Equivariantly normal varieties for diagonalizable group actions

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Abstract

Given a finite group scheme G over a field and a G-variety X, we obtain a criterion for X to be G-normal in the sense of [Br24]. When G is diagonalizable, we describe the local structure of G-normal varieties in codimension 1 and their dualizing sheaf. As an application, we obtain a version of the Hurwitz formula for G-normal varieties, where G is linearly reductive.

1 Introduction

This paper is a sequel to [Br24], which considered actions of finite group schemes over fields of positive characteristic. In this setting, an action on a variety X does not necessarily lift to an action on the normalization $\tilde{X}$; as a remedy, the notion of equivariant normalization was introduced and explored. The present paper focuses on G-normal varieties, where G is a finite diagonalizable group scheme, with two main motivations in mind: firstly, finite diagonalizable group schemes have an especially simple structure and representation theory, which makes them very accessible. Secondly, they are quite ubiquitous; for example, the study of G-normal varieties with G linearly reductive reduces to some extent to the diagonalizable case (see Remark 4.9 for details, and Section 9 for an illustration).

Given a finite diagonalizable group scheme G acting faithfully on a G-normal variety X with categorical quotient $\pi : X \to Y$, the variety $Y = X/G$ is known to be normal and π is a G-torsor over a dense open subset of Y. We may thus view X as a "ramified G-cover" of Y (as a partial converse, every variety obtained as a G-torsor over a normal variety is G-normal). When G is constant, π is indeed a ramified Galois cover with group G and X is normal; this gives back the classically studied abelian covers, see [Pa91]. But the general case has quite different geometric features: if G is infinitesimal, then π is purely inseparable, and hence ramified everywhere.

Moreover, X is generally singular in codimension 1, with cuspidal singularities only. Still, X satisfies Serre's property (S_2), as well as an equivariant version of (R_1): every effective G-stable divisor is Cartier at its generic points.

For simplicity, we now state our main results in the setting of curves over an algebraically closed field k. Let X be such a curve, equipped with a faithful G-action with quotient $\pi : X \to Y$. We assume that X is G-normal; this is equivalent to the (schematic) orbit $G \cdot x \subset X$ being a Cartier divisor for any $x \in X(k)$, and implies that the curve Y is smooth. Denote by $\mathrm{Stab}_G(x) \subset G$ the stabilizer of x; it acts linearly on the fibre at x of the conormal sheaf to $G \cdot x$ via a character $\nu(x) : \mathrm{Stab}_G(x) \to \mathbb{G}_m$. We may now describe the equivariant local structure of X at x:

Theorem 1. (i) *The group scheme $H = \mathrm{Stab}_G(x)$ is cyclic, and $\nu(x)$ generates its character group.*

(ii) *There exists an open G-stable neighborhood $U = U(x) \subset X$ of x such that the quotient morphism $U \to U/G = V$ factors as $U \xrightarrow{\varphi} U/H \xrightarrow{\psi} V$, where φ is a cyclic cover of degree $|H|$, and ψ is a G/H-torsor.*

By a cyclic cover of degree n in this local setting, we mean a morphism of affine schemes $\varphi : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$, where $A = B[T]/(T^n - g)$ and $g \in B$ is a nonzero-divisor. Then φ is the quotient by the cyclic group scheme $\boldsymbol{\mu}_n$ acting on $\mathrm{Spec}(A)$ via the $\mathbb{Z}/n\mathbb{Z}$-grading $A = \bigoplus_{m=0}^{n-1} B\bar{T}^m$; also, φ is a $\boldsymbol{\mu}_n$-torsor outside of the zero scheme of g.

As a consequence of Theorem 1, every G-normal curve is a tamely ramified G-torsor in the sense of [BB22, Def. 2.1]. But equivariant normality imposes additional conditions; for example, a cyclic cover as above is $\boldsymbol{\mu}_n$-normal if and only if B is an integrally closed domain and the zero scheme of g is reduced.

As a further consequence, the finite morphism π is flat and a local complete intersection. So its dualizing sheaf is isomorphic to the relative canonical sheaf $\omega_{X/Y}$, and hence is invertible.

Theorem 2. *The sheaf $\omega_{X/Y}$ is equipped with a G-linearization and a G-invariant global section $s_{X/Y}$ such that*

$$\mathrm{div}(s_{X/Y}) = \sum \big(|\,\mathrm{Stab}_G(x)| - 1 \big)\, G \cdot x$$

(sum over the G-orbits of k-rational points of X).

For a projective G-normal curve X, this readily yields the relation between arithmetic genera

$$2p_a(X) - 2 = |G| \big(2p_a(Y) - 2\big) + \sum \big(|\operatorname{Stab}_G(x)| - 1\big), \tag{1.1}$$

where the sum runs again over the orbits. (We have of course $p_a(Y) = g(Y)$ as Y is smooth; also, $g(X) = g(Y)$ if G is infinitesimal, see [Br24, Rem. 5.2]).

The above results are used in the preprint [FM24] by P. Fong and M. Maccan, which studies a remarkable class of smooth projective surfaces in positive characteristics: relatively minimal surfaces equipped with an isotrivial elliptic fibration. It turns out that every such surface S is obtained as a quotient $(E \times X)/G$, where E is an elliptic curve, $G \subset E$ a finite subgroup scheme, and X a projective G-normal curve. Thus, S is equipped with an action of the algebraic group E for which the quotient is the elliptic fibration $f : S \to X/G = Y$. When G is diagonalizable, the multiple fibers and relative canonical bundle of f are described in terms of the quotient $\pi : X \to Y$ in [FM24, §3], based on Theorems 1 and 2.

Theorem 1 is well-known if G is constant; then X is a smooth curve, the morphism $\psi : U/H \to V$ is étale, and π is a tamely ramified G-cover. In this setting, Theorem 2 and the equality (1.1) follow from the classical Hurwitz formula. For an arbitrary finite group scheme G, the morphism ψ may well be purely inseparable. Still, its dualizing sheaf is equipped with a natural section, and this reduces the proof of Theorem 2 to a local computation. In view of this theorem, the branch locus of π (i.e., the locus where it fails to be a G-torsor) can be read off the relative canonical sheaf $\omega_{X/Y}$.

In the general setting of a G-normal variety X over an arbitrary field, where G is diagonalizable, we obtain versions of Theorems 1 and 2 over an open G-stable subset of X with complement of codimension at least 2; see Theorems 6.2 and 7.4. In this setting, the relative dualizing sheaf ω_π is torsion-free and satisfies (S_2), so that it suffices to determine it in codimension 1; see [Kol22] for further aspects of duality for torsion-free (S_2) sheaves.

The layout of this paper is as follows. Sections 2 and 3 collect preliminary notions and results on actions of finite group schemes over a field k, especially diagonalizable ones. For these, we discuss torsors and uniform cyclic covers in the sense of [AV04] (Remarks 3.2 and 3.3), which form the main ingredients of our local structure results. Our presentation has some overlap with those of [Hau20, Sec. 3] and [CDL24, Sec. 0.2]; we have provided details in order to be self-contained.

Section 4 begins with basic classification and finiteness results for equivariantly normal varieties having a prescribed quotient (Propositions 4.3 and 4.5). We then obtain a G-normality criterion for a finite group scheme G (Proposition 4.10), which takes a simpler form when G is diagonalizable.

Under this assumption, we obtain our local structure theorem by steps: in Section 5, we first describe the fibres of the quotient morphism π at points of codimension 1 for a G-normal variety X, and then the corresponding semi-local rings of X; these may be viewed as equivariant analogs of discrete valuation rings (see Propositions 5.1 and 5.5). From this, we deduce a more global description of X in codimension 1 (Theorem 6.2, the main result of Section 6). In turn, this yields in Section 7 a formula à la Hurwitz for the dualizing sheaf of the quotient morphism π.

Section 8 contains applications to curves; in particular, we determine the arithmetic genus of any projective G-normal curve X, where G is diagonalizable. More generally, we obtain a version of the Chevalley–Weil formula by computing the equivariant Euler characteristic of $\omega_X^{\otimes n}$ for any positive integer n (Proposition 8.3). We refer to [CW34] for the original version of this formula, over the complex numbers, and to [EL80, §3] for a broad generalization to finite groups of order prime to p.

In Section 9, we extend our description of the dualizing sheaf to the case that G is linearly reductive (Theorem 9.1 and Proposition 9.4). We then show that there are finitely many G-normal curves over a prescribed smooth curve and having a prescribed branch divisor, if k is algebraically closed (Theorem 9.3). Finally, we obtain a characterization of the finite linearly reductive group schemes that admit a faithful action on a curve (Proposition 9.5).

Our version of the Chevalley-Weil formula yields numerical information on the canonical ring $R(\omega_X) = \bigoplus_{n=0}^{\infty} H^0(X, \omega_X^{\otimes n})$ and its invariant subring $R(\omega_X)^G$, where G is diagonalizable and X is a projective G-normal curve; see Remark 8.5 for details. If G is a subgroup scheme of an elliptic curve E and k is algebraically closed, then $R(\omega_X)^G$ is the canonical ring of the smooth projective surface $S = (E \times X)/G$ (see [FM24, Lem. 3.7]). This motivates a further exploration of these rings. It would also be interesting to describe the G-equivariant K-theory of such a curve X by extending results of [EL80, Lo83, EL84, DK19], where X is smooth and G is a finite group of order prime to p.

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2 Finite group schemes and their actions

Throughout this paper, we work over a field k of characteristic $p > 0$. Schemes are assumed to be separated and of finite type over k unless explicitly mentioned. A *variety* is a geometrically integral scheme. We say that an open subset U of a variety X is *big* if its complement $X \setminus U$ has codimension at least 2.

We denote by G a finite group scheme and by $|G|$ its order, i.e., the dimension of the algebra $\mathcal{O}(G)$ viewed as a k-vector space. Recall that G lies in a unique exact sequence

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \pi_0(G) \longrightarrow 1, \tag{2.1}$$

where G^0 is infinitesimal and $\pi_0(G)$ is finite and étale. If k is perfect, then this sequence has a unique splitting (see e.g. [DG70, II.5.2.4].

We say that G is *linearly reductive* if every G-module is semi-simple. By Nagata's theorem, this is equivalent to G^0 being of multiplicative type and $|\pi_0(G)|$ being prime to p (see [DG70, IV.3.3.6]).

A *G-scheme* is a scheme X equipped with a G-action

$$\alpha : G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x.$$

This action is said to be *faithful* if every nontrivial subgroup scheme of G acts nontrivially.

For any G-scheme X and any closed subscheme X', the morphism $G \times X' \to X$, $(g, x) \mapsto g \cdot x$ is finite. We denote its schematic image by $G \cdot X'$; this is the smallest closed G-stable subscheme of X containing X'. Also, we denote by $C_G(X')$ the *centralizer* of X', i.e., the largest subgroup scheme of G that acts trivially on X'. In particular, for any closed point $x \in X$, we obtain the (schematic) *orbit* $G \cdot x$ and the centralizer $C_G(x)$. These will be discussed at the end of the next paragraph.

Given a field extension K/k and a scheme X, we denote by X_K the K-scheme obtained from X by the base change $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$. Then G_K is a finite K-group scheme and the formation of (2.1) commutes with any such base change. Moreover, every G-action α on X yields a G_K-action α_K on X_K, which may be identified with a G-action on the k-scheme X_K via the isomorphism

$$G_K \times_{\mathrm{Spec}(K)} X_K \xrightarrow{\sim} G \times X_K.$$

The canonical projection $\mathrm{pr} : X_K \to X$ is equivariant relative to the homomorphism $G_K \to G$. If $x \in X$ is a closed point with residue field $K = \kappa(x)$, then viewing x as a K-point of X_K, the orbit $G_K \cdot x \subset X_K$ has schematic image $G \cdot x$ under the above projection. Moreover, we have an isomorphism of G_K-schemes $G_K \cdot x \simeq G_K/C_{G_K}(x)$ and the equality $C_{G_K}(x) = \mathrm{Stab}_G(x)$, the fibre at x of the stabilizer $\mathrm{Stab}_G \subset G \times X$.

Next, we discuss quotients by G; for this, we only consider G-schemes X that admit a covering by open affine G-stable subsets. This assumption is satisfied if G is infinitesimal (then every open subset is G-stable), or if X is quasi-projective (e.g., a curve). Every such scheme X admits a categorical quotient $\pi : X \to Y = X/G$, where Y is a scheme, π is finite and surjective, and the morphism

$$\gamma = (\alpha, \mathrm{pr}_2) : G \times X \longrightarrow X \times_Y X, \quad (g, x) \longmapsto (g \cdot x, x) \tag{2.2}$$

is surjective as well. Therefore, $\mathcal{A} = \pi_*(\mathcal{O}_X)$ is a coherent algebra over $\mathcal{B} = \mathcal{O}_Y$, equipped with a G-linearization. Moreover, the natural map $\mathcal{B} \to \mathcal{A}^G$ is an isomorphism (see [DG70, III.2.6.1] for these results). If X is a variety, then so is Y.

Given a G-scheme X with quotient $\pi : X \to Y$, and a point $x \in X$ with image $y \in Y$, the (schematic) fiber X_y is a finite $G_{\kappa(y)}$-scheme containing x as a closed point. This defines the orbit $G_{\kappa(y)} \cdot x$; it has the same underlying topological space as X_y, see [DG70, III.2.6.1] again. In particular, this orbit depends only on y; we will denote it by Ω_y. If x is closed in X, then Ω_y may be identified with $G \cdot x$, as $\mathcal{O}(G \cdot x)$ is the image of the natural homomorphism $\mathcal{O}(X_y) \to \mathcal{O}(G) \otimes_k \kappa(x)$, and the right-hand side is identified with $\mathcal{O}(G_{\kappa(y)}) \otimes_{\kappa(y)} \kappa(x)$.

Also, recall that X has a largest open G-stable subset X_{fr} on which G acts freely. Moreover, π restricts to a G-torsor $X_{\mathrm{fr}} \to Y_{\mathrm{fr}}$, where Y_{fr} is open in Y (see loc. cit.). We say that a G-variety X is *generically free* if X_{fr} is nonempty.

The quotient morphism π is the composition

$$X \xrightarrow{\varphi} Z = X/G^0 \xrightarrow{\psi} Y, \tag{2.3}$$

where φ (resp. ψ) denotes the quotient by G^0 (resp. $\pi_0(G)$). Moreover, φ is purely inseparable, and hence a universal homeomorphism. The formation of π, φ and ψ commutes with flat base change $Y' \to Y$; in particular, with field extensions.

Lemma 2.1. *Assume that X is a G-variety. Then the G-action on X is faithful (resp. generically free, free) if and only if the G^0-action on X is faithful (resp. generically free, free) and the $\pi_0(G)$-action on X/G^0 is faithful (resp. free).*

Proof. By fpqc descent, we may assume k algebraically closed. Then $G = G^0 \rtimes \pi_0(G)$ and hence $G(k) \xrightarrow{\sim} \pi_0(G)(k)$; moreover, $\pi_0(G)$ is constant.

Denote by H the kernel of the G-action on X. Then also $H \simeq H^0 \rtimes \pi_0(H)$, and $\pi_0(H)$ is constant. Moreover, H^0 (resp. $\pi_0(H)$) is the kernel of the action of G^0

(resp. $\pi_0(G)$) on X. Since φ is bijective, it follows that $\pi_0(H)$ is the kernel of the $\pi_0(G)$-action on Z. This readily implies the assertion on faithful actions.

The assertion on free actions is obtained similarly by considering the stabilizers of k-rational points of X. This assertion implies that the G-action on X is generically free if and only if so are the G^0-action on X and the $\pi_0(G)$-action on Z. But for an action of a finite group (that is, a finite constant group scheme) on a variety, being generically free is equivalent to being faithful; this completes the proof. □

Lemma 2.2. *Let X be a G-variety with quotient $\pi : X \to Y$, and let $X' \subset X$ be a closed G-stable subscheme.*

(i) *The quotient morphism $\pi' : X' \to Y' = X'/G$ exists and lies in a commutative square*

$$\begin{array}{ccc} X' & \xrightarrow{\;i\;} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{\;j\;} & Y, \end{array}$$

where i denotes the inclusion, and j is finite and purely inseparable. If G is linearly reductive, then j is a closed immersion.

(ii) *For any open subset $U \subset X$ such that $U \cap X'$ is dense in X', there exists an open G-stable subset $V \subset U$ such that $V \cap X'$ is dense in X'.*

Proof. (i) Since $X' \subset X$ is covered by G-stable open affine subsets, it admits a categorical quotient $\pi' : X' \to Y'$. Moreover, the universal property of π yields a unique morphism $j : Y' \to Y$ such that $j \circ \pi' = i \circ \pi$; as a consequence, $j \circ \pi'$ is finite. Since π' is dominant, it follows that j is finite. Also, for any algebraically closed field extension K/k, the map $j(K) : Y'(K) \to Y(K)$ is injective, since $Y'(K)$ is the orbit space $X'(K)/G(K)$ and likewise $Y(K) = X(K)/G(K)$. So j is purely inseparable.

To show the final assertion, we may assume X affine. Then the restriction map $\mathcal{O}(X) \to \mathcal{O}(X')$ is surjective, and hence so is $\mathcal{O}(X)^G \to \mathcal{O}(X')^G$ by linear reductivity. Equivalently, $j^\# : \mathcal{O}(Y) \to \mathcal{O}(Y')$ is surjective as desired.

(ii) If G is constant, then it permutes the generic points $x_1, \ldots, x_n$ of X'. By assumption, $U \cap X' \supset \{x_1, \ldots, x_n\}$ and hence $V = \bigcap_{g \in G} g \cdot U$ is an open G-stable subset of X such that $V \cap X' \supset \{x_1, \ldots, x_n\}$. This yields our assertion in this case, and hence in the case that G is étale by Galois descent. The general case follows from this by using the factorization (2.3) of π. □

With the assumption of Lemma 2.2, we denote by $x_1, \ldots, x_n$ the generic points of X' and let

$$\mathcal{O}_{X,X'} = \mathcal{O}_{X,x_1} \cap \cdots \cap \mathcal{O}_{X,x_n}. \tag{2.4}$$

This is a semi-local ring, and its maximal ideals correspond bijectively to $x_1, \ldots, x_n$.

Lemma 2.3. *With the above notation, the ring $\mathcal{O}_{X,X'}$ is equipped with a (functorial) G-action, and is a union of finite-dimensional G-submodules.*

Proof. We have $\mathcal{O}_{X,X'} = \varinjlim \mathcal{O}(U)$, where U runs over the open subsets of X containing $x_1, \ldots, x_n$. By Lemma 2.2, it follows that $\mathcal{O}_{X,X'} = \varinjlim \mathcal{O}(V)$, where V runs over the above open subsets which are G-stable. This readily yields the assertions. □

In what follows, we will consider G-varieties with a prescribed quotient Y, that is, pairs (X, π), where X is a G-variety and $\pi : X \to Y$ is the quotient morphism. Assuming in addition that the G-action is faithful, we say that X is a *G-variety over Y*. Morphisms of such pairs are G-equivariant morphisms of varieties over Y; in particular, the automorphism group of (X, π) is the group $\mathrm{Aut}_Y^G(X)$ of equivariant relative automorphisms. We now obtain a description of this group, under assumptions that will be fulfilled by diagonalizable groups.

