HOMOLOGICAL DIMENSION OF ISOGENY CATEGORIES OF COMMUTATIVE ALGEBRAIC GROUPS

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Abstract. We determine the homological dimension of various isogeny categories of commutative algebraic groups over a field k, in terms of the cohomological dimension of k at certain primes. This generalizes results of Serre, Oort and Milne, by an alternative approach.

1. Introduction

The commutative group schemes of finite type over a field k form an abelian category $\mathcal{C}$. When k is algebraically closed, the homological properties of $\mathcal{C}$ have been investigated by Serre and Oort (see [Se60, Oo66]); in particular, the homological dimension turns out to be 1 in characteristic 0, and 2 otherwise. Building on these results, Milne determined the homological dimension of $\mathcal{C}$ when k is perfect: then

$$\mathrm{hd}(\mathcal{C}) = \begin{cases} \mathrm{cd}(k)+1 & \text{if } \mathrm{char}(k)=0, \\ \max(2, \mathrm{cd}(k)+1) & \text{otherwise} \end{cases}$$

(see [Mi70]). Here $\mathrm{cd}(k)$ denotes the cohomological dimension of k, i.e., the supremum of the ℓth cohomological dimensions $\mathrm{cd}_\ell(k)$ over all primes ℓ.

The aim of this paper is to generalize these results to various isogeny categories, obtained from $\mathcal{C}$ by formally inverting isogenies of a certain type; these may also be viewed as quotient categories $\mathcal{C}/\mathcal{B}$, where $\mathcal{B}$ is a Serre subcategory of $\mathcal{C}$ all of whose objects are finite group schemes. The first result in this direction is again due to Serre in [Se60]: if k is algebraically closed of positive characteristic, then $\mathrm{hd}(\mathcal{C}/\mathcal{I}) = 2$, where $\mathcal{I}$ denotes the full subcategory of $\mathcal{C}$ with objects the infinitesimal group schemes. Note that $\mathcal{C}/\mathcal{I}$ is equivalent to the category of "quasi-algebraic groups" considered in [Se60]; it is obtained from $\mathcal{C}$ by inverting the radicial (or purely inseparable) isogenies.

More recently, it was shown in [Br17] that $\mathrm{hd}(\mathcal{C}/\mathcal{F}) = 1$ for an arbitrary field k, where $\mathcal{F}$ denotes the Serre subcategory of $\mathcal{C}$ with objects the finite group schemes. For this, we adapted the original approach of Serre and Oort, in the somewhat simpler setting of the full isogeny category $\mathcal{C}/\mathcal{F}$. A more conceptual proof is presented in [Br18a], which also contains a representation-theoretic description of $\mathcal{C}/\mathcal{F}$.

In this paper, we consider the radicial isogeny category $\mathcal{C}/\mathcal{I}$ over an arbitrary field, as well as the S-isogeny category $\mathcal{C}/\mathcal{F}_S$ and the étale S-isogeny category $\mathcal{C}/\mathcal{E}_S$, where S denotes a set of prime numbers, and $\mathcal{F}_S$ (resp. $\mathcal{E}_S$) is the full subcategory of $\mathcal{F}$ with objects the S-primary torsion groups (resp. the étale S-primary torsion groups). Our main result can be formulated as follows:

Theorem. *Let k be a field of characteristic $p \geq 0$, and S a subset of the set $\mathbb{P}$ of prime numbers. Set $\mathrm{cd}_{S'}(k) := \sup_{\ell \notin S}(\mathrm{cd}_\ell(k))$ if $S \neq \mathbb{P}$, and $\mathrm{cd}_{S'}(k) = 0$ if $S = \mathbb{P}$.*

(i) *If $p = 0$ or $p \in S$, then $\mathrm{hd}(\mathcal{C}/\mathcal{F}_S) = \mathrm{cd}_{S'}(k) + 1$.*
(ii) *If k is perfect and $p > 0$, then $\mathrm{hd}(\mathcal{C}/\mathcal{E}_S) = \max(2, \mathrm{cd}_{S'}(k) + 1)$.*
(iii) *If $p > 0$, then $\mathrm{hd}(\mathcal{C}/\mathcal{I}) = \max(2, \mathrm{cd}(k) + 1)$.*

This yields the values of $\mathrm{hd}(\mathcal{C}/\mathcal{F}_S)$ and $\mathrm{hd}(\mathcal{C}/\mathcal{E}_S)$ for an arbitrary set S when k is perfect, since $\mathcal{E}_S = \mathcal{F}_S$ if $p \notin S$. Also, this gives back Milne's formula: just take $S = \emptyset$. But the determination of $\mathrm{hd}(\mathcal{C})$ when k is imperfect seems to be an open question, see Remark 3.11 for more details.

Our approach is independent from the results of [Se60, Oo66] (which describe all higher extension groups for "elementary" algebraic groups) and [Mi70] (which constructs a spectral sequence relating the higher extension groups over the field k with those over its algebraic closure, via the cohomology of the absolute Galois group). We follow a suggestion of Oort (see [Oo66, II.15.2]), albeit in a slightly different setting which yields a more uniform result. Specifically, we show that

$$\mathrm{hd}(\mathcal{A}) = \max(\mathrm{hd}(\mathcal{A}_{\mathrm{tors}}), \mathrm{hd}(\mathcal{A}/\mathcal{A}_{\mathrm{tors}})) \tag{1.0.1}$$

for any artinian abelian category $\mathcal{A}$, where $\mathcal{A}_{\mathrm{tors}}$ denotes the full subcategory of $\mathcal{A}$ consisting of those objects X such that the multiplication map n_X is zero for some positive integer $n = n(X)$. When $\mathcal{A}$ is one of the isogeny categories considered above (which are indeed artinian), it turns out that $\mathcal{A}/\mathcal{A}_{\mathrm{tors}}$ has homological dimension at most 1, and $\mathcal{A}_{\mathrm{tors}}$ can be explicitly determined in each case.

This approach also applies to the full subcategory $\mathcal{L}$ of $\mathcal{C}$ with objects the linear (or equivalently, affine) group schemes. Note that $\mathcal{L}$ contains $\mathcal{F}$; thus, we may consider the isogeny category $\mathcal{L}/\mathcal{B}$, where $\mathcal{B}$ equals $\mathcal{F}_S$, $\mathcal{E}_S$ or $\mathcal{I}$. It turns out that $\mathrm{hd}(\mathcal{L}/\mathcal{B}) = \mathrm{hd}(\mathcal{C}/\mathcal{B})$ in all cases, see Remarks 3.11, 3.16 and 3.18 for details.

The layout of this paper is as follows. In Section 2, we gather preliminary results on artinian abelian categories and the corresponding pro-artinian categories. These results are closely related to classical work of Gabriel in [Ga62, Chap. III]. Some of them may be well known (for example, the fact that the quotient of any artinian abelian category by a Serre subcategory is artinian), but we could not locate appropriate references. The main result of this section is Theorem 2.22, which yields a more general version of (1.0.1).

In Section 3, we first describe the three subcategories $\mathcal{F}_S$, $\mathcal{E}_S$ and $\mathcal{I}$ of $\mathcal{C}$, and the corresponding isogeny categories. Then we prove the main theorem under the assumption that k is perfect, for each of these three cases (Theorems 3.7, 3.14 and 3.15). Finally, we show that the categories $\mathcal{C}/\mathcal{I}$ and $\mathcal{C}/\mathcal{F}_S$, where $p \in S$, are invariant under purely inseparable extension of the base field k, thereby completing the proof of the main theorem.

Further developments and applications to the fundamental group are presented in [Br18b].

2. Preliminaries on artinian abelian categories

2.1. **Quotients of artinian categories.** Throughout this subsection, $\mathcal{A}$ denotes an abelian category. We assume that $\mathcal{A}$ is *artinian*, i.e., for any $X \in \mathcal{A}$, every descending chain of subobjects of X is stationary.

Let $\mathcal{B}$ be a full subcategory of $\mathcal{A}$. We assume that $\mathcal{B}$ is *a Serre subcategory*, i.e., is stable under taking subobjects, quotients and extensions in $\mathcal{A}$; then $\mathcal{B}$ is an abelian subcategory of $\mathcal{A}$. We denote by ${}^{\perp}\mathcal{B}$ the full subcategory of $\mathcal{A}$ consisting of those objects X for which $\mathrm{Hom}_{\mathcal{A}}(X, Y) = 0$ for all $Y \in \mathcal{B}$; thus, $\mathcal{B}$ and ${}^{\perp}\mathcal{B}$ are disjoint.

Lemma 2.1. (i) *The subcategory ${}^{\perp}\mathcal{B}$ is stable under taking quotients and extensions in $\mathcal{A}$.*

(ii) *For any $X \in \mathcal{A}$, there exists an exact sequence*

$$0 \longrightarrow X^{\mathcal{B}} \longrightarrow X \longrightarrow X_{\mathcal{B}} \longrightarrow 0, \tag{2.1.1}$$

where $X^{\mathcal{B}} \in {}^{\perp}\mathcal{B}$ and $X_{\mathcal{B}} \in \mathcal{B}$, and such a sequence is unique up to isomorphism.

Proof. (i) This follows readily from the fact that every exact sequence in $\mathcal{A}$

$$0 \longrightarrow X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X_2, Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X_1, Y).$$

(ii) Since $\mathcal{A}$ is artinian, there exists a subobject Z of X such that $X/Z \in \mathcal{B}$ and Z is minimal for this property. Assume that $Z \notin {}^{\perp}\mathcal{B}$; then we may choose a non-zero morphism $f : Z \to W$, where $W \in \mathcal{B}$. Since $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$, we have that $Z/\mathrm{Ker}(f) \in \mathcal{B}$ and then $X/\mathrm{Ker}(f) \in \mathcal{B}$. But $\mathrm{Ker}(f) \subsetneq Z$, a contradiction. This shows the existence of the exact sequence (2.1.1).

For the uniqueness, consider another exact sequence

$$0 \longrightarrow Z' \longrightarrow X \longrightarrow Y' \longrightarrow 0,$$

where $Z' \in {}^\perp\mathcal{B}$ and $Y' \in \mathcal{B}$. View Z and Z' as subobjects of X. By the definition of ${}^\perp\mathcal{B}$, the composition $Z' \to X \to Y$ is zero; thus, Z' is a subobject of Z. Then Z/Z' is in ${}^\perp\mathcal{B}$ (as a quotient of $Z \in {}^\perp\mathcal{B}$) and in $\mathcal{B}$ (as a subobject of $X/Z' = Y'$), and hence is zero. Thus, $Z' = Z$, and $Y' \cong Y$. □

Remark 2.2. By Lemma 2.1 (ii), the pair $({}^\perp\mathcal{B}, \mathcal{B})$ of subcategories of $\mathcal{A}$ is a torsion pair, as defined e.g. in [BR07, Sec. 1.1]. As a consequence, the assignment $X \mapsto X_{\mathcal{B}}$ extends to an additive functor $\mathcal{A} \to \mathcal{B}$ which is left adjoint to the inclusion $\mathcal{B} \to \mathcal{A}$, and hence right exact. Likewise, the assignment $X \mapsto X^{\mathcal{B}}$ extends to an additive functor $\mathcal{A} \to {}^\perp\mathcal{B}$ which is right adjoint to the inclusion ${}^\perp\mathcal{B} \to \mathcal{A}$, and hence left exact (see [loc. cit.] for these results).

Next, consider the quotient category $\mathcal{A}/\mathcal{B}$: its objects are those of $\mathcal{A}$ and we have

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y) = \lim_{\to} \big(\mathrm{Hom}_{\mathcal{A}}(X', Y/Y')\big)$$

for all $X, Y \in \mathcal{A}$, where the direct limit is taken over all subobjects $X' \subset X$ such that $X/X' \in \mathcal{B}$, and all subobjects $Y' \subset Y$ such that $Y' \in \mathcal{B}$. We denote by

$$Q : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{B}$$

the quotient functor (which is exact). In particular, we have a canonical morphism of abelian groups

$$Q = Q(X, Y) : \mathrm{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$$

for any $X, Y \in \mathcal{A}$.

We now obtain a simpler description of the quotient category $\mathcal{A}/\mathcal{B}$, which generalizes [Br17, Lem. 3.1]:

Lemma 2.3. *Let $\mathcal{C}$ be the full subcategory of $\mathcal{A}/\mathcal{B}$ with objects those of ${}^\perp\mathcal{B}$.*

(i) *The inclusion of $\mathcal{C}$ in $\mathcal{A}/\mathcal{B}$ is an equivalence of categories.*

(ii) *For any $X, Y \in \mathcal{C}$, we have*

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \lim_{\to} \big(\mathrm{Hom}_{\mathcal{A}}(X, Y/Y')\big),$$

where the direct limit is taken over all $Y' \subset Y$ such that $Y' \in \mathcal{B}$.

(iii) *With the assumptions of* (ii)*, let $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ with representative $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y/Y')$. Then $\varphi = 0$ (resp. φ is a monomorphism, an epimorphism) if and only if $f = 0$ (resp.* $\mathrm{Ker}(f) \in \mathcal{B}$*, f is an epimorphism).*

Proof. (i) Let $X \in \mathcal{A}$, then the morphism $X^{\mathcal{B}} \to X$ is an isomorphism in $\mathcal{A}/\mathcal{B}$ with the notation of Lemma 2.1 (ii). Thus, the inclusion $\mathcal{C} \to \mathcal{A}/\mathcal{B}$ is essentially surjective.

(ii) Let $X \in {}^\perp\mathcal{B}$ and $X' \subset X$ such that $X/X' \in \mathcal{B}$. Then the quotient morphism $X \to X/X'$ is zero by the definition of ${}^\perp\mathcal{B}$. Thus, $X' = X$.

(iii) By [Ga62, III.1.Lem. 2], φ is zero (resp. a monomorphism, an epimorphism) if and only if $\mathrm{Im}(f)$ (resp. $\mathrm{Ker}(f)$, $\mathrm{Coker}(f)$) is in $\mathcal{B}$. As $X \in {}^{\perp}\mathcal{B}$, we have $\mathrm{Im}(f) \in \mathcal{B}$ if and only if $f = 0$. Likewise, as $Y \in {}^{\perp}\mathcal{B}$, we have $\mathrm{Coker}(f) \in \mathcal{B}$ if and only if f is an epimorphism. □

Lemma 2.4. *Let $X, Y \in \mathcal{A}$ and let $\varphi \in \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$ be a monomorphism. Then there exists a commutative triangle in $\mathcal{A}/\mathcal{B}$*

$$\begin{array}{ccc} \tilde{X} & & \\ {\scriptstyle \psi}\downarrow & \searrow^{Q(f)} & \\ X & \xrightarrow{\varphi} & Y, \end{array}$$

where ψ is an isomorphism and $f \in \mathrm{Hom}_{\mathcal{A}}(\tilde{X}, Y)$ is a monomorphism.

