}_m$-bundle f is isomorphic to $\pi^\times : L^\times \to Y$, where L denotes the line bundle on Y that corresponds to the invertible sheaf $\mathcal{L}$. The isomorphism of line bundles $L \cong X \times^{\mathbb{G}_m} \mathbb{A}^1$ is checked similarly, by using the isomorphism (4). □

This proposition and the preceding discussion yield bijective correspondences between line bundles, invertible sheaves, and principal $\mathbb{G}_m$-bundles over a variety X.

Example 3.1.4. As in Example 2.1.7, we consider a vector space V of finite dimension $n \geq 2$, and denote by $\mathbb{P}(V)$ the projective space of lines in V.

Let $L \subset \mathbb{P}(V) \times V$ be the incidence variety, consisting of those pairs (x, v) such that v lies on the line x, and let $\pi : L \to \mathbb{P}(V)$ denote the restriction of the first projection $p_1 : \mathbb{P}(V) \times V \to \mathbb{P}(V)$. By using an affine open covering of $\mathbb{P}(V)$ as in Example 2.1.7 again, one easily checks that π is a line bundle: the *tautological line bundle on* $\mathbb{P}(V)$, denoted by $O_{\mathbb{P}(V)}(-1)$. Every line bundle on $\mathbb{P}(V)$ is isomorphic to the tensor power $L^{\otimes n} =: O_{\mathbb{P}(V)}(n)$, for a unique $n \in \mathbb{Z}$. The invertible sheaf on $\mathbb{P}(V)$ that corresponds to $O_{\mathbb{P}(V)}(n)$ is the sheaf $\mathcal{O}_{\mathbb{P}(V)}(n)$ defined in [Ha74, II.5].

The second projection $p_2 : \mathbb{P}(V) \times V \to V$ restricts to a proper morphism $f : L \to V$ that sends the zero section to the origin, and restricts to an isomorphism $L^\times \cong V \setminus \{0\}$ of varieties over $\mathbb{P}(V) \cong (V \setminus \{0\})/\mathbb{G}_m$. Thus, we have $f_*(\mathcal{O}_L) = \mathcal{O}_V$, and hence $\mathcal{O}(L) = \mathcal{O}(V) \cong \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(V^\vee)$. This yields isomorphisms

$$\Gamma(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(n)) \cong \mathrm{Sym}^n(V^\vee)$$

for all $n \geq 0$; also, $\Gamma(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(n)) = 0$ for all $n < 0$.

Next, consider a variety X and a morphism $f : X \to \mathbb{P}(V)$. Then we obtain a line bundle $L := f^* O_{\mathbb{P}(V)}(1)$ on X, and a finite-dimensional subspace $W \subseteq \Gamma(X, L)$ (the image of $V^\vee = \Gamma(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$) under the pull-back map f^*) such that W *generates* L in the sense that the sections of elements of W have no common zero; equivalently, W generates the associated invertible sheaf $\mathcal{L}$ in the sense of [Ha74, II.7].

Conversely, given a line bundle L on X, generated by a finite-dimensional subspace $W \subseteq \Gamma(X, L)$, we obtain a morphism

$$f = f_{L,W} : X \longrightarrow \mathbb{P}(W^\vee)$$

such that $L = f^*\mathcal{O}_{\mathbb{P}(W^\vee)}(1)$ and this identifies the inclusion $W \to \Gamma(X, L)$ with the pull-back map f^* on global sections (see [Ha74, Thm. II.7.1]). The map f sends every point $x \in X$ to the hyperplane of W consisting of those sections of L that vanish at x.

The two constructions above are mutually inverse by [Ha74, Thm. II.7.1] again. Also, note that given nested subspaces $V \subseteq W \subseteq \Gamma(X, L)$ such that V generates L, the map f_V factors as f_W followed by the linear projection $\mathbb{P}(W^\vee) \dashrightarrow \mathbb{P}(V^\vee)$ (a rational map, defined on the image $f_W(X)$).

3.2 G-quasi-projective varieties and G-linearized line bundles

In this subsection, we fix an algebraic group G.

Definition 3.2.1. We say that a G-variety X is G-*quasi-projective* if there exists a (locally closed) immersion $\iota : X \to \mathbb{P}(V)$, where V is a finite-dimensional G-module and ι is G-equivariant.

Examples 3.2.2. (i) Every affine G-variety X is G-quasi-projective: indeed, X admits an equivariant immersion into some G-module V (Proposition 2.2.5), and hence into the projectivization $\mathbb{P}(V \oplus k)$, where G acts trivially on k.

(ii) When G is linear and $H \subset G$ is a closed subgroup, the homogeneous space G/H is G-quasi-projective (see [Bo91, Thm. 6.8] and its proof).

(iii) Let C be an elliptic curve, acting on itself by translation. Then C is not C-quasi-projective: otherwise, we obtain a nonconstant homomorphism $C \to \mathrm{GL}(V)$, a contradiction as C is complete and irreducible, and $\mathrm{GL}(V)$ is affine.

(iv) More generally, if G acts faithfully on a G-quasi-projective variety X, then G has a faithful projective representation, and hence is linear.

Given a finite-dimensional G-module V, the action of G on $\mathbb{P}(V)$ lifts to an action on the tautological line bundle $L := O_{\mathbb{P}(V)}(-1)$. Moreover, G acts linearly on the fibres of L, i.e., every $g \in G$ induces a linear map $L_x \to L_{g \cdot x}$ for any $x \in \mathbb{P}(V)$. This motivates the following:

Definition 3.2.3. Let X be a G-variety, and $\pi : L \to X$ a line bundle. A G-*linearization* of L is an action of G on the variety L such that π is equivariant and the action on fibres is linear.

One may easily check that a G-linearization of L is an action of G on that variety, which commutes with the $\mathbb{G}_m$-action by multiplication on fibres.

We will see in Proposition 3.2.6 that the G-quasi-projective varieties are exactly those admitting an ample G-linearized line bundle. For this, we need some preliminary results. We first show that the above notion of linearization is equivalent to that introduced by Mumford in terms of cocycles, see [MFK94, Def. 1.6].

Denote as usual by $\alpha : G \times X \to X$ the action, and by $p_2 : G \times X \to X$ the projection. Every point $g \in G$ yields a map

$$g \times \mathrm{id} : X \longrightarrow G \times X, \quad x \longmapsto (g, x),$$

which satisfies $(g \times \mathrm{id})^* \alpha^*(L) = g^*(L)$ and $(g \times \mathrm{id})^* p_2^*(L) = L$. Thus, every morphism of line bundles $\Phi : \alpha^*(L) \to p_2^*(L)$ induces morphisms $\Phi_g : g^*(L) \to L$ for all $g \in G$.

Lemma 3.2.4. *With the above notation and assumptions, there is a bijective correspondence between the G-linearizations of L and those isomorphisms*

$$\Phi : \alpha^*(L) \longrightarrow p_2^*(L)$$

of line bundles over $G \times X$ such that $\Phi_{gh} = \Phi_h \circ h^*(\Phi_g)$ for all $g, h \in G$.

Proof. Let $\beta : G \times L \to L$ be a G-linearization. Then the square

(6)
$$\begin{array}{ccc} G \times L & \xrightarrow{\beta} & L \\ {\scriptstyle \mathrm{id}\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\alpha} & X \end{array}$$

is commutative, and hence induces a morphism

$$\gamma : G \times L \longrightarrow \alpha^*(L)$$

of varieties over $G \times X$ (as $\alpha^*(L)$ is the fibred product $(G \times X) \times_X L$). Also, note that $G \times L = p_2^*(L)$. Since β is linear on fibres, so is γ; hence we obtain a morphism of line bundles $\gamma : p_2^*(L) \to \alpha^*(L)$. In fact, γ is an isomorphism, since so are the induced morphisms $\gamma_g : L \to g^*(L)$ for all $g \in G$. Moreover, the associativity property of the action β on $G \times L$ translates into the condition that $\gamma_{gh} = h^*(\gamma_g) \circ \gamma_h$ for all $g, h \in G$. Thus, $\Phi := \gamma^{-1}$ is an isomorphism satisfying the desired cocycle condition. Conversely, any such isomorphism yields a linearization by reversing the above arguments. □

By using the correspondence of Lemma 3.2.4, one may easily check that the tensor product of any two G-linearized line bundles over a G-variety X is equipped with a linearization; also, the dual of any G-linearized line bundle over X is G-linearized as well. Thus, the isomorphism classes of G-linearized line bundles form an abelian group relative to the tensor product. We call this group the *equivariant Picard group*, and denote it by $\mathrm{Pic}_G(X)$. It comes with a homomorphism

(7)
$$\phi : \mathrm{Pic}_G(X) \longrightarrow \mathrm{Pic}(X)$$

that forgets the linearization.

Lemma 3.2.5. *Let X be a G-variety, and L a G-linearized line bundle over X. Then the space of global sections $\Gamma(X, L)$ has a natural structure of G-module.*

Proof. Since L^{-1} is also equipped with a G-linearization, it is a $G \times \mathbb{G}_m$-variety, where $\mathbb{G}_m$ acts by scalar multiplication on fibres. Moreover, the space of those global regular functions on L^{-1} that are eigenvectors of weight 1 for the $\mathbb{G}_m$-action is identified with $\Gamma(X, L)$. So the assertion follows from Proposition 2.2.4. □

Proposition 3.2.6. *Let X be a G-variety. Then X is G-quasi-projective if and only if it admits an ample G-linearized line bundle.*

Proof. Assume that X is G-quasi-projective and choose an equivariant immersion $\iota : X \to \mathbb{P}(V)$, where V is a finite-dimensional G-module. Then $\iota^* O_{\mathbb{P}(V)}(1)$ is an ample G-linearized line bundle over X.

Conversely, assume that X has such a bundle L. Replacing L with a positive power, we may assume that L is very ample. Thus, we may choose a finite-dimensional subspace $V \subseteq \Gamma(X, L)$ that generates L, and such that the map $f_{V,L} : X \to \mathbb{P}(V^\vee)$ is an immersion. By Lemma 3.2.5, V is contained in some finite-dimensional G-submodule $W \subseteq \Gamma(X, L)$. Then of course W generates L; moreover, the map $f_{L,W} : X \to \mathbb{P}(W^\vee)$ is G-equivariant, and factors through f_V. Hence f_W yields the desired immersion. □

Example 3.2.7. Let V be a finite-dimensional vector space, and consider the projective space $X := \mathbb{P}(V)$ equipped with the action of its full automorphism group, $\mathrm{PGL}(V)$. Then one can show that the line bundle $L := O_{\mathbb{P}(V)}(1)$ is not $\mathrm{PGL}(V)$-linearizable (see Example 4.2.4). But L has a natural $\mathrm{GL}(V)$-linearization.

We claim that $L^{\otimes n}$ is $\mathrm{PGL}(V)$-linearizable, where n denotes the dimension of V. Indeed, $L^{\otimes n} = O_{\mathbb{P}(V)}(n)$ is the pull-back of $O_{\mathbb{P}(\mathrm{Sym}^n(V))}(1)$ under the nth Segre embedding $\iota : \mathbb{P}(V) \to \mathbb{P}(\mathrm{Sym}^n(V))$; moreover, the representation of $\mathrm{GL}(V)$ in $\mathrm{Sym}^n(V)$ factors through a representation of $\mathrm{PGL}(V)$. Alternatively, one may observe that L is the canonical bundle of $\mathbb{P}(V)$ (the top exterior power of the cotangent bundle), and hence is equipped with a canonical $\mathrm{PGL}(V)$-linearization.

Example 3.2.8. Let X be the image of the morphism

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^2, \quad [s : t] \longmapsto [s^2 t : (s+t)^3 : st^2].$$

Then X is the nodal cubic curve with equation $(x+z)^3 = xyz$, where x, y, z denote the homogeneous coordinates on $\mathbb{P}^2$. Moreover, the map $\eta : \mathbb{P}^1 \to X$ induced by f is the normalization; η sends 0 and ∞ to the nodal point $P := [0 : 1 : 0]$ of X, and restricts to an isomorphism on $\mathbb{P}^1 \setminus \{0, \infty\}$. One may check that the scheme-theoretic fibre of η at P is $\{0, \infty\}$ (viewed as a reduced subscheme of $\mathbb{P}^1$). In other words, X is obtained from $\mathbb{P}^1$ by identifying 0 and ∞ (see [Se88, IV.4] for a general construction of singular curves by identifying points in smooth curves).

The action of $\mathbb{G}_m$ on $\mathbb{P}^1$ by multiplication fixes 0 and ∞, and hence yields an action on X for which P is the unique fixed point; the complement $X \setminus \{P\}$ is a $\mathbb{G}_m$-orbit. We claim that the projective curve X is not $\mathbb{G}_m$-projective. Otherwise, X is isomorphic to the orbit closure $\overline{\mathbb{G}_m \cdot x} \subseteq \mathbb{P}(V)$ for some finite-dimensional G-module V and some $x \in \mathbb{P}(V)$. By Example 2.2.2, we have a decomposition

$$V = \bigoplus_{n \in \mathbb{Z}} V_n,$$

where $t \cdot v = t^n v$ for all $t \in \mathbb{G}_m$ and $v \in V_n$, i.e., V_n is the weight subspace of V with weight n. Write accordingly $x = [v_{n_0} + \cdots + v_{n_1}]$, where $n_0 \leq n_1$ and $v_{n_0} \neq 0 \neq v_{n_1}$. Then $v_{n_0} \neq v_{n_1}$, since x is not fixed by $\mathbb{G}_m$. The orbit map

$$\mathbb{G}_m \longrightarrow \mathbb{P}(V), \quad t \longmapsto t \cdot x$$

extends to a morphism $g : \mathbb{P}^1 \to X$, which is equivariant for the above actions of $\mathbb{G}_m$. Moreover, $g(0) = [v_{n_0}]$ and $g(\infty) = [v_{n_1}]$. Thus, $[v_{n_0}]$ and $[v_{n_1}]$ are distinct $\mathbb{G}_m$-fixed points in $\overline{\mathbb{G}_m \cdot x}$, a contradiction. This proves our claim.