Lemma 2.4. *Let Y be a variety, and X a G-variety over Y.*

(i) *The quotient morphism $\varphi : X \to X/G^0 = Z$ induces an injective homomorphism $\varphi_* : \mathrm{Aut}_Y^{G^0}(X) \to \mathrm{Aut}_Y(Z)$.*

(ii) *The natural homomorphism $\pi_0(G)(k) \to \mathrm{Aut}_Y(Z)$ is bijective.*

(iii) *If G is commutative and the exact sequence (2.1) splits, then*

$$G(k) \xrightarrow{\sim} \mathrm{Aut}_Y^{G^0}(X) = \mathrm{Aut}_Y^G(X).$$

Proof. The existence of the homomorphism φ_* follows from the universal property of the categorical quotient. Clearly, the kernel of φ_* is isomorphic to $\mathrm{Aut}_Z(X)$, and hence is a subgroup of $\mathrm{Aut}_{k(Z)}(k(X))$. Since the extension of function fields $k(X)/k(Z)$ is purely inseparable, it follows that φ_* is injective, proving (i).

By Lemma 2.1, the action of $\pi_0(G)$ on Z is generically free. Therefore, we may identify $\mathrm{Aut}_Y(Z)$ with a subgroup of $\mathrm{Aut}_{Y_{\mathrm{fr}}}(Z_{\mathrm{fr}})$, where $Z_{\mathrm{fr}} \to Y_{\mathrm{fr}}$ is a $\pi_0(G)$-torsor. Moreover, we have natural isomorphisms

$$\mathrm{Aut}_{Y_{\mathrm{fr}}}(Z_{\mathrm{fr}}) \simeq \mathrm{Hom}(Z_{\mathrm{fr}}, \pi_0(G)) \simeq \mathrm{Hom}(\pi_0(Z_{\mathrm{fr}}), \pi_0(G)) \simeq \pi_0(G)(k),$$

where the second one follows from [DG70, I.4.6.5], and the third one holds since Z_{fr} is a variety and hence $\pi_0(Z_{\mathrm{fr}}) = \mathrm{Spec}(k)$. This implies (ii).

Under the assumptions of (iii), the group $G(k)$ acts on X by G-equivariant automorphisms over Y, and $G(k) \xrightarrow{\sim} \pi_0(G)(k)$. This yields the desired assertion by combining (i) and (ii). □

3 Diagonalizable group schemes

We now assume that G is diagonalizable, and denote by $\Lambda = \Lambda(G) = \mathrm{Hom}_{\mathrm{gp}}(G, \mathbb{G}_m)$ its character group. Recall that Λ is a finite abelian group, and we have a Λ-grading

$$\mathcal{O}(G) = \bigoplus_{\lambda \in \Lambda} k\lambda. \tag{3.1}$$

More generally, every G-module M has a Λ-grading

$$M = \bigoplus_{\lambda \in \lambda} M_\lambda,$$

where the λ-weight space M_λ consists of the $m \in M$ such that $g \cdot m = \lambda(g)m$ identically on $g \in G$. This gives an equivalence between the category of G-modules and that of Λ-graded vector spaces.

We say that G is *cyclic* if $\Lambda \simeq \mathbb{Z}/n\mathbb{Z}$ for some positive integer n; equivalently, G is isomorphic to the group scheme $\boldsymbol{\mu}_n$ of nth roots of unity.

Every diagonalizable group scheme G is isomorphic (non-canonically) to a finite product of cyclic group schemes. It follows that the exact sequence (2.1) has a unique splitting: $G = G^0 \times \pi_0(G)$, where G^0 (resp. $\pi_0(G)$) is a finite product of cyclic group schemes of order a power of p (resp. prime to p). Also, the subgroup schemes H of G correspond bijectively to the subgroups of Λ via $H \mapsto \Lambda(G/H) = \Lambda^H$. As a consequence, for any field extension K/k, the map $H \mapsto H_K$ yields a bijection between the subgroup schemes of G and those of G_K.

Given a G-scheme X with quotient $\pi : X \to Y$, the coherent G-linearized sheaf $\mathcal{A}$ is Λ-graded:

$$\mathcal{A} = \bigoplus_{\lambda \in \Lambda} \mathcal{A}_\lambda, \tag{3.2}$$

where $\mathcal{A}_0 = \mathcal{B}$ and each $\mathcal{A}_\lambda$ is a coherent $\mathcal{B}$-module. Moreover, the multiplication of $\mathcal{A}$ induces morphisms of $\mathcal{B}$-modules

$$\mathrm{mult}_{\lambda,\mu} : \mathcal{A}_\lambda \otimes_{\mathcal{B}} \mathcal{A}_\mu \longrightarrow \mathcal{A}_{\lambda+\mu} \tag{3.3}$$

for all $\lambda, \mu \in \Lambda$, and

$$\sigma_{\lambda,n} : \mathcal{A}_\lambda^{\otimes n} \longrightarrow \mathcal{A}_{n\lambda} \tag{3.4}$$

for all such λ and all integers $n \geq 1$. The formation of $\mathcal{A}_\lambda$, $\mathrm{mult}_{\lambda,\mu}$ and $\sigma_{\lambda,n}$ commutes with arbitrary base change $Y' \to Y$.

Lemma 3.1. *With the above notation, π is a G-torsor if and only if* $\mathrm{mult}_{\lambda,\mu}$ *is an isomorphism for all $\lambda, \mu \in \Lambda$. Under these assumptions, each $\sigma_{\lambda,n}$ is an isomorphism as well.*

Proof. The first assertion follows from [SGA3, Exp. VIII, Prop. 4.1]; we present a proof for completeness. If π is the trivial torsor $\mathrm{pr}_Y : G \times Y \to Y$, then $\mathcal{A}$ is identified with $\mathcal{B} \otimes_k \mathcal{O}(G)$ as a $\mathcal{B}$-G-algebra, where G acts on $\mathcal{O}(G)$ via its action on itself by multiplication. Using the decomposition (3.1), this identifies each multiplication map $\mathrm{mult}_{\lambda,\mu}$ with the map $\mathcal{B} \otimes_k k\lambda \otimes_k k\mu \to \mathcal{B} \otimes_k k(\lambda + \mu)$ induced by the multiplication in $\mathcal{O}(G)$. As a consequence, $\mathrm{mult}_{\lambda,\mu}$ is an isomorphism. By fpqc descent, this still holds if π is an arbitrary G-torsor.

Conversely, assume that each $\mathrm{mult}_{\lambda,\mu}$ is an isomorphism. Then the multiplication map induces an isomorphism $\mathcal{A}_\lambda \otimes_{\mathcal{B}} \mathcal{A}_{-\lambda} \xrightarrow{\sim} \mathcal{B}$ for any $\lambda \in \Lambda$. In particular, every $\mathcal{A}_\lambda$ is invertible, and hence π is finite and flat. To show that it is a G-torsor, it suffices to check that the morphism (2.2) is an isomorphism. As in the proof of Lemma 2.4, we may assume that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Then γ corresponds to the algebra homomorphism

$$\gamma^\# : A \otimes_B A \longrightarrow \mathcal{O}(G) \otimes_k A, \quad a_1 \otimes a_2 \longmapsto (g \mapsto (g \cdot a_1)a_2),$$

where $\mathcal{O}(G) \otimes_k A$ is identified with the space of morphisms $G \to A$. Therefore, $\gamma^\# = \bigoplus_{\lambda,\mu \in \Lambda} \gamma^\#_{\lambda,\mu}$, where $\gamma^\#_{\lambda,\mu} : A_\lambda \otimes_B A_\mu \to k\lambda \otimes_k A_{\lambda+\mu}$ satisfies $\gamma^\#_{\lambda,\mu} = \lambda \otimes \mathrm{mult}_{\lambda,\mu}$. Thus, each $\gamma^\#_{\lambda,\mu}$ is an isomorphism, and hence so is γ.

This completes the proof of the first assertion ; the second assertion is obtained by a similar descent argument. □

Remark 3.2. The above lemma takes a much more concrete form when $G = \boldsymbol{\mu}_n$. Then for any G-torsor $\pi : X \to Y$, we have

$$\mathcal{A} = \bigoplus_{m=0}^{n-1} \mathcal{A}_m,$$

where each $\mathcal{A}_m$ is an invertible $\mathcal{B}$-module. The multiplication map yields isomorphisms $\sigma_{1,m} : \mathcal{A}_1^{\otimes m} \xrightarrow{\sim} \mathcal{A}_m$ for $m = 1, \dots, n-1$, together with an isomorphism

$\sigma = \sigma_{1,n} : \mathcal{A}_1^{\otimes n} \xrightarrow{\sim} \mathcal{B}$. Moreover, one may easily check that the resulting homomorphism of $\mathcal{B}$-algebras $\mathrm{Sym}_{\mathcal{B}}(\mathcal{A}_1) \to \mathcal{A}$ factors through an isomorphism

$$\mathrm{Sym}_{\mathcal{B}}(\mathcal{A}_1)/\mathcal{I} \xrightarrow{\sim} \mathcal{A},$$

where $\mathcal{I}$ denotes the ideal of $\mathrm{Sym}_{\mathcal{B}}(\mathcal{A}_1)$ generated by the $a_1 \cdots a_n - \sigma(a_1 \otimes \cdots \otimes a_n)$, where $a_1, \ldots, a_n \in \mathcal{A}_1$.

Conversely, given an invertible sheaf $\mathcal{L}$ on Y together with a trivializing section $\sigma \in \Gamma(Y, \mathcal{L}^{\otimes n}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{L}^{\otimes -n}, \mathcal{O}_Y)$, the $\mathcal{B}$-algebra $\mathcal{A} = \mathrm{Sym}_{\mathcal{B}}(\mathcal{L}^{\otimes -1})/\mathcal{I}$ (where $\mathcal{I}$ is defined as above) corresponds to a $\boldsymbol{\mu}_n$-torsor satisfying $\mathcal{A}_1 = \mathcal{L}^{\otimes -1}$. This gives back a well-known classification of $\boldsymbol{\mu}_n$-torsors, see e.g. [DG70, III.4.5.6]. It may be extended to torsors under diagonalizable group schemes with only notational difficulties.

Remark 3.3. More generally, consider a scheme Y equipped with an invertible sheaf $\mathcal{L}$ and a section $\sigma \in \Gamma(Y, \mathcal{L}^{\otimes n})$, where n is a positive integer (we no longer assume that σ trivializes $\mathcal{L}$). Denote again by $\mathcal{I}$ the ideal of $\mathrm{Sym}_{\mathcal{B}}(\mathcal{L}^{\otimes -1})$ generated by the $a_1 \cdots a_n - \sigma(a_1 \otimes \cdots \otimes a_n)$, where $a_1, \ldots, a_n \in \mathcal{L}^{\otimes -1}$, and let $\mathcal{A} = \mathrm{Sym}_{\mathcal{B}}(\mathcal{L}^{\otimes -1})/\mathcal{I}$. Then $\mathcal{A}$ is $\mathbb{Z}/n\mathbb{Z}$-graded and $\mathcal{A}_m = \mathcal{L}^{\otimes -m}$ for $m = 0, \ldots, n-1$. So $X = \mathrm{Spec}_{\mathcal{B}}(\mathcal{A})$ is a $\boldsymbol{\mu}_n$-scheme with quotient Y, and $\sigma = \sigma_{1,n}$; moreover, the quotient morphism $\pi : X \to Y$ is flat.

Denote by E the zero scheme of σ; this is an effective Cartier divisor on Y. Moreover, the ideal sheaf $\mathcal{J} \subset \mathcal{A}$ generated by $\mathcal{L}^{\otimes -1}$ is homogeneous and satisfies $\mathcal{A}/\mathcal{J} \simeq \mathcal{O}_Y/\sigma\mathcal{O}_Y = \mathcal{O}_Y/\mathcal{O}_Y(-E)$. Thus, $\mathcal{J}$ corresponds to an effective $\boldsymbol{\mu}_n$-stable Cartier divisor D on X, sent isomorphically to E by π. Moreover, D is the fixed point subscheme $X^{\boldsymbol{\mu}_n}$, and π restricts to a $\boldsymbol{\mu}_n$-torsor $X \setminus \mathrm{Supp}(D) \to Y \setminus \mathrm{Supp}(E)$. We have $\pi^*\mathcal{O}_Y(-E) = \mathcal{O}_X(-nD)$ in view of the definition of $\mathcal{A}$; equivalently, $\pi^*(E) = nD$.

The above construction is classical if n is prime to p; then $\pi : X \to Y$ is étale outside of D, and totally ramified along D. For an arbitrary n, this yields exactly the *uniform cyclic covers* considered in [AV04]. These are also known as *simple cyclic covers*, see [CDL24, Sec. 0.2].

Lemma 3.4. *Let X be a G-variety.*

(i) *Every nonzero $\mathcal{A}_\lambda$ is a torsion-free $\mathcal{B}$-module of rank 1, and the set $\Lambda(X) = \{\lambda \in \Lambda \mid \mathcal{A}_\lambda \neq 0\}$ is a subgroup of Λ.*

(ii) *The quotient morphism π is flat if and only if every nonzero $\mathcal{A}_\lambda$ is invertible.*

(iii) *The G-action on X is faithful if and only if $\Lambda(X) = \Lambda$; under these assumptions, the action is generically free.*

Proof. (i) We may assume again that $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. Then $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$ is a Λ-graded domain, hence B is a domain and each A_λ is a torsion-free B-module. Denoting by K (resp. L) the fraction field of A (resp. B), we have $K = A \otimes_B L$ (see e.g. [Br24, Lem. 2.3 (ii)]) and hence $K = \bigoplus_{\lambda \in \Lambda} K_\lambda$, where $K_0 = L$ and $K_\lambda = A_\lambda \otimes_B L$. Thus, $\lambda \in \Lambda(X)$ if and only if $K_\lambda \neq 0$; then K_λ is a 1-dimensional L-vector space. Moreover, if K_λ and K_μ are nonzero, then also $K_{\lambda-\mu}$ is nonzero. This implies the assertion.

(ii) Since π is finite, its flatness is equivalent to the local freeness of the $\mathcal{B}$-module $\mathcal{A}$, and hence to the local freeness of each $\mathcal{A}_\lambda$. Together with (i), this yields the assertion.

(iii) Since X is affine, the G-action is faithful if and only if so is its linear representation in A; equivalently, the intersection of the kernels of the characters $\lambda \in \Lambda(X)$ is trivial. In turn, this is equivalent to the equality $\Lambda(X) = \Lambda$. We may then choose finitely many homogeneous elements $f_1, \ldots, f_n \in A$ of respective weights $\lambda_1, \ldots, \lambda_n$ which generate the group Λ. Then $U = X_{f_1} \cap \cdots \cap X_{f_n}$ is a dense open G-stable subset of X, equipped with a G-equivariant morphism $(f_1, \ldots, f_n) : U \to \mathbb{G}_m^n$ where G acts on $\mathbb{G}_m^n$ via $g \cdot (t_1, \ldots, t_n) = (\lambda_1(g)t_1, \ldots, \lambda_n(g)t_n)$. Since the latter action is free, we see that G acts freely on U. □

4 Equivariantly normal varieties

Let G be a finite group scheme, and X a G-variety. Following [Br24, Def. 4.1], we say that X is G-*normal* if every finite birational morphism of G-varieties $X' \to X$ is an isomorphism.

By loc. cit., §4, this notion of equivariant normality is related to the usual notion of normality as follows: we may view G as a closed subgroup scheme of a smooth connected algebraic group $G^\#$ (for example, we may embed G in GL_n via the regular representation, where $n = |G|$). Given a G-variety X with quotient $\pi : X \to Y$, the product $G^\# \times X$ is a variety equipped with an action of $G^\# \times G$ via $(a, g) \cdot (b, x) = (abg^{-1}, g \cdot x)$. Since $G^\#$ and X are covered by open affine G-stable subsets, the quotient $X^\# = (G^\# \times X)/G$ exists. This is a variety equipped with an action of $G^\#$ and two morphisms $\varphi : X^\# \to Y$, $\psi : X^\# \to G^\#/G$. Moreover, φ is the categorical quotient by $G^\#$, and ψ is faithfully flat and $G^\#$-equivariant; the fibre of ψ at the base point of the homogeneous space $G^\#/G$ is G-equivariantly isomorphic to X. We say that $X^\#$ is the *induced variety* $G^\# \times^G X$.

Lemma 4.1. *With the above notation, X is G-normal if and only if $X^\#$ is normal; then Y is normal as well.*

This is proved by the same argument as [Br24, Lem. 4.7], which deals with the quotient $(G^{\#} \times X)/G^0$. As an application, we show:

Lemma 4.2. *Let Y be a variety, and $u : X \to X'$ a morphism of G-varieties over Y. If X' is G-normal and generically free, then u is an isomorphism.*

Proof. Denote by η the generic point of Y; then u induces a $G_{\kappa(y)}$-equivariant morphism $u_\eta : X_\eta \to X'_\eta$ between generic fibres. Moreover, X'_η is a $G_{\kappa(y)}$-torsor by assumption, and X_η is $G_{\kappa(y)}$-homogeneous in view of [Br24, Cor. 2.5]. Thus, u_η is an isomorphism, and hence u is birational. Also, u is quasi-finite, since $\pi = \pi' \circ u$ is finite. We now consider the induced varieties $X^{\#}$, $X'^{\#}$ and the induced morphism $u^{\#} : X^{\#} \to X'^{\#}$, which is still birational and quasi-finite. Since $X'^{\#}$ is normal (Lemma 4.1), $u^{\#}$ is an open immersion by Zariski's Main Theorem. Thus, u is an open immersion.

To complete the proof, it suffices to check that u is finite. For this, we may assume that Y is affine. Then X and X' are affine, and hence so is u. Moreover, $\mathcal{O}(X)$ is a finite $\mathcal{O}(X')$-module via $u^{\#}$, since it is a finite module over $\mathcal{O}(Y)$. This yields the desired assertion. □

We now consider a generically free G-normal variety X over a prescribed variety Y (which must be normal by Lemma 4.1). Denote by $L = k(Y)$ the corresponding function field, and by $\eta = \mathrm{Spec}(L)$ the generic point of Y. Then the generic fiber X_η is a G_L-torsor over η; moreover, we have $X_\eta = \mathrm{Spec}(K)$, where K denotes the function field $k(X)$ (see e.g. [Br24, Lem. 2.3]).