Proof. By Lemma 2.3, we may assume that $X \in {}^{\perp}\mathcal{B}$; then φ is represented by a morphism $g : X \to Y/Y'$ in $\mathcal{A}$, where $Y' \subset Y$ and $Y' \in \mathcal{B}$. Moreover, $\mathrm{Ker}(g) \in \mathcal{B}$ as φ is a monomorphism in $\mathcal{A}/\mathcal{B}$. We may thus replace X by $X/\mathrm{Ker}(g)$, and assume that g is a monomorphism in $\mathcal{A}$. Form the fibered square in $\mathcal{A}$

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & Y \\ {\scriptstyle \tilde{q}}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{g} & Y/Y', \end{array}$$

where q denotes the quotient morphism. Then f is a monomorphism in $\mathcal{A}$; also, q and $\tilde{q}$ yield isomorphisms in $\mathcal{A}/\mathcal{B}$. □

Proposition 2.5. *The quotient category $\mathcal{A}/\mathcal{B}$ is artinian.*

Proof. Let $X \in \mathcal{A}$ and consider a descending chain $(X_n)_{n \geq 1}$ of subobjects of X in $\mathcal{A}/\mathcal{B}$, i.e., $X_1 = X$ and there exist monomorphisms $\varphi_n \in \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X_{n+1}, X_n)$ for all $n \geq 1$. In view of Lemma 2.4, there is a monomorphism $f_1 : \tilde{X}_2 \to X_1$ in $\mathcal{A}$ such that $\tilde{X}_2 \cong X_2$ in $\mathcal{A}/\mathcal{B}$ and this isomorphism identifies $Q(f_1)$ with φ_1. Thus, we may replace φ_1 with $Q(f_1)$ and assume that φ_1 is the image of a monomorphism in $\mathcal{A}$. Iterating this construction, we may assume that $(X_n)_{n \geq 1}$ is a descending chain of subobjects of X in $\mathcal{A}$. Hence this sequence is stationary. □

2.2. **The lifting property.** We continue to consider an abelian category $\mathcal{A}$ and a Serre subcategory $\mathcal{B}$.

Definition 2.6. The pair $(\mathcal{A}, \mathcal{B})$ satisfies the *lifting property* if for any epimorphism $X \to Y$ in $\mathcal{A}$, where $Y \in \mathcal{B}$, there exists a subobject $X' \subset X$ such that $X' \in \mathcal{B}$ and the composition $X' \to X \to Y$ is an epimorphism in $\mathcal{B}$.

Equivalently, given an exact sequence

$$0 \longrightarrow Z \longrightarrow X \longrightarrow Y \longrightarrow 0$$

in $\mathcal{A}$, where $Y \in \mathcal{B}$, there exists a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \longrightarrow & V & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \scriptstyle{\mathrm{id}} & & \\ 0 & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0, \end{array}$$

where $V, W \in \mathcal{B}$.

The lifting property is often used to show that the bounded derived category of $\mathcal{B}$ is naturally equivalent to the full subcategory of the bounded derived category of $\mathcal{A}$ with objects having cohomology in $\mathcal{B}$ (see [KS05, Thm. 13.2.8] and its applications). We now obtain a handy description of morphisms in the quotient category $\mathcal{A}/\mathcal{B}$:

Lemma 2.7. *Assume that $(\mathcal{A}, \mathcal{B})$ satisfies the lifting property. Then the natural map*

$$\lim_{\rightarrow} \big(\mathrm{Hom}_{\mathcal{A}}(X, Y/Y')\big) \longrightarrow \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$$

is an isomorphism for all $X, Y \in \mathcal{A}$ (where the direct limit is taken over all $Y' \subset Y$ such that $Y' \in \mathcal{B}$).

Proof. Let $\varphi \in \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y)$ be represented by $f \in \mathrm{Hom}_{\mathcal{A}}(X', Y/Y')$, where $X' \subset X$, $Y' \subset Y$ and $X/X', Y' \in \mathcal{B}$. By the lifting property, there exists $Z \subset X$ such that $Z \in \mathcal{B}$ and $X = X' + Z$. We then have a commutative diagram in $\mathcal{A}$

$$\begin{array}{ccccc} X & \xleftarrow{\;i\;} & X' & \xrightarrow{\;f\;} & Y/Y' \\ \downarrow & & \downarrow & & \downarrow \\ X/Z & \xleftarrow{\;j\;} & X'/X' \cap Z & \xrightarrow{\;g\;} & Y/Y'', \end{array}$$

where i denotes the inclusion, j the canonical isomorphism, $X' \cap Z \in \mathcal{B}$, $Y'' := Y' + f(X' \cap Z) \in \mathcal{B}$, and the vertical arrows are quotients. It follows that φ is represented by a morphism $h : X \to Y/Y''$ in $\mathcal{A}$. This shows the surjectivity of the considered map.

For the injectivity, let $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y/Y')$ such that $Q(f) = 0$. Then $\mathrm{Im}(f) \in \mathcal{B}$ by [Ga62, III.1.Lem. 2]. Denote by Y'' the preimage of $\mathrm{Im}(f)$ in Y; then $Y'' \in \mathcal{B}$ and the composition $X \to Y/Y' \to Y/Y''$ is zero in $\mathcal{A}$. □

The following lemma yields a dual statement to that of Lemma 2.1, in the presence of the lifting property:

Lemma 2.8. *Assume that $\mathcal{A}$ is artinian and $(\mathcal{A}, \mathcal{B})$ satisfies the lifting property. Then every object $X \in \mathcal{A}$ lies in an exact sequence*

$$0 \longrightarrow Z \longrightarrow X \longrightarrow Y \longrightarrow 0$$

in $\mathcal{A}$, where $Z \in \mathcal{B}$ and $Y \in {}^{\perp}\mathcal{B}$.

Proof. Consider the exact sequence (2.1.1) and use the lifting property to choose a $Z \subset X$ such that $Z \in \mathcal{B}$ and the composition $Z \to X \to X_{\mathcal{B}}$ is an epimorphism in $\mathcal{A}$. Then $X = Z + X^{\mathcal{B}}$ and hence $X/Z \cong X^{\mathcal{B}}/X^{\mathcal{B}} \cap Z \in {}^{\perp}\mathcal{B}$. $\square$

Next, we show that every exact sequence in $\mathcal{A}/\mathcal{B}$ can be lifted to an exact sequence in $\mathcal{A}$, thereby generalizing [Br17, Prop. 3.5] with a simpler proof:

Lemma 2.9. *Assume that $\mathcal{A}$ is artinian and $(\mathcal{A}, \mathcal{B})$ satisfies the lifting property. Consider a complex*

$$0 \longrightarrow X_n \xrightarrow{\varphi_{n-1}} X_{n-1} \xrightarrow{\varphi_{n-2}} \cdots \xrightarrow{\varphi_1} X_1 \longrightarrow 0 \tag{2.2.1}$$

in $\mathcal{A}/\mathcal{B}$.

(i) *There exists a complex*

$$0 \longrightarrow Y_n \xrightarrow{f_{n-1}} Y_{n-1} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} Y_1 \longrightarrow 0 \tag{2.2.2}$$

in $\mathcal{A}$, together with epimorphisms $g_i : X_i \to Y_i$ in $\mathcal{A}$ for $i = 1, \ldots, n$, such that $Y_i \in {}^{\perp}\mathcal{B}$ and $\mathrm{Ker}(g_i) \in \mathcal{B}$ for $i = 1, \ldots, n$, and the diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & X_n & \xrightarrow{\varphi_{n-1}} & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_2 & \xrightarrow{\varphi_1} & X_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Q(g_n)} & & \downarrow{\scriptstyle Q(g_{n-1})} & & & & \downarrow{\scriptstyle Q(g_2)} & & \downarrow{\scriptstyle Q(g_1)} & & \\ 0 & \longrightarrow & Y_n & \xrightarrow[Q(f_{n-1})]{} & Y_{n-1} & \longrightarrow & \cdots & \longrightarrow & Y_2 & \xrightarrow[Q(f_1)]{} & Y_1 & \longrightarrow & 0 \end{array} \tag{2.2.3}$$

commutes in $\mathcal{A}/\mathcal{B}$.

(ii) *If the complex (2.2.1) is exact in $\mathcal{A}/\mathcal{B}$, then (2.2.2) may be chosen to be exact in $\mathcal{A}$.*

Proof. (i) If $n = 1$, then we just have an object $X \in \mathcal{A}$ and the assertion follows from Lemma 2.8.

For an arbitrary n, this lemma yields subobjects $X'_i \subset X_i$ such that $X'_i \in \mathcal{B}$ and $X_i/X'_i \in {}^{\perp}\mathcal{B}$ for $i = 1, \ldots, n$. Replacing each X_i with X_i/X'_i, we may thus assume that $X_i \in {}^{\perp}\mathcal{B}$. In view of Lemma 2.3, φ_{n-1} is represented by an $f_{n-1} \in \mathrm{Hom}_{\mathcal{A}}(X_n, X_{n-1}/X'_{n-1})$ for some $X'_{n-1} \subset X_{n-1}$ such that $X'_{n-1} \in \mathcal{B}$. Then $X_{n-1}/X'_{n-1} \in {}^{\perp}\mathcal{B}$ and the quotient morphism $X_{n-1} \to X_{n-1}/X'_{n-1}$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. Therefore, we may replace X_{n-1} with X_{n-1}/X'_{n-1}, and hence assume that $\varphi_{n-1} = Q(f_{n-1})$ for some $f_{n-1} \in \mathrm{Hom}_{\mathcal{A}}(X_n, X_{n-1})$. Arguing similarly with $\varphi_{n-2}, \ldots, \varphi_1$, we reduce to the case where there exist

$f_i \in \mathrm{Hom}_{\mathcal{A}}(X_i, X_{i-1})$ such that $\varphi_i = Q(f_i)$ for $i = 1, \ldots, n-1$. As $\varphi_i \circ \varphi_{i+1} = 0$ for $i = 1, \ldots, n-2$, we obtain $f_i \circ f_{i+1} = 0$ by Lemma 2.3 again.

(ii) By (i), we may assume that $X_i \in {}^{\perp}\mathcal{B}$ and $\varphi_i = Q(f_i)$ for $i = 1, \ldots, n-1$, where

$$0 \longrightarrow X_n \xrightarrow{f_{n-1}} X_{n-1} \cdots \xrightarrow{f_1} X_1 \longrightarrow 0 \tag{2.2.4}$$

is a complex in $\mathcal{A}$ with homology objects in $\mathcal{B}$. Then f_1 is an epimorphism in $\mathcal{A}$ by Lemma 2.3 again. Also, since $\mathrm{Ker}(f_1)/\mathrm{Im}(f_2) \in \mathcal{B}$, we may choose an $X_2' \subset \mathrm{Ker}(f_1) \subset X_2$ such that $X_2' \in \mathcal{B}$ and $\mathrm{Ker}(f_1) = X_2' + \mathrm{Im}(f_2)$. This yields a commutative diagram in $\mathcal{A}$

$$\begin{array}{ccccccc} X_3 & \xrightarrow{f_2} & X_2 & \xrightarrow{f_1} & X_1 & \longrightarrow & 0 \\ \downarrow \mathrm{id} & & \downarrow h & & \downarrow \mathrm{id} & & \\ X_3 & \xrightarrow{g_2} & X_2/X_2' & \xrightarrow{g_1} & X_1 & \longrightarrow & 0, \end{array}$$

where h denotes the quotient. Since

$$\mathrm{Ker}(g_1) = h(\mathrm{Ker}(f_1)) = h(\mathrm{Im}(f_2)) = \mathrm{Im}(g_2),$$

the bottom sequence is exact in $\mathcal{A}$. Thus, we may replace (2.2.4) with the complex

$$0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_3 \xrightarrow{g_2} X_2/X_2' \xrightarrow{g_1} X_1 \longrightarrow 0,$$

and hence assume that (2.2.4) is exact at X_2 and X_1. We now argue as above at X_3; this yields a new complex which is exact at X_3, and where the tail $\mathrm{Im}(f_2) \to X_2 \to X_1 \to 0$ is unchanged. Iterating this construction completes the proof. □

2.3. **Pro-artinian categories.** We continue to consider an artinian abelian category $\mathcal{A}$. Recall the definition of the *pro category* $\mathrm{Pro}(\mathcal{A})$: its objects are the filtered projective systems of objects of $\mathcal{A}$, and we have

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(\{X_i\}, \{Y_j\}) = \lim_{\leftarrow j} \lim_{\rightarrow, i} \big(\mathrm{Hom}_{\mathcal{A}}(X_i, Y_j)\big). \tag{2.3.1}$$

The category $\mathcal{A}$ is equivalent to a Serre subcategory of $\mathrm{Pro}(\mathcal{A})$; the latter is abelian, has enough projectives, and every artinian object of $\mathrm{Pro}(\mathcal{A})$ is isomorphic to an object of $\mathcal{A}$. Moreover, $\mathrm{Pro}(\mathcal{A})$ has exact projective limits (in particular, it has arbitrary products), and every object of $\mathrm{Pro}(\mathcal{A})$ is the filtered inverse limit of its artinian quotients (see [Oo64, §1] for these results).

Thus, $\mathrm{Pro}(\mathcal{A})$ is a *pro-artinian category* as defined in [DG70, V.2.2]; this is the dual notion to that of a locally noetherian category introduced in [Ga62, II.4]. Conversely, every pro-artinian category $\mathcal{C}$ is equivalent to $\mathrm{Pro}(\mathcal{A})$, where $\mathcal{A}$ denotes the full subcategory of $\mathcal{C}$ consisting of the artinian objects (see [DG70, V.2.3.1] or [Ga62, II.4.Thm. 1]).

Also, recall the natural isomorphisms

$$\mathrm{Ext}^i_{\mathcal{A}}(X,Y) \xrightarrow{\cong} \mathrm{Ext}^i_{\mathrm{Pro}(\mathcal{A})}(X,Y) \tag{2.3.2}$$

for all $X, Y \in \mathcal{A}$ and all $i \geq 0$ (see [Oo64, Thm. 3.5]). Here the higher extension groups $\mathrm{Ext}^i_{\mathcal{A}}$ are defined via equivalence classes of Yoneda extensions in the abelian category $\mathcal{A}$.

For later use, we record a characterization of projective objects of $\mathrm{Pro}(\mathcal{A})$:

Lemma 2.10. *The following are equivalent for an object $P \in \mathrm{Pro}(\mathcal{A})$:*

(i) *P is projective in $\mathrm{Pro}(\mathcal{A})$.*
(ii) *$\mathrm{Ext}^1_{\mathrm{Pro}(\mathcal{A})}(P, Z) = 0$ for any $Z \in \mathcal{A}$.*
(iii) *For any epimorphism $X \to Y$ in $\mathcal{A}$, the induced map*

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(P, X) \longrightarrow \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(P, Y)$$

is surjective.

Proof. (i) ⇒ (ii) This is well known.

(ii) ⇒ (iii) This follows from the long exact sequence for higher extension groups.

(iii) ⇒ (i) This is a consequence of [DG70, V.2.3.5]. □

Next, let $\mathcal{B}$ be a Serre subcategory of the artinian abelian category $\mathcal{A}$. By (2.3.1), we may identify $\mathrm{Pro}(\mathcal{B})$ with the full subcategory of $\mathrm{Pro}(\mathcal{A})$ consisting of the filtered projective systems of objects of $\mathcal{B}$.

Lemma 2.11. *With the above notation and assumptions, the following conditions are equivalent for $X \in \mathrm{Pro}(\mathcal{A})$:*

(i) *$X \in \mathrm{Pro}(\mathcal{B})$.*
(ii) *Every artinian quotient of X is an object of $\mathcal{B}$.*

Moreover, $\mathrm{Pro}(\mathcal{B})$ is a Serre subcategory of $\mathrm{Pro}(\mathcal{A})$, stable under inverse limits.

Proof. We find it easier to check the dual statement, where $\mathcal{A}$ is a noetherian abelian category, $\mathcal{B}$ a Serre subcategory, and $\mathrm{Pro}(\mathcal{A})$ is replaced with the ind category $\mathrm{Ind}(\mathcal{A})$ (a locally noetherian category).

Let $X \in \mathrm{Ind}(\mathcal{B})$. Then X is a direct limit of objects of $\mathcal{B}$, and hence a sum of subobjects in $\mathcal{B}$ as the latter is a Serre subcategory. Thus, every noetherian subobject Y of X is a finite sum of subobjects in $\mathcal{B}$, and hence $Y \in \mathcal{B}$. Conversely, if every noetherian subobject of X is in $\mathcal{B}$, then $X \in \mathrm{Ind}(\mathcal{B})$ as X is the sum of its noetherian subobjects. This shows the equivalence (i)⇔(ii).

