Commutative algebraic groups up to isogeny. II

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Abstract. This paper develops a representation-theoretic approach to the isogeny category $\underline{\mathcal{C}}$ of commutative group schemes of finite type over a field k, studied in [**Br16**]. We construct a ring R such that $\underline{\mathcal{C}}$ is equivalent to the category R-mod of all left R-modules of finite length. We also construct an abelian category of R-modules, $R\text{-}\widetilde{\text{mod}}$, which is hereditary, has enough projectives, and contains R-mod as a Serre subcategory; this yields a more conceptual proof of the main result of [**Br16**], asserting that $\underline{\mathcal{C}}$ is hereditary. We show that $R\text{-}\widetilde{\text{mod}}$ is equivalent to the isogeny category of commutative quasi-compact k-group schemes.

Contents

1. Introduction

In this paper, we develop a representation-theoretic approach to the isogeny category of commutative algebraic groups over a field k, studied in [**Br16**]. This

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abelian category, that we denote by $\underline{\mathcal{C}}$, is equivalent to the quotient of the abelian category $\mathcal{C}$ of group schemes of finite type over k by the Serre subcategory $\mathcal{F}$ of finite k-group schemes. The main result of [**Br16**] asserts that $\underline{\mathcal{C}}$ is *hereditary*, i.e., $\mathrm{Ext}^i_{\underline{\mathcal{C}}}(G,H) = 0$ for all $i \geq 2$ and all $G, H \in \underline{\mathcal{C}}$. By a theorem of Serre (see [**Se60**, 10.1]) which was the starting point of our work, this also holds for the original category $\mathcal{C}$ when k is algebraically closed of characteristic 0. But for certain fields k, the extension groups in $\mathcal{C}$ can be non-zero in arbitrarily large degrees, as a consequence of [**Mi70**, Thm. 1].

To prove that $\underline{\mathcal{C}}$ is hereditary, the approach of [**Br16**] is similar to that of Serre in [**Se60**], generalized by Oort in [**Oo66**] to determine the extension groups in $\mathcal{C}$ when k is algebraically closed of positive characteristic. As $\underline{\mathcal{C}}$ is easily seen to be a finite length category, it suffices to check the vanishing of higher extension groups for all simple objects $G, H \in \underline{\mathcal{C}}$. These are the additive group, the simple tori and the simple abelian varieties, and one may then adapt the case-by-case analysis of [**Se60, Oo66**] to the easier setting of the isogeny category.

In this paper, we obtain a more conceptual proof, by constructing a ring R such that $\underline{\mathcal{C}}$ is equivalent to the category R-mod of all left R-modules of finite length; moreover, R-mod is a Serre category of an abelian category $R\text{-}\widetilde{\mathrm{mod}}$ of left R-modules, which is hereditary and has enough projectives. For a more precise statement, we refer to Theorem 3.5 in §3.2.5, which can be read independently of the rest of the paper. Our result generalizes, and builds on, the equivalence of the isogeny category of unipotent groups over a perfect field of positive characteristic with the category of modules of finite length over a localization of the Dieudonné ring (see [**DG70**, V.3.6.7]).

More specifically, the abelian category $\mathcal{C}$ has very few projectives: the unipotent groups in characteristic 0, and the trivial group otherwise (see [**Br16**, Thm. 2.9, Cor. 5.15])). This drawback was remedied in [**Se60**] by considering the abelian category $\widehat{C}$ of *pro-algebraic groups*. If k is algebraically closed of characteristic 0, then $\widehat{\mathcal{C}}$ is hereditary, has enough projectives, and contains $\mathcal{C}$ as a Serre subcategory. We obtain a similar result for the isogeny category $\underline{\mathcal{C}}$ over an arbitrary field; this category has more projectives than $\mathcal{C}$ (e.g., the tori), but still not enough of them. We show that $\underline{\mathcal{C}}$ is a Serre subcategory of the isogeny category $\widetilde{\underline{\mathcal{C}}}$ of *quasi-compact group schemes*, which is a hereditary abelian category having enough projectives. In addition, $\widetilde{\underline{\mathcal{C}}}$ is equivalent to the category $R\text{-}\widetilde{\mathrm{mod}}$ mentioned above (see again §3.2.5 for a more precise statement). The quasi-compact group schemes, studied by Perrin in [**Pe75, Pe76**], form a restricted class of pro-algebraic groups, discussed in more details in §3.2.2.

Since $\underline{\mathcal{C}}$ is a length category, it is equivalent to the category of all left modules of finite length over a basic pseudo-compact ring A, which is then uniquely determined (this result is due to Gabriel, see [**Ga62**, IV.4] and [**Ga71**, 7.2]). The ring R that we construct is also basic, but not pseudo-compact; it may be viewed as a dense subring of A. Its main advantage for our purposes is that the above category $R\text{-}\widetilde{\mathrm{mod}}$ consists of R-modules but not of A-modules.

This paper is organized as follows. In Section 2, we study homological properties of abelian categories equipped with a torsion pair. This setting turns out to be very useful when dealing with algebraic groups, since these are obtained as extensions of groups of special types: for example, every connected algebraic group is an extension of an abelian variety by a affine algebraic group, and these are unique up to isogeny. The main result of Section 2 is Theorem 2.13, which explicitly describes certain abelian categories equipped with a torsion pair, in terms of modules over triangular matrix rings.

Section 3 begins with a brief survey of the structure theory for commutative algebraic groups, with emphasis on categorical aspects. We also treat in parallel the affine group schemes (which form the pro-completion of the abelian category of affine algebraic groups) and the quasi-compact group schemes. We then obtain our main Theorem 3.5 by combining all these structure results with Theorem 2.13. Next, after some auxiliary developments in Subsection 3.3, we study the finiteness properties of the spaces of morphisms and extensions in the isogeny category $\underline{\mathcal{C}}$. In particular, we show that $\underline{\mathcal{C}}$ is $\mathbb{Q}$-linear, Hom- and Ext-finite if and only if k is a number field (Proposition 3.15). The final Subsection 3.5 initiates the study of the indecomposable objects of $\underline{\mathcal{C}}$, by considering a very special situation: extensions of abelian varieties with prescribed simple factors by unipotent groups, over the field of rational numbers. Using a classical result of Dlab and Ringel on representations of species (see [**DR76**]), we obtain a characterization of finite representation type in that setting (Proposition 3.16).

Notation and conventions. All considered categories are assumed to be small. We denote categories by calligraphic letters, e.g., $\mathcal{X}, \mathcal{Y}$, and functors by boldface letters, e.g., $\mathbf{L}, \mathbf{R}$. By abuse of notation, we write $X \in \mathcal{X}$ if X is an object of $\mathcal{X}$. Also, we say that $\mathcal{X}$ contains $\mathcal{Y}$ if $\mathcal{Y}$ is a full subcategory of $\mathcal{X}$.

For any ring R, we denote by R-Mod the category of left R-modules, and by $R\text{-Mod}^{\text{fg}}$ (resp. $R\text{-Mod}^{\text{ss}}$, R-mod) the full subcategory of finitely generated modules (resp. of semi-simple modules, of modules of finite length).

2. A construction of hereditary categories

2.1. Two preliminary results. Let $\mathcal{C}$ be an abelian category. Recall that the *homological dimension* of $\mathcal{C}$ is the smallest non-negative integer $n =: \text{hd}(\mathcal{C})$ such that $\text{Ext}^{n+1}_{\mathcal{C}}(X,Y) = 0$ for all $X, Y \in \mathcal{C}$; equivalently, $\text{Ext}^m_{\mathcal{C}}(X,Y) = 0$ for all $X, Y \in \mathcal{C}$ and all $m > n$. If there is no such integer, then $\text{hd}(\mathcal{C})$ is understood to be infinite.

Also, recall that $\mathcal{C}$ is said to be *semi-simple* if $\text{hd}(\mathcal{C}) = 0$; equivalently, every short exact sequence in $\mathcal{C}$ splits. If $\text{hd}(\mathcal{C}) \leq 1$, then $\mathcal{C}$ is said to be *hereditary.*

We now record two easy lemmas, for which we could not locate appropriate references.

Lemma 2.1. *The following conditions are equivalent for an abelian category $\mathcal{C}$ and a non-negative integer n:*

(i) $\text{hd}(\mathcal{C}) \leq n$.
(ii) *The functor* $\text{Ext}^n_{\mathcal{C}}(X, ?)$ *is right exact for any* $X \in \mathcal{C}$.
(iii) *The functor* $\text{Ext}^n_{\mathcal{C}}(?, Y)$ *is right exact for any* $Y \in \mathcal{C}$.

Proof. (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii) Both assertions follow from the vanishing of $\text{Ext}^{n+1}_{\mathcal{C}}(X, ?)$ in view of the long exact sequence of Ext groups.

(ii) $\Rightarrow$ (i) Let $\xi \in \text{Ext}^{n+1}_{\mathcal{C}}(X,Y)$ be the class of an exact sequence

$$0 \longrightarrow Y \longrightarrow X_{n+1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X \longrightarrow 0$$

in $\mathcal{C}$. We cut this sequence in two short exact sequences

$$0 \longrightarrow Y \longrightarrow X_{n+1} \longrightarrow Z \longrightarrow 0, \quad 0 \longrightarrow Z \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X \longrightarrow 0$$

with classes $\xi_1 \in \mathrm{Ext}^1_{\mathcal{C}}(Z,Y)$, $\xi_2 \in \mathrm{Ext}^n_{\mathcal{C}}(X,Z)$ respectively. Then ξ is the Yoneda product $\xi_1 \cdot \xi_2$. Since the natural map $\mathrm{Ext}^n_{\mathcal{C}}(X,X_{n+1}) \to \mathrm{Ext}^n_{\mathcal{C}}(X,Z)$ is surjective, there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & X_{n+1} & \longrightarrow & X'_n & \longrightarrow & \cdots & \longrightarrow & X'_1 & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \mathrm{id}\downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & X_n & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & X & \longrightarrow & 0.
\end{array}$$

Also, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X'_{n+1} & \xrightarrow{f'} & X_{n+1} & \longrightarrow & 0 \\
 & & \mathrm{id}\downarrow & & \downarrow & & f\downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X_{n+1} & \xrightarrow{f} & Z & \longrightarrow & 0,
\end{array}$$

where the top exact sequence is split (as $\mathrm{id}_{X_{n+1}}$ yields a section of f'). Thus, $f^*(\xi_1) = 0$ in $\mathrm{Ext}^1_{\mathcal{C}}(X_{n+1},Y)$. By concatenating both diagrams, we obtain a morphism of extensions

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & Y & \longrightarrow & X'_{n+1} & \longrightarrow & \cdots & \longrightarrow & X'_1 & \longrightarrow & X & \longrightarrow & 0 \\
 & & \mathrm{id}\downarrow & & \downarrow & & & & \downarrow & & \mathrm{id}\downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X_{n+1} & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & X & \longrightarrow & 0.
\end{array}$$

Thus, ξ is also represented by the top exact sequence, and hence $\xi = f^*(\xi_1)\cdot\xi_2 = 0$. This completes the proof of (ii) $\Rightarrow$ (i).

A dual argument shows that (iii) $\Rightarrow$ (i). □

Next, recall that a subcategory $\mathcal{D}$ of an abelian category $\mathcal{C}$ is said to be a *Serre subcategory* if $\mathcal{D}$ is non-empty, strictly full in $\mathcal{C}$, and stable under taking subobjects, quotients and extensions.

LEMMA 2.2. *Let $\mathcal{C}$ be a hereditary abelian category, and $\mathcal{D}$ a Serre subcategory. Then $\mathcal{D}$ is hereditary.*

PROOF. In view of the assumption on $\mathcal{D}$, the natural map $\mathrm{Ext}^1_{\mathcal{D}}(X,Y) \to \mathrm{Ext}^1_{\mathcal{C}}(X,Y)$ is an isomorphism for any $X, Y \in \mathcal{D}$. So the assertion follows from Lemma 2.1. □

2.2. Torsion pairs. Throughout this subsection, we consider an abelian category $\mathcal{C}$ equipped with a *torsion pair*, that is, a pair of strictly full subcategories $\mathcal{X}$, $\mathcal{Y}$ satisfying the following conditions:

(i) $\mathrm{Hom}_{\mathcal{C}}(X,Y) = 0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$.

(ii) For any $C \in \mathcal{C}$, there exists an exact sequence in $\mathcal{C}$

$$0 \longrightarrow X_C \xrightarrow{f_C} C \xrightarrow{g_C} Y_C \longrightarrow 0, \tag{2.1}$$

where $X_C \in \mathcal{X}$ and $Y_C \in \mathcal{Y}$.

Then $\mathcal{X}$ is stable under quotients, extensions and coproducts, and $\mathcal{Y}$ is stable under subobjects, extensions and products. Moreover, the assignment $C \mapsto X_C$ extends to an additive functor $\mathbf{R} : \mathcal{C} \to \mathcal{X}$, right adjoint to the inclusion. Dually, the assignment $C \mapsto Y_C$ extends to an additive functor $\mathbf{L} : \mathcal{C} \to \mathcal{Y}$, left adjoint to the inclusion (see e.g. [**BR07**, Sec. 1.1] for these results).

Lemma 2.3. *Assume that* $\mathcal{X}$, $\mathcal{Y}$ *are Serre subcategories of* $\mathcal{C}$. *Then:*

(i) $\mathbf{R}$, $\mathbf{L}$ *are exact.*
(ii) $\mathrm{Ext}^n_{\mathcal{C}}(X,Y) = 0$ *for all* $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ *and* $n \geq 1$.
(iii) $\mathrm{hd}(\mathcal{C}) \leq \mathrm{hd}(\mathcal{X}) + \mathrm{hd}(\mathcal{Y}) + 1$.

Proof. (i) Let $C \in \mathcal{C}$ and consider a subobject $i : C_1 \hookrightarrow C$. Denote by $q : C \to C_2 := C/C_1$ the quotient map, and by $C_1 \cap X_C$ the kernel of the map $(q, g_C) : C \to C_2 \times Y_C$. Then $C_1 \cap X_C \hookrightarrow X_C$, and hence $C_1 \cap X_C \in \mathcal{C}$. Moreover, $C_1/C_1 \cap X_C \hookrightarrow C/X_C \cong Y_C$, and hence $C_1/C_1 \cap X_C \in \mathcal{Y}$. Thus, $C_1 \cap X_C = \mathbf{R}(C_1)$ and $C_1/C_1 \cap X_C = \mathbf{L}(C_1)$. So we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{R}(C_1) & \xrightarrow{\mathbf{R}(i)} & \mathbf{R}(C) & \longrightarrow & \mathbf{R}(C)/\mathbf{R}(C_1) & \longrightarrow & 0 \\
 & & {\scriptstyle f_{C_1}}\downarrow & & {\scriptstyle f_C}\downarrow & & {\scriptstyle g}\downarrow & & \\
0 & \longrightarrow & C_1 & \xrightarrow{i} & C & \xrightarrow{q} & C_2 & \longrightarrow & 0.
\end{array}$$

As we just showed, the left square is cartesian; it follows that g is a monomorphism. This yields a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{R}(C_1) & \xrightarrow{\mathbf{R}(i)} & \mathbf{R}(C) & \longrightarrow & \mathbf{R}(C)/\mathbf{R}(C_1) & \longrightarrow & 0 \\
 & & {\scriptstyle f_{C_1}}\downarrow & & {\scriptstyle f_C}\downarrow & & {\scriptstyle g}\downarrow & & \\
0 & \longrightarrow & C_1 & \xrightarrow{i} & C & \xrightarrow{q} & C_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{L}(C_1) & \longrightarrow & \mathbf{L}(C) & \longrightarrow & C_2/\mathrm{Im}(g) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Moreover, $\mathbf{R}(C)/\mathbf{R}(C_1) \in \mathcal{X}$ and $C_2/\mathrm{Im}(g) \in \mathcal{Y}$ by our assumption on $\mathcal{X}$, $\mathcal{Y}$. It follows that $\mathbf{R}(C)/\mathbf{R}(C_1) = \mathbf{R}(C_2)$ and $C_2/\mathrm{Im}(g) = \mathbf{L}(C_2)$. Thus, $\mathbf{R}, \mathbf{L}$ are exact.

(ii) We first show that $\mathrm{Ext}^1_{\mathcal{C}}(X,Y) = 0$. Consider an exact sequence

$$0 \longrightarrow Y \longrightarrow C \longrightarrow X \longrightarrow 0$$

in $\mathcal{C}$. Then the induced map $\mathbf{R}(C) \to \mathbf{R}(X) = X$ is an isomorphism, since $\mathbf{R}$ is exact and $\mathbf{R}(Y) = 0$. Thus, the above exact sequence splits; this yields the assertion.

Next, we show the vanishing of any $\xi \in \mathrm{Ext}^n_{\mathcal{C}}(X,Y)$. For this, we adapt the argument of Lemma 2.1. Choose a representative of ξ by an exact sequence

$$0 \longrightarrow Y \longrightarrow C_n \longrightarrow \cdots \longrightarrow C_1 \longrightarrow X \longrightarrow 0$$

in $\mathcal{C}$ and cut it in two short exact sequences

$$0 \longrightarrow Y \longrightarrow C_n \longrightarrow Z \longrightarrow 0, \quad 0 \longrightarrow Z \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow X \longrightarrow 0.$$

This yields an exact sequence

$$0 \longrightarrow \mathbf{R}(Z) \longrightarrow \mathbf{R}(C_{n-1}) \longrightarrow \cdots \longrightarrow \mathbf{R}(C_1) \longrightarrow X \longrightarrow 0.$$

Also, we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & C'_n & \longrightarrow & \mathbf{R}(Z) & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & C_n & \longrightarrow & Z & \longrightarrow & 0, \end{array}$$

where the top sequence splits by the above step. So ξ is also represented by the exact sequence

$$0 \longrightarrow Y \longrightarrow C'_n \longrightarrow \mathbf{R}(C_{n-1}) \longrightarrow \cdots \longrightarrow \mathbf{R}(C_1) \longrightarrow X \longrightarrow 0,$$

which has a trivial class in $\mathrm{Ext}^n_{\mathcal{C}}(X, Y)$.