Proposition 4.3. *With the above notation, the assignement $X \mapsto X_\eta$ yields an equivalence between the category of generically free G-normal varieties over Y and the category of G_L-torsors $\mathrm{Spec}(K) \to \mathrm{Spec}(L)$, where K/L is a field extension.*

Proof. Clearly, the assignement $F : X \mapsto X_\eta$ is functorial. We show that it is essentially surjective. Let K/L be a field extension such that $\mathrm{Spec}(K) \to \mathrm{Spec}(L)$ is a G-torsor; equivalently, G acts freely on the k-scheme $\mathrm{Spec}(K)$, and L is the invariant subfield K^G. By [Br24, Cor. 3.3, Cor. 3.4], there exists a G-variety X_0 and a G-equivariant isomorphism $K \simeq k(X_0)$. Replacing X_0 with a dense open G-stable subset, we may assume that it admits a quotient $Y_0 = X_0/G$. Then Y_0 is birational to Y, and hence (replacing again X_0 with a dense G-stable open subset) we may further assume that Y_0 is an open affine subset of Y. We now use the G-normalization X of Y in K, which can be defined as follows. If Y is affine, consider the integral closure $\widetilde{\mathcal{O}(Y)}$ of $\mathcal{O}(Y)$ in K. By [Br24, Rem. 4.11], there exists a largest

G-stable subalgebra $R \subset \widetilde{\mathcal{O}(Y)}$; then $\mathcal{O}(Y) \subset R$ and X is a G-normal variety over Y via the corresponding morphism $\pi : X = \mathrm{Spec}(R) \to Y$. Moreover, the formation of R commutes with localization by multiplicative subsets of $\mathcal{O}(Y)$, and hence we have $\pi^{-1}(Y_0) = X_0$. For an arbitrary Y, we apply the above construction to an open affine covering, and get a G-normal variety X over Y such that $X_\eta = \mathrm{Spec}(K)$.

Next, F is faithful as η is dense in Y. To complete the proof, we show that F is full. Equivalently, given two generically free G-normal varieties $\pi : X \to Y$, $\pi' : X' \to Y$ and a G_L-equivariant morphism $u : X_\eta \to X'_\eta$ over η, we show that u extends to a G-equivariant morphism $X \to X'$ over Y.

Note that u is an isomorphism, and extends to a G-equivariant birational map $f : X \dashrightarrow X'$ over Y. Taking the graph of f and then its G-normalization yields a G-normal variety X'' equipped with birational G-equivariant morphisms $\varphi : X'' \to X$, $\varphi' : X'' \to X'$ such that $\varphi' = f \circ \varphi$ as rational maps. Moreover, the composition $\pi \circ \varphi = \pi' \circ \varphi' : X'' \to Y$ is finite, since $X \times_Y X'$ is finite over Y. So $\pi \circ \varphi$ factors through a finite birational morphism $X''/G \to Y$, which is an isomorphism as Y is normal. Thus, X'' is a G-normal variety over Y, and hence φ, φ' are isomorphisms (Lemma 4.2). This gives the desired extension. □

Likewise, given a finite group scheme G, the assignement $X \mapsto k(X)$ yields an anti-equivalence between the category of G-normal curves and that of function fields of one variable equipped with a (functorial) G-action.

Returning to the setting of Proposition 4.3, we obtain a bijection between the set of isomorphism classes of G-normal varieties over Y and the subset of $H^1_{\mathrm{fppf}}(L, G)$ parameterizing G_L-torsors $\mathrm{Spec}(K) \to \mathrm{Spec}(L)$, where K/L is a field extension. Here $H^1_{\mathrm{fppf}}(L, G)$ denotes the set of isomorphism classes of G_L-torsors over $\mathrm{Spec}(L)$; it has a natural structure of abelian group if G is commutative.

We now state a variant of Proposition 4.3, which is proved by similar arguments:

Proposition 4.4. *Let Y be a normal variety, and $V \subset Y$ a dense open subset. Then the assignement $X \to X_V$ yields an equivalence between the category of G-normal varieties X over Y such that $V \subset Y_{\mathrm{fr}}$ and the category of G-torsors over V.*

As a consequence, the isomorphism classes of G-normal varieties over Y such that $V \subset Y_{\mathrm{fr}}$ are in bijection with those of G-torsors $U \to V$, where U is a variety. Since G is commutative, the set $H^1_{\mathrm{fppf}}(V, G)$ (consisting of the isomorphism classes of G-torsors over V) has a natural structure of an abelian group, see e.g. [DG70, III.4.4.3].

Proposition 4.5. *Let G be a finite diagonalizable group scheme, and V a normal variety. Then the group $H^1_{\mathrm{fppf}}(V, G)$ is finite if either k is algebraically closed, or k is perfect and G is infinitesimal.*

Proof. Since G is isomorphic to a finite product of cyclic groups, we may assume that $G = \boldsymbol{\mu}_n$. Then there is an exact sequence

$$0 \longrightarrow \mathcal{O}(V)^\times/(\mathcal{O}(V)^\times)^n \longrightarrow H^1_{\mathrm{fppf}}(V, G) \longrightarrow \mathrm{Pic}(V)[n] \longrightarrow 0,$$

where $\mathrm{Pic}(V)[n]$ denotes the n-torsion subgroup of the Picard group (see [DG70, III.4.5.6]). Moreover, we have an exact sequence

$$0 \longrightarrow k^\times \longrightarrow \mathcal{O}(V)^\times \longrightarrow \mathrm{U}(V) \longrightarrow 0,$$

where the abelian group $\mathrm{U}(V)$ is free of finite rank: indeed, this holds for k algebraically closed by [Ro61, Thm. 1], and the general case follows as $\mathcal{O}(V)^\times \subset \mathcal{O}(V_{\bar{k}})^\times$ and $k^\times = \bar{k}^\times \cap \mathcal{O}(V)^\times$.

If k is algebraically closed, then $k^\times = (k^\times)^n$ and hence

$$\mathcal{O}(V)^\times/(\mathcal{O}(V)^\times)^n \xrightarrow{\sim} \mathrm{U}(V)/n\mathrm{U}(V).$$

As the latter group is finite, it remains to prove that $\mathrm{Pic}(V)[n]$ is finite. For this, we show the finiteness of the n-torsion in the divisor class group $\mathrm{Cl}(V)$. We may choose a normal projective completion $V \subset \bar{V}$. Then we have an exact sequence

$$0 \longrightarrow \Gamma \longrightarrow \mathrm{Cl}(\bar{V}) \longrightarrow \mathrm{Cl}(V) \longrightarrow 0,$$

where the abelian group Γ is finitely generated (by the classes of irreducible components of codimension 1 of $\bar{V} \setminus V$). This yields an exact sequence

$$0 \longrightarrow \Gamma[n] \longrightarrow \mathrm{Cl}(\bar{V})[n] \longrightarrow \mathrm{Cl}(V)[n] \longrightarrow \Gamma/n\Gamma,$$

where $\Gamma/n\Gamma$ is finite. So it suffices to check that $\mathrm{Cl}(\bar{V})[n]$ is finite. In other words, we may assume that V is projective. We then have an exact sequence

$$0 \longrightarrow A(k) \longrightarrow \mathrm{Cl}(V) \longrightarrow \mathrm{NS}(V) \longrightarrow 0,$$

where A is an abelian variety (the dual of the Albanese variety of V) and the Néron-Severi group $\mathrm{NS}(V)$ is finitely generated; see [Be13, p. 3] and the references therein. So we get an exact sequence

$$0 \longrightarrow A(k)[n] \longrightarrow \mathrm{Cl}(V)[n] \longrightarrow \mathrm{NS}(V)[n] \longrightarrow A(k)/nA(k).$$

As $A(k)[n]$ is finite and $A(k) = nA(k)$, this completes the proof in the case that k is algebraically closed.

If k is perfect and G is infinitesimal, then we may assume that $G = \boldsymbol{\mu}_{p^r}$ and we argue similarly by using the equality $k^\times = (k^\times)^{p^r}$ and the injectivity of the pull-back map $\mathrm{Pic}(V) \to \mathrm{Pic}(V_{\bar{k}})$. □

Combining Propositions 4.4 and 4.5, we obtain:

Corollary 4.6. *Let G be a diagonalizable finite group scheme, Y a normal variety, and $V \subset Y$ a dense open subset. Assume either that k is algebraically closed, or that it is perfect and G is infinitesimal. Then there are finitely many isomorphism classes of G-normal varieties over Y that are free over V.*

Under the above assumptions, the automorphism group of any G-variety over Y is isomorphic to $G(k)$ by Lemma 2.4. In particular, this group is trivial if G is infinitesimal.

Also, note that Corollary 4.6 does not extend to infinitesimal unipotent group schemes over a perfect field k. For example, if $f \in \mathcal{O}(Y)$ then the zero subscheme of $T^p - f \in \mathcal{O}(Y \times \mathbb{A}^1) = \mathcal{O}(Y)[T]$ yields an $\boldsymbol{\alpha}_p$-torsor $X \to Y$, where $\boldsymbol{\alpha}_p$ denotes the Frobenius kernel of the additive group. If in addition $f \notin \mathcal{O}(Y)^p$ then X is a variety, and hence is $\boldsymbol{\alpha}_p$-normal. As the resulting map $\mathcal{O}(Y) \to H^1_{\mathrm{fppf}}(Y, \boldsymbol{\alpha}_p)$ is a group homomorphism with kernel $\mathcal{O}(Y)^p$ (see e.g. [DG70, III.4.5.1]), we obtain infinitely many isomorphism classes of $\boldsymbol{\alpha}_p$-torsors $X \to Y$ where X is a variety, by taking for Y an affine variety of positive dimension.

Remark 4.7. With the notation and assumptions of Proposition 4.3, the group $H^1_{\mathrm{fppf}}(L, G)$ can be explicitly determined if (say) k is algebraically closed. More specifically, we may assume that $G = \boldsymbol{\mu}_n$, so that $H^1_{\mathrm{fppf}}(L, G) = L^\times/(L^\times)^n$ by [DG70, III.4.5.6]; we may further assume that Y is projective. We then have an exact sequence

$$0 \longrightarrow L^\times/k^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(Y) \longrightarrow \mathrm{Cl}(Y) \longrightarrow 0,$$

where $\mathrm{Div}(Y)$ denotes the free abelian group of Weil divisors on Y. Since the groups $L^\times/k^\times$ and $\mathrm{Div}(Y)$ are torsion-free, this yields an exact sequence

$$0 \longrightarrow \mathrm{Cl}(Y)[n] \longrightarrow L^\times/(L^\times)^n \xrightarrow{\mathrm{div}_n} \mathrm{Div}(Y)/n\,\mathrm{Div}(Y) \longrightarrow \mathrm{Cl}(Y)/n\,\mathrm{Cl}(Y) \longrightarrow 0$$

with an obvious notation. Moreover, we obtain an isomorphism

$$\mathrm{Cl}(Y)/n\,\mathrm{Cl}(Y) \xrightarrow{\sim} \mathrm{NS}(Y)/n\,\mathrm{NS}(Y)$$

by arguing as in the proof of Proposition 4.5. So we get an exact sequence

$$0 \longrightarrow \mathrm{Cl}(Y)[n] \longrightarrow L^\times/(L^\times)^n \xrightarrow{\mathrm{div}_n} \mathrm{Div}(Y)/n\,\mathrm{Div}(Y) \longrightarrow \mathrm{NS}(Y)/n\,\mathrm{NS}(Y) \longrightarrow 0,$$

where the first and last groups are finite.

In the case that Y is a normal projective curve, this yields an exact sequence

$$0 \longrightarrow \mathrm{Pic}(Y)[n] \xrightarrow{i} L^\times/(L^\times)^n \xrightarrow{\mathrm{div}_n} \bigoplus_{y \in Y(k)} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\deg} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

where i may be viewed as the pull-back map $H^1_{\mathrm{fppf}}(Y, \boldsymbol{\mu}_n) \to H^1_{\mathrm{fppf}}(L, \boldsymbol{\mu}_n)$. This map is described as follows: to each $\boldsymbol{\mu}_n$-torsor $\pi : X \to Y$, one associates an invertible sheaf $\mathcal{L}$ on Y and a trivializing section $\sigma \in \Gamma(Y, \mathcal{L}^n)$ as in Remark 3.2. A trivialization of $\mathcal{L}$ at η identifies σ with some $f \in L^\times$, and changing the trivialization replaces f with $f\,g^n$ for some $g \in L^\times$.

Moreover, div_n is the reduction mod n of the divisor map $L^\times \to \bigoplus_{y \in Y(k)} \mathbb{Z}$, $f \mapsto (\mathrm{ord}_y(f), y \in Y(k))$, and deg is the sum map.

Finally, given $f \in L^\times$, the associated $\boldsymbol{\mu}_n$-torsor (as in [DG70, III.4.5.6] again) is the generic fiber of a $\boldsymbol{\mu}_n$-normal variety if and only if the polynomial $T^n - f$ is irreducible in $L[T]$.

Remark 4.8. A finite group scheme G is diagonalizable if and only if it is isomorphic to a subgroup scheme of a torus $T \simeq \mathbb{G}_m^n$. Then the assignement $X \mapsto X^\# = T \times^G X$ defines a bijection between G-normal varieties and normal T-varieties equipped with an equivariant morphism φ to the quotient torus T/G, whose fibre at the base point is geometrically integral. The dimension of X is the *complexity* of the T-variety $X^\#$ (the transcendence degree of the field of T-invariant rational functions).

Given a torus T, there is a classification of normal T-varieties in terms of objects mixing convex and algebraic geometry, called proper polyhedral divisors and divisorial fans, which generalize the cones and fans associated with toric varieties. Over an algebraically closed field k of characteristic 0, this classification is obtained in [AH06] for affine varieties; the general case is treated in [AHS08]. Over an arbitrary field, normal affine T-varieties of complexity 1 are also classified by proper polyhedral divisors, as shown in [Lan15]. Yet we do not know how to deduce from this a full classification of G-normal varieties, which incorporates the additional datum of the morphism φ.

Remark 4.9. Returning to an arbitrary finite group scheme G, note that a G-variety X is G-normal if and only if X_K is G_K^0-normal for some separable field extension

K/k (see [Br24, Cor. 4.8, Prop. 4.9]). If G is linearly reductive, we may take for K/k a finite Galois extension such that G_K^0 is diagonalizable and $\pi_0(G)_K$ is constant. Then the quotient morphism $X_K \to Y_K$ factors uniquely as $X_K \to Z_K \to Y_K$, where $Z_K = X_K/G_K^0$ is a normal variety and $Z_K \to Y_K$ is a finite Galois cover with group $\pi_0(G)(K)$, of order prime to p. This reduces a number of questions on equivariantly normal varieties under linearly reductive group schemes to the diagonalizable case; see the final Section 9 for an illustration.

Next, we obtain a criterion for G-normality of a G-variety X in terms of the $\mathcal{B}$-algebra $\mathcal{A}$, with the notation of Section 2. Clearly, we may assume that G acts faithfully on X.

Let D be a prime divisor on X. Then $G \cdot D$ is a closed G-stable subscheme of X, of pure codimension 1. We denote by $\mathcal{O}_{X,G\cdot D}$ the corresponding semi-local ring; it has Krull dimension 1 and is equipped with a (functorial) G-action (Lemma 2.3). We have $\mathcal{O}_{X,G\cdot D}^G = \mathcal{O}_{Y,E}$, where $E = \pi(D)$ is a prime divisor on Y. Moreover, $\mathcal{O}_{X,G\cdot D}$ has a largest G-stable ideal $I = I_{G\cdot D}$, since $G \cdot D$ is the smallest G-stable subscheme of X containing D. (We may think of $\mathcal{O}_{X,G\cdot D}$ as a "G-local" ring of dimension 1).

We say that D is *free* if it meets the free locus X_{fr}, and *nonfree* otherwise. If G acts generically freely on X, then there are only finitely many nonfree divisors: the irreducible components of codimension 1 of the nonfree locus $X \setminus X_{\mathrm{fr}}$.

If G is diagonalizable with character group Λ, then the algebra $\mathcal{O}_{X,G\cdot D}$ is Λ-graded by Lemma 2.3 again. Thus, $I_{G\cdot D}$ is the largest Λ-graded ideal of $\mathcal{O}_{X,G\cdot D}$. Moreover, the G-action on X is generically free (Lemma 3.4).

Proposition 4.10. *Let G be a finite group scheme, Y a variety, and X a G-variety over Y. Then X is G-normal if and only if it satisfies the following conditions:*

(i) *The variety Y is normal.*

(ii) *The coherent sheaf of $\mathcal{B}$-algebras $\mathcal{A}$ satisfies (S_2).*

(iii) *For any nonfree divisor D, the ideal $I_{G\cdot D}$ is principal.*

If G is diagonalizable, then (ii) *is equivalent to*

(ii)' *The sheaf $\mathcal{A}_\lambda$ is divisorial for any $\lambda \in \Lambda$.*

Recall that a coherent sheaf $\mathcal{F}$ satisfies Serre's condition (S_2) if it has depth at least 2 at every point of codimension at least 2 of its support. Before proving Proposition 4.10, we record two preliminary results which can be extracted from [EGAIV_2].

Lemma 4.11. *Let $\mathcal{F}$ be a coherent torsion-free sheaf on a variety Z.*

(i) *$\mathcal{F}$ satisfies (S_2) if and only if the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is an isomorphism for any open subsets $U \subset V \subset Z$ such that V is big in U.*

(ii) *Given a finite surjective morphism of varieties $\pi : Z \to W$, the sheaf $\mathcal{F}$ satisfies (S_2) if and only if so does $\varphi_*(\mathcal{F})$.*

Proof. (i) This is a consequence of [EGAIV$_2$, Prop. 5.9.8 and Prop. 5.10.14] (see also [Kol22, §1]).

(ii) This follows from [EGAIV$_2$, Cor. 5.7.11] (see also [Kol22, Lem. 18]). □

Proof of Proposition 4.10. By [Br24, Thm. 4.12], X is G-normal if and only if $\mathcal{O}_X$ satisfies (S_2) and every G-stable ideal is invertible in codimension 1. Together with the above lemma, it follows that (ii) holds for every G-normal variety X. Moreover, (i) holds by Lemma 4.1, and (iii) follows from the fact that every invertible ideal of a semi-local ring is principal.

Conversely, assume (i), (ii) and (iii). Then $\mathcal{O}_X$ satisfies (S_2) by Lemma 4.11 again. So it suffices to check that every G-stable ideal is invertible in codimension 1. In other words, for any prime divisor D on X, every G-stable ideal I of $\mathcal{O}_{X,G\cdot D}$ is principal.

If D is free, we may replace X with its dense open G-stable subset X_{fr}, and hence assume that G acts freely on X. Then by fpqc descent, the assignement $Z \mapsto Z \times_Y X$ yields a bijection between closed subschemes of Y and closed G-stable subschemes of X. As a consequence, we have $I = I^G \mathcal{O}_X$. Moreover, I^G is identified with an ideal of the discrete valuation ring $\mathcal{O}_{Y,E}$ (where $E = \pi(D)$), and hence is invertible. This yields our assertion in this case.