By this equivalence, $\mathrm{Ind}(\mathcal{B})$ is stable under taking subobjects. We show that it is stable under taking quotients as well: let $f : X \to Y$ be an epimorphism in $\mathrm{Ind}(\mathcal{A})$, where $X \in \mathrm{Ind}(\mathcal{B})$. Consider a noetherian subobject Y' of Y; then the pull-back $f^{-1}(Y')$ is an object of $\mathrm{Ind}(\mathcal{B})$. Thus, Y' is a sum of subobjects in $\mathcal{B}$ (since so is $f^{-1}(Y')$). So Y' is a finite such sum, and $Y' \in \mathcal{B}$.

Next, consider an exact sequence $0 \to X_1 \to X \to X_2 \to 0$ in $\mathrm{Ind}(\mathcal{A})$, where $X_1, X_2 \in \mathrm{Ind}(\mathcal{B})$. Let X' be a noetherian subobject of X, then we have an exact sequence $0 \to X_1' \to X' \to X_2' \to 0$, where X_i' is a noetherian subobject of X_i for $i = 1, 2$. Thus, $X_i' \in \mathcal{B}$, and $X' \in \mathcal{B}$ as well. So $\mathrm{Ind}(\mathcal{B})$ is stable under extensions.

Finally, to show that $\mathrm{Ind}(\mathcal{B})$ is stable under direct limits, it suffices to check the stability under direct sums. Let $(X_i)_{i \in I}$ be a family of objects in $\mathrm{Ind}(\mathcal{B})$, and Y a noetherian subobject of $\oplus_{i \in I} X_i$. Note that Y is the union of its subobjects $Y_J := Y \cap \oplus_{j \in J} X_j$, where J runs over the finite subsets of I. Choosing J such that Y_J is maximal, we easily obtain that $Y = Y_J$; then Y is a subobject of $\oplus_{j \in J} X_j$. Hence $Y \in \mathcal{B}$, since $\mathrm{Ind}(\mathcal{B})$ is stable under finite direct sums. □

Given an artinian abelian category $\mathcal{A}$ and a Serre subcategory $\mathcal{B}$, the quotient functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ extends uniquely to an exact functor

$$\mathrm{Pro}(Q) : \mathrm{Pro}(\mathcal{A}) \longrightarrow \mathrm{Pro}(\mathcal{A}/\mathcal{B})$$

which commutes with filtered inverse limits, in view of the dual statements to [KS05, Prop. 6.1.9, Cor. 8.6.8]. Since $\mathrm{Pro}(Q)$ sends every object of $\mathrm{Pro}(\mathcal{B})$ to zero, it factors uniquely through an exact functor

$$R : \mathrm{Pro}(\mathcal{A})/\mathrm{Pro}(\mathcal{B}) \longrightarrow \mathrm{Pro}(\mathcal{A}/\mathcal{B}).$$

Proposition 2.12. *With the above notation and assumptions, R is an equivalence of categories.*

Proof. By the dual statement of [Ga62, III.4.Prop. 8], the subcategory $\mathrm{Pro}(\mathcal{B})$ of $\mathrm{Pro}(\mathcal{A})$ is *colocalizing*, i.e., $\mathrm{Pro}(Q)$ has a left adjoint. In view of the dual statement of [Ga62, III.4.Cor. 1], it follows that the quotient category $\mathrm{Pro}(\mathcal{A})/\mathrm{Pro}(\mathcal{B})$ is pro-artinian. By the structure theorem for these categories (the dual statement of [Ga62, II.4.Thm. 1]), it suffices to show that R restricts to an equivalence of the full subcategories consisting of artinian objects. Also, the isomorphism classes of artinian objects of $\mathrm{Pro}(\mathcal{A}/\mathcal{B})$ are exactly those of objects of $\mathcal{A}/\mathcal{B}$. The latter may be viewed as a full subcategory of $\mathrm{Pro}(\mathcal{A})/\mathrm{Pro}(\mathcal{B})$ by Lemma 2.11; moreover, R restricts to the identity functor on $\mathcal{A}/\mathcal{B}$. Thus, it suffices in turn to show that the isomorphism classes of artinian objects of $\mathrm{Pro}(\mathcal{A})/\mathrm{Pro}(\mathcal{B})$ are exactly those of $\mathcal{A}/\mathcal{B}$.

Let X be an object of $\mathcal{A}/\mathcal{B}$, or equivalently of $\mathcal{A}$. We show that X is artinian in $\mathrm{Pro}(\mathcal{A})/\mathrm{Pro}(\mathcal{B})$ by adapting arguments from Subsection 2.1. Consider a monomorphism $\varphi : Y \to X$ in $\mathrm{Pro}(\mathcal{A})/\mathrm{Pro}(\mathcal{B})$. Then φ is represented by a morphism $f : Y' \to X/X'$ in $\mathrm{Pro}(\mathcal{A})$, where $Y' \subset Y$, $Y/Y' \in \mathrm{Pro}(\mathcal{B})$, $X' \subset X$ and $X' \in \mathrm{Pro}(\mathcal{B})$; moreover, $\mathrm{Ker}(f) \in \mathrm{Pro}(\mathcal{B})$. Replacing Y with an isomorphic object in $\mathrm{Pro}(\mathcal{A})/\mathrm{Pro}(\mathcal{B})$, we may thus assume that φ is represented by $f : Y \to X/X'$ for some $X' \in \mathrm{Pro}(\mathcal{B})$. Then X' is an object of $\mathcal{A}$, and hence of $\mathcal{B}$. So φ is represented by a monomorphism in $\mathcal{A}/\mathcal{B}$. Since the latter category is artinian (Proposition 2.5), it follows that X is indeed artinian in $\mathrm{Pro}(\mathcal{A})/\mathrm{Pro}(\mathcal{B})$.

Conversely, let X be an object of $\mathrm{Pro}(\mathcal{A})$ such that $\mathrm{Pro}(X)$ is artinian in $\mathrm{Pro}(\mathcal{A})/\mathrm{Pro}(\mathcal{B})$. Consider the family $(X_i)_{i\in I}$ of subobjects of X in $\mathrm{Pro}(\mathcal{A})$ such that $X/X_i \in \mathcal{A}$. Then $\cap_{i\in I} X_i = 0$ in $\mathrm{Pro}(\mathcal{A})$, hence $\cap_{i\in I} \mathrm{Pro}(Q)(X_i) = 0$ in $\mathrm{Pro}(\mathcal{A})/\mathrm{Pro}(\mathcal{B})$, since $\mathrm{Pro}(Q)$ is exact and commutes with inverse limits (by the dual statement to [Ga62, III.4.Prop. 9]). But $(\mathrm{Pro}(Q)(X_i))_{i\in I}$ has a minimal element in $\mathrm{Pro}(\mathcal{A})/\mathrm{Pro}(\mathcal{B})$, say $\mathrm{Pro}(Q)(X_{i_0})$. Hence $\mathrm{Pro}(Q)(X_{i_0})$ is zero in that quotient category, i.e., $X_{i_0} \in \mathrm{Pro}(\mathcal{B})$. Thus, X/X_{i_0} is an object of $\mathcal{A}$, isomorphic to X in $\mathrm{Pro}(\mathcal{A})/\mathrm{Pro}(\mathcal{B})$. □

Remark 2.13. In view of the dual statement to [Ga62, III.4.Prop. 10], every colocalizing subcategory $\mathcal{C}$ of $\mathrm{Pro}(\mathcal{A})$ is pro-artinian; moreover, assigning to $\mathcal{C}$ its full subcategory of artinian objects defines a bijective correspondence between colocalizing subcategories of $\mathrm{Pro}(\mathcal{A})$ and Serre subcategories of $\mathcal{A}$. One may check that the inverse bijection is $\mathcal{B} \mapsto \mathrm{Pro}(\mathcal{B})$, by using Lemma 2.11 and [Ga62, III.3.Cor. 1].

Lemma 2.14. *Let $\mathcal{A}$ be an artinian abelian category, and $\mathcal{B}$ a Serre subcategory. Then the pair $(\mathcal{A}, \mathcal{B})$ satisfies the lifting property if and only if every projective object of* $\mathrm{Pro}(\mathcal{B})$ *is projective in* $\mathrm{Pro}(\mathcal{A})$*. Under this assumption, every projective object of* $\mathrm{Pro}(\mathcal{A})$ *is sent by* $\mathrm{Pro}(Q)$ *to a projective object of* $\mathrm{Pro}(\mathcal{A}/\mathcal{B})$.

Proof. Assume that the pair $(\mathcal{A}, \mathcal{B})$ satisfies the lifting property. Let P be a projective object of $\mathrm{Pro}(\mathcal{B})$. By Lemma 2.10, to show that P is projective in $\mathrm{Pro}(\mathcal{A})$, it suffices to check that for any epimorphism $f : X \to Y$ in $\mathcal{A}$ and any morphism $h : P \to Y$ in $\mathrm{Pro}(\mathcal{A})$, there exists $g : P \to X$ in $\mathrm{Pro}(\mathcal{A})$ such that $g \circ f = h$. Write P as a projective system (P_i, f_{ij}), where $P_i \in \mathcal{B}$ and $f_{ij} \in \mathrm{Hom}_{\mathcal{B}}(P_j, P_i)$ for $i \leq j$; then $h \in \lim_{\to,i}(\mathrm{Hom}_{\mathcal{A}}(P_i, Y))$. Choose a representative $h_i : P_i \to Y$ and let $Y' := \mathrm{Im}(h_i)$. Then Y' is a subobject of Y and $Y' \in \mathcal{B}$; hence there exists a subobject X' of $f^{-1}(Y')$ such that $X' \in \mathcal{B}$ and f pulls back to an epimorphism $f' : X' \to Y'$ in $\mathcal{B}$. Since P is projective in $\mathrm{Pro}(\mathcal{B})$, there exist $j \geq i$ and $g_j : P_j \to X'$ such that $g_j \circ f' = h_i \circ f_{ij}$. This yields the desired lift $g : P \to X$.

Conversely, assume that every projective object in $\mathrm{Pro}(\mathcal{B})$ is projective in $\mathrm{Pro}(\mathcal{A})$. Let $f : X \to Y$ be an epimorphism in $\mathcal{A}$, where $Y \in \mathcal{B}$. Since $\mathrm{Pro}(\mathcal{B})$ has enough projectives, there exists an epimorphism $g : P \to Y$ in that category, where P is projective. By assumption, g lifts to a morphism $h : P \to X$. Denote by X' the image of h; then X' is an artinian object of $\mathrm{Pro}(\mathcal{B})$, and hence an object of $\mathcal{B}$ by Lemma 2.11. Moreover, the composition

$X' \to X \to Y$ is an epimorphism. Thus, the pair $(\mathcal{A}, \mathcal{B})$ satisfies the lifting property.

The remaining assertion is a consequence of the dual statement to [Ga62, III.3.Cor. 3]. Alternatively, it can be proved by adapting the above argument: let $P = (P_i, f_{ij})$ be a projective object of $\mathrm{Pro}(\mathcal{A})$. Consider an epimorphism $\varphi : X \to Y$ in $\mathcal{A}/\mathcal{B}$ and a morphism $\psi_i : P_i \to Y$ in $\mathcal{A}/\mathcal{B}$. Replacing X (resp. Y) with $X^{\mathcal{B}}$ (resp. $Y^{\mathcal{B}}$), we may assume that φ is represented by an epimorphism $f : X \to Y/Y'$ in $\mathcal{A}$, for some $Y' \subset Y$ such that $Y' \in \mathcal{B}$ (Lemma 2.3). Also, in view of Lemma 2.7, ψ_i is represented by a morphism $h_i : P_i \to Y/Y''$ in $\mathcal{A}$, for some $Y'' \subset Y$ such that $Y'' \in \mathcal{B}$. By considering $Y/(Y' + Y'')$, we may assume that $Y'' = Y'$. Since P is projective in $\mathrm{Pro}(\mathcal{A})$, there exist $j \geq i$ and a morphism $h_j : P_j \to X$ in $\mathcal{A}$ that lifts $h_i : P_i \to Y/Y'$. Thus, $Q(h_j) \in \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(P_j, X)$ lifts $\psi_i \in \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(P_i, Y)$. □

2.4. **Homological dimension.** Recall that the homological dimension of an abelian category $\mathcal{A}$ is the smallest non-negative integer n (if it exists) such that $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = 0$ for all $X, Y \in \mathcal{A}$ and $i > n$. We then set $n =: \mathrm{hd}(\mathcal{A})$. If no such integer n exists, then we set $\mathrm{hd}(\mathcal{A}) := \infty$.

When $\mathcal{A}$ has enough projectives, its homological dimension is the supremum of the lengths of minimal projective resolutions of all objects (see e.g. [We94, Lem. 4.1.6]). This holds in particular when $\mathcal{A}$ is the pro category of an artinian abelian category (see [Oo66, p. 229].

Given a family of abelian categories $(\mathcal{A}_i)_{i \in I}$, we clearly have

$$\mathrm{hd}\big(\textstyle\bigoplus_{i \in I} \mathcal{A}_i\big) = \sup_{i \in I} \big(\mathrm{hd}(\mathcal{A}_i)\big). \tag{2.4.1}$$

We will also need the following observation:

Lemma 2.15. *Let $\mathcal{A}$ be an artinian abelian category. Then*

$$\mathrm{hd}(\mathcal{A}) = \mathrm{hd}(\mathrm{Pro}(\mathcal{A})).$$

Proof. In view of the isomorphism (2.3.2), we have $\mathrm{hd}(\mathcal{A}) \leq \mathrm{hd}(\mathrm{Pro}(\mathcal{A}))$.

To show the opposite inequality, we may assume that $\mathrm{hd}(\mathcal{A})$ is finite, say n. Let $X \in \mathrm{Pro}(\mathcal{A})$, then we may write X as a filtered projective system (X_i, φ_{ij}), where $X_i \in \mathcal{A}$. By [DG70, V.2.3.9], the natural map

$$\varinjlim \big(\mathrm{Ext}^m_{\mathrm{Pro}(\mathcal{A})}(X_i, Y)\big) \longrightarrow \mathrm{Ext}^m_{\mathrm{Pro}(\mathcal{A})}(X, Y)$$

is an isomorphism for any $m \geq 0$ and any $Y \in \mathcal{A}$. As a consequence, we get that $\mathrm{Ext}^m_{\mathrm{Pro}(\mathcal{A})}(X, Y) = 0$ for any $m > n$ and $Y \in \mathcal{A}$.

If $n = 0$, then it follows that X is projective, in view of Lemma 2.10. Thus, $\mathrm{hd}(\mathrm{Pro}(\mathcal{A})) = 0$ as desired.

If $n \geq 1$, then we consider an exact sequence in $\mathrm{Pro}(\mathcal{A})$

$$0 \longrightarrow X_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0,$$

where P_0 is projective. Then the natural map

$$\mathrm{Ext}^m_{\mathrm{Pro}(\mathcal{A})}(X_1, Y) \longrightarrow \mathrm{Ext}^{m+1}_{\mathrm{Pro}(\mathcal{A})}(X, Y)$$

is an isomorphism for any $m > 0$ and $Y \in \mathrm{Pro}(\mathcal{A})$. Thus $\mathrm{Ext}^m_{\mathrm{Pro}(\mathcal{A})}(X_1, Y) = 0$ for any $m > \max(n-1, 0)$ and $Y \in \mathcal{A}$. By induction on n, it follows that X admits a projective resolution in $\mathrm{Pro}(\mathcal{A})$

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

Thus, $\mathrm{hd}(\mathrm{Pro}(\mathcal{A})) \leq n$. □

Our next statement provides a partial answer to a long-standing question of Oort (see [Oo66, II.15.2]):

Proposition 2.16. *Let $\mathcal{A}$ be an artinian abelian category, and $\mathcal{B}$ a Serre subcategory. If $(\mathcal{A}, \mathcal{B})$ satisfies the lifting property, then*

$$\mathrm{hd}(\mathcal{A}) \geq \max(\mathrm{hd}(\mathcal{B}), \mathrm{hd}(\mathcal{A}/\mathcal{B})). \tag{2.4.2}$$

Proof. By Lemma 2.14 and the isomorphism (2.3.2), the natural map

$$\mathrm{Ext}^i_{\mathcal{B}}(X, Y) \longrightarrow \mathrm{Ext}^i_{\mathcal{A}}(X, Y)$$

is an isomorphism for all $X, Y \in \mathcal{B}$ and $i \geq 0$. Thus, $\mathrm{hd}(\mathcal{B}) \leq \mathrm{hd}(\mathcal{A})$.

