Cox rings of spherical varieties: Lecture 3

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Introduction to Lecture 3

As a continuation of Lecture 2, we will first obtain a presentation of the Cox ring of a wonderful variety by generators and relations, and apply it to the wonderful compactification of an adjoint semi-simple group.

To make this presentation more explicit in the general case, one needs to describe the product of two simple submodules in the coordinate ring of a spherical homogeneous space. This is an open question that we will discuss.

Next, we will turn to the Cox ring of an arbitrary spherical variety. For this, we will first introduce the equivariant Cox ring, discuss its relation to the ordinary Cox ring, and present some general results on the equivariant Cox ring of an almost homogeneous variety.

We will then consider the equivariant Cox ring of an arbitrary spherical variety, and show how to reduce its structure to that of the Cox ring of an associated wonderful variety.

We will illustrate the various notions and results for the class of horospherical varieties.

Recap on wonderful varieties

Let G be a simply-connected semi-simple group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus. Denote by U the unipotent part of B.

Let X be a wonderful G-variety. Denote by $D_1, \dots, D_r$ the boundary divisors and by $\gamma_1, \dots, \gamma_r \in \mathrm{X}^*(T)$ the corresponding spherical roots. For $i = 1, \dots, r$, the canonical section $s_i \in H^0(X, \mathcal{O}_X(D_i))$ is G-invariant.

Denote by G/H the open G-orbit in X, and by $\mathcal{D}$ the set of colors. To any $D \in \mathcal{D}$, one assigns a $B \times H$-eigenvector $f_D \in \mathcal{O}(G)$ with B-weight ω_D and H-weight χ_D. The canonical section $s_D \in H^0(X, \mathcal{O}_X(D))$ is a B-eigenvector of weight ω_D.

The character group $\mathrm{X}^*(H)$ is generated by the χ_D, $D \in \mathcal{D}$. The intersection of the kernels of the χ_D is a closed normal subgroup $H_1 \lhd H$. Moreover, H/H_1 is diagonalizable with character group $\mathrm{X}^*(H)$.

We have $\mathrm{Cox}(G/H) = \mathcal{O}(G/H_1)$ and the algebra $\mathcal{O}(G/H_1)^U$ is the polynomial ring in the f_D, $D \in \mathcal{D}$.

A presentation of their Cox ring

Recall from Lecture 2 that $\mathrm{Cox}(X)^U$ is the polynomial ring in the canonical sections $s_1, \ldots, s_r$ and s_D, $D \in \mathcal{D}$.

So the $\mathrm{Cl}(X)$-graded ring $\mathrm{Cox}(X)$ is generated by $s_1, \ldots, s_r$ and simple G-modules $V_D \simeq V(\omega_D)$, where $D \in \mathcal{D}$. Moreover, each s_i is homogeneous of degree $[D_i]$ and each V_D is homogeneous of degree $[D]$.

In other terms, we have $\mathrm{Cox}(X) = S/I$, where

$$S = \mathbb{C}[s_1, \ldots, s_r] \otimes \mathrm{Sym}(\bigoplus_{D \in \mathcal{D}} V_D)$$

and $I \subset S$ is a homogeneous ideal.

To determine I, it suffices to describe the multiplication map

$$m_{D_1,D_2} : V_{D_1} \otimes V_{D_2} \longrightarrow V_{D_1} V_{D_2}$$

for any two $D_1, D_2 \in \mathcal{D}$, where the right-hand side denotes the product in $\mathrm{Cox}(X)$. Indeed, it is easy to show that the ideal I is generated by the kernels of the m_{D_1,D_2}.

We view each $V_D \subset \mathcal{O}(G)^{(H)}_{\chi_D}$ as the G-submodule of $\mathcal{O}(G/H_1)$ generated by f_D. More generally, given integers $n_D \geq 0$ ($D \in \mathcal{D}$), we denote by $V_{\sum n_D D}$ the G-submodule of $\mathcal{O}(G/H_1)$ generated by $\prod_{D \in \mathcal{D}} f_D^{n_D}$.

Since $\mathcal{O}(G/H_1)^U = \mathbb{C}[f_D, D \in \mathcal{D}]$, we have a canonical isomorphism of G-modules

$$\mathcal{O}(G/H_1) = \bigoplus_{(n_D)} V_{\sum n_D D},$$

where the sum runs over all families of non-negative integers indexed by $\mathcal{D}$.

Thus, $V_{D_1} V_{D_2}$ is a partial sum of simple modules $V_{\sum n_D D}$, where

$$\sum_{D \in \mathcal{D}} n_D \omega_D \leq_X \omega_{D_1} + \omega_{D_2} \text{ and } \sum_{D \in \mathcal{D}} n_D \chi_D = \chi_{D_1} + \chi_{D_2}.$$

So each simple component of a product $V_{D_1} V_{D_2}$ is uniquely determined by its highest weight λ, and is the Cartan component of the tensor product $\bigotimes_{D \in \mathcal{D}} V_D^{\otimes n_D}$, where $\lambda = \sum_D n_D \omega_D$. Moreover, we have

$$\omega_{D_1} + \omega_{D_2} - \lambda = \sum_{i=1}^{r} n_i \gamma_i$$

for some non-negative integers $n_1, \ldots, n_r$.

