Cox rings of spherical varieties: Lecture 1

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Overview

- Lecture 1: Introduction to Cox rings: definitions, first classes of examples (toric varieties, flag varieties, Fano varieties).
  Relation to birational geometry via Mori dream spaces.
- Lecture 2: Wonderful varieties, their Picard group and monoid of effective divisors.
  Descriptions of their Cox ring via representation theory and commutative algebra.
  Cox rings of wonderful group compactifications and their relation to the Vinberg monoid.
- Lecture 3: Equivariant Cox rings: construction and first properties.
  Description of the equivariant Cox rings of spherical varieties by reduction to the wonderful case.
  Generators and relations for the Cox rings of wonderful varieties.
  Cox rings of horospherical varieties.

Throughout these lectures, we work over the field of complex numbers $\mathbb{C}$.

Introduction to Lecture 1

Cox rings are important invariants of algebraic varieties, which encode much information and are generally hard to determine.

This lecture will first give an introduction to Cox rings, by successive generalizations beginning with section rings. The emphasis is on smooth varieties.

We will then use equivariant methods to treat two broad classes of examples: toric varieties (which form the starting point of the theory) and flag varieties.

After reviewing the relation of Cox rings to birational geometry via Mori dream spaces, we will present a definition of Cox rings of normal varieties via rigidified line bundles, and illustrate it for toric varieties.

Section rings

Definition

Let X be a projective variety, and L an ample line bundle on X. The associated **section ring** is

$$R = R(X; L) = \bigoplus_{n=-\infty}^{\infty} H^0(X, L^{\otimes n}).$$

Then R is a finitely generated $\mathbb{Z}$-graded algebra satisfying $R_0 = \mathbb{C}$ and $R_n = 0$ for all $n < 0$.

Example

If $L = \mathcal{O}_X(1)$ for a projective embedding of X into $\mathbb{P}^N$, and

$$S = \mathbb{C}[x_0, \ldots, x_N]/I_X$$

is the homogeneous coordinate ring of X, then we have an injective homomorphism of graded rings $S \to R(X; L)$, which is surjective in large degrees. If in addition X is normal, then $R(X; L)$ is the normalization of S.

We return to the general case of a projective variety X equipped with an ample line bundle L. Then $\mathrm{Spec}(R) = \tilde{X}$ is an affine variety.

The grading of R translates into an action of the multiplicative group $\mathbb{G}_m$ on $\tilde{X}$ with a unique fixed point 0 (corresponding to the maximal graded ideal). Moreover,

$$\tilde{X} \setminus \{0\} = L \setminus L_0 = \mathrm{Spec}_X\Big(\bigoplus_{n=-\infty}^{\infty} \mathcal{L}^{\otimes n}\Big),$$

where $L_0 \subset L$ is the zero section, and $\mathcal{L}$ the sheaf of local sections of L.

We have a diagram

$$\begin{array}{ccc} L \setminus L_0 & \xrightarrow{\iota} & \tilde{X} \\ {\scriptstyle \pi}\big\downarrow & & \\ X & & \end{array}$$

where ι is an open immersion, and π is a principal bundle for $\mathbb{G}_m$ acting by multiplication on fibers.

Example

Take $X = \mathbb{P}^N$ and $L = \mathcal{O}(1)$. Then $R(X; L)$ is the graded polynomial ring $\mathbb{C}[x_0, \ldots, x_N]$, where the variables x_i have degree 1.

We get the diagram

$$\begin{array}{ccc} \mathbb{A}^{N+1} \setminus \{0\} & \xrightarrow{\iota} & \mathbb{A}^{N+1} \\ \downarrow{\scriptstyle \pi} & & \\ \mathbb{P}^N & & \end{array}$$

where ι is the inclusion, and π the projection.

The notion of section ring extends to several line bundles $L_1, \ldots, L_r$ on an arbitrary variety X.

Definition

Let

$$R = R(X; L_1, \ldots, L_r) = \bigoplus_{(n_1, \ldots, n_r) \in \mathbb{Z}^r} H^0(X, L_1^{\otimes n_1} \otimes \cdots \otimes L_r^{\otimes n_r}).$$

This is a $\mathbb{Z}^r$-graded algebra with $R_0 = \mathbb{C}$: the **section ring** of $L_1, \ldots, L_r$.