The inequality $\mathrm{hd}(\mathcal{A}/\mathcal{B}) \leq \mathrm{hd}(\mathcal{A})$ is obtained by combining Lemmas 2.14 and 2.15, and using the characterization of the homological dimension by lengths of projective resolutions in the pro category. That inequality can also be deduced from Lemma 2.9 as follows: we may again assume that $\mathrm{hd}(\mathcal{A}) =: n$ is finite. Let $X, Y \in \mathcal{A}$ and $\xi \in \mathrm{Ext}^{n+1}_{\mathcal{A}/\mathcal{B}}(X, Y)$. Then ξ is represented by an exact sequence in $\mathcal{A}/\mathcal{B}$

$$0 \longrightarrow Y \longrightarrow X_{n+1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X \longrightarrow 0. \tag{2.4.3}$$

In view of Lemma 2.9 (ii), we may assume (possibly by replacing X, Y with isomorphic objects in $\mathcal{A}/\mathcal{B}$) that (2.4.3) is represented by an exact sequence in $\mathcal{A}$. Since $\mathrm{Ext}^{n+1}_{\mathcal{A}}(X, Y) = 0$ and the quotient functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is exact, it follows that $\xi = 0$. □

We will show in Theorem 2.22 that the equality holds in (2.4.2) when $\mathcal{B}$ is the S-primary torsion subcategory of $\mathcal{A}$ for some set S of prime numbers. But the inequality (2.4.2) is generally strict, as shown by the following:

Example 2.17. We freely use some notions and results from the representation theory of quivers, which can be found e.g. in [Be91, 4.1]. Consider the quiver Q: $1 \longrightarrow 2$ and let $\mathcal{A}$ be the category of finite-dimensional representations of Q over a field k. Then $\mathcal{A}$ is an artinian and noetherian abelian category, and $\mathrm{hd}(\mathcal{A}) = 1$. The simple objects of $\mathcal{A}$ are isomorphic to S_1: $k \longrightarrow 0$ or S_2: $0 \longrightarrow k$; moreover, S_2 is projective and S_1 is not. Let $\mathcal{B}$ be the full subcategory of $\mathcal{A}$ with objects the sums of copies of S_2. Then $\mathcal{B}$

is a Serre subcategory, equivalent to the category k-mod of finite-dimensional k-vector spaces; thus, $\mathrm{hd}(\mathcal{B}) = 0$. Since $\mathcal{B}$ consists of projective objects of $\mathcal{A}$, it satisfies the lifting property. Moreover, the quotient category $\mathcal{A}/\mathcal{B}$ is again equivalent to k-mod, and hence $\mathrm{hd}(\mathcal{A}/\mathcal{B}) = 0$.

2.5. **S-primary torsion subcategories.** Let $\mathbb{P}$ be the set of all prime numbers, and S a subset of $\mathbb{P}$. We denote by $\langle S \rangle$ the set of positive integers n such that all prime factors of n are in S. We say that an object X of an abelian category is S-*torsion* (resp. S-*divisible*) if the multiplication

$$n_X : X \longrightarrow X, \quad x \longmapsto nx$$

is zero for some $n \in \langle S \rangle$ (resp. is an epimorphism for all $n \in \langle S \rangle$). When S is the whole $\mathbb{P}$, we just say that X is torsion, resp. divisible.

Next, let $\mathcal{A}$ be an artinian abelian category. Denote by $\mathcal{A}_S$ (resp. $\mathcal{A}^S$) the full subcategory of $\mathcal{A}$ formed by the S-primary torsion (resp. S-divisible) objects. One may readily check that $\mathcal{A}_S$ is a Serre subcategory of $\mathcal{A}$; moreover, $\mathcal{A}^S = {}^{\perp}\mathcal{A}_S$ with the notation of Subsection 2.1. In view of Lemma 2.1, it follows that every $X \in \mathcal{A}$ lies in a unique exact sequence

$$0 \longrightarrow X^S \longrightarrow X \longrightarrow X_S \longrightarrow 0, \tag{2.5.1}$$

where X^S is S-divisible and X_S is S-primary torsion. With this convention, the torsion subcategory $\mathcal{A}_{\mathrm{tors}}$ considered in the introduction is denoted by $\mathcal{A}_{\mathbb{P}}$.

Also, note the equivalence of categories

$$\bigoplus_{p \in S} \mathcal{A}_p \xrightarrow{\cong} \mathcal{A}_S, \tag{2.5.2}$$

which yields equivalences of categories $\mathcal{A}_{S_1} \times \mathcal{A}_{S_2} \cong \mathcal{A}_{S_1 \cup S_2}$ for any two disjoint subsets S_1, S_2 of $\mathbb{P}$.

For later use, we record the following observation:

Lemma 2.18. (i) *Let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}_S$. Then $\mathcal{A}_S/\mathcal{B}$ is equivalent to $(\mathcal{A}/\mathcal{B})_S$.*
(ii) *Let T be a subset of S. Then $\mathcal{A}_S/\mathcal{A}_T \cong \mathcal{A}_{S \setminus T}$.*

Proof. (i) We may view $\mathcal{A}_S/\mathcal{B}$ and $(\mathcal{A}/\mathcal{B})_S$ as full subcategories of $\mathcal{A}/\mathcal{B}$; thus, it suffices to show that they have the same objects. Let $X \in (\mathcal{A}/\mathcal{B})_S$, then there exists $n \in \langle S \rangle$ such that $n_X = 0$ in $\mathcal{A}/\mathcal{B}$. By [Ga62, III.1.Lem. 2], this is equivalent to the condition that $\mathrm{Im}_{\mathcal{A}}(n_X) \in \mathcal{B}$. Then $\mathrm{Im}_{\mathcal{A}}(n_X)$ is S-primary torsion, and hence X is S-torsion as well, i.e., $X \in \mathcal{A}_S/\mathcal{B}$. The converse implication is obvious.

(ii) This follows readily from the decomposition (2.5.2). □

Lemma 2.19. *With the above notation, the pair $(\mathcal{A}, \mathcal{A}_S)$ satisfies the lifting property.*

Proof. Consider an exact sequence in $\mathcal{A}$

$$0 \longrightarrow Z \longrightarrow X \longrightarrow Y \longrightarrow 0,$$

where $Y \in \mathcal{A}_S$. Then Y is a quotient of X_S, and hence it suffices to check the lifting property for the exact sequence (2.5.1).

We may choose $n \in \langle S \rangle$ such that $n_{X_S} = 0$; then n_{X^S} is an epimorphism as X^S is n-divisible. Applying the snake lemma to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^S & \longrightarrow & X & \longrightarrow & X_S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle n_{X^S}} & & \downarrow{\scriptstyle n_X} & & \downarrow{\scriptstyle n_{X_S}} & & \\ 0 & \longrightarrow & X^S & \longrightarrow & X & \longrightarrow & X_S & \longrightarrow & 0, \end{array}$$

we obtain an epimorphism $\mathrm{Ker}(n_X) \to X_S$. Since $\mathrm{Ker}(n_X) \in \mathcal{A}_S$, this completes the proof. □

The following result extends [Br17, Prop. 3.6 (ii)] to our categorical setting:

Lemma 2.20. *Let $X, Y \in \mathcal{A}$. Then the natural map*

$$Q : \mathrm{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_S}(X, Y)$$

induces an isomorphism

$$S^{-1}\mathrm{Hom}_{\mathcal{A}}(X, Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_S}(X, Y).$$

Proof. For any $n \in \langle S \rangle$, the multiplication map n_X yields an isomorphism in $\mathcal{A}/\mathcal{A}_S$, since $\mathrm{Ker}_{\mathcal{A}}(n_X)$ and $\mathrm{Coker}_{\mathcal{A}}(n_X)$ are S-primary torsion. Thus, $\mathrm{End}_{\mathcal{A}/\mathcal{A}_S}(X)$ is an algebra over the localization $S^{-1}\mathbb{Z}$, and $\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_S}(X, Y)$ is a module over $S^{-1}\mathbb{Z}$. This yields a natural map

$$\gamma : S^{-1}\mathrm{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_S}(X, Y).$$

To show that γ is an isomorphism, we may replace X with X^S. Indeed, the map $\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_S}(X, Y) \to \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_S}(X^S, Y)$ is an isomorphism, since the inclusion $X^S \to X$ is an isomorphism in $\mathcal{A}/\mathcal{A}_S$; also, the map $S^{-1}\mathrm{Hom}_{\mathcal{A}}(X, Y) \to S^{-1}\mathrm{Hom}_{\mathcal{A}}(X^S, Y)$ is an isomorphism as well, since $\mathrm{Ext}^i_{\mathcal{A}}(X_S, Y)$ is S-primary torsion for all $i \geq 0$. Likewise, we may replace Y with Y^S.

Since now X is S-divisible, we have an isomorphism

$$X/\mathrm{Ker}_{\mathcal{A}}(n_X) \xrightarrow{\cong} X$$

in $\mathcal{A}$ for any $n \in \langle S \rangle$. Denote its inverse by

$$n_X^{-1} : X \xrightarrow{\cong} X/\mathrm{Ker}_{\mathcal{A}}(n_X).$$

Then $\gamma(n^{-1}f) = Q(n_Y^{-1} \circ f)$ for any such n and $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$.

We now show that γ is injective. Given n and f as above such that $\gamma(n^{-1}f) = 0$, we have $Q(n_Y^{-1} \circ f) = 0$ in $\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_S}(X, Y)$, and hence $n_Y^{-1} \circ f = 0$

in $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ in view of Lemma 2.3. As n_Y is an epimorphism, it follows that $f=0$.

Finally, we show that γ is surjective. Let $f \in \mathrm{Hom}_{\mathcal{A}/\mathcal{A}_S}(X,Y)$. By Lemma 2.3 again, f is represented by $\varphi \in \mathrm{Hom}_{\mathcal{A}}(X, Y/Y')$ for some S-primary torsion subobject $Y' \subset Y$. Choose $n \in \langle S \rangle$ such that $n_{Y'} = 0$; then $Y' \subset \mathrm{Ker}_{\mathcal{A}}(n_Y)$ and hence f is also represented by some $\psi \in \mathrm{Hom}_{\mathcal{A}}(X, Y/\mathrm{Ker}_{\mathcal{A}}(n_Y))$. It follows that $f = \gamma(n^{-1}\psi)$. □

Lemma 2.21. *We have natural isomorphisms*

$$S^{-1}\mathrm{Ext}^i_{\mathcal{A}}(X,Y) \xrightarrow{\cong} \mathrm{Ext}^i_{\mathcal{A}/\mathcal{A}_S}(X,Y)$$

for all $i \geq 0$ and all $X, Y \in \mathcal{A}$.

Proof. We may choose a projective resolution of X in $\mathrm{Pro}(\mathcal{A})$

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

Then $\mathrm{Ext}^i_{\mathcal{A}}(X,Y)$ is the ith homology group of the complex

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(P_0, Y) \longrightarrow \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(P_1, Y) \longrightarrow \cdots$$

in view of the isomorphism (2.3.2). Since $\mathrm{Pro}(Q)(P_0), \mathrm{Pro}(Q)(P_1), \ldots$ are projective in $\mathrm{Pro}(\mathcal{A}/\mathcal{A}_S)$ (Lemmas 2.14 and 2.19) and $\mathrm{Pro}(Q)$ is exact, we see that $\mathrm{Ext}^i_{\mathcal{A}/\mathcal{A}_S}(X,Y)$ is the ith homology group of the complex

$$0 \to \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A}/\mathcal{A}_S)}(\mathrm{Pro}(Q)(P_0), Y) \to \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A}/\mathcal{A}_S)}(\mathrm{Pro}(Q)(P_1), Y) \to \cdots$$

(we identify $\mathrm{Pro}(Q)(Y) = Q(Y)$ with Y). As localizing by S is exact, it suffices to construct a natural isomorphism

$$S^{-1}\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(Z,Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A}/\mathcal{A}_S)}(\mathrm{Pro}(Q)(Z), Y)$$

for any $Z \in \mathrm{Pro}(\mathcal{A})$ and $Y \in \mathcal{A}$. Write Z as the projective system (Z_i, φ_{ij}), then

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(Z,Y) = \lim_{\to} \big(\mathrm{Hom}_{\mathcal{A}}(Z_i, Y)\big)$$

and hence

$$S^{-1}\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}(Z,Y) = S^{-1} \lim_{\to} \big(\mathrm{Hom}_{\mathcal{A}}(Z_i, Y)\big) = \lim_{\to} \big(S^{-1}\mathrm{Hom}_{\mathcal{A}}(Z_i, Y)\big).$$

By Lemma 2.20, this is naturally isomorphic to

$$\lim_{\to} \big(\mathrm{Hom}_{\mathcal{A}/\mathcal{A}_S}(Z_i, Y)\big) = \mathrm{Hom}_{\mathrm{Pro}(\mathcal{A}/\mathcal{A}_S)}(\mathrm{Pro}(Q)(Z), Y).$$

□

We now come to the main result of this section:

Theorem 2.22. *Let $\mathcal{A}$ be an artinian abelian category, S a set of prime numbers, and $\mathcal{A}_S$ the S-primary torsion subcategory of $\mathcal{A}$. Then*

$$\mathrm{hd}(\mathcal{A}) = \max(\mathrm{hd}(\mathcal{A}_S), \mathrm{hd}(\mathcal{A}/\mathcal{A}_S)).$$

Proof. Combining Proposition 2.16 and Lemma 2.19 yields the inequality $\mathrm{hd}(\mathcal{A}) \geq \max(\mathrm{hd}(\mathcal{A}_S), \mathrm{hd}(\mathcal{A}/\mathcal{A}_S))$.

To show the opposite inequality, we may of course assume that $\mathrm{hd}(\mathcal{A}_S) = d$ and $\mathrm{hd}(\mathcal{A}/\mathcal{A}_S) = e$ are both finite. Let $X, Y \in \mathcal{A}$. In view of the exact sequence (2.5.1) for X and Y, it suffices to show that $\mathrm{Ext}^i_{\mathcal{A}}(X', Y') = 0$ for all $i > \max(d, e)$ and all $X' \in \{X_S, X^S\}$, $Y' \in \{Y_S, Y^S\}$. This vanishing holds when $X' = X_S$ and $Y' = Y_S$, since we then have $\mathrm{Ext}^i_{\mathcal{A}}(X', Y') = \mathrm{Ext}^i_{\mathcal{A}_S}(X', Y')$ by Lemmas 2.14 and 2.19.