We will obtain another proof of that claim in Example 4.2.5, by showing that every $\mathbb{G}_m$-linearizable line bundle on X has degree zero.

Example 3.2.9. Let Y be the image of the morphism

$$g : \mathbb{P}^1 \longrightarrow \mathbb{P}^2, \quad [s : t] \longmapsto [s^3 : s^2 t : t^3].$$

Then Y is the cuspidal cubic curve with equation $y^3 = x^2 z$. Again, the map $\eta : \mathbb{P}^1 \to Y$ induced by f is the normalization; it sends ∞ to the cuspidal point $Q := [0 : 0 : 1]$ of Y, and restricts to an isomorphism on $\mathbb{P}^1 \setminus \{\infty\}$. The scheme-theoretic fibre of η at Q is $Z := \mathrm{Spec}(\mathcal{O}_{\mathbb{P}^1,\infty}/\mathfrak{m}^2)$ (a fat point of order 2). Thus, Y is obtained from $\mathbb{P}^1$ by sending Z to the reduced point Q (see again [Se88, IV.4] for this construction). The action of $\mathbb{G}_a$ on $\mathbb{P}^1$ by translation fixes ∞, and hence yields an action on Y for which Q is the unique fixed point.

If $\mathrm{char}(k) = 0$, then one can show that Y is not $\mathbb{G}_a$-projective, see Example 4.2.6. But Y is $\mathbb{G}_a$-projective if $\mathrm{char}(k) = p > 0$: indeed, when $p \geq 3$, the morphism

$$h : \mathbb{P}^1 \longrightarrow \mathbb{P}^{p-1}, \quad [s : t] \longmapsto [s^p : s^{p-2} t^2 : s^{p-3} t^3 : \cdots : t^p]$$

factors through an immersion $Y \hookrightarrow \mathbb{P}^{p-1}$. Moreover, h is equivariant for the $\mathbb{G}_a$-action on $\mathbb{P}^1$ given by $u \cdot (s,t) = (s + tu, t)$, and for the induced action on $\mathbb{P}^{p-1} \subset \mathbb{P}(k[s,t]_p) \cong \mathbb{P}^p$, where $k[s,t]_p$ denotes the space of homogeneous polynomials of degree p in s, t; the hyperplane of that space spanned by $s^p, s^{p-2}t^2, s^{p-3}t^3, \ldots, t^p$ is stable under this action. When $p = 2$, the above morphism h has degree 2; we replace it with the birational morphism

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3, \quad [s : t] \longmapsto [s^4 : s^2 t^2 : s t^3 : t^4],$$

and argue similarly.

3.3 Associated fibre bundles

In this subsection, G denotes an algebraic group, and $f : X \to Y$ a G-bundle.

Lemma 3.3.1. *The pull-back by f yields isomorphisms*

$$\mathcal{O}(Y) \xrightarrow{\cong} \mathcal{O}(X)^G, \quad \mathrm{Pic}(Y) \xrightarrow{\cong} \mathrm{Pic}_G(X).$$

Proof. The first isomorphism follows from Proposition 2.3.3 (ii).

Since f is G-invariant, the pull-back of any line bundle on Y is equipped with a G-linearization. The converse follows from the theory of faithfully flat descent, for which we refer to [BLR90, Chap. 6]. More specifically, recall that a G-linearization of a line bundle L on X is exactly an isomorphism $\alpha^*(L) \xrightarrow{\cong} p_2^*(L)$ of line bundles over $G \times X$, which satisfies a cocycle condition (Lemma 3.2.4). We now use the isomorphism $G \times X \xrightarrow{\cong} X \times_Y X$ given by the graph of the action, which identifies the two projections $p_1, p_2 : X \times_Y X \to X$ with $\alpha, p_2 : G \times X \to X$. This yields an isomorphism $p_1^*(L) \xrightarrow{\cong} p_2^*(L)$ which satisfies the assumptions of a descent data (as may be checked by arguing as in [BLR90, 6.2 Ex. B]). So the assertion follows from descent for invertible sheaves, which is in turn a consequence of [BLR90, 6.1 Thm. 4]. □

Next, we obtain a very useful descent result for G-bundles, after [MFK94, Prop. 7.1]. To state it, recall that a line bundle L on a variety Z equipped with a morphism $g : Z \to W$ is called *ample relative to g*, or just *g-ample*, if the pull-back of L to $g^{-1}(V)$ is ample for any affine open subset V of W.

Proposition 3.3.2. *Let X' be a G-variety, and $\varphi : X' \to X$ a G-equivariant morphism. Assume that X' is equipped with a φ-ample G-linearized line bundle L. Then there exists a cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\psi} & Y, \end{array}$$

where Y' is a variety, and f' a G-bundle. Moreover, there exists a ψ-ample line bundle M over Y' such that $L = f^*(M)$.

Proof. Arguing as in the proof of Lemma 3.3.1, one checks that the X-scheme X' is equipped with a descent data for the faithfully flat morphism $f : X \to Y$; moreover, the G-linearization of L yields a descent data for the associated invertible sheaf $\mathcal{L}$ on X'. Thus, [BLR90, 6.2 Thm. 7] yields the statements. □

Corollary 3.3.3. *The associated fibre bundle $\varphi : X \times^G Z \to Y$ exists for any G-quasi-projective variety Z.*

Proof. By assumption, we may choose an ample G-linearized line bundle L on Z. Then the projection $p_1 : X \times Z \to X$ and the line bundle $p_2^*(L)$ on $X \times Z$ satisfy the assumptions of Proposition 3.3.2. So the desired statement follows from that proposition. □

Corollary 3.3.3 may be applied to any *affine* G-variety in view of Example 3.2.2 (i), and hence to any finite-dimensional G-module V (viewed as an affine space). The resulting morphism

$$\varphi : X \times^G V \longrightarrow Y$$

is then a vector bundle over Y (as this holds after pull-back by the faithfully flat morphism f), called the *associated vector bundle*. In particular, given a linear algebraic group G, a closed subgroup H, and a finite-dimensional H-module V, we can form the associated vector bundle $\varphi : G \times^H V \to G/H$, also called a *homogeneous vector bundle.*

As another application of Corollary 3.3.3, we obtain a factorization property of principal bundles:

Corollary 3.3.4. *Let G be a linear algebraic group, $f : X \to Y$ a G-bundle, and $H \subset G$ a closed subgroup. Then f factors uniquely as $\psi \circ \varphi$, where $\varphi : X \to Z$ is an H-bundle, and $\psi : Z \to Y$ is smooth with fibres isomorphic to G/H. If H is a normal subgroup of G, then ψ is a G/H-bundle.*

Proof. Applying Corollary 3.3.3 to the G-variety G/H, we obtain a cartesian square

$$\begin{array}{ccc} X \times G/H & \xrightarrow{p_1} & X \\ \gamma \downarrow & & \downarrow f \\ Z & \xrightarrow{\psi} & Y, \end{array}$$

where ψ is smooth with fibres isomorphic to G/H. Moreover, the base point $\xi \in G/H$ yields a morphism

$$\varphi : X \longrightarrow Z, \quad x \longmapsto \gamma(x, \xi),$$

which is H-invariant (as γ is H-invariant), and satisfies $\psi \circ \varphi = f$ (as $\psi \circ \gamma = p_1 \circ f$). Also, the pull-back of φ by f is identified with the map

$$G \times X \longrightarrow X \times G/H, \quad (g, x) \longmapsto (x, g \cdot \xi),$$

which is an H-bundle. Since f is faithfully flat, φ is an H-bundle as well.

Next, assume that H is normal in G; then the composition $\varphi \circ \alpha : G \times X \to Z$ is invariant under the action of $H \times H$ given by $(h_1, h_2) \cdot (g, x) := (gh_1^{-1}, h_2 \cdot x)$. Also, denoting by $q : G \to G/H$ the quotient map, the product map $q \times \varphi : G \times X \to G/H \times Z$ is an $H \times H$-torsor. Thus, $\varphi \circ \alpha$ factors through a unique morphism $G/H \times Z \to Z$, which is clearly a group action such that ψ is invariant. Moreover, the pull-back of ψ under the faithfully flat morphism $f : X \to Y$ is the trivial G/H-torsor, since the pull-back of f is the trivial G-torsor. By Remark 2.3.6 (iii), it follows that ψ is a G/H-torsor. □

3.4 Invariant line bundles and lifting groups

Let X be a variety, and $\pi : L \to X$ a line bundle. We denote by $\mathrm{Aut}(X)$ the automorphism group of X, and by $\mathrm{Aut}(X)_L$ the stabilizer of L in that group:

$$\mathrm{Aut}_L(X) = \{g \in \mathrm{Aut}(X) \mid g^*(L) \cong L\}.$$

Also, we denote by $\mathrm{Aut}^{\mathbb{G}_m}(L)$ the group of automorphisms of the variety L that commute with the action of $\mathbb{G}_m$ by multiplication on fibres. These groups are related as follows:

Lemma 3.4.1. (i) *With the above notation and assumptions, every $\gamma \in \mathrm{Aut}^{\mathbb{G}_m}(L)$ induces an automorphism $g \in \mathrm{Aut}(X)$ such that the square*

$$\begin{array}{ccc} L & \xrightarrow{\gamma} & L \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{g} & X \end{array}$$

commutes; in particular, $g \in \mathrm{Aut}(X)_L$.

(ii) *The assignement*

$$\pi_* : \mathrm{Aut}^{\mathbb{G}_m}(L) \longrightarrow \mathrm{Aut}(X)_L, \quad \gamma \longmapsto g$$

yields an exact sequence of groups

$$1 \longrightarrow \mathcal{O}(X)^* \longrightarrow \mathrm{Aut}^{\mathbb{G}_m}(L) \longrightarrow \mathrm{Aut}(X)_L \longrightarrow 1. \tag{8}$$

(iii) *For any* $\gamma \in \mathrm{Aut}^{\mathbb{G}_m}(L)$ *and* $f \in \mathcal{O}(X)^*$, *we have* $\gamma f \gamma^{-1} = \pi_*(\gamma) \cdot f$, *where the conjugation in the left-hand side takes place in* $\mathrm{Aut}^{\mathbb{G}_m}(L)$, *and the action in the right-hand side arises from the natural action of* $\mathrm{Aut}(X)$ *on* $\mathcal{O}(X)$.

Proof. (i) Since γ preserves the zero section of L, it restricts to an automorphism g of X. As the fibres of π are exactly the $\mathbb{G}_m$-orbit closures, γ sends the fibre L_x to $L_{g(x)}$ for any $x \in X$. This yields the assertions.

(ii) Clearly, π_* is a group homomorphism. To show that it is surjective, let $g \in \mathrm{Aut}(X)_L$. Then there exists an isomorphism $\varphi : L \xrightarrow{\cong} g^*(L)$. Moreover, $g^*(L)$ lies in a cartesian square

$$\begin{array}{ccc} g^*(L) & \xrightarrow{\psi} & L \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{g} & X, \end{array}$$

where ψ is a $\mathbb{G}_m$-equivariant morphism of varieties. Thus, $\gamma := \psi \circ \varphi$ is a $\mathbb{G}_m$-equivariant automorphism of L that lifts g.

Also, the kernel of π_* consists of the automorphisms of L viewed as a line bundle, i.e., of the multiplications by regular invertible functions on X.

(iii) Since γ is linear on fibres, we have $\gamma(f(z)) = f(\pi(z))\,\gamma(z)$ for any $z \in L$. Thus, $\gamma f \gamma^{-1}(z) = f(\pi(\gamma^{-1}(z))) = f(\pi_*(\gamma)^{-1})\,\pi(z)$. This yields the statement. □

Definition 3.4.2. Let G be an algebraic group, X a G-variety and L a line bundle over X. We say that L is *G-invariant* if $g^*(L) \cong L$ for all $g \in G$.

In other words, L is G-invariant if and only if the image of the homomorphism $G \to \mathrm{Aut}(X)$ is contained in $\mathrm{Aut}(X)_L$. We may then take the pull-back of the exact sequence (8) by the resulting homomorphism $G \to \mathrm{Aut}(X)_L$; this yields an exact sequence of groups

$$1 \longrightarrow \mathcal{O}(X)^* \longrightarrow \mathcal{G}(L) \longrightarrow G \longrightarrow 1 \tag{9}$$

for some subgroup $\mathcal{G}(L) \subseteq \mathrm{Aut}^{\mathbb{G}_m}(L)$, which will be called the *lifting group* associated with L.

Remarks 3.4.3. (i) A line bundle L on a G-variety X is invariant if and only if its class in $\mathrm{Pic}(X)$ lies in the G-fixed subgroup, $\mathrm{Pic}(X)^G$. Also, note that L is G-linearizable if and only if it is G-invariant and the extension (9) admits a splitting which defines an algebraic action of G on L. In particular, the forgetful homomorphism (7) sends $\mathrm{Pic}_G(X)$ to $\mathrm{Pic}(X)^G$.