Also, let

$$\mathcal{R} = \mathcal{R}(X; L_1, \ldots, L_r) = \bigoplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^r} \mathcal{L}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{L}_r^{\otimes n_r}.$$

This is a sheaf of $\mathbb{Z}^r$-graded algebras over X, and the corresponding morphism

$$\pi : \hat{X} = \mathrm{Spec}_X(\mathcal{R}) \longrightarrow X$$

is a principal bundle for the torus $\mathbb{G}_m^r$. Moreover, we have $R = H^0(X, \mathcal{R})$ and this yields an "affinization" morphism

$$\iota : \hat{X} \to \mathrm{Spec}(R) = \tilde{X}.$$

The section ring $R(X; L_1, \ldots, L_r)$ is finitely generated if $L_1, \ldots, L_r$ are globally generated, but not in general.

Example

Let X be an elliptic curve, L_1 a non-torsion line bundle of degree 0, and L_2 a line bundle of positive degree. Then $H^0(X; L_1^{\otimes n_1} \otimes L_2^{\otimes n_2}) \neq 0$ if and only if $n_2 \geq 1$. Thus, the graded algebra $R(X; L_1, L_2)$ is not finitely generated, since its monoid of non-zero degrees is not finitely generated.

The Cox ring

We now assume that X is smooth and its Picard group $\mathrm{Pic}(X)$ is free of finite rank r. Choose line bundles $L_1, \ldots, L_r$ whose classes form a basis of $\mathrm{Pic}(X)$.

Definition

The section ring $R(X; L_1, \ldots, L_r)$ is called the **Cox ring** $\mathrm{Cox}(X)$. The sheaf $\mathcal{R} = \mathcal{R}(X; L_1, \ldots, L_r)$ is called the **Cox sheaf**.

This is a $\mathrm{Pic}(X)$-graded ring, and one may check that it is independent of the choice of a basis of $\mathrm{Pic}(X)$. Also, $\mathcal{R}$ and the corresponding morphism $\pi : \hat{X} \to X$ are independent of the above choice.

Denote by $S_X \simeq \mathbb{G}_m^r$ the torus with character group $\mathrm{Pic}(X) \simeq \mathbb{Z}^r$. Then π is a principal bundle under S_X. In particular, $\hat{X}$ is a smooth variety.

Definition

The variety $\hat{X}$ is called the **characteristic space** of X.

We have $\mathrm{Cox}(X) = \mathcal{O}(\hat{X})$ and hence $\mathrm{Cox}(X)$ is a normal domain.

Definition

A line bundle is **effective** if it has a non-zero global section.

The classes of effective line bundles in $\mathrm{Pic}(X)$ form a submonoid, denoted by $\mathrm{Eff}(X)$.
We now assume that the ring $\mathrm{Cox}(X)$ is finitely generated. Then the monoid $\mathrm{Eff}(X)$ is finitely generated as well. One then shows that the morphism $\iota : \hat{X} \to \tilde{X}$ is an open immersion, and the complement of its image has codimension ≥ 2.

Definition

Let Z be a variety, and U an open subset. We say that U is **big** (in Z) if $\mathrm{codim}_Z(Z \setminus U) \geq 2$.

Thus, $\hat{X}$ is identified with a big open subset of $\tilde{X}$.

Also, for any big open subset $U \subset X$, we have isomorphisms

$$\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(U) \text{ and } \mathrm{Cox}(X) \xrightarrow{\sim} \mathrm{Cox}(U)$$

via restriction.

First examples of Cox rings

(i) If X is the **affine space** $\mathbb{A}^n$, then $\mathrm{Pic}(X) = 0$ and hence $\mathrm{Cox}(X) = \mathcal{O}(X) = \mathbb{C}[x_1, \ldots, x_n]$.

(ii) If X is the **projective space** $\mathbb{P}^n$, then $\mathrm{Pic}(X) = \mathbb{Z}$ with generator $\mathcal{O}_X(1)$, and hence $\mathrm{Cox}(X) = R(X; \mathcal{O}_X(1)) = \mathbb{C}[x_0, \ldots, x_n]$.

(iii) Let X be an n-dimensional **quadric**, i.e., the zero locus in $\mathbb{P}^{n+1}$ of a non-degenerate quadratic form $q = q(x_0, \ldots, x_n)$.
If $n \geq 3$ then again, $\mathrm{Pic}(X) = \mathbb{Z}$ with generator $\mathcal{O}_X(1)$. As a consequence, $\mathrm{Cox}(X) = \mathbb{C}[x_0, \ldots, x_n]/(q)$.
If $n = 2$ then $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and hence $\mathrm{Pic}(X) = \mathbb{Z}^2$ with basis the classes of $\mathcal{O}(1,0), \mathcal{O}(0,1)$. Moreover, $\mathrm{Cox}(X) = \mathbb{C}[x, y, z, w]$, where x, y (resp. z, w) form a basis of $H^0(X, \mathcal{O}(1,0))$, resp. $H^0(X, \mathcal{O}(0,1))$.