(iii) We may assume that $\mathrm{hd}(\mathcal{X}) := m$ and $\mathrm{hd}(\mathcal{Y}) := n$ are both finite. In view of the exact sequence (2.1) and the long exact sequence for Ext groups, it suffices to show that $\mathrm{Ext}^{m+n+2}_{\mathcal{C}}(C, C') = 0$ for all C, C' in $\mathcal{X}$ or $\mathcal{Y}$. By (ii), this holds whenever $C \in \mathcal{X}$ and $C' \in \mathcal{Y}$. Also, if $C, C' \in \mathcal{X}$, then $\mathrm{Ext}^{m+1}_{\mathcal{C}}(C, C') = 0$: indeed, every exact sequence

$$0 \longrightarrow C' \longrightarrow C_{m+1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C \longrightarrow 0$$

is Yoneda equivalent to the exact sequence

$$0 \longrightarrow C' \longrightarrow \mathbf{R}(C_{m+1}) \longrightarrow \cdots \longrightarrow \mathbf{R}(C_1) \longrightarrow C \longrightarrow 0,$$

which in turn is equivalent to 0 by assumption. Likewise, $\mathrm{Ext}^{n+1}_{\mathcal{C}}(C, C') = 0$ for all $C, C' \in \mathcal{Y}$. So we are reduced to checking that $\mathrm{Ext}^{m+n+2}_{\mathcal{C}}(Y, X) = 0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$.

For this, we adapt again the argument of Lemma 2.1. Let $\xi \in \mathrm{Ext}^{m+n+2}_{\mathcal{C}}(Y, X)$ be represented by an exact sequence

$$0 \longrightarrow X \longrightarrow C_{m+n+2} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow Y \longrightarrow 0$$

in $\mathcal{C}$. This yields two exact sequences

$$0 \longrightarrow X \longrightarrow C_{m+n+2} \longrightarrow \cdots \longrightarrow C_{n+2} \longrightarrow Z \longrightarrow 0,$$

$$0 \longrightarrow Z \longrightarrow C_{n+1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow Y \longrightarrow 0.$$

As $\mathrm{Ext}^{n+1}_{\mathcal{C}}(Y, \mathbf{L}(Z)) = 0$ by the above step, the exact sequence

$$0 \longrightarrow \mathbf{R}(Z) \longrightarrow Z \longrightarrow \mathbf{L}(Z) \longrightarrow 0$$

yields a surjection $\mathrm{Ext}^{n+1}_{\mathcal{C}}(Y, \mathbf{R}(Z)) \to \mathrm{Ext}^{n+1}_{\mathcal{C}}(Y, Z)$. Thus, there exists a commutative diagram of exact sequences

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathbf{R}(Z) & \longrightarrow & C'_{n+1} & \longrightarrow & \cdots & \longrightarrow & C'_1 & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \\ 0 & \longrightarrow & Z & \longrightarrow & C_{n+1} & \longrightarrow & \cdots & \longrightarrow & C_1 & \longrightarrow & Y & \longrightarrow & 0. \end{array}$$

Also, we have an exact sequence

$$0 \longrightarrow X \longrightarrow \mathbf{R}(C_{m+n+2}) \longrightarrow \cdots \longrightarrow \mathbf{R}(C_{n+2}) \longrightarrow \mathbf{R}(Z) \longrightarrow 0,$$

with trivial class as $\mathrm{Ext}^{m+1}_{\mathcal{C}}(\mathbf{R}(Z), X) = 0$. Hence ξ is also represented by the exact sequence

$$0 \to X \to \mathbf{R}(C_{m+n+2}) \to \cdots \to \mathbf{R}(C_{n+2}) \to C'_{n+1} \to \cdots \to C'_1 \to Y \to 0,$$

which has a trivial class as well. $\square$

Corollary 2.4. *Assume that $\mathcal{X}$ and $\mathcal{Y}$ are semi-simple Serre subcategories of $\mathcal{C}$. Then:*

(i) *Every object of $\mathcal{X}$ is projective in $\mathcal{C}$.*
(ii) *Every object of $\mathcal{Y}$ is injective in $\mathcal{C}$.*
(iii) *$\mathcal{C}$ is hereditary.*

Proof. (i) Let $X \in \mathcal{X}$. Then $\mathrm{Ext}^1_{\mathcal{C}}(X, Y) = 0$ for all $Y \in \mathcal{Y}$, by Lemma 2.3. Moreover, $\mathrm{Ext}^1_{\mathcal{C}}(X, X') = 0$ for all $X' \in \mathcal{X}$ by our assumption. In view of the exact sequence (2.1), it follows that $\mathrm{Ext}^1_{\mathcal{C}}(X, C) = 0$ for all $C \in \mathcal{C}$, i.e., X is projective in $\mathcal{C}$.

(ii) This is checked similarly.

(iii) This follows from Lemma 2.3 (iii).

□

2.3. The category of extensions. We still consider an abelian category $\mathcal{C}$ equipped with a torsion pair $(\mathcal{X}, \mathcal{Y})$. Let $\mathcal{E}$ be the category with objects the triples (X, Y, ξ), where $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and $\xi \in \mathrm{Ext}^1_{\mathcal{C}}(Y, X)$; the morphisms from (X, Y, ξ) to (X', Y', ξ') are the pairs of morphisms $(u : X \to X', v : Y \to Y')$ such that $u_*(\xi) = v^*(\xi')$ in $\mathrm{Ext}^1_{\mathcal{C}}(Y, X')$. We say that $\mathcal{E}$ is the *category of extensions* associated with the triple $(\mathcal{C}, \mathcal{X}, \mathcal{Y})$.

We may assign to any $C \in \mathcal{C}$, the triple

$$\mathbf{T}(C) := (\mathbf{R}(C), \mathbf{L}(C), \xi(C)),$$

where $\xi(C) \in \mathrm{Ext}^1_{\mathcal{C}}(\mathbf{L}(C), \mathbf{R}(C))$ denotes the class of the extension (2.1),

$$0 \longrightarrow \mathbf{R}(C) \longrightarrow C \longrightarrow \mathbf{L}(C) \longrightarrow 0.$$

Lemma 2.5. *Assume that $\mathcal{X}, \mathcal{Y}$ are Serre subcategories of $\mathcal{C}$, and $\mathrm{Hom}_{\mathcal{C}}(Y, X) = 0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$. Then the above assignment extends to a covariant functor $\mathbf{T} : \mathcal{C} \to \mathcal{E}$, which is an equivalence of categories.*

Proof. Consider a morphism $f : C \to C'$ in $\mathcal{C}$. Then f lies in a unique commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_C & \xrightarrow{f_C} & C & \xrightarrow{g_C} & Y_C & \longrightarrow & 0 \\ & & {\scriptstyle u}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle v}\downarrow & & \\ 0 & \longrightarrow & X_{C'} & \longrightarrow & C' & \longrightarrow & Y_{C'} & \longrightarrow & 0. \end{array}$$

Denote by $\xi \in \mathrm{Ext}^1_{\mathcal{C}}(Y_C, X_C)$, $\xi' \in \mathrm{Ext}^1_{\mathcal{C}}(Y_{C'}, X_{C'})$ the classes of the above extensions and set $u =: \mathbf{R}(f)$, $v =: \mathbf{L}(f)$. These fit into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{R}(C) & \longrightarrow & C & \longrightarrow & \mathbf{L}(C) & \longrightarrow & 0 \\ & & {\scriptstyle \mathbf{R}(f)}\downarrow & & \downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \\ 0 & \longrightarrow & \mathbf{R}(C') & \longrightarrow & D & \longrightarrow & \mathbf{L}(C) & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & {\scriptstyle \mathbf{L}(f)}\downarrow & & \\ 0 & \longrightarrow & \mathbf{R}(C') & \longrightarrow & C' & \longrightarrow & \mathbf{L}(C') & \longrightarrow & 0 \end{array}$$

It follows that $\mathbf{R}(f)_*(\xi) = \mathbf{L}(f)^*(\xi')$. Thus, the assignment $f \mapsto (\mathbf{R}(f), \mathbf{L}(f))$ defines the desired covariant functor $\mathbf{T}$. We now show that $\mathbf{T}$ is an equivalence of categories.

Since $(\mathcal{X}, \mathcal{Y})$ is a torsion pair, $\mathbf{T}$ is essentially surjective. We check that it is faithful. Let $C, C' \in \mathcal{C}$ and consider $f \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ such that $\mathbf{R}(f) = 0 = \mathbf{L}(f)$. Then the composition $X_C \xrightarrow{f_C} C \xrightarrow{f} C'$ is zero, and hence f factors through $g : Y_C \to C'$. Moreover, the composition $Y_C \xrightarrow{g} C' \xrightarrow{g_{C'}} Y_{C'}$ is zero, and hence g factors through $h : Y_C \to X_{C'}$. By our assumption, $h = 0$; thus, $f = 0$.

Finally, we show that $\mathbf{T}$ is full. Let again $C, C' \in \mathcal{C}$ and consider $u : X_C \to X_{C'}$, $v : Y_C \to Y_{C'}$ such that $u_*(\xi) = v^*(\xi')$, where ξ (resp. ξ') denotes the class of the extension (2.1) for C (resp. C'). Since $\mathrm{Hom}_{\mathcal{C}}(Y, X) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, these extensions are uniquely determined by their classes, and in turn by C, C'. Thus, we have a commutative diagram of extensions in $\mathcal{C}$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_C & \xrightarrow{f_C} & C & \xrightarrow{g_C} & Y_C & \longrightarrow & 0 \\
 & & {\scriptstyle u}\downarrow & & \downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \\
0 & \longrightarrow & X_{C'} & \xrightarrow{i} & D & \xrightarrow{q} & Y_C & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & {\scriptstyle v}\downarrow & & \\
0 & \longrightarrow & X_{C'} & \xrightarrow{f_{C'}} & C' & \xrightarrow{g_{C'}} & Y_{C'} & \longrightarrow & 0.
\end{array}$$

This yields a morphism $f : C \to C'$ such that $\mathbf{R}(f) = u$ and $\mathbf{L}(f) = v$. □

With the assumptions of Lemma 2.5, the subcategory $\mathcal{X}$ (resp. $\mathcal{Y}$) of $\mathcal{C}$ is identified via $\mathbf{T}$ with the full subcategory of $\mathcal{E}$ with objects the triples of the form $(X, 0, 0)$ (resp. $(0, Y, 0)$). Assuming in addition that *$\mathcal{X}$ and $\mathcal{Y}$ are semi-simple*, we now obtain a description of homomorphism and extension groups in $\mathcal{E}$:

Proposition 2.6. *With the above assumptions, there is an exact sequence*

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{E}}(Z, Z') \xrightarrow{\iota} \mathrm{Hom}_{\mathcal{X}}(X, X') \times \mathrm{Hom}_{\mathcal{Y}}(Y, Y') \longrightarrow$$

$$\xrightarrow{\varphi} \mathrm{Ext}^1_{\mathcal{C}}(Y, X') \longrightarrow \mathrm{Ext}^1_{\mathcal{E}}(Z, Z') \longrightarrow 0$$

for any $Z = (X, Y, \xi), Z' = (X', Y', \xi') \in \mathcal{E}$, where ι denotes the inclusion, and $\varphi(u, v) := u_*(\xi') - v^*(\xi)$.

Proof. We have $\mathrm{Ker}(\iota) = 0$ and $\mathrm{Im}(\iota) = \mathrm{Ker}(\varphi)$ by the definition of the morphisms in $\mathcal{E}$. Thus, it suffices to check that $\mathrm{Coker}(\varphi) \cong \mathrm{Ext}^1_{\mathcal{E}}(Z, Z')$.

Consider the exact sequence

$$0 \longrightarrow X' \longrightarrow Z' \longrightarrow Y' \longrightarrow 0$$

in $\mathcal{E}$, with class $\xi' \in \mathrm{Ext}^1_{\mathcal{E}}(Y', X') = \mathrm{Ext}^1_{\mathcal{C}}(Y', X')$. This yields an exact sequence

$$\mathrm{Hom}_{\mathcal{E}}(Y, Y') \xrightarrow{\partial'} \mathrm{Ext}^1_{\mathcal{E}}(Y, X') \longrightarrow \mathrm{Ext}^1_{\mathcal{E}}(Y, Z') \longrightarrow \mathrm{Ext}^1_{\mathcal{E}}(Y, Y'),$$

where $\partial'(v) := v^*(\xi')$ for any $v \in \mathrm{Hom}_{\mathcal{E}}(Y, Y')$. Moreover, since $\mathcal{Y}$ is a semi-simple Serre subcategory of $\mathcal{E}$, we have $\mathrm{Hom}_{\mathcal{E}}(Y, Y') = \mathrm{Hom}_{\mathcal{Y}}(Y, Y')$ and $\mathrm{Ext}^1_{\mathcal{E}}(Y, Y') = 0$. So we obtain a natural isomorphism

$$\mathrm{Ext}^1_{\mathcal{E}}(Y, Z') \cong \mathrm{Ext}^1_{\mathcal{E}}(Y, X')/\{v^*(\xi') \mid v \in \mathrm{Hom}_{\mathcal{Y}}(Y, Y')\}. \tag{2.2}$$

Similarly, the exact sequence

$$0 \longrightarrow X \longrightarrow Z \longrightarrow Y \longrightarrow 0$$

in $\mathcal{E}$, with class $\xi \in \mathrm{Ext}^1_{\mathcal{C}}(Y, X)$, yields an exact sequence

$$\mathrm{Hom}_{\mathcal{E}}(X, Z') \xrightarrow{\partial} \mathrm{Ext}^1_{\mathcal{E}}(Y, Z') \longrightarrow \mathrm{Ext}^1_{\mathcal{E}}(Z, Z') \longrightarrow \mathrm{Ext}^1_{\mathcal{E}}(X, Z'),$$

where $\partial(u) := u_*(\xi)$ for any $u \in \mathrm{Hom}_{\mathcal{E}}(X, Z')$. Moreover, the natural map

$$\mathrm{Hom}_{\mathcal{X}}(X, X') \longrightarrow \mathrm{Hom}_{\mathcal{E}}(X, Z')$$

is an isomorphism, since $\mathrm{Hom}_{\mathcal{E}}(X, Y') = 0$. Also, $\mathrm{Ext}^1_{\mathcal{E}}(X, Z') = 0$ by Corollary 2.4. Hence we obtain a natural isomorphism

$$\mathrm{Ext}^1_{\mathcal{E}}(Z, Z') \cong \mathrm{Ext}^1_{\mathcal{E}}(Y, Z')/\{u_*(\xi) \mid u \in \mathrm{Hom}_{\mathcal{X}}(X, X')\}. \tag{2.3}$$

Putting together the isomorphisms (2.2) and (2.3) yields the desired assertion. □

2.4. Universal extensions. We still consider an abelian category $\mathcal{C}$ equipped with a torsion pair $(\mathcal{X}, \mathcal{Y})$, and make the following assumptions:

(a) $\mathcal{X}, \mathcal{Y}$ are Serre subcategories of $\mathcal{C}$.
(b) $\mathcal{X}, \mathcal{Y}$ are semi-simple.
(c) $\mathrm{Hom}_{\mathcal{C}}(Y, X) = 0$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$.
(d) There exists a covariant exact functor $\mathbf{F} : \mathcal{Y} \to \widetilde{\mathcal{X}}$, where $\widetilde{\mathcal{X}}$ is a semi-simple abelian category containing $\mathcal{X}$ as a Serre subcategory, and a bifunctorial isomorphism

$$\mathrm{Ext}^1_{\mathcal{C}}(Y, X) \xrightarrow{\cong} \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{F}(Y), X) \tag{2.4}$$

for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$.

Remark 2.7. The above assumptions are satisfied by the isogeny category of algebraic groups and some natural Serre subcategories, as we will see in Subsection 3.2. Also, assumptions (a), (b) and (c) are just those of Corollary 2.4 and Proposition 2.6. Note that a weak version of (d) always holds, where we only require $\widetilde{\mathcal{X}}$ to be a category containing $\mathcal{X}$: take $\widetilde{\mathcal{X}}$ to be the opposite category of covariant functors from $\mathcal{X}$ to sets; then the functor $\mathrm{Ext}^1_{\mathcal{C}}(Y, ?)$ is an object of $\widetilde{\mathcal{X}}$ for any $Y \in \mathcal{Y}$, and the isomorphism (2.4) follows from Yoneda's lemma. But requiring $\widetilde{\mathcal{X}}$ to be abelian and semi-simple is a restrictive assumption.

Under the four above assumptions, every $C \in \mathcal{C}$ defines an extension class $\xi(C) \in \mathrm{Ext}^1_{\mathcal{C}}(Y_C, X_C)$, and in turn a morphism $\eta(C) \in \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{F}(Y_C), X_C)$. Moreover, every morphism $f : C \to C'$ in $\mathcal{C}$ induces morphisms $u : X_C \to X_{C'}$, $v : Y_C \to Y_{C'}$ such that the push-forward $u_*\xi(C) \in \mathrm{Ext}^1_{\mathcal{C}}(Y, X')$ is identified with $u \circ \eta(C) \in \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{F}(Y_C), X_{C'})$, and the pull-back $v^*\xi(C') \in \mathrm{Ext}^1_{\mathcal{C}}(Y, X')$ is identified with $\eta(C') \circ F(v) \in \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{F}(Y_C), X_{C'})$.

It follows that the category of extensions $\mathcal{E}$ (considered in Subsection 2.3) is equivalent to the category $\mathcal{F}$ with objects the triples (X, Y, η), where $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and $\eta \in \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{F}(Y), X)$; the morphisms from (X, Y, η) to (X', Y', η') are the pairs of morphisms $(u : X \to X',\ v : Y \to Y')$ such that the diagram

$$\begin{array}{ccc} \mathbf{F}(Y) & \xrightarrow{F(v)} & \mathbf{F}(Y') \\ \eta\downarrow & & \eta'\downarrow \\ X & \xrightarrow{u} & X' \end{array}$$

commutes. With this notation, Lemma 2.5 yields readily:

LEMMA 2.8. *The assignment* $C \mapsto (\mathbf{R}(C), \mathbf{L}(C), \eta(C))$ *extends to an equivalence of categories* $\mathcal{C} \overset{\cong}{\to} \mathcal{F}$.

Next, consider the category $\widetilde{\mathcal{F}}$ with objects the triples (X, Y, η), where $X \in \widetilde{\mathcal{X}}$, $Y \in \mathcal{Y}$ and $\eta \in \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{F}(Y), X)$; the morphisms are defined like those of $\mathcal{F}$. Then one may readily check the following:

LEMMA 2.9. *With the above notation,* $\mathcal{F}$ *is a Serre subcategory of* $\widetilde{\mathcal{F}}$. *Moreover, the triple* $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{X}}, \mathcal{Y})$ *satisfies the assumptions* (a), (b), (c), *and* (d) *with the same functor* $\mathbf{F}$. *For any* $Z = (X, Y, \eta)$ *and* $Z' = (X', Y', \eta') \in \widetilde{\mathcal{F}}$, *we have an exact sequence*

$$0 \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{F}}}(Z, Z') \overset{\iota}{\longrightarrow} \mathrm{Hom}_{\mathcal{X}}(X, X') \times \mathrm{Hom}_{\mathcal{Y}}(Y, Y') \longrightarrow$$

$$\overset{\psi}{\longrightarrow} \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\widetilde{X}, X') \longrightarrow \mathrm{Ext}^1_{\widetilde{\mathcal{F}}}(Z, Z') \longrightarrow 0,$$

where $\widetilde{X} := \mathbf{F}(Y)$ *and* $\psi(u, v) := u \circ \eta - \eta' \circ \mathbf{F}(v)$.