If D is nonfree, then we may choose a generator f of the ideal $I_{G\cdot D}$. Then $I \subset f\mathcal{O}_{X,G\cdot D}$, and hence $I = fJ$ where J is a G-stable ideal of $\mathcal{O}_{X,G\cdot D}$; in particular, $I \subsetneq J$. So we have an exact sequence $0 \to J/I \to \mathcal{O}_{X,G\cdot D}/I \to \mathcal{O}_{X,G\cdot D}/J \to 0$ of $\mathcal{O}_{X,G\cdot D}$-modules of finite length. This implies our assertion by induction on the length of $\mathcal{O}_{X,G\cdot D}/I$ (this length is minimal if and only if $I = I_{G\cdot D}$).

It remains to prove the equivalence (ii)⇔(ii)' when G is diagonalizable. In view of the decomposition (3.2), $\mathcal{A}$ satisfies (S_2) if and only if each $\mathcal{A}_\lambda$ satisfies (S_2). As Y is normal and $\mathcal{A}_\lambda$ has rank 1, this amounts to $\mathcal{A}_\lambda$ being divisorial (see e.g. [Har80, Cor. 1.4 and Prop. 1.6]).

□

Combining Lemma 3.4 with Proposition 4.10, we readily obtain:

Corollary 4.12. *Let G be a finite diagonalizable group scheme, and X a G-normal variety over a regular variety Y. Then the quotient morphism π is flat.*

5 Local structure in codimension 1

Throughout this section, we consider a finite diagonalizable group G and a G-variety X over a normal variety Y. Let again D be a prime divisor on X, with image E in Y. Then E is a prime divisor on the normal variety Y, and hence $B = \mathcal{O}_{Y,E}$ is a discrete valuation ring. Moreover, the semi-local ring $A = \mathcal{O}_{X,G \cdot D}$ is a finitely generated, torsion-free B-module, and hence is free of finite rank. Also, G acts on A, and $B = A^G$.

Denote by $\mathfrak{m}$ the maximal ideal of A corresponding to D; then $\mathfrak{n} = \mathfrak{m} \cap B$ is the maximal ideal of B. The residue field $A/\mathfrak{m}$ is the function field $k(D) = \kappa(x)$, where x denotes the generic point of D. Likewise, $B/\mathfrak{n} = k(E) = \kappa(y)$ for the generic point $y = \pi(x)$ of D; the extension $\kappa(x)/\kappa(y)$ is finite. Let $\bar{A} = A/\mathfrak{n}A$; this is an artinian $\kappa(y)$-algebra, equipped with a action of $G_{\kappa(y)}$. We have $\bar{A} = \mathcal{O}(X_y)$, where X_y denotes the (schematic) fibre of π at y; recall that X_y contains $\Omega_y = G_{\kappa(y)} \cdot x$ as its unique (schematic) orbit.

We will obtain a description of the G-algebra $\bar{A}$, or equivalently of the G-scheme X_y, under the assumption that G is diagonalizable. Then A is a Λ-graded B-module:

$$A = \bigoplus_{\lambda \in \Lambda} A_\lambda, \tag{5.1}$$

where $A_0 = B$ and each A_λ is a free B-module of rank 1 (Lemma 3.4). This yields a Λ-grading

$$\bar{A} = \bigoplus_{\lambda \in \Lambda} \bar{A}_\lambda \tag{5.2}$$

such that each $\bar{A}_\lambda$ is a one-dimensional $\kappa(y)$-vector space. Let $I = I_{G \cdot D}$ be the largest Λ-graded ideal of A. Clearly, we have $\mathfrak{n}A \subset I$, and hence I is the preimage of the largest Λ-graded ideal $\bar{I}$ of $\bar{A}$ under the quotient map $A \to \bar{A}$. Moreover, there is an exact sequence of $G_{\kappa(y)}$-modules

$$0 \longrightarrow \bar{I} \longrightarrow \bar{A} \longrightarrow \mathcal{O}(\Omega_y) \longrightarrow 0, \tag{5.3}$$

and $\bar{I}$ is nilpotent.

Proposition 5.1. (i) *There exists a unique subgroup scheme $H \subset G$ such that $\bar{A} = \bar{A}^{H_{\kappa(y)}} \oplus \bar{I}$.*

(ii) *The morphism* $X_y/H_{\kappa(y)} = \mathrm{Spec}(\bar{A}^{H_{\kappa(y)}}) \to \mathrm{Spec}(\kappa(y))$ *is a torsor under* $(G/H)_{\kappa(y)}$.

(iii) *We have* $H_{\kappa(y)} = C_{G_{\kappa(y)}}(x)$ *and* $\mathrm{Spec}(\bar{A}^{H_{\kappa(y)}}) \simeq \Omega_y$ *as* $(G/H)_{\kappa(y)}$*-schemes.*

Remark 5.2. The subgroup scheme H only depends on $y \in Y$, or equivalently on the prime divisor E; we will denote it by $H(E)$ or by $H(D)$ according to the setup.

If the restriction $\pi|_D : D \to E$ is birational, then $\kappa(y) = \kappa(x)$ and hence $H_{\kappa(x)}$ is just the stabilizer $\mathrm{Stab}_G(x)$. In the general case, as the above proposition is rather technical, we illustrate it with two examples before giving the proof.

Let $G = \boldsymbol{\mu}_{p^2}$ so that $\Lambda = \mathbb{Z}/p^2\mathbb{Z}$. Consider the polynomial ring $k[T_0, T_1, T_2]$ equipped with the Λ-grading such that T_0, T_1, T_2 are homogeneous of respective weights $0, 1, p$. Let

$$X = \mathrm{Spec}\big(k[T_0, T_1, T_2]/(T_1^p - T_0 T_2)\big).$$

This is a normal affine surface containing the origin as its unique nonsmooth point. Moreover, G acts on X (since $T_1^p - T_0 T_2$ is homogeneous of weight p) with quotient

$$Y = \mathrm{Spec}(k[T_0, T_2^p]) \simeq \mathbb{A}^2.$$

The zero subscheme of (T_0, T_1) in X is a G-stable nonfree divisor D, and its image E in Y is the zero subscheme of T_0. With the above notation, we obtain

$$\kappa(x) = k(T_2) \supset k(T_2^p) = \kappa(y), \quad \bar{A} = k(T_2)[T_1]/(T_1^p),$$

so that $H = \boldsymbol{\mu}_p$. The orbit Ω_y equals the point x, which is not fixed by $G_{\kappa(y)}$ (or equivalently, by G).

Next, let $X' \subset X$ be the zero subscheme of $T_2^p - T_0 - 1 \in \mathcal{O}(X)^G = \mathcal{O}(Y)$. Then X' is a G-stable curve containing $x' = (0, 0, 1)$ as its unique nonsmooth point. Moreover, X' has quotient $Y' = \mathrm{Spec}(k[T_0]) \simeq \mathbb{A}^1$, and x' has image $y' = 0$. We obtain with an obvious notation

$$\kappa(x') = \kappa(y'), \quad \bar{A}' = k[T_1, T_2]/(T_1^p, T_2^p - 1),$$

so that $H' = \boldsymbol{\mu}_p$ again; also, x' is neither free nor G-stable. One may check that X' is G-normal.

The latter example will be generalized in Section 8 to all G-normal curves, where G is diagonalizable and infinitesimal.

Proof of Proposition 5.1. (i) For any $\lambda \in \Lambda$, we have either $\bar{A}_\lambda \subset \bar{I}$, or every nonzero element of $\bar{A}_\lambda$ is invertible. (Indeed, the ideal $\bar{A}_\lambda \bar{A}$ is Λ-graded, and hence is either contained in $\bar{I}$ or equal to $\bar{A}$). Also, note that every nonzero element of $\bar{A}_\lambda$ is invertible if and only if $\bar{A}_\lambda$ meets $\bar{A}^\times$. As a consequence, the characters satisfying this condition form a subgroup of Λ. Thus, there exists a unique subgroup scheme H of G such that $\bar{A}_\lambda$ meets $\bar{A}^\times$ if and only if $\bar{A}_\lambda \subset \bar{A}^{H_{\kappa(y)}}$. In view of (5.2), this yields $\bar{A} = \bar{A}^{H_{\kappa(y)}} \oplus \bar{I}$.

(ii) This follows from the definition of H in view of Lemma 3.1.

(iii) By (i) together with the exact sequence (5.3), we obtain a $G_{\kappa(y)}$-equivariant isomorphism $\bar{A}^{H_{\kappa(y)}} \xrightarrow{\sim} \mathcal{O}(\Omega_y)$. As a consequence, $H_{\kappa(y)}$ acts trivially on Ω_y. In particular, $H_{\kappa(y)} \subset C_{G_{\kappa(y)}}(x)$. On the other hand, $C_{G_{\kappa(y)}}(x)$ fixes the image of x in $\operatorname{Spec}(\bar{A}^{H_{\kappa(y)}})$. In view of (ii), this gives the opposite inclusion $C_{G_{\kappa(y)}}(x) \subset H_{\kappa(y)}$. $\square$

We may reformulate the above proposition in geometric terms by considering the factorization of $\pi : X \to Y$ as

$$X \xrightarrow{\varphi} Z = X/H \xrightarrow{\psi} X/G = Y,$$

where φ (resp. ψ) is the quotient by H (resp. G/H). Then φ yields a morphism between fibres $\varphi_y : X_y \to Z_y$. Moreover, $Z_y = X_y/H_{\kappa(y)}$ by Lemma 2.2 together with the linear reductivity of H.

Corollary 5.3. (i) *The restriction of φ to D is birational to its image.*

(ii) *The morphism ψ is a G/H-torsor in a neighborhood of y.*

(iii) *The fixed point subscheme $X_y^{H_{\kappa(y)}}$ yields a section of φ_y.*

(iv) *The divisor D is free if and only if H is trivial.*

Proof. (i) The decomposition $\bar{A} = \bar{A}^H \oplus \bar{I}$ induces a decomposition $\bar{\mathfrak{m}} = \bar{\mathfrak{m}}^H \oplus \bar{I}$, so that $\kappa(x) = A/\mathfrak{m} = \bar{A}/\bar{\mathfrak{m}} = \bar{A}^H/\bar{\mathfrak{m}}^H = A^H/\mathfrak{m}^H$, where the latter equality follows from the inclusion $\mathfrak{n}A^H \subset \mathfrak{m}^H$. This yields our assertion.

(ii) This follows from Proposition 5.1 together with the openness of the free locus of the G/H-variety Z.

(iii) We identify the algebra homomorphism $\varphi_y^\# : \mathcal{O}(Z_y) \to \mathcal{O}(X_y)$ with the inclusion $\bar{A}^{H_{\kappa(y)}} \subset \bar{A}$. The latter inclusion has a retraction $\bar{A} \to \bar{A}/\bar{I} = \bar{A}^{H_{\kappa(y)}}$, where $\bar{I}$ is the largest $H_{\kappa(y)}$-stable ideal of $\bar{A}$ such that $H_{\kappa(y)}$ acts trivially on $\bar{A}/\bar{I}$, i.e., the ideal of $X_y^{H_{\kappa(y)}}$.

(iv) If H is trivial, then $X_y \to \operatorname{Spec}(\kappa(y))$ is a $G_{\kappa(y)}$-torsor, and hence $x \in X_{\mathrm{fr}}$. Conversely, if H is nontrivial then D is nonfree by (iii). $\square$

For a G-normal variety X, we will lift the above description of $\bar{A} = \mathcal{O}(X_y)$ to a description of $A = \mathcal{O}_{X,G\cdot D}$. For this, we need:

Lemma 5.4. *If X is G-normal, then the largest Λ-graded ideal I is generated by a homogeneous element.*

Proof. By Proposition 4.10, there exists $f \in A$ such that $I = Af$. Write $f = \sum_\lambda f_\lambda$, where $f_\lambda \in I_\lambda$ for all λ. We claim that $I = Af_\nu$ for some ν.

Indeed, $f \notin \mathfrak{m}I$, where $\mathfrak{m}$ denotes the maximal ideal of x in A. So there exists ν such that $f_\nu \notin \mathfrak{m}I$. We have $Af_\nu \subset I$; if this inclusion is strict, then the annihilator J of the Λ-graded A-module I/Af_ν is a proper Λ-graded ideal, and hence is contained in I. But we have the equality of localizations $I_\mathfrak{m} = (Af_\nu)_\mathfrak{m}$ and hence $J_\mathfrak{m} = A_\mathfrak{m}$, which contradicts the fact that $I_\mathfrak{m} = \mathfrak{m}_\mathfrak{m} \neq A_\mathfrak{m}$. This proves our claim. □

Proposition 5.5. *Assume that X is G-normal and let f be a homogeneous generator of I, with weight $\nu \in \Lambda(G)$.*

(i) *The restriction $\nu|_H$ generates $\Lambda(H)$, and is uniquely determined by I.*

(ii) *We have $H \simeq \boldsymbol{\mu}_n$ for some $n \geq 1$, and $A = \bigoplus_{m=0}^{n-1} A^H f^m \simeq A^H[T]/(T^n - g)$, where $g = f^n$ satisfies $g \in A^H$ and $gA = \mathfrak{n}A$.*

(iii) *We have the equality $X_y^H = \Omega_y$, and the inclusion $G \cdot D \subset X^H$ with equality in a G-stable neighborhood of x.*

Proof. We show (i) and (ii) simultaneously. Let $g = f^n$, where $n = |H|$. Then $g \in A^H$, since the character group $\Lambda(H)$ is killed by n. Thus, the image $\bar{f}$ of f in $\bar{A}$ satisfies $\bar{f}^n \in \bar{A}^{H_{\kappa(y)}}$. But $\bar{f} \in \bar{I}$ and $\bar{A} = \bar{A}^{H_{\kappa(y)}} \oplus \bar{I}$, so that $\bar{f}^n = 0$; equivalently, $g \in \mathfrak{n}A$. Moreover, Proposition 5.1 yields the equalities $\bar{A} = \bar{A}^{H_{\kappa(y)}} \oplus \bar{I} = \bar{A}^{H_{\kappa(y)}} \oplus \bar{f}\bar{A}$ and hence

$$\bar{A} = \bar{A}^{H_{\kappa(y)}} + \bar{f}\bar{A} = \bar{A}^{H_{\kappa(y)}} + \bar{f}\bar{A}^{H_{\kappa(y)}} + \bar{f}^2\bar{A} = \cdots = \sum_{m=0}^{n-1} \bar{f}^m \bar{A}^{H_{\kappa(y)}}.$$

Taking weights, we obtain $\Lambda = \Lambda(G/H) + \mathbb{Z}\nu$, where $\Lambda(G/H) = \Lambda^H$. Also, recall that $\Lambda(H) = \Lambda/\Lambda^H$; thus, this group is generated by $\nu|_H$. As a consequence, $\nu|_H$ has order n, and hence $H \simeq \boldsymbol{\mu}_n$; moreover, Λ is the disjoint union of the translates $\Lambda^H + m\nu$ where $m = 0, \ldots, n-1$. Also, $\bar{f}$ is uniquely determined up to multiplication by a homogeneous unit of $\bar{A}$, so that ν is unique up to translation by Λ^H; equivalently, $\nu|_H$ is unique.

For any $\lambda \in \Lambda^H$ and $m = 0, \ldots, n-1$, we have $\bar{A}_{\lambda+m\nu} = \bar{A}_\lambda \bar{f}^m$, and hence $A_{\lambda+m\nu} = A_\lambda f^m$ by Nakayama's lemma. As a consequence, $A = \sum_{m=0}^{n-1} A^H f^m$. Moreover, this sum is direct in view of the above decomposition of Λ into cosets of Λ^H.

It remains to show that $gA = \mathfrak{n}A$, or equivalently, $gA^H = \mathfrak{n}A^H$. Clearly, we have $gA^H \subset \mathfrak{n}A^H$. Thus, it suffices to check that $\dim_{\kappa(y)} A^H/gA^H = \dim_{\kappa(y)} A^H/\mathfrak{n}A^H$. But we have isomorphisms of $\kappa(y)$-algebras

$$A^H/\mathfrak{n}A^H \simeq \bar{A}^{H_{\kappa(y)}} \simeq \bar{A}/\bar{I} \simeq A/I = A/fA \simeq A^H/gA^H,$$

where the latter isomorphism follows from the decomposition of A as an A^H-module. This yields the desired equality of dimensions.

(iii) Note that X^H is G-stable, and contains D by Proposition 5.1; therefore, $G \cdot D \subset X^H$. Denote by $\mathcal{J} \subset \mathcal{O}_X$ the ideal sheaf of X^H, and by $J \subset A$ the corresponding ideal. Then $f \in J$, as it is an H-eigenvector of nonzero weight. Thus, we have $I(G \cdot D) \subset J$; this implies our assertions. □

Remark 5.6. Given $\lambda \in \Lambda$, consider the multiplication map

$$\sigma_{\lambda,|G|} : A_\lambda^{\otimes |G|} \longrightarrow A_{|G|\lambda} = A_0 = B$$

as in (3.4). With the notation of Proposition 5.5, write $\lambda = \mu + m\nu$, where $\mu \in \Lambda^H$ and $0 \leq m \leq n-1$. Then the image of $\sigma_{\lambda,|G|}$ is the ideal of B generated by $f^{|G|m} = g^{[G:H]m}$.

Indeed, we have $A_\lambda = A_\mu f^m$ by the above proposition, and the multiplication map $\sigma_{\mu,|G|} : A_\mu^{\otimes|G|} \to B$ is an isomorphism in view of Remark 3.2 and Corollary 5.3.

6 Local structure in codimension 1 (continued)

We still consider a finite diagonalizable group scheme G and a G-normal variety X over Y (a normal variety). To describe the local structure of X in codimension 1, we may freely replace it by any big open subset U which is G-stable and satisfies some additional assumptions. Here are those we will need at this stage.

Lemma 6.1. *There exists a largest open subset $U \subset X$ satisfying the following properties:*

(i) *The quotient U/G is regular.*

(ii) *The nonfree locus $U \setminus U_{\mathrm{fr}}$ is a disjoint union of prime divisors D.*

(iii) *For any such D, the ideal $\mathcal{I}_{G\cdot D}$ is invertible.*

Moerover, U is G-stable and big in X.

Proof. These properties are open, and hold in codimension 1; this readily yields that U exists and is big in X. To complete the proof, it suffices to show that each of these properties defines a G-stable open subset. This clearly holds for (i) and (ii). For (iii), this follows from fpqc descent as this property is stable by automorphisms arising from k-points of G. □

We now assume that X satisfies the above properties (i), (ii), (iii); we then say that X is *strongly G-normal*.

To every prime divisor D on X, we associate a cyclic subgroup scheme $H(D) \subset G$ (where x denotes the generic point of D) and a weight $\nu(D) \in \Lambda$ (the weight of a homogeneous generator of the largest Λ-graded ideal of $\mathcal{O}_{X,G\cdot D}$), as in Proposition 5.5; recall that $\nu(D)|_{H(D)}$ generates $\Lambda(H(D))$.