Denote by $p^{\lambda}_{D_1,D_2}$ the composition of the maps

$$V_{D_1} \otimes V_{D_2} \longrightarrow V_{D_1} V_{D_2} \longrightarrow V(\lambda) \longrightarrow \bigotimes_{D \in \mathcal{D}} V_D^{\otimes n_D} \longrightarrow \mathrm{Sym}(\bigoplus_{D \in \mathcal{D}} V_D).$$

Then $p^{\lambda}_{D_1,D_2}$ is uniquely determined up to a non-zero scalar.

Proposition

There exists unique normalizations of the $p^{\lambda}_{D_1,D_2}$ such that the ideal I of relations is generated by

$$v_1 \otimes v_2 - \sum_{\lambda} p^{\lambda}_{D_1,D_2}(v_1 \otimes v_2) \prod_{i=1}^{r} s_i^{n_i},$$

where $v_1 \in V_{D_1}$, $v_2 \in V_{D_2}$ and the sum runs over the dominant weights of the form $\omega_{D_1} + \omega_{D_2} - \sum_{i=1}^{r} n_i \gamma_i$ for some non-negative integers $n_1, \ldots, n_r$.

Indeed, the above relations hold by the description of $\mathrm{Cox}(X)$ in Lecture 2. They generate the kernels of the multiplication maps m_{D_1,D_2}, and hence the ideal of relations.

Example: wonderful group compactifications

Let X be the wonderful compactification of the adjoint group G_{ad}. Then the graded ring $\mathrm{Cox}(X)$ is generated by $s_1, \ldots, s_r$ and the $G \times G$-submodules $V(\varpi_i)^* \otimes V(\varpi_i) = \mathrm{End}(V(\varpi_i))$ for $i = 1, \ldots, r$, where the ϖ_i are the fundamental weights.

We have isomorphisms of G-modules

$$V(\varpi_i) \otimes V(\varpi_j) = \bigoplus_{\lambda \in \Lambda^+} c_{i,j}^{\lambda} \, V(\lambda)$$

for $1 \leq i \leq j \leq r$, where the multiplicities $c_{i,j}^{\lambda}$ are certain "Littlewood-Richardson" coefficients.

By the lemma below, it follows that we have in $\mathcal{O}(G)$

$$\mathrm{End}(V(\varpi_i)) \; \mathrm{End}(V(\varpi_j)) = \bigoplus_{\lambda} \mathrm{End}(V(\lambda)),$$

where the sum runs over those $\lambda \in \Lambda^+$ such that $c_{i,j}^{\lambda} \neq 0$. Then $\lambda = \varpi_i + \varpi_j - \sum_{k=1}^{r} n_k \alpha_k$ for unique non-negative integers $n_1, \ldots, n_r$.

So the ideal of relations of $\mathrm{Cox}(X)$ is generated by the

$$v_i \otimes v_j - \sum_{\lambda} p_{i,j}^{\lambda}(v_i \otimes v_j) \prod_{k=1}^{r} s_k^{n_k},$$

where $v_i \in \mathrm{End}(V(\varpi_i))$; $v_j \in \mathrm{End}(V(\varpi_j))$; $\lambda \in \Lambda^+$ satisfies $c_{i,j}^{\lambda} \neq 0$; and

$$p_{i,j}^{\lambda} : \mathrm{End}(V(\varpi_i)) \otimes \mathrm{End}(V(\varpi_j)) \longrightarrow \mathrm{Sym}(\bigoplus_{i=1}^{r} \mathrm{End}(V(\varpi_i))$$

is the map defined above.

If $G = \mathrm{SL}_n$ then $V(\varpi_i) = \bigwedge^i \mathbb{C}^n$ for $i = 1, \ldots, n-1$. Moreover,

$$V(\varpi_i) \otimes V(\varpi_j) = \bigoplus V(\varpi_{i'} + \varpi_{j'}),$$

where the sum runs over those (i', j') such that $0 \leq i' \leq i \leq j \leq j' \leq n$, and we set $\varpi_0 = \varpi_n = 0$.

As a consequence, the relations of $\mathrm{Cox}(X)$ are quadratic for the $\mathbb{Z}$-grading in which each color has degree 1. This also holds for $G = \mathrm{Sp}_{2n}$ but for no other simple group.

The multiplication in the coordinate ring of G

For any $\lambda, \mu \in \Lambda^+$, we have an isomorphism of G-modules
$$V(\lambda) \otimes V(\mu) \simeq \bigoplus_{\nu \in \Lambda^+, \nu \leq \lambda+\mu} c_{\lambda,\mu}^{\nu} \, V(\nu).$$

Lemma

With the above notation, we have in $\mathcal{O}(G)$

$$\mathrm{End}(V(\lambda)) \ \mathrm{End}(V(\mu)) = \bigoplus_{\nu} \mathrm{End}(V(\nu)),$$

where the sum runs over those $\nu \in \Lambda^+$ such that $c_{\lambda,\mu}^{\nu} \neq 0$.

Proof.