We now consider the covariant exact functors like in Lemma 2.3:

$$\widetilde{\mathbf{R}} : \widetilde{\mathcal{F}} \longrightarrow \widetilde{\mathcal{X}}, \quad (X, Y, \eta) \longmapsto X, \quad (u, v) \longmapsto u,$$

$$\widetilde{\mathbf{L}} : \widetilde{\mathcal{F}} \longrightarrow \mathcal{Y}, \quad (X, Y, \eta) \longmapsto Y, \quad (u, v) \longmapsto v.$$

LEMMA 2.10. *With the above notation, the assignment* $Y \mapsto (\mathbf{F}(Y), Y, \mathrm{id}_{\mathbf{F}(Y)})$ *extends to a covariant exact functor* $\mathbf{E} : \mathcal{Y} \to \widetilde{\mathcal{X}}$, *which is left adjoint to* $\widetilde{\mathbf{L}}$.

PROOF. For any morphism $v : Y_1 \to Y_2$ in $\mathcal{Y}$, the induced morphism $\mathbf{F}(v) : \mathbf{F}(Y_1) \to \mathbf{F}(Y_2)$ satisfies $(\mathbf{F}(v), v) \in \mathrm{Hom}_{\widetilde{\mathcal{F}}}(\mathbf{E}(Y_1), \mathbf{E}(Y_2))$ by the definition of the morphisms in $\widetilde{\mathcal{F}}$. We may thus set $\mathbf{E}(v) := (\mathbf{F}(v), v)$. As $\mathbf{F}$ is a covariant exact functor, so is $\mathbf{E}$. We now check the adjunction assertion: let $Y \in \mathcal{Y}$ and $(X, Y', \eta) \in \widetilde{\mathcal{F}}$. Then $\mathrm{Hom}_{\widetilde{\mathcal{F}}}(\mathbf{E}(Y), (X, Y', \eta))$ consists of the pairs $(u \in \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{F}(Y), X)$, $v \in \mathrm{Hom}_{\mathcal{Y}}(Y, Y'))$ such that $u = \eta \circ F(v)$. Thus, the map induced by $\widetilde{\mathbf{L}}$,

$$\mathrm{Hom}_{\widetilde{\mathcal{F}}}(\mathbf{E}(Y), (X, Y', \eta)) \longrightarrow \mathrm{Hom}_{\mathcal{Y}}(Y, Y'), \quad (u, v) \longmapsto v$$

is an isomorphism. $\square$

For any $Y \in \mathcal{Y}$, there is a tautological exact sequence

$$0 \longrightarrow \mathbf{F}(Y) \overset{\iota}{\longrightarrow} \mathbf{E}(Y) \overset{\pi}{\longrightarrow} Y \longrightarrow 0 \tag{2.5}$$

in $\widetilde{\mathcal{F}}$, which is universal in the following sense:

PROPOSITION 2.11. *Let* $Z = (X, Y, \eta) \in \widetilde{\mathcal{F}}$.

(i) *There exists a unique morphism* $\mu : \mathbf{E}(Y) \to Z$ *in* $\widetilde{\mathcal{F}}$ *such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{F}(Y) & \overset{\iota}{\longrightarrow} & \mathbf{E}(Y) & \overset{\pi}{\longrightarrow} & Y & \longrightarrow & 0 \\ & & {\scriptstyle\eta}\downarrow & & {\scriptstyle\mu}\downarrow & & {\scriptstyle\mathrm{id}_Y}\downarrow & & \\ 0 & \longrightarrow & X & \overset{f}{\longrightarrow} & Z & \overset{g}{\longrightarrow} & Y & \longrightarrow & 0 \end{array}$$

commutes, and the left square is cartesian.

(ii) *The resulting exact sequence*

$$0 \longrightarrow \mathbf{F}(Y) \stackrel{(\eta,\iota)}{\longrightarrow} X \times \mathbf{E}(Y) \stackrel{f-\mu}{\longrightarrow} Z \longrightarrow 0 \tag{2.6}$$

is a projective resolution of Z *in* $\widetilde{\mathcal{F}}$. *In particular,* $\widetilde{\mathcal{F}}$ *has enough projectives.*

(iii) Z *is projective if and only if* η *is a monomorphism; then* $Z \cong X' \times \mathbf{E}(Y)$ *for some subobject* $X' \hookrightarrow X$.

(iv) $\mathbf{E}(Y)$ *is a projective cover of* Y.

Proof. (i) This follows from the isomorphism (2.5) together with Yoneda's lemma.

(ii) By Corollary 2.4, X and $\mathbf{F}(Y)$ are projective objects of $\widetilde{\mathcal{F}}$. Moreover, $\mathbf{E}(Y)$ is projective as well, since $\mathrm{Hom}_{\widetilde{\mathcal{F}}}(\mathbf{E}(Y), ?) \cong \mathrm{Hom}_{\mathcal{Y}}(Y, \widetilde{\mathbf{L}}(?))$, where $\widetilde{\mathbf{L}}$ is exact and $\mathcal{Y}$ is semi-simple.

(iii) If Z is projective, then of course $\mathrm{Ext}^1_{\widetilde{\mathcal{F}}}(Z, X') = 0$ for all $X' \in \widetilde{\mathcal{X}}$. In view of the projective resolution (2.6), it follows that the map

$$\mathrm{Hom}_{\widetilde{\mathcal{X}}}(X, X') \times \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{E}(Y), X') \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{F}(Y), X'), \quad (u, v) \longmapsto u \circ \eta + v \circ \iota$$

is surjective. As $\mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{E}(Y), X') = 0$ by Lemma 2.10, this just means that the map

$$\mathrm{Hom}_{\widetilde{\mathcal{X}}}(X, X') \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{F}(Y), X'), \quad (u, v) \longmapsto u \circ \eta$$

is surjective (alternatively, this follows from the exact sequence of Lemma 2.9). Since $\widetilde{\mathcal{X}}$ is semi-simple, the pull-back map

$$\mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{F}(Y), X') \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathrm{Ker}(\eta), X')$$

is surjective as well. Thus, $\mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathrm{Ker}(\eta), X') = 0$ for all X', i.e., $\mathrm{Ker}(\eta) = 0$. Using the semi-simplicity of $\widetilde{\mathcal{X}}$ again, we may choose a subobject $X' \hookrightarrow X$ such that $X = X' \oplus \mathrm{Im}(\eta)$; then the natural map $X' \times \mathbf{E}(Y) \to Z$ is an isomorphism. Conversely, $X' \times \mathbf{E}(Y)$ is projective by (ii).

(iv) This follows from (iii), since we have $\mathrm{Hom}_{\widetilde{\mathcal{F}}}(X', Y) = 0$ for any $X' \in \widetilde{\mathcal{X}}$, and $\mathrm{Hom}_{\widetilde{\mathcal{F}}}(\mathbf{E}(Y'), Y) \cong \mathrm{Hom}_{\mathcal{Y}}(Y', Y)$ for any $Y' \in \mathcal{Y}$. □

Finally, we obtain two homological characterizations of the universal objects $\mathbf{E}(Y)$, the first one being somewhat analogous to the notion of exceptional objects:

Lemma 2.12. *The following conditions are equivalent for* $Z = (X, Y, \eta) \in \widetilde{\mathcal{F}}$:

(i) $Z \cong \mathbf{E}(Y)$.

(ii) $\mathrm{End}_{\widetilde{\mathcal{F}}}(Z) \cong \mathrm{End}_{\mathcal{Y}}(Y)$ *via* $\widetilde{\mathbf{L}}$, *and* $\mathrm{Ext}^1_{\widetilde{\mathcal{F}}}(Z, Z) = 0$.

(iii) $\mathrm{Hom}_{\widetilde{\mathcal{F}}}(Z, X') = 0 = \mathrm{Ext}^1_{\widetilde{\mathcal{F}}}(Z, X')$ *for all* $X' \in \widetilde{\mathcal{X}}$.

Proof. (i) ⇒ (ii), (i) ⇒ (iii) This follows from Lemma 2.10 and Proposition 2.11.

(ii) ⇒ (i) In view of the exact sequence of Lemma 2.9, we may rephrase the assumption as follows: for any $v \in \mathrm{End}_{\mathcal{Y}}(Y)$, there exists a unique $u \in \mathrm{End}_{\widetilde{\mathcal{X}}}(X)$ such that $u \circ \eta = \eta \circ F(v)$; moreover, the map $\psi : (f, g) \mapsto f \circ \eta - \eta \circ F(g)$ is surjective. As a consequence, η is an epimorphism (by the uniqueness of u), and $\mathrm{Ker}(\eta)$ is stable under $F(v)$ for any $v \in \mathrm{End}_{\mathcal{Y}}(Y)$ (by the existence of u). Then $\psi(u, v)$ vanishes identically on $\mathrm{Ker}(\eta)$ for any $u \in \mathrm{End}_{\widetilde{\mathcal{X}}}(X)$, $v \in \mathrm{End}_{\mathcal{Y}}(Y)$. As ψ is surjective, this forces $\mathrm{Ker}(\eta) = 0$. Thus, η is an isomorphism, and hence the pair $(\eta : \mathbf{F}(Y) \to X, \mathrm{id} : Y \to Y)$ yields an isomorphism $\mathbf{E}(Y) \to Z$ in $\widetilde{\mathcal{F}}$.

(iii) $\Rightarrow$ (i) In view of the long exact exact sequence of Lemma 2.9 again, the map

$$\psi : \mathrm{Hom}_{\widetilde{\mathcal{X}}}(X, X') \longrightarrow \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathbf{F}(Y), X'), \quad u \longmapsto u \circ \eta$$

is an isomorphism for any $\eta \in \widetilde{\mathcal{X}}$. It follows that η is an isomorphism as well. □

2.5. Relation to module categories. Let $\mathcal{C}$ be an abelian category equipped with a torsion pair $(\mathcal{X}, \mathcal{Y})$ satisfying the assumptions (a), (b), (c), (d) of Subsection 2.4. We assume in addition that $\mathcal{C}$ is a *finite length* category, i.e., every object has a composition series. Then the semi-simple categories $\mathcal{X}$, $\mathcal{Y}$ are of finite length as well; as a consequence, each of them is equivalent to the category of all left modules of finite length over a ring, which can be constructed as follows.

Denote by I the set of isomorphism classes of simple objects of $\mathcal{X}$. Choose a representative S for each class, and let

$$D_S := \mathrm{End}_{\mathcal{X}}(S)^{\mathrm{op}};$$

then D_S is a division ring. Given $X \in \mathcal{X}$, the group $\mathrm{Hom}_{\mathcal{X}}(S, X)$ is a left D_S-vector space of finite dimension; moreover, $\mathrm{Hom}_{\mathcal{X}}(S, X) = 0$ for all but finitely many $S \in I$. Thus,

$$\mathbf{M}(X) := \bigoplus_{S \in I} \mathrm{Hom}_{\mathcal{X}}(S, X)$$

is a left module of finite length over the ring

$$R_{\mathcal{X}} := \bigoplus_{S \in I} D_S.$$

(Notice that every $R_{\mathcal{X}}$-module of finite length is semi-simple; moreover, the ring $R_{\mathcal{X}}$ is semi-simple if and only if I is finite). The assignment $X \mapsto \mathbf{M}(X)$ extends to a covariant functor

$$\mathbf{M}_{\mathcal{X}} : \mathcal{X} \xrightarrow{\cong} R_{\mathcal{X}}\text{-mod}, \tag{2.7}$$

which is easily seen to be an equivalence of categories. Likewise, we have an equivalence of categories

$$\mathbf{M}_{\mathcal{Y}} : \mathcal{Y} \longrightarrow R_{\mathcal{Y}}\text{-mod},$$

where $R_{\mathcal{Y}} := \bigoplus_{T \in J} D_T$.

We now make a further (and final) assumption:

(e) The equivalence (2.7) extends to an equivalence of categories

$$\mathbf{M}_{\widetilde{\mathcal{X}}} : \widetilde{\mathcal{X}} \xrightarrow{\cong} R_{\mathcal{X}}\text{-Mod}^{\mathrm{ss}}. \tag{2.8}$$

The right-hand side of (2.8) is a semi-simple category containing $\mathcal{X}$ as a Serre subcategory, as required by assumption (d). When the set I is finite, assumption (e) just means that $\widetilde{\mathcal{X}} = R_{\mathcal{X}}$-Mod. For an arbitrary set I, the objects of $\widetilde{\mathcal{X}}$ are the direct sums $X = \bigoplus_{S \in I} X_S$, where each X_S is a left D_S-vector space. We say that X_S is the *isotypical component* of X of type S; its dimension (possibly infinite) is the *multiplicity* of S in X.

We now turn to the covariant exact functor $\mathbf{F} : \mathcal{Y} \to \widetilde{\mathcal{X}}$. We may identify each simple object T of $\mathcal{Y}$ with D_T, on which $R_{\mathcal{Y}}$ acts via left multiplication. Then $\mathbf{F}(D_T)$ is a semi-simple left $R_{\mathcal{X}}$-module. Also, $\mathbf{F}(D_T)$ is a right D_T-module, via the action of D_T on itself via right multiplication, which yields an isomorphism

$$D_T \xrightarrow{\cong} \mathrm{End}_{R_{\mathcal{Y}}}(D_T),$$

and the ring homomorphism $\mathrm{End}_{R_{\mathcal{Y}}}(D_T) \to \mathrm{End}_{R_{\mathcal{X}}}(\mathbf{F}(D_T))$ induced by $\mathbf{F}$. As the left and right actions of D_T on itself commute, $\mathbf{F}(D_T)$ is a D_T-$R_{\mathcal{Y}}$-bimodule, which we view as a $R_{\mathcal{X}}$-$R_{\mathcal{Y}}$-bimodule. Let

$$M := \bigoplus_{T \in J} \mathbf{F}(D_T).$$

This is again a $R_{\mathcal{X}}$-$R_{\mathcal{Y}}$-bimodule, semi-simple as a $R_{\mathcal{X}}$-module and as a $R_{\mathcal{Y}}$-module. So we may form the triangular matrix ring

$$R := \begin{pmatrix} R_{\mathcal{Y}} & 0 \\ M & R_{\mathcal{X}} \end{pmatrix}$$

as in [**ARS97**, §III.2]. More specifically, R consists of the triples (x, y, m), where $x \in R_{\mathcal{X}}$, $y \in R_{\mathcal{Y}}$, and $m \in M$; the addition and multiplication are those of the matrices $\left(\begin{smallmatrix} y & 0 \\ m & x \end{smallmatrix}\right)$, using the bimodule structure of M. Note the decomposition $R = (R_{\mathcal{X}} \oplus R_{\mathcal{Y}}) \oplus M$, where $R_{\mathcal{X}} \oplus R_{\mathcal{Y}}$ is a subring, and M is an ideal of square 0.

We say that a R-module Z is *locally finite*, if $Z = X \oplus Y$ as an $R_{\mathcal{X}} \oplus R_{\mathcal{Y}}$-module, where X is a semi-simple $R_{\mathcal{X}}$-module and Y is an $R_{\mathcal{Y}}$-module of finite length. We denote by $R\text{-}\widetilde{\mathrm{mod}}$ the full subcategory of R-Mod with objects the locally finite modules; then R-mod is a Serre subcategory of $R\text{-}\widetilde{\mathrm{mod}}$. We may now state our main homological result:

THEOREM 2.13. *With the above notation and assumptions (a), (b), (c), (d), (e), the abelian categories $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are hereditary, and $\widetilde{\mathcal{F}}$ has enough projectives. Moreover, there are compatible equivalences of categories*

$$\mathbf{M} : \mathcal{F} \xrightarrow{\cong} R\text{-mod}, \quad \widetilde{\mathbf{M}} : \widetilde{\mathcal{F}} \xrightarrow{\cong} R\text{-}\widetilde{\mathrm{mod}}.$$

PROOF. The first assertion follows by combining Corollary 2.4, Lemma 2.5 and Proposition 2.11.

To show the second assertion, we can freely replace $\mathcal{X}$, $\widetilde{\mathcal{X}}$, $\mathcal{Y}$ with compatibly equivalent categories in the construction of $\mathcal{F}$, $\widetilde{\mathcal{F}}$. Thus, we may assume that $\mathcal{X} = R_{\mathcal{X}}$-mod, $\widetilde{\mathcal{X}} = R_{\mathcal{X}}\text{-Mod}^{\mathrm{ss}}$ and $\mathcal{Y} = R_{\mathcal{Y}}$-mod.

The category of all left R-modules is equivalent to the category of triples (X, Y, f), where X is a $R_{\mathcal{X}}$-module, Y a $R_{\mathcal{Y}}$-module, and $f : M \otimes_{R_{\mathcal{Y}}} Y \to X$ a morphism of $R_{\mathcal{X}}$-modules. The morphisms from (X, Y, f) to (X', Y', f') are the pairs (u, v), where $u \in \mathrm{Hom}_{\mathcal{X}}(X, X')$, $v \in \mathrm{Hom}_{\mathcal{Y}}(Y, Y')$, and the following diagram commutes:

$$\begin{array}{ccc} M \otimes_{R_{\mathcal{Y}}} Y & \xrightarrow{\mathrm{id}_M \otimes v} & M \otimes_{R_{\mathcal{Y}}} Y' \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow \\ X & \xrightarrow{u} & X'. \end{array}$$

(This result is obtained in [**ARS97**, Prop. III.2.2] for modules of finite length over an Artin algebra. The proof adapts without change to the present setting). Moreover, the full subcategory R-mod (resp. $R\text{-}\widetilde{\mathrm{mod}}$) is equivalent to the full subcategory of triples (X, Y, f), where X and Y have finite length (resp. X is semi-simple and Y has finite length).

To complete the proof, it suffices to show that the covariant exact functor $\mathbf{F} : \mathcal{Y} \to \widetilde{\mathcal{X}}$ is isomorphic to $M \otimes_{R_{\mathcal{Y}}} -$. As $\mathbf{F}$ commutes with finite direct sums, we have $\mathbf{F} = \bigoplus_{T \in J} \mathbf{F}_T$ for covariant exact functors $\mathbf{F}_T : D_T\text{-mod} \to \widetilde{\mathcal{X}}$. We now argue as in the proof of the Eilenberg-Watts theorem (see [**Ba68**, Thm. II.2.3]).