Given a cyclic subgroup scheme H of G and a weight $\nu \in \Lambda$ such that $\nu|_H$ generates $\Lambda(H)$, let

$$U(H,\nu) = X_{\rm fr} \cup \bigcup_D G\cdot D,$$

where the union runs over the nonfree divisors D on X such that $H(D) = H$ and $\nu(D)|_{H(D)} = \nu|_H$. Then $U(H,\nu)$ is an open G-stable subset of X, and its quotient $V(H,\nu) = U(H,\nu)/G$ is identified with an open subset of Y. Also, note that $\nu|_H$ identifies H with the cyclic group scheme $\boldsymbol{\mu}_n$, where $n = n(H,\nu)$; moreover, we have $\Lambda = \Lambda^H + \sum_{m=0}^{n-1} m\nu$ and $n\nu \in \Lambda^H$. The multiplication in $\mathcal{A}$ yields a morphism of $\mathcal{B}$-modules

$$\sigma = \sigma_{\nu,n} : \mathcal{A}_\nu^{\otimes n} \longrightarrow \mathcal{A}_{n\nu}.$$

The restriction $\sigma|_{V(H,\nu)}$ may be viewed as a section of the invertible sheaf $\mathcal{A}_\nu^{\otimes -n} \otimes_{\mathcal{B}} \mathcal{A}_{n\nu}$ on $V(H,\nu)$. We denote its divisor of zeroes by $\Delta(H,\nu)$; this is an effective divisor on the regular variety $V(H,\nu)$, that we call the *branch divisor*.

Theorem 6.2. *Let X be a strongly G-normal variety.*

(i) *We have $X = \bigcup_{H,\nu} U(H,\nu)$, and $U(H,\nu) \cap U(H',\nu') = X_{\rm fr}$ unless $H' = H$ and $\nu'|_H = \nu|_H$; then $U(H',\nu') = U(H,\nu)$.*

(ii) *The quotient $U = U(H,\nu) \to V = V(H,\nu)$ factors as $U \xrightarrow{\varphi_U} U/H \xrightarrow{\psi_U} U/G$, where ψ_U is a G/H-torsor. Moreover, the natural homomorphism of $\mathcal{A}^H$-algebras*

$$\Phi : \big(\mathcal{A}^H \otimes_{\mathcal{B}} \mathrm{Sym}_{\mathcal{B}}(\mathcal{A}_\nu)\big)/\mathcal{I} \longrightarrow \mathcal{A}$$

restricts to an isomorphism on V, where $\mathcal{I}$ denotes the ideal of $\mathcal{A}^H \otimes_{\mathcal{B}} \mathrm{Sym}_{\mathcal{B}}(\mathcal{A}_\nu)$ generated by the $a_1 \cdots a_n - \sigma(a_1 \otimes \cdots \otimes a_n)$ with $a_1, \ldots, a_n \in \mathcal{A}_\nu$.

(iii) *The branch divisor $\Delta_V = \Delta(H, \nu)$ is reduced and its support is $V \setminus V_{\mathrm{fr}}$.*

(iv) *The fixed point subscheme U^H is an effective Cartier divisor on U, and we have $U^H = \bigcup G \cdot D$ (union over the G-orbits of nonfree divisors of U). Moreover, $\pi_U^*(\Delta_V) = nU^H$ as Cartier divisors on U.*

Proof. (i) All of this follows readily from the definition of $U(H, \nu)$.

(ii) For any $\lambda, \mu \in \Lambda$, the multiplication map $\mathrm{mult}_{\lambda,\mu}$ (3.3) restricts to a nonzero homomorphism of invertible $\mathcal{B}_V$-modules, and hence yields an isomorphism

$$(\mathcal{A}_\lambda \otimes_{\mathcal{B}} \mathcal{A}_\mu)\,|_V \xrightarrow{\sim} \mathcal{A}_{\lambda+\mu}|_V(-D_{\lambda,\mu}),$$

where $D_{\lambda,\mu}$ is an effective divisor on V. By Lemma 3.1, $D_{\lambda,\mu}$ is a sum of nonfree divisors. Combining this lemma with Proposition 5.5, it follows that $D_{\lambda,\mu} = 0$ for all $\lambda, \mu \in \Lambda^H$. This yields the assertion on ψ_U by using again Lemma 3.1.

We now show that Φ_V is an isomorphism. Note that the natural map

$$\bigoplus_{m=0}^{n-1} \mathcal{A}^H \otimes_{\mathcal{B}} \mathcal{A}_{m\nu} \longrightarrow \mathcal{A}^H \otimes_{\mathcal{B}} \mathrm{Sym}_{\mathcal{B}}(\mathcal{A}_\nu)/\mathcal{I}$$

is an isomorphism on V. So it suffices to show that the multiplication map

$$\mathrm{mult}_{\lambda,m\nu} : \mathcal{A}_\lambda \otimes_{\mathcal{B}} \mathcal{A}_{m\nu} \longrightarrow \mathcal{A}_{\lambda+m\nu}$$

is an isomorphism on V for all $\lambda \in \Lambda^H$ and $m = 0, \ldots, n-1$. But this map is an isomorphism at the generic point of every prime divisor D meeting U, in view of Proposition 5.5. This yields the desired assertion by arguing as in the above paragraph.

(iii) This follows from Proposition 5.5, since σ is identified with g at the generic point of every D as above.

(iv) The ideal of U^H is generated by the homogeneous components of $\mathcal{A}^H$ of nonzero weight relative to H, and hence by $\mathcal{A}_\nu$ in view of (ii). In particular, this ideal is invertible; this yields the first assertion.

By Proposition 5.5 again, we have $U^H \subset \bigcup G \cdot D$ with equality at the generic points. As U^H and $\bigcup_D G \cdot D$ are Cartier divisors, they must coincide. The equality of Cartier divisors $\pi_U^*(\Delta_V) = nU^H$ follows similarly from Proposition 5.5. □

Remark 6.3. The above theorem has a partial converse: let U be a G-variety with quotient $\pi : U \to V$, where V is regular. If U satisfies the conditions (ii) and (iii) for some H, ν, σ and Δ, then U is G-normal.

Indeed, by Proposition 4.10, it suffices to show that the largest Λ-graded ideal of $A = \mathcal{O}_{U,G\cdot D}$ is principal for any nonfree divisor D. But we have $A = A^H[f]$, where $f \in A$ is homogeneous and satisfies $f^n \in A^H$. Thus, $\bar{A} = \bar{A}^H[\bar{f}]$, where $\bar{f}^n = 0$. Moreover, the natural morphism $\mathrm{Spec}(A^H) \to \mathrm{Spec}(B)$ is a G/H-torsor, hence the morphism $\mathrm{Spec}(\bar{A}^H) \to \mathrm{Spec}(B/\mathfrak{n}) = \mathrm{Spec}(\kappa(y))$ is a $(G/H)_{\kappa(y)}$-torsor. As a consequence, $\bar{f}$ generates the largest Λ-graded ideal of $\bar{A} = A/\mathfrak{n}A$. Since $fA \supset f^n A = \mathfrak{n}A$, it follows that f generates the largest Λ-graded ideal of A.

Remark 6.4. For any $\lambda \in \Lambda$, the multiplication map $\sigma_{\lambda,|G|} : \mathcal{A}_\lambda^{\otimes |G|} \to \mathcal{B}$ restricts to a global section of $(\mathcal{A}_\lambda|_{Y_{\mathrm{reg}}})^{\otimes -|G|}$. By Remark 5.6, the divisor of this section on each $U(H,\nu)$ equals $[G : H]m\Delta(H,\nu)$, where $m = m(H,\nu)$ is the unique integer such that $0 \leq m \leq |H| - 1$ and $\lambda - m\nu \in \Lambda^H$. As a consequence, the class of the divisorial sheaf $\mathcal{A}_\lambda$ in $\mathrm{Cl}(Y)_{\mathbb{Q}}$ (the divisor class group of Y with rational coefficients) satisfies

$$[\mathcal{A}_\lambda] = -\sum_{H,\nu} \frac{m(H,\nu)}{|H|} \Delta(H,\nu) \tag{6.1}$$

with the above notation.

With the notation of Theorem 6.2 (ii), the morphism $\varphi_U : U \to U/H$ is a uniform cyclic cover with branch divisor Δ, as defined in [AV04]. We say that a G-normal variety X is *uniform*, if it is the closure of some $U = U(H,\nu)$; equivalently, the equality $X = U(H,\nu)$ holds in codimension 1. (We do not assume that X is strongly G-normal). Then the closure of the branch divisor Δ in Y is a reduced effective divisor that we will still call the branch divisor, and denote by Δ for simplicity.

Proposition 6.5. *Let $X = \overline{U(H,\nu)}$ be a uniform G-normal variety over Y. Assume that $\mathcal{O}(Y)^\times = k^\times$ and the divisor class group $\mathrm{Cl}(Y)$ has no n-torsion, where $n = |G|$. Then $G = H \simeq \boldsymbol{\mu}_n$, the G-variety X over Y is uniquely determined by Δ, and the class $[\Delta]$ is divisible by n in $\mathrm{Cl}(Y)$. Moreover, every reduced effective divisor on Y with class divisible by n is obtained from a uniform $\boldsymbol{\mu}_n$-normal variety.*

Proof. If $H \neq G$ then G/H has a nontrivial quotient isomorphic to $\boldsymbol{\mu}_d$ for some d dividing n, and hence the G/H-torsor $\psi : U/H \to V$ factors through a $\boldsymbol{\mu}_d$-torsor $\phi : W \to V$, where W is a variety. Since $\mathrm{Pic}(V) = \mathrm{Cl}(V) = \mathrm{Cl}(Y)$ has no d-torsion, we have $\phi_*(\mathcal{O}_W) = \mathcal{O}_V[T]/(T^d - f)$ for some $f \in \mathcal{O}(V)^\times$ (Remark 3.2). But $\mathcal{O}(V)^\times = \mathcal{O}(Y)^\times = k^\times$ and hence W is not geometrically integral, a contradiction.

Thus, $G = H$ and we may assume that $G = \boldsymbol{\mu}_n$ and ν is the defining character. Then $\mathcal{A} = \bigoplus_{m=0}^{n-1} \mathcal{A}_m$, where each $\mathcal{A}_m$ is a divisorial $\mathcal{B}$-module (Proposition 4.10). By Theorem 6.2, the multiplication map $\mathrm{mult}_{\ell,m} : \mathcal{A}_\ell \otimes_{\mathcal{B}} \mathcal{A}_m \to \mathcal{A}_{\ell+m}$ is an isomorphism on V whenever $\ell + m \leq n - 1$. So we may assume that $\mathcal{A}_m = \mathcal{O}_Y(-mD)$ for $m = 0, \ldots, n-1$, where D is a Weil divisor on Y. Moreover, the multiplication map $\sigma_{1,n} : \mathcal{A}_1^{\otimes n} \to \mathcal{B}$ yields an isomorphism $\mathcal{O}_Y(-nD) \xrightarrow{\sim} \mathcal{O}_Y(-\Delta)$ in view of Theorem 6.2 again. Equivalently, we have a section $\sigma \in \Gamma(Y, nD)$ with divisor Δ, which uniquely determines the multiplication of the $\mathcal{B}$-algebra $\mathcal{A}$. Thus, the G-variety X over Y is uniquely determined by the pair (D, σ) up to isomorphism given by linear equivalence of divisors; moreover, we have $[\Delta] = n[D]$ in $\mathrm{Cl}(Y)$. In turn, X is uniquely determined by Δ in view of our assumptions on Y. This proves the second assertion; the final one follows from Theorem 6.2 once more. □

Remark 6.6. Keep the asssumptions of the above proposition, and assume in addition that Y is regular. We may further assume that $G = \boldsymbol{\mu}_n$ and ν is the defining character. Then the sheaf $\mathcal{L} = \mathcal{A}_1$ is invertible; denote by

$$\varphi : L = \mathbb{V}(\mathcal{L}) = \mathrm{Spec}\,\mathrm{Sym}_{\mathcal{B}}(\mathcal{L}) \longrightarrow Y$$

the line bundle with sheaf of local sections $\mathcal{L}^{\otimes -1}$. By the above proof, we may view X as the zero scheme in L of the section

$$\tau^n - \sigma \in \Gamma(L, \varphi^*(L^{\otimes n})) = \bigoplus_{m=0}^{\infty} \Gamma(Y, \mathcal{L}^{\otimes m-n}),$$

where $\tau \in \Gamma(L, \varphi^*(L)) = \bigoplus_{m=0}^{\infty} \Gamma(Y, \mathcal{L}^{\otimes m-1})$ denotes the canonical section, corresponding to $1 \in \Gamma(Y, \mathcal{O}_Y)$, and $\sigma \in \Gamma(Y, \mathcal{L}^{\otimes -n})$. This gives back a classical construction of $\boldsymbol{\mu}_n$-covers, see e.g. [Laz04, Prop. 4.1.6].

Also, note that for $p = 2$, every $\boldsymbol{\mu}_2$-normal variety is uniform. But there exist nonuniform $\boldsymbol{\mu}_p$-normal varieties for any $p \geq 3$: for example, $\mathbb{P}^1$ where $\boldsymbol{\mu}_p$ acts by multiplication.

Example 6.7. Let X be a uniform G-normal variety over the projective space $\mathbb{P}^N$. By Proposition 6.5, we have $G \simeq \boldsymbol{\mu}_n$ and X is classified by its branch divisor, a reduced hypersurface $\Delta \subset \mathbb{P}^N$ of degree d divisible by n. Moreover, X is realized as a hypersurface in the line bundle $O_{\mathbb{P}^N}(d) = \mathbb{V}(\mathcal{O}_{\mathbb{P}^N}(-d))$ via the construction of Remark 6.6.

If X is regular, then so is X^G in view of [Hau20, Lem. 3.5.2]. Thus, taking for Δ a singular hypersurface yields many examples of singular G-normal varieties.

7 The relative dualizing sheaf

Let G be a finite group scheme acting on a scheme X with quotient morphism $\pi : X \to Y$. Since π is finite, it has a (0)-dualizing sheaf $\omega_\pi = \pi^! \mathcal{O}_X$; this is the $\mathcal{O}_X$-module that corresponds to the $\mathcal{A}$-module $\mathcal{H}\mathrm{om}_{\mathcal{B}}(\mathcal{A}, \mathcal{B})$. It is equipped with a trace map $\mathrm{tr}_\pi : \pi_*(\omega_\pi) \to \mathcal{B}$ via evaluation at 1. If π is flat and a local complete intersection (l.c.i.), then ω_π is isomorphic to the canonical sheaf $\omega_{X/Y}$; in particular, ω_π is invertible (see e.g. [Li06, §6.4], which will be our reference for duality theory). In any case, ω_π is equipped with a G-linearization for which the trace map is invariant.

If G is linearly reductive, then the unique G-invariant projection $\mathcal{A} \to \mathcal{B}$ (the "Reynolds operator") yields a global section $s_\pi \in \Gamma(X, \omega_\pi) = \mathrm{Hom}_{\mathcal{B}}(\mathcal{A}, \mathcal{B})$. Note that s_π is G-invariant and satisfies $\mathrm{tr}_\pi(s_\pi) = 1$. Also, the formation of ω_π and s_π commutes with flat base change $Y' \to Y$. If the morphism π is flat l. c. i, then we denote s_π by $s_{X/Y}$.

Given a subgroup scheme $H \subset G$ and the corresponding factorization of π as $X \xrightarrow{\varphi} Z = X/H \xrightarrow{\psi} Y = X/G$, we may identify ω_π with $\omega_\varphi \otimes_{\mathcal{O}_X} \varphi^*(\omega_\psi)$ (see [Li06, Lem. 6.4.26]). If G is linearly reductive, then we have

$$s_\pi = s_\varphi \otimes \varphi^*(s_\psi), \tag{7.1}$$

where $s_\psi \in \Gamma(Z, \omega_\psi) = \mathrm{Hom}_{\mathcal{B}}(\mathcal{A}^H, \mathcal{B})$ is the projection $\mathcal{A}^H \to \mathcal{B}$. If in addition H is a normal subgroup of G, then ψ is the quotient morphism by G/H, and s_ψ is the corresponding canonical section.

We will need two further observations, certainly well-known but for which we could not locate any reference.

Lemma 7.1. *Let G be a finite group scheme, and $\pi : X \to Y$ a G-torsor.*

(i) *The (finite flat) morphism π is l.c.i. and satisfies $\omega_{X/Y} \simeq \mathcal{O}_X$.*

(ii) *If G is linearly reductive, then $s_{X/Y}$ is a trivializing section of $\omega_{X/Y}$.*

Proof. (i) We use the construction at the beginning of Section 4. Choose an embedding of G in some GL_n. Then π is the composition

$$X \xrightarrow{\iota} Z = \mathrm{GL}_n \times^G X \xrightarrow{\psi} Y,$$

where the morphism ι is a closed immersion with image $G \times^G X$, and ψ is the projection $\mathrm{GL}_n \times^G X \to X/G = Y$. Thus, ψ is a GL_n-torsor. In particular, it is smooth and its relative tangent sheaf is isomorphic to $\mathrm{Lie}(\mathrm{GL}_n) \otimes_k \mathcal{O}_{Z/Y}$. As a consequence, $\det(\Omega^1_{Z/Y}) \simeq \mathcal{O}_{Z/Y}$. Also, recall that X is the fibre at the base point of the other projection $\varphi : \mathrm{GL}_n \times^G X \to \mathrm{GL}_n/G$. Thus, ι is a regular immersion and its conormal sheaf $\mathcal{C}_{X/Z}$ is trivial (see [Li06, Prop. 6.3.11]). This gives an isomorphism $\omega_{X/Y} \simeq \mathcal{O}_X$ in view of loc. cit., Def. 6.4.7.

(ii) We use the commutative diagram of flat morphisms

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Spec}(k) \\ \uparrow{\scriptstyle \mathrm{pr}_G} & & \uparrow \\ G \times X & \xrightarrow{\mathrm{pr}_X} & X \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & Y, \end{array}$$

where both squares are cartesian. Since the formation of $s_{X/Y}$ commutes with flat base change, this yields $\alpha^*(s_{X/Y}) = s_{G\times X/X} = \mathrm{pr}_G^*(s_G)$. By fpqc descent, we are reduced to checking that $s_G \in \Gamma(G, \omega_G) = \mathrm{Hom}_k(\mathcal{O}(G), k)$ is a free generator of the $\mathcal{O}(G)$-module ω_G.

In the case that G is diagonalizable, the k-vector space $\mathrm{Hom}_k(\mathcal{O}(G), k)$ has basis $(e_\lambda)_{\lambda\in\Lambda}$ dual to the basis Λ of $\mathcal{O}(G)$, and the $\mathcal{O}(G)$-module structure of $\Gamma(G, \omega_G)$ is given by $\mu \cdot e_\lambda = e_{\lambda-\mu}$. Therefore, this module is freely generated by $e_0 = s_G$.