(iv) Let X be the **Grassmannian** of linear subspaces of dimension m in $\mathbb{C}^n$, where $1 \leq m \leq n-1$. Then $X \subset \mathbb{P}(\bigwedge^m \mathbb{C}^n)$ via the Plücker embedding, and $\mathrm{Pic}(X) = \mathbb{Z}$ with generator $\mathcal{O}_X(1)$. The graded ring $\mathrm{Cox}(X)$ is generated by $H^0(X, \mathcal{O}_X(1)) = (\bigwedge^m \mathbb{C}^n)^*$, and the ideal of relations is generated by the Plücker relations (of degree 2).
When $n = 4$ and $m = 2$ one gets back the quadric of dimension 4.

A long exact sequence

Let G be a connected linear algebraic group, and X a G-variety. Consider the **equivariant Picard group** $\mathrm{Pic}_G(X)$ consisting of isomorphism classes of G-linearized line bundles on X. We have a homomorphism of abelian groups $\mathrm{Pic}_G(X) \to \mathrm{Pic}(X)$ which forgets the linearization, and another homomorphism of abelian groups $\mathrm{X}^*(G) \to \mathrm{Pic}_G(X)$, $\lambda \mapsto \mathcal{O}_X[\lambda]$ (the trivial line bundle on which G acts via multiplication by λ).

Proposition

Assume that X is normal and $\mathrm{Pic}(G) = 0$. Then we have an exact sequence

$$0 \longrightarrow \mathcal{O}(X)^{*G} \longrightarrow \mathcal{O}(X)^* \longrightarrow \mathrm{X}^*(G) \longrightarrow \mathrm{Pic}_G(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

The condition that $\mathrm{Pic}(G) = 0$ is equivalent to the coordinate ring $\mathcal{O}(G)$ being a unique factorization domain. This holds after replacing G with a finite cover.

If in addition X is complete, or a big open subset of a complete variety, then $\mathcal{O}(X)^* = \mathbb{C}^*$ and we obtain a short exact sequence

$$0 \longrightarrow \mathrm{X}^*(G) \longrightarrow \mathrm{Pic}_G(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

Next, let G be a connected algebraic group such that $\mathrm{Pic}(G) = 0$, and let $\pi : X \to Y$ be a principal G-bundle.

Then the pull-backs $\mathcal{O}(Y)^* \longrightarrow \mathcal{O}(X)^{*G}$ and $\mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}_G(X)$ are isomorphisms by descent.

If X is normal, then the above proposition yields an exact sequence

$$0 \longrightarrow \mathcal{O}(Y)^* \longrightarrow \mathcal{O}(X)^* \longrightarrow \mathrm{X}^*(G) \xrightarrow{c} \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0,$$

where c is the **characteristic homomorphism** which sends every character λ to the associated line bundle

$$L(\lambda) = (X \times \mathbb{C})/G \longrightarrow X/G = Y.$$

Here G acts on $X \times \mathbb{C}$ via $g \cdot (x, t) = (g \cdot x, \lambda(g)t)$.

As an application, we will prove:

Proposition

Let X be a smooth variety such that $\mathrm{Pic}(X)$ is free of finite rank. Then $\mathcal{O}(\hat{X})^* = \mathcal{O}(X)^*$ and $\mathrm{Pic}(\hat{X}) = 0$. If in addition the ring $\mathrm{Cox}(X)$ is finitely generated, then it is a unique factorization domain with unit group $\mathcal{O}(X)^*$.

Proof.

The first assertion follows from the above exact sequence for the principal S_X-bundle $\hat{X} \to X$, as the characteristic homomorphism

$$c : X^*(S_X) \to \mathrm{Pic}(X)$$

is the identity of $\mathrm{Pic}(X)$.