Given a left D_T-vector space V, every $v \in V$ yields a D_T-morphism $v : D_T \to V$, and hence a $R_{\mathcal{X}}$-morphism $\mathbf{F}_T(v) : \mathbf{F}_T(D_T) = \mathbf{F}(D_T) \to \mathbf{F}_T(V)$. The resulting map $V \to \mathrm{Hom}_{R_{\mathcal{X}}}(\mathbf{F}(D_T), \mathbf{F}_T(V))$ is easily checked to be a D_T-morphism. In view of the natural isomorphism

$$\mathrm{Hom}_{D_T}(V, \mathrm{Hom}_{R_{\mathcal{X}}}(\mathbf{F}(D_T), \mathbf{F}_T(V))) \cong \mathrm{Hom}_{R_{\mathcal{X}}}(\mathbf{F}(D_T) \otimes_{D_T} V, \mathbf{F}_T(V)),$$

this yields a functorial map

$$f_V : \mathbf{F}(D_T) \otimes_{D_T} V \longrightarrow \mathbf{F}_T(V).$$

When $V = D_T$, one checks that f_V is identified to the identity map of $\mathbf{F}(D_T)$; moreover, the formation of f_V commutes with finite direct sums, since so does $\mathbf{F}_T$. So f_V yields an isomorphism of functors $\mathbf{F}_T \cong \mathbf{F}(D_T) \otimes_{D_T} -$. □

Remark 2.14. Instead of assumption (e), we may make the stronger and much simpler assumption that $\widetilde{\mathcal{X}} = \mathcal{X}$. (This holds for the isogeny category of vector extensions of abelian varieties, as we will see in §3.2.3). Then we obtain as in the proof of Theorem 3.5 that $\widetilde{\mathcal{F}} = \mathcal{F}$ *is hereditary, has enough projectives, and is equivalent to* R-mod.

Next, we obtain a separation property of the above ring R, and we describe its center $Z(R)$ as well as the center of the abelian category R-mod. We denote by Z_S (resp. Z_T) the center of the division ring D_S (resp. D_T) for any $S \in I$, $T \in J$.

Proposition 2.15. (i) *The intersection of all the left ideals of finite colength in R is zero.*

(ii) *The center $Z(R)$ consists of the triples $(x, y, 0)$, where $x = \sum_S x_S \in \bigoplus_{S \in I} Z_S$, $y = \sum_T y_T \in \bigoplus_{T \in J} Z_T$ and $x_S m = m y_T$ for all $m \in \mathbf{F}(D_T)$.*

(iii) *The center of R-mod is the completion of $Z(R)$, consisting of the pairs (x, y), where $x = (x_S) \in \prod_{S \in I} Z_S$, $y = (y_T) \in \prod_{T \in J} Z_T$ and $x_S m = m y_T$ for all $m \in \mathbf{F}(D_T)$.*

Proof. (i) Given $S \in I$ and $T \in J$, we may form the triangular matrix ring

$$R_{S,T} := \begin{pmatrix} D_T & 0 \\ \mathbf{F}(D_T)_S & D_S \end{pmatrix},$$

where $\mathbf{F}(D_T)_S$ denotes the isotypical component of type S of the $R_{\mathcal{X}}$-module $\mathbf{F}(D_T)$. Clearly, $R_{S,T}$ is the quotient of R by a two-sided ideal $I_{S,T}$, and $\bigcap_{S,T} I_{S,T} = 0$. Thus, it suffices to show the assertion for R replaced with $R_{S,T}$.

The left D_S-vector space $\mathbf{F}(D_T)_S$ contains a family of subspaces $(M_a)_{a \in A}$ such that the dimension of each quotient $\mathbf{F}(D_T)/M_a$ is finite, and $\bigcap_{a \in A} M_a = 0$. Then $\left(\begin{smallmatrix} 0 & 0 \\ M_a & 0 \end{smallmatrix}\right)$ is a left ideal of $R_{S,T}$, as well as $\left(\begin{smallmatrix} D_T & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & D_S \end{smallmatrix}\right)$. Moreover, all these left ideals have finite colength, and their intersection is zero.

(ii) This is a direct verification.

(iii) Recall that the center of R-mod consists of the families $z = (z_N)_{N \in R\text{-mod}}$ such that $z_N \in \mathrm{End}_R(N)$ and $f \circ z_N = z_{N'} \circ f$ for any $f \in \mathrm{Hom}_R(N, N')$; in particular, z_N is central in $\mathrm{End}_R(N)$. Thus, $z_S \in Z_S$ and $z_T \in Z_T$ for all S, T. Since $N = X \oplus Y$ as a $R_{\mathcal{X}} \oplus R_{\mathcal{Y}}$-module, we see that z is uniquely determined by the families $(z_S)_{S \in I}$, $(z_T)_{T \in J}$. Moreover, we have $z_S m = m z_T$ for all $m \in \mathbf{F}(D_T)_S$, as follows e.g. from Lemma 2.9. Thus, the center of R-mod is contained in the completion of $Z(R)$. The opposite inclusion follows from (i). □

REMARK 2.16. Assume that each D_S-D_T-bimodule $\mathbf{F}(D_T)_S$ contains a family of sub-bimodules $(N_a)_{a\in A}$ such that each quotient $\mathbf{F}(D_T)_S$ has finite length as a D_S-module, and $\bigcap_{a\in A} N_a = 0$. Then R satisfies a stronger separation property, namely, its two-sided ideals of finite colength (as left modules) have zero intersection.

The above assumption obviously holds if $\mathbf{F}(D_T)_S$ has finite length as a D_S-module.It also holds if both D_S and D_T are (say) of characteristic 0 and finite-dimensional over $\mathbb{Q}$; indeed, $\mathbf{F}(D_T)_S$ is a module over $D_S \otimes_{\mathbb{Q}} D_T^{\mathrm{op}}$, and the latter is a finite-dimensional semi-simple $\mathbb{Q}$-algebra.

3. Applications to commutative algebraic groups

3.1. Some isogeny categories.

3.1.1. *Algebraic groups* [**DG70, SGA3, Br16**]. Throughout this section, we fix a ground field k, with algebraic closure $\bar{k}$ and characteristic $\mathrm{char}(k)$. An *algebraic group* G is a group scheme of finite type over k. A *subgroup* $H \subset G$ is a k-subgroup scheme; then H is a closed subscheme of G. When $\mathrm{char}(k) = 0$, every algebraic group is smooth.

Unless otherwise mentioned, all algebraic groups will be assumed *commutative*. They form the objects of an abelian category $\mathcal{C}$, with morphisms the homomorphisms of k-group schemes (see [**SGA3**, VIA, Thm. 5.4.2]). Every object in $\mathcal{C}$ is artinian (since every decreasing sequence of closed subschemes of a scheme of finite type eventually terminates), but generally not noetherian: in the multiplicative group $\mathbb{G}_m$, the subgroups of roots of unity of order ℓ^n, where ℓ is a fixed prime and n a non-negative integer, form an infinite ascending chain.

The finite group schemes form a Serre subcategory $\mathcal{F}$ of $\mathcal{C}$. The quotient category $\mathcal{C}/\mathcal{F}$ is equivalent to the localization of $\mathcal{C}$ with respect to the multiplicative system of *isogenies*, i.e., of morphisms with finite kernel and cokernel. Also, $\mathcal{C}/\mathcal{F}$ is equivalent to its full subcategory $\underline{\mathcal{C}}$ with objects the smooth connected algebraic groups (see [**Br16**, Lem. 3.1]). We say that $\underline{\mathcal{C}}$ is the *isogeny category of algebraic groups*. Every object of $\underline{\mathcal{C}}$ is artinian and noetherian, i.e., $\underline{\mathcal{C}}$ *is a finite length category* (see [**Br16**, Prop. 3.2]).

Let G be an algebraic group, with group law denoted additively. For any integer n, we have the multiplication map

$$n_G : G \longrightarrow G, \quad x \longmapsto nx.$$

We say that G is *divisible*, if n_G is an epimorphism for any $n \neq 0$. When $\mathrm{char}(k) = 0$, this is equivalent to G being connected; when $\mathrm{char}(k) > 0$, the divisible algebraic groups are the semi-abelian varieties (these will be discussed in detail in §3.2.4).

If G is divisible, then the natural map

$$\mathbb{Z} \longrightarrow \mathrm{End}_{\mathcal{C}}(G), \quad n \longmapsto n_G$$

extends to a homomorphism $\mathbb{Q} \to \mathrm{End}_{\underline{\mathcal{C}}}(G)$; in other terms, $\mathrm{End}_{\underline{\mathcal{C}}}(G)$ is a $\mathbb{Q}$-algebra. As a consequence, $\mathrm{Ext}^n_{\underline{\mathcal{C}}}(G, G')$ is a $\mathbb{Q}$-vector space for any divisible algebraic groups G, G' and any integer $n \geq 0$. By [**Br16**, Prop. 3.6]), the induced maps

$$\mathrm{Hom}_{\mathcal{C}}(G, G')_{\mathbb{Q}} \longrightarrow \mathrm{Hom}_{\underline{\mathcal{C}}}(G, G'), \quad \mathrm{Ext}^1_{\mathcal{C}}(G, G')_{\mathbb{Q}} \longrightarrow \mathrm{Ext}^1_{\underline{\mathcal{C}}}(G, G'), \tag{3.1}$$

are isomorphisms, where we set $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ for any abelian group M.

In particular, *the isogeny category* $\underline{\mathcal{C}}$ *is* $\mathbb{Q}$*-linear when* $\mathrm{char}(k) = 0$; then its objects are just the connected algebraic groups, and its morphisms are the rational multiples of morphisms in $\mathcal{C}$.

Given an extension of fields K/k and an algebraic group G over k, we obtain an algebraic group over K,

$$G_K := G \otimes_k K = G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K),$$

by extension of scalars. The assignment $G \mapsto G_K$ extends to the *base change functor*

$$\otimes_k K : \mathcal{C} = \mathcal{C}_k \longrightarrow \mathcal{C}_K,$$

which is exact and faithful. Also, G is finite if and only if G_K is finite. As a consequence, we obtain a base change functor, still denoted by

$$\otimes_k K : \underline{\mathcal{C}}_k \longrightarrow \underline{\mathcal{C}}_K,$$

and which is still exact and faithful. When K/k is purely inseparable, the above functor is an equivalence of categories (see [**Br16**, Thm. 3.11]). We say that $\underline{\mathcal{C}}_k$ is *invariant under purely inseparable field extensions.*

Thus, to study $\underline{\mathcal{C}}_k$ when $\mathrm{char}(k) = p > 0$, we may replace k with its *perfect closure*, $k_i := \bigcup_{n \geq 0} k^{1/p^n} \subset \bar{k}$, and hence assume that k is perfect. This will be very useful, since the structure of algebraic groups is much better understood over a perfect ground field (see e.g. §3.1.5).

3.1.2. *Linear algebraic groups, affine group schemes* [**DG70**, III.3]. A (possibly non-commutative) algebraic group G is called *linear* if G is isomorphic to a subgroup scheme of the general linear group GL_n for some integer $n > 0$; this is equivalent to G being affine (see e.g. [**Br15**, Prop. 3.1.1]). The smooth linear algebraic groups are the "linear algebraic groups defined over k" in the sense of [**Bo91**].

For any exact sequence

$$0 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 0$$

in $\mathcal{C}$, the group G is affine if and only if G_1 and G_2 are affine (see e.g. [**Br15**, Prop. 3.1.2]). Thus, the (commutative) linear algebraic groups form a Serre subcategory $\mathcal{L}$ of $\mathcal{C}$, which contains $\mathcal{F}$.

The property of being affine is also invariant under field extensions and isogenies, in the following sense: an algebraic group G is affine if and only if G_K is affine for some field extension K of k, if and only if H is affine for some isogeny $f : G \to H$. It follows that the quotient category $\mathcal{L}/\mathcal{F}$ is equivalent to its full subcategory $\underline{\mathcal{L}}$ with objects the smooth connected linear algebraic groups. Moreover, $\underline{\mathcal{L}}$ is invariant under purely inseparable field extensions.

The affine k-group schemes (not necessarily of finite type) form an abelian category $\widetilde{\mathcal{L}}$, containing $\mathcal{L}$ as a Serre subcategory. Moreover, every affine group scheme G is the filtered inverse limit of linear algebraic groups, quotients of G (see [**DG70**, III.3.7.4, III.3.7.5]). In fact, $\widetilde{\mathcal{L}}$ is the pro-completion of the abelian category $\mathcal{L}$, in the sense of [**DG70**, V.2.3.1].

We say that a group scheme G is *pro-finite*, if G is an inverse limit of finite group schemes; equivalently, G is affine and every algebraic quotient group of G is finite. The pro-finite group schemes form a Serre subcategory $\widetilde{\mathcal{FL}}$ of $\widetilde{\mathcal{L}}$. The quotient category $\widetilde{\mathcal{L}}/\widetilde{\mathcal{FL}}$ is the *isogeny category of affine group schemes*; it contains $\mathcal{L}/\mathcal{F}$ as a Serre subcategory.

3.1.3. *Groups of multiplicative type* [**DG70**, IV.1]. The invertible diagonal matrices form a subgroup scheme $\mathrm{D}_n \subset \mathrm{GL}_n$, which is commutative, smooth and connected; moreover, $\mathrm{D}_1 = \mathrm{GL}_1$ is isomorphic to the multiplicative group $\mathbb{G}_m$, and

$\mathrm{D}_n \cong \mathbb{G}_m^n$ (the product of n copies of $\mathbb{G}_m$). An algebraic group G is said to be *diagonalizable*, if G is isomorphic to a subgroup of D_n for some n. Also, G is called *of multiplicative type* (resp. a *torus*), if the base change $G_{\bar{k}}$ is diagonalizable (resp. isomorphic to some $\mathrm{D}_{n,\bar{k}}$). Both properties are invariant under field extensions, but not under isogenies. Also, the tori are the smooth connected algebraic groups of multiplicative type. The diagonalizable algebraic groups (resp. the algebraic groups of multiplicative type) form a Serre subcategory $\mathcal{D}$ (resp. $\mathcal{M}$) of $\mathcal{C}$.

For any diagonalizable algebraic group G, the *character group*

$$\mathbf{X}(G) := \mathrm{Hom}_{\mathcal{C}}(G, \mathbb{G}_m)$$

is a finitely generated abelian group. Moreover, the assignment $G \mapsto \mathbf{X}(G)$ extends to an anti-equivalence of categories

$$\mathbf{X} : \mathcal{D} \longrightarrow \mathbb{Z}\text{-Mod}^{\mathrm{fg}},$$

where the right-hand side denotes the category of finitely generated abelian groups (see [**DG70**, IV.1.1] for these results).

Given an algebraic group of multiplicative type G, there exists a finite Galois extension of fields K/k such that G_K is diagonalizable. Thus, G_{k_s} is diagonalizable, where k_s denotes the *separable closure* of k in $\bar{k}$. Let

$$\Gamma := \mathrm{Gal}(k_s/k) = \mathrm{Aut}(\bar{k}/k)$$

denote the *absolute Galois group* of k. Then Γ is a pro-finite topological group, the inverse limit of its finite quotients $\mathrm{Gal}(K/k)$, where K runs over the finite Galois field extensions of k. Also, Γ acts on the character group,

$$\mathbf{X}(G) := \mathrm{Hom}_{\mathcal{C}_{k_s}}(G_{k_s}, \mathbb{G}_{m,k_s}),$$

and the stabilizer of any character is an open subgroup. Thus, $\mathbf{X}(G)$ is a *discrete Galois module* in the sense of [**Se97**, §2.1]. Moreover, G is diagonalizable if and only if Γ fixes $\mathbf{X}(G)$ pointwise; then the base change map

$$\mathrm{Hom}_{\mathcal{C}_k}(G, \mathbb{G}_{m,k}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_{k_s}}(G_{k_s}, \mathbb{G}_{m,k_s})$$

is an isomorphism, i.e., the two notions of character groups are compatible. Furthermore, G is a torus (resp. finite) if and only if the abelian group $\mathbf{X}(G)$ is free (resp. finite); also, note that the tori are the divisible algebraic groups of multiplicative type.

The above assignment $G \mapsto \mathbf{X}(G)$ yields an anti-equivalence of categories

$$\mathbf{X} : \mathcal{M} \longrightarrow \mathbb{Z}\Gamma\text{-Mod}^{\mathrm{fg}} \tag{3.2}$$

(*Cartier duality*), where the right-hand side denotes the category of discrete Γ-modules which are finitely generated as abelian groups. Moreover, the abelian group $\mathrm{Hom}_{\mathcal{C}}(T, T')$ is free of finite rank, for any tori T and T' (see [**DG70**, IV.1.2, IV.1.3] for these results).

Consider the full subcategory $\mathcal{FM}$ of $\mathcal{M}$ with objects the finite group schemes of multiplicative type. Then $\mathcal{FM}$ is a Serre subcategory of $\mathcal{M}$, anti-equivalent via $\mathbf{X}$ to the category of finite discrete Γ-modules. Moreover, the quotient category $\mathcal{M}/\mathcal{FM}$ is equivalent to its full subcategory $\underline{\mathcal{T}}$ with objects the tori, and we have an anti-equivalence of categories

$$\mathbf{X}_{\mathbb{Q}} : \underline{\mathcal{T}} \longrightarrow \mathbb{Q}\Gamma\text{-mod}, \quad T \longmapsto \mathbf{X}(T)_{\mathbb{Q}}. \tag{3.3}$$

Here $\mathbb{Q}\Gamma$-mod denotes the category of finite-dimensional $\mathbb{Q}$-vector spaces equipped with a discrete linear action of Γ; note that $\mathbb{Q}\Gamma$-mod is semi-simple, $\mathbb{Q}$-linear and invariant under purely inseparable field extensions. In view of (3.1), this yields natural isomorphisms

$$\mathrm{Hom}_{\underline{\mathcal{T}}}(T, T') \cong \mathrm{Hom}_{\mathcal{C}}(T, T')_{\mathbb{Q}} \cong \mathrm{Hom}^{\Gamma}(\mathbf{X}(T')_{\mathbb{Q}}, \mathbf{X}(T)_{\mathbb{Q}})$$

for any tori T, T'. As a consequence, *the isogeny category* $\underline{\mathcal{T}}$ *is semi-simple,* $\mathbb{Q}$*-linear, Hom-finite, and invariant under purely inseparable field extensions.*

Next, we extend the above results to affine k-group schemes, not necessarily algebraic, by using again results of [**DG70**, IV.1.2, IV.1.3]. We say that an affine group scheme G is of multiplicative type, if so are all its algebraic quotient groups. Denote by $\widetilde{\mathcal{M}}$ the full subcategory of $\widetilde{\mathcal{L}}$ with objects the group schemes of multiplicative type; then $\widetilde{\mathcal{M}}$ is a Serre subcategory of $\widetilde{\mathcal{L}}$. Moreover, the Cartier duality (3.2) extends to an anti-equivalence of categories

$$\mathbf{X} : \widetilde{\mathcal{M}} \longrightarrow \mathbb{Z}\Gamma\text{-Mod}, \tag{3.4}$$

where $\mathbb{Z}\Gamma$-Mod stands for the category of all discrete Γ-modules. Note that $\mathbb{Z}\Gamma$-Mod is an abelian category, containing $\mathbb{Z}\Gamma\text{-Mod}^{\mathrm{fg}}$ as a Serre subcategory.