In the general case of a linearly reductive group G, we may assume k algebraically closed by fpqc descent again. Then G^0 is diagonalizable and $G \simeq G^0 \rtimes \pi_0(G)$, where $\pi_0(G)$ is constant of order N prime to p. Thus, $s_G : \mathcal{O}(G) \to k$ satisfies

$$s_G(\varphi) = \frac{1}{N} \sum_{g\in G(k)} g \cdot \varphi_0$$

for any $\varphi \in \mathcal{O}(G)$, where φ_0 denotes the homogeneous component of weight 0 relative to G^0; note that $g\varphi_0 = (g \cdot \varphi)_0$ as g induces an automorphism of $\Lambda(G^0)$. Moreover, we have an isomorphism of algebras

$$\mathcal{O}(G) \simeq \prod_{g\in G(k)} \mathcal{O}(gG^0)$$

and $s_G(\varphi) = \frac{1}{N}\varphi_0$ for any $g \in G(k)$ and $\varphi \in \mathcal{O}(gG^0)$. As gG^0 is a trivial G^0-torsor, s_G restricts to a nonzero scalar multiple of s_{gG^0}. This implies our assertion. □

Lemma 7.2. *Let G be a finite group scheme, and X a G-normal variety with quotient $\pi : X \to Y$. Then the sheaf ω_π is torsion-free and satisfies (S_2).*

Proof. By local duality, we have an isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_\pi) \simeq \mathrm{Hom}_{\mathcal{B}}(\pi_*(\mathcal{F}), \mathcal{B})$ for any coherent sheaf $\mathcal{F}$ on X. If $\mathcal{F}$ is torsion then so is $\pi_*(\mathcal{F})$; this readily implies that ω_π is torsion-free. To show that ω_π is (S_2), it suffices to check that so is $\pi_*(\omega_\pi)$ (Lemma 4.11). As $\pi_*(\omega_\pi) \simeq \mathcal{H}om_{\mathcal{B}}(\mathcal{A}, \mathcal{B})$, the desired assertion follows from the normality of Y and [Har80, Cor. 1.2 and Prop. 1.3]. □

We also record a preliminary result, to be used in Sections 8 and 9.

Lemma 7.3. *Consider again a finite group scheme G, and a G-normal variety X with quotient $\pi : X \to Y$. Then there exists a G-stable big open subset $U \subset X$ such that the quotient $V = U/G$ is regular and the morphism $\pi|_U : U \to V$ is flat l.c.i.*

Proof. We may replace X with $\pi^{-1}(Y_{\mathrm{reg}})$, and hence assume that Y is regular. Then π is flat in codimension 1, so that we may further assume that it is flat everywhere. We now argue as in the proof of Lemma 7.1 (i). With the notation of that proof, the variety Z is normal by Lemma 4.1, and Z_{reg} is stable by GL_n as the latter is smooth. Thus, we have $Z_{\mathrm{reg}} = \mathrm{GL}_n \times^G U$ for a unique G-stable open subset $U \subset X$. Moreover, U is big in X, since Z_{reg} is big in Z. To complete the proof, it suffices to show that $\pi|_U$ is l.c.i. But $\pi|_U = \psi|_{Z_{\mathrm{reg}}} \circ \iota$, where $\iota : U \to Z_{\mathrm{reg}}$ is a regular immersion and $\psi|_{Z_{\mathrm{reg}}} : Z_{\mathrm{reg}} \to Y$ is a morphism between regular varieties, and hence is l.c.i. by [Li06, Ex. 6.3.18]. So the desired assertion follows from loc. cit., Prop. 6.3.20. □

Next, we assume that G is diagonalizable and X is a G-normal variety on which G acts faithfully. Denote by $\iota : U \subset X$ the inclusion of the largest strongly G-normal subset (Lemma 6.1). Then we have $\omega_\pi = \iota_*(\omega_{\pi|_U})$ in view of Lemmas 4.11 and 7.2. Therefore, to determine ω_π, we may further assume that X *is strongly G-normal.*

Theorem 7.4. *With the above assumptions, π is flat and l.c.i. Moreover, the canonical G-invariant section $s_{X/Y}$ of $\omega_{X/Y}$ satisfies*

$$\mathrm{div}(s_{X/Y}) = \sum \big(|H(D)| - 1\big)\, G \cdot D, \tag{7.2}$$

where the sum runs over the G-orbits of nonfree divisors D.

Proof. The first assertion follows from Corollary 4.12 and Theorem 6.2 (ii). With the notation of this theorem, X is covered by the G-stable open subsets $U(H, \nu)$, and hence we may assume that $X = U(H, \nu)$. Then $\pi = \psi \circ \varphi$, where $\psi : Z = X/H \to Y = X/G$ is a G/H-torsor. Thus, $s_{Z/Y}$ is a trivializing section of $\omega_{Z/Y}$ (Lemma 7.1). Also, $\omega_{X/Y} \simeq \omega_{X/Z}$ by [Li06, Thm. 6.4.9]; this identifies $s_{X/Y}$ with $s_{X/Z}$ in view of (7.1). So it suffices to check that the divisor of $s_{X/Z}$ equals $(|H| - 1) \sum_D G \cdot D$ (sum over the G-orbits of nonfree divisors).

Let D be such a divisor, with generic point x. With the notation of Proposition 5.5, the A^H-module A is free with basis $1, f, \ldots, f^{n-1}$, where f is a local equation of Ω_y at x, and $n = |H|$. Denote by $e_0, \ldots, e_{n-1}$ the dual basis of the A^H-module $\varphi^! A = \mathrm{Hom}_{A^H}(A, A^H)$; then e_0 is the projection $A \to A^H$. Moreover, the A-module structure of $\varphi^! A$ is given by $f \cdot e_i = e_{i-1}$ for $i = 1, \ldots, n-1$, and $f \cdot e_0 = e_{n-1}$. So $\varphi^! A$ is freely generated by e_{n-1}, and $e_0 = f^{n-1} e_{n-1}$. In view of Proposition 5.5 again, it follows that $\mathrm{div}(s_{X/Y}) = (n-1) G \cdot D$ in a neighborhood of x. □

Remark 7.5. Still assuming G diagonalizable and X strongly G-normal, we may rewrite (7.2) as the equality of Cartier divisors

$$\mathrm{div}(s_{X/Y}) = \pi^*(\Delta_Y) - G \cdot \Delta_X, \tag{7.3}$$

where $\Delta_Y = Y \setminus Y_{\mathrm{fr}}$ and $\Delta_X = X \setminus X_{\mathrm{fr}}$ are viewed are reduced effective divisors, so that Δ_Y is the branch divisor. Indeed,

$$G \cdot \Delta_X = \sum G \cdot D$$

(sum over the G-orbits of nonfree divisors); this is a G-stable effective Cartier divisor on X. Moreover,

$$\pi^*(\Delta_Y) = \sum |H(D)| \, G \cdot D$$

in view of Theorem 6.2 (iv).

In turn, (7.3) gives an isomorphism of G-linearized sheaves

$$\omega_X(G \cdot \Delta_X) \simeq \pi^*\big(\omega_Y(\Delta_Y)\big), \tag{7.4}$$

where the dualizing sheaf ω_X is equipped with a G-linearization via the canonical isomorphism

$$\omega_X \simeq \omega_{X/Y} \otimes \pi^*(\omega_Y). \tag{7.5}$$

The isomorphism (7.4) will be used in the next section to obtain a version of the Chevalley–Weil formula for projective G-normal curves. The equality (7.3) will be generalized to linearly reductive group schemes in Section 9.

8 Curves

Throughout this section, G denotes a finite diagonalizable group scheme, Y a curve, and X a G-normal curve over Y. In this situation, all the above developments take a much simpler form: the G-action on X is generically free (Lemma 3.4) and Y is regular (Lemma 4.1). Moreover, the quotient $\pi : X \to Y$ is flat l. c. i. (Lemma 7.3). Also, recall that the orbit $G \cdot x$ is an effective Cartier divisor on X for any closed point $x \in X$, and the nonfree locus $X \setminus X_{\mathrm{fr}}$ consists of finitely many closed points; in particular, X is strongly G-normal. For any nonfree point $x \in X$, the subgroup scheme $H = H(x) \subset G$ is cyclic (Proposition 5.1). Moreover, we have $X = \bigcup_{H,\nu} U(H,\nu)$, where $U(H,\nu)$ is the union of X_{fr} and those nonfree x such that $H(x) = H$ and $G \cdot x$ has an equation of weight ν in $\mathcal{O}_{X,G\cdot x}$ (see Theorem 6.2 for these results).

By Proposition 4.3, the G-normal curve X is uniquely determined by its fiber at the generic point of Y. Also, Theorem 7.4 readily yields a version of the Hurwitz formula:

Corollary 8.1. *There is an isomorphism of G-linearized sheaves*

$$\omega_{X/Y} \simeq \mathcal{O}_X \Big(\sum (|H(x)| - 1)\, G \cdot x \Big) \tag{8.1}$$

(sum over the G-orbits of nonfree points), which identifies $s_{X/Y}$ with the canonical section of the right-hand side.

We now assume X projective. Taking degrees in (7.4) and using the equality $\pi^*(\pi(x)) = |H(x)|\, G \cdot x$ (which follows from Theorem 6.2), we obtain

$$2p_a(X) - 2 = |G| \big(2p_a(Y) - 2 + \sum_{y \in Y \setminus Y_{\mathrm{fr}}} (1 - \frac{1}{n(y)}) \deg(y) \big), \tag{8.2}$$

where $n(y) = |H(x)|$ for any $y = \pi(x) \in Y \setminus Y_{\mathrm{fr}}$. This generalizes (1.1) to an arbitrary field.

We may also determine the degree of the invertible sheaves $\mathcal{A}_\lambda$ on Y, where $\lambda \in \Lambda$. By Remark 6.4, we have the equality in $\mathrm{Pic}(Y)_{\mathbb{Q}}$

$$[\mathcal{A}_\lambda] = - \sum_{y \in Y \setminus Y_{\mathrm{fr}}} \frac{m(y,\lambda)}{n(y)} y,$$

where $m(y,\lambda)$ denotes the unique integer such that $0 \leq m(y,\lambda) \leq n(y) - 1$ and $\lambda - m(y,\lambda)\nu(y)$ restricts trivially to $H(y) = H(x)$. As a consequence,

$$\deg(\mathcal{A}_\lambda) = - \sum_{y \in Y \setminus Y_{\mathrm{fr}}} \frac{m(y,\lambda)}{n(y)} \deg(y). \tag{8.3}$$

Still assuming X projective, we consider the G-linearized sheaf $\omega_X^{\otimes n}$, where n is any positive integer. We will determine its equivariant Euler characteristic, that is, the character

$$\operatorname{ch}\chi(\omega_X^{\otimes n}) = \operatorname{ch} H^0(X, \omega_X^{\otimes n}) - \operatorname{ch} H^1(X, \omega_X^{\otimes n}),$$

where the character of a finite-dimensional G-module $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is defined as the formal sum $\operatorname{ch}(M) = \sum_{\lambda\in\Lambda} \dim(M_\lambda)\, e^\lambda$ in the group algebra $\mathbb{Z}[\Lambda] = \bigoplus_{\lambda\in\Lambda} \mathbb{Z}\, e^\lambda$. Our key ingredient is the following observation.

Lemma 8.2. *Let $s = s_{X/Y}$ be the canonical G-invariant section of $\omega_{X/Y}$. For any integer $n \geq 1$, consider the exact sequence of coherent G-linearized sheaves on X*

$$0 \longrightarrow \omega_X^{\otimes n} \stackrel{s^n}{\longrightarrow} \pi^*\big(\omega_Y^{\otimes n}(n\Delta_Y)\big) \longrightarrow \mathcal{F}_n \longrightarrow 0. \tag{8.4}$$

Then $\mathcal{F}_n$ has a filtration by G-linearized subsheaves with subquotients being $e^{m\nu(y)}\mathcal{O}_{\Omega_y}$ ($m = 0, \ldots, n-1$, $y \in Y \setminus Y_{\mathrm{fr}}$).

Here $\mathcal{O}_{\Omega_y}$ denotes the structure sheaf of the G-orbit Ω_y, equipped with its natural G-linearization, and $e^{m\nu(y)}\mathcal{O}_{\Omega_y}$ stands for the same sheaf with G-linearization twisted by the character $m\nu(y)$.

Proof. We work in the category of coherent G-linearized sheaves. We then have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-n\, G\cdot\Delta_X) \stackrel{s^n}{\longrightarrow} \mathcal{O}_X \longrightarrow \mathcal{G}_n \longrightarrow 0,$$

where $\mathcal{G}_n$ has support $X \setminus X_{\mathrm{fr}}$. Also, there is an isomorphism

$$\omega_X^{\otimes n}(n\, G\cdot\Delta_X) \simeq \pi^*\big(\omega_Y^{\otimes n}(n\Delta_Y)\big)$$

in view of (7.4). This gives the exact sequence (8.4) with

$$\mathcal{F}_n = \mathcal{G}_n \otimes_{\mathcal{O}_Y} \pi^*\big(\omega_Y^{\otimes n}(n\Delta_Y)\big).$$

Moreover, decomposing $X \setminus X_{\mathrm{fr}}$ into the disjoint union of the set-theoretic fibers $\pi^{-1}(y)$ where $y \in Y \setminus Y_{\mathrm{fr}}$, we obtain an isomorphism

$$\mathcal{G}_n \simeq \bigoplus_{y\in Y\setminus Y_{\mathrm{fr}}} \mathcal{O}_X/\mathcal{O}_X(-n\,\Omega_y).$$

Every summand $\mathcal{O}_X/\mathcal{O}_X(-n\,\Omega_y)$ has a canonical filtration with subquotients being $\mathcal{O}_X(-m\,\Omega_y)/\mathcal{O}_X\big(-(m+1)\,\Omega_y\big)$, where $m = 0, \ldots, n-1$. Furthermore, Proposition 5.5 yields isomorphisms

$$\mathcal{O}_X(-m\,\Omega_y)/\mathcal{O}_X\big(-(m+1)\,\Omega_y\big) \simeq e^{m\nu(y)}\,\mathcal{O}(\Omega_y).$$

This implies our statement by using the isomorphism

$$\pi^*(\mathcal{M})|_{\Omega_y} \simeq \mathcal{O}_{\Omega_y}$$

for any invertible sheaf $\mathcal{M}$ on Y (which holds as Ω_y is contained in the schematic fiber of π at y). □

Proposition 8.3. *For any $n \geq 1$ and $\lambda \in \Lambda$, we have*

$$\operatorname{ch}\chi(\omega_X^{\otimes n}) = (2n-1)\,(p_a(Y)-1)\operatorname{ch}\mathcal{O}(G) + \sum_{y \in Y \setminus Y_{\mathrm{fr}}} \gamma(y)\operatorname{ch}\mathcal{O}\big(G/H(y)\big)\deg(y),$$

where $\gamma(y) = \sum_{m=0}^{n(y)-1}(n-1-m)\,e^{m\nu(y)}$.

Proof. This follows from Lemma 8.2 by taking cohomology. More specifically, the short exact sequence (8.4) yields a long exact sequence of cohomology G-modules, and hence the equality

$$\operatorname{ch}\chi(\omega_X^{\otimes n}) = \operatorname{ch}\chi\big(\pi^*(\omega_Y^{\otimes n}(n\Delta_Y))\big) - \operatorname{ch}\chi(\mathcal{F}_n).$$

Moreover, we obtain

$$\operatorname{ch}\chi\big(\pi^*(\omega_Y^{\otimes n}(n\Delta_Y))\big) = \sum_{\lambda \in \Lambda} \chi\big(Y, \omega_Y^{\otimes n}(n\Delta_Y) \otimes \mathcal{A}_\lambda\big)\,e^\lambda$$

by using the projection formula. So the Riemann–Roch theorem yields

$$\operatorname{ch}\chi\big(\pi^*(\omega_Y^{\otimes n}(n\Delta_Y))\big) = \big((2n-1)\,(p_a(Y)-1) + n\deg(\Delta_Y)\big)\operatorname{ch}\mathcal{O}(G) + \sum_{\lambda \in \Lambda}\deg(\mathcal{A}_\lambda)\,e^\lambda. \tag{8.5}$$

Also, we obtain

$$\operatorname{ch}\chi(\mathcal{F}_n) = \operatorname{ch}H^0(X, \mathcal{F}_n) = \sum_{y \in Y \setminus Y_{\mathrm{fr}}} \Big(\sum_{m=0}^{n-1} e^{m\nu(y)}\Big)\operatorname{ch}\mathcal{O}(\Omega_y),$$

by using the filtration of $\mathcal{F}_n$ from Lemma 8.2, and

$$\operatorname{ch}\mathcal{O}(\Omega_y) = \deg(y)\operatorname{ch}\mathcal{O}\big(G/H(y)\big)$$

as follows from Proposition 5.1. Together with (8.3) and (8.5), this gives the desired formula. □

Remark 8.4. By Serre duality, the G-module $H^1(X, \omega_X)$ is the trivial module k. So Proposition 8.3 for $n = 1$ determines the character of the G-module $H^0(X, \omega_X)$. It gives back the Hurwitz formula (8.2) by specializing characters to dimensions. But the latter formula has a much more direct proof.

For $n \geq 2$, the character of the G-module $H^0(X, \omega_X^{\otimes n})$ is still determined by the above proposition if $p_a(Y) \geq 2$, since $H^1(X, \omega_X^{\otimes n}) = 0$ under this assumption. Indeed, we then have $\deg(\omega_X) > 0$ in view of (8.2), so the desired vanishing follows from Serre duality again.

Also, if X is a G-torsor over Y then Proposition 8.3 boils down to the equality

$$\operatorname{ch} \chi(\omega_X^{\otimes n}) = (2n-1)\,(p_a(Y)-1)\operatorname{ch} \mathcal{O}(G),$$

which is easily checked directly. For an arbitrary projective G-normal curve X, the proposition expresses the equivariant Euler characteristic of $\omega_X^{\otimes n}$ as the sum of the above "main term" and "correcting terms" associated with the branch points.

As a consequence of Proposition 8.3, we obtain

$$\chi(\omega_X^{\otimes n})_0 = (2n-1)(p_a(Y)-1) + \sum_{y \in Y \setminus Y_{\mathrm{fr}}} \big(n - \lfloor \frac{n-1}{n(y)} \rfloor\big) \deg(y), \tag{8.6}$$

where the "virtual multiplicity" $\chi(\omega_X^{\otimes n})_0$ is defined via the decomposition

$$\operatorname{ch} \chi(\omega_X^{\otimes n}) = \sum_{\lambda \in \Lambda} \chi(\omega_X^{\otimes n})_\lambda \, e^\lambda.$$

Remark 8.5. The *canonical ring*

$$R(\omega_X) = \bigoplus_{n=0}^{\infty} H^0(X, \omega_X^{\otimes n})$$

is a graded algebra equipped with a G-action. One may check that $R(\omega_X)$ is a finitely generated domain of Krull dimension at most 2, which is G-normal and hence Cohen–Macaulay. As a consequence, the G-invariant subring

$$R(\omega_X)^G = \bigoplus_{n=0}^{\infty} H^0(X, \omega_X^{\otimes n})_0$$

is a normal domain of Krull dimension at most 2, and thus is Cohen–Macaulay as well.