When $X' = X^S$ and $Y' = Y_S$, we choose $n \in \langle S \rangle$ such that $n_{Y_S} = 0$; then $\mathrm{Ext}^i_{\mathcal{A}}(Z, Y_S)$ is killed by n for all $i \geq 0$ and $Z \in \mathcal{A}$. Thus, the exact sequence

$$0 \longrightarrow \mathrm{Ker}_{\mathcal{A}}(n_{X^S}) \longrightarrow X^S \xrightarrow{n} X^S \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathrm{Ext}^i_{\mathcal{A}}(X^S, Y_S) \longrightarrow \mathrm{Ext}^i_{\mathcal{A}}(\mathrm{Ker}_{\mathcal{A}}(n_{X^S}), Y_S).$$

Since $\mathrm{Ker}_{\mathcal{A}}(n_{X^S}) \in \mathcal{A}_S$, it follows that $\mathrm{Ext}^i_{\mathcal{A}}(X^S, Y_S) = 0$ for all $i > d$.

When $X' = X_S$ and $Y' = Y^S$, we obtain similarly an exact sequence

$$\mathrm{Ext}^i_{\mathcal{A}}(X_S, \mathrm{Ker}_{\mathcal{A}}(n_{Y^S})) \longrightarrow \mathrm{Ext}^i_{\mathcal{A}}(X_S, Y^S) \longrightarrow 0$$

for all $i \geq 0$ and $n \in \langle S \rangle$ such that $n_{X_S} = 0$. Thus, $\mathrm{Ext}^i_{\mathcal{A}}(X_S, Y^S) = 0$ for all $i > d$.

Finally, we consider the case where $X' = X^S$ and $Y' = Y^S$. For all $i > e$, we have

$$0 = \mathrm{Ext}^i_{\mathcal{A}/\mathcal{A}_S}(X^S, Y^S) = S^{-1}\mathrm{Ext}^i_{\mathcal{A}}(X^S, Y^S)$$

in view of Lemma 2.21. Thus, for any $\xi \in \mathrm{Ext}^i_{\mathcal{A}}(X^S, Y^S)$, we may find some $n \in \langle S \rangle$ (depending on ξ) such that $n\xi = 0$. Then ξ lies in the image of $\mathrm{Ext}^i_{\mathcal{A}}(X^S, \mathrm{Ker}_{\mathcal{A}}(n_{Y^S}))$, and the latter group vanishes if $i \geq d$. □

3. Isogeny categories of algebraic groups

3.1. S-isogeny categories. We return to the setting of the introduction, and consider a field k of characteristic $p \geq 0$. We choose an algebraic closure $\bar{k}$ of k, and denote by k_s the separable closure of k in $\bar{k}$. The Galois group of k_s/k is denoted by Γ; this is a profinite group.

By an *algebraic k-group*, we mean a commutative group scheme of finite type over k. The algebraic k-groups are the objects of an abelian category $\mathcal{C} = \mathcal{C}_k$ with morphisms being the homomorphisms of k-group schemes. Since every descending chain of closed subschemes of a k-scheme of finite type is stationary, the category $\mathcal{C}$ is artinian. The finite (resp. finite étale) k-group schemes form a full subcategory $\mathcal{F} = \mathcal{F}_k$ (resp. $\mathcal{E} = \mathcal{E}_k$) of $\mathcal{C}$; one may easily check that $\mathcal{F}$ and $\mathcal{E}$ are Serre subcategories. By Cartier duality, $\mathcal{E}$ is anti-equivalent to the category Γ-mod of finite abelian groups equipped with a discrete action of Γ (see [DG70, II.5.1.7]).

As in Subsection 2.5, we consider a set S of prime numbers, and the S-primary torsion subcategory $\mathcal{C}_S$ (resp. $\mathcal{F}_S$, $\mathcal{E}_S$) of $\mathcal{C}$ (resp. $\mathcal{F}$, $\mathcal{E}$). We say that the quotient category $\mathcal{C}/\mathcal{F}_S$ is the *S-isogeny category*, and $\mathcal{C}/\mathcal{E}_S$ is the *étale S-isogeny category*. Both are artinian abelian categories in view of Proposition 2.5; the quotient category $\mathcal{C}/\mathcal{F}_{\mathbb{P}}$ is the (full) isogeny category studied in [Br17] and [Br18a].

By Lemma 2.19, the pair $(\mathcal{C}, \mathcal{C}_S)$ has the lifting property. Also, $(\mathcal{C}, \mathcal{F}_{\mathbb{P}})$ has the lifting property as well, in view of the main result of [Br15] (see also [LA15, Thm. 3.2]). As an easy consequence, we obtain:

Lemma 3.1. *The pair $(\mathcal{C}, \mathcal{F}_S)$ has the lifting property. If k is perfect, then $(\mathcal{C}, \mathcal{E}_S)$ has the lifting property as well.*

Proof. Consider an epimorphism $G \to H \to 0$ in $\mathcal{C}$, where $H \in \mathcal{F}_S$. As just recalled, there exists a finite subgroup scheme $G' \subset G$ such that the composition $G' \to G \to H$ is an epimorphism. Since $G' \cong G'_S \times G'_{S'}$ and the composition $G'_{S'} \to G \to H$ is zero, it follows that the composition $G'_S \to G \to H$ is an epimorphism as well.

When k is perfect, the reduced subscheme $G'_{S,\mathrm{red}} \subset G'_S$ is an étale subgroup scheme (as follows from [DG70, II.5.2.3]), and the composition $G'_{S,\mathrm{red}} \to G' \to G \to H$ is surjective on $\bar{k}$-rational points. Thus, this composition is an epimorphism whenever $H \in \mathcal{E}_S$. □

(If k is imperfect, then the lifting property fails for the pair $(\mathcal{C}, \mathcal{E}_{\mathbb{P}})$, see [Br15, Rem. 3.3]).

We now describe the subcategories $\mathcal{C}_S$, $\mathcal{F}_S$ and $\mathcal{E}_S$ of $\mathcal{C}$. Note first that $\mathcal{E}_S$ is anti-equivalent to the S-primary torsion subcategory $\Gamma\text{-mod}_S$ of Γ-mod. Also, (2.5.2) yields equivalences of categories

$$(3.1.1) \qquad \bigoplus_{\ell \in S} \mathcal{C}_\ell \cong \mathcal{C}_S, \quad \bigoplus_{\ell \in S} \mathcal{F}_\ell \cong \mathcal{F}_S, \quad \bigoplus_{\ell \in S} \mathcal{E}_\ell \cong \mathcal{E}_\ell.$$

As a consequence, we have equivalences

$$\mathcal{C}_{\mathbb{P}} \cong \mathcal{C}_S \times \mathcal{C}_{S'}, \quad \mathcal{F}_{\mathbb{P}} \cong \mathcal{F}_S \times \mathcal{F}_{S'}, \quad \mathcal{E}_{\mathbb{P}} \cong \mathcal{E}_S \times \mathcal{E}_{S'},$$

where $S' := \mathbb{P} \setminus S$.

Given $G \in \mathcal{C}$ and $n \in \mathbb{Z}$, recall that the multiplication map n_G is étale if n is prime to p (see [SGA3, VIIA.8.4]); then $\mathrm{Ker}(n_G) \in \mathcal{E}$. Also, recall that $n_G = 0$ when G is finite of order n, where the *order* of G is the dimension of the k-vector space $\mathcal{O}(G)$ (see [SGA3, VIIA.8.5]). In particular, every finite group scheme of order prime to p is étale. Together with (3.1.1), this yields

$$(3.1.2) \qquad \mathcal{C}_S \cong \begin{cases} \mathcal{F}_S = \mathcal{E}_S & \text{if } p \notin S, \\ \mathcal{C}_p \times \mathcal{F}_{S \setminus \{p\}} = \mathcal{C}_p \times \mathcal{E}_{S \setminus \{p\}} & \text{if } p \in S. \end{cases}$$

To describe the p-primary torsion category $\mathcal{C}_p$, we need some structure results for affine algebraic groups. Recall from [DG70, IV.3.1] that every such group G lies in a unique exact sequence

$$0 \longrightarrow M \longrightarrow G \longrightarrow U \longrightarrow 0, \tag{3.1.3}$$

where M is of multiplicative type and U is unipotent; moreover, we have $\mathrm{Hom}_{\mathcal{C}}(M, U) = 0 = \mathrm{Hom}_{\mathcal{C}}(U, M)$. We denote by $\mathcal{L}$ (resp. $\mathcal{M}$, $\mathcal{U}$) the full subcategory of $\mathcal{C}$ with objects the affine algebraic groups (resp. the groups of multiplicative type, the unipotent groups); then $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{U}$ are Serre subcategories of $\mathcal{C}$. If k is perfect, then the exact sequence (3.1.3) has a unique splitting by [DG70, IV.3.1] again; this yields an equivalence of categories

$$\mathcal{M} \times \mathcal{U} \xrightarrow{\cong} \mathcal{L}. \tag{3.1.4}$$

Lemma 3.2. *Assume that $p > 0$.*

(i) *The objects of $\mathcal{C}_p$ are exactly the algebraic groups obtained as extensions (3.1.3), where $M \in \mathcal{M}_p$ and $U \in \mathcal{U}$.*
(ii) *If k is perfect, then $\mathcal{C}_p \cong \mathcal{M}_p \times \mathcal{U}$.*
(iii) *For an arbitrary field k, we have $\mathcal{C}_p/\mathcal{F}_p \cong \mathcal{U}/(\mathcal{F}_p \cap \mathcal{U})$.*

Proof. Let $G \in \mathcal{C}_p$. Since abelian varieties are divisible, every such variety obtained as a subquotient of G is zero. It follows that G is affine (e.g., by using [Br17, Prop. 2.8]), and hence an extension of a unipotent group by a p-primary torsion group of multiplicative type. Conversely, every such extension is p-primary torsion, since so is every unipotent group. This proves (i).

The assertions (ii) and (iii) are direct consequences of (i) in view of the above structure results for affine algebraic groups. □

Next, we describe the subcategories $\mathcal{C}^S = {}^{\perp}\mathcal{C}_S$ (the S-divisible algebraic groups), ${}^{\perp}\mathcal{F}_S$ and ${}^{\perp}\mathcal{E}_S$, with the notation of Subsections 2.1 and 2.5; here $\mathcal{F}_S$ and $\mathcal{E}_S$ are viewed as Serre subcategories of $\mathcal{C}$. For this, recall that every algebraic group G lies in a unique exact sequence

$$0 \longrightarrow G^0 \longrightarrow G \longrightarrow \pi_0(G) \longrightarrow 0, \tag{3.1.5}$$

where G^0 is connected and $\pi_0(G) \in \mathcal{E}$ (see e.g. [DG70, II.5.1.8]). We may now treat the easy case where $p \notin S$:

Lemma 3.3. *If $p \notin S$, then $\mathcal{C}^S = {}^{\perp}\mathcal{F}_S = {}^{\perp}\mathcal{E}_S$ consists of the algebraic groups G such that $\pi_0(G) \in \mathcal{E}_{S'}$.*

Proof. Note that $\mathcal{C}^S$ consists of the algebraic groups G such that every quotient S-primary torsion group of G is trivial. Since such a quotient group is étale, it is a quotient of $\pi_0(G)$, and thus of $\pi_0(G)_S$. So $G \in \mathcal{C}^S$ if and only if $\pi_0(G)$ is S'-primary torsion. □

By contrast, the case where $p \in S$ is much more involved:

Lemma 3.4. *Assume that $p \in S$.*

(i) *The objects of $\mathcal{C}_S$ (resp. $\mathcal{C}^S$) are exactly the algebraic groups G obtained as extensions*

$$0 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 0,$$

where G_1 is an S-primary torsion group of multiplicative type and G_2 is unipotent (resp. G_1 is a semi-abelian variety and $G_2 \in \mathcal{E}_{S'}$). Moreover, ${}^\perp\mathcal{E}_S$ consists of the algebraic groups G such that $\pi_0(G) \in \mathcal{E}_{S'}$.

(ii) *When k is perfect, ${}^\perp\mathcal{F}_S$ consists of the smooth algebraic groups G such that $\pi_0(G) \in \mathcal{E}_{S'}$.*

(iii) *Returning to an arbitrary field k, the exact sequence*

$$0 \longrightarrow G^S \longrightarrow G \longrightarrow G_S \longrightarrow 0$$

(where G^S is S-divisible and G_S is S-primary torsion) has a unique splitting in $\mathcal{C}/\mathcal{F}_S$, for any $G \in \mathcal{C}$.

Proof. (i) Let $G \in \mathcal{C}$. By [Br17, Thm. 2.11], there exist two exact sequences in $\mathcal{C}$

$$0 \longrightarrow H \longrightarrow G \longrightarrow U \longrightarrow 0, \quad 0 \longrightarrow M \longrightarrow H \longrightarrow A \longrightarrow 0,$$

where U is unipotent, A an abelian variety, and M of multiplicative type. In particular, $U \in \mathcal{C}_S$. Thus, $G \in \mathcal{C}_S$ if and only if $A = 0$ and $M \in \mathcal{C}_S$. The latter condition is equivalent to $M \in \mathcal{F}_S$, in view of the structure of groups of multiplicative type (see [DG70, IV.1.3]). This shows the assertion on $\mathcal{C}_S$.

Also, if $G \in \mathcal{C}^S$, then $U = 0$ as $\mathcal{C}^S$ is stable under taking quotients. Let T be the largest subtorus of M, then we obtain an exact sequence in $\mathcal{C}$

$$0 \longrightarrow M/T \longrightarrow G/T \longrightarrow A \longrightarrow 0,$$

where M/T is finite and of multiplicative type. In particular, M/T is n-primary torsion for some positive integer n, and hence the class of the above extension in $\mathrm{Ext}^1_{\mathcal{C}}(A, M/T)$ is n-primary torsion as well. Thus, the pull-back of this extension under n_A is trivial. Since n_A is an epimorphism with finite kernel, we obtain a map

$$f : M/T \times A \longrightarrow G/T,$$

which is an epimorphism with finite kernel as well. As a consequence, there exists an exact sequence in $\mathcal{C}$

$$0 \longrightarrow B \longrightarrow G/T \longrightarrow G_2 \longrightarrow 0,$$

where B is an abelian variety (the image of A under f) and G_2 is finite. This yields in turn an exact sequence in $\mathcal{C}$

$$0 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 0,$$

where G_1 is a semi-abelian variety (extension of B by T). Using again the stability of $\mathcal{C}^S$ under taking quotients, it follows that $G_2 \in \mathcal{F}_{S'} = \mathcal{E}_{S'}$. Conversely, if $G_2 \in \mathcal{E}_{S'}$, then $G_2 \in \mathcal{C}^S$; also, $G_1 \in \mathcal{C}^S$ since it is divisible. As $\mathcal{C}^S$ is stable under extensions, we get that $G \in \mathcal{C}^S$. This shows the assertion on $\mathcal{C}^S$. That on ${}^{\perp}\mathcal{E}_S$ is obtained by similar arguments.

(ii) Let $G \in {}^{\perp}\mathcal{F}_S$. Since k is perfect, the reduced subscheme G_{red} is a smooth subgroup of G (see [DG70, II.5.2.3]). Moreover, G/G_{red} is infinitesimal, hence finite and p-primary torsion. As $p \in S$, it follows that $G = G_{\mathrm{red}}$ is smooth. Moreover, $\pi_0(G) \in \mathcal{E}_{S'}$. Conversely, every smooth connected algebraic group has no non-zero finite quotient, and hence is an object of ${}^{\perp}\mathcal{F}_S$. Also, every finite S'-primary torsion group is in ${}^{\perp}\mathcal{F}_S$. This yields the assertion by using again the stability of ${}^{\perp}\mathcal{F}_S$ under extensions.