(ii) The extension (9) is generally not central: by Lemma 3.4.1 (iii), the action of $\mathcal{G}(L)$ on $\mathcal{O}(X)^*$ by conjugation factors through the action of G via its natural action on $\mathcal{O}(X)$. Moreover, the latter action of G on $\mathcal{O}(X)^*$ may be nontrivial, e.g., when $X = \mathbb{G}_m \times \mathbb{G}_m$ and G is the group of order 2 acting via $(x, y) \mapsto (y, x)$.

Also, $\mathcal{O}(X)^*$ is not necessarily an algebraic group: for example, if X is the torus $\mathbb{G}_m^r$, then $\mathcal{O}(X) \cong k[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}]$ and hence

$$\mathcal{O}(X)^* \cong \{ct_1^{a_1} \cdots t_r^{a_r} \mid c \in k^*, (a_1, \ldots, a_r) \in \mathbb{Z}^r\} \cong k^* \times \mathbb{Z}^r.$$

The structure of the unit group $\mathcal{O}(X)^*$, where X is an irreducible variety, will be analyzed in §4.1.

Lemma 3.4.4. *Let G be an algebraic group, X a G-variety, and L, M two G-invariant line bundles over X.*

(i) *There is an isomorphism of groups over G*

$$\mathcal{G}(L) \times_G \mathcal{G}(M) \xrightarrow{\cong} \mathcal{G}(L \otimes M).$$

(ii) *For any integer n, the extension*

$$1 \longrightarrow \mathcal{O}(X)^* \longrightarrow \mathcal{G}(L^{\otimes n}) \longrightarrow G \longrightarrow 1$$

is the push-out of the extension (9) by the nth power map of $\mathcal{O}(X)^*$.

Proof. (i) For any line bundle L on X, there is a natural isomorphism $\mathrm{Aut}^{\mathbb{G}_m}(L) \cong \mathrm{Aut}^{\mathbb{G}_m}(L^\times)$, as follows from the correspondence between line bundles and $\mathbb{G}_m$-bundles. Also, there is a natural homomorphism

$$\mathrm{Aut}^{\mathbb{G}_m}(L^\times) \times_{\mathrm{Aut}(X)} \mathrm{Aut}^{\mathbb{G}_m}(M^\times) \longrightarrow \mathrm{Aut}^{\mathbb{G}_m \times \mathbb{G}_m}(L^\times \times_X M^\times)$$

of groups over $\mathrm{Aut}(X)$, where L, M are arbitrary line bundles over X. As $(L \otimes M)^\times$ is the quotient of $L^\times \times_X M^\times$ by the anti-diagonal subgroup in $\mathbb{G}_m \times \mathbb{G}_m$, we obtain a homomorphism

$$u : \mathrm{Aut}^{\mathbb{G}_m}(L) \times_{\mathrm{Aut}(X)} \mathrm{Aut}^{\mathbb{G}_m}(M) \longrightarrow \mathrm{Aut}^{\mathbb{G}_m}((L \otimes M)^\times) \cong \mathrm{Aut}^{\mathbb{G}_m}(L \otimes M)$$

of groups over $\mathrm{Aut}(X)$.

Next, assume that L, M are G-invariant. By the definition of the lifting groups, u yields a homomorphism

$$v : \mathcal{G}(L) \times_G \mathcal{G}(M) \longrightarrow \mathcal{G}(L \otimes M)$$

of groups over G, which restricts to the product map $\mathcal{O}(X)^* \times \mathcal{O}(X)^* \to \mathcal{O}(X)^*$. Since the kernel of the latter map is the anti-diagonal, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}(X)^* & \longrightarrow & \mathcal{G}(L) \times_G \mathcal{G}(M) & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & \mathcal{O}(X)^* & \longrightarrow & \mathcal{G}(L \otimes M) & \longrightarrow & G & \longrightarrow & 1, \end{array}$$

and hence an isomorphism of extensions. This yields the desired statement.

(ii) follows readily from (i) by induction on n. □

With the above notation and assumptions, denote by $\mathrm{Ext}^1(G, \mathcal{O}(X)^*)$ the group of isomorphism classes of extensions (of abstract groups). By Lemma 3.4.4, assigning with any invariant line bundle L the extension (9) yields a homomorphism

$$\epsilon : \mathrm{Pic}(X)^G \longrightarrow \mathrm{Ext}^1(G, \mathcal{O}(X)^*).$$

Moreover, by Remark 3.4.3 (i), the kernel of ϵ contains the image of $\mathrm{Pic}_G(X)$ under the forgetful homomorphism, and equality holds when G is finite: in that case, ϵ yields the obstruction to the existence of a G-linearization. This implies readily the following:

Proposition 3.4.5. *Let G be a finite group acting on a variety X, and L a line bundle over X. If L is G-invariant, then some positive power $L^{\otimes n}$ admits a G-linearization; we may take for n the order of G.*

Proof. It suffices to check that $\epsilon(L^{\otimes n}) = 0$, where n denotes the order of G. But this follows from the isomorphism $\mathrm{Ext}^1(G, M) \cong H^2(G, M)$ for any $\mathbb{Z}G$-module M (see [Bro94, Thm. 3.12]), combined with the fact that the cohomology groups $H^i(G, M)$, where $i \geq 1$, are killed by n (see [Bro94, Cor. 10.2]). □

Remarks 3.4.6. (i) The assumption that L is G-invariant cannot be omitted in Proposition 3.4.5. Consider indeed $X := \mathbb{P}^1 \times \mathbb{P}^1$ equipped with the action of the group G of order 2, generated by the automorphism $(x, y) \mapsto (y, x)$. Denote by $p_1, p_2 : X \to \mathbb{P}^1$ the two projections and let $L := p_1^*\mathcal{O}_{\mathbb{P}^1}(m_1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(m_2)$, where $m_1 \neq m_2$. Then $L^{\otimes n} = p_1^*\mathcal{O}_{\mathbb{P}^1}(m_1 n) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(m_2 n)$ is G-linearizable if and only if $n = 0$.

(ii) For an arbitrary algebraic group G, analyzing the obstruction to linearizability via the above map ϵ is more complicated. Indeed, one has to take into account the condition that the splitting of the extension (9) yields an *algebraic* action of G on X. We will develop an alternative approach to linearizability in the next section.

Corollary 3.4.7. *Let X be a quasi-projective variety equipped with the action of a finite group G.*

(i) *X is G-quasi-projective.*

(ii) *X admits a covering by open affine G-stable subsets.*

Proof. (i) By assumption, X admits an ample line bundle L. Then the line bundle $M := \bigotimes_{g \in G} g^*(L)$ is also ample, and clearly G-invariant. By Proposition 3.4.5, there exists a positive integer n such that $M^{\otimes n}$ is G-linearizable. Since $M^{\otimes n}$ is ample, X is G-quasi-projective in view of Proposition 3.2.6.

(ii) We may assume that X is a G-stable locally closed subvariety of $\mathbb{P}(V)$, where V is a finite-dimensional G-module. Then the closure, $\bar{X} \subseteq \mathbb{P}(V)$, and the boundary, $Y := \bar{X} \setminus X$, are closed G-stable subvarieties. Hence Y corresponds to a closed G-stable subvariety $Z \subseteq V$, also stable by the $\mathbb{G}_m$-action via scalar multiplication. It suffices to show that for any $x = [v] \in X$, there exists $f \in \mathcal{O}(V)^G$ homogeneous such that $f(v) \neq 0$ and f vanishes identically on Z: then the complement of the zero locus of f in $\bar{X}$ is an open affine G-stable subset, containing x, and contained in X. Since the orbit $G \cdot v$ is a finite set and does not meet the cone Z, we may find $F \in \mathcal{O}(V)$ homogeneous such that $F|_Z = 0$ and $F(g \cdot v) \neq 0$ for all $g \in G$. Then $f := \prod_{g \in G} g \cdot F$ satisfies the desired properties. □

Still considering a finite group G, we now characterize those G-varieties that are covered by affine G-stable open subsets, after [Mum08, Chap. II, §7]:

Proposition 3.4.8. *The following conditions are equivalent for a variety X equipped with an action of a finite group G:*

(i) *Every G-orbit in X is contained in some affine open subset.*

(ii) *X admits a covering by open affine G-stable subsets.*

(iii) *There exists a quotient morphism $\pi : X \to Y$, where Y is a variety.*

Proof. (i)⇒(ii) Let $x \in X$. By assumption, we may choose an affine open subset U of X that contains the orbit $G \cdot x$. Then $\bigcap_{g \in G} g \cdot U$ is an affine G-stable open subset, and still contains $G \cdot x$.

(ii)⇒(iii) When X is affine, the existence of π is obtained in [Bo91, Prop. 6.15]. The general case follows by a glueing argument.

(iii)⇒(i) Let $x \in X$ and choose an affine neighborhood V of $\pi(X)$ in Y. Since π is affine, $\pi^{-1}(V)$ is affine as well; clearly, $\pi^{-1}(V)$ is open, G-stable, and contains $G \cdot x$. □

One may readily check that the morphism π is finite if it exists, see e.g. the proof of [Bo91, Prop. 6.15]. Also, note that condition (i) is satisfied when X is quasi-projective, since every finite subset of points is contained in some open affine subset. But (i) already fails for the group G of order 2 and a smooth complete threefold X, as shown by a construction of Hironaka (presented in Example 5.3.5 below).

Some further developments.

As for §2.1 and §2.2, the contents of §3.1 are classical. We have included them in order to get a unified and coordinate-free presentation of notions and results that are somehow scattered in [Ha74].

The notion of G-linearization of a line bundle features prominently in Mumford's work on geometric invariant theory (see [MFK94]). Given an ample G-linearized line bundle L on an irreducible projective G-variety X, the section ring,

$$R(X, L) := \bigoplus_{n=0}^{\infty} \Gamma(X, L^{\otimes n}),$$

is a finitely generated, positively graded algebra on which G acts by automorphisms of graded algebra; we have an isomorphism of G-varieties $X \cong \operatorname{Proj} R(X, L)$. When G is reductive, the subalgebra of G-invariants,

$$R(X, L)^G := \bigoplus_{n=0}^{\infty} \Gamma(X, L^{\otimes n})^G,$$

is finitely generated as well; its Proj is by definition the geometric invariant theory quotient, $X//G$. The inclusion of graded algebras $R(X, L)^G \subseteq R(X, L)$ yields a rational map

$$\pi : X \dashrightarrow X//G.$$

The largest open subset of X on which π is defined is the *semistable locus*, denoted by $X^{ss}(L)$. Thus, a point $x \in X$ is semistable if and only if there exist a positive integer n and a section $s \in \Gamma(X, L^{\otimes n})^G$ such that $s(x) \neq 0$. The resulting morphism

$$\pi : X^{ss}(L) \longrightarrow X/\!/G$$

is clearly G-invariant; one shows that π is affine, surjective, and satisfies $\pi_*(\mathcal{O}_{X^{ss}(L)})^G \cong \mathcal{O}_{X/\!/G}$. In particular, π is a categorical quotient. We refer to [MFK94] for these results and many further developments and applications; see also [Muk03] for a self-contained introduction.

One may define linearizations of sheaves of $\mathcal{O}_X$-modules for any scheme X equipped with an action of a group scheme G; see [SGA3, I.6], where the resulting objects are called G-*equivariant*. In particular, a G-linearization of a quasi-coherent sheaf $\mathcal{F}$ on X is equivalent to the data of a G-action on the scheme

$$\mathrm{V}(\mathcal{F}) := \operatorname{Spec} \operatorname{Sym}_{\mathcal{O}_X}(\mathcal{F}) \longrightarrow X$$

(the relative spectrum of the symmetric algebra of $\mathcal{F}$), which commutes with the $\mathbb{G}_m$-action associated with the grading of $\operatorname{Sym}_{\mathcal{O}_X}(\mathcal{F})$, and lifts the given G-action on X.

Given a homogeneous space $X = G/H$, where G is a linear algebraic group and H a closed subgroup, the coherent G-linearized sheaves on X are exactly the sheaves of local sections of homogeneous vector bundles $G \times^H V \to G/H$, where V is an H-module; this defines an equivalence of categories between coherent G-linearized sheaves on G/H and H-modules. This important relation between the geometry of homogeneous spaces and representation theory is developed e.g. in [Ja03].

The lifting group $\mathcal{G}(L)$ was introduced by Mumford when G is an abelian variety acting on itself by translation, and called the theta group of L (see [Mum08, Chap. III, §23]). Note that $\mathcal{O}(X)^* = k^*$ in this situation; also, by [Ram64, Cor. 2], the group $\operatorname{Aut}^{\mathbb{G}_m}(L)$ is locally algebraic (possibly with infinitely many components) and acts algebraically on L. So we obtain a central extension of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(L) \longrightarrow G \longrightarrow 1.$$

This holds, more generally, for the lifting group associated with any algebraic group action on a complete irreducible variety; see Proposition 5.2.4 below for an application to linearization.