If $p_a(Y) \geq 2$ then $\deg(\omega_X) > 0$ and $\operatorname{ch} R(\omega_X)_n = \operatorname{ch} \chi(\omega_X^{\otimes n})$ by Remark 8.4; in particular, $R(\omega_X)^G$ has Krull dimension 2 and we have $\dim R(\omega_X)^G_n = \chi(\omega_X^{\otimes n})_0$. So Proposition 8.3 and (8.6) yield numerical information on the canonical ring and its G-invariant subring. For instance, one obtains the following formula for the Hilbert series of the graded algebra $R(\omega_X)^G$:

$$\frac{(1+z)\big(1+(p_a(Y)-3)z+z^2\big)}{(1-z)^2} - \sum_{y \in Y \setminus Y_{\mathrm{fr}}} \deg(y)\, \frac{z^2(1-z^{n(y)-1})}{(1-z)^2(1-z^{n(y)})}.$$

In the remainder of this section, we assume that G is infinitesimal; equivalently, $G \simeq \boldsymbol{\mu}_{p^r}$ by [Br24, Lem. 5.3]. (The general case somehow reduces to this in view of the factorization (2.3) of $\pi : X \to Y$ as $X \xrightarrow{\varphi} Z = X/G^0 \xrightarrow{\psi} Y$, where ψ is a finite abelian cover).

We identify G with $\boldsymbol{\mu}_{p^r}$, and hence the character group Λ with $\mathbb{Z}/p^r\mathbb{Z}$. So the subgroup schemes of G are exactly the $\boldsymbol{\mu}_{p^s}$ for $s = 0, \ldots, r$. Choose a nonfree point $x \in X$ and let $A = \mathcal{O}_{X,x}$, $y = \pi(x)$ and $B = \mathcal{O}_{Y,y}$ as in Section 5; then we have $H(x) = \boldsymbol{\mu}_{p^s}$ for some positive integer $s = s(x) \leq r$. We may now state a refinement of Proposition 5.5:

Proposition 8.6. *With the above notation, we have an isomorphism of $\mathbb{Z}/p^r\mathbb{Z}$-graded B-algebras*

$$A \simeq B[T_1, T_2]/(T_1^{p^{r-s}} - u, T_2^{p^s} - t\, T_1^{\nu})$$

where T_1 (resp. T_2) is homogeneous of weight p^s (resp. ν prime to p), $u \in B$ is a unit, and $t \in B$ generates the maximal ideal.

Proof. By Corollary 5.3, the morphism $X/H \to Y$ is a torsor under $G/H = \boldsymbol{\mu}_{p^{r-s}}$ in a neighborhood of y. Together with Remark 3.2, this yields an isomorphism of $\mathbb{Z}/p^r\mathbb{Z}$-graded B-algebras

$$A^H \simeq B[T_1]/(T_1^{p^{r-s}} - u),$$

where T_1 is homogeneous of weight p^s and $u \in B^\times$. We also have an isomorphism of $\mathbb{Z}/p^r\mathbb{Z}$-graded B-algebras

$$A \simeq A^H[T_2]/(T_2^{p^s} - g),$$

where T_2 is homogeneous of weight ν prime to p, and $g \in A^H$ is homogeneous of weight $p^s\nu$; moreover, $gA = \mathfrak{n}A$ (Proposition 5.5). Thus, $g = t\, T_1^{\nu}$, where $t \in B$ generates $\mathfrak{n}$. Combining both displayed isomorphisms yields the statement. □

Remark 8.7. The above proposition takes a much simpler form in the case that $r = s$, i.e., x is a G-fixed point: then $A \simeq B[T]/(T^{p^r} - t)$ for a uniformizer $t \in B$. As a consequence, the homomorphism $B/Bt \to A/Af$ is bijective, where f denotes the image of T in A. Equivalently, X is regular at x and $\kappa(x) = B/Bt = \kappa(y)$. This yields a refinement of [Br24, Prop. 5.5].

On the other hand, if $r > s$ then we have with the notation of Section 5:

$$\bar{A} = A/tA \simeq \kappa(y)[\bar{T_1}, \bar{T_2}]/(\bar{T_1}^{p^{r-s}} - \bar{u}, \bar{T_2}^{p^s}),$$

where $\bar{u} \in \kappa(y)^\times$ denotes the image of $u \in A^\times$. If in addition k is perfect, then so is $\kappa(y)$ (as the extension $\kappa(y)/k$ is finite), and hence we obtain

$$\bar{A} \simeq \kappa(y)[U_1, U_2]/(U_1^{p^{r-s}}, U_2^{p^s}).$$

Thus, U_1, U_2 form a minimal system of generators of the ideal $\bar{\mathfrak{m}}$. So X is not regular at x, and $\kappa(x) = \kappa(y)$. But if $\kappa(y)$ is imperfect, then we may choose $u \in A^\times$ such that $\bar{u} \notin \kappa(y)^p$, and hence $A/Af \simeq \kappa(y)[T]/(T^{p^{r-s}} - \bar{u})$ is a field, where f denotes the image of T_2 in A. In that case, X is regular but not smooth at x, and $\kappa(y) \subsetneq \kappa(x)$; the morphism $\operatorname{Spec}(\kappa(x)) \to \operatorname{Spec}(\kappa(y))$ is the torsor under $(\boldsymbol{\mu}_{p^{r-s}})_{\kappa(y)}$ considered in Proposition 5.1.

Next, let s, ν be integers such that $1 \leq s \leq r$ and $0 \leq \nu \leq p^r - 1$, and let $U = U(s, \nu)$ be the open subset of X consisting of the free locus X_{fr} together with the closed points $x \in X$ such that $H(x) = \boldsymbol{\mu}_{p^s}$ and the largest $\mathbb{Z}/p^r\mathbb{Z}$-graded ideal of $\mathcal{O}_{X,x}$ is generated by a homogeneous element of weight ν. Then $V = V(s, \nu) = U/G$ is an open subset of Y containing Y_{fr}, and $\Delta = \Delta(s, \nu) = V \setminus V_{\mathrm{fr}}$ is a reduced effective divisor on V. With the notation of the beginning of Section 6, we have $U = U(H, \nu)$ and $\Delta = \Delta(H, \nu)$, where $H = \boldsymbol{\mu}_{p^s}$. So the multiplication in $\mathcal{A}|_V$ yields isomorphisms of $\mathcal{B}|_V$-modules

$$\sigma = \sigma_{\nu, p^s} : \mathcal{A}_\nu^{\otimes p^s}|_V \xrightarrow{\sim} \mathcal{A}_{\nu p^s}|_V(-\Delta), \quad \tau = \sigma_{p^s \nu, p^{r-s}} : \mathcal{A}_{p^s}^{\otimes p^{r-s}}|_V \xrightarrow{\sim} \mathcal{B}|_V,$$

since $\mathcal{A}_{p^s} \subset \mathcal{A}^H$ and the morphism $U/H = \operatorname{Spec}_{\mathcal{B}|_V}(\mathcal{A}^H) \to V$ is a G/H-torsor (Theorem 6.2). We may now state:

Proposition 8.8. *There is an isomorphism of $\mathbb{Z}/p^r\mathbb{Z}$-graded $\mathcal{B}|_V$-algebras*

$$\mathcal{A}|_V \simeq \operatorname{Sym}_{\mathcal{B}|_V}(\mathcal{A}_\nu \oplus \mathcal{A}_{p^s})/\mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by the $a_1 \cdots a_{p^s} - \sigma(a_1 \otimes \cdots \otimes a_{p^s})$ with $a_1, \ldots, a_{p^s} \in \mathcal{A}_\nu$, and the $b_1 \cdots b_{p^{r-s}} - \tau(b_1 \otimes \cdots \otimes b_{p^{r-s}})$ with $b_1, \ldots, b_{p^{r-s}} \in \mathcal{A}_{p^s}$.

Proof. Using the fact that $\mathrm{Spec}_{\mathcal{B}|_V}(\mathcal{A}^H) \to V$ is a G/H-torsor together with Remark 3.2, we obtain an isomorphism

$$\mathcal{A}^H|_V \simeq \mathrm{Sym}_{\mathcal{B}|_V}(\mathcal{A}_{p^s})/\mathcal{J},$$

where $\mathcal{J}$ denotes the ideal generated by the $b_1 \cdots b_{p^{r-s}} - \tau(b_1 \otimes \cdots \otimes b_{p^{r-s}})$ with $b_1, \ldots, b_{p^{r-s}} \in \mathcal{A}_{p^s}$. The assertion follows by combining this isomorphism with that of Theorem 6.2 (ii). □

The above proposition takes again a much simpler form in the case that $r = s$: then we just have an isomorphism of $\mathcal{B}|_V$-modules $\sigma : \mathcal{A}_\nu^{\otimes p^r}|_V \xrightarrow{\sim} \mathcal{O}_V(-\Delta)$, and the algebra $\mathcal{A}|_V$ is the quotient of $\mathrm{Sym}_{\mathcal{B}|_V}(\mathcal{A}_\nu)$ by the ideal generated by the $a_1 \cdots a_{p^r} - \sigma(a_1 \otimes \cdots \otimes a_{p^r})$. Moreover, U is regular along U^H by Remark 8.7.

Finally, we consider the tangent sheaf $\mathcal{T}_X$ consisting of the k-linear derivations of the structure sheaf $\mathcal{O}_X$, and its relative version $\mathcal{T}_{X/Y}$ consisting of the $\mathcal{O}_Y$-linear derivations. Recall that $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ is equipped with a G-linearization, since so is Ω^1_X; likewise, $\mathcal{T}_{X/Y}$ is G-linearized as well. As a consequence, for any $x \in X$, the fibre $\mathcal{T}_X(x)$ is a linear representation of $H(x)$. By [Br24, Prop. 5.4], the sheaf $\mathcal{T}_X$ is invertible; in particular, its fibre at x is a vector space of dimension 1 over the residue field $\kappa(x)$, for any x as above. We will recover the latter result in a more explicit way:

Corollary 8.9. *With the above notation, we have $\mathcal{T}_U = \mathcal{T}_{U/V}$. Moreover, $\mathcal{T}_U(x)$ is a $\kappa(x)$-vector space of dimension 1 on which $H(x)$ acts with weight $-\nu$, for any nonfree point $x \in U$.*

Proof. Since U is affine, the fraction field of $\mathcal{O}(U)$ is the function field $k(X)$. Moreover, the invariant subfield $k(X)^G$ is generated by k and $k(X)^{p^r}$ (see e.g. [Br24, Prop. 5.1]). Thus, every k-linear derivation of $\mathcal{O}(U)$ vanishes on $\mathcal{O}(U)^G = \mathcal{O}(V)$. This shows the first assertion.

For the second assertion, we use the structure of the $\mathbb{Z}/p^r\mathbb{Z}$-graded B-algebra A (Proposition 8.6), which gives an isomorphism of graded A-modules

$$\Omega^1_{A/B} \simeq (A\,dT_1 \oplus A\,dT_2)/(A\,t\,T_1^{\nu-1}dT_1)$$

with the notation of this proposition. Taking A-duals, we obtain an isomorphism of graded A-modules $\mathrm{Der}_B(A) \simeq A\,\partial_2$, where ∂_2 arises from the derivation $\partial/\partial T_2$ of $B[T_1, T_2]$. As T_2 has weight ν, this yields the result. □

For instance, if $x \in X(k)$ is G-fixed, then it is a smooth point of X (e.g. by Remark 8.7) and hence the Zariski tangent space $T_x(X)$ is a line on which G acts with weight $-\nu$.

Remark 8.10. The tangent sheaf $\mathcal{T}_X$ is equipped with a G-invariant global section: the generator ξ of the Lie algebra $\mathrm{Lie}(G) = \mathrm{Lie}(\boldsymbol{\mu}_{p^r}) = \mathrm{Lie}(\mathbb{G}_m) = k$. We determine the divisor of ξ, which is clearly supported at the nonfree points. Given such a point x, we have $\xi(T_1) = 0$ and $\xi(T_2) = \nu T_2$ with the notation of Proposition 8.6; thus, $\xi = \nu \partial_2$. As a consequence, the ideal of zeroes of ξ in A equals AT_2. In turn, this equals $I_{G \cdot x}$ by Proposition 5.5. Thus, $A/AT_2 = \mathcal{O}(G \cdot x)$, and hence

$$\mathrm{div}(\xi) = \sum G \cdot x = G \cdot \Delta_X,$$

where the sum runs over the G-orbits of nonfree points.

Given x as above and $y = \pi(x)$, the orbit $G \cdot x = G_{\kappa(y)} \cdot x$ is a $(G/H)_{\kappa(y)}$-torsor over $\mathrm{Spec}(\kappa(y))$ (Proposition 5.1). Thus, we have $\dim_k \mathcal{O}(G \cdot x) = p^{r-s} \deg(y)$ and hence the degree of the tangent sheaf satisfies

$$\deg(\mathcal{T}_X) = \sum_{y \in Y \setminus Y_{\mathrm{fr}}} p^{r-s(y)} \deg(y) = |G| \sum_{y \in Y \setminus Y_{\mathrm{fr}}} \frac{\deg(y)}{n(y)}.$$

In particular, we have $\deg(\mathcal{T}_X) \geq 0$, and equality holds if and only if X is a G-torsor over Y. Also, note that the fixed point subscheme X^G is reduced and satisfies $\deg(\mathcal{T}_X) \equiv \deg(X^G) \pmod{p}$.

9 Linearly reductive group schemes

Throughout this section, we denote by G a finite linearly reductive group scheme, and by X a G-normal variety on which G acts faithfully with quotient $\pi : X \to Y$. Our first aim is to determine the relative dualizing sheaf ω_π as a G-linearized sheaf. Using Lemmas 7.2 and 7.3, we may assume that Y is regular and π is flat l. c. i. Then $\omega_\pi = \omega_{X/Y}$ is invertible and has a canonical G-invariant section $s_{X/Y}$, which trivializes it over the free locus and whose formation commutes with flat base change on Y.

Theorem 9.1. *With the above assumptions, we have* $\mathrm{div}(s_{X/Y}) = \pi^*(\Delta_Y) - G \cdot \Delta_X$.

Proof. We start with a reduction to the case that G^0 *is diagonalizable and* $\pi_0(G)$ *is constant.* As in Remark 4.9, we may choose a finite Galois extension K/k such that G^0_K is diagonalizable and $\pi_0(G)_K$ is constant. Then the G_K-variety X_K satisfies our assumptions; moreover, $\Delta_{Y_K} = (\Delta_Y)_K$ as the base change to K preserves the free locus and reducedness, and $(G \cdot \Delta_X)_K = G_K \cdot (\Delta_X)_K$ as the formation of the schematic image commutes with flat base change. By Galois descent, it thus suffices to prove the theorem for the G_K-variety X_K.

If $G = G^0$ then the desired assertion follows from Remark 7.5. We now prove this assertion when $G = \pi_0(G)$, i.e., G *is constant*. Since G is linearly reductive, its order is prime to p; also, recall that X is normal.

We will need a folklore result for which we could not find any reference. Consider a point $x \in X$ with centralizer $C_G(x)$. Then the quotient morphism $\pi : X \to Y$ factors as

$$X \xrightarrow{\varphi} Z = X/C_G(x) \xrightarrow{\psi} Y = X/G, \quad x \longmapsto z \longmapsto y.$$

In turn, ψ factors as

$$Z \xrightarrow{\tau} W = X/N_G(x) \xrightarrow{\eta} Y, \quad z \longmapsto w \longmapsto y,$$

where $N_G(x)$ denotes the largest subgroup of G that stabilizes x, and τ is the quotient by $N_G(x)/C_G(x)$.

Lemma 9.2. *With the above notation, we have:*

(i) $\kappa(z) = \kappa(x)$.

(ii) *τ is a torsor under $N_G(x)/C_G(x)$ in a neighborhood of z.*

(iii) *η induces an isomorphism of completed local rings $\widehat{\mathcal{O}}_{Y,y} \simeq \widehat{\mathcal{O}}_{W,w}$.*

(iv) *ψ is étale at z and s_ψ is a trivializing section of ω_ψ at that point.*

(v) *If x is the generic point of a prime divisor, then the linear representation of $C_G(x)$ in the cotangent space at x is faithful. In particular, $C_G(x)$ is cyclic.*

Proof. (i) This follows from the isomorphism of local rings $\mathcal{O}_{Z,z} \simeq \mathcal{O}_{X,x}^{C_G(x)}$ in view of the linear reductivity of $C_G(x)$.

(ii) Likewise, the linear reductivity of $N_G(x)$ and the isomorphism $\mathcal{O}_{W,w} \simeq \mathcal{O}_{X,x}^{N_G(x)}$ imply that $\kappa(w) = \kappa(x)^{N_G(x)} = \kappa(x)^{N_G(x)/C_G(x)}$. As $N_G(x)/C_G(x)$ acts faithfully on x, the desired assertion follows from Galois theory.

(iii) This is a special case of [SGA1, Exp. V, Prop. 2.2]) and can be proved directly as follows: π induces an isomorphism $\widehat{\mathcal{O}}_{Y,y} \simeq \widehat{\mathcal{O}}_{X,G\cdot x}^G$. Moreover, $G \cdot x$ is the disjoint union of the $g \cdot x$, where $g \in G/N_G(x)$. Thus, we have a G-equivariant isomorphism of semi-local rings $\widehat{\mathcal{O}}_{X,G\cdot x} \simeq \prod_{g \in G/N_G(x)} \widehat{\mathcal{O}}_{X,g\cdot x}$. Taking G-invariants, we obtain $\widehat{\mathcal{O}}_{Y,y} \simeq \widehat{\mathcal{O}}_{X,x}^{N_G(x)}$. Moreover, $\mathcal{O}_{X,x}^{N_G(x)} \simeq \mathcal{O}_{W,w}$; this yields the assertion.

(iv) The morphism τ is étale at z by (ii), and hence ω_τ is invertible at that point; moreover, s_τ is a trivializing section in view of Lemma 7.1. Also, η is étale at w by (iii), so that ω_η is invertible at that point. Moreover, s_η is a trivializing section, since its formation commutes with flat base change. This implies the assertions in view of the isomorphism $\omega_\psi \simeq \omega_\tau \otimes \tau^*(\omega_\eta)$ identifying s_ψ with $s_\tau \otimes \tau^*(s_\eta)$.

(v) The group $C_G(x)$ acts faithfully on the discrete valuation ring $\mathcal{O}_{X,x}$ and stabilizes the powers $\mathfrak{m}_x^n$ of the maximal ideal. Since $\bigcap_{n\geq 1} \mathfrak{m}_x^n = 0$, the induced action on some quotient $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is faithful as well. By linear reductivity, it follows that the induced action on some subquotient $\mathfrak{m}_x^m/\mathfrak{m}_x^{m+1}$ is faithful. As $\mathfrak{m}_x^m/\mathfrak{m}_x^{m+1}$ is the mth symmetric power of the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ (a vector space of dimension 1 over the residue field $\kappa(x)$), this yields our assertion. □

We may now prove Theorem 9.1 in the case that G constant. Let D be a prime divisor in X, with generic point x. By (7.1) and Lemma 9.2 (iv), the multiplicity of D in $s_{X/Y}$ equals that in $s_{X/Z}$. The latter multiplicity equals $|C_G(x)| - 1$ in view of Theorem 7.4 together with Lemma 9.2 (i), (v). On the other hand, denoting by E (resp. F) the image of D in Y (resp. Z), we see that F occurs in $\psi^*(E)$ with multiplicity 1 (as ψ is étale at z), and D occurs in $\varphi^*(F)$ with multiplicity $|C_G(x)|$ (by Theorem 6.2). So D occurs in $\pi^*(\Delta_Y) - G \cdot \Delta_X$ with multiplicity $|C_G(x)| - 1$ as desired.

