Cox rings of spherical varieties
Lecture 2: Wonderful varieties

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Introduction to Lecture 2

After a brief recap on Cox rings and sheaves, we will consider wonderful varieties, and describe their Picard group and their monoid of effective divisors. We will also determine the extremal rays of their effective cone.

We will then describe the structure of the Cox ring of a wonderful variety as a module over the invariant subring. In geometric terms, this yields a degeneration of certain affine almost homogeneous varieties.

Finally, we will obtain a description of the Cox ring in terms of commutative algebra and representation theory, with applications to the fibers of the above degeneration.

When applied to the wonderful compactification of an adjoint semi-simple group (which forms our main class of examples), this gives back a remarkable algebraic monoid introduced by Vinberg.

Recap on Cox rings and sheaves

Let X be a normal variety, and U its smooth locus. Assume that the divisor class group $\mathrm{Cl}(X) = \mathrm{Pic}(U)$ is finitely generated, and $\mathcal{O}(X)^* = \mathbb{C}^*$. Then one may define a ring structure on the vector space

$$\bigoplus_{[D]\in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D)) = \bigoplus_{[L]\in \mathrm{Pic}(U)} H^0(U, L).$$

This yields the **Cox ring** $\mathrm{Cox}(X)$. It is graded by $\mathrm{Cl}(X)$ and satisfies $\mathrm{Cox}(X) = \mathrm{Cox}(U)$.

Likewise, one may define an $\mathcal{O}_X$-algebra structure on the **Cox sheaf**

$$\mathcal{R}_X = \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{O}_X(D).$$

This yields the **characteristic space** $\hat{X} = \mathrm{Spec}_X(\mathcal{R}_X)$.

Denote by S_X the diagonalizable group with character group $\mathrm{Cl}(X)$. Then the $\mathrm{Cl}(X)$-grading of $\mathcal{R}_X$ yields an action of S_X on $\hat{X}$, and the canonical map

$$q : \hat{X} \longrightarrow X$$

is a good quotient by this action.

Assume that the algebra $\mathrm{Cox}(X)$ is finitely generated and let

$$\tilde{X} = \mathrm{Spec}(\mathrm{Cox}(X)).$$

This is an affine variety equipped with an action of S_X.

We have a canonical map

$$\iota : \hat{X} \longrightarrow \tilde{X}$$

which is S_X-equivariant, and identifies $\hat{X}$ with a big open subset of $\tilde{X}$.

These constructions are gathered in the following diagram:

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\iota} & \tilde{X} \\ {\scriptstyle q}\downarrow & & \\ X & & \end{array}$$

If X is smooth and $\mathrm{Pic}(X)$ is a free abelian group of rank r, then q is a principal bundle under $S_X \simeq \mathbb{G}_m^r$. One may show that q is a universal principal bundle with structure group a torus and basis X.

Likewise, one may think of $\mathrm{Cox}(X)$ as the "universal homogeneous coordinate ring" of X.

Wonderful varieties

Let G be a connected reductive algebraic group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus. We identify the character group $\mathrm{X}^*(B)$ with $\mathrm{X}^*(T) = \Lambda$.

Definition

A G-variety X is **wonderful** if it satisfies the following conditions:

(i) X is smooth, projective, and contains an open orbit X_G^0.

(ii) The boundary $X \setminus X_G^0$ is a union of prime divisors $D_1, \dots, D_r$ intersecting transversally (the **boundary divisors**).

(iii) The G-orbit closures in X are exactly the partial intersections $D_{i_1} \cap \cdots \cap D_{i_s}$, where $1 \leq i_1 < \cdots < i_s \leq r$.

In particular, X contains a unique closed G-orbit $Y = D_1 \cap \cdots \cap D_r$.

By a result of Luna, X is a spherical G-variety of rank r.

Also, one may show that the largest central torus of G acts trivially on X, and hence we may assume that G is *semi-simple*. Replacing G with a finite cover, we may further assume that it is *simply-connected*.

Choose $x \in X$ such that the orbit $B \cdot x = X_B^0$ is open in X. Then we have $X_G^0 = G \cdot x = X \setminus (D_1 \cup \cdots \cup D_r)$. Moreover, $X_G^0 \setminus X_B^0$ is a union of finitely many prime B-stable divisors, the **colors**. We identify the colors with their closures in X; these are the prime B-stable divisors which are not G-stable.

Let $\mathcal{D}$ be the set of colors. Then

$$X \setminus X_B^0 = D_1 \cup \cdots \cup D_r \cup \bigcup_{D \in \mathcal{D}} D.$$

By the local structure of spherical varieties, the open B-stable subset

$$X_{Y,B} = X \setminus \bigcup_{D \in \mathcal{D}} D$$

is isomorphic to an affine space $\mathbb{A}^n$ where $n = \dim(X)$. Thus, $X_{Y,B}$ is called the **big cell**. It meets each G-orbit along its open B-orbit.

Each intersection $D_i \cap X_{Y,B}$ has an equation $f_i \in \mathcal{O}(X_{Y,B})$, unique up to a non-zero scalar, and hence an eigenvector of B. Let $\gamma_i \in \Lambda$ be the opposite of the weight of f_i; then $\gamma_1, \ldots, \gamma_r$ are the **spherical roots**. They form a basis of the **weight group** $M = \Lambda(X) \subset \Lambda$ (the set of weights of B-eigenvectors in the function field $\mathbb{C}(X)$).