For the second assertion, recall that $\tilde{X} = \mathrm{Spec}(\mathrm{Cox}(X))$ is a normal variety containing $\hat{X}$ as a big open subset. Thus, the restriction maps $\mathcal{O}(\tilde{X})^* \to \mathcal{O}(\hat{X})^*$ and $\mathrm{Cl}(\tilde{X}) \to \mathrm{Cl}(\hat{X})$ are isomorphisms, where Cl denotes the divisor class group. Moreover, the natural map $\mathrm{Pic}(\hat{X}) \to \mathrm{Cl}(\hat{X})$ is an isomorphism as $\hat{X}$ is smooth. So $\mathrm{Cl}(\tilde{X}) = 0$. □

This can be made more explicit: let $D \subset X$ be an effective divisor, and $s_D \in H^0(X, \mathcal{O}_X(D))$ its canonical section. This yields an exact sequence $0 \to \mathcal{O}_X(-D) \xrightarrow{s_D} \mathcal{O}_X \to \mathcal{O}_D \to 0$ on X, and hence an exact sequence $0 \to \mathcal{O}_{\hat{X}}(-\pi^{-1}(D)) \xrightarrow{s_D} \mathcal{O}_{\hat{X}} \to \mathcal{O}_{\pi^{-1}(D)} \to 0$ on $\hat{X}$, where $s_D \in \mathcal{O}(\hat{X})$.

So s_D yields an equation of $\pi^{-1}(D)$ in $\hat{X}$, and hence of its closure in $\tilde{X}$. Moreover, every divisor in $\tilde{X}$ is linearly equivalent to $\overline{\pi^{-1}(D)}$ for some divisor D on X.

Toric varieties

Let X be a toric variety with torus $T \simeq \mathbb{G}_m^n$ and fan $\Sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^n$. Recall that the rays $\rho_1, \ldots, \rho_N \in \Sigma(1)$ correspond bijectively to the prime T-stable divisors $D_1, \ldots, D_N$ in X. The following result is due to Cox:

Theorem

If X is smooth and complete, then $\mathrm{Pic}(X)$ *is a free abelian group of rank $N - n$. Moreover,* $\mathrm{Cox}(X)$ *is the polynomial ring $\mathbb{C}[s_{D_1}, \ldots, s_{D_N}]$.*

Proof.

Choose a T-fixed point $x \in X$. Then x has a unique T-stable open affine neighborhood $U_x \simeq \mathbb{A}^n$ on which T acts linearly via n linearly independent characters. It follows that U_x meets n prime T-stable divisors $D_1, \ldots, D_n$, and $D_{n+1}, \ldots, D_N$ form a basis of the divisor class group $\mathrm{Cl}(X) \simeq \mathrm{Pic}(X)$. For any $(a_{n+1}, \ldots, a_N) \in \mathbb{Z}^{N-n}$, the line bundle $L = \mathcal{O}_X(\sum_{i=n+1}^N a_i D_i)$ is T-linearized. Thus, $H^0(X, L)$ is a T-module, and hence the direct sum of its T-weight spaces. Moreover, every T-eigenvector $s \in H^0(X, L)$ satisfies $\mathrm{div}(s) = \sum_{i=1}^N b_i D_i$ for unique non-negative integers $b_1, \ldots, b_N$. Thus, $s = c \prod_{i=1}^N s_{D_i}^{b_i}$ for a unique $c \in \mathbb{C}^*$.

The above theorem can be reformulated in geometric terms: given a smooth complete toric T-variety X, it yields an affine space $\tilde{X} = \mathbb{A}^N$ on which the torus $T \times S_X \simeq \mathbb{G}_m^n \times \mathbb{G}_m^{N-n} = \mathbb{G}_m^N$ acts by pointwise multiplication, and a $\mathbb{G}_m^N$-stable open subset $\hat{X} \subset \mathbb{A}^N$ such that the quotient $\pi : \hat{X} \to X$ exists and is a principal S_X-bundle. Thus, the complement $\mathbb{A}^N \setminus \hat{X}$ is the union of coordinate subspaces of $\mathbb{A}^N$, all of codimension ≥ 2.

This **quotient presentation** extends to any toric variety X such that $\mathcal{O}(X)^* = \mathbb{C}^*$; equivalently, the vector space $N_{\mathbb{R}}$ is spanned by Σ.
We then have an exact sequence of abelian groups

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^N \longrightarrow \mathrm{Cl}(X) \longrightarrow 0,$$

where $M = \mathrm{X}^*(T)$.