4 Linearization of line bundles

4.1 Unit groups and character groups

We first record the following preliminary result:

Lemma 4.1.1. *Let X be an affine irreducible variety.*

(i) *There exists an irreducible projective variety $\bar{X}$ that contains X as an open subset. If X is normal, then $\bar{X}$ may be taken normal as well.*

(ii) *For any variety Y containing X as an open subset, the complement $Y \setminus X$ has pure codimension 1 in Y.*

Proof. (i) We may view X as a closed subvariety of some affine space $\mathbb{A}^n$. Consider the closure Y of X in the projective completion $\mathbb{P}^n \supset \mathbb{A}^n$. Then Y is projective, irreducible, and contains X as an open subset.

If X is normal, then consider the normalization map

$$\eta_Y : \tilde{Y} \longrightarrow Y$$

(see [Ha74, Ex. II.3.8]). Then $\tilde{Y}$ is projective by [Ha74, Cor. II.4.8, Ex. II.4.1, Ex. III.5.7], and still contains X as an open subset.

(ii) Consider again the normalization map η_Y. Then the pull-back map $\eta_Y^{-1}(X) \to X$ is the normalization map of X. As a consequence, $\eta_Y^{-1}(X)$ is affine. Also, since η_Y is finite, we have $\dim(\eta_Y^{-1}(Z)) = \dim(Z)$ for any irreducible subvariety Z of X. Thus, we may replace Y (resp. X) with $\tilde{Y}$ (resp. $\eta_Y^{-1}(X)$), and hence assume that Y is normal.

Next, we may remove from Y the union of all the irreducible components of codimension 1 of $Y \setminus X$, so that $\operatorname{codim}_Y(Y \setminus X) \geq 2$. Assuming that $Y \neq X$, we may choose a point $y \in Y \setminus X$ and an affine neigborhood V of y in Y. Then $U := V \cap X$ is affine, normal, and $\operatorname{codim}_V(V \setminus U) \geq 2$. Thus, every regular function on U has no pole on V, and hence extends to a regular function on V. In other words, the restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$ is an isomorphism. So the evaluation map at y yields a homomorphism $\mathcal{O}(V) \to k$, which must be of the form $f \mapsto f(x)$ for some $x \in U$ (since U is affine). Thus, $f(y) = f(x)$ for all $f \in \mathcal{O}(V)$. Since V is affine, it follows that $y = x \in X$, a contradiction. □

Next, we obtain versions of three results of Rosenlicht (see [Ros61]):

Proposition 4.1.2. *Let X be an irreducible variety. Then the quotient group $\mathcal{O}(X)^*/k^*$ is free of finite rank, where k^* is viewed as the subgroup of $\mathcal{O}(X)^*$ consisting of nonzero constant functions.*

Proof. Since $\mathcal{O}(X)^* \subseteq \mathcal{O}(U)^*$ for any nonempty open subset $U \subseteq X$, we may assume that X is affine. Consider the normalization map $\eta_X : \tilde{X} \to X$. Then $\tilde{X}$ is affine, and $\eta_X^{\#}$ identifies $\mathcal{O}(X)^*/k^*$ with a subgroup of $\mathcal{O}(\tilde{X})^*/k^*$. Thus, we may assume in addition that X is normal.

Choose a normal completion $\bar{X} \supseteq X$ as in Lemma 4.1.1, and denote by $D_1, \ldots, D_r$ the irreducible components of $\bar{X} \setminus X$; these are prime divisors of $\bar{X}$. As $\bar{X}$ is normal, we may view $\mathcal{O}(X)$ as the algebra consisting of those rational functions on $\bar{X}$ that have poles along $D_1, \ldots, D_r$ only. This identifies $\mathcal{O}(X)^*$ with the multiplicative group of rational functions on $\bar{X}$ having zeroes and poles along $D_1, \ldots, D_r$ only. Let $v_1, \ldots, v_r$ be the discrete valuations of the function field $k(\bar{X}) = k(X)$ associated with $D_1, \ldots, D_r$, so that $v_i(f)$ is the order of the zero or pole of f along D_i, for any $f \in k(\bar{X})$ and $i = 1, \ldots, r$. Then the map

$$\mathcal{O}(X)^* \longrightarrow \mathbb{Z}^r, \quad f \longmapsto (v_1(f), \ldots, v_r(f))$$

is a group homomorphism with kernel k^*, since every rational function on $\bar{X}$ having no zero or pole is constant. Thus, $\mathcal{O}(X)^*/k^*$ is isomorphic to a subgroup of $\mathbb{Z}^r$. □

Proposition 4.1.3. *Let X, Y be irreducible varieties. Then the product map*

$$\mathcal{O}(X)^* \times \mathcal{O}(Y)^* \longrightarrow \mathcal{O}(X \times Y)^*, \quad (f,g) \longmapsto ((x,y) \mapsto f(x)g(y))$$

is surjective.

Proof. It suffices to show that every $f \in \mathcal{O}(X \times Y)^*$ can be written as $(x,y) \mapsto g(x)h(y)$ for some $g \in k(X)$ and $h \in k(Y)$. Indeed, for any $y \in Y$, the map $f_y : x \mapsto f(x,y)$ is a regular invertible function on X. Taking y such that h is defined at y, we see that $g \in \mathcal{O}(X)^*$; likewise, $h \in \mathcal{O}(Y)^*$.

Therefore, we may replace X and Y with any non-empty open subsets, and hence assume that they are both smooth and affine; then $X \times Y$ is smooth and affine, too. As in the proof of Proposition 4.1.2, choose a normal completion $\bar{X}$ of X and denote by $D_1, \ldots, D_r$ the irreducible components of $\bar{X} \setminus X$. Let $f \in \mathcal{O}(X \times Y)^*$ and view f as a rational function on $\bar{X} \times Y$. Then the divisor $\mathrm{div}(f)$ is supported in $(\bar{X} \setminus X) \times Y$, and hence we have $\mathrm{div}(f) = \sum_{i=1}^r n_j D_j \times Y$ for some integers $n_1, \ldots, n_r$. Thus, $\mathrm{div}(f_y) = \sum_{i=1}^r n_j D_j$ for all y in a nonempty open subset $V \subseteq Y$. Choose $y_0 \in V$; then $\mathrm{div}(f_y f_{y_0}^{-1}) = 0$ for all $y \in V$. Since $\bar{X}$ is complete and normal, it follows that $f_y f_{y_0}^{-1}$ is constant on X. Thus, $f(x,y) = f(x,y_0)h(y)$ for some rational function h on Y. □

Remarks 4.1.4. (i) For any irreducible variety X, consider the exact sequence

$$0 \longrightarrow k^* \longrightarrow \mathcal{O}(X)^* \longrightarrow \mathrm{U}(X) \longrightarrow 0. \tag{10}$$

Then Proposition 4.1.2 asserts that *the abelian group* $\mathrm{U}(X)$ *is free of finite rank.* Moreover, any point $x \in X$ defines a splitting of (10), since the subgroup of $\mathcal{O}(X)^*$ consisting of functions with value 1 at x is sent isomorphically to $\mathrm{U}(X)$.

(ii) Given two irreducible varieties X, Y, Proposition 4.1.3 yields an exact sequence

$$0 \longrightarrow k^* \longrightarrow \mathcal{O}(X)^* \times \mathcal{O}(Y)^* \longrightarrow \mathcal{O}(X \times Y)^* \longrightarrow 0,$$

where k^* is sent to $\mathcal{O}(X)^* \times \mathcal{O}(Y)^*$ via $t \mapsto (t, t^{-1})$. It follows that

$$\mathrm{U}(X \times Y) \cong \mathrm{U}(X) \times \mathrm{U}(Y).$$

(iii) The above isomorphism does not hold for arbitrary schemes X, Y, nor is the group $\mathrm{U}(X)$ finitely generated in this generality. For example, let X be an affine variety and $Y := \mathrm{Spec}(k[t]/(t^2))$. Then $\mathcal{O}(Y) \cong k \oplus k\varepsilon$, where ϵ denotes the image of t in $\mathcal{O}(Y)$, so that $\varepsilon^2 = 0$. Thus,

$$\mathcal{O}(X \times Y) \cong \mathcal{O}(X) \otimes \mathcal{O}(Y) \cong \mathcal{O}(X) \oplus \varepsilon\mathcal{O}(X).$$

As a consequence,

$$\mathcal{O}(X \times Y)^* = \mathcal{O}(X)^*(1 + \varepsilon\mathcal{O}(X)) \cong \mathcal{O}(X)^* \times \mathcal{O}(X).$$

In particular, $\mathcal{O}(Y)^* \cong k^* \times k$ and $\mathrm{U}(Y) \cong k$, while $\mathrm{U}(X \times Y) \cong \mathrm{U}(X) \times \mathcal{O}(X)$. So the natural map $\mathrm{U}(X) \times \mathrm{U}(Y) \to \mathrm{U}(X \times Y)$ is not an isomorphism when X has positive dimension.

We now apply Propositions 4.1.2 and 4.1.3 to the (multiplicative) *characters* of an algebraic group G, that is, to the homomorphisms of algebraic groups $\chi : G \to \mathbb{G}_m$. The characters of G form a commutative group (under pointwise multiplication) that we denote by $\widehat{G}$. We may view $\widehat{G}$ as a subgroup of $\mathcal{O}(G)^*$, consisting of functions with value 1 at the neutral element e.

Proposition 4.1.5. *Let G be a connected algebraic group, and $f \in \mathcal{O}(G)^*$ such that $f(e) = 1$. Then $f \in \widehat{G}$.*

Proof. The map
$$f \circ m : G \times G \longrightarrow \mathbb{A}^1, \quad (g, h) \longmapsto f(gh)$$
lies in $\mathcal{O}(G \times G)^*$. By Proposition 4.1.3 (which may be applied, since G is an irreducible variety), it follows that there exist $\varphi, \psi \in \mathcal{O}(G)^*$ such that $f(gh) = \varphi(g)\psi(h)$ for all $g, h \in G$. Replacing φ with a scalar multiple, we may assume that $\varphi(e) = 1$. Taking $g = e$, we obtain $\psi = f$; then taking $h = e$ yields $\varphi = f$. Thus, $f(gh) = f(g)f(h)$, i.e., f is a character. □

It follows that the character group of a connected algebraic group G satisfies $\widehat{G} \cong \mathrm{U}(G)$, and hence is free of finite rank in view of Proposition 4.1.2. Moreover, the product map $k^* \times \widehat{G} \to \mathcal{O}(G)^*$ is an isomorphism, and so is the natural map $\widehat{G} \times \widehat{H} \to \widehat{G \times H}$ for any connected algebraic group H. We will need the following generalization of these results:

Lemma 4.1.6. *Let G be a connected algebraic group, and X an irreducible variety. Then the product map $\widehat{G} \times \mathcal{O}(X)^* \to \mathcal{O}(G \times X)^*$ is an isomorphism.*

If X is equipped with a G-action, then for any $f \in \mathcal{O}(X)^*$, there exists $\chi = \chi(f) \in \widehat{G}$ such that $f(g \cdot x) = \chi(g) f(x)$ for all $g \in G$ and $x \in X$. Moreover, the assignement $f \mapsto \chi(f)$ defines an exact sequence

$$0 \longrightarrow \mathcal{O}(X)^{*G} \longrightarrow \mathcal{O}(X)^* \xrightarrow{\chi} \widehat{G}, \tag{11}$$

where $\mathcal{O}(X)^{*G}$ denotes the subgroup of G-invariants in $\mathcal{O}(X)^*$.

Proof. The first assertion is a consequence of Proposition 4.1.3, Remark 4.1.4 and Proposition 4.1.5. The second assertion follows similarly by considering the map $f \circ \alpha \in \mathcal{O}(G \times X)^*$. □

As an application, we describe all the linearizations of the trivial bundle:

Lemma 4.1.7. *Let G be a connected algebraic group, and X an irreducible G-variety. Then every G-linearization of the trivial line bundle $p_1 : X \times \mathbb{A}^1 \to X$ is of the form $g \cdot (x, z) = (g \cdot x, \chi(g)z)$ for a unique $\chi \in \widehat{G}$.*

Proof. Let $\beta : G \times X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ be a G-linearization. Since β lifts the G-action on X and commutes with the $\mathbb{G}_m$-action by multiplication on $\mathbb{A}^1$, we must have

$$\beta(g, x, z) = (g \cdot x, f(g, x)z)$$

for some $f \in \mathcal{O}(G \times X)^*$. Moreover, $f(e, x) = 1$ for all $x \in X$, since $\beta(e, x, z) = z$ for all $z \in \mathbb{A}^1$. Using Lemma 4.1.6, it follows that $f(g, x) = \chi(g)$ for some character χ of G. □

With the assumptions of Lemma 4.1.7, we denote by $\mathcal{O}_X(\chi)$ the trivial line bundle over X equipped with the G-linearization as in that lemma, i.e., G acts via χ on each fibre. We then have isomorphisms of G-linearized line bundles for all $\chi, \eta \in \widehat{G}$:

$$\mathcal{O}_X(-\chi) \cong \mathcal{O}_X(\chi), \quad \mathcal{O}_X(\chi) \otimes \mathcal{O}_X(\eta) \cong \mathcal{O}_X(\chi + \eta).$$

Lemma 4.1.8. *Keep the above assumptions and let $\chi, \eta \in \widehat{G}$. Then the G-linearized line bundles $\mathcal{O}_X(\chi), \mathcal{O}_X(\eta)$ are isomorphic if and only if $\eta - \chi = \chi(f)$ for some $f \in \mathcal{O}(X)^*$.*