Denote by H the isotropy group G_x, then $X_G^0 = G/H$ and $X_B^0 = BH/H = B/B \cap H$. Let $v_1, \ldots, v_r$ be the valuations of the field $\mathbb{C}(X) = \mathbb{C}(G/H)$ associated with $D_1, \ldots, D_r$; then $v_i(f_j) = \delta_{i,j}$ for $1 \leq i, j \leq r$.

By the embedding theory of spherical homogeneous spaces, $v_1, \ldots, v_r$ generate the valuation cone $\mathcal{V} = \mathcal{V}(G/H) \subset N_{\mathbb{R}}$, where N is the dual lattice of M. Thus, $\mathcal{V}$ is the dual cone of the cone generated by $-\gamma_1, \ldots, -\gamma_r$.

Another ingredient of embedding theory is the map

$$\rho : \mathcal{D} \longrightarrow N_{\mathbb{R}}$$

defined as follows: each $D \in \mathcal{D}$ yields a valuation v_D of $\mathbb{C}(X)$. Moreover, $v_D(f)$ depends only on the weight of f for any $f \in \mathbb{C}(X)^{(B)}$.

So we have $\rho(v_D) + \sum_{i=1}^r \langle D, \gamma_i \rangle v_i = 0$, where $\langle D, \gamma_i \rangle = -v_D(f_i)$.

For later use, we record a combinatorial result:

Lemma

The cone generated by the $\rho(v_D)$, where $D \in \mathcal{D}$, contains $-\mathcal{V}$ (the opposite of the valuation cone).

Denote by $a_x : G \to X$ the orbit map $g \mapsto g \cdot x$. For any color $D \in \mathcal{D}$, the pull-back $a_x^{-1}(D)$ is an effective divisor in G, stable by B (acting by left multiplication) and by H (acting by right multiplication). Since $\mathcal{O}(G)$ is a UFD and $\mathcal{O}(G)^* = \mathbb{C}^*$, this pull-back has an equation $f_D \in \mathcal{O}(G)$, unique up to a non-zero scalar. We normalize f_D by requiring that $f_D(e) = 1$.

Note that f_D is an eigenvector of $B \times H$, and is uniquely determined by its weight $(\omega_D, \chi_D) \in \mathrm{X}^*(B) \times \mathrm{X}^*(H)$. Moreover, $\omega_D|_{B\cap H} = \chi_D|_{B\cap H}$, i.e., (ω_D, χ_D) lies in the fiber product $\mathrm{X}^*(B) \times_{\mathrm{X}^*(B\cap H)} \mathrm{X}^*(H)$.

Lemma

The abelian group $\mathrm{X}^*(B) \times_{\mathrm{X}^*(B\cap H)} \mathrm{X}^*(H)$ *is free with basis the pairs* (ω_D, χ_D)*, where* $D \in \mathcal{D}$*. Moreover,* $\mathrm{X}^*(H)$ *is generated by the* χ_D*.*

We may also normalize $f_1, \ldots, f_r \in \mathcal{O}(X_{Y,B})$ by requiring that $f_i(x) = 1$ for $i = 1, \ldots, r$. Identifying each f_i with its pull-back to $BH \subset G$, we have $f_i = \prod_{D\in\mathcal{D}} f_D^{-\langle D,\gamma_i\rangle}$. Equivalently,

$$\sum_{D\in\mathcal{D}} \langle D, \gamma_i\rangle \omega_D = \gamma_i \quad \text{and} \quad \sum_{D\in\mathcal{D}} \langle D, \gamma_i\rangle \chi_D = 0.$$

Example: wonderful varieties under SL_2

Let $G = \mathrm{SL}_2$ with Borel subgroup B the subgroup of upper triangular matrices, and maximal torus T the subgroup of diagonal matrices. Denote by α the positive root.

There are three wonderful G-varieties up to isomorphism:

(i) $\mathbb{P}^1 = G/B$,
(ii) $\mathbb{P}^1 \times \mathbb{P}^1$ on which G acts diagonally,
(iii) $\mathbb{P}^2 = \mathbb{P}(V_2)$.

Here V_2 denotes the space of quadratic forms in two variables (this is the simple G-module with highest weight 2).

(i) The G-variety $\mathbb{P}^1$ has a unique color D: the B-fixed point ∞. We have $(\omega_D, \chi_D) = (1, 1)$. The big cell is $\mathbb{P}^1 \setminus \{\infty\} = \mathbb{A}^1$. There is no spherical root in this case.

(ii) For the G-variety $\mathbb{P}^1 \times \mathbb{P}^1$, the closed G-orbit is $Y = \mathrm{diag}(\mathbb{P}^1) = G/B$, and the open G-orbit is G/H where $H \simeq T$.

The colors are $D = \mathbb{P}^1 \times \{\infty\}$ and $E = \{\infty\} \times \mathbb{P}^1$. They satisfy $(\omega_D, \chi_D) = (1, 1)$ and $(\omega_E, \chi_E) = (1, -1)$.