(iii) Choose $n \in \langle S \rangle$ such that $n_{G_S} = 0$. Then n_{G^S} is an epimorphism, and its kernel is finite for dimension reasons. So n_{G^S} yields an isomorphism in $\mathcal{C}/\mathcal{F}_S$. Thus, $\mathrm{Hom}_{\mathcal{C}/\mathcal{F}_S}(G_S, G^S) = 0 = \mathrm{Ext}^1_{\mathcal{C}/\mathcal{F}_S}(G_S, G^S)$. This yields our statement. □

Remark 3.5. If k is imperfect, then ${}^{\perp}\mathcal{F}$ contains non-smooth algebraic groups. Indeed, by [To13, Lem. 6.3], there exists an exact sequence in $\mathcal{C}$

$$0 \longrightarrow \alpha_p \longrightarrow G \longrightarrow \mathbb{G}_a \longrightarrow 0 \tag{3.1.6}$$

such that every smooth connected subgroup scheme of G is zero. In particular, the extension (3.1.6) is nontrivial; also, G is unipotent and non-smooth. If $f : G \to H$ is an epimorphism with $H \in \mathcal{F}$, then f induces an epimorphism $\mathbb{G}_a \to H/f(\alpha_p)$. Since every finite quotient of $\mathbb{G}_a$ is zero, we see that H is a quotient of α_p, and hence must be zero by the nontriviality of the extension (3.1.6). Thus, $G \in {}^{\perp}\mathcal{F}$.

We now obtain a description of the S-isogeny category, which generalizes [Br17, Prop. 3.6, Prop. 5.10]:

Proposition 3.6. (i) *If $p \notin S$, then $\mathcal{C}/\mathcal{C}_S = \mathcal{C}/\mathcal{F}_S = \mathcal{C}/\mathcal{E}_S$ is equivalent to the category $\underline{\mathcal{C}}^S$ with objects the algebraic groups G such that $\pi_0(G) \in \mathcal{E}_{S'}$, and with morphisms given by*

$$\mathrm{Hom}_{\underline{\mathcal{C}}^S}(G, H) = S^{-1}\mathrm{Hom}_{\mathcal{C}}(G, H).$$

(ii) *If $p \in S$ and k is perfect, then*

$$\mathcal{C}/\mathcal{F}_S \cong \mathcal{U}/(\mathcal{F}_p \cap \mathcal{U}) \times \mathcal{C}/\mathcal{C}_S.$$

Moreover, $\mathcal{C}/\mathcal{C}_S$ is equivalent to the category with objects the extensions of finite étale S'-primary torsion groups by semiabelian varieties, and with morphisms as in (i).

Proof. (i) This follows from Lemmas 2.20 and 3.3.

(ii) Let $G \in \mathcal{C}$. We have a unique decomposition $G = G^S \times G_S$ in $\mathcal{C}/\mathcal{F}_S$, as a consequence of Lemma 3.4 (iii). Moreover, we have a unique decomposition $G_S = G_p \times G_{S\setminus\{p\}}$ in $\mathcal{C}$ by (3.1.2). As $G_{S\setminus\{p\}} \in \mathcal{F}_S$, it follows that $G = G^S \times G_p$ in $\mathcal{C}/\mathcal{F}_S$. This yields the assertion in view of Lemmas 3.2 (iii) and 3.4 (ii). □

3.2. **Homological dimension of S-isogeny categories.** We keep the notation of Subsection 3.1. For any prime number ℓ, we denote by $\mathrm{cd}_\ell(k) = \mathrm{cd}_\ell(\Gamma)$ the ℓth cohomological dimension, i.e., the smallest integer n (if it exists) such that $H^i(\Gamma, M) = 0$ for all $i > n$ and all ℓ-primary torsion, discrete Γ-modules M; if no such n exists, then $\mathrm{cd}_\ell(k) := \infty$ (see [GS06, 6.1] for further developments on this notion). We may restrict to objects M of $\Gamma\text{-mod}_\ell$ in the above definition, since every ℓ-primary torsion, discrete Γ-module is the filtered direct limit of its finite submodules, and cohomology commutes with such limits. We set $\mathrm{cd}_S(k) := \sup_{\ell\in S}(\mathrm{cd}_\ell(k))$ for any nonempty subset $S \subset \mathbb{P}$; also, $\mathrm{cd}_\emptyset(k) := 0$ and $\mathrm{cd}(k) := \mathrm{cd}_\mathbb{P}(k)$.

Theorem 3.7. *Let k be a perfect field of characteristic $p \geq 0$. Let S be a subset of the set $\mathbb{P}$ of prime numbers, and $S' := \mathbb{P} \setminus S$. Then*

$$\mathrm{hd}(\mathcal{C}/\mathcal{F}_S) = \begin{cases} \mathrm{cd}_{S'}(k) + 1 & \text{if } p = 0 \text{ or } p \in S, \\ \max(2, \mathrm{cd}_{S'}(k) + 1) & \text{otherwise.} \end{cases}$$

To prove this result, we will apply Theorem 2.22 to the artinian abelian category $\mathcal{A} := \mathcal{C}/\mathcal{F}_S$ and to the set $\mathbb{P}$ of all primes. By Lemma 2.18, we have equivalences of categories $\mathcal{A}_\mathbb{P} \cong \mathcal{C}_\mathbb{P}/\mathcal{F}_S$ and $\mathcal{A}/\mathcal{A}_\mathbb{P} \cong \mathcal{C}/\mathcal{C}_\mathbb{P}$; this yields

$$\mathrm{hd}(\mathcal{C}/\mathcal{F}_S) = \max(\mathrm{hd}(\mathcal{C}_\mathbb{P}/\mathcal{F}_S), \mathrm{hd}(\mathcal{C}/\mathcal{C}_\mathbb{P})). \tag{3.2.1}$$

We will describe the categories $\mathcal{C}_\mathbb{P}/\mathcal{F}_S$ and $\mathcal{C}/\mathcal{C}_\mathbb{P}$ (Lemma 3.8), and determine their homological dimensions (Lemmas 3.9 and 3.10). The assumption that k is perfect will be used in the description of $\mathcal{C}_\mathbb{P}/\mathcal{F}_S$ when $p > 0$ and $p \notin S$ (Lemma 3.8 (iii), based on Lemma 3.2 (ii)).

In what follows, we set $\underline{\mathcal{C}} := \mathcal{C}/\mathcal{F}$ (the full isogeny category of algebraic groups) and $\underline{\mathcal{U}} := \mathcal{U}/(\mathcal{F} \cap \mathcal{U}) = \mathcal{U}(/\mathcal{F}_p \cap \mathcal{U})$ (the isogeny category of unipotent algebraic groups). Also, we denote by $\underline{\mathcal{S}}$ the isogeny category of semi-abelian varieties, i.e., the full subcategory of $\underline{\mathcal{C}}$ consisting of semi-abelian varieties.

Lemma 3.8. (i) *If $p = 0$ then $\mathcal{A}_\mathbb{P} \cong \mathcal{E}_{S'}$ and $\mathcal{A}/\mathcal{A}_\mathbb{P} \cong \underline{\mathcal{C}}$.*

(ii) *If $p \in S$ then $\mathcal{A}_\mathbb{P} \cong \underline{\mathcal{U}} \times \mathcal{E}_{S'}$ and $\mathcal{A}/\mathcal{A}_\mathbb{P} \cong \underline{\mathcal{S}}$.*

(iii) *If $p > 0$ and $p \notin S$ then $\mathcal{A}_\mathbb{P} \cong \mathcal{M}_p \times \mathcal{U} \times \mathcal{E}_{S'\setminus\{p\}}$ and $\mathcal{A}/\mathcal{A}_\mathbb{P} \cong \underline{\mathcal{S}}$.*

Proof. (i) Since $p = 0$, we have $\mathcal{C}_\mathbb{P} = \mathcal{F}_\mathbb{P} = \mathcal{E}_\mathbb{P}$, hence $\mathcal{A}_\mathbb{P} \cong \mathcal{E}_{S'}$ (in view of Lemma 2.18) and $\mathcal{A}/\mathcal{A}_\mathbb{P} \cong \mathcal{C}/\mathcal{F}_\mathbb{P}$.

(ii) Using the isomorphism (3.1.2), we obtain

$$\mathcal{A}_\mathbb{P} \cong \mathcal{C}_\mathbb{P}/\mathcal{F}_S \cong \mathcal{C}_p/\mathcal{F}_p \times \mathcal{C}_{\mathbb{P}\setminus\{p\}}/\mathcal{F}_{S\setminus\{p\}}.$$

Also, $\mathcal{C}_{\mathbb{P}\setminus\{p\}} = \mathcal{E}_{\mathbb{P}\setminus\{p\}}$, hence $\mathcal{C}_{\mathbb{P}\setminus\{p\}}/\mathcal{F}_{S\setminus\{p\}} \cong \mathcal{E}_{S'}$. Moreover, $\mathcal{C}_p/\mathcal{F}_p \cong \underline{\mathcal{U}}$ by Lemma 3.2. This yields an equivalence $\mathcal{A}_{\mathbb{P}} \cong \underline{\mathcal{U}} \times \mathcal{E}_{S'}$. Furthermore, $\mathcal{A}/\mathcal{A}_{\mathbb{P}} \simeq \mathcal{C}/\mathcal{C}_{\mathbb{P}}$ is equivalent to $\underline{\mathcal{S}}$ by Proposition 3.6.

(iii) Using (3.1.2) again, we obtain $\mathcal{A}_{\mathbb{P}} \cong \mathcal{C}_p \times \mathcal{E}_{S'\setminus\{p\}}$. By Lemma 3.2, it follows that $\mathcal{A}_{\mathbb{P}} \cong \mathcal{M}_p \times \mathcal{U} \times \mathcal{E}_{S'\setminus\{p\}}$. Finally, the equivalence $\mathcal{A}/\mathcal{A}_{\mathbb{P}} \cong \underline{\mathcal{S}}$ is obtained as in (ii). □

Lemma 3.9. (i) $\mathrm{hd}(\underline{\mathcal{U}}) = \mathrm{hd}(\underline{\mathcal{C}}) = 1$.

(ii) $\mathrm{hd}(\underline{\mathcal{S}}) = \begin{cases} 0 & \textit{if } k \textit{ is an algebraic extension of a finite field,} \\ 1 & \textit{otherwise.} \end{cases}$

(iii) $\mathrm{hd}(\mathcal{M}_\ell) = \mathrm{hd}(\mathcal{E}_\ell) = \mathrm{hd}(\Gamma\text{-mod}_\ell)$ *for any prime number* ℓ.

(iv) $\mathrm{hd}(\mathcal{U}) = 2$.

Proof. (i) The assertion on $\underline{\mathcal{U}}$ is a consequence of results in [DG70, V.3.6], as explained in [Br18a, 3.1.5]. The assertion on $\underline{\mathcal{C}}$ is the main result of [Br17].

(ii) Since $\underline{\mathcal{S}}$ is a Serre subcategory of $\underline{\mathcal{C}}$, we have $\mathrm{hd}(\underline{\mathcal{S}}) \leq 1$ (see e.g. [Oo64, §3]). Moreover, $\mathrm{hd}(\underline{\mathcal{S}}) = 0$ if and only if k is an algebraic extension of a finite field, as follows from [Br17, Prop. 5.8].

(iii) Just observe that $\mathcal{M}_\ell$ is anti-equivalent to $\mathcal{E}_\ell$ by Cartier duality (see [DG70, IV.3.5]).

(iv) This follows from [DG70, V.1.5.3, V.1.5.5]. □

Next, recall that

$$\mathrm{hd}(\mathcal{E}_S) = \sup_{\ell \in S} \big(\mathrm{hd}(\mathcal{E}_\ell)\big) = \sup_{\ell \in S} \big(\mathrm{hd}(\Gamma\text{-mod}_\ell)\big),$$

where the first equality follows from (2.4.1) and (3.1.1). We now show:

Lemma 3.10. $\mathrm{hd}(\Gamma\text{-mod}_\ell) = \mathrm{cd}_\ell(\Gamma) + 1$ *for any prime number* ℓ.

Proof. This follows from the results of [Mi70, Sec. 1], but we have been unable to understand some of the arguments given there. Thus, we provide an independent proof.

We begin with some observations on the category $\Gamma\text{-mod}_\ell$ of finite ℓ-groups equipped with a discrete action of Γ. This is an artinian and noetherian category equipped with a forgetful functor

$$\Gamma\text{-mod}_\ell \longrightarrow \mathrm{mod}_\ell, \quad X \longmapsto \bar{X}$$

and with a duality

$$X \longmapsto \mathrm{Hom}_{\mathrm{Mod}_\ell}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell),$$

where Γ acts on the right-hand side via its given action on X and the trivial action on $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. We denote by $\mathrm{Ind}(\Gamma\text{-mod}_\ell)$ the associated ind category; this is a locally noetherian category and its noetherian objects are exactly those of $\Gamma\text{-mod}_\ell$. By [Ga62, II.4.Thm. 1], it follows that $\mathrm{Ind}(\Gamma\text{-mod}_\ell)$ is equivalent to the category $\Gamma\text{-Mod}_\ell$ of discrete ℓ-primary torsion Γ-modules (indeed, the latter category is locally noetherian, and its full subcategory consisting of noetherian objects is Γ-mod$_\ell$ as well). Also, Ind(Γ-mod$_\ell$) has enough injectives, and the natural map

$$\mathrm{Ext}^i_{\Gamma\text{-mod}_\ell}(X,Y) \longrightarrow \mathrm{Ext}^i_{\mathrm{Ind}(\Gamma\text{-mod}_\ell)}(X,Y)$$

is an isomorphism for all $X, Y \in \Gamma$-mod$_\ell$ (as follows from the dual statement of (2.3.2)).

We now adapt the argument of [Mi70, Prop. p. 437]. We claim that there exists a spectral sequence

$$H^i(\Gamma, \mathrm{Ext}^j_{\mathrm{Mod}_\ell}(\bar{X},\bar{Y})) \Rightarrow \mathrm{Ext}^{i+j}_{\Gamma\text{-Mod}_\ell}(X,Y)$$

for all $X \in \Gamma$-mod$_\ell$ and $Y \in \Gamma$-Mod$_\ell$. This can be proved by adapting the argument of [Mi86, Thm. I.0.3], which yields a similar spectral sequence in the setting of discrete Γ-modules; alternatively, the claim can be derived from [We94, 10.8.7] and the subsequent discussion. We provide a direct argument: first note that

$$H^0(\Gamma, \mathrm{Hom}_{\mathrm{Mod}_\ell}(\bar{X},\bar{Y})) = \mathrm{Hom}_{\Gamma\text{-Mod}_\ell}(X,Y).$$

Also, the functor $H^0(\Gamma, ?)$ is left exact and the forgetful functor is exact. So the claim will follow from the spectral sequence of a composite functor (see [We94, 5.8.3]), once we show that the endofunctor $Y \mapsto \mathrm{Hom}_{\mathrm{Mod}_\ell}(\bar{X},\bar{Y})$ of Γ-Mod$_\ell$ takes injectives to Γ-acyclics.

Let I be an injective object of Γ-Mod$_\ell$. Then the map

$$\iota : I \longrightarrow \mathrm{Hom}(\Gamma, I), \quad x \mapsto (g \mapsto g \cdot x)$$

is injective and Γ-equivariant, where Γ acts on the right-hand side via right multiplication on itself. Moreover, the image of ι is contained in the subgroup

$$\mathrm{Hom}_{\mathrm{cont}}(\Gamma, I) := \varinjlim \big(\mathrm{Hom}(\Gamma/U, I^U)\big),$$

where the direct limit is over all open normal subgroups U of Γ. This subgroup is an object of Γ-Mod$_\ell$, and is injective in that category: indeed,

$$\mathrm{Hom}_{\Gamma\text{-Mod}_\ell}(Z, \mathrm{Hom}_{\mathrm{cont}}(\Gamma, I)) \cong \mathrm{Hom}^{\Gamma}_{\mathrm{cont}}(Z \times \Gamma, I) \cong \mathrm{Hom}_{\mathrm{Mod}_\ell}(\bar{Z}, \bar{I})$$

for any $Z \in \Gamma$-Mod$_\ell$, and $\bar{I}$ is injective in Mod$_\ell$ by (i). Thus, ι identifies I with a summand of $\mathrm{Hom}(\Gamma, I)$. The latter is Γ-acyclic in view of [We94, 6.3.3, 6.11.13]. This completes the proof of the claim.