Proof. Let $F : \mathcal{O}_X(\chi) \to \mathcal{O}_X(\eta)$ be an isomorphism of G-linearized line bundles. Then F yields an automorphism of the trivial bundle, i.e., the multiplication by some $f \in \mathcal{O}(X)^*$. By Lemma 4.1.6, we then have $f(g \cdot x) = \chi(f)(g)f(x)$ for all $g \in G$ and $x \in X$. Thus, $\eta - \chi = \chi(f)$. The converse follows by reversing this argument. □

4.2 The equivariant Picard group

In this subsection, we denote by G a connected algebraic group, and by X an irreducible G-variety. We first obtain a key criterion for the existence of a linearization :

Lemma 4.2.1. *Let $\pi : L \to X$ be a line bundle. Then L admits a G-linearization if and only if the line bundles $\alpha^*(L)$ and $p_2^*(L)$ on $G \times X$ are isomorphic.*

Proof. If L admits a linearization, then $\alpha^*(L) \cong p_2^*(L)$ by Lemma 3.2.4. For the converse, let $\Phi : \alpha^*(L) \to p_2^*(L)$ be an isomorphism. Since $\alpha(e, x) = p_2(e, x) = x$ for all $x \in X$, the pull-back of Φ to $\{e\} \times X$ is identified with an automorphism of the line bundle L, i.e., with the multiplication by some $f \in \mathcal{O}(X)^*$. Replacing Φ with $\Phi \circ p_2^{\#}(f^{-1})$, we may assume that $f = 1$. Then, as in the proof of Lemma 3.2.4, Φ corresponds to a morphism $\beta : G \times L \to L$ such that the square (6) commutes; moreover, $\beta(e, z) = z$ for all $z \in L$. It remains to show that β satisfies the associativity condition of a group action. But the obstruction to associativity is an automorphism of the line bundle

$$\mathrm{id} \times \pi : G \times G \times L \to G \times G \times X,$$

i.e., the multiplication by some $\varphi \in \mathcal{O}(G \times G \times X)^*$. Moreover, since $\beta(g, \beta(e, z)) = \beta(g, z) = \beta(e, \beta(g, z))$ for all $g \in G$ and $z \in L$, we have $\varphi(g, e, x) = 1 = \varphi(e, g, x)$ for all $g \in G$ and $x \in X$. To complete the proof, it suffices to show that $\varphi = 1$.

By Lemma 4.1.6, there exist $\chi \in \widehat{G \times G}$ and $\psi \in \mathcal{O}(X)^*$ such that $\varphi(g, h, x) = \chi(g, h)\psi(x)$ for all $g, h \in G$ and $x \in X$. Evaluating at $g = h = e$, we obtain $\psi = 1$; then evaluating at $h = e$, we see that $\chi(g, e) = \chi(e, g) = 1$ for all $g \in G$. Since the natural map $\widehat{G} \times \widehat{G} \to \widehat{G \times G}$ is an isomorphism, it follows that $\chi = 1$. Thus, $\varphi(g, h, x) = 1$, as desired. □

Next, we consider the equivariant Picard group $\mathrm{Pic}_G(X)$ and the forgetful homomorphism $\phi : \mathrm{Pic}_G(X) \to \mathrm{Pic}(X)$ defined in §3.2. We also have homomorphisms

$$\gamma : \widehat{G} \longrightarrow \mathrm{Pic}_G(X), \quad \chi \longmapsto \mathcal{O}_X(\chi),$$

$$\chi : \mathcal{O}(X)^* \longrightarrow \widehat{G}, \quad f \longmapsto \chi(f),$$

defined in §4.1. We may now state one of the main results of this text:

Theorem 4.2.2. *There is an exact sequence*

$$0 \to \mathcal{O}(X)^{*G} \to \mathcal{O}(X)^* \xrightarrow{\chi} \widehat{G} \xrightarrow{\gamma} \mathrm{Pic}_G(X) \xrightarrow{\phi} \mathrm{Pic}(X) \xrightarrow{\alpha^* - p_2^*} \mathrm{Pic}(G \times X). \tag{12}$$

Proof. In view of Lemma 4.1.6, it suffices to show that the above sequence is exact at $\widehat{G}$, $\mathrm{Pic}_G(X)$ and $\mathrm{Pic}(X)$. The exactness at $\widehat{G}$ follows from Lemma 4.1.8. Since the kernel of ϕ consists of the isomorphism classes of G-linearized line bundles which are trivial as line bundles, the exactness at $\mathrm{Pic}_G(X)$ is a consequence of Lemmas 4.1.7 and 4.1.8. Finally, the exactness at $\mathrm{Pic}(X)$ is equivalent to Lemma 4.2.1. □

By the above theorem, the obstruction to the existence of a linearization is given by the map $\alpha^* - p_2^* : \mathrm{Pic}(X) \to \mathrm{Pic}(G \times X)$. We now modify this obstruction map to make it simpler to use:

Proposition 4.2.3. *With the notation and assumptions of Theorem 4.2.2, the sequence*

$$\mathrm{Pic}_G(X) \xrightarrow{\phi} \mathrm{Pic}(X) \xrightarrow{\psi} \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$$

is exact, where ψ sends every $L \in \mathrm{Pic}(X)$ to $\alpha^*(L)$ mod $p_2^*\mathrm{Pic}(X)$.

Proof. Consider the morphism

$$e \times \mathrm{id} : X \to G \times X, \quad x \mapsto (e, x).$$

Since $\alpha \circ (e \times \mathrm{id}) = p_2 \circ (e \times \mathrm{id}) = \mathrm{id}$, we have $(e \times \mathrm{id})^* \circ (\alpha^* - p_2^*) = 0$ on $\mathrm{Pic}(G \times X)$. Also, since $e \times \mathrm{id}$ is a section of p_2, the map $p_2^* : \mathrm{Pic}(X) \to \mathrm{Pic}(G \times X)$ is a section of $(e \times \mathrm{id})^*$. As a consequence, the kernel of $(e \times \mathrm{id})^* : \mathrm{Pic}(G \times X) \to \mathrm{Pic}(X)$ is sent isomorphically to $\mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$ by the quotient map $\mathrm{Pic}(G \times X) \to \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$. Combining these observations yields the statement. □

Example 4.2.4. Continuing with Example 3.2.7, we show that $L := O_{\mathbb{P}(V)}(1)$ admits no linearization relative to $G := \mathrm{PGL}(V)$. Indeed, L admits an $\mathrm{SL}(V)$-linearization, which is unique as the character group of $\mathrm{SL}(V)$ is trivial. If L is G-linearized, then G acts on $\Gamma(\mathbb{P}(V), L) = V^\vee$ by lifting the natural action of $\mathrm{SL}(V)$. This yields a section of the quotient homomorphism $\mathrm{SL}(V) \to \mathrm{PGL}(V)$, a contradiction.

Example 4.2.5. Continuing with Example 3.2.8, we show that no line bundle of nonzero degree on X is $\mathbb{G}_m$-linearizable. Consider indeed the normalization $\eta : \mathbb{P}^1 \to X$; recall that $\eta^{-1}(P) = \{0, \infty\}$ (as schemes), and η restricts to an isomorphism $\mathbb{P}^1 \setminus \{0, \infty\} \to X \setminus \{P\}$. For any line bundle L on X, the pull-back $\eta^*(L)$ is equipped with isomorphisms of fibres

$$\eta^{-1}(L)_0 \cong \eta^{-1}(L)_\infty \cong L_P.$$

If L is $\mathbb{G}_m$-linearized, then $\mathbb{G}_m$ acts on these fibres and the above isomorphisms are equivariant. Thus, the $\mathbb{G}_m$-actions on the lines $\eta^{-1}(L)_0$ and $\eta^{-1}(L)_\infty$ have the same weight. On the other hand, the $\mathbb{G}_m$-linearized line bundle $\eta^{-1}(L)$ on $\mathbb{P}^1$ is isomorphic to some $O_{\mathbb{P}^1}(n)$ equipped with its natural linearization twisted by some weight m, as follows from Theorem 4.2.2; then n is the degree of L. Thus, $\eta^{-1}(L)_0$ has weight $n + m$, and $\eta^{-1}(L)_\infty$ has weight m. So we conclude that $n = 0$ if L is linearizable.

One can show that the group $\mathrm{Pic}(\mathbb{G}_m \times X)/p_2^*\mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}$, and this identifies the obstruction map ψ with the degree map $\mathrm{Pic}(X) \to \mathbb{Z}$ (see [Bri15, Ex. 2.15]). As a consequence, every line bundle of degree 0 on X is $\mathbb{G}_m$-linearizable.

Example 4.2.6. Continuing with Example 3.2.9, we show that no line bundle of nonzero degree on Y is $\mathbb{G}_a$-linearizable, if $\mathrm{char}(k) = 0$; it follows that Y is not $\mathbb{G}_a$-projective. We adapt the argument of Example 4.2.5: the normalization $\eta : \mathbb{P}^1 \to Y$ satisfies $\eta^{-1}(Q) = \mathrm{Spec}(\mathcal{O}_{\mathbb{P}^1,\infty}/\mathfrak{m}^2)$ (as schemes), and η restricts to an isomorphism $\mathbb{P}^1 \setminus \{\infty\} \to X \setminus \{Q\}$. The pull-back of any line bundle L on X restricts to the trivial bundle over $Z := \eta^{-1}(Q)$, and this also holds for a $\mathbb{G}_a$-linearized line bundle. On the other hand, $\eta^{-1}(L) \cong O_{\mathbb{P}^1}(n)$ equipped with its natural linearization, since $\mathbb{G}_a$ has trivial character group. If $n \neq 0$, then we may assume that $n \geq 1$ by replacing L with L^{-1}. Note that Z is the zero scheme of the $\mathbb{G}_a$-invariant section y^2 of $\mathcal{O}_{\mathbb{P}^1}(2)$; thus, we have an exact sequence of $\mathbb{G}_a$-linearized sheaves on $\mathbb{P}^1$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(n-2) \xrightarrow{y^2} \mathcal{O}_{\mathbb{P}^1}(n) \longrightarrow \mathcal{O}_Z(n) \longrightarrow 0$$

and hence an exact sequence of $\mathbb{G}_a$-modules

$$0 \longrightarrow \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-2)) \xrightarrow{y^2} \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) \longrightarrow \Gamma(Z, \mathcal{O}_Z(n)) \longrightarrow 0,$$

since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-2)) = 0$. This yields an isomorphism of $\mathbb{G}_a$-modules

$$\Gamma(Z, \mathcal{O}_Z(n)) \cong k[x,y]_n / y^2 k[x,y]_{n-2},$$

where, as in Example 3.2.9, we denote by $k[x,y]_m$ the space of homogeneous polynomials of degree m in x, y, on which $\mathbb{G}_a$ acts via $t \cdot (x,y) = (x+ty, y)$. Thus, the subspace of $\mathbb{G}_a$-invariants in $\Gamma(Z, \mathcal{O}_Z(n))$ is the line spanned by the image of $x^{n-1}y$. Likewise, the fibre $\mathcal{O}_{\mathbb{P}^1}(n)_\infty$ is the line spanned by the image of x^n. It follows that $\mathcal{O}_Z(n)$ has no $\mathbb{G}_a$-invariant trivialization, and hence that L is not $\mathbb{G}_a$-linearizable.

In characteristic $p > 0$, the action of $\mathbb{G}_a$ on $\Gamma(Z, \mathcal{O}_Z(p))$ is trivial. Hence x^p yields a $\mathbb{G}_a$-invariant trivialization of $\mathcal{O}_Z(p)$, in agreement with Example 3.2.9.

One can show that $\mathrm{Pic}(\mathbb{G}_a \times X)/p_2^* \mathrm{Pic}(X)$ is isomorphic to $k[t]/k$ (viewed as an additive group), and this identifies the obstruction map ψ with the map $L \mapsto \deg(L)t$ (see [Bri15, Ex. 2.16]). As a consequence, a line bundle over X is $\mathbb{G}_a$-linearizable if and only if its degree is a multiple of $\mathrm{char}(k)$.

4.3 Picard groups of principal bundles

In this subsection, G denotes a connected algebraic group, and $f : X \to Y$ a G-bundle, where X, Y are irreducible varieties.

By combining Lemma 3.3.1, Theorem 4.2.2 and Proposition 4.2.3, we readily obtain the following:

Proposition 4.3.1. *There is an exact sequence*

$$(13) \qquad 0 \to \mathrm{U}(Y) \xrightarrow{f^*} \mathrm{U}(X) \xrightarrow{\chi} \widehat{G} \xrightarrow{\gamma} \mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \xrightarrow{\psi} \mathrm{Pic}(G \times X)/p_2^* \mathrm{Pic}(X),$$

where γ assigns to any character of G, the class of the associated line bundle over Y.

The above map γ is called the *characteristic homomorphism* of the G-bundle f.

We will obtain a cohomological interpretation of most of the exact sequence (13). For this, we need the following preliminary results:

Lemma 4.3.2. *There exists a smallest closed normal subgroup H of G such that G/H is a torus. Moreover, H is connected, $\widehat{H}$ is trivial, and the pull-back map $\widehat{G/H} \to \widehat{G}$ is an isomorphism.*

Proof. We first reduce to the case where G is linear, by using the affinization theorem (see [Ros56, Sec. 5] and [DG70, III.3.8]): G has a smallest closed normal subgroup N such that G/N is linear; moreover, $\mathcal{O}(N) = k$. In particular, N is connected, $\widehat{N}$ is trivial, and the pull-back map $\widehat{G/N} \to \widehat{G}$ is an isomorphism. We may thus replace G with G/N, and assume that G is linear.