The big cell is $(\mathbb{P}^1 \setminus \{\infty\}) \times (\mathbb{P}^1 \setminus \{\infty\}) \simeq \mathbb{A}^2$ on which T acts linearly with weight $-\alpha$. The spherical root is $\gamma = \alpha$, and $\langle D, \gamma \rangle = \langle E, \gamma \rangle = 1$. Also, $\rho(v_D) = \rho(v_E) = -v_Y$, where v_Y denotes the order of zero or pole along Y.

(iii) For the G-variety $X = \mathbb{P}^2 = \mathbb{P}(V_2)$, the open orbit is G/H, where $H \simeq N_G(T)$; it consists of the non-degenerate quadratic forms. The group of components $H/H^0 \simeq N_G(T)/T$ has order 2.

The closed orbit Y is isomorphic to G/B. This is the conic in $\mathbb{P}^2$ consisting of the degenerate forms.

There is a unique color D: the tangent line to the conic Y at the B-fixed point. We have $\mathrm{X}^*(B) = \mathbb{Z}$ and $\mathrm{X}^*(H) = \mathbb{Z}/2\mathbb{Z}$; then $(\omega_D, \chi_D) = (2, 0)$.

The big cell is $X \setminus D \simeq \mathbb{P}^2 \setminus \mathbb{P}^1 = \mathbb{A}^2$ on which T acts linearly with weights $-\alpha, -2\alpha$. The spherical root is $\gamma = 2\alpha$, and $\langle D, \gamma \rangle = 2$. We have $\rho(v_D) = -v_Y$.

In the cases (ii) and (iii), X is the unique non-trivial embedding of G/H.

Example: wonderful group compactifications

Let $G_{\mathrm{ad}} = G/Z$, where Z denotes the center of G. We view G_{ad} as a homogeneous space under $G \times G$ acting by left and right multiplication. Then $G_{\mathrm{ad}} = (G \times G)/(Z \times Z)\mathrm{diag}(G)$.

As a base point, we take the neutral element $e \in G_{\mathrm{ad}}$; then $(B^- \times B) \cdot e = B^- B/Z$ is open in G_{ad}, where B and B^- are opposite Borel subgroups of G with common torus T.

We have $B = UT$ and $B^- = U^- T$, where U (resp. U^-) denotes the unipotent part of B (resp. B^-). Moreover, $B^- B/Z \simeq U^- \times T_{\mathrm{ad}} \times U$, where $T_{\mathrm{ad}} = T/Z$. Thus, the weight lattice M is $X^*(T_{\mathrm{ad}})$ (this is the root lattice of (G, T)), and $N = X_*(T_{\mathrm{ad}})$ (the lattice of coweights of (G, T)).

The spherical roots are the pairs $(-\alpha_i, \alpha_i)$, where $\alpha_1, \ldots, \alpha_r$ denote the simple roots of (B, T). The colors are the "Schubert divisors" $\overline{B^- r_i B}/Z$, where $r_1, \ldots, r_r$ denote the simple reflections. Their respective $B^- \times B$-weights are $(-\varpi_i, \varpi_i)$ where ϖ_i denote the fundamental weights.

The valuation cone $\mathcal{V}$ is the negative Weyl chamber. Moreover, the map $\rho : \mathcal{D} \to N_{\mathbb{R}}$ is injective and its image consists of the simple coroots. The integers $\langle D, \gamma_i \rangle$ are the entries of the Cartan matrix.

The Picard group

Since X is smooth, we identify $\mathrm{Pic}(X)$ with the divisor class group $\mathrm{Cl}(X)$. Recall that $\mathrm{Cl}(X)$ is freely generated by the classes of the colors; moreover, the divisor $\sum_{D\in\mathcal{D}} n_D D$ is globally generated (resp. ample) if and only if $n_D \geq 0$ (resp. $n_D > 0$) for all D. In particular, every nef divisor is globally generated. Also, rational equivalence coincides with numerical equivalence.

We have $\mathrm{div}(f_i) = D_i - \sum_{D\in\mathcal{D}} \langle D, \gamma_i\rangle D$, and hence the relation in $\mathrm{Cl}(X)$

$$[D_i] = \sum_{D\in\mathcal{D}} \langle D, \gamma_i\rangle [D].$$

Next, recall that every line bundle L on X has a unique G-linearization, since $\mathrm{X}^*(G) = 0 = \mathrm{Pic}(G)$. In particular, $H^0(X, L)$ is a finite-dimensional G-module.

Taking $L = \mathcal{O}_X(D)$ where $D \in \mathcal{D}$, the canonical section s_D is a B-eigenvector of weight ω_D (since its pull-back under $a_x : G \to X$ is f_D). Also, taking $L = \mathcal{O}_X(D_i)$, the canonical section s_i is G-invariant.

We have identifications $\mathrm{Cl}(G/H) = \mathrm{Pic}(G/H) = \mathrm{Pic}^G(G/H) = \mathrm{X}^*(H)$ and an exact sequence

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{f} \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(G/H) \longrightarrow 0,$$

where $f(n_1, \ldots, n_r) = n_1[D_1] + \cdots + n_r[D_r]$.