The inclusion "$\subset$" is clear. To show the opposite inclusion, recall that $\mathrm{End}(V(\lambda))$ is identified with a $G \times G$-submodule of $\mathcal{O}(G)$ by sending any endomorphism u to the map $g \mapsto \mathrm{Tr}(u \circ g)$. In particular, the identity map of $V(\lambda)$ is sent to the character χ_λ of this simple G-module. Since we have
$$\chi_\lambda \, \chi_\mu = \sum_{\nu \in \Lambda^+} c_{\lambda,\mu}^{\nu} \, \chi_\nu$$
in $\mathcal{O}(G)$, this yields the statement. □

The multiplication in the coordinate ring of a spherical homogeneous space

For any arbitrary spherical homogeneous space G/H, describing the product of any two simple G-submodules in $\mathcal{O}(G/H)$ is an open question.

We discuss this question in the case where H is reductive. Then each simple G-submodule $V(\lambda) \subset \mathcal{O}(G/H)$ contains a unique H-fixed line $\mathbb{C}f_\lambda$, where f_λ is a **spherical function**.

The spherical functions form a basis of the algebra of bi-invariant regular functions $\mathcal{O}(G)^{H\times H}$. Thus, we have

$$f_\lambda\ f_\mu = \sum_\nu a^\nu_{\lambda,\mu} f_\nu,$$

where the sum runs over the dominant weights in $\Lambda(G/H)$, and $a^\nu_{\lambda,\mu} \in \mathbb{C}$.

By a result of Ruitenburg, $V(\nu)$ occurs in the product $V(\lambda)\ V(\mu)$ if and only if $a^\nu_{\lambda,\mu} \neq 0$. This reduces the above question to the decomposition of the product of spherical functions into a sum of spherical functions.

This decomposition is unknown in general. When G/H is a symmetric space with restricted root system of type A, a conjecture of Bravi and Gandini predicts that

$$V(\nu) \subset V(\lambda)\, V(\mu) \Leftrightarrow V(\nu) \subset V(\lambda) \otimes V(\mu) \text{ and } \nu \leq_{G/H} \lambda + \mu$$

for all $\lambda, \mu, \nu \in \Lambda^+ \cap \Lambda(G/H)$, where $\leq_{G/H}$ is defined via the restricted root system $\Phi_{G/H}$.

The implication "$\Rightarrow$" follows from the definition of $\leq_{G/H}$.
The converse implication is a consequence of a conjecture of Stanley on the multiplication of Jack symmetric functions.

Bravi and Gandini establish the conjecture when $\Phi_{G/H}$ is a product of root systems of type A_1. They also propose generalizations of the conjecture to further classes of spherical homogeneous spaces G/H with H reductive, including Hermitian symmetric spaces.

The equivariant Cox sheaf of a normal G-variety

Let X be a normal variety equipped with an action of a linear algebraic group G. Define the **equivariant class group** $\mathrm{Cl}_G(X)$ as the group of isomorphism classes of G-linearized reflexive sheaves $\mathcal{F}$ of rank 1. Then $\mathrm{Cl}_G(X) \simeq \mathrm{Pic}_G(U)$, where U denotes the smooth locus. We assume that this group is finitely generated, and denote by $S_{X,G}$ the "dual" diagonalizable group.

Choose a base point $x \in U$. Then the construction of the Cox sheaf adapts to this equivariant setting, by considering G-linearized sheaves equipped with a rigidification at x. The assumption that $\mathcal{O}(X)^* = \mathbb{C}^*$ is replaced with $\mathcal{O}(X)^{*G} = \mathbb{C}^*$, since $\mathrm{Aut}^G(\mathcal{F}) = \mathcal{O}(X)^{*G}$ for any $\mathcal{F}$ as above. This yields the **equivariant Cox sheaf**

$$\mathcal{R}_{X,G} = \bigoplus_{[\mathcal{F}] \in \mathrm{Cl}_G(X)} \mathcal{F}^x$$

and the morphism

$$q : \hat{X}_G = \mathrm{Spec}_X(\mathcal{R}_{X,G}) \to X.$$

The group $G \times S_{X,G}$ acts on $\hat{X}_G$. Moreover, q is G-equivariant and is a good quotient by $S_{X,G}$.

The equivariant Cox ring

With the above notation and assumptions, define the **equivariant Cox ring**

$$\mathrm{Cox}_G(X) = H^0(X, \mathcal{R}_{X,G}).$$

This is a $\mathrm{Cl}_G(X)$-graded ring equipped with a compatible action on G. Moreover, $\mathrm{Cox}_G(X) = \mathrm{Cox}_G(X_0)$ for any G-stable big open subset $X_0 \subset X$. Taking for X_0 the smooth locus U, we obtain

$$\mathrm{Cox}_G(X) = \mathrm{Cox}_G(U) = \bigoplus_{[\mathcal{L}] \in \mathrm{Pic}_G(U)} H^0(U, \mathcal{L}^\times).$$

If $\mathrm{Cox}_G(X)$ is finitely generated, then we have a diagram

$$\begin{array}{ccl} \hat{X}_G & \xrightarrow{\iota} & \tilde{X}_G = \mathrm{Spec}(\mathrm{Cox}_G(X)) \\ {\scriptstyle q}\downarrow & & \\ X & & \end{array}$$

where ι is the inclusion of a G-stable big open subset.