Consider the full subcategory $\mathbb{Z}\Gamma\text{-Mod}^{\mathrm{tors}} \subset \mathbb{Z}\Gamma\text{-Mod}$ with objects the discrete Γ-modules which are torsion as abelian groups. Then $\mathbb{Z}\Gamma\text{-Mod}^{\mathrm{tors}}$ is a Serre subcategory of $\mathbb{Z}\Gamma$-Mod, anti-equivalent via $\mathbf{X}$ to the full subcategory $\widetilde{\mathcal{FM}} \subset \widetilde{\mathcal{M}}$ with objects the pro-finite group schemes of multiplicative type.

For any $M \in \mathbb{Z}\Gamma$-Mod, the kernel and cokernel of the natural map $M \to M_{\mathbb{Q}}$ are torsion. It follows readily that the induced covariant functor $\mathbb{Z}\Gamma\text{-Mod} \to \mathbb{Q}\Gamma\text{-Mod}$ yields an equivalence of categories

$$(\mathbb{Z}\Gamma\text{-Mod})/(\mathbb{Z}\Gamma\text{-Mod}^{\mathrm{tors}}) \xrightarrow{\cong} \mathbb{Q}\Gamma\text{-Mod},$$

where $\mathbb{Q}\Gamma$-Mod denotes the category of all $\mathbb{Q}$-vector spaces equipped with a discrete linear action of Γ. Thus, the category $\mathbb{Q}\Gamma$-Mod is anti-equivalent to $\widetilde{\mathcal{M}}/\widetilde{\mathcal{FM}}$. In turn, the latter category is equivalent to its full subcategory with objects the inverse limits of tori: the *isogeny category of pro-tori*, that we denote by $\underline{\widetilde{\mathcal{T}}}$. Clearly, the category $\mathbb{Q}\Gamma$-Mod is semi-simple. Thus, $\underline{\widetilde{\mathcal{T}}}$ is semi-simple as well; its simple objects are the simple tori, i.e., the tori G such that every subgroup $H \subsetneq G$ is finite.

As in Subsection 2.5, choose representatives T of the set I of isomorphism classes of simple tori, and let $D_T := \mathrm{End}_{\underline{\mathcal{T}}}(T)^{\mathrm{op}}$; then each D_T is a division ring of finite dimension over $\mathbb{Q}$. Let

$$R_{\underline{\mathcal{T}}} := \bigoplus_{T \in I} D_T,$$

then we have an equivalence of categories

$$\underline{\mathcal{T}} \xrightarrow{\cong} R_{\underline{\mathcal{T}}}\text{-mod}$$

which extends to an equivalence of categories

$$\underline{\widetilde{\mathcal{T}}} \xrightarrow{\cong} R_{\underline{\mathcal{T}}}\text{-Mod}^{\mathrm{ss}}.$$

Note finally that $\underline{\widetilde{\mathcal{T}}}$ is equivalent to a Serre subcategory of the isogeny category $\underline{\widetilde{\mathcal{L}}}$.

3.1.4. *Unipotent groups, structure of linear groups* [**DG70**, IV.2, IV.3]. The upper triangular matrices with all diagonal entries equal to 1 form a subgroup scheme $\mathrm{U}_n \subset \mathrm{GL}_n$, which is smooth and connected; moreover, U_1 is isomorphic to the additive group $\mathbb{G}_a$. An algebraic group G is called *unipotent* if G is isomorphic to a subgroup of U_n for some n. The (commutative) unipotent algebraic groups form a Serre subcategory $\mathcal{U}$ of $\mathcal{L}$. Also, the property of being unipotent is invariant under field extensions, in the sense of §3.1.2.

We say that an affine group scheme G is unipotent, if so are all algebraic quotients of G (this differs from the definition given in [**DG70**, IV.2.2.2], but both notions are equivalent in view of [**DG70**, IV.2.2.3]). The unipotent group schemes form a Serre subcategory $\widetilde{\mathcal{U}}$ of $\widetilde{\mathcal{L}}$, which is the pro-completion of $\mathcal{U}$ (as defined in [**DG70**, V.2.3.1]).

By [**DG70**, IV.2.2.4, IV.3.1.1], every affine group scheme G lies in a unique exact sequence

$$0 \longrightarrow M \longrightarrow G \longrightarrow U \longrightarrow 0, \tag{3.5}$$

where M is of multiplicative type and U is unipotent; moreover, $\mathrm{Hom}_{\widetilde{\mathcal{L}}}(M, U) = 0 = \mathrm{Hom}_{\widetilde{\mathcal{L}}}(U, M)$. Thus, $(\widetilde{\mathcal{M}}, \widetilde{\mathcal{U}})$ (resp. $(\mathcal{M}, \mathcal{U})$) is a torsion pair of Serre subcategories of $\widetilde{\mathcal{L}}$ (resp. $\mathcal{L}$), as considered in Subsection 2.2.

If the field k is perfect, then the exact sequence (3.5) has a unique splitting (see [**DG70**, IV.3.1.1]). It follows that the assignment $(M, U) \mapsto M \times U$ yields equivalences of categories

$$\widetilde{\mathcal{M}} \times \widetilde{\mathcal{U}} \xrightarrow{\cong} \widetilde{\mathcal{L}}, \quad \mathcal{M} \times \mathcal{U} \xrightarrow{\cong} \mathcal{L}.$$

In turn, this yields equivalences of isogeny categories

$$\underline{\widetilde{\mathcal{T}}} \times \underline{\widetilde{\mathcal{U}}} \xrightarrow{\cong} \underline{\widetilde{\mathcal{L}}}, \quad \underline{\mathcal{T}} \times \underline{\mathcal{U}} \xrightarrow{\cong} \underline{\mathcal{L}}. \tag{3.6}$$

In fact, *the latter equivalences hold over an arbitrary field*, as $\underline{\mathcal{T}}$, $\underline{\mathcal{U}}$ and $\underline{\mathcal{L}}$ are invariant under purely inseparable field extensions.

We now assume that $\mathrm{char}(k) = 0$. Then every unipotent algebraic group G is isomorphic to the direct sum of n copies of $\mathbb{G}_a$, where $n := \dim(G)$. In particular, G is isomorphic as a scheme to the affine space $\mathbb{A}^n$, and hence is smooth and connected. Moreover, every morphism of unipotent groups $f : G \to H$ is linear in the corresponding coordinates $x_1, \ldots, x_n$ on G. Thus, *the category $\mathcal{U}$ is equivalent to the category k-*mod *of finite-dimensional k-vector spaces.* This extends to an equivalence of $\widetilde{\mathcal{U}}$ to the category k-Mod of all k-vector spaces (see [**DG70**, IV.2.4.2]).

As every finite unipotent group is trivial, $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ are their own isogeny categories; they are obviously semi-simple and k-linear, and $\mathcal{U}$ is Hom-finite. In view of the equivalence (3.6), it follows that $\underline{\mathcal{L}}$ *and* $\underline{\widetilde{\mathcal{L}}}$ *are semi-simple.*

3.1.5. *Unipotent groups in positive characteristics* [**DG70**, V.1, V.3]. Throughout this paragraph, we assume that $\mathrm{char}(k) = p > 0$; then every unipotent algebraic group G is p-torsion. The structure of these groups is much more complicated than in characteristic 0. For example, the additive group $\mathbb{G}_a$ admits many finite subgroups, e.g., the (schematic) kernel of the Frobenius endomorphism

$$F : \mathbb{G}_a \longrightarrow \mathbb{G}_a, \quad x \longmapsto x^p.$$

The ring $\mathrm{End}_{\mathcal{U}}(\mathbb{G}_a)$ is generated by k (acting by scalar multiplication) and F, with relations $Fx - x^p F = 0$ for any $x \in k$ (see [**DG70**, II.3.4.4]).

Assume in addition that k is perfect. Then the categories $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ may be described in terms of modules over the *Dieudonné ring* $\mathbb{D}$ (see [**DG70**, V.1]). More specifically, $\mathbb{D}$ is a noetherian domain, generated by the ring of Witt vectors $W(k)$, the Frobenius F and the Verschiebung V; also, R is non-commutative unless $k = \mathbb{F}_p$. The left ideal $\mathbb{D}V \subset \mathbb{D}$ is two-sided, and the quotient ring $\mathbb{D}/\mathbb{D}V$ is isomorphic to $\mathrm{End}_{\mathcal{U}}(\mathbb{G}_a)$. More generally, for any positive integer n, the left ideal $\mathbb{D}V^n$ is two-sided and $\mathbb{D}/\mathbb{D}V^n \cong \mathrm{End}_{\mathcal{U}}(W_n)$, where W_n denotes the *group of Witt vectors of length* n; this is a smooth connected unipotent group of dimension n, which lies in an exact sequence

$$0 \longrightarrow W_n \longrightarrow W_{n+1} \longrightarrow \mathbb{G}_a \longrightarrow 0.$$

The $\mathrm{End}_{\mathcal{U}}(\mathbb{G}_a)$-module $\mathrm{Ext}^1_{\mathcal{U}}(\mathbb{G}_a, W_n)$ is freely generated by the class of the above extension. Moreover, the assignment

$$G \longmapsto \mathbf{M}(G) := \varinjlim \mathrm{Hom}_{\mathcal{U}}(G, W_n)$$

extends to an anti-equivalence $\mathbf{M}$ of $\widetilde{\mathcal{U}}$ with the full subcategory of $\mathbb{D}$-Mod with objects V-torsion modules. Also, G is algebraic (resp. finite) if and only if $\mathbf{M}(G)$ is finitely generated (resp. of finite length); we have $\mathbf{M}(W_n) = \mathbb{D}/\mathbb{D}V^n$ for all n. As a consequence, $\mathbf{M}$ restricts to an anti-equivalence of $\mathcal{U}$ with the full subcategory of $\mathbb{D}$-Mod with objects the finitely generated modules M which are V-torsion.

This yields a description of the isogeny categories $\underline{\mathcal{U}}, \underline{\widetilde{\mathcal{U}}}$ in terms of module categories. Let $S := \mathbb{D} \setminus \mathbb{D}V$; then we may form the left ring of fractions

$$S^{-1}\mathbb{D} =: R = R_{\underline{\mathcal{U}}}$$

by [**DG70**, V.3.6.3]. This is again a (generally non-commutative) noetherian domain; its left ideals are the two-sided RV^n in view of [**DG70**, V.3.6.11]. In particular, R has a unique maximal ideal, namely, RV; moreover, the quotient ring R/RV is isomorphic to the division ring of fractions of $\mathrm{End}_{\mathcal{U}}(\mathbb{G}_a)$. Thus, R is a discrete valuation domain (not necessarily commutative), as considered in [**KT07**]. By [**DG70**, V.3.6.7], a morphism of unipotent group schemes $f : G \to H$ is an isogeny if and only if the associated morphism $S^{-1}\mathbf{M}(f) : S^{-1}\mathbf{M}(H) \to S^{-1}\mathbf{M}(G)$ is an isomorphism. As a consequence, $S^{-1}\mathbf{M}$ yields an anti-equivalence of $\underline{\mathcal{U}}$ (resp. $\underline{\widetilde{\mathcal{U}}}$) with R-mod (resp. R-$\mathrm{Mod}^{\mathrm{tors}}$), where the latter denotes the full subcategory of R-Mod with objects the V-torsion modules.

We now show that the abelian category R-$\mathrm{Mod}^{\mathrm{tors}}$ is hereditary, and has enough projectives and a unique indecomposable projective object. Let $M \in R$-$\mathrm{Mod}^{\mathrm{tors}}$ and choose an exact sequence in R-Mod

$$0 \longrightarrow M \longrightarrow I \longrightarrow J \longrightarrow 0,$$

where I is injective in R-Mod; equivalently, the multiplication by V in I is surjective. Thus, J is injective in R-Mod as well. Let $I^{\mathrm{tors}} \subset I$ be the largest V-torsion submodule. Then we have an exact sequence in R-Mod

$$0 \longrightarrow M \longrightarrow I^{\mathrm{tors}} \longrightarrow J_{\mathrm{tors}} \longrightarrow 0.$$

Moreover, I^{tors}, J^{tors} are injective in R-Mod, and hence in R-$\mathrm{Mod}^{\mathrm{tors}}$ as well. As the injective objects of R-Mod are direct sums of copies of the division ring of fractions $K := \mathrm{Fract}(R)$ and of the quotient K/R (see e.g. [**KT07**, Thm. 6.3]), it follows that the abelian category R-$\mathrm{Mod}^{\mathrm{tors}}$ is hereditary and has enough injectives; moreover, it has a unique indecomposable injective object, namely,

$$K/R = \varinjlim RV^{-n}/R \cong \varinjlim R/RV^n$$

(the injective hull of the simple module). Also, note that R-Mod$^{\mathrm{tors}}$ is equipped with a duality (i.e., an involutive contravariant exact endofunctor), namely, the assignment $M \mapsto \mathrm{Hom}_R(M, K/R)$. As a consequence, we obtain an equivalence of $\widetilde{\underline{\mathcal{U}}}$ with R-Mod$^{\mathrm{tors}}$, which restricts to an equivalence of $\underline{\mathcal{U}}$ with R-mod.

Thus, $\widetilde{\underline{\mathcal{U}}}$ *is hereditary and has enough projectives; its unique indecomposable projective object is* $W := \varprojlim W_n$. Also, by [**DG70**, V.3.6.11] (see also [**KT07**, Thm. 4.8]), every unipotent algebraic group is isogenous to $\bigoplus_{n\geq 1} a_n W_n$ for uniquely determined integers $a_n \geq 0$. In other terms, *every indecomposable object of* $\underline{\mathcal{U}}$ *is isomorphic to* W_n *for a unique* $n \geq 1$.

Note finally that the above structure results for $\underline{\mathcal{U}}$ extend to an arbitrary field k of characteristic p, by invariance under purely inseparable field extensions. More specifically, $\underline{\mathcal{U}}$ is equivalent to R-mod, where R denotes the ring constructed as above from the perfect closure k_i.

3.2. More isogeny categories.

3.2.1. *Abelian varieties* [**Mi86**]. An *abelian variety* is a smooth, connected algebraic group A which is proper as a k-scheme. Then A is a projective variety and a divisible commutative group scheme; its group law will be denoted additively. Like for tori, the abelian group $\mathrm{Hom}_{\mathcal{C}}(A, A')$ is free of finite rank for any abelian varieties A and A'. Moreover, we have the Poincaré complete reducibility theorem: for any abelian variety A and any abelian subvariety $B \subset A$, there exists an abelian subvariety $C \subset A$ such that the map

$$B \times C \longrightarrow A, \quad (x, y) \longmapsto x + y$$

is an isogeny.

We denote by $\mathcal{P}$ the full subcategory of $\mathcal{C}$ with objects the proper algebraic groups; then $\mathcal{P}$ is a Serre subcategory of $\mathcal{C}$, containing $\mathcal{F}$ and invariant under field extensions. Moreover, the quotient category $\mathcal{P}/\mathcal{F}$ is equivalent to its full subcategory $\underline{\mathcal{A}}$ with objects the abelian varieties.

By (3.1), we have an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(A, A')_{\mathbb{Q}} \xrightarrow{\cong} \mathrm{Hom}_{\underline{\mathcal{A}}}(A, A')$$

for any abelian varieties A, A'. Also, the base change map

$$\mathrm{Hom}_{\mathcal{C}_k}(A, A') \longrightarrow \mathrm{Hom}_{\mathcal{C}_K}(A_K, A'_K)$$

is an isomorphism for any extension of fields K/k such that k is separably closed in K (see [**Co06**, Thm. 3.19] for a modern version of this classical result of Chow).

In view of the above results, *the abelian category* $\underline{\mathcal{A}}$ *is semi-simple,* $\mathbb{Q}$*-linear, Hom-finite, and invariant under purely inseparable field extensions. Also,* $\underline{\mathcal{A}}$ *is a Serre subcategory of* $\underline{\mathcal{C}}$.

Like for tori again, $\underline{\mathcal{A}}$ is equivalent to the category of all left modules of finite length over the ring

$$R_{\underline{\mathcal{A}}} := \bigoplus_{A \in J} D_A,$$

where J denotes the set of isogeny classes of simple abelian varieties, and we set $D_A := \mathrm{End}_{\underline{\mathcal{A}}}(A)^{\mathrm{op}}$ for chosen representatives A of the classes in J. Moreover, each D_A is a division algebra of finite dimension over $\mathbb{Q}$. Such an endomorphism algebra is a classical object, considered e.g. in [**Mu08**, Chap. IV] and [**Oo88**] where it is denoted by $\mathrm{End}_k^0(A)$. The choice of a polarization of A yields an involutory anti-automorphism of A (the Rosati involution), and hence an isomorphism of D_A with its opposite algebra.

3.2.2. *General algebraic groups, quasi-compact group schemes* [**Br16, Pe75, Pe76**]. The linear algebraic groups form the building blocks for all connected algebraic groups, together with the abelian varieties. Indeed, we have *Chevalley's structure theorem*: for any connected algebraic group G, there exists an exact sequence

$$0 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where L is linear and A is an abelian variety. Moreover, there is a unique smallest such subgroup $L \subset G$, and this group is connected. If G is smooth and k is perfect, then L is smooth as well (see [**Co02, Br15**] for modern expositions of this classical result).

Returning to an arbitrary ground field k, it is easy to see that $\mathrm{Hom}_{\mathcal{C}}(A, L) = 0$ for any abelian variety A and any linear algebraic group L; also, the image of any morphism $L \to A$ is finite (see e.g. [**Br16**, Prop. 2.5]).