Proposition

There is a natural isomorphism $\mathrm{Cl}(X) \simeq \mathrm{X}^*(B) \times_{\mathrm{X}^*(B \cap H)} \mathrm{X}^*(H)$ *which identifies the restriction* $\mathrm{Cl}(X) \to \mathrm{Cl}(G/H)$ *with the projection* $\mathrm{X}^*(B) \times_{\mathrm{X}^*(B \cap H)} \mathrm{X}^*(H) \to \mathrm{X}^*(H)$.

Proof.

We have a restriction map $\mathrm{Cl}(X) = \mathrm{Cl}^G(X) \to \mathrm{Cl}^B(X_{Y,B})$. Since $X_{Y,B}$ is an affine space, the canonical map $\mathrm{X}^*(B) \to \mathrm{Cl}^B(X_{Y,B})$ is an isomorphism. Together with the restriction map $\mathrm{Cl}(X) \to \mathrm{X}^*(H)$, this yields a map $\mathrm{Cl}(X) \to \mathrm{X}^*(B) \times \mathrm{X}^*(H)$. One checks that this map sends any $D \in \mathcal{D}$ to (ω_D, χ_D). Hence it is an isomorphism by the previous lemma. □

The effective divisors

Recall that $\mathrm{Eff}(X) \subset \mathrm{Pic}(X)$ denotes the set of classes of effective divisors. This is a monoid under addition.

Proposition

The monoid $\mathrm{Eff}(X)$ *is generated by the classes of the colors and the boundary divisors.*

Proof.

Let L be an effective line bundle on X. Then $H^0(X, L)$ is a non-zero G-module, and hence contains a B-eigenvector s. Thus, $L \simeq \mathcal{O}_X(D)$, where $D = \mathrm{div}(s)$ is an effective B-stable divisor. □

More generally, the monoid of effective cycles of any dimension d on any spherical variety Z is generated by the classes of the B-orbit closures of dimension d.

In particular, if Z is projective then its cone of effective curves is rational polyhedral, and its nef cone is rational polyhedral as well.

The extremal rays of the effective cone

We say that an effective divisor E on X is **fixed** if $H^0(X, \mathcal{O}_X(nE)) = \mathbb{C}s_E^n$ for all $n \geq 1$.

Every fixed effective divisor is G-invariant.

Theorem

(i) *The extremal rays of* $\mathrm{Eff}(X)_\mathbb{R}$ *consist of the rays generated by the fixed divisors, and the rays generated by pull-backs of Schubert divisors under the G-equivariant morphisms $X \to G/P$, where $P \supset H$ is a maximal parabolic subgroup of G. Both types of rays are disjoint.*

(ii) *The fixed divisors are exactly the boundary divisors D_i such that $H^0(X, \mathcal{O}_X(D_i)) = \mathbb{C}s_i$.*

(iii) *A boundary divisor D_i is not fixed if and only if $\langle D, \gamma_i \rangle \geq 0$ for all $D \in \mathcal{D}$. Then there is a unique G-equivariant morphism $\varphi : X \to X'$, where X' is a wonderful variety of rank 1 such that the boundary divisor D_1' is ample, and $D_i = \varphi^*(D_1')$.*

In particular, any boundary divisor is either fixed or globally generated.

Example: wonderful group compactifications (continued)

We consider again the wonderful compactification X of the adjoint group G_{ad}. Then $\mathrm{Pic}(X)$ is identified with the weight lattice of (G, T). The classes of the colors (resp. the boundary divisors) are identified with the fundamental weights (resp. the simple roots).

The nef cone $\mathrm{Nef}(X)_{\mathbb{R}}$ is the positive Weyl chamber. The effective cone $\mathrm{Eff}(X)_{\mathbb{R}}$ is generated by the simple roots (as follows from the above theorem).

The globally generated boundary divisors D_i correspond to the simple roots α_i that are dominant; equivalently, to the isolated points of the Dynkin diagram. We then have $G_{\mathrm{ad}} \simeq \mathrm{PSL}(2) \times G'$ for some adjoint semi-simple group G'. This yields an isomorphism $X \simeq \mathbb{P}(M_2) \times X'$ where $\mathbb{P}(M_2)$ denotes the projectivization of the space of 2×2 matrices (the wonderful compactification of $\mathrm{PSL}(2)$), and X' is the wonderful compactification of G'. Then D_i is the pull-back of the smooth quadric $(\det = 0) \simeq \mathbb{Q}^2$ under the projection $X \to \mathbb{P}(M_2) \simeq \mathbb{P}^3$.

Thus, if G is simple and $G \neq \mathrm{PSL}(2)$ then every boundary divisor is fixed.

The Cox sheaf

We still consider a simply-connected semi-simple group G, and a wonderful G-variety X with set of colors $\mathcal{D}$. We identify $\mathrm{Pic}(X)$ with $\mathbb{Z}^{\mathcal{D}}$ (the free abelian group on the colors), and denote by $S_X = \mathbb{G}_m^{\mathcal{D}}$ the torus with character group $\mathrm{Pic}(X)$.