Clearly, every character $\chi \in \widehat{G}$ restricts trivially to the unipotent radical $R_u(G)$, and also to the commutator subgroup (G, G). Thus, χ restricts trivially to the subgroup $H := R_u(G) \cdot (G, G) \subset G$, which is closed, connected, and normal in G. Moreover, G/H is the quotient of the connected reductive group $G/R_u(G)$ by its commutator subgroup, and hence is a torus in view of [Bo91, Prop. 14.2]. This readily yields the assertions. □

Lemma 4.3.3. *There is an exact sequence of sheaves*

$$0 \longrightarrow \mathcal{O}_Y^* \longrightarrow f_*(\mathcal{O}_X^*) \longrightarrow \widehat{G} \longrightarrow 0, \tag{14}$$

where $\widehat{G}$ is viewed as a constant sheaf on Y.

Proof. Let $V \subseteq Y$ be an open subset, and $U := f^{-1}(V) \subseteq X$. Then U is an irreducible G-variety, and the restriction $f_V : U \to V$ is a G-bundle. By Lemmas 3.3.1 (i) and 4.1.6, we have an exact sequence

$$0 \longrightarrow \mathcal{O}(V)^* \longrightarrow \mathcal{O}(U)^* \longrightarrow \widehat{G}.$$

This yields the complex of sheaves (14), and its left exactness. To check its right exactness, it suffices to show the following claim: for any $\chi \in \widehat{G}$ and $y \in Y$, there exist an open neighborhood V of y and $h \in \mathcal{O}(U)^*$ such that $\chi(h) = \chi$.

Consider the closed normal subgroup $H \subseteq G$ introduced in Lemma 4.3.2, and the quotient torus $G/H =: T$; then $\widehat{G} \cong \widehat{T}$ and $\widehat{H} = 0$. Since T is affine, we may form the fibre bundle associated with the G-bundle f and the G-variety T; this yields a T-bundle $\varphi : Z \to Y$ (see Corollary 3.3.4). Also, $f : X \to Y$ factors as $\varphi \circ \psi$, where $\psi : X \to Z$ is a H-bundle. Moreover, the natural map $\mathcal{O}_Z^* \to \psi_*(\mathcal{O}_X^*)$ is an isomorphism by the first step and the vanishing of $\widehat{H}$. Thus, $f_*(\mathcal{O}_X^*) \cong \varphi_*(\mathcal{O}_Z)^*$, and the left exact sequence (14) may be identified with

$$0 \longrightarrow \mathcal{O}_Y^* \longrightarrow \varphi_*(\mathcal{O}_Z^*) \longrightarrow \widehat{T} \longrightarrow 0.$$

In other words, we may assume that $G = T$. Then every G-bundle is locally trivial, as follows from Proposition 3.1.3; moreover, the claim holds obviously for any trivial bundle. □

Proposition 4.3.4. (i) *The long exact sequence of cohomology associated with the short exact sequence (14) begins with*

$$0 \to \mathcal{O}(Y)^* \xrightarrow{f^*} \mathcal{O}(X)^* \xrightarrow{\chi} \widehat{G} \xrightarrow{\gamma} \mathrm{Pic}(Y) \xrightarrow{f^*} H^1(Y, f_*(\mathcal{O}_X^*)).$$

(ii) *The group* $H^1(Y, f_*(\mathcal{O}_X^*))$ *is isomorphic to the subgroup of* $\mathrm{Pic}(X)$ *consisting of the classes of those line bundles that are trivial on some open covering of* X *of the form* $f^{-1}(\mathcal{V})$*, where* $\mathcal{V}$ *is an open covering of* Y*.*

Proof. Both assertions may be readily checked by using the isomorphisms $\mathrm{Pic}(Y) \cong H^1(Y, \mathcal{O}_Y^*)$, $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$ (see [Ha74, Ex. III.4.5]). □

Proposition 4.3.4 yields an exact sequence

$$0 \to \mathrm{U}(Y) \xrightarrow{f^*} \mathrm{U}(X) \xrightarrow{\chi} \widehat{G} \longrightarrow \mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X)$$

which gives back part of (13). But we do not know how to recover the obstruction map $\psi : \mathrm{Pic}(X) \to \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$ via this cohomological approach.

5 Normal G-varieties

5.1 Picard groups of products

Consider two varieties X, Y, and the projections $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$. This yields a map

$$p_1^* \times p_2^* : \mathrm{Pic}(X) \times \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X \times Y), \quad (L, M) \longrightarrow p_1^*(L) \otimes p_2^*(M),$$

which is injective as $L \cong (\mathrm{id} \times y)^*(p_1^*(L) \otimes p_2^*(M))$ and $M \cong (x \times \mathrm{id})^*(p_1^*(L) \otimes p_2^*(M))$ for any points $x \in X$, $y \in Y$ and for any line bundles L on X, and M on Y. In general, this map is not surjective, as shown by the following:

Example 5.1.1. Let C be an elliptic curve. Consider the diagonal, $\mathrm{diag}(C) \subset C \times C$, and the associated line bundle $L(\mathrm{diag}(C))$ on $C \times C$. Then there exist no line bundles L, M on C such that $L(\mathrm{diag}(C)) \cong p_1^*(L) \otimes p_2^*(M)$. Otherwise, we have

$$\mathrm{diag}(C) \sim p_1^*(D) + p_2^*(E),$$

where D (resp. E) is a divisor on C associated with L (resp. M), and $\sim$ stands for linear equivalence of divisors. Taking intersection numbers with the divisors $\{x\} \times C$ and $C \times \{y\}$ for $x, y \in C$, we obtain $\deg(D) = \deg(E) = 1$. Thus, the self-intersection number $\mathrm{diag}(C)^2$ equals 1. But $\mathrm{diag}(C)$ is the scheme-theoretic fibre at the origin of the map

$$C \times C \longrightarrow C, \quad (x, y) \longmapsto x - y$$

defined by the group law on C. Therefore, the normal bundle to $\mathrm{diag}(C)$ in $C \times C$ is trivial. Hence $\mathrm{diag}(C)^2 = 0$ in view of [Ha74, Ex. V.1.4.1], a contradiction.

One can show that the cokernel of $p_1^* \times p_2^* : \mathrm{Pic}(C) \times \mathrm{Pic}(C) \to \mathrm{Pic}(C \times C)$ is the endomorphism group of the algebraic group C, see [Ha74, Ex. IV.4.10].

We will show that the above map $p_1^* \times p_2^*$ is an isomorphism when X, Y are normal irreducible varieties, and one of them is *rational* (i.e., it contains a nonempty open subset isomorphic to an open subset of some affine space $\mathbb{A}^n$; equivalently, its function field is a purely transcendental extension of k). For this, we will view the Picard group of (say) X as the group of isomorphism classes of Cartier divisors on X, as in the above example. This identifies $\mathrm{Pic}(X)$ with a subgroup of the *divisor class group*, $\mathrm{Cl}(X)$, consisting of classes of Weil divisors on X under the relation of linear equivalence, $\sim$ (see [Ha74, Prop. II.6.13, Prop. II.6.15]).

Note that the pull-back of Weil divisors under an arbitrary morphism of normal varieties is not defined in general. But the pull-back under any open immersion $\iota : U \to X$ is defined, and yields a surjective map $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ (see [Ha74, Prop. II.6.5]). Also, the projections p_1, p_2 yield well-defined pull-backs.

With these observations at hand, we may now state:

Proposition 5.1.2. *Let X, Y be normal irreducible varieties, and assume that X is rational. Then the map*

$$p_1^* \times p_2^* : \mathrm{Cl}(X) \times \mathrm{Cl}(Y) \longrightarrow \mathrm{Cl}(X \times Y)$$

is an isomorphism, and restricts to an isomorphism

$$\mathrm{Pic}(X) \times \mathrm{Pic}(Y) \cong \mathrm{Pic}(X \times Y).$$

Proof. By assumption, X contains a nonempty open affine subset U that is isomorphic to an open subset of some $\mathbb{A}^n$. Since the pull-back map $\mathrm{Cl}(Y) \to \mathrm{Cl}(\mathbb{A}^n \times Y)$ is an isomorphism (see [Ha74, Prop. II.6.6]) and the pull-back map $\mathrm{Cl}(\mathbb{A}^n \times Y) \to \mathrm{Cl}(U \times Y)$ is surjective, the map $p_2^* : \mathrm{Cl}(Y) \to \mathrm{Cl}(U \times Y)$ is surjective. Also, the kernel of the pull-back map $\mathrm{Cl}(X \times Y) \to \mathrm{Cl}(U \times Y)$ is generated by the classes $[D_i \times Y]$, where D_i denote the irrreducible components of $X \setminus U$. Since $[D_i \times Y] = p_1^*([D_i])$, it follows that the map $p_1^* \times p_2^*$ is surjective on divisor class groups.

To show that this map is injective, consider Weil divisors D on X and E on Y such that $p_1^*(D) + p_2^*(E) \sim 0$. In other words, there exists $f \in k(X \times Y)$ such that $\mathrm{div}(f) = p_1^*(D) + p_2^*(E)$. For a general point $y \in Y$, the rational function $f_y : x \mapsto f(x, y)$ is defined and satisfies $\mathrm{div}(f_y) = D$; thus, $D \sim 0$. Likewise, $E \sim 0$; this completes the proof of the assertion on divisor class groups.

For the assertion on Picard groups, it suffices to show the following claim: given two Weil divisors D on X and E on Y such that $p_1^*(D) + p_2^*(E)$ is Cartier, D and E must be Cartier as well. But this claim follows by pulling back to $X \times \{y\}$, $\{x\} \times Y$ for general points $x \in X$, $y \in Y$, as in the above step. □

Next, we show that Proposition 5.1.2 may be applied to the product of a connected linear algebraic group with a normal variety:

Proposition 5.1.3. *Let G be a connected linear algebraic group. Then the variety G is rational, and its Picard group is finite.*

Proof. Choose a Borel subgroup $B \subseteq G$. Then G is birationally isomorphic to $B \times G/B$ in view of [Bo91, Cor. 15.8]. Moreover, the variety B is isomorphic to a product of copies of $\mathbb{A}^1$ and $\mathbb{A}^1 \setminus \{0\}$; also, G/B is rational in view of the Bruhat decomposition (see [Bo91, §14] for these results). Thus, G is rational as well.

To show the finiteness of $\mathrm{Pic}(G)$, we use the exact sequence

$$0 \longrightarrow \widehat{G} \longrightarrow \widehat{B} \xrightarrow{\gamma} \mathrm{Pic}(G/B) \longrightarrow \mathrm{Pic}(G) \longrightarrow \mathrm{Pic}(G \times B)/p_2^*\mathrm{Pic}(G)$$

which follows from Proposition 4.3.1 in view of the isomorphisms $\mathrm{U}(G) \cong \widehat{G}$, $\mathrm{U}(B) \cong \widehat{B}$ (Proposition 4.1.5) and $\mathrm{U}(G/B) = 0$ (as G/B is a complete irreducible variety). Since G is smooth, the pull-back map $\mathrm{Pic}(G) \to \mathrm{Pic}(G \times \mathbb{A}^1)$ is an isomorphism, and the analogous map $\mathrm{Pic}(G \times \mathbb{A}^1) \to \mathrm{Pic}(G \times (\mathbb{A}^1 \setminus \{0\}))$ is surjective (see [Ha74, II.6.5, II.6.6, II.6.11]). It follows that $p_2^* : \mathrm{Pic}(G) \to \mathrm{Pic}(G \times B)$ is surjective. So it suffices to show that the characteristic homomorphism $\gamma : \widehat{B} \to \mathrm{Pic}(G/B)$ has finite cokernel.

Denote by $R(G)$ the radical of G; then the quotient $G' := G/R(G)$ is semisimple. Also, $R(G) \subseteq B$ and hence B is the pull-back of a unique Borel subgroup $B' \subseteq G'$; moreover, $G/B \cong G'/B'$ and γ factors through the analogous map $\gamma' : \widehat{B'} \to \mathrm{Pic}(G'/B')$. So we may assume that G is semisimple. Then $\widehat{G} = \{0\}$; moreover, the rank of $\widehat{B}$ is the rank of G, say r. Thus, it suffices to show that the group $\mathrm{Pic}(G/B)$ is generated by r elements. But this follows from the Bruhat decomposition again, since G/B contains the open Bruhat cell isomorphic to an affine space, and its complement is the union of r prime divisors. □

Proposition 5.1.3 does not extend to arbitrary fields, as shown by the following:

Example 5.1.4. Let K be an imperfect field; then $\mathrm{char}(K) = p > 0$ and there exists $a \in K$ such that $a \notin K^p$. Consider the closed subscheme $G \subset \mathbb{A}^2$ defined by the equation $y^p = x + ax^p$. Over the extension $L := K(a^{\frac{1}{p}})$, this equation may be written as $(y - a^{\frac{1}{p}}x)^p = x$; thus, the base change $G_L := G \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$ is isomorphic to the affine line $\mathbb{A}^1_L$. In particular, G is geometrically integral. Also, G is a subgroup scheme of $\mathbb{A}^2$ viewed as $\mathbb{G}_a \times \mathbb{G}_a$, and hence G is a connected linear algebraic group (a form of $\mathbb{G}_a$). We claim that *the variety G is not rational if $p \geq 3$; if in addition the group $G(K)$ is infinite (e.g., if K is separably closed), then* $\mathrm{Pic}(G)$ *is infinite as well.*

The closure of $G \subset \mathbb{A}^2$ in the projective plane $\mathbb{P}^2$ is the curve C with homogeneous equation $y^p = xz^{p-1} + ax^p$. The complement $C \setminus G$ consists of a unique point P_∞, with homogeneous coordinates $[1 : a^{\frac{1}{p}} : 0]$; in particular, the residue field of P_∞ is L and hence $\deg(P_\infty) = p$. Also, P_∞ is a regular point of C; indeed, one can check that the maximal ideal of the local ring $\mathcal{O}_{C,P_\infty}$ is generated by $\frac{z}{x}$. As a consequence, C is regular. But C is not smooth when $p \geq 3$, since the base change C_L has homogeneous equation $(y - a^{\frac{1}{p}}x)^p = xz^{p-1}$ and hence is singular at P_∞.