It follows that $(\underline{\mathcal{L}}, \underline{\mathcal{A}})$ is a torsion pair of Serre subcategories in $\underline{\mathcal{C}}$, and we have $\mathrm{Hom}_{\underline{\mathcal{C}}}(A, L) = 0$ for all $A \in \underline{\mathcal{A}}$, $L \in \underline{\mathcal{L}}$. Therefore, $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(L, A) = 0$ for all such A, L by Lemma 2.3. In view of Chevalley's structure theorem and the vanishing of $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A', A)$ for all $A, A' \in \underline{\mathcal{A}}$, we obtain that $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(G, A) = 0$ for all $G \in \underline{\mathcal{C}}$ and $A \in \underline{\mathcal{A}}$. Thus, *every abelian variety is injective in* $\underline{\mathcal{C}}$ (see [**Br16**, Thm. 5.16] for the determination of the injective objects of $\underline{\mathcal{C}}$).

By (3.6), we have $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(T, U) = 0 = \mathrm{Ext}^1_{\underline{\mathcal{C}}}(U, T)$ for all $T \in \underline{\mathcal{T}}$, $U \in \underline{\mathcal{U}}$. Also, recall that $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(T, A) = 0$ for all $A \in \underline{\mathcal{A}}$ and $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(T, T') = 0$ for all $T' \in \underline{\mathcal{T}}$. By Chevalley's structure theorem again, it follows that $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(T, G) = 0$ for all $G \in \underline{\mathcal{C}}$. Thus, *every torus is projective in* $\underline{\mathcal{C}}$. If $\mathrm{char}(k) = 0$, then $\underline{\mathcal{L}}$ is semi-simple, as seen in §3.1.4. In view of Corollary 2.4, it follows that every linear algebraic group is projective in $\underline{\mathcal{C}}$ (see [**Br16**, Thm. 5.14] for the determination of the projective objects of $\underline{\mathcal{C}}$ in arbitrary characteristics).

We now adapt the above results to the setting of *quasi-compact* group schemes. Recall that a scheme is quasi-compact if every open covering admits a finite refinement. Every affine scheme is quasi-compact, as well as every scheme of finite type (in particular, every algebraic group). Also, every connected group scheme is quasi-compact (see [**Pe75**, II.2.4, II.2.5] or [**SGA3**, VIA, Thm. 2.6.5]). The quasi-compact (commutative) group schemes form an abelian category $\widetilde{\mathcal{C}}$, containing $\mathcal{C}$ as a Serre subcategory. Moreover, every $G \in \widetilde{\mathcal{C}}$ is the limit of a filtered inverse system $((G_i)_{i \in I}, (u_{ij} : G_j \to G_i)_{i \leq j})$ such that the G_i are algebraic groups and the u_{ij} are affine morphisms (see [**Pe75**, V.3.1, V.3.6]). Also, there is a unique exact sequence in $\widetilde{\mathcal{C}}$

$$0 \longrightarrow G^0 \longrightarrow G \longrightarrow F \longrightarrow 0,$$

where G^0 is connected and F is *pro-étale* (i.e., a filtered inverse limit of finite étale group schemes); see [**Pe75**, II.2.4, V.4.1]. Finally, there is an exact sequence as in Chevalley's structure theorem

$$0 \longrightarrow H \longrightarrow G^0 \longrightarrow A \longrightarrow 0,$$

where H is an affine group scheme, and A an abelian variety (see [**Pe75**, V.4.3.1]).

Note that $\widetilde{\mathcal{C}}$ is not the pro-completion of the abelian category $\mathcal{C}$, as infinite products do not necessarily exist in $\widetilde{\mathcal{C}}$. For example, the product of infinitely many copies of a non-zero abelian variety A is not represented by a scheme (this may be checked by arguing as in [**SP16**, 91.48], with the morphism $\mathrm{SL}_2 \to \mathbb{P}^1$ replaced by the surjective smooth affine morphism $V \to A$, where V denotes the disjoint union of finitely many open affine subschemes covering A).

We define the *isogeny category of quasi-compact group schemes*, $\widetilde{\underline{\mathcal{C}}}$, as the quotient category of $\widetilde{\mathcal{C}}$ by the Serre subcategory $\widetilde{\mathcal{FC}} = \widetilde{\mathcal{FL}}$ of pro-finite group schemes. Every object of $\widetilde{\underline{\mathcal{C}}}$ is isomorphic to an extension of an abelian variety A by an affine group scheme H. Moreover, $\mathrm{Hom}_{\widetilde{\underline{\mathcal{C}}}}(A, H) = 0$ and the image of every morphism $f : H \to A$ is finite (indeed, f factors through a closed immersion $H/\mathrm{Ker}(f) \to A$ by [**Pe75**, V.3.3]). As a consequence, $(\widetilde{\underline{\mathcal{L}}}, \underline{\mathcal{A}})$ is a torsion pair of Serre subcategories of $\widetilde{\underline{\mathcal{C}}}$; moreover, $\mathrm{Hom}_{\widetilde{\underline{\mathcal{C}}}}(A, H) = 0$ for all $A \in \underline{\mathcal{A}}$, $H \in \widetilde{\underline{\mathcal{L}}}$.

Like for the category $\underline{\mathcal{C}}$, it follows that every abelian variety is projective in $\widetilde{\underline{\mathcal{C}}}$, and every pro-torus is injective; when $\mathrm{char}(k) = 0$, every affine group scheme is projective.

3.2.3. *Vector extensions of abelian varieties* [**Br16**, 5.1]. The objects of the title are the algebraic groups G obtained as extensions

$$0 \longrightarrow U \longrightarrow G \longrightarrow A \longrightarrow 0, \tag{3.7}$$

where A is an abelian variety and U is a *vector group*, i.e., $U \cong n\mathbb{G}_a$ for some n. As $\mathrm{Hom}_{\mathcal{C}}(U, A) = 0 = \mathrm{Hom}_{\mathcal{C}}(A, U)$ (see §3.1.2), the data of G and of the extension (3.9) are equivalent. Also, we have a bi-functorial isomorphism

$$\mathrm{Ext}^1_{\mathcal{C}}(A, \mathbb{G}_a) \xrightarrow{\cong} H^1(A, \mathcal{O}_A), \tag{3.8}$$

where the right-hand side is a k-vector space of dimension $\dim(A)$ (see [**Oo66**, III.17]).

If $\mathrm{char}(k) = p > 0$, then $p_U = 0$ and hence the class of the extension (3.7) is killed by p. Thus, this extension splits after pull-back by the isogeny $p_A : A \to A$.

From now on, we assume that $\mathrm{char}(k) = 0$; then the vector extensions of A are the extensions by unipotent groups. We denote by $\underline{\mathcal{V}}$ the full subcategory of $\underline{\mathcal{C}}$ with objects the vector extensions of abelian varieties. By the Chevalley structure theorem (§3.1.2) and the structure of linear algebraic groups (§3.1.4), a connected algebraic group G is an object of $\underline{\mathcal{V}}$ if and only if $\mathrm{Hom}_{\underline{\mathcal{C}}}(T, G) = 0$. As the functor $\mathrm{Hom}_{\underline{\mathcal{C}}}(T, ?)$ is exact, it follows that $\underline{\mathcal{V}}$ is a Serre subcategory of $\underline{\mathcal{C}}$. Moreover, $(\mathcal{U}, \underline{\mathcal{A}})$ is a torsion pair of Serre subcategories of $\underline{\mathcal{V}}$; they are both semi-simple in view of §§3.1.4 and 3.2.1.

By (3.1), we have an isomorphism

$$\mathrm{Ext}^1_{\mathcal{C}}(A, U)_{\mathbb{Q}} \xrightarrow{\cong} \mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, U).$$

In view of (3.8), this yields bi-functorial isomorphisms

$$\mathrm{Ext}^1_{\underline{\mathcal{V}}}(A, U) \cong H^1(A, \mathcal{O}_A) \otimes_k U \cong \mathrm{Hom}_k(H^1(A, \mathcal{O}_A)^*, U), \tag{3.9}$$

where $H^1(A, \mathcal{O}_A)^*$ denotes of course the dual k-vector space. Moreover, the assignment $A \mapsto H^1(A, \mathcal{O}_A)^*$ extends to a covariant exact functor

$$\mathbf{U} : \underline{\mathcal{A}} \longrightarrow \mathcal{U},$$

as follows e.g. from [**Br16**, Cor. 5.3].

So the triple $(\underline{\mathcal{V}}, \mathcal{U}, \underline{\mathcal{A}})$ satisfies the assumptions (a), (b), (c), (d) of Subsection 2.4, with $\widetilde{\mathcal{X}} = \mathcal{U}$ and $\mathbf{F} = \mathbf{U}$. With the notation of §3.2.1, set

$$M_A := H^1(A, \mathcal{O}_A)^*$$

for any $A \in J$; then M_A is a k-D_A-bimodule.

Proposition 3.1. *With the above notation, there is an equivalence of categories*

$$\mathbf{M}_{\mathcal{V}} : \underline{\mathcal{V}} \xrightarrow{\cong} R_{\underline{\mathcal{V}}}\text{-mod},$$

where $R_{\underline{\mathcal{V}}}$ stands for the triangular matrix ring

$$\begin{pmatrix} \bigoplus_{A \in J} D_A & 0 \\ \bigoplus_{A \in J} M_A & k \end{pmatrix}.$$

Moreover, the center of $\underline{\mathcal{V}}$ is $\mathbb{Q}$.

Proof. The first assertion follows from Theorem 2.13 and Remark 2.14.

By Proposition 2.15, the center of $\underline{\mathcal{V}}$ consists of the pairs $z = (x, (y_A)_{A \in J})$ where $x \in k$, $y_A \in Z_A$ (the center of D_A) and $xm = my_A$ for all $m \in M_A$ and $A \in J$. In particular, if the simple abelian variety A satisfies $D_A = \mathbb{Q}$, then $x = y_A \in \mathbb{Q}$. As such abelian varieties exist (e.g., elliptic curves without complex multiplication), it follows that $x \in \mathbb{Q}$. Then for any $A \in J$, we obtain $y_A = x$ as $M_A \neq 0$. So $z \in \mathbb{Q}$. □

Next, recall that every abelian variety A has a universal vector extension $\mathbf{E}(A)$, by the vector group $\mathbf{U}(A)$. In view of Proposition 2.11, the projective objects of $\underline{\mathcal{V}}$ are the products of unipotent groups and universal vector extensions; moreover, every $G \in \underline{\mathcal{V}}$ has a canonical projective resolution,

$$0 \longrightarrow \mathbf{U}(A) \longrightarrow U \times \mathbf{E}(A) \longrightarrow G \longrightarrow 0, \tag{3.10}$$

where A denotes of course the abelian variety quotient of G. In particular, the abelian category $\underline{\mathcal{V}}$ is hereditary and has enough projectives. This recovers most of the results in [**Br16**, Sec. 5.1].

3.2.4. *Semi-abelian varieties* [**Br15**, 5.4]. These are the algebraic groups G obtained as extensions

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0, \tag{3.11}$$

where A is an abelian variety and T is a torus. Like for vector extensions of abelian varieties, we have $\mathrm{Hom}_{\mathcal{C}}(T, A) = 0 = \mathrm{Hom}_{\mathcal{C}}(A, T)$; thus, the data of G and of the extension (3.11) are equivalent.

The Weil-Barsotti formula (see e.g. [**Oo66**, III.17, III.18]) yields a bi-functorial isomorphism

$$\mathrm{Ext}^1_{\mathcal{C}}(A, T) \xrightarrow{\cong} \mathrm{Hom}^{\Gamma}(\mathbf{X}(T), \widehat{A}(k_s)).$$

Here $\widehat{A}$ denotes the dual of A; this is an abelian variety with dimension $\dim(A)$ and with Lie algebra $H^1(A, \mathcal{O}_A)$. In view of (3.1), this yields in turn a bi-functorial isomorphism

$$\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, T) \xrightarrow{\cong} \mathrm{Hom}^{\Gamma}(\mathbf{X}(T), \widehat{A}(k_s)_{\mathbb{Q}}). \tag{3.12}$$

We denote by $\underline{\mathcal{S}}$ the full subcategory of $\underline{\mathcal{C}}$ with objects the semi-abelian varieties. By [**Br15**, Lem. 5.4.3, Cor. 5.4.6], $\underline{\mathcal{S}}$ is a Serre subcategory of $\underline{\mathcal{C}}$, invariant under purely inseparable field extensions. Moreover, $(\underline{\mathcal{T}}, \underline{\mathcal{A}})$ is a torsion pair of Serre subcategories of $\underline{\mathcal{S}}$. The assignment $A \mapsto \widehat{A}(k_s)_{\mathbb{Q}}$ extends to a contravariant exact functor $\underline{\mathcal{A}} \to \mathbb{Q}\Gamma$-Mod in view of [**Br16**, Rem. 4.8] (see also Lemma 3.7 below). Since $\mathbb{Q}\Gamma$-Mod is anti-equivalent to $\widetilde{\underline{\mathcal{T}}}$ (§3.1.3), this yields a covariant exact functor

$$\widetilde{\mathbf{T}} : \underline{\mathcal{A}} \longrightarrow \widetilde{\underline{\mathcal{T}}}$$

together with a bi-functorial isomorphism

$$\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, T) \xrightarrow{\cong} \mathrm{Hom}_{\widetilde{\underline{\mathcal{T}}}}(\widetilde{\mathbf{T}}(A), T).$$

Thus, the triple $(\underline{\mathcal{S}}, \underline{\mathcal{T}}, \underline{\mathcal{A}})$ satisfies the assumptions (a), (b), (c), (d) of Subsection 2.4 with $\widetilde{\mathcal{X}} = \widetilde{\underline{\mathcal{T}}}$ and $\mathbf{F} = \widetilde{\mathbf{T}}$; moreover, the assumption (e) of Subsection 2.5 holds by construction. With the notation of §§3.1.3 and 3.2.1, let

$$M_{T,A} := \mathrm{Hom}^{\Gamma}(\mathbf{X}(T), \widehat{A}(k_s)_{\mathbb{Q}})$$

for all $T \in I$, $A \in J$; then $M_{T,A}$ is a D_T-D_A-bimodule. In view of Theorem 2.13, we obtain:

Proposition 3.2. *There is an equivalence of categories*

$$\mathbf{M}_{\mathcal{S}} : \underline{\mathcal{S}} \xrightarrow{\cong} R_{\underline{\mathcal{S}}}\text{-mod},$$

where $R_{\underline{\mathcal{S}}}$ stands for the triangular matrix ring

$$\begin{pmatrix} \bigoplus_{A\in J} D_A & 0 \\ \bigoplus_{T\in I, A\in J} M_{T,A} & \bigoplus_{T\in I} D_T. \end{pmatrix}.$$

Remark 3.3. If k is locally finite (i.e., the union of its finite subfields), then the abelian group $A(k)$ is torsion for any abelian variety A. It follows that $\underline{\mathcal{S}} \cong \underline{\mathcal{T}} \times \underline{\mathcal{A}}$ and $R_{\underline{\mathcal{S}}} = R_{\underline{\mathcal{T}}} \times R_{\underline{\mathcal{A}}}$. In particular, the center of $R_{\underline{\mathcal{S}}}$ is an infinite direct sum of fields.

On the other hand, if k is not locally finite, then the abelian group $A(k_s)$ has infinite rank for any non-zero abelian variety A (see [**FJ74**, Thm. 9.1]). As a consequence, A admits no universal extension in $\underline{\mathcal{S}}$. If in addition k is separably closed, then the center of $R_{\underline{\mathcal{S}}}$ is $\mathbb{Q}$, as follows by arguing as in the proof of Proposition 3.1 with $\mathbb{G}_a$ replaced by $\mathbb{G}_m$. We do not know how to determine the center of $R_{\underline{\mathcal{S}}}$ for an arbitrary (not locally finite) field k.

Next, consider the isogeny category $\widetilde{\underline{\mathcal{C}}}$ of quasi-compact group schemes, and denote by $\widetilde{\underline{\mathcal{S}}} \subset \widetilde{\underline{\mathcal{C}}}$ the full subcategory with objects the group schemes obtained as extensions of abelian varieties by pro-tori. Then $\underline{\mathcal{S}}$ is a Serre subcategory of $\widetilde{\underline{\mathcal{S}}}$; moreover, $(\widetilde{\underline{\mathcal{T}}}, \underline{\mathcal{A}})$ is a torsion pair of Serre subcategories of $\widetilde{\underline{\mathcal{S}}}$, and $\mathrm{Hom}_{\widetilde{\underline{\mathcal{S}}}}(A, T) = 0$ for any abelian variety A and any pro-torus T. Thus, $\widetilde{\underline{\mathcal{S}}}$ is equivalent to the category of extensions of abelian varieties by pro-tori (as defined in Subsection 2.3), and in turn to the category $\widetilde{R_{\underline{\mathcal{S}}}\text{-mod}}$ by Theorem 2.13.

In view of the results of Subsection 2.4, every abelian variety A has a universal extension in $\widetilde{\underline{\mathcal{S}}}$, by the pro-torus with Cartier dual $\widehat{A}(k_s)_{\mathbb{Q}}$. Moreover, the projective objects of $\widetilde{\underline{\mathcal{S}}}$ are the products of pro-tori and universal extensions; also, every $G \in \widetilde{\underline{\mathcal{S}}}$ has a canonical projective resolution, similar to (3.10). In particular, the abelian category $\widetilde{\underline{\mathcal{S}}}$ is hereditary and has enough projectives.