The Cox sheaf is

$$\mathcal{R}_X = \bigoplus_{(n_D)\in \mathrm{Pic}(X)} \mathcal{O}_X(\sum_{D\in\mathcal{D}} n_D D).$$

This is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras, graded by $\mathrm{Pic}(X)$ and compatibly G-linearized. Therefore, the characteristic space

$$\hat{X} = \mathrm{Spec}_X(\mathcal{R}_X)$$

is a scheme equipped with an action of $G \times S_X = \tilde{G}$ (a connected reductive group) and with a morphism

$$q : \hat{X} \longrightarrow X.$$

Moreover, q is G-equivariant and is a principal bundle under S_X. Thus, $\hat{X}$ is a smooth spherical $\tilde{G}$-variety.

The Cox ring

Keep the above notation and consider the Cox ring

$$\mathrm{Cox}(X) = \mathcal{O}(\hat{X}) = \bigoplus_{(n_D) \in \mathrm{Pic}(X)} H^0(X, \mathcal{O}_X(\sum_{D \in \mathcal{D}} n_D D)).$$

Then the ring $\mathrm{Cox}(X)$ is graded by the monoid $\mathrm{Eff}(X)$ and equipped with a compatible G-action. It is finitely generated in view of the following:

Lemma

Let Z be a spherical variety. Then the algebra $\mathcal{O}(Z)$ is finitely generated.

Proof.

The G-module $\mathcal{O}(Z)$ is multiplicity-free and its highest weights form a submonoid $\mathcal{M} \subset \Lambda(Z)$. It suffices to show that $\mathcal{M}$ is generated by finitely many weights $\lambda_1, \ldots, \lambda_m$: then the corresponding simple submodules $V(\lambda_1), \ldots, V(\lambda_m)$ will generate $\mathcal{O}(Z)$.

We have $\mathcal{O}(Z) = \mathcal{O}(Z_G^0) \cap \bigcap_{i=1}^n \mathcal{O}_{v_i}$, where $v_1, \ldots, v_n$ denote the G-invariant valuations associated with the prime G-stable divisors in Z. Thus, $\mathcal{M} = \{\lambda \in \Lambda(Z) \mid v_i(\lambda) \geq 0 \ (i = 1, \ldots, n)\}$. So $\mathcal{M}$ is finitely generated by Gordan's lemma. □

We denote by $s_1, \ldots, s_r$ the canonical sections of the boundary divisors, and by s_D ($D \in \mathcal{D}$) those of the colors. Recall that the s_i are G-invariant, and the s_D are B-eigenvectors; in particular, U-invariant.

Proposition

(i) *The invariant ring* $\mathrm{Cox}(X)^U$ *is a polynomial ring in the* s_i *and* s_D.

(ii) *The subring* $\mathrm{Cox}(X)^G$ *is a polynomial ring in the* s_i.

(iii) *The ring* $\mathrm{Cox}(X)$ *is a free module over* $\mathrm{Cox}(X)^G$.

Proof.

(i) This is proved by the same argument as Cox's theorem in Lecture 1.

(ii) This follows readily from (i).

(iii) Denote by M the G-submodule of $\mathrm{Cox}(X)$ generated by the monomials in the s_D. Then the multiplication of $\mathrm{Cox}(X)$ yields a morphism of G-modules

$$m : \mathrm{Cox}(X)^G \otimes M \longrightarrow \mathrm{Cox}(X).$$

By (i), m restricts to an isomorphism $\mathrm{Cox}(X)^G \otimes M^U \to \mathrm{Cox}(X)^U$. Thus, m is an isomorphism. □

Let $\tilde{X} = \mathrm{Spec}(\mathrm{Cox}(X))$. Then $\tilde{X}$ is an affine spherical $\tilde{G}$-variety containing $\hat{X}$ as a $\tilde{G}$-stable big open subset.

One may show that the base point $x \in X$ lifts to a unique base point $\tilde{x} \in \tilde{X}$ such that $s_D(x) = 1$ for all $D \in \mathcal{D}$. Moreover, the isotropy group $\tilde{G}_{\tilde{x}}$ is isomorphic to H embedded in $\tilde{G} = G \times \mathbb{G}_m^{\mathcal{D}}$ via $h \mapsto (h, \chi_D(h)\ (D \in \mathcal{D}))$.

For any $D \in \mathcal{D}$, denote by

$$\varepsilon_D : \mathbb{G}_m^{\mathcal{D}} \longrightarrow \mathbb{G}_m$$

the D-component; then the characters ε_D form a basis of the abelian group $\mathrm{X}^*(\mathbb{G}_m^{\mathcal{D}}) = \mathrm{X}^*(\tilde{G})$.

Moreover, each s_D is an eigenvector of the Borel subgroup $\tilde{B} = B \times \mathbb{G}_m^{\mathcal{D}}$ of $\tilde{G}$, with weight $(\omega_D, \varepsilon_D)$. Each s_i is a $\tilde{G}$-eigenvector of weight $\sum_{D \in \mathcal{D}} \langle D, \gamma_i \rangle \varepsilon_D$.