Finally, we handle the general case (where G^0 is diagonalizable and $\pi_0(G)$ is constant of order prime to p). Recall the factorization of π as

$$X \xrightarrow{\varphi} Z = X/G^0 \xrightarrow{\psi} Y = X/G = Z/\pi_0(G). \tag{9.1}$$

Moreover, X is a G^0-normal variety on which G^0 acts generically freely, and Z is a normal variety on which $\pi_0(G)$ acts generically freely (Lemmas 2.1 and 3.4). Thus, ψ is a tamely ramified Galois cover with group $\pi_0(G)(k)$. We may further assume that X is strongly reductive relative to G^0.

Let again D be a prime divisor in X with generic point x and set $E = \pi(D)$, $F = \varphi(D)$ with generic points y, z. Then F occurs in $\psi^*(E)$ with multiplicity $|C_{\pi_0(G)}(z)|$ by the above step. Moreover, we have

$$\varphi^*(F) = |H(x)|\, G^0 \cdot D$$

by Theorem 6.2. Thus, $\pi^*(\Delta_Y) - G \cdot \Delta_X$ equals

$$\big(|H(x)|\, |C_{\pi_0(G)}(z)| - 1\big)\, G^0 \cdot D$$

in a neighborhood of x. On the other hand, we have

$$\mathrm{div}(s_{X/Y}) = \mathrm{div}(s_{X/Z}) + \varphi^* \mathrm{div}(s_{Y/Z})$$

by (7.1), and

$$\mathrm{div}(s_{X/Z}) = \big(|H(x)| - 1\big) G^0 \cdot D$$

in a neighborhood of x (Theorem 7.4), whereas

$$\mathrm{div}(s_{Y/Z}) = \big(|C_{\pi_0(G)}(z)| - 1\big)\, F$$

in a neighborhood of z (by the above step again). As a consequence, $G^0 \cdot D$ occurs in $\mathrm{div}(s_{X/Y})$ with multiplicity

$$|H(x)| - 1 + |H(x)| \,\big(|C_{\pi_0(G)}(z)| - 1\big) = |H(x)|\, |C_{\pi_0(G)}(z)| - 1. \tag{9.2}$$

So $\mathrm{div}(s_{X/Y})$ and $\pi^*(\Delta_Y) - G \cdot \Delta_X$ have the same multiplicity along $G^0 \cdot D$.

To complete the proof, it suffices to show that $G^0 \cdot D$ coincides with $G \cdot D$ in a neighborhood of x. Since $\pi_0(G)$ is constant, there exists a finite purely inseparable extension K/k such that $G_K = G^0_K \rtimes \pi_0(G)_K = \coprod_{g \in G(K)} G^0_K\, g$. Therefore, we have $G_K \cdot D_K = \bigcup_{g \in G(K)} G^0_K \cdot g \cdot D_K$. Moreover, D_K is irreducible (possibly nonreduced). Denoting by x' its generic point, the generic points of $G_K \cdot D_K$ are exactly the $g \cdot x'$, where $g \in G(K)$. As a consequence, $G^0_K \cdot D_K$ coincides with $G_K \cdot D_K$ in a neighborhood of x'. Taking the schematic image under the projection $\mathrm{pr} : X_K \to X$, $D_K \mapsto D$, $x' \mapsto x$ yields the desired assertion. □

From now on, we assume that k is algebraically closed and consider the case that X is a curve. Then $Y = X/G$ is a smooth curve, and $\pi : X \to Y$ is flat l. c. i. by Lemma 7.3. We first record a basic finiteness result:

Theorem 9.3. *Let G be a finite linearly reductive group scheme over an algebraically closed field, and Y a smooth curve. Then there are finitely many isomorphism classes of G-normal curves over Y having a prescribed branch divisor.*

Proof. This is a direct consequence of Corollary 4.6 if G is infinitesimal, and of [SGA1, Exp. XII, Cor. 2.12] if G is constant of order prime to p. The general case follows by using once more the factorization (9.1) of the quotient morphism. □

Next, we extend parts of Theorems 1 and 2 to our current setting:

Proposition 9.4. (i) *For any $x \in X(k)$, the group $\mathrm{Stab}_G(x)$ is cyclic, and the character associated with its linear action on the fiber $\mathcal{I}_{G\cdot x}(x)$ generates its character group.*

(ii) *We have*

$$\operatorname{div}(s_{X/Y}) = \sum \big(|\operatorname{Stab}_G(x)| - 1\big)\, G\cdot x$$

(sum over the G-orbits of k-rational points).

Proof. (i) The assertion follows from Proposition 5.5 in the case that G is infinitesimal, and from Lemma 9.2 (v) in the case that G is constant. The general case will follow by combining these cases and using some additional arguments. We first show that $\operatorname{Stab}_G(x)$ is cyclic.

Since k is algebraically closed, we have $G = G^0 \rtimes \pi_0(G)$ and $\operatorname{Stab}_G(x) = \operatorname{Stab}_{G^0}(x) \rtimes \operatorname{Stab}_{\pi_0(G)}(x)$. Moreover, we may identify G^0 with $\boldsymbol{\mu}_{p^r}$ for some $r \geq 0$, and this identifies $\operatorname{Stab}_{G^0}(x)$ with $\boldsymbol{\mu}_{p^s}$ for some $s \leq r$. Consider again the factorization (9.1) of π, where $Z = X/G^0$ is a smooth curve. Let $z = \varphi(x)$; then $\operatorname{Stab}_{\pi_0(G)}(x) = \operatorname{Stab}_{\pi_0(G)}(z)$ as φ is bijective and $\pi_0(G)$-equivariant. Also, $\operatorname{Stab}_{\pi_0(G)}(z)$ is cyclic by Lemma 9.2 (v).

To complete the proof, it suffices to check that $\operatorname{Stab}_{\pi_0(G)}$ centralizes $\operatorname{Stab}_{G^0}(x)$, since both groups are cyclic of coprime orders. Consider the subgroup scheme $H = G^0 \rtimes \operatorname{Stab}_{\pi_0(G)}(x)$ of G, and the corresponding factorization $X \to X/H \to Y$ of π. Note that X is an H-normal curve (since $H \supset G^0$); in particular, X/H is a smooth curve. Also, $\operatorname{Stab}_G(x) \subset H$. Thus, we may replace G with H, and assume that $\pi_0(G)$ *fixes* x.

We now use some constructions and results from Section 5. Clearly, we have $G\cdot x = G^0 \cdot x$, and G fixes z. Thus, $A = \mathcal{O}_{X,G\cdot x} = \mathcal{O}_{X,x}$ is a local algebra equipped with a G-action, as well as its quotient algebra $\bar{A} = \mathcal{O}(X_z)$. The ideal I of $G\cdot x$ in A is G-stable and generated by a G^0-eigenvector f of weight $\nu \in \Lambda(G^0)$. The action of $\pi_0(G)$ on G^0 by conjugation yields an action on $\Lambda(G^0)$ by group automorphisms, which stabilizes the subgroup $\Lambda\big(G^0/\operatorname{Stab}_{G^0}(x)\big)$ (as $\pi_0(G)$ normalizes $\operatorname{Stab}_{G^0}(x)$). For any $g \in \pi_0(G)(k) = G(k)$, we have $g\cdot f = u_g f$ for some $u_g \in A^\times$. Moreover, $g \cdot f$ is a G^0-eigenvector of weight $g\cdot\nu$, and hence u_g is a G^0-eigenvector of weight $g\cdot\nu - \nu$. By Proposition 5.5, we have $g\cdot\nu - \nu \in \Lambda\big(G^0/\operatorname{Stab}_{G^0}(x)\big)$ and the character group of $\operatorname{Stab}_{G^0}(x)$ is generated by the restriction of ν. As a consequence, $\pi_0(G)$ acts trivially on this character group. Equivalently, $\pi_0(G)$ centralizes $\operatorname{Stab}_{G^0}(x)$ as desired. This shows that $\operatorname{Stab}_G(x)$ is cyclic indeed.

Next, observe that the map $G(k) \to \Lambda\big(G^0/\operatorname{Stab}_{G^0}(x)\big)$, $g \mapsto g\cdot\nu - \nu$ is a 1-cocycle from a finite group of order prime to p, to a finite p-group. Thus, this map is a 1-coboundary: there exists $\nu_0 \in \Lambda\big(G^0/\operatorname{Stab}_{G^0}(x)\big)$ such that $g\cdot\nu - \nu = g\cdot\nu_0 - \nu_0$ for all $g \in G(k)$. We may choose a G^0-eigenvector $\bar{f}_0 \in \bar{A}$ of weight ν_0, and lift it to a G^0-eigenvector $f_0 \in A$ (of the same weight). Then $\bar{f}_0 \in \bar{A}^\times$, and hence $f_0 \in A^\times$.

Replacing f with f/f_0, we may thus assume that $g \cdot \nu = \nu$ for all $g \in G(k)$. Then $\overline{u_g} \in \bar{A}^{G^0} = k$ for all such g, and hence $\bar{f}$ is a G-eigenvector of weight $\chi \in \Lambda(G)$.

Since G is linearly reductive, we may lift $\bar{f}$ to a G-eigenvector $\varphi \in I$ (of the same weight). Then $I = A\,\varphi$ by Nakayama's lemma, and hence $\mathrm{Stab}_G(x)$ acts on the fiber $I(x) = \mathcal{I}_{G \cdot x}(x)$ by the character $\chi|_{\mathrm{Stab}_G(x)}$. By construction, we have $\chi|_{\mathrm{Stab}_{G^0}(x)} = \nu|_{\mathrm{Stab}_{G^0}(x)}$; recall that the latter generates $\Lambda\big(\mathrm{Stab}_{G^0}(x)\big)$. To complete the proof, it suffices to show that $\chi|_{\pi_0(G)}$ generates $\Lambda\big(\pi_0(G)\big)$.

Let $n = |\,\mathrm{Stab}_{G^0}(x)|$; then $\varphi^n \in A^{\mathrm{Stab}_{G^0}(x)} = \mathcal{O}_{Z,z}$, and φ^n generates its maximal ideal $\mathfrak{m}_z$ by Proposition 5.5. Thus, $n\,\chi|_{\pi_0(G)}$ is the weight of $\pi_0(G)$ in the cotangent space of Z at z. This yields the desired assertion (as $\pi_0(G)$ is constant).

(ii) By Equation (9.2) in the proof of Theorem 9.1, the orbit $G \cdot x$ (viewed as a Cartier divisor) occurs in $\mathrm{div}(s_{X/Y})$ with multiplicity $|\,\mathrm{Stab}_{G^0}(x)|\,|\,\mathrm{Stab}_{\pi_0(G)}(z)| - 1$. Moreover, $|\,\mathrm{Stab}_{G^0}(x)|\,|\,\mathrm{Stab}_{\pi_0(G)}(z)| = |\,\mathrm{Stab}_{G^0}(x)|\,|\,\mathrm{Stab}_{\pi_0(G)}(x)| = |\,\mathrm{Stab}_G(x)|$. □

Finally, we obtain a characterization of the finite linearly reductive group schemes that act faithfully on some curve, by an argument essentially due to Giancarlo Lucchini Arteche.

Proposition 9.5. *The following conditions are equivalent for a finite linearly reductive group scheme G over an algebraically closed field k:*

(i) *G admits a faithful action on a curve.*

(ii) *$G \simeq \boldsymbol{\mu}_{p^r} \rtimes H$ for some integer $r \geq 0$ and some finite (constant) group H of order prime to p.*

Proof. (i) ⇒ (ii) This follows from the structure of linearly reductive group schemes together with [Br24, Lem. 5.3].

(ii) ⇒ (i) There exists a smooth projective curve X on which H acts faithfully (we may even assume that $H = \mathrm{Aut}(X)$ by the main result of [MV83]). We will extend this action to a faithful *rational* action of G; by [Br24, Cor. 4.4], this will give a faithful G-action on some projective curve, equivariantly birational to X.

Let K be the function field of X, and choose $t \in K \setminus K^p$. Then t forms a p-basis of K, i.e., we have a decomposition $K = \bigoplus_{m=0}^{p-1} K^p\, t^m$ which is a $\mathbb{Z}/p\mathbb{Z}$-grading of the K^p-algebra K. By induction on r, we obtain a $\mathbb{Z}/p^r\mathbb{Z}$-grading

$$K = \bigoplus_{m=0}^{p^r-1} K^{p^r}\, t^m,$$

of the K^{p^r}-algebra K, such that t has weight 1. In turn, this gives a (functorial) faithful action of $\boldsymbol{\mu}_{p^r}$ on $\mathrm{Spec}(K)$, or equivalently a faithful rational $\boldsymbol{\mu}_{p^r}$-action on X in view of [Br24, Cor. 3.4] (one may check that every such rational action is obtained by this construction).

We also have a faithful H-action on $K = k(X)$, and an H-action on $\boldsymbol{\mu}_{p^r}$ by conjugation. The latter action is given by a character

$$\chi : H \longrightarrow (\mathbb{Z}/p^r\mathbb{Z})^\times,$$

as the automorphism group of $\boldsymbol{\mu}_{p^r}$ is isomorphic to $(\mathbb{Z}/p^r\mathbb{Z})^\times$. So the H-action on X extends to an action of $G = \boldsymbol{\mu}_{p^r} \rtimes H$ if and only if we have the equality

$$h^{-1}zh = z^{\chi(h)}$$

as automorphisms of the R-algebra $R \otimes K$, for any test algebra R and any $h \in H(R)$, $z \in \boldsymbol{\mu}_{p^r}(R) \subset R^\times$. Clearly, this equality holds on the R-subalgebra $R \otimes K^{p^r}$, and hence it suffices to check that

$$(h^{-1}zh)(t) = z^{\chi(h)}(t) = z^{\chi(h)} \otimes t \in R \otimes K,$$

where the second equality holds as t is a $\boldsymbol{\mu}_{p^r}$-eigenvector of weight 1. This is equivalent to the equality

$$z\big(h(t)\big) = z^{\chi(h)} \otimes h(t),$$

and hence to the condition that

$$h(t) \in K^{p^r} t^{\chi(h)} \quad \text{for all } h \in H \tag{9.3}$$

(note that $K^{p^r} t^{\chi(h)}$ makes sense as $\chi(h)$ is an integer mod p^r).

We now show the existence of a p-basis t satisfying (9.3). For this, we need some additional notation. Denote by C the image of χ, which is a finite cyclic group of order n. We may identify C with a subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$, since $|H|$ is prime to p. We then have an exact sequence of groups

$$1 \longrightarrow I \longrightarrow H \longrightarrow C \longrightarrow 1,$$

where $I = \mathrm{Ker}(\chi)$. Also, denote by $L \subset K$ the subfield of invariants K^I. Then C acts on L, and hence on the smooth projective curve Y with function field L. As seen in Remark 4.7, we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}(Y)[p^r] \longrightarrow L^\times/(L^\times)^{p^r} \longrightarrow \bigoplus_{y \in Y(k)} \mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\deg} \mathbb{Z}/p^r\mathbb{Z} \longrightarrow 0, \tag{9.4}$$

which is clearly C-equivariant (where C permutes the k-rational points of Y). In other words, (9.4) is an exact sequence of modules over the group algebra $\mathbb{Z}/p^r\mathbb{Z}[C]$.

We claim that the module $L^\times/(L^\times)^{p^r}$ contains a copy of $\mathbb{Z}/p^r\mathbb{Z}[C]$. Indeed, choosing a free point $y \in Y_{\mathrm{fr}}(k)$ yields a copy of $\mathbb{Z}/p^r\mathbb{Z}[C]$ in $\bigoplus_{y\in Y(k)} \mathbb{Z}/p^r\mathbb{Z}$, and hence a copy of the augmentation ideal $J \subset \mathbb{Z}/p^r\mathbb{Z}[C]$ in the $\mathbb{Z}/p^r\mathbb{Z}[C]$-module $\mathrm{Ker}(\deg) \subset \bigoplus_{y\in Y(k)} \mathbb{Z}/p^r\mathbb{Z}$. Moreover, we have an exact sequence of $\mathbb{Z}/p^r\mathbb{Z}[C]$-modules

$$0 \longrightarrow J \longrightarrow \mathbb{Z}/p^r\mathbb{Z}[C] \longrightarrow \mathbb{Z}/p^r\mathbb{Z} \longrightarrow 0,$$

where C acts trivially on $\mathbb{Z}/p^r\mathbb{Z}$. This exact sequence is split as $|C|$ is prime to p. Also, $\mathrm{Ker}(\deg)$ contains a copy of the trivial C-module $\mathbb{Z}/p^r\mathbb{Z}$, e.g., that generated by $\sum_{c\in C} c \cdot (y_1 - y_2)$, where $y_1, y_2 \in Y_{\mathrm{fr}}(k)$ are distinct. All of this yields a copy of $\mathbb{Z}/p^r\mathbb{Z}[C]$ in $\mathrm{Ker}(\deg)$, and hence implies our claim in view of (9.4).

By this claim and the fact that $\mathbb{Z}/p^r\mathbb{Z}$ contains the nth roots of unity, the $\mathbb{Z}/p^r\mathbb{Z}[C]$-module $L^\times/(L^\times)^{p^r}$ contains a copy of $\mathbb{Z}/p^r\mathbb{Z}$ on which C acts via the character χ. In other words, there exists $t \in L \setminus L^p$ such that $h(t) \in L^{p^r} t^{\chi(h)}$ for all $h \in H$. Then t satisfies (9.3), and $t \in K \setminus K^p$ as $K^p \cap L = (K^p)^I = L^p$. □

Remark 9.6. Let H be a finite group of order prime to p, acting on $\boldsymbol{\mu}_{p^r}$ via a character $\chi : H \to (\mathbb{Z}/p^r\mathbb{Z}]^\times$. By the above proof, given a smooth projective curve X on which H acts faithfully, there exists a projective curve X_r which is H-equivariantly birational to X and on which $\mu_{p^r} \rtimes H$ acts faithfully.

If $g(X) \geq 2$ then $X_r \neq X$, as the automorphism group scheme Aut_X is étale. Under this assumption, there exists no projective curve Y which is $\boldsymbol{\mu}_{p^r}$-normal for infinitely many values of r, and birational to X: otherwise, the connected automorphism group scheme Aut^0_Y is not finite, and hence Y must have infinitely many automorphisms, a contradiction. In particular, we obtain infinitely many pairwise nonisomorphic projective models X_r.

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