As $\mathrm{Ext}^j_{\mathrm{Mod}_\ell}(?, ?) = 0$ for all $j \geq 2$, this claim yields the inequality

$$\mathrm{hd}(\Gamma\text{-mod}_\ell) \leq \mathrm{cd}_\ell(\Gamma) + 1.$$

To show the opposite inequality, consider first the case where $\mathrm{cd}_\ell(\Gamma) =: n$ is finite. We may then find an $X \in \Gamma$-mod$_\ell$ such that $H^n(\Gamma, X) \neq 0$ and X is killed by ℓ. We have $X = \mathrm{Hom}_{\mathrm{Mod}_\ell}(Y, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ for a unique $Y \in \Gamma$-mod$_\ell$; then

Y is killed by ℓ as well. By the claim and the fact that $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ is injective in Mod_ℓ, there is an injection

$$H^n(\Gamma, X) \hookrightarrow \mathrm{Ext}^n_{\Gamma\text{-}\mathrm{Mod}_\ell}(Y, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Also, the exact sequence

$$0 \longrightarrow \mathbb{Z}/\ell\mathbb{Z} \longrightarrow \mathbb{Q}_\ell/\mathbb{Z}_\ell \xrightarrow{\ \ell\ } \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow 0$$

yields an injection

$$\mathrm{Ext}^n_{\Gamma\text{-}\mathrm{Mod}_\ell}(Y, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \hookrightarrow \mathrm{Ext}^{n+1}_{\Gamma\text{-}\mathrm{Mod}_\ell}(Y, \mathbb{Z}/\ell\mathbb{Z}).$$

This completes the proof in this case. In the case where $\mathrm{cd}_\ell(\Gamma)$ is infinite, we apply the above arguments to an infinite sequence of positive integers n such that $H^n(\Gamma, ?) \neq 0$. □

We may now complete the proof of Theorem 3.7 by freely using the above three lemmas.

If $p = 0$, then $\mathrm{hd}(\mathcal{A}_\mathbb{P}) = \mathrm{hd}(\mathcal{E}_{S'}) = \mathrm{cd}_{S'}(k) + 1$ and $\mathrm{hd}(\mathcal{A}/\mathcal{A}_\mathbb{P}) = \mathrm{hd}(\underline{\mathcal{C}}) = 1$.

If $p \in S$, then $\mathrm{hd}(\mathcal{A}_\mathbb{P}) = \max(\mathrm{hd}(\underline{\mathcal{U}}), \mathrm{hd}(\mathcal{E}_{S'})) = \mathrm{cd}_{S'}(k)+1$ and $\mathrm{hd}(\mathcal{A}/\mathcal{A}_\mathbb{P}) = \mathrm{hd}(\underline{\mathcal{S}}) \leq 1$.

Finally, if $p > 0$ and $p \notin S$, then

$$\mathrm{hd}(\mathcal{A}_\mathbb{P}) = \max(\mathrm{hd}(\mathcal{U}), \mathrm{hd}(\mathcal{M}_p), \mathrm{hd}(\mathcal{E}_{S'\setminus\{p\}})) = \max(2, \mathrm{cd}_{S'}(k) + 1)$$

(where the second equality follows from the fact that $\mathrm{hd}(\mathcal{M}_p) = \mathrm{cd}_p(k) \leq 1$, see [GS06, Prop. 6.1.9]) and $\mathrm{hd}(\mathcal{A}/\mathcal{A}_\mathbb{P}) = \mathrm{hd}(\underline{\mathcal{S}}) \leq 1$.

In either case, we conclude by using (3.2.1).

Remark 3.11. (i) Still assuming k perfect, the homological dimension of the subcategory $\mathcal{L}$ of $\mathcal{C}$ may be determined along similar lines. Indeed, $\mathcal{L}$ contains the torsion subcategory $\mathcal{C}_\mathbb{P}$, as follows from Lemma 3.2. Thus, for any Serre subcategory $\mathcal{B}$ of $\mathcal{C}_\mathbb{P}$, we have $(\mathcal{L}/\mathcal{B})_{\mathrm{tors}} = \mathcal{C}_\mathbb{P}/\mathcal{B}$ in view of Lemma 2.18. Using Theorem 2.22, it follows that

$$\mathrm{hd}(\mathcal{L}/\mathcal{B}) = \max(\mathrm{hd}(\mathcal{C}_\mathbb{P}/\mathcal{B}), \mathrm{hd}(\mathcal{L}/\mathcal{C}_\mathbb{P})) \tag{3.2.2}$$

Also, $\mathrm{hd}(\mathcal{L}/\mathcal{C}_\mathbb{P}) \leq 1$ as $\mathcal{L}/\mathcal{C}_\mathbb{P}$ is a Serre subcategory of $\mathcal{C}/\mathcal{C}_\mathbb{P}$. Taking $\mathcal{B} = \mathcal{F}_S$ and using the fact that $\mathrm{hd}(\mathcal{C}_\mathbb{P}/\mathcal{F}_S) \geq 1$ (Lemmas 3.8 and 3.9), we obtain

$$\mathrm{hd}(\mathcal{L}/\mathcal{F}_S) = \mathrm{hd}(\mathcal{C}/\mathcal{F}_S)$$

under the assumptions of Theorem 3.7.

Alternatively, one may determine directly $\mathrm{hd}(\mathcal{L}/\mathcal{F}_S)$ by using the equivalence of categories (3.1.4).

(ii) When k is imperfect, the homological dimension of $\mathcal{C}$ seems to be unknown, already in the case where k is separably closed. Under that assumption, it is asserted in [Kr75, p. 75] that the category Ac_k of (commutative) affine k-group schemes has homological dimension 3. Since Ac_k is equivalent to $\mathrm{Pro}(\mathcal{L})$

(see [DG70, V.2.2.2, V.2.3.1]), this yields $\mathrm{hd}(\mathcal{L}) = 3$ in view of Lemma 2.15. Using the methods developed above, one can deduce from this that $\mathrm{hd}(\mathcal{C}) = 3$ as well. But the proof of the equality $\mathrm{hd}(\mathrm{Ac}_k) = 3$, left to the reader as an exercise ("Aufgabe"), could not be reconstituted so far.

3.3. **The radicial isogeny category.** We keep the notation of Subsections 3.1 and 3.2, and assume that $p > 0$. Let $\mathcal{I}$ be the full subcategory of $\mathcal{C}$ with objects the infinitesimal group schemes; then $\mathcal{I}$ is a Serre subcategory of $\mathcal{C}_p$, as follows from [DG70, IV.3.5.3]. We say that $\mathcal{C}/\mathcal{I}$ is the *radicial isogeny category.*

By [Br17, Lem. 2.2], the pair $(\mathcal{C}, \mathcal{I})$ satisfies the lifting property. Also, recall that an algebraic group is infinitesimal if and only if it is finite and connected. In view of the exact sequence (3.1.5), it follows that every finite group scheme G lies in a unique exact sequence

$$0 \longrightarrow I \longrightarrow G \longrightarrow E \longrightarrow 0, \tag{3.3.1}$$

where $I := G^0$ is infinitesimal, and $E := \pi_0(G)$ is finite étale; moreover, $\mathrm{Hom}_{\mathcal{F}}(I, E) = 0 = \mathrm{Hom}_{\mathcal{F}}(E, I)$. When k is perfect, the extension (3.3.1) has a unique splitting (as follows from [DG70, II.5.2.4]). Thus, for perfect k, the product map $\mathcal{I} \times \mathcal{E} \to \mathcal{F}$ is an equivalence of categories (see [DG70, IV.3.5] for further developments).

Lemma 3.12. *If $G \in \mathcal{C}$ is smooth, then $G \in {}^{\perp}\mathcal{I}$. The converse holds when k is perfect.*

Proof. If G is smooth, then so is every quotient of G. In particular, every infinitesimal quotient is trivial, that is, $G \in {}^{\perp}\mathcal{I}$.

Conversely, assume that k is perfect and let $G \in {}^{\perp}\mathcal{I}$. Recall that the reduced subscheme G_{red} is a smooth subgroup of G and G/G_{red} is infinitesimal. It follows that G is smooth. □

(If k is imperfect, then ${}^{\perp}\mathcal{I}$ contains non-smooth algebraic groups, as follows from Remark 3.5.)

We now obtain a more explicit description of the category $\mathcal{C}/\mathcal{I}$, over an arbitrary field k of characteristic $p > 0$. To state it, we recall some standard results on Frobenius kernels. For any algebraic group G and any positive integer n, we denote by $F^n_{G/k} : G \to G^{(p^n)}$ the nth iterated relative Frobenius morphism. Then the nth Frobenius kernel $G_n := \mathrm{Ker}(F^n_{G/k})$ is an infinitesimal subgroup scheme of G; moreover, G/G_n is smooth for $n \gg 0$, and every infinitesimal subgroup scheme of G is contained in some G_n (see [SGA3, VIIA.8]). This implies readily:

Proposition 3.13. *With the above notation, $\mathcal{C}/\mathcal{I}$ is equivalent to its full subcategory with objects the smooth algebraic groups. Moreover, we have*

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(G,H) = \lim_{\to} \big(\mathrm{Hom}_{\mathcal{C}}(G, H/H_n)\big)$$

for all smooth algebraic groups G, H, where the direct limit is taken over all positive integers.

Next, we determine the homological dimension of the radicial isogeny category over a perfect field:

Theorem 3.14. *Let k be a perfect field of characteristic $p > 0$. Then*

$$\mathrm{hd}(\mathcal{C}/\mathcal{I}) = \max(2, \mathrm{cd}(k) + 1).$$

Proof. We argue as in the proof of Theorem 3.7. Let $\mathcal{A} := \mathcal{C}/\mathcal{I}$ and consider the set $\mathbb{P}$ of all primes. Then $\mathcal{A}$ is an artinian abelian category and $\mathcal{A}/\mathcal{A}_{\mathbb{P}} \cong \mathcal{C}/\mathcal{C}_{\mathbb{P}} \cong \underline{\mathcal{S}}$ by Lemmas 2.18 and 3.8. Thus, $\mathrm{hd}(\mathcal{A}/\mathcal{A}_{\mathbb{P}}) \leq 1$ (Lemma 3.9). Also, $\mathcal{A}_{\mathbb{P}} \cong \mathcal{C}_{\mathbb{P}}/\mathcal{I} \cong (\mathcal{C}_p \times \mathcal{E}_{p'})/\mathcal{I}$ by Lemma 2.18 again combined with (3.1.2). Moreover, the decomposition $\mathcal{C}_p = \mathcal{M}_p \times \mathcal{U}$ (Lemma 3.2) induces a decomposition $\mathcal{I} = \mathcal{M}_p \times \mathcal{J}$, where $\mathcal{J}$ denotes the (Serre) subcategory of $\mathcal{U}$ with objects the infinitesimal unipotent groups. Thus, $\mathcal{A}_{\mathbb{P}} \cong \mathcal{U}/\mathcal{J} \times \mathcal{E}_{p'}$ and hence

$$\mathrm{hd}(\mathcal{A}_{\mathbb{P}}) = \max(\mathrm{hd}(\mathcal{U}/\mathcal{J}), \mathrm{cd}_{p'}(k) + 1)$$

by (2.4.1) and Lemma 3.10. Also, recall that $\mathrm{cd}_p(k) \leq 1$ by [GS06, Prop. 6.1.9]. Thus, it suffices to show that

$$\mathrm{hd}(\mathcal{U}/\mathcal{J}) = 2$$

in view of Theorem 2.22.

We have $\mathrm{hd}(\mathcal{U}/\mathcal{J}) \leq \mathrm{hd}(\mathcal{U})$ by Proposition 2.16, since the pair $(\mathcal{U}, \mathcal{J})$ satisfies the lifting property. This yields $\mathrm{hd}(\mathcal{U}/\mathcal{J}) \leq 2$ in view of Lemma 3.9. So it suffices in turn to check that

$$\mathrm{Ext}^2_{\mathcal{U}/\mathcal{J}}(\mathbb{G}_a, \nu_p) = k, \tag{3.3.2}$$

where $\mathbb{G}_a$ denotes the additive group, and ν_p denotes the constant group scheme associated with $\mathbb{Z}/p\mathbb{Z}$ (as in [Oo66, I.2.15]).

The short exact sequence

$$0 \longrightarrow \nu_p \longrightarrow \mathbb{G}_a \stackrel{F-\mathrm{id}}{\longrightarrow} \mathbb{G}_a \longrightarrow 0$$

yields a long exact sequence

$$\mathrm{Ext}^1_{\mathcal{U}/\mathcal{J}}(\mathbb{G}_a, \mathbb{G}_a) \stackrel{F_*-\mathrm{id}}{\longrightarrow} \mathrm{Ext}^1_{\mathcal{U}/\mathcal{J}}(\mathbb{G}_a, \mathbb{G}_a) \to \mathrm{Ext}^2_{\mathcal{U}/\mathcal{J}}(\mathbb{G}_a, \nu_p) \to \mathrm{Ext}^2_{\mathcal{U}/\mathcal{J}}(\mathbb{G}_a, \mathbb{G}_a),$$

where F_* denotes the push-out. As $\mathbb{G}_a/I \cong \mathbb{G}_a$ for any infinitesimal subgroup $I \subset \mathbb{G}_a$, Lemma 2.9 implies that the canonical map

$$Q^i : \mathrm{Ext}^i_{\mathcal{U}}(\mathbb{G}_a, \mathbb{G}_a) \longrightarrow \mathrm{Ext}^i_{\mathcal{U}/\mathcal{J}}(\mathbb{G}_a, \mathbb{G}_a)$$

is surjective for any $i \geq 0$.

We now claim that $\mathrm{Ext}^2_{\mathcal{U}}(\mathbb{G}_a, \mathbb{G}_a) = 0$. This is a direct consequence of results of [DG70]. More specifically, the pro category $\mathrm{Pro}(\mathcal{C})$ is equivalent with that of commutative affine group schemes (see [DG70, V.2.2.c]) and this induces an equivalence of $\mathrm{Pro}(\mathcal{U})$ with the category of commutative unipotent group schemes (as follows from Lemma 2.11 and [DG70, IV.2.2.5]). The latter category is anti-equivalent to that of V-primary torsion modules over the Dieudonné ring $\mathbb{D}$, where V denotes the Verschiebung; this anti-equivalence takes a commutative unipotent group scheme G to its Dieudonné module $M(G)$, and $\mathbb{G}_a$ to $\mathbb{D}/\mathbb{D}V$ (see [DG70, V.1.4.2, V.1.4.3]). By combining Lemma 2.15 and [DG70, V.1.5.1], this leads to isomorphisms

$$\mathrm{Ext}^i_{\mathcal{U}}(G, H) \xrightarrow{\cong} \mathrm{Ext}^i_{\mathbb{D}\text{-Mod}}(M(H), M(G))$$

for all $G, H \in \mathcal{U}$ and all $i \geq 0$, where $\mathbb{D}$-Mod denotes the category of all left $\mathbb{D}$-modules. In view of the projective resolution

$$0 \longrightarrow \mathbb{D} \xrightarrow{V} \mathbb{D} \longrightarrow \mathbb{D}/\mathbb{D}V \longrightarrow 0$$

in $\mathbb{D}$-Mod, we thus obtain that $\mathrm{Ext}^i_{\mathcal{U}}(G, \mathbb{G}_a) = 0$ for all $G \in \mathcal{U}$ and $i \geq 2$. This proves the claim.