The inclusion $\mathbb{C}[s_1, \ldots, s_r] = \mathrm{Cox}(X)^G \subset \mathrm{Cox}(X)$ corresponds to the map

$$f = (s_1, \ldots, s_r) : \tilde{X} \longrightarrow \tilde{X}//G \simeq \mathbb{A}^r,$$

which is a good quotient by G. Moreover, f is equivariant under $\mathbb{G}_m^{\mathcal{D}}$ acting linearly on $\mathbb{A}^r$ via the above weights.

Theorem

With the above notation, the morphism f is flat and its (schematic) fibers are normal affine G-varieties.

Proof.

The flatness of f follows from the freeness of $\mathrm{Cox}(X)$ as a module over $\mathrm{Cox}(X)^G$.

Every fiber Z of f is an affine G-scheme of finite type, and $\mathcal{O}(Z)^U$ is a polynomial ring. Thus, Z is a normal variety by the next result. □

Proposition

Let R be a finitely generated G-algebra. If R^U is a normal domain, then R is a normal domain as well.

To show that R is a domain, consider its minimal prime ideals $I_1, \ldots, I_m$. Then these ideals are G-stable, and the set of zero-divisors in R is $I_1 \cup \cdots \cup I_m$. If this set is non-zero, then we may assume that $I_1 \neq 0$. Thus, we may choose a non-zero $r \in I_1^U$. So the annihilator of r is a non-zero U-stable ideal: it contains an U-fixed point, contradiction.

To show that R is normal, consider its integral closure S in its fraction field K. Then $R \subset S \subset K$ and S is G-stable; moreover, $R^U \subset S^U \subset K^U$ and K^U is the fraction field of R^U. Also, the conductor of S in R is G-stable, and hence we may choose a non-zero $r \in R^U$ such that $rS \subset R$. Thus, $rS^U \subset R^U$, so that S^U is integral over R^U. As R^U is normal, we get $S^U = R^U$ and $S = R$. □

Next, we will obtain an algebraic description of $\mathrm{Cox}(X)$ with applications to the geometry of the fibers of f.

For this, we may choose the maximal torus $T \subset B$ so that $T \cap H$ is the intersection of the kernels of the spherical roots $\gamma_1, \ldots, \gamma_r$ (as follows from the local structure theorem). Equivalently,

$$(\gamma_1, \ldots, \gamma_r) : T/T \cap H \xrightarrow{\sim} \mathbb{G}_m^r.$$

We consider the action of $G \times T$ on $\tilde{X}$, where G acts naturally and T acts via the homomorphism

$$u : T \longrightarrow \mathbb{G}_m^{\mathcal{D}}, \quad t \longmapsto (\omega_D(t), D \in \mathcal{D}).$$

The above action stabilizes the open orbit $\tilde{X}^0_{\tilde{G}}$.

Lemma

There is an isomorphism of $G \times T$-varieties

$$\tilde{X}^0_{\tilde{G}} \simeq G/H_1 \times^{T \cap H} T,$$

where H_1 denotes the intersection of the kernels of the characters of H, and $T \cap H$ acts on G/H_1 via the action of H by right multiplication.

This identifies the restriction $f^0 : \tilde{X}^0_{\tilde{G}} \longrightarrow \mathbb{A}^r$ with the projection $G/H_1 \times^{T \cap H} T \longrightarrow T/T \cap H \simeq \mathbb{G}_m^r \subset \mathbb{A}^r$.

In particular, f^0 is a fibration with fiber G/H_1.

Also, $H_1 \lhd H$ and H/H_1 is a diagonalizable group with character group $\mathrm{X}^*(H)$; moreover, $\mathcal{O}(G/H_1) = \bigoplus_{\chi \in \mathrm{X}^*(H)} \mathcal{O}(G)^{(H)}_\chi$.

Thus, $\mathcal{O}(G/H_1)$ is a multiplicity-free module under $G \times H/H_1$, and the algebra $\mathcal{O}(G/H_1)^U$ is the polynomial ring $\mathbb{C}[f_D, D \in \mathcal{D}]$.

It follows that $\mathcal{O}(G/H_1) = \mathrm{Cox}(G/H)$ is a finitely generated normal domain, and $G/H_1 = \widehat{G/H}$ is quasi-affine.

The coordinate ring of the open orbit

The above lemma yields an algebraic description of the ring $\mathcal{O}(\tilde{X}^0_{\tilde{G}})$. To formulate it, we need further notation.

For any G-module M and any dominant weight $\lambda \in \Lambda^+$, we denote by $M_{(\lambda)}$ the **isotypical component** of M with type λ, i.e., the sum of all simple submodules of highest weight λ.

Also, we denote by e^λ ($\lambda \in \Lambda$) the basis of $\mathcal{O}(T)$ consisting of characters.

Lemma

There is an isomorphism of $G \times T$-algebras

$$\mathcal{O}(\tilde{X}^0_{\tilde{G}}) \simeq \bigoplus \mathcal{O}(G/H_1)_{(\lambda)}\, e^\mu,$$

where the sum runs over the pairs (λ, μ) such that $\lambda \in \Lambda^+$, $\mu \in \Lambda$, and $\lambda - \mu \in \Lambda(X)$, and the right-hand side is a subalgebra of

$$\mathcal{O}(G \times T) = \bigoplus_{\lambda \in \Lambda^+, \mu \in \Lambda} \mathcal{O}(G)_{(\lambda)}\, e^\mu.$$

An algebraic description of the Cox ring

Given $\lambda, \mu \in \Lambda$, we say that $\lambda \leq_X \mu$ if $\mu - \lambda$ is a linear combination of the spherical roots $\gamma_1, \ldots, \gamma_r$ with non-negative integer coefficients. Then $\mu - \lambda \in \Lambda(X)$ and $\lambda \leq \mu$ for the usual partial order on weights.