Some basic properties of the equivariant Cox ring

Assume in addition that G is connected and $\mathrm{Pic}(G) = 0$. Then $\mathrm{Cl}_G(X)$ is finitely generated if and only if so is $\mathrm{Cl}(X)$, in view of the exact sequence

$$0 \longrightarrow \mathrm{X}^*(G) \longrightarrow \mathrm{Cl}_G(X) \longrightarrow \mathrm{Cl}(X) \longrightarrow 0$$

(which follows from the corresponding exact sequence for Picard groups).

Proposition

Let X be a normal variety such that $\mathcal{O}(X)^{*G} = \mathbb{C}^*$, equipped with an action of a connected linear algebraic group G such that $\mathrm{Pic}(G) = 0$.

If the ring $\mathrm{Cox}_G(X)$ is finitely generated, then it is a normal domain, graded by $\mathrm{Cl}_G(X)$ and on which G acts by graded automorphisms.

Under these assumptions, $\hat{X}_G$ is a variety as well. Moreover, the natural map $\hat{U}_G \to U$ is a principal bundle under $S_{X,G} = S_{U,G}$.

As a consequence, $\hat{U}_G$ is a smooth variety with ring of regular functions being $\mathrm{Cox}_G(X)$.

Relation to the ordinary Cox ring

We now assume that $\mathcal{O}(X)^* = \mathbb{C}^*$, so that the equivariant Cox ring $\mathrm{Cox}_G(X)$ and the ordinary Cox ring $\mathrm{Cox}(X)$ are defined.

For any $\lambda \in \mathrm{X}^*(G)$ and any G-linearized invertible sheaf $\mathcal{L}$ on the smooth locus U, we obtain a G-linearized sheaf $\mathcal{L}[\lambda] = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X[\lambda]$. If $\mathcal{L}$ is rigidified at $x \in U$, then so is $\mathcal{L}[\lambda]$. This yields an action of $\mathrm{X}^*(G)$ on $\mathrm{Cox}_G(X)$. Also, recall that the abelian group $\mathrm{X}^*(G)$ is free of finite rank; choose a basis $\lambda_1, \ldots, \lambda_s$ of this group.

Theorem

With the above notation, $\mathrm{Cox}_G(X)$ is a free module over the Laurent polynomial ring $\mathbb{C}[\mathrm{X}^*(G)] \simeq \mathbb{C}[\lambda_1^{\pm 1}, \ldots, \lambda_s^{\pm 1}]$.
Moreover, there is a canonical isomorphism of graded rings

$$\mathrm{Cox}_G(X)/(\lambda_1 - 1, \ldots, \lambda_s - 1) \simeq \mathrm{Cox}(X)$$

compatible with the isomorphism $\mathrm{Cl}_G(X)/\mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_s \simeq \mathrm{Cl}(X)$.

In particular, $\mathrm{Cox}_G(X) = \mathrm{Cox}(X)$ whenever every character of G is trivial (e.g., if G is semi-simple and simply-connected, or is unipotent).

The equivariant Cox ring of a G-stable open subset

Let $X_0 \subset X$ be a G-stable open subset and $D_1, \ldots, D_m$ the irreducible components of codimension 1 of $X \setminus X_0$. Then each D_i is G-invariant, and hence $\mathcal{O}_X(D_i)$ is G-linearized. Denote by $s_i \in H^0(X, \mathcal{O}_X(D_i))^G$ the canonical section. Also, assume that $\mathcal{O}(X_0)^{*G} = \mathbb{C}^*$, so that both $\mathrm{Cox}_G(X_0)$ and $\mathrm{Cox}_G(X)$ are well-defined.

Proposition

With the above notation, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}^m \xrightarrow{f} \mathrm{Cl}_G(X) \longrightarrow \mathrm{Cl}_G(X_0) \longrightarrow 0$$

where $f(n_1, \ldots, n_m) = [\mathcal{O}_X(n_1 D_1 + \cdots + n_m D_m)]$.

Moreover, $s_1 - 1, \ldots, s_r - 1$ *form a regular sequence in* $\mathrm{Cox}_G(X)$, *and*

$$\mathrm{Cox}_G(X_0) \simeq \mathrm{Cox}_G(X)/(s_1 - 1, \ldots, s_r - 1).$$

If in addition X_0 *is a unique* G*-orbit, then* $\mathrm{Cox}_G(X)^G = \mathbb{C}[s_1, \ldots, s_r]$.

Homogeneous spaces

Let $X = G/H$ be a homogeneous space with base point x.

The G-linearized line bundles on X equipped with a rigidification at x are exactly the homogeneous line bundles $L(\lambda)$, where $\lambda \in \mathrm{X}^*(H)$.

Moreover, there is a G-equivariant isomorphism $H^0(X, L(\lambda)) \simeq \mathcal{O}(G)^{(H)}_{\lambda}$, where the right-hand side denotes the space of right H-eigenvectors of weight λ in $\mathcal{O}(G)$, on which G acts via left multiplication.