To show that G is not rational if $p \geq 3$, it suffices to check that the arithmetic genus $p_a(C) := \dim H^1(C, \mathcal{O}_C)$ is nonzero. For this, note that the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-p) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

yields an isomorphism $H^1(C, \mathcal{O}_C) \cong H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-p))$. Moreover, by Serre duality,

$$H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-p)) \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(p-3))^\vee \cong K[x, y, z]^\vee_{p-3}.$$

Thus, $p_a(C) = \frac{(p-1)(p-2)}{2}$ is indeed nonzero.

We now turn to the Picard group of $G = C \setminus \{P_\infty\}$. As C is regular, $\mathrm{Pic}(C) = \mathrm{Cl}(C)$ and we have a right exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Cl}(C) \longrightarrow \mathrm{Cl}(G) \longrightarrow 0,$$

where $1 \in \mathbb{Z}$ is sent to the class $[P_\infty]$ (see [Ha74, Prop. II.6.5]. The latter class is not torsion (since $\deg(P_\infty) = p$) and hence the above sequence is also left exact. Also, since C has points of degree 1 (for example, the origin, $[0 : 0 : 1] =: P_0$), we have an exact sequence

$$0 \longrightarrow \mathrm{Cl}^0(C) \longrightarrow \mathrm{Cl}(C) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$

where $\mathrm{Cl}^0(C)$ denotes the group of divisor classes of degree 0. Combining both exact sequences readily yields an exact sequence

$$0 \longrightarrow \mathrm{Cl}^0(C) \longrightarrow \mathrm{Cl}(G) \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0,$$

which is split by the subgroup of $\mathrm{Cl}(G)$ generated by $[P_0]$.

To complete the proof of the claim, it suffices to check that $\mathrm{Cl}^0(C)$ is infinite. Consider the map $f : G(K) \to \mathrm{Cl}^0(C)$ that sends any point P to the class $[P] - [P_0]$. Then f is injective: with an obvious notation, if $[Q] - [P_0] \sim [P] - [P_0]$, then there exists $f \in K(C)$ such that $\mathrm{div}(f) = [Q] - [P]$. If in addition $Q \neq P$, then f yields a birational morphism $C \to \mathbb{P}^1$, which contradicts the fact that C is not rational.

5.2 Linearization of powers of line bundles

In this subsection, G denotes a connected linear algebraic group. We obtain a key result on the existence of linearizations:

Theorem 5.2.1. *Let X be a normal irreducible G-variety, and L a line bundle on X. Then L is G-invariant. Moreover, there exists a positive integer n such that $L^{\otimes n}$ admits a G-linearization; we may take for n the exponent of the finite abelian group* $\mathrm{Pic}(G)$.

Proof. By Proposition 5.1.2 applied to $G \times X$, we have $\alpha^*(L) \cong p_1^*(M) \otimes p_2^*(N)$ for some line bundles M on G and N on X. Pulling back to $\{e\} \times X$, we obtain $L \cong N$; then pulling back to $\{g\} \times X$, we obtain $g^*(L) \cong L$. Thus, L is G-invariant.

Let n be the exponent of $\mathrm{Pic}(G)$. Then $M^{\otimes n}$ is trivial, hence we have an isomorphism $\alpha^*(L^{\otimes n}) \cong p_2^*(L^{\otimes n})$. In view of Lemma 4.2.1, it follows that $L^{\otimes n}$ is G-linearizable. □

Next, we present a simpler version of the obstruction map to linearization, for principal bundles over normal varieties:

Corollary 5.2.2. *Let $f : X \to Y$ be a principal G-bundle, where X, Y are normal irreducible varieties. Then the sequence*

$$0 \longrightarrow \mathrm{U}(Y) \xrightarrow{f^*} \mathrm{U}(X) \xrightarrow{\chi} \widehat{G} \xrightarrow{\gamma} \mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \xrightarrow{\alpha_x^*} \mathrm{Pic}(G),$$

is exact for any $x \in X$, where $\alpha_x : G \to X$ denotes the orbit map.

Proof. By Proposition 4.3.1, it suffices to check exactness at $\mathrm{Pic}(X)$. In view of Propositions 5.1.2 and 5.1.3, the map $p_1^* : \mathrm{Pic}(G) \to \mathrm{Pic}(G \times X)$ induces an isomorphism

$$\mathrm{Pic}(G) \xrightarrow{\cong} \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X),$$

with inverse the map induced by $(\mathrm{id} \times x)^*$ for any $x \in X$. Moreover, the composition $(\mathrm{id} \times x)^* \circ \psi : \mathrm{Pic}(X) \to \mathrm{Pic}(G)$ is just the orbit map α_x^*; this yields the statement in view of Proposition 4.3.1 again. □

Prominent examples of principal bundles are quotients of algebraic groups by closed subgroups. For these, Proposition 4.1.5 and Corollary 5.2.2 imply readily the following result, due to Raynaud in a greater generality (see [Ray70, Prop. VII.1.5]):

Corollary 5.2.3. *Let H be a closed connected subgroup of G. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{U}(G/H) \longrightarrow \widehat{G} \longrightarrow \widehat{H} \xrightarrow{\gamma} \mathrm{Pic}(G/H) \longrightarrow \mathrm{Pic}(G) \longrightarrow \mathrm{Pic}(H),$$

where γ denotes the characteristic homomorphism, and all other maps are pull-backs.

Finally, we obtain an existence result for linearizations in the setting of complete (not necessarily normal) varieties, after [MFK94, Prop. 1.5]:

Proposition 5.2.4. *Let X be a complete irreducible G-variety, and L a G-invariant line bundle over X. Then there exists a positive integer n such that $L^{\otimes n}$ is G-linearizable.*

Proof. Note that $\mathcal{O}(X)^* = k^*$, since every regular function on X is constant. Also, recall from [Ram64, Cor. 2] that the group $\mathrm{Aut}^{\mathbb{G}_m}(L)$ has a natural structure of locally algebraic group (possibly with infinitely many components) acting algebraically on L. Since G is connected, it follows that $\mathcal{G}(L)$ is an algebraic group acting algebraically on L; moreover, (9) yields a central extension of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(L) \longrightarrow G \longrightarrow 1. \tag{15}$$

In particular, the variety $\mathcal{G}(L)$ is a $\mathbb{G}_m$-bundle over G. Since G is affine, so is $\mathcal{G}(L)$ by Proposition 2.3.3. Thus, the algebraic group $\mathcal{G}(L)$ is linear, in view of Corollary 2.2.6. By Lemma 5.2.5 below, there exist a character $\chi \in \widehat{\mathcal{G}(L)}$ and a positive integer n such that $\chi(t) = t^n$ for all $t \in \mathbb{G}_m \subseteq \mathcal{G}(L)$. Equivalently, the push-out of the extension (15) by the nth power map of $\mathbb{G}_m$ is the trivial extension. In view of Lemma 3.4.4, this means that the extension of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(L^{\otimes n}) \longrightarrow G \longrightarrow 1$$

is trivial. Thus, $L^{\otimes n}$ is linearizable by Remark 3.4.3 (ii). □

Lemma 5.2.5. *Consider a central extension of algebraic groups*

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G} \xrightarrow{f} G \longrightarrow 1,$$

where G is connected. Then there exist a character $\chi \in \widehat{\mathcal{G}}$ and a positive integer n such that $\chi(t) = t^n$ for all $t \in \mathbb{G}_m$ (identified with a closed subgroup of $\mathcal{G}$).

Proof. Note that $\mathcal{G}$ is connected. If G is reductive, then so is $\mathcal{G}$. Thus, $\mathcal{G} = \mathcal{C} \cdot (\mathcal{G}, \mathcal{G})$, where $\mathcal{C}$ is a central torus, and the commutator subgroup $(\mathcal{G}, \mathcal{G})$ is semi-simple; moreover, $\mathcal{C} \cap (\mathcal{G}, \mathcal{G})$ is finite (see [Bo91, Prop. 14.2]). Likewise, $G = C \cdot (G, G)$ and $C \cap (G, G)$ is finite. Moreover, f restricts to an exact sequence of tori

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{C} \longrightarrow C \longrightarrow 1,$$

and to an isogeny of semi-simple groups

$$\varphi : (\mathcal{G}, \mathcal{G}) \longrightarrow (G, G).$$

So there exists $\chi \in \widehat{\mathcal{C}}$ such that $\chi(t) = t$ for all $t \in \mathbb{G}_m$; also, a positive multiple $n\chi$ extends to a character of $\mathcal{G}$ if and only if $n\chi$ vanishes identically on the scheme-theoretic intersection $\mathcal{C} \cap (\mathcal{G}, \mathcal{G})$, a finite group scheme. This yields the existence of n in this case.

In the general case, the unipotent radical $R_u(\mathcal{G})$ is sent isomorphically to $R_u(G)$. Thus, we obtain a central extension of connected reductive groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}/R_u(\mathcal{G}) \longrightarrow G/R_u(G) \longrightarrow 1.$$

We conclude by the above step, since $\widehat{\mathcal{G}/R_u(\mathcal{G})} = \widehat{\mathcal{G}}$. □

5.3 Local G-quasi-projectivity and applications

Throughout this subsection, we still denote by G a connected linear algebraic group. We first obtain two preliminary results:

Lemma 5.3.1. *Let X be a normal irreducible G-variety, and D a Weil divisor on X. Then $g^*(D) \sim D$ for any $g \in G$.*

Proof. Consider the regular locus $X_{\rm reg} \subseteq X$; this is a nonempty open G-stable subset of X, with complement of codimension at least 2. Thus, it suffices to show the statement for the pull-back $D_{X_{\rm reg}}$. So we may assume that X is regular; then D is a Cartier divisor. Denote by L the associated line bundle over X. Then L is G-invariant by Theorem 5.2.1; this is equivalent to the desired statement. □

Lemma 5.3.2. *Let X be a normal irreducible G-variety, and $D \subset X$ a subvariety of pure codimension 1, viewed as an effective Weil divisor on X. If D contains no G-orbit, then D is a Cartier divisor, generated by global sections s_g ($g \in G$), such that $\operatorname{div}(s_g) = g^*(D)$. If in addition $X \setminus D$ is affine, then D is ample.*

Proof. Let $U := X \setminus D$; this is an open subset of X, and the translates $g \cdot U$, where $g \in G$, cover X (since D contains no G-orbit). Also, the pull-back D_U is trivial. For any $g \in G$, there exists $f = f_g \in k(X)^*$ such that $g^*(D) = D + \operatorname{div}(f)$, in view of Lemma 5.3.1. Thus, the pull-back $g^*(D)_U$ is trivial as well; equivalently, $D_{g \cdot U}$ is trivial. It follows that D is Cartier. Moreover, for any g as above, there exists a global section $s = s_g \in \Gamma(X, \mathcal{O}_X(D))$ with divisor $g^*(D)$. Since the supports of these divisors have no common point, D is generated by the global sections s_g ($g \in G$).

Next, assume that U is affine; then so is of course each translate $g \cdot U$. Note that $\mathcal{O}(g \cdot U)$ is the increasing union of the vector spaces $\Gamma(X, \mathcal{O}_X(nD))s_g^{-n}$, where n runs over the positive integers. Since the algebra $\mathcal{O}(g \cdot U)$ is finitely generated, there exist n and a finite-dimensional subspace $V \subseteq \Gamma(X, \mathcal{O}_X(nD))$ such that $s_g^n \in V$ and the subspace $V s_g^{-n}$ generates $\mathcal{O}(g \cdot U)$. As X is covered by finitely many translates $g_1 \cdot U, \ldots, g_m \cdot U$, we may choose n and V so that each algebra $\mathcal{O}(g_i \cdot U)$ is generated by $V s_{g_i}^{-n}$. This yields a morphism

$$f_V : X \longrightarrow \mathbb{P}(V^\vee),$$

which restricts to an immersion on each $g_i \cdot U = f_V^{-1}(\mathbb{P}(V^\vee \setminus H_i)$, where H_i denotes the hyperplane of $\mathbb{P}(V^\vee)$ with equation $s_{g_i}^n \in V$. Thus, f_V is an immersion, and $\mathcal{O}_X(nD) = f_V^* \mathcal{O}_{\mathbb{P}(V^\vee)}(1)$ is very ample. □

Next, we come to Sumihiro's theorem presented in the introduction:

Theorem 5.3.3. *Let X be a normal irreducible G-variety. Then X admits a covering by G-quasi-projective open subsets.*

Proof. Let $x \in X$ and choose an affine open neighborhood U of x. Then $G \cdot U$ is a G-stable neighborhood of x containing U. We may thus replace X with $G \cdot U$, and it suffices to show that X is then G-quasi-projective.