By this claim and the surjectivity of Q^2, we have $\mathrm{Ext}^2_{\mathcal{U}/\mathcal{J}}(\mathbb{G}_a, \mathbb{G}_a) = 0$ and hence

$$\mathrm{Ext}^2_{\mathcal{U}/\mathcal{J}}(\mathbb{G}_a, \nu_p) \cong \mathrm{Coker}(F_* - \mathrm{id}). \tag{3.3.3}$$

We now claim that Q^1 is injective. Let $\xi \in \mathrm{Ext}^1_{\mathcal{U}}(\mathbb{G}_a, \mathbb{G}_a)$ be represented by an exact sequence in $\mathcal{U}$

$$0 \longrightarrow \mathbb{G}_a \xrightarrow{\alpha} G \xrightarrow{\beta} \mathbb{G}_a \longrightarrow 0,$$

which splits in $\mathcal{U}/\mathcal{J}$. Then we may choose $\gamma \in \mathrm{Hom}_{\mathcal{U}/\mathcal{J}}(G, \mathbb{G}_a)$ such that $\gamma \circ \alpha = \mathrm{id}$ in $\mathrm{End}_{\mathcal{U}/\mathcal{J}}(\mathbb{G}_a)$. In view of Lemma 2.7, γ is represented by $\delta \in \mathrm{Hom}_{\mathcal{U}}(G, \mathbb{G}_a/I)$ for some infinitesimal subgroup $I \subset \mathbb{G}_a$. Then $I \subset \mathrm{Ker}(F^n)$ for some $n \geq 1$; thus, we may assume that $I = \mathrm{Ker}(F^n)$, and identify the quotient map $\mathbb{G}_a \to \mathbb{G}_a/I$ with $F^n : \mathbb{G}_a \to \mathbb{G}_a$. Then $\delta \circ \alpha - F^n \in \mathrm{End}_{\mathcal{U}}(\mathbb{G}_a)$ represents zero in $\mathrm{End}_{\mathcal{U}/\mathcal{J}}(\mathbb{G}_a)$. As $\mathbb{G}_a$ is smooth, it follows from Lemma 2.3 that $\delta \circ \alpha - F^n = 0$ in $\mathrm{End}_{\mathcal{U}}(\mathbb{G}_a)$. This yields a commutative diagram of exact sequences in $\mathcal{U}$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{G}_a & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & \mathbb{G}_a & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle F^n} & \swarrow{\scriptstyle \delta} & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathbb{G}_a & \longrightarrow & G/I & \longrightarrow & \mathbb{G}_a & \longrightarrow & 0.
\end{array}$$

Thus, the bottom exact sequence splits, i.e. $F^n_*(\xi) = 0$ in $\mathrm{Ext}^1_{\mathcal{U}}(\mathbb{G}_a, \mathbb{G}_a)$. But $\mathrm{Ext}^1_{\mathcal{U}}(\mathbb{G}_a, \mathbb{G}_a)$ is a free module over $\mathrm{End}_{\mathcal{U}}(\mathbb{G}_a)$ (acting either on the left or on the right) in view of [DG70, V.1.5.2]. Also, recall that $\mathrm{End}_{\mathcal{U}}(\mathbb{G}_a)$ is the noncommutative polynomial ring $k[F]$ (see [DG70, II.3.4.4]). It follows that $\xi = 0$, proving the claim.

By that claim, Q^1 is an isomorphism. In view of (3.3.3), it follows that $\mathrm{Ext}^2_{\mathcal{U}/\mathcal{J}}(\mathbb{G}_a, \nu_p)$ is the cokernel of the map

$$F_* - \mathrm{id} : \mathrm{Ext}^1_{\mathcal{U}}(\mathbb{G}_a, \mathbb{G}_a) \longrightarrow \mathrm{Ext}^1_{\mathcal{U}}(\mathbb{G}_a, \mathbb{G}_a).$$

This yields the isomorphism (3.3.2) by using the structure of $\mathrm{Ext}^1_{\mathcal{U}}(\mathbb{G}_a, \mathbb{G}_a)$ recalled above. □

By Theorems 3.7 and 3.14, we have $\mathrm{hd}(\mathcal{C}/\mathcal{I}) = \mathrm{hd}(\mathcal{C})$ if k is perfect. Under that assumption, we now determine the homological dimension of the étale isogeny category $\mathcal{C}/\mathcal{E}_S$. If $p \notin S$, then $\mathcal{E}_S = \mathcal{F}_S$ and Theorem 3.7 applies again. Thus, we may assume that $p \in S$:

Theorem 3.15. *Let k be a perfect field of characteristic $p > 0$, and S a set of prime numbers containing p. Then*

$$\mathrm{hd}(\mathcal{C}/\mathcal{E}_S) = \max(2, \mathrm{cd}_{S'}(k) + 1).$$

Proof. We argue again as in the proof of Theorem 3.7; we omit the details. Let $\mathcal{A} := \mathcal{C}/\mathcal{E}_S$, then $\mathcal{A}/\mathcal{A}_{\mathbb{P}} \cong \mathcal{C}/\mathcal{C}_{\mathbb{P}} \cong \underline{\mathcal{S}}$ has homological dimension 1, and

$$\mathcal{A}_{\mathbb{P}} \cong \mathcal{C}_p/\mathcal{E}_p \times \mathcal{E}_{S'} \cong \mathcal{U}/\mathcal{E}_p \times \mathcal{M}_p \times \mathcal{E}_{S'}$$

has homological dimension

$$\max(\mathrm{hd}(\mathcal{U}/\mathcal{E}_p), \mathrm{cd}_p(k), \mathrm{cd}_{S'}(k) + 1).$$

Since $\mathrm{cd}_p(k) \leq 1$, it suffices to show that $\mathrm{hd}(\mathcal{U}/\mathcal{E}_p) = 2$. By Lemma 3.1, the pair $(\mathcal{U}, \mathcal{E}_p)$ satisfies the lifting property. Using Proposition 2.16 and Lemma 3.9, it follows that $\mathrm{hd}(\mathcal{U}/\mathcal{E}_p) \leq \mathrm{hd}(\mathcal{U}) = 2$. Also, by adapting the proof of Theorem 3.14, one may check that

$$\mathrm{Ext}^2_{\mathcal{U}/\mathcal{E}_p}(\mathbb{G}_a, \alpha_p) = k,$$

where α_p lies in the exact sequence

$$0 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \longrightarrow 0.$$

More specifically, this exact sequence yields an exact sequence

$$\mathrm{Coker}(F_*) \longrightarrow \mathrm{Ext}^2_{\mathcal{U}/\mathcal{E}_p}(\mathbb{G}_a, \alpha_p) \longrightarrow \mathrm{Ext}^2_{\mathcal{U}/\mathcal{E}_p}(\mathbb{G}_a, \mathbb{G}_a),$$

where F_* is viewed as an endomorphism of $\mathrm{Ext}^1_{\mathcal{U}/\mathcal{E}_p}(\mathbb{G}_a, \mathbb{G}_a)$. Moreover, the canonical map $Q^i : \mathrm{Ext}^i_{\mathcal{U}}(\mathbb{G}_a, \mathbb{G}_a) \to \mathrm{Ext}^i_{\mathcal{U}/\mathcal{E}_p}(\mathbb{G}_a, \mathbb{G}_a)$ is still surjective for any $i \geq 0$. As $\mathrm{Ext}^2_{\mathcal{U}}(\mathbb{G}_a, \mathbb{G}_a) = 0$, it follows that $\mathrm{Ext}^2_{\mathcal{U}/\mathcal{E}_p}(\mathbb{G}_a, \alpha_p) \cong \mathrm{Coker}(F_*)$. Also, Q^1 is injective as seen by the argument of Theorem 3.14, where the infinitesimal subgroup $I \subset \mathbb{G}_a$ is replaced by a finite étale subgroup E, and

F^n by an endomorphism of $\mathbb{G}_a$ with kernel E. It follows that $\mathrm{Ext}^2_{\mathcal{U}/\mathcal{E}_p}(\mathbb{G}_a, \alpha_p)$ is isomorphic to the cokernel of F_* viewed as an endomorphism of $\mathrm{Ext}^1_{\mathcal{U}}(\mathbb{G}_a, \mathbb{G}_a)$. We conclude by using again the structure of that extension group as a module over $\mathrm{End}_{\mathcal{U}}(\mathbb{G}_a) \cong k[F]$. □

Remark 3.16. By combining Remark 3.11 (i) with Theorems 3.14 and 3.15, one can show that

$$\mathrm{hd}(\mathcal{L}/\mathcal{B}) = \mathrm{hd}(\mathcal{F}/\mathcal{B})$$

when $\mathcal{B}$ equals $\mathcal{I}$ or $\mathcal{E}_S$.

3.4. **Invariance under purely inseparable field extensions.** Consider a field extension k'/k. For any algebraic k-group G, we obtain an algebraic k'-group $G_{k'} = G \otimes_k k'$ by base change. This yields a functor

$$\otimes_k k' : \mathcal{C} = \mathcal{C}_k \longrightarrow \mathcal{C}_{k'},$$

which is exact and faithful.

Theorem 3.17. *Let k'/k be a purely inseparable field extension, and S a set of prime numbers containing $p := \mathrm{char}(k)$. Then the above base change functor induces equivalences of categories*

$$\mathcal{C}_k/\mathcal{F}_{S,k} \xrightarrow{\cong} \mathcal{C}_{k'}/\mathcal{F}_{S,k'} \quad \textit{and} \quad \mathcal{C}_k/\mathcal{I}_k \xrightarrow{\cong} \mathcal{C}_{k'}/\mathcal{I}_{k'}.$$

Proof. We adapt the argument of [Br17, Thm. 3.11]. Denote by

$$Q' : \mathcal{C}_{k'} \longrightarrow \mathcal{C}_{k'}/\mathcal{F}_{S,k'}$$

the quotient functor. Then the composite functor

$$\mathcal{C}_k \xrightarrow{\otimes_k k'} \mathcal{C}_{k'} \xrightarrow{Q'} \mathcal{C}_{k'}/\mathcal{F}_{S,k'}$$

is exact and takes every object of $\mathcal{F}_{S,k}$ to zero. Thus, this functor factors uniquely through an exact functor

$$\mathcal{C}_k/\mathcal{F}_{S,k} \longrightarrow \mathcal{C}_{k'}/\mathcal{F}_{S,k'}$$

that we still denote by $\otimes_k k'$. We will check that $\otimes_k k'$ is essentially surjective, faithful, and full.

Let $G' \in \mathcal{C}_{k'}$. By [Br17, Lem. 3.10], there exist a $G \in \mathcal{C}_k$ and an epimorphism $f : G' \to G_{k'}$ with kernel in $\mathcal{I}_k$, and hence in $\mathcal{F}_{S,k}$ since $p \in S$. Thus, $\otimes_k k'$ is essentially surjective.

Next, let $G, H \in \mathcal{C}_k$ and $\varphi \in \mathrm{Hom}_{\mathcal{C}_k/\mathcal{F}_{S,k}}(G, H)$ such that $\varphi_{k'}$ is zero. By Lemmas 2.7 and 3.1, we have $\varphi = Q(f)$ for some $f \in \mathrm{Hom}_{\mathcal{C}_k}(G, H/H')$, where $H' \subset H$ and $H' \in \mathcal{F}_{S,k}$. Then $\mathrm{Im}(f_{k'}) \in \mathcal{F}_{S,k'}$ in view of [Ga62, III.1.Lem. 2]. It follows that $\mathrm{Im}(f) \in \mathcal{F}_{S,k}$ and hence that $\varphi = 0$. So $\otimes_k k'$ is faithful.

To show that this functor is full, it suffices to check that every $\varphi' \in \mathrm{Hom}_{\mathcal{C}_{k'}/\mathcal{F}_{S,k'}}(G_{k'}, H_{k'})$ can be written as $\varphi_{k'}$ for some φ as above. By Lemmas 2.7 and 3.1 again, we have $\varphi' = Q'(f')$ for some $f' \in \mathrm{Hom}_{\mathcal{C}_{k'}}(G_{k'}, H_{k'}/H')$, where $H' \subset H_{k'}$ and $H_{k'}/H' \in \mathcal{F}_{S,k'}$. Applying [Br17, Lem. 3.10] once more, we obtain a k-subgroup $K \subset H$ such that $H' \subset K_{k'}$ and $K_{k'}/H' \in \mathcal{I}_{k'}$. Then $K_{k'} \in \mathcal{F}_{S,k'}$ and hence $K \in \mathcal{F}_{S,k}$. Thus, we may replace H' with $K_{k'}$, and assume that $f' \in \mathrm{Hom}_{\mathcal{C}_{k'}}(G_{k'}, H_{k'}/K_{k'})$. Replacing H with H/K yields a further reduction to the case where $f' \in \mathrm{Hom}_{\mathcal{C}_{k'}}(G_{k'}, H_{k'})$.

Consider the graph $\Gamma(f')$, a k'-subgroup of $G_{k'} \times H_{k'}$. By [Br17, Lem. 3.10] again, there exists a k-subgroup $\Delta \subset G \times H$ such that $\Gamma(f') \subset \Delta_{k'}$ and $\Delta_{k'}/\Gamma(f') \in \mathcal{I}_{k'}$. Since the projection $\Gamma(f') \to G_{k'}$ is an isomorphism in $\mathcal{C}_{k'}$, we see that the projection $\Delta_{k'} \to G_{k'}$ is an epimorphism in that category, with kernel $\Delta_{k'} \cap H_{k'} \in \mathcal{I}_{k'}$. Thus, the projection $\Delta \to G$ is an epimorphism in $\mathcal{C}_k$, with kernel $\Delta \cap H \in \mathcal{I}_k$. Now let Γ be the image of Δ under

$$\mathrm{id} \times q : G \times H \longrightarrow G \times H/(H \cap \Delta),$$

where q denotes the quotient. Then the projection $\Gamma \to G$ is an isomorphism, i.e., Γ is the graph of some $f \in \mathrm{Hom}_{\mathcal{C}_k}(G, H/(\Delta \cap H))$. As $\Gamma(f') \subset \Delta_{k'}$, we obtain $f_{k'} = q_{k'} \circ f'$. So $\varphi' = Q'(f_{k'}) = Q(f)_{k'}$ as desired.

This completes the proof for the S-isogeny category $\mathcal{C}/\mathcal{F}_S$. The argument for the radicial isogeny category $\mathcal{C}/\mathcal{I}$ is entirely similar. □

Our main theorem now follows readily by combining Theorems 3.7, 3.14, 3.15 and 3.17.

Remark 3.18. With the assumptions of Theorem 3.17, the base change functor also induces equivalences of categories

$$\mathcal{L}_k/\mathcal{F}_{S,k} \xrightarrow{\cong} \mathcal{L}_{k'}/\mathcal{F}_{S,k'} \quad \text{and} \quad \mathcal{L}_k/\mathcal{I}_k \xrightarrow{\cong} \mathcal{L}_{k'}/\mathcal{I}_{k'}.$$

as follows by the same argument.

Taking for k' the perfect closure of k and using the equivalence of categories $\mathcal{L}_{k'} \cong \mathcal{U}_{k'} \times \mathcal{M}_{k'}$ (3.1.4), this yields further equivalences (with our original notation $\mathcal{L} = \mathcal{L}_k$, $\mathcal{I} = \mathcal{I}_k$, ...)

$$\mathcal{L}/\mathcal{F}_S \cong \mathcal{U}/(\mathcal{F}_p \cap U) \times \mathcal{M}/\mathcal{M}_S, \quad \mathcal{L}/\mathcal{I} \cong \mathcal{U}/(\mathcal{I} \cap \mathcal{U}) \times \mathcal{M}/\mathcal{M}_p.$$

The former equivalence also follows from Proposition 3.6.

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