Thus, we have $\mathrm{Cl}_G(X) = \mathrm{Pic}_G(X) = \mathrm{X}^*(H)$ and

$$\mathrm{Cox}_G(X) = \bigoplus_{\lambda \in \mathrm{X}^*(H)} \mathcal{O}(G)^{(H)}_{\lambda} \simeq \mathcal{O}(G/H_1),$$

where H_1 denotes the intersection of the kernels of the $\lambda \in \mathrm{X}^*(H)$ (so that $H_1 \triangleleft H$ and H/H_1 is diagonalizable with character group $\mathrm{X}^*(H)$). Moreover, $q : \hat{X}_G \to X$ is the projection $G/H_1 \to G/H$. This is a principal bundle under H/H_1.

Example

If G is semi-simple and simply-connected, and $H = P$ is a parabolic subgroup, then $H_1 = [P, P]$ and we recover the example of flag varieties.

Spherical varieties: some notation

Let G be a connected reductive group, and X a spherical G-variety.

Replacing G with a finite cover, we may assume that $G = \mathbf{G} \times C$, where $\mathbf{G}$ is semi-simple and simply-connected, and C is the largest central subtorus of G. Then $\mathrm{X}^*(G) = \mathrm{X}^*(C)$ and $\mathrm{Pic}(G) = 0$. Thus, every line bundle L on X has a G-linearization, and any two linearizations of L differ by a character of C.

We may further replace C with any quotient group, and hence assume that C acts faithfully on X.

We denote by $X^0 = X^0_G$ the open G-orbit, and by $D_1, \ldots, D_m$ the prime G-stable divisors in X (the **boundary divisors**). These are the irreducible components of codimension 1 of $X \setminus X^0_G$.

Also, denote by $X^{\leq 1} = X^{\leq 1}_G$ the union of X^0_G and the G-orbits of codimension 1. Then $X^{\leq 1}$ is a G-stable big open subset of X, which meets each boundary divisor along its open G-orbit. Moreover, $X^{\leq 1}$ is a smooth toroidal G-variety.

Fom now on, we assume that $\mathcal{O}(X)^* = \mathbb{C}^*$. Then the Cox ring $\mathrm{Cox}(X)$ is defined. Moreover, $\mathcal{O}(X^{\leq 1})^* = \mathbb{C}^*$ and $\mathrm{Cox}(X) = \mathrm{Cox}(X^{\leq 1})$ as rings graded by $\mathrm{Cl}(X) = \mathrm{Cl}(X^{\leq 1}) = \mathrm{Pic}(X^{\leq 1})$.

Likewise, $\mathrm{Cox}_G(X) = \mathrm{Cox}_G(X^{\leq 1})$ as rings graded by $\mathrm{Cl}_G(X) = \mathrm{Pic}_G(X^{\leq 1})$.

Thus, we may replace X with $X^{\leq 1}$. In particular, *we may assume X smooth and toroidal.*

Example

Let X be a toric variety under $G = T$, with fan Σ. Then the fan of $X^{\leq 1}$ consists of the origin (which corresponds to the open orbit) and the rays $\rho_1, \ldots, \rho_m \in \Sigma(1)$ (which correspond to the boundary divisors $D_1, \ldots, D_m$).

The group $\mathrm{Cl}_T(X) = \mathrm{Pic}_T(X^{\leq 1})$ is freely generated by the classes of the sheaves $\mathcal{O}_X(D_i)$ equipped with their natural linearization. In particular, $\mathrm{Cl}_T(X)$ is torsion-free.

But the divisor class group $\mathrm{Cl}(X)$ may have "arbitrarily large" torsion. For example, let $X = \mathrm{Spec}\, R(\mathbb{P}^1; \mathcal{O}(m))$ (the affine cone over $\mathbb{P}^1$ embedded in $\mathbb{P}^{m+1}$ as a rational normal curve). Then one easily checks that X is an affine toric surface and $\mathrm{Cl}(X) \simeq \mathbb{Z}/m\mathbb{Z}$.

A further example

Take for G a simply-connected semi-simple group of rank ≥ 2.
Choose two distinct fundamental weights ϖ_1, ϖ_2 and denote by P^1, P^2 the corresponding maximal parabolic subgroups. Also, choose a positive integer m and let

$$v = v_{m\varpi_1} + v_{m\varpi_2} \in V = V(m\varpi_1) \oplus V(m\varpi_2),$$

where $v_{m\varpi_1}, v_{m\varpi_2}$ are highest weight vectors. Finally, let

$$X = \overline{G \cdot [v]} \subset \mathbb{P}(V).$$

Then we have

$$X = G \cdot [v] \cup G \cdot [v_{m\varpi_1}] \cup G \cdot [v_{m\varpi_2}] = G/H \cup G/P^1 \cup G/P^2,$$

where $H \subset P^1 \cap P^2$ is the kernel of the character $m(\varpi_1 - \varpi_2)$.