3.2.5. *General algebraic groups (continued).* We return to the setting of §3.2.2, and consider the isogeny category of algebraic groups, $\underline{\mathcal{C}}$, as a Serre category of the isogeny category of quasi-compact group schemes, $\underline{\widetilde{\mathcal{C}}}$. Also, recall from §§3.1.4, 3.1.5 the isogeny category of unipotent algebraic groups, $\underline{\mathcal{U}}$, a Serre subcategory of that of unipotent group schemes, $\underline{\widetilde{\mathcal{U}}}$. Likewise, we have the isogeny category of semi-abelian varieties, $\underline{\mathcal{S}}$, a Serre subcategory of the isogeny category of extensions of abelian varieties by pro-tori, $\underline{\widetilde{\mathcal{S}}}$. These are the ingredients of a structure result for $\underline{\mathcal{C}}$, $\underline{\widetilde{\mathcal{C}}}$ in positive characteristics:

PROPOSITION 3.4. *If* $\mathrm{char}(k) = p > 0$*, then the assignment* $(S, U) \mapsto S \times U$ *extends to equivalences of categories*

$$\underline{\mathcal{S}} \times \underline{\mathcal{U}} \xrightarrow{\cong} \underline{\mathcal{C}}, \quad \underline{\widetilde{\mathcal{S}}} \times \underline{\widetilde{\mathcal{U}}} \xrightarrow{\cong} \underline{\widetilde{\mathcal{C}}}.$$

PROOF. The first equivalence is obtained in [**Br16**, Prop. 5.10]. We provide an alternative proof: by Chevalley's structure theorem, every $G \in \underline{\mathcal{C}}$ lies in an exact sequence in $\underline{\mathcal{C}}$

$$0 \longrightarrow U \longrightarrow G \longrightarrow S \longrightarrow 0,$$

where $U \in \underline{\mathcal{U}}$ and $S \in \underline{\mathcal{S}}$. Moreover, we have $\mathrm{Ext}^n_{\underline{\mathcal{C}}}(U, S) = 0 = \mathrm{Ext}^n_{\underline{\mathcal{C}}}(S, U)$ for all $n \geq 0$, since the multiplication map p_S is an isomorphism in $\underline{\mathcal{C}}$, while $p_U^n = 0$ for $n \gg 0$. In particular, the above exact sequence has a unique splitting, which is functorial in U, S.

The second equivalence follows from the first one, as every quasi-compact group scheme is the inverse limit of its algebraic quotient groups. □

Next, we obtain the main result of this paper; for this, we gather some notation. Define a ring $R = R_{\underline{\mathcal{C}}}$ by

$$R = \begin{cases} R_{\underline{\mathcal{S}}} \times R_{\underline{\mathcal{U}}}, & \text{if } \mathrm{char}(k) > 0 \\ R_{\underline{\mathcal{S}}} \times_{R_{\underline{\mathcal{A}}}} R_{\underline{\mathcal{V}}}, & \text{if } \mathrm{char}(k) = 0. \end{cases}$$

More specifically,

$$R = \begin{pmatrix} \bigoplus_{A \in J} D_A & 0 \\ \bigoplus_{T \in I, A \in J} M_{T,A} & \bigoplus_{T \in I} D_T \end{pmatrix} \times (\mathbb{D} \setminus \mathbb{D}V)^{-1}\mathbb{D}$$

if $\mathrm{char}(k) > 0$, where I (resp. J) denotes the set of isogeny classes of simple tori (resp. of simple abelian varieties), $T \in I$ (resp. $A \in J$) denote representatives of their classes, $D_T := \mathrm{End}_{\mathcal{T}}(T)^{\mathrm{op}}_{\mathbb{Q}}$, $D_A := \mathrm{End}_{\mathcal{A}}(A)^{\mathrm{op}}_{\mathbb{Q}}$, and $M_{T,A} := \mathrm{Hom}^{\Gamma}(\mathbf{X}(T), \widehat{A}(k_s)_{\mathbb{Q}})$. Moreover, $\mathbb{D}$ denotes the Dieudonné ring over the perfect closure of k, and $V \in \mathbb{D}$ the Verschiebung as in §3.1.5.

If $\mathrm{char}(k) = 0$, then

$$R = \begin{pmatrix} \bigoplus_{A \in J} D_A & 0 \\ (\bigoplus_{T \in I, A \in J} M_{T,A}) \oplus (\bigoplus_{A \in J} M_A) & (\bigoplus_{T \in I} D_T) \oplus k \end{pmatrix},$$

where $M_A := H^1(A, \mathcal{O}_A)^*$. We may now state:

THEOREM 3.5. *With the above notation, the abelian categories* $\underline{\mathcal{C}}$ *and* $\underline{\widetilde{\mathcal{C}}}$ *are hereditary, and* $\underline{\widetilde{\mathcal{C}}}$ *has enough projectives. Moreover, there are compatible equivalences of categories*

$$\underline{\mathcal{C}} \xrightarrow{\cong} R\text{-mod}, \quad \underline{\widetilde{\mathcal{C}}} \xrightarrow{\cong} R\text{-}\widetilde{\text{mod}}.$$

PROOF. When $\mathrm{char}(k) > 0$, this follows by combining Proposition 3.4 with the structure results for $\underline{\mathcal{U}}$, $\underline{\widetilde{\mathcal{U}}}$ recalled in §3.1.5, and those for $\underline{\mathcal{S}}$, $\underline{\widetilde{\mathcal{S}}}$ obtained in Proposition 3.2. When $\mathrm{char}(k) = 0$, recall from §3.2.2 that $(\underline{\mathcal{L}}, \underline{\mathcal{A}})$ (resp. $(\underline{\widetilde{\mathcal{L}}}, \underline{\mathcal{A}})$) is a torsion pair of Serre subcategories of $\underline{\mathcal{C}}$ (resp. $\underline{\widetilde{\mathcal{C}}}$); moreover, $\underline{\mathcal{L}} \cong \underline{\mathcal{T}} \times \mathcal{U}$ and $\underline{\widetilde{\mathcal{L}}} \cong \underline{\widetilde{\mathcal{T}}} \times \widetilde{\mathcal{U}}$ by §3.1.4. In view of the bi-functorial isomorphisms (3.9) and (3.12), the assertions follow from Theorem 2.13 like in the cases of vector extensions of abelian varieties (Proposition 3.1) and of semi-abelian varieties (Proposition 3.2). □

Also, recall from Proposition 2.15 that the intersection of the left ideals of finite colength in R is zero. One may check (by using Remark 2.16) that this also holds for the two-sided ideals of finite colength as left modules.

We will show in Subsection 3.4 that the center of $\underline{\mathcal{C}}$ equals $\mathbb{Q}$ if $\mathrm{char}(k) = 0$, and contains $\mathbb{Q} \times \mathbb{Z}_p$ if $\mathrm{char}(k) = p > 0$.

3.3. Functors of points. Let G be an algebraic group. Then the group of $\bar{k}$-points, $G(\bar{k})$, is equipped with an action of the absolute Galois group Γ. We also have the subgroup of k_s-points, $G(k_s)$, which is stable under Γ. Since every $x \in G(k_s)$ lies in $G(K)$ for some finite Galois extension of fields K/k, we see that the stabilizer of x in Γ is open, i.e., $G(k_s)$ is a discrete Γ-module. Likewise, the Γ-module $G(\bar{k})$ is discrete as well.

Clearly, the assignment $G \mapsto G(\bar{k})$ extends to a covariant exact functor

$$(\bar{k}) : \mathcal{C} \longrightarrow \mathbb{Z}\Gamma\text{-Mod}, \quad \mathcal{F} \longrightarrow \mathbb{Z}\Gamma\text{-Mod}^{\mathrm{tors}}.$$

The assignment $G \mapsto G(k_s)$ also extends to a covariant functor

$$(k_s) : \mathcal{C} \longrightarrow \mathbb{Z}\Gamma\text{-Mod}, \quad \mathcal{F} \longrightarrow \mathbb{Z}\Gamma\text{-Mod}^{\mathrm{tors}}$$

which is additive and left exact. But the functor (k_s) is not exact when k is an imperfect field, as seen from the exact sequence

$$0 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \longrightarrow 0,$$

where $p := \mathrm{char}(k)$ and F denotes the Frobenius endomorphism, $x \mapsto x^p$. Yet the functors $(\bar{k})$ and (k_s) are closely related:

LEMMA 3.6. (i) *Let K/k be an extension of fields of characteristic $p > 0$ and assume that $K^{p^n} \subset k$. Then $p^n x \in G(k)$ for any $x \in G(K)$.*
(ii) *For any $x \in G(\bar{k})$, there exists $n = n(x) \geq 0$ such that $p^n x \in G(k_s)$.*

PROOF. (i) By assumption, we have $x \in G(k^{1/p^n})$. Consider the nth relative Frobenius morphism,

$$F^n_{G/k} : G \longrightarrow G^{(p^n)}$$

(see e.g. [**CGP15**, A.3]). Then $F^n_{G/k}(x) \in G^{(p^n)}(k)$: indeed, this holds with G replaced with any scheme of finite type, since this holds for the affine space $\mathbb{A}^n$ and the formation of the relative Frobenius morphism commutes with immersions. We also have the nth Verschiebung,

$$V^n_{G/k} : G^{(p^n)} \longrightarrow G,$$

which satisfies $V^n_{G/K} \circ F^n_{G/k} = p^n_G$ (see [**SGA3**, VIIA.4.3]). It follows that $p^n x = V^n_{G/k}(F^n_{G/k}(x))$ is in $G(k)$.

(ii) Just apply (i) to k_s instead of k, and use the fact that $\bar{k} = \bigcup_{n \geq 0} k_s^{1/p^n}$. □

As a direct consequence, we obtain:

Lemma 3.7. (i) *The natural map* $G(k_s)_{\mathbb{Q}} \to G(\bar{k})_{\mathbb{Q}}$ *is an isomorphism for any algebraic group* G.

(ii) *The covariant exact functor*

$$(\bar{k})_{\mathbb{Q}} : \mathcal{C} \longrightarrow \mathbb{Q}\Gamma\text{-Mod}, \quad G \longmapsto G(\bar{k})_{\mathbb{Q}} := G(\bar{k})_{\mathbb{Q}}$$

yields a covariant exact functor, also denoted by

$$(\bar{k})_{\mathbb{Q}} : \underline{\mathcal{C}} \longrightarrow \mathbb{Q}\Gamma\text{-Mod}.$$

Remarks 3.8. (i) Assume that G is unipotent. If $\mathrm{char}(k) = p > 0$, then $G(\bar{k})$ is p-torsion and hence $G(\bar{k})_{\mathbb{Q}} = 0$. On the other hand, if $\mathrm{char}(k) = 0$, then $G \cong n\mathbb{G}_a$ and hence $G(\bar{k})_{\mathbb{Q}} \cong n\bar{k}$ as a Γ-module. Using the normal basis theorem, it follows that the multiplicity of any simple discrete Γ-module M in $G(\bar{k})_{\mathbb{Q}}$ equals $n \dim(M)$; in particular, all these multiplicities are finite.

The latter property does not extend to the case where G is a torus. For example, the multiplicity of the trivial Γ-module in $\mathbb{G}_m(\bar{k})_{\mathbb{Q}}$ is the rank of the multiplicative group k^*, which is infinite when k is not locally finite. Indeed, under that assumption, k contains either $\mathbb{Q}$ or the field of rational functions in one variable $\mathbb{F}_p(t)$; so the assertion follows from the infiniteness of prime numbers and of irreducible polynomials in $\mathbb{F}_p[t]$.

Whenn G is an abelian variety, the finiteness of multiplicities of the Γ-module $G(k)_{\mathbb{Q}}$ will be discussed in the next subsection.

(ii) For any abelian variety A and any torus T, we have a bi-functorial isomorphism

$$\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, T) \xrightarrow{\cong} \mathrm{Hom}^{\Gamma}(\mathbf{X}(T), \widehat{A}(\bar{k})_{\mathbb{Q}}),$$

as follows from (3.12) together with Lemma 3.7. Also, recall that $\widehat{A}$ is isogenous to A (see e.g. [**Mi86**]) and hence the Γ-module $\widehat{A}(\bar{k})_{\mathbb{Q}}$ is isomorphic (non-canonically) to $A(\bar{k})_{\mathbb{Q}}$.

Next, we associate an endofunctor of $\underline{\mathcal{C}}$ with any discrete Γ-module M, which is a free abelian group of finite rank. We may choose a finite Galois extension $K/k \subset k_s/k$ such that Γ acts on M via its finite quotient $\Gamma' := \mathrm{Gal}(K/k)$. For any $G \in \mathcal{C}$, consider the tensor product of commutative group functors $G_K \otimes_{\mathbb{Z}} M$. This group functor is represented by an algebraic group over K (isomorphic to the product of r copies of G_K, where r denotes the rank of M as an abelian group), equipped with an action of Γ' such that the structure map $G_K \otimes_{\mathbb{Z}} M \to \mathrm{Spec}(K)$ is equivariant. By Galois descent (see e.g. [**Co06**, Cor. 3.4]), the quotient

$$G(M) := (G_K \otimes_{\mathbb{Z}} M)/\Gamma'$$

is an algebraic group over k, equipped with a natural Γ'-equivariant isomorphism

$$G_K \otimes_{\mathbb{Z}} M \xrightarrow{\cong} G(M)_K.$$

The assignment $G \mapsto G(M)$ extends to a covariant endofunctor (M) of $\mathcal{C}$, which is exact as the base change functor $\otimes_k K$ is faithful. Moreover, one may easily check that (M) is independent of the choice of K, and hence comes with a natural Γ-equivariant isomorphism

$$G_{k_s} \otimes_{\mathbb{Z}} M \xrightarrow{\cong} G(M)_{k_s}$$

for any $G \in \mathcal{C}$. In particular, we have an isomorphism of Γ-modules

$$G(k_s) \otimes_{\mathbb{Z}} M \xrightarrow{\cong} G(M)(k_s).$$

This implies readily:

LEMMA 3.9. *With the above notation and assumptions, the endofunctor* (M) *of* $\mathcal{C}$ *yields a covariant exact endofunctor* $(M)_{\mathbb{Q}}$ *of* $\underline{\mathcal{C}}$ *that stabilizes* $\underline{\mathcal{T}}$, $\underline{\mathcal{U}}$, $\underline{\mathcal{A}}$ *and* $\underline{\mathcal{S}}$. *Moreover,* $(M)_{\mathbb{Q}}$ *only depends on* $M_{\mathbb{Q}} \in \mathbb{Q}\Gamma$-mod, *and there is a natural isomorphism of* Γ*-modules*

$$G(M)(k_s)_{\mathbb{Q}} \cong G(k_s) \otimes_{\mathbb{Z}} M_{\mathbb{Q}} = G(k_s)_{\mathbb{Q}} \otimes_{\mathbb{Q}} M_{\mathbb{Q}}$$

for any $G \in \mathcal{C}$.

REMARKS 3.10. (i) If G is a torus, then one readily checks that $G(M)$ is the torus with character group $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbf{X}(G))$. As a consequence, the endofunctor (M) of $\underline{\mathcal{T}}$ is identified with the tensor product by the dual module M^*, under the anti-equivalence of categories of $\underline{\mathcal{T}}$ with $\mathbb{Q}\Gamma$-mod.

On the other hand, the endofunctor (M) of $\underline{\mathcal{U}}$ is just given by the assignment $G \mapsto rG$, where r denotes the rank of the free abelian group M. Indeed, we may assume that G is indecomposable and (using the invariance of $\underline{\mathcal{U}}$ under purely inseparable field extensions) that k is perfect. Then $G \cong W_n$ and hence $G(M)_{k_s} = W_{n,k_s} \otimes_{\mathbb{Z}} M \cong rW_{n,k_s}$ in $\underline{\mathcal{U}}_{k_s}$. In view of the uniqueness of the decomposition in $\underline{\mathcal{U}}$ as a direct sum of groups of Witt vectors, it follows that $G(M) \cong rW_n$ as desired.

(ii) The endofunctor (M) can be interpreted in terms of Weil restriction when M is a permutation Γ-module, i.e., M has a $\mathbb{Z}$-basis which is stable under Γ. Denote by $\Delta \subset \Gamma$ the isotropy group of some basis element, and by $K \subset k_s$ the fixed point subfield of Δ. Then K/k is a finite separable field extension, and one may check that there is a natural isomorphism $G(M) \cong \mathrm{R}_{K/k}(G_K)$ with the notation of [**CGP15**, A.5].

(iii) The assignment $G \mapsto G(M)$ is in fact a special case of a tensor product construction introduced by Milne in the setting of abelian varieties (see [**Mi72**]) and systematically studied by Mazur, Rubin and Silverberg in [**MRS07**]. More specifically, the tensor product $M \otimes_{\mathbb{Z}} G$, defined there in terms of Galois cohomology, is isomorphic to $G(M)$ in view of [**MRS07**, Thm. 1.4].

3.4. Finiteness conditions for Hom and Ext groups. Recall from §§3.1.3, 3.2.1 that the abelian categories $\underline{\mathcal{T}}$, $\underline{\mathcal{A}}$ are $\mathbb{Q}$-linear and Hom-finite. Also, recall from §3.1.4 that $\underline{\mathcal{U}} \cong \mathcal{U} \cong k$-mod is k-linear, semi-simple and Hom-finite when $\mathrm{char}(k) = 0$.

PROPOSITION 3.11. (i) *The abelian categories* $\underline{\mathcal{T}}$ *and* $\underline{\mathcal{A}}$ *are not* K*-linear for any field* K *strictly containing* $\mathbb{Q}$.

(ii) *The abelian category* $\underline{\mathcal{U}}$ *is not* K*-linear for any field* K, *when* $\mathrm{char}(k) = p > 0$.

(iii) *The center of* $\underline{\mathcal{C}}$ *is* $\mathbb{Q}$ *when* $\mathrm{char}(k) = 0$.

PROOF. (i) The assertion clearly holds for $\underline{\mathcal{T}}$, as $\mathrm{End}_{\underline{\mathcal{T}}}(\mathbb{G}_m) = \mathbb{Q}$. For $\underline{\mathcal{A}}$, we replace $\mathbb{G}_m$ with appropriate elliptic curves E. Given any $t \in k$, there exists such a curve with j-invariant t (see e.g. [**Si86**, Prop. III.1.4]). If $\mathrm{char}(k) = 0$, then we choose $t \in \mathbb{Q} \setminus \mathbb{Z}$; in particular, t is not an algebraic integer. By [**Si86**, Thm. C.11.2], we have $\mathrm{End}_{\mathcal{A}}(E) = \mathbb{Z}$ and hence $\mathrm{End}_{\underline{\mathcal{A}}}(E) = \mathbb{Q}$. If $\mathrm{char}(k) = p$

and k is not algebraic over $\mathbb{F}_p$, then we may choose t transcendental over $\mathbb{F}_p$. By [**Mu08**, §22], we then have again $\mathrm{End}_{\underline{\mathcal{A}}}(E) = \mathbb{Q}$. Finally, if k is algebraic over $\mathbb{F}_p$, then every abelian variety A is defined over a finite subfield of k, and hence the associated Frobenius endomorphism lies in $\mathrm{End}_{\underline{\mathcal{A}}}(A) \setminus \mathbb{Q}$. In that case, it follows from [**Oo88**, (2.3)] that $\mathrm{End}_{\underline{\mathcal{A}}}(E)$ is an imaginary quadratic number field in which p splits, if E is ordinary. On the other hand, if E is supersingular, then $\mathrm{End}_{\underline{\mathcal{A}}}(E)$ contains no such field. It follows that the largest common subfield to all rings $\mathrm{End}_{\underline{\mathcal{A}}}(E)$ is $\mathbb{Q}$.