Denoting by S_X the diagonalizable group with character group $\mathrm{Cl}(X)$, this translates into an exact sequence of diagonalizable groups

$$1 \longrightarrow S_X \longrightarrow \mathbb{G}_m^N \longrightarrow T \longrightarrow 1.$$

Definition

Let G be an algebraic group, X a G-variety, and $f : X \to Y$ a morphism of varieties. We say that f is a **good quotient** by G if f is affine and G-invariant, and the induced map $\mathcal{O}_Y \to f_*(\mathcal{O}_X)^G$ is an isomorphism.

Proposition

Let X be a toric variety such that $\mathcal{O}(X)^* = \mathbb{C}^*$. Then there exists a big open subset $\hat{X} \subset \mathbb{A}^N$, stable by the action of $\mathbb{G}_m^N$, and an equivariant morphism $q : \hat{X} \to X$ which is a good quotient by S_X.

This yields a diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\ \iota\ } & \mathbb{A}^N \\ {\scriptstyle q}\big\downarrow & & \\ X & & \end{array}$$

where ι denotes the inclusion.

Flag varieties

Let $X = G/P$ be a flag variety, where G is a semi-simple algebraic group, and $P \subset G$ a parabolic subgroup. We choose a Borel subgroup $B \subset P$ and a maximal torus $T \subset B$.

Replacing G with a finite cover, we may assume that it is simply-connected; then $\mathrm{Pic}(G) = 0$. Using the Bruhat decomposition, it follows that $\mathrm{Pic}(P) = 0$. So the principal P-bundle $G \to G/P$ yields isomorphisms

$$\mathrm{X}^*(P) \xrightarrow{\sim} \mathrm{Pic}_P(G) \xrightarrow{\sim} \mathrm{Pic}(G/P), \quad \lambda \longmapsto L(\lambda).$$

We may identify $\mathrm{X}^*(P)$ with a subgroup of $\mathrm{X}^*(T)$.

Denote by $\alpha_1, \ldots, \alpha_n$ the simple roots of (G, T). For $i = 1, \ldots, n$, let P^i be the maximal parabolic subgroup of G containing B such that $-\alpha_i$ is not a root of (P, T). Then $\mathrm{X}^*(P^i) = \mathbb{Z}\varpi_i$, where ϖ_i denotes the ith fundamental weight.

We have

$$P = \bigcap_{i \in I} P^i$$

for a unique subset I of $\{1, \ldots, n\}$. Moreover, the ϖ_i ($i \in I$) form a basis of $\mathrm{X}^*(P)$. They also generate the monoid $\mathrm{Eff}(X)$ of effective line bundles.

Recall a version of the Borel-Weil theorem: for any dominant weight $\lambda \in \mathrm{X}^*(P)$, we have an isomorphism of G-modules

$$H^0(X, L(\lambda)) \simeq V(\lambda)^*,$$

where $V(\lambda)$ denotes the simple G-module with highest weight λ.

In particular, $H^0(X, L(\varpi_i)) = V(\varpi_i)^*$ for all $i \in I$. Moreover, we have an isomorphism of G-modules $V(\varpi_i)^* \simeq V(\varpi_j)$, where $j \in \{1, \ldots, n\}$ satisfies $\alpha_j = -w_0(\alpha_i)$.

The canonical sections of the Schubert divisors $D_i \subset X$ yield highest weight vectors $s_{D_i} \in V(\varpi_i)^*$.

Given two dominant weights $\lambda, \mu \in \mathrm{X}^*(P)$, the isomorphism

$$L(\lambda) \otimes L(\mu) \xrightarrow{\sim} L(\lambda + \mu)$$

induces a surjective map of G-modules

$$V(\lambda)^* \otimes V(\mu)^* \longrightarrow V(\lambda + \mu)^*$$

which is dual to the injective map

$$V(\lambda + \mu) \longrightarrow V(\lambda) \otimes V(\mu)$$

(the "Cartan component").

Together with a theorem of Kostant, this yields:

Proposition

(i) *The Cox ring of* $X = G/P$ *has a presentation with generators the* G*-modules* $V(\varpi_i)^*(i \in I)$*, and relations the kernels of the natural maps*

$$\operatorname{Sym}^2 V(\varpi_i)^* \to V(2\varpi_i)^* \quad (i \in I), \text{ and}$$

$$V(\varpi_i)^* \otimes V(\varpi_j)^* \to V(\varpi_i + \varpi_j)^* \quad (i, j \in I, i \neq j).$$

(ii) *The variety* $\tilde{X}$ *is the* G*-orbit closure of the sum of highest weight vectors,* $v = \sum_{i \in I} v_{\varpi_i} \in \bigoplus_{i \in I} V(\varpi_i)$.