Thus, X is a horospherical variety of rank 1. Moreover, one may check that the open orbit G/H is a big open subset of X. As a consequence,

$$\mathrm{Cl}(X) = \mathrm{Cl}_G(X) = \mathrm{Cl}_G(G/H) = \mathrm{X}^*(H) \simeq (\mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2)/\mathbb{Z}m(\varpi_1 - \varpi_2)$$

and this group has torsion subgroup $\mathbb{Z}/m\mathbb{Z}$.

We have $\mathrm{Cox}(X) = \mathrm{Cox}_G(X) = \mathrm{Cox}_G(G/H) = \mathcal{O}(G/H_1)$, where H_1 is the stabilizer of v in G. So $\hat{X} = G/H_1$ and $\tilde{X} = \overline{G \cdot v} \subset V$.

Further notation on spherical varieties

Choose a Borel subgroup $B \subset G$; then $B = \mathbf{B} \times C$, where $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$.

Denote by $X_B^0 \subset X_G^0$ the open B-orbit, and by $\mathcal{D}$ the set of prime B-stable divisors in X_G^0 (the **colors**). Then $X_G^0 \setminus X_B^0 = \bigcup_{D \in \mathcal{D}} D$.

Choose $x \in X_B^0$ with isotropy group H in G. Then $X_G^0 = G \cdot x = G/H$ and $X_B^0 = B \cdot x = B/B \cap H$.

The pull-back of each $D \in \mathcal{D}$ under the quotient map $G \to G/H$ is an effective divisor, stable by $B \times H$: it has an equation $f_D \in \mathcal{O}(G)$, unique up to multiplication in $\mathcal{O}(G)^* = \mathbb{C}^* \, \mathrm{X}^*(C)$. So we may normalize f_D by requiring that $f_D(e) = 1$ and f_D is invariant by C. Then f_D is unique, and is a $B \times H$-eigenvector. We denote its weight by (ω_D, χ_D).

Let $V_D \subset \mathcal{O}(G)$ be the span of the $g \cdot f_D$, where $g \in G$ acts by left multiplication. Then $V_D \subset \mathcal{O}(G)^{(H)}_{\chi_D}$ is a simple G-module with highest weight ω_D. This defines a G-morphism $G/H \to \mathbb{P}(V_D^*)$ which extends uniquely to a G-morphism

$$\varphi_D : X \longrightarrow \mathbb{P}(V_D)^*.$$

The projective space $\mathbb{P}(V_D^*)$ contains a unique B-stable hyperplane, with equation $f_D \in V_D$. Moreover, the pull-back of this hyperplane under φ_D is the color D identified with its closure in X.

This defines a canonical G-linearization of the invertible sheaf $\mathcal{O}_X(D) = \varphi_D^* \mathcal{O}_{\mathbb{P}(V_D^*)}(1)$. We denote by $[D]_G$ the corresponding class in $\mathrm{Cl}_G(X) = \mathrm{Pic}_G(X)$. The canonical section $s_D \in H^0(X, \mathcal{O}_X(D))$ is a B-eigenvector of weight ω_D.

Also, the sheaf $\mathcal{O}_X(D_i)$ associated with any boundary divisor D_i is canonically G-linearized. We denote by $[D_i]_G \in \mathrm{Cl}_G(X)$ the corresponding class. The canonical section $s_i \in H^0(X, \mathcal{O}_X(D_i))$ is G-invariant.

By a general result on equivariant Cox rings, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}^m \xrightarrow{f} \mathrm{Cl}_G(X) \longrightarrow \mathrm{Cl}_G(G/H) \longrightarrow 0,$$

where $f(n_1, \ldots, n_m) = \sum_{i=1}^m a_i [D_i]_G$. Moreover, $s_1 - 1, \ldots, s_m - 1$ form a regular sequence in $\mathrm{Cox}_G(X)$, and we have an isomorphism of graded rings

$$\mathrm{Cox}_G(G/H) \simeq \mathrm{Cox}_G(X)/(s_1 - 1, \ldots, s_m - 1).$$

Finally, the invariant ring $\mathrm{Cox}_G(X)^G$ is freely generated by $s_1, \ldots, s_m$.

The equivariant Cox ring of a spherical variety

Keep the above notation, and denote by U the unipotent part of B. Choose a basis $\lambda_1, \ldots, \lambda_s$ of the free abelian group $\mathrm{X}^*(C)$.

Proposition

The invariant ring $\mathrm{Cox}_G(X)^U$ *is the Laurent polynomial ring* $\mathbb{C}[\lambda_1^{\pm}, \ldots, \lambda_m^{\pm}, s_1, \ldots, s_m, s_D \ (D \in \mathcal{D})] = \mathcal{O}(C)[s_1, \ldots, s_m, s_D \ (D \in \mathcal{D})]$.
Moreover, $\mathrm{Cox}_G(X)$ *is a free module over* $\mathrm{Cox}_G(X)^G = \mathbb{C}[s_1, \ldots, s_m]$.

This generalizes a result of Lecture 2 (where G is semi-simple and simply-connected, and X is wonderful), and is proved by similar arguments.

For example, if G is a torus T, then X is a toric variety and we obtain $\mathrm{Cox}_T(X) = \mathcal{O}(T)[s_1, \ldots, s_m]$, where the s_i are the canonical sections of the toric divisors. This gives back Cox's theorem in view of the relation between the equivariant and ordinary Cox rings.