Theorem

There is an isomorphism of $G \times T$-algebras

$$\mathrm{Cox}(X) \simeq \bigoplus \mathcal{O}(G/H_1)_{(\lambda)}\ e^{\mu},$$

where the sum runs over the pairs (λ, μ) such that $\lambda \in \Lambda^+$, $\mu \in \Lambda$ and $\lambda \leq_X \mu$; the right-hand side is a subalgebra of $\mathcal{O}(G \times T)$.

This isomorphism identifies each s_i with e^{γ_i} and each s_D with $f_D e^{\omega_D}$.

Corollary

For any $\lambda, \mu \in \Lambda^+$, we have

$$\mathcal{O}(G/H_1)_{(\lambda)}\ \mathcal{O}(G/H_1)_{(\mu)} \subset \bigoplus_{\nu \in \Lambda^+, \nu \leq_X \lambda + \mu} \mathcal{O}(G/H_1)_{(\nu)}.$$

The fibers of the quotient map

Let $z = (z_1, \dots, z_r) \in \mathbb{C}^r$, and $I = I(z) = \{i \mid z_i \neq 0\} \subset \{1, \dots, r\}$. Denote by $\tilde{X}_z$ the fiber of the quotient map $f : \tilde{X} \to \mathbb{A}^r$ at z, so that

$$\mathcal{O}(\tilde{X}_z) = \mathrm{Cox}(X)/(s_1 - z_1, \dots, s_r - z_r).$$

Proposition

(i) *The G-variety $\tilde{X}_z$ depends only on I.*

(ii) *The general fibers, where $I = \{1, \dots, r\}$, satisfy $\mathcal{O}(\tilde{X}_z) = \mathcal{O}(G/H_1)$.*

(iii) *The special fiber $\tilde{X}_0$, where $I = \emptyset$, is horospherical.*

Thus, the general fibers of f are isomorphic to $\mathrm{Spec}(\mathcal{O}(G/H_1)) = \widetilde{G/H}$. This is a normal affine G-variety containing G/H_1 as a G-stable big open subset. It is called the **canonical embedding** of G/H_1.

Moreover, the quotient map f realizes a G-equivariant flat degeneration of $\widetilde{G/H}$ to a normal affine horospherical G-variety (its "horospherical contraction").

Proof.

By the theorem, the G-algebra $\mathcal{O}(\tilde{X}_z)$ is isomorphic to $\mathcal{O}(G/H_1)$ as a G-module; moreover, the multiplication is the linear G-equivariant map

$$m_I : \mathcal{O}(G/H_1) \otimes \mathcal{O}(G/H_1) \longrightarrow \mathcal{O}(G/H_1)$$

such that the restriction of m_I to any $\mathcal{O}(G/H_1)_{(\lambda)} \otimes \mathcal{O}(G/H_1)_{(\mu)}$ is the multiplication in $\mathcal{O}(G/H_1)$,

$$m : \mathcal{O}(G/H_1)_{(\lambda)} \otimes \mathcal{O}(G/H_1)_{(\mu)} \longrightarrow \bigoplus_{\nu \in \Lambda^+, \nu \leq_X \lambda + \mu} \mathcal{O}(G/H_1)_{(\nu)},$$

followed by the projection onto the partial sum of the $\mathcal{O}(G/H_1)_{(\nu)}$ such that $\lambda + \mu - \nu$ is a linear combination of the γ_i, $i \in I$. This readily yields the statements. □

Example: wonderful varieties under SL_2 (continued)

(i) Let $X = \mathbb{P}^1 = \mathbb{P}(V_1)$, where V_1 denotes the standard G-module $\mathbb{C}^2$.
Then $\tilde{X} = V_1$, $\hat{X} = V_1 \setminus \{0\} = G/U$ and $S_X = \mathbb{G}_m$.
The quotient map is constant.

(ii) Let $X = \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}(V_1) \times \mathbb{P}(V_1)$,
Then $\tilde{X} = V_1 \times V_1$ and $\hat{X} = (V_1 \setminus \{0\}) \times (V_1 \setminus \{0\})$ as G-varieties.
Also, $S_X = \mathbb{G}_m^2$ acting on $V_1 \times V_1$ by $(t, u) \cdot (v, w) = (tv, uw)$.
The quotient map $f : V_1 \times V_1 \to \mathbb{A}^1$ is given by the determinant. Its general fibers are isomorphic to $G = G/H_1$. The special fiber is the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1$. It contains infinitely many G-orbits, isomorphic to G/U.