Since $X = G \cdot U$, the complement $D := X \setminus U$ contains no G-orbit; also, D has pure codimension 1 in X in view of Lemma 4.1.1. By Lemma 5.3.2, it follows that D is an ample Cartier divisor. Let L be the associated line bundle; then some positive power of L is G-linearizable in view of Theorem 5.2.1. By Proposition 3.2.6, we conclude that X is G-quasi-projective. □

Combining Theorem 5.3.3 and Corollary 3.3.3, we obtain readily the following:

Corollary 5.3.4. *Let $f : X \to Y$ be a principal G-bundle, and Z a normal G-variety. Then the associated fibre bundle $X \times^G Z \to Y$ exists.*

The above corollary does not extend to an arbitrary algebraic group G, even to the group of order 2, as shown by the following example due to Hironaka (see [Hi62] and also [Ha74, App. B, Ex. 3.4.1]).

Example 5.3.5. In the projective space $\mathbb{P}^3$ with homogeneous coordinates x_0, x_1, x_2, x_3, consider the two smooth conics

$$C_1 := (x_3 = x_0x_1 + x_1x_2 + x_0x_2 = 0), \quad C_2 := (x_2 = x_0x_1 + x_1x_3 + x_0x_3 = 0).$$

They intersect transversally at the two points

$$P_1 := [1 : 0 : 0 : 0], \quad P_2 := [0 : 1 : 0 : 0].$$

The involution

$$\sigma : \mathbb{P}^3 \longrightarrow \mathbb{P}^3, \quad [x_0 : x_1 : x_2 : x_3] \longmapsto [x_1 : x_0 : x_3 : x_2]$$

exchanges the curves C_1, C_2, and the points P_1, P_2.

Now blow up $\mathbb{P}^3 \setminus \{P_2\}$, first along $C_1 \setminus \{P_2\}$, and then along the strict transform of $C_2 \setminus \{P_2\}$. Likewise, blow up $\mathbb{P}^3 \setminus \{P_1\}$ first along $C_2 \setminus \{P_1\}$, and then along the strict transform of $C_1 \setminus \{P_1\}$. We may glue these two blown-up varieties along their common open subset obtained by blowing up $\mathbb{P}^3 \setminus \{P_1, P_2\}$ along $(C_1 \cup C_2) \setminus \{P_1, P_2\}$. This yields a smooth complete variety Z equipped with a morphism

$$f : Z \longrightarrow \mathbb{P}^3.$$

Moreover, σ lifts to a unique involution τ of Z.

The fibre of f at P_1 is the union of two projective lines ℓ_1, m_1, where m_1 is contracted by the second blow-up. Likewise, the fibre of f at P_2 is the union of two lines ℓ_2, m_2, where m_2 is contracted by the second blow-up. Denote by F_1 (resp. F_2) the fibre of f at a general point of C_1 (resp. C_2). Then we have rational equivalences

$$F_1 \sim \ell_1 + m_1, \quad F_2 \sim m_1, \quad F_1 \sim m_2, \quad F_2 \sim \ell_2 + m_2.$$

As a consequence,

$$\ell_1 + \ell_2 \sim 0. \tag{16}$$

Also, note that τ exchanges ℓ_1 and ℓ_2.

We claim that there exists no affine τ-stable open subset of Z that meets ℓ_1. Indeed, if U is such an open subset, then the complement $D := Z \setminus U$ has pure codimension 1 in Z by Lemma 4.1.1. Moreover, $D \cap (\ell_1 \cup \ell_2)$ is nonempty (since the open affine subset U does not contain the projective line ℓ_1, nor ℓ_2) and finite (since ℓ_1 and $\ell_2 = \tau(\ell_1)$ meet U). Viewing D as a reduced Weil divisor, it follows that the intersection number $D \cdot (\ell_1 + \ell_2)$ is positive. But this contradicts the equivalence (16).

Consider the action of the group H of order 2 on Z via τ. By the claim and Proposition 3.4.8, there exists no quotient morphism $Z \to Y$, where Y is a variety.

Let G be a connected linear group containing H; for example, we may take $G = \mathbb{G}_m$ if $\mathrm{char}(k) \neq 2$, and $G = \mathbb{G}_a$ if $\mathrm{char}(k) = 2$. We claim that the associated fibre bundle $G \times^H Z \to G/H$ does not exist. Otherwise, $G \times^H Z$ is a smooth G-variety, and hence is covered by G-quasi-projective open subsets in view of Theorem 5.3.3. Intersecting these open subsets with $Z \subset G \times^H Z$, we obtain a covering of Z by H-stable quasi-projective open subsets. But then Z is covered by H-stable affine open subsets by Corollary 3.4.7; this yields a contradiction in view of Proposition 3.4.8.

Next, we present a further remarkable consequence of Sumihiro's theorem:

Corollary 5.3.6. *Let X be a normal variety equipped with an action of a torus T. Then X admits a covering by T-stable affine open subsets.*

Proof. In view of Theorem 5.3.3, we may assume that X is a T-stable subvariety of $\mathbb{P}(V)$ for some finite-dimensional T-module V. By Example 2.2.2, the dual module $V^\vee$ admits a basis consisting of T-eigenvectors, say $f_1, \ldots, f_n$. Thus, the complements of the hyperplanes $(f_i = 0)$ in $\mathbb{P}(V)$ form a covering by T-stable affine open subsets. As a consequence, we may assume that X is a T-subvariety of an affine T-variety Z.

We now argue as in the proof of Corollary 3.4.7. Consider the closure $\bar{X}$ of X in Z, and the complement $Y := \bar{X} \setminus X$. Let $x \in X$; then by Example 2.2.2 again, there exists a

T-eigenvector $f \in \mathcal{O}(\bar{X})$ such that f which vanishes identically on Y, and $f(x) \neq 0$. Then the complement of the zero locus of f in $\bar{X}$ is an open affine T-stable subset, containing x and contained in X. □

Finally, we obtain an equivariant version of Chow's Lemma, which asserts that every complete variety is the image of a projective variety under a projective birational morphism:

Corollary 5.3.7. *Let X be a normal complete irreducible G-variety. Then there exist a normal projective irreducible G-variety X' and a projective birational G-equivariant morphism $f : X' \to X$.*

Proof. We adapt the argument sketched in [Ha74, Ex. II.4.10]. By Theorem 5.3.3, X admits an open covering $(U_1, \dots, U_n)$, where each U_i is nonempty, G-stable and G-quasi-projective. Thus, we may view each U_i as a G-stable open subset of a projective G-variety Y_i. Then

$$U := U_1 \cap \cdots \cap U_n$$

is a nonempty G-stable open subset of X, equipped with a diagonal morphism

$$\varphi : U \longrightarrow X \times Y_1 \times \cdots \times Y_n, \quad x \longmapsto (x, x, \dots, x).$$

Denote by X' the closure of the image of U, by

$$f : X' \longrightarrow X$$

the restriction of the projection to X, and by

$$g : X' \longrightarrow Y_1 \times \cdots \times Y_n =: Y$$

the restriction of the projection to the remaining factors. Then the morphism f is projective, since Y is projective.

We claim that the restriction $f^{-1}(U) \to U$ is an isomorphism. For this, note that $\varphi(U)$ is closed in $U \times Y$, as the graph of the diagonal. As a consequence, $\varphi(U) = (U \times Y) \cap X' = f^{-1}(U)$. Since the projection to X induces an isomorphism $\varphi(U) \to U$, this proves the claim.

By that claim, f is birational. To complete the proof, it suffices to show that g is a closed immersion, since the projectivity of X follows from this. As X' is covered by the open subsets $f^{-1}(U_i)$ $(1 \leq i \leq n)$, it suffices in turn to check that f restricts to closed immersions

$$f_i : f^{-1}(U_i) \longrightarrow p_i^{-1}(U_i)$$

for all i, where $p_i : Y \to Y_i$ denotes the projection to the ith factor. Since $f = p_i \circ g$ on U, this also holds on the whole X', and hence f sends $f^{-1}(U_i)$ to $p_i^{-1}(U_i) \cong U_i \times \prod_{j \neq i} Y_j$. Moreover, one may check that $f^{-1}(U_i) \subset U_i \times U_i \times \prod_{j \neq i} Y_j$ is contained in $\mathrm{diag}(U_i) \times \prod_{j \neq i} Y_j$, by a graph argument as in the above step. It follows that f_i is indeed a closed immersion. □

Some further developments.

Most results of Section 4 extend to an arbitrary base field, see [Bri15, 2.2, 2.3] and its references.

The original motivation for Sumihiro's theorem is the question whether there exists an equivariant completion of a given G-variety X, i.e., a complete G-variety containing X as a G-stable dense open subset. By work of Nagata (see [Na62, Na63], and [Lu93] for a modern presentation), every variety admits a completion. It is shown in [Su74, Su75] that every normal G-variety admits an equivariant completion, if G is linear (possibly non-connected). The proof combines the local structure results presented here with valuation-theoretic methods adapted from Nagata's work.

Proposition 5.1.2 holds over an arbitrary base field K, with the same proof. Also, Proposition 5.1.3 extends as follows: let G be a connected linear algebraic group over K. Assume that G is reductive or K is perfect. Then the K-variety G is unirational, and its Picard group is finite (see e.g. [Bo91, Thm. 18.2] for the former assertion, and [Sa81, Lem. 6.9] for the latter). Here a K-variety X is called unirational if there exist an open subvariety U of some affine space $\mathbb{A}^n_K$ and a dominant morphism $U \to X$; equivalently, the function field $K(X)$ is a subfield of a purely transcendental extension of K.

The above assumptions cannot be suppressed in view of Example 5.1.4, which also shows that there exist nontrivial forms of the additive group over any imperfect base field K. A classification of these forms has been obtained by Russell, see [Ru70]. More generally, unipotent algebraic groups over an arbitrary field have been studied e.g. by Kambayashi, Miyanishi, and Takeuchi in [KMT74]. In particular, they showed that some forms of the additive group (including the one presented in Example 5.1.4) have an infinite Picard group, see [KMT74, 6.11, 6.12].

Still, the Picard group of a connected linear algebraic group G over a field K is always torsion, see [Ray70, Cor. VII.1.6]. More specifically, there exists a positive integer $n = n(G)$ such that $n\mathrm{Pic}(G_L) = 0$ for any field extension L of K, see [Bri15, Prop. 2.5]. It follows that $L^{\otimes n}$ is G-linearizable for any line bundle L on a normal G-variety X, see [Bri15, Thm. 2.14].

As a consequence, Corollary 5.2.2 holds over an arbitrary field K, provided that X has a K-rational point; in particular, Corollary 5.2.3 extends without any change. For torsors over smooth varieties, a much more precise result is due to Sansuc (see [Sa81, Prop. 6.10]): for any G-torsor $f : X \to Y$, where G is a connected linear algebraic group (assumed to be reductive if K is imperfect) and X, Y are smooth, there is a natural exact sequence

$$0 \to \mathrm{U}(Y) \to \mathrm{U}(X) \to \widehat{G}(K) \to \mathrm{Pic}(Y) \to \mathrm{Pic}(X) \to \mathrm{Pic}(G) \to \mathrm{Br}(Y) \to \mathrm{Br}(X),$$

where Br denotes the cohomological Brauer group. The proof uses the fppf cohomology.

In another direction, the results of Subsections 5.2 and 5.3 can be extended to non-normal varieties, see [Bri15]. For this, one adapts a cohomological approach like in Subsection 4.3, with the Zariski topology replaced by the étale topology.

The above results can also be extended to actions of possibly non-linear algebraic groups. A key ingredient is Chevalley's structure theorem, which asserts that every connected algebraic group G over an algebraically closed field lies in a unique exact sequence of algebraic groups

$$1 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where L is linear and connected, and A an abelian variety (see [Co02] for a modern proof of this classical result). A version of Chevalley's structure theorem for G-varieties has been obtained by Nishi and Matsumura in [Mat63]: every *smooth* variety equipped with a faithful action of G is equivariantly isomorphic to the associated fibre bundle $G \times^H Y$, for some closed subgroup scheme $H \subset G$ such that $H \supset L$ and H/L is finite, and some H-scheme Y. This reduces somehow the G-action on X to the action of the affine group scheme H on Y. Note that H and Y are not unique, since one may replace H with a larger subgroup scheme $H' \subset G$ such that H'/H is finite, and Y with $H' \times^H Y$. Also, Y is smooth if k has characteristic 0. In positive characteristics, it is an open question whether one may choose H and Y to be smooth.

The Nishi-Matsumura theorem has been extended in [Bri10] to actions of connected algebraic groups (possibly non-linear) on normal varieties. It turns out that every such variety admits an open equivariant covering by associated fibre bundles as above; like in Sumihiro's theorem, examples show that the normality assumption cannot be removed. The existence of an equivariant completion in this setting is an open question.

We mention finally that the equivariant geometry of algebraic spaces or stacks is an active research area; see e.g. [Bi02] for a survey on quotients of algebraic spaces by actions of algebraic groups, and [AHR15] for a local structure theorem on algebraic stacks, which yields in particular a stacky version of Sumihiro's theorem on torus actions.

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