Returning to the spherical setting, one shows as in Lecture 2:

Proposition

$\mathrm{Cox}_G(X)$ *is a finitely generated normal domain.*

The spherical closure

The normalizer $N_G(H)$ acts on G/H via right multiplication. This action commutes with the G-action, and hence permutes the colors $D \in \mathcal{D}$. (Also, recall that the $N_G(H)$-action on G/H factors through an action of the quotient group $N_G(H)/H$, and this group is diagonalizable).

Definition

The largest subgroup of $N_G(H)$ which stabilizes each color is called the **spherical closure** of H and denoted by $\bar{H}$.

Clearly, we have $H\,C \subset \bar{H} \subset N_G(H)$ and $N_G(H)/\bar{H}$ is finite. Thus, $\bar{H} = \mathbf{H} \times C \subset \mathbf{G} \times C = G$, where $\mathbf{H} \subset \mathbf{G}$ is a sober spherical subgroup (i.e., $N_{\mathbf{G}}(\mathbf{H})/\,\mathbf{H}$ is finite). Moreover, $\mathcal{D}(\mathbf{G}\,/\,\mathbf{H}) = \mathcal{D}$.

By a result of Knop, $\mathbf{G}\,/\,\mathbf{H}$ admits a wonderful compactification $\mathbf{X}$. The natural map $G/H \to G/\bar{H} = \mathbf{G}\,/\,\mathbf{H}$ extends uniquely to a morphism

$$\varphi : X \longrightarrow \mathbf{X},$$

since X is toroidal. Moreover, φ is $\mathbf{G}$-equivariant and C-invariant.

Since **G** is semi-simple and simply-connected, each line bundle on **X** has a unique **G**-linearization. This yields a homomorphism

$$\varphi^* : \mathrm{Pic}(\mathbf{X}) = \mathrm{Pic}_{\mathbf{G}}(\mathbf{X}) \longrightarrow \mathrm{Pic}_G(X) = \mathrm{Cl}_G(X).$$

For any $D \in \mathcal{D}$ with image **D** in **X**, we have $\varphi^*([\mathbf{D}]) = [D]_G$.

Moreover, for any line bundle L on **X**, we have a morphism of G-modules $\varphi^* : H^0(\mathbf{X}, L) \to H^0(X, \varphi^*(L))$. This yields a homomorphism of graded rings

$$\varphi^* : \mathrm{Cox}(\mathbf{X}) = \mathrm{Cox}_{\mathbf{G}}(\mathbf{X}) \longrightarrow \mathrm{Cox}_G(X)$$

compatible with the G-actions.

In particular, φ^* sends the invariant ring $\mathrm{Cox}(\mathbf{X})^{\mathbf{G}}$ to $\mathrm{Cox}_G(X)^{\mathbf{G}}$.
Recall from Lecture 2 that $\mathrm{Cox}(\mathbf{X})^{\mathbf{G}} = \mathbb{C}[\mathbf{s}_1, \ldots, \mathbf{s}_r]$, where $r = \mathrm{rk}(\mathbf{G} / \mathbf{H})$ and the $\mathbf{s}_i$ are the canonical sections of the boundary divisors of **X**. Also, $\mathrm{Cox}_G(X)^{\mathbf{G}} = \mathcal{O}(C)[s_1, \ldots, s_m]$. One may easily check that

$$\varphi^*(\mathbf{s}_i) = \prod_{j=1}^{m} s_j^{-v_j(\gamma_i)},$$

where v_j denotes the order of zero or pole along D_j, and γ_i is the ith spherical root.

Structure of the equivariant Cox ring

Theorem

The map

$$\psi : \mathrm{Cox}(\mathbf{X}) \otimes_{\mathrm{Cox}(\mathbf{X})^{\mathbf{G}}} \mathrm{Cox}_G(X)^{\mathbf{G}} \longrightarrow \mathrm{Cox}_G(X), \quad a \otimes b \longmapsto \varphi^*(a)\, b$$

is a ring isomorphism.

Proof.

Since ψ is a morphism of G-modules, it suffices to show that the induced map

$$\psi^U : \mathrm{Cox}(\mathbf{X})^U \otimes_{\mathrm{Cox}(\mathbf{X})^{\mathbf{G}}} \mathrm{Cox}_G(X)^{\mathbf{G}} \longrightarrow \mathrm{Cox}_G(X)^U$$

is an isomorphism.

The left-hand side is generated as an $\mathcal{O}(C)$-algebra by the $s_{\mathbf{D}} \otimes 1$ $(D \in \mathcal{D})$ and the $1 \otimes s_i$ $(i = 1, \ldots, m)$. Moreover, we have $\psi(s_{\mathbf{D}} \otimes 1) = s_D$ and $\psi(1 \otimes s_i) = s_i$.