(iii) Let $X = \mathbb{P}^2 = \mathbb{P}(V_2)$, where V_2 denotes the simple G-module of quadratic forms in two variables. Then $\tilde{X} = V_2$, $\hat{X} = V_2 \setminus \{0\}$ and $S_X = \mathbb{G}_m$.
The quotient map $f : V_2 \to \mathbb{A}^1$ is given by the discriminant. Its general fibers are isomorphic to $G/T = G/H_1$. The special fiber $\tilde{X}_0$ is the cone of degenerate quadratic forms. We have $\tilde{X}_0 = G/H_0 \cup \{0\}$, where $H_0 = U \rtimes \mathrm{Ker}(\alpha)$.

Example: wonderful group compactifications (continued)

Let X be the wonderful compactification of G_{ad}. Then G (resp. B, T) is replaced with $G \times G$ (resp. $B^- \times B$, $T \times T$).

Moreover, $H = (Z \times Z)\mathrm{diag}(G)$. Thus, $H_1 = \mathrm{diag}(G)$, so that $(G \times G)/H_1 = G$ on which G acts by left and right multiplication.

Also, $S_X = T$ and hence we have an action of $G \times G \times T$ on $\tilde{X}$. Moreover, $\tilde{X}^0_{\tilde{G}} = G \times^Z T$.

The morphism $f : \tilde{X} \to \mathbb{A}^r$ restricts to the fibration

$$f^0 : G \times^Z T \longrightarrow T/Z = T_{\mathrm{ad}} = \mathbb{G}_m^r,$$

where the latter identification is given by the simple roots $\alpha_1, \ldots, \alpha_r$.

In particular, the general fiber of f is G. The special fiber $\tilde{X}_0$ is the affine horospherical $G \times G$-variety containing $(G \times G)/(U^- \times U)\mathrm{diag}(T)$ as a big open orbit. Also, $\tilde{X}_0$ contains a unique fixed point of $G \times G$.

By the Peter-Weyl theorem, we have

$$\mathcal{O}(G) = \bigoplus_{\lambda \in \Lambda^+} \mathrm{End}(V(\lambda))$$

as $G \times G$-modules. So we obtain an isomorphism of $G \times G \times T$-algebras

$$\mathrm{Cox}(X) \simeq \bigoplus_{\lambda \in \Lambda^+, \mu \in \Lambda, \lambda \leq \mu} \mathrm{End}(V(\lambda))\, e^{\mu}$$

as a subalgebra of $\mathcal{O}(G \times G \times T)$, where $\leq$ denotes the usual partial order on weights.

This identifies the subalgebra $\mathrm{Cox}(X)^{G \times G}$ with the algebra

$$\bigoplus_{\mu \in \Lambda, \mu \geq 0} \mathbb{C}[e^{\mu}] = \mathbb{C}[e^{\alpha_1}, \cdots, e^{\alpha_r}]$$

and hence each s_i with e^{α_i}.

Also, the s_D are identified with the $u_{\varpi_i} e^{\varpi_i}$, where $u_{\varpi_i} \in \mathrm{End}(V(\varpi_i))^{B^- \times B}$ is the tensor product of highest weight vectors $f_{-\varpi_i} \otimes v_{\varpi_i}$, normalized so that $f_{-\varpi_i}(v_{\varpi_i}) = 1$.

Algebraic monoids

Definition

An **algebraic monoid** is an algebraic variety M equipped with an associative binary operation $m : M \times M \to M$ which is a morphism of varieties, and with an identity element $e \in M$ for the multiplication m.

The **unit group** $G(M)$ is the group of elements of M having a two-sided inverse.

One shows that $G(M)$ is an algebraic group, open in M. Thus, the action of $G(M) \times G(M)$ on M by left and right multiplication has an open orbit, namely $G(M)$. So we may view M as a (possibly non-normal) equivariant embedding of the group $G(M)$, viewed as the homogeneous space $G(M) \times G(M)/\mathrm{diag}(G(M))$.

As a partial converse, we have:

Lemma

Let Z be an affine equivariant embedding of a connected linear algebraic group L. Then Z has a unique structure of algebraic monoid with identity element being that of L. Moreover, $G(Z) = L$.

The Vinberg monoid

Proposition

(i) *The affine variety* $\tilde{X} = \mathrm{Spec}(\mathrm{Cox}(X))$ *is an algebraic monoid with unit group* $G \times^Z T$ *and having a zero element.*

(ii) *The quotient map* $f : \tilde{X} \to \mathbb{A}^r$ *is a homomorphism of algebraic monoids, where* $\mathbb{A}^r$ *is equipped with pointwise multiplication.*

(iii) f *is the universal homomorphism from* $\tilde{X}$ *to a commutative algebraic monoid.*

Also, recall that $\tilde{X}$ is normal and f is flat, with schematic fibers being spherical $G \times G$-varieties.

The monoid $\tilde{X}$ was first constructed by Vinberg; it is called the **enveloping monoid** and denoted by $\mathrm{Env}(G)$.

The quotient map f is called the **abelianization**. Its special fiber $\tilde{X}_0$ is the **asymptotic semigroup** and denoted by $\mathrm{As}(G)$. (It is indeed a closed subsemigroup of $\mathrm{Env}(G)$). Moreover, f may be viewed as the universal $G \times G$-equivariant deformation of $\mathrm{As}(G)$ in a specific sense.

Some references for Lecture 2

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