(ii) Assume that $\underline{\mathcal{U}}$ is K-linear for some field K. Then K is a subfield of $\mathrm{End}_{\underline{\mathcal{U}}}(\mathbb{G}_a)$, and hence $\mathrm{char}(K) = p$. More generally, for any $n \geq 1$, we have a ring homomorphism $K \to \mathrm{End}_{\underline{\mathcal{U}}}(W_n) = R/RV^n$ with the notation of §3.1.5. As these homomorphisms are compatible with the natural maps $R/RV^{n+1} = \mathrm{End}_{\underline{\mathcal{U}}}(W_{n+1}) \to \mathrm{End}_{\underline{\mathcal{U}}}(W_n) = R/RV^n$, we obtain a ring homomorphism $K \to \varprojlim R/RV^n$. Since the right-hand side has characteristic 0, this yields a contradiction.

(iii) By Proposition 2.15, it suffices to show that the center of R is $\mathbb{Q}$. Moreover, every central element of R is of the form

$$z = \begin{pmatrix} \sum_{A \in J} y_A & 0 \\ 0 & (\sum_{T \in I} x_T) + x \end{pmatrix},$$

where each y_A is central in D_A, each x_T is central in D_T, and $x \in k$; also, $x m_A = m_A y_A$ for all $m \in M_A$, and $x_T m_{T,A} = m_{T,A} y_A$ for all $m_{T,A} \in M_{T,A}$. Like in the proof of Proposition 3.1, it follows that $x \in \mathbb{Q}$ and $y_A = x$ for all A. As a consequence, $x_T = y_A$ whenever $M_{T,A} \neq 0$.

To complete the proof, it suffices to show that for any $T \in I$, there exists $A \in J$ such that $M_{T,A} \neq 0$; equivalently, the Γ-module $A(\bar{k})_{\mathbb{Q}}$ contains $\mathbf{X}(T)_{\mathbb{Q}}$. We may choose an elliptic curve E such that $E(k)$ is not torsion, i.e., $E(k)_{\mathbb{Q}} \neq 0$; then the Γ-module $E(\mathbf{X}(T))(\bar{k})$ contains $\mathbf{X}(T)$. Thus, the desired assertion holds for some simple factor A of $E(\mathbf{X}(T))$. □

Remark 3.12. The above statement (iii) does not extend to the case where $\mathrm{char}(k) = p > 0$, since we then have $\underline{\mathcal{C}} \cong \underline{\mathcal{S}} \times \underline{\mathcal{U}}$. One may then show that the center of $\underline{\mathcal{U}}$ contains the ring of p-adic integers, $\mathbb{Z}_p = W(\mathbb{F}_p)$, with equality if and only if k is infinite. In view of Remark 3.3, it follows that the center of $\underline{\mathcal{C}}$ contains $\mathbb{Q} \times \mathbb{Z}_p$, with equality if k is separably closed.

Proposition 3.13. *The abelian category $\underline{\mathcal{S}}$ is $\mathbb{Q}$-linear and Hom-finite. It is Ext-finite if and only if k satisfies the following condition:*

(MW) *The vector space $A(k)_{\mathbb{Q}}$ is finite-dimensional for any abelian variety A.*

Proof. Recall that every semi-abelian variety is divisible. In view of (3.1), it follows that $\underline{\mathcal{S}}$ is $\mathbb{Q}$-linear. It is Hom-finite in view of the Hom-finiteness of $\underline{\mathcal{T}}$ and $\underline{\mathcal{A}}$, combined with Proposition 2.6.

By that proposition, $\underline{\mathcal{S}}$ is Ext-finite if and only if the $\mathbb{Q}$-vector space $\mathrm{Ext}^1_{\underline{\mathcal{S}}}(A, T)$ is finite-dimensional for any abelian variety A and any torus T. In view of the isomorphism (3.12) and of the anti-equivalence of categories (3.3), this amounts to the condition that the vector space $\mathrm{Hom}^{\Gamma}(M, \widehat{A}(k_s)_{\mathbb{Q}})$ be finite-dimensional for any $M \in \mathbb{Q}\Gamma\text{-mod}$. The latter condition is equivalent to (MW) by Lemma 3.9. □

REMARKS 3.14. (i) The above condition (MW) is a weak version of the Mordell-Weil theorem, which asserts that the abelian group $A(k)$ is finitely generated for any abelian variety A over a number field k.

(ii) The condition (MW) holds trivially if k is locally finite, as the abelian group $A(k)$ is torsion under that assumption.

(iii) Let K/k be a finitely generated regular extension of fields (recall that the regularity assumption means that k is algebraically closed in K, and K is separable over k). If (MW) holds for k, then it also holds for K in view of the Lang-Néron theorem (see [**Co06**] for a modern proof of this classical result). As a consequence, (MW) holds whenever k is finitely generated over a number field or a locally finite field.

One can also show that (MW) is invariant under purely transcendental extensions (not necessarily finitely generated), by using the fact that every rational map from a projective space to an abelian variety is constant.

PROPOSITION 3.15. *Assume that* $\mathrm{char}(k) = 0$. *Then the* $\mathbb{Q}$*-linear category* $\underline{\mathcal{V}}$ *(resp.* $\underline{\mathcal{C}}$*) is Hom-finite if and only if* k *is a number field. Under that assumption,* $\underline{\mathcal{V}}$ *and* $\underline{\mathcal{C}}$ *are Ext-finite as well.*

PROOF. If $\underline{\mathcal{V}}$ is Hom-finite, then $\mathrm{End}_{\underline{\mathcal{V}}}(\mathbb{G}_a)$ is finite-dimensional as a $\mathbb{Q}$-vector space. Since $\mathrm{End}_{\underline{\mathcal{V}}}(\mathbb{G}_a) = \mathrm{End}_{\mathcal{U}}(\mathbb{G}_a) = k$, this means that k is a number field.

Conversely, if k is a number field, then $\mathcal{U}$ is Hom-finite, and hence so is $\underline{\mathcal{L}}$. In view of Proposition 2.6, it follows that $\underline{\mathcal{C}}$ is Hom-finite, and hence so is $\underline{\mathcal{V}}$. By that proposition again, to prove that $\underline{\mathcal{C}}$ is Ext-finite, it suffices to check that the $\mathbb{Q}$-vector space $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, L)$ is finite-dimensional for any abelian variety A and any connected linear algebraic group L. Since $L = U \times T$ for a unipotent group U and a torus T, this finiteness assertion follows by combining the isomorphisms (3.9), (3.12) and the Mordell-Weil theorem. □

3.5. Finiteness representation type: an example. As in §3.2.3, we consider the abelian category $\underline{\mathcal{V}}$ of vector extensions of abelian varieties over a field k of characteristic 0. Recall that $\underline{\mathcal{V}}$ is $\mathbb{Q}$-linear and hereditary, and has enough projectives; its simple objects are the additive group $\mathbb{G}_a$ and the simple abelian varieties. In particular, $\underline{\mathcal{V}}$ has infinitely many isomorphism classes of simple objects. Also, by Proposition 3.15, $\underline{\mathcal{V}}$ is Hom-finite if and only if k is a number field; then $\underline{\mathcal{V}}$ is Ext-finite as well.

We now assume that k is a number field. Choose a finite set $F = \{A_1, \ldots, A_r\}$ of simple abelian varieties, pairwise non-isogenous. Denote by $\underline{\mathcal{V}}_F$ the Serre subcategory of $\underline{\mathcal{V}}$ generated by F. More specifically, the objects of $\underline{\mathcal{V}}_F$ are the algebraic groups obtained as extensions

$$0 \longrightarrow m_0\mathbb{G}_a \longrightarrow G \longrightarrow \bigoplus_{i=1}^{r} m_i A_i \longrightarrow 0,$$

where $m_0, m_1, \ldots, m_r$ are non-negative integers. The morphisms of $\underline{\mathcal{V}}_F$ are the homomorphisms of algebraic groups.

By Theorem 2.13 and Remark 2.14, we have an equivalence of categories

$$\underline{\mathcal{V}}_F \xrightarrow{\cong} R_F\text{-mod},$$

where R_F denotes the triangular matrix ring

$$\begin{pmatrix} D_1 \oplus \cdots \oplus D_r & 0 \\ M_1 \oplus \cdots \oplus M_r & k \end{pmatrix}.$$

Here $D_i := \mathrm{End}_{\underline{\mathcal{V}}}(A_i)^{\mathrm{op}}$ is a division ring of finite dimension as a $\mathbb{Q}$-vector space, and $M_i := H^1(A_i, \mathcal{O}_{A_i})^*$ is a k-D_i-bimodule, of finite dimension as a k-vector space. Thus, R_F is a finite-dimensional $\mathbb{Q}$-algebra. Also, R_F is hereditary, since so is $\underline{\mathcal{V}}_F$.

The $\mathbb{Q}$-*species* of R_F is the directed graph Γ_F with vertices $0, 1, \ldots, r$ and edges $\varepsilon_i := (0, i)$ for $i = 1, \ldots, r$. The vertex 0 is labeled with the field k, and each vertex $i = 1, \ldots, r$ is labeled with the division ring D_i; each edge ε_i is labeled with the k-D_i-bimodule M_i. The category R_F-mod is equivalent to that of representations of the $\mathbb{Q}$-species Γ_F, as defined in [**DR76**] (see also [**Le12**]).

The *valued graph* of Γ_F is the underlying non-directed graph Δ_F, where each edge $\{0, i\}$ is labeled with the pair $(\dim_k(M_i), \dim_{D_i}(M_i))$. As all edges contain 0, we say that 0 is a *central vertex*; in particular, Δ_F is connected.

Recall that an Artin algebra is said to be *of finite representation type* if it has only finitely many isomorphism classes of indecomposable modules of finite length. In view of the main result of [**DR76**], R_F is of finite representation type if and only if Δ_F is a Dynkin diagram. By inspecting such diagrams having a central vertex, this is equivalent to Δ_F being a subgraph (containing 0 as a central vertex) of one of the following graphs:

$\mathbf{B}_3$: $1 \text{———} 0 \overset{(1,2)}{\text{———}} 2$ $\mathbf{C}_3$: $1 \text{———} 0 \overset{(2,1)}{\text{———}} 2$

$\mathbf{D}_4$: $\begin{array}{ccccc} 1 & \text{———} & 0 & \text{———} & 2 \\ & & | & & \\ & & 3 & & \end{array}$

$\mathbf{G}_2$: $0 \overset{(3,1)}{\text{———}} 1$ $\mathbf{G}_2^{\mathrm{op}}$: $0 \overset{(1,3)}{\text{———}} 1$

The subgraphs obtained in this way are as follows:

$\mathbf{A}_2$: $0 \text{———} 1$ $\mathbf{B}_2$: $0 \overset{(1,2)}{\text{———}} 1$

$\mathbf{C}_2$: $0 \overset{(2,1)}{\text{———}} 1$ $\mathbf{A}_3$: $1 \text{———} 0 \text{———} 2$

Here all unmarked edges have value $(1, 1)$. We set

$$g_i := \dim(A_i) = \dim_k(M_i), \quad n_i := [D_i : \mathbb{Q}], \quad n := [k : \mathbb{Q}].$$

Then the label of each edge $\{0, i\}$ is $(g_i, \frac{g_i n}{n_i})$. The above list entails restrictions on these labels, and hence on the simple abelian varieties A_i and the associated division rings D_i. We will work out the consequences of these restrictions in the case where k is the field of rational numbers, which yields an especially simple result:

PROPOSITION 3.16. *When $k = \mathbb{Q}$, the algebra R_F is of finite representation type if and only if $\Delta_F = \mathbf{D}_4, \mathbf{C}_3, \mathbf{G}_2$ or a subgraph containing 0 as a central vertex (i.e., $\mathbf{A}_2, \mathbf{A}_3, \mathbf{C}_2$), and the abelian varieties A_i satisfy the following conditions:*

$\mathbf{D}_4$: *A_1, A_2, A_3 are elliptic curves.*
$\mathbf{C}_3$: *A_1 is an elliptic curve and A_2 is a simple abelian surface with $[D_2 : \mathbb{Q}] = 2$.*
$\mathbf{G}_2$: *A_1 is a simple abelian threefold with $[D_1 : \mathbb{Q}] = 3$.*

PROOF. Since Δ_F is a Dynkin diagram, we have $\dim_{\mathbb{Q}}(M_i) = 1$ or $\dim_{D_i}(M_i) = 1$. In the former case, we have $g_i = 1$, that is, A_i is an elliptic curve. Moreover, $D_i \hookrightarrow \mathrm{End}_{\mathbb{Q}}(M_i)$ as M_i is a k-D_i-bimodule; thus, $D_i = \mathbb{Q}$. In the latter case, we have $g_i = n_i$. The result follows from these observations via a case-by-case checking. □

REMARKS 3.17. (i) We may view the Dynkin diagrams $\mathbf{D}_4$ and $\mathbf{C}_3$ as unfoldings of $\mathbf{G}_2$. In fact, a similar picture holds for the abelian varieties under consideration: let $A := A_1 \oplus \cdots \oplus A_r$, then $A_1, \ldots, A_r$ satisfy the assertion of Proposition 3.16 if and only if $\dim(A) = 3 = \dim \mathrm{End}_{\underline{\mathcal{V}}}(A)_{\mathbb{Q}}$. In the "general" case where A is simple, this yields type $\mathbf{G}_2$; it "specializes" to types $\mathbf{C_3}$ and $\mathbf{D}_4$.

(ii) When R_F is of finite representation type, its indecomposable modules of finite length are described by the main result of [**DR76**]: the isomorphism classes of such modules correspond bijectively to the positive roots of the root system with Dynkin diagram Δ_F, by assigning with each module its dimension type (the sequence of multiplicities of the simple modules). This yields a case-by-case construction of the indecomposable objects of $\underline{\mathcal{V}}_F$. For example, in type $\mathbf{D}_4$, the indecomposable object associated with the highest root (i.e., with the sequence of multiplicities $m_0 = 2$, $m_1 = m_2 = m_3 = 1$) is the quotient of the universal vector extension $E(A_1 \oplus A_2 \oplus A_3)$ by a copy of $\mathbb{G}_a$ embedded diagonally in $U(A_1 \oplus A_2 \oplus A_3) \cong 3\mathbb{G}_a$. But we do not know any uniform construction of indecomposable objects for all types, along the lines of (i).

(iii) All the abelian varieties A_i over $\mathbb{Q}$ that occur in Proposition 3.16 satisfy the condition that $\mathrm{End}(A_i)_{\mathbb{Q}}$ is a field of dimension equal to $\dim(A_i)$. This condition defines the class of *abelian varieties of* GL_2*-type*, introduced by Ribet in [**Ri92**]; it includes all elliptic curves over $\mathbb{Q}$, and also the abelian varieties associated with certain modular forms via a construction of Shimura (see [**Sh71**, Thm. 7.14]). Assuming a conjecture of Serre on Galois representations, Ribet showed in [**Ri92**] that this construction yields all abelian varieties of GL_2-type up to isogeny.

Examples of abelian varieties of GL_2-type have been obtained by González, Guárdia and Rotger in dimension 2 (see [**GGR05**, Cor. 3.10]), and by Baran in dimension 3 (see [**Ba14**]).

(iv) Still assuming that k is a number field, the question of characterizing finite representation type makes sense, more generally, for the Serre subcategory $\underline{\mathcal{C}}_{E,F} \subset \underline{\mathcal{C}}$ generated by a finite set E of simple linear algebraic groups and a finite set F of simple abelian varieties, pairwise non-isogenous (so that $\underline{\mathcal{C}}_{\mathbb{G}_a,F} = \underline{\mathcal{V}}_F$). The abelian category $\underline{\mathcal{C}}_{E,F}$ is equivalent to $R_{E,F}$-mod, where $R_{E,F}$ is a triangular matrix algebra of finite dimension over $\mathbb{Q}$, constructed as above. The $\mathbb{Q}$-species associated with $R_{E,F}$ is the directed graph $\Gamma_{E,F}$ with vertices $E \sqcup F$ and edges (i,j) for all $i \in E$, $j \in F$ such that $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A_j, L_i) \neq 0$; here A_j (resp. L_i) denotes the corresponding simple abelian variety (resp. linear algebraic group). In particular, if $\mathbb{G}_a \in E$ then the associated vertex is linked to all vertices in F, but some simple tori need not be linked to some simple abelian varieties. Each vertex v is labeled with the division ring D_v opposite to the endomorphism ring of the corresponding simple module, and each edge (i,j) is labeled with the D_i-D_j-bimodule $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A_j, L_i)$. Then again, the category $R_{E,F}$ is equivalent to that of representations of the $\mathbb{Q}$-species $\Gamma_{E,F}$; it is of finite representation type if and only if each connected component of the associated valued graph $\Delta_{E,F}$ is a Dynkin diagram. Note that such a diagram comes with a bipartition (by vertices in E, F).

To obtain a full characterization of finite representation type in this generality, we would need detailed information on the structure of Γ-module of $A(\bar{k})_{\mathbb{Q}}$ for any $A \in F$. But it seems that very little is known on this topic. For example, just take $E := \{\mathbb{G}_m\}$; recall that $\mathrm{Ext}^1_{\underline{\mathcal{C}}}(A, \mathbb{G}_m) \cong A(k)_{\mathbb{Q}}$ for any abelian variety A. We may thus assume that the finitely generated abelian group $A(k)$ is infinite for any $A \in F$; then $\mathbb{G}_m$ is a central vertex of $\Gamma_{\mathbb{G}_m,F}$. Arguing as in the proof of Proposition 3.16, one obtains a similar characterization of finite representation type in terms of Dynkin diagrams satisfying the following conditions:

$\mathbf{D}_4$: $[D_i : \mathbb{Q}] = 3 = \dim A_i(k)_{\mathbb{Q}}$ for $i = 1, 2, 3$.
$\mathbf{C}_3$: $[D_1 : \mathbb{Q}] = 1 = \dim A_1(k)_{\mathbb{Q}}$ and $[D_2 : \mathbb{Q}] = 2 = \dim A_2(k)_{\mathbb{Q}}$.
$\mathbf{G}_2$: $[D_1 : \mathbb{Q}] = 3 = \dim A_1(k)_{\mathbb{Q}}$.

As a consequence, all simple abelian varieties A occuring in F must satisfy $[D_A : \mathbb{Q}] \leq 3$ (in particular, D_A is commutative) and $\dim A(k)_{\mathbb{Q}} = [D_A : \mathbb{Q}]$. We do not know whether such abelian varieties exist in arbitrary large dimensions.

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