Also, the right-hand side is the polynomial ring over $\mathcal{O}(C)$ in the variables s_D $(D \in D)$ and $s_1, \ldots, s_m$. This yields the assertion. □

Some applications

By combining the theorem with the isomorphism

$$\mathcal{O}(C) \otimes \mathrm{Cox}_G(X)^G \xrightarrow{\sim} \mathrm{Cox}_G(X)^{\mathbf{G}}$$

given by the multiplication map in $\mathrm{Cox}_G(X)$, we obtain:

Corollary

The multiplication map induces a ring isomorphism

$$\mathcal{O}(C) \otimes \mathrm{Cox}_G(X)^C \xrightarrow{\sim} \mathrm{Cox}_G(X).$$

Moreover, the G-module $\mathrm{Cox}_G(X)^C$ has highest weight vectors the monomials in the s_D ($D \in \mathcal{D}$) and $s_1, \ldots, s_m$.

These algebraic results can be reformulated geometrically, in terms of the quotient map

$$f : \tilde{X}_G \longrightarrow \tilde{X}_G // \mathbf{G}$$

and its analogue $\mathbf{f} : \tilde{\mathbf{X}} \to \mathbb{A}^r$.

By the theorem, we have a cartesian square

$$\begin{array}{ccc} \tilde{X}_G & \xrightarrow{f} & \tilde{X}_G /\!/ \mathbf{G} \\ \downarrow & & \downarrow \\ \tilde{\mathbf{X}} & \xrightarrow{\mathbf{f}} & \mathbb{A}^r, \end{array}$$

where the vertical maps are induced by φ.

Since $\mathbf{f}$ is flat and its (schematic) fibers are normal varieties, it follows that *f is flat and its fibers are normal varieties* as well.

Also, by the corollary, $\tilde{X}_G /\!/ \mathbf{G} \simeq C \times \mathbb{A}^m$ and the map $\tilde{X}_G \to \mathbb{A}^r$ factors through a monomial map $\mathbb{A}^m \to \mathbb{A}^r$.

As a consequence, *the quotient map $\tilde{X}_G \to \tilde{X}_G /\!/ G$ is flat, and its fibers are products of C with fibers of* $\mathbf{f}$.

One can show that the general fibers are isomorphic to the affine variety with coordinate ring $\mathcal{O}(G/H_1)$, where H_1 denotes the intersection of the kernels of the characters of H. (This ring is isomorphic to $\mathrm{Cox}_G(G/H)$).

Example: horospherical varieties

We assume that $H \subset G$ is a **horospherical subgroup**, i.e., H contains a maximal unipotent subgroup U of G.

Then $N_G(H)$ is a standard parabolic subgroup P of G, and P/H is a torus. In particular, P is the spherical closure of H. We have $P = \mathbf{P} \times C$ for a unique parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. Thus, $\mathbf{X} = \mathbf{G} \,/\, \mathbf{P}$ and the morphism $\varphi : X \to \mathbf{X}$ is a fibration with fiber a toric variety Y under P/H.

Each boundary divisor $D_i \subset X$ intersects Y along a toric divisor E_i. Moreover, the assignment $D_i \mapsto E_i$ yields a bijection between boundary divisors in X and toric divisors in Y.

Let $\mathbf{P} = \mathbf{P}^I = \bigcap_{i \in I} \mathbf{P}^i$, where P^i denotes the standard maximal parabolic subgroup with character group generated by the ith fundamental weight ϖ_i. Then the colors of $\mathbf{G} \,/\, \mathbf{P}$ are exactly the Schubert divisors $\mathbf{D}_i$ $(i \in I)$ with respective $\mathbf{P}$-weight ϖ_i. Their pull-backs D_i under φ are the colors of G/H (identified with their closures in X). They satisfy $V_{D_i} = V(\varpi_i)^*$.

The theorem yields a ring isomorphism

$$\mathrm{Cox}_G(X) \simeq \mathbb{C}[s_1, \ldots, s_m] \otimes \mathrm{Cox}(\mathbf{G} / \mathbf{P})$$

as $\mathrm{Cox}(\mathbf{G} / \mathbf{P})^{\mathbf{G}} = \mathbb{C}$.

Also, $\mathrm{Cox}(\mathbf{G} / \mathbf{P}) = \mathcal{O}(\mathbf{G} / [\mathbf{P}, \mathbf{P}])$ as seen in Lecture 1, and hence

$$\mathrm{Cox}_G(X) \simeq \mathbb{C}[s_1, \ldots, s_m] \otimes \mathcal{O}(\mathbf{G} / [\mathbf{P}, \mathbf{P}]).$$

This ring is generated by $s_1, \ldots, s_m$ together with the simple G-modules $V(\varpi_i)^*$, where $i \in I$.

The relations between these generators are consequences of quadratic relations, which express the fact that

$$V(\varpi_i)^* V(\varpi_j)^* = V(\varpi_i + \varpi_j)^* \quad (i, j \in I)$$

in $\mathcal{O}(\mathbf{G} / [\mathbf{P}, \mathbf{P}])$.

Since $\mathbb{C}[s_1, \ldots, s_m] \simeq \mathrm{Cox}(Y)$ by Cox's theorem, we obtain a ring isomorphism $\mathrm{Cox}_G(X) \simeq \mathrm{Cox}(Y) \otimes \mathrm{Cox}(\mathbf{G} / \mathbf{P})$. In loose terms, the fibration φ is "Cox trivial".

Some references for Lecture 3

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