(iii) *We have* $\hat{X} = G \cdot v \simeq G/[P, P]$.

Example

If X is the Grassmannian of m-dimensional subspaces of $\mathbb{C}^n$, then $\tilde{X} \subset \bigwedge^m \mathbb{C}^n$ is the cone of decomposable (or pure) tensors. The above presentation gives back the Plücker relations.

Returning to the general case, consider the unipotent part U of B. Then the ring of invariants $\operatorname{Cox}(X)^U$ is the polynomial ring $\mathbb{C}[s_{D_i}, i \in I]$.

Fano varieties

Definition

A **Fano variety** is a smooth projective variety X such that the anticanonical line bundle $\mathcal{O}_X(-K_X) = \det(T_X)$ is ample.

Proposition

The Picard group of any Fano variety is free of finite rank.

Proof.

Let L be a numerically trivial line bundle on a Fano variety X. Then $L(-K_X)$ is ample by the Nakai-Moishezon criterion. In view of the Kodaira vanishing theorem, it follows that $H^i(X, L) = 0$ for all $i > 0$. In particular, $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$. By the Riemann-Roch theorem, this yields $\chi(X, L) = \chi(X, \mathcal{O}_X) = 1$ and hence $H^0(X, L) \simeq \mathbb{C}$. Thus, $L = \mathcal{O}_X(D)$ for some effective divisor D, and $H^0(X, L) = \mathbb{C}s_D$. Then $H^0(X, L^{\otimes n}) = \mathbb{C}s_D^n$ for any integer n, and hence the class of L is non-torsion. Also, the exponential sheaf sequence yields an isomorphism $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z})$. As a consequence, $\mathrm{Pic}(X)$ is finitely generated and torsion-free. □

The following result is due to Birkar, Cascini, Hacon and McKernan:

Theorem

The Cox ring of any Fano variety is finitely generated.

The Fano varieties of dimension 2 are called **Del Pezzo surfaces**. These are either $\mathbb{P}^1 \times \mathbb{P}^1$, or the surfaces obtained from $\mathbb{P}^2$ by blowing up at most 8 points in general position in the following sense:

- No three points lie on a line.
- No six points lie on a conic.
- No eight points lie on a cubic curve which is singular at some of these points.

By the above theorem, the cone of effective curves on every Del Pezzo surface is rational polyhedral (this can be checked directly).

In contrast, every surface X obtained by blowing up 9 points in general position in $\mathbb{P}^2$ contains infinitely many exceptional curves of the first kind. Such curves generate extremal rays of the cone of effective curves, which is therefore not polyhedral.

Some notions of birational geometry

Let X be a normal projective variety, and L a line bundle on X.

We say that L is **numerically effective** (nef) if $L \cdot C \geq 0$ for any curve $C \subset X$. The numerical equivalence classes of the nef line bundles form a closed convex cone $\mathrm{Nef}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$ (the closure of the ample cone).

We say that L is *semi-ample* if the base locus $\mathrm{Bs}(L^{\otimes n})$ is empty for some positive integer n. Then L is nef, but the converse fails.

We say that L is **movable** if the stable base locus $\bigcap_{n \geq 1} \mathrm{Bs}(L^{\otimes n})$ is empty. Denote by $\mathrm{Mov}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$ the convex cone generated by the movable classes; then $\mathrm{Nef}(X) \subset \mathrm{Mov}(X) \subset \mathrm{Eff}(X)_{\mathbb{R}}$.

We say that X is $\mathbb{Q}$-**factorial** if for any Weil divisor D on X, there exists a positive integer n such that nD is Cartier. Equivalently, the quotient $\mathrm{Cl}(X)/\mathrm{Pic}(X)$ is torsion.

Under this assumption, a **small** $\mathbb{Q}$-**factorial modification** (SQM) of X is a birational map $f : X \dashrightarrow X'$ such that X' is a normal projective $\mathbb{Q}$-factorial variety, and f is an isomorphism in codimension 1. Then the pull-back $f^* : \mathrm{N}^1(X')_{\mathbb{R}} \to \mathrm{N}^1(X)_{\mathbb{R}}$ is well-defined, and preserves the movable and effective cones.

Cox rings and Mori dream spaces

A **Mori dream space** (MDS) is a normal projective variety X satisfying the following conditions:

(i) The group $\mathrm{Pic}(X)$ is finitely generated and X is $\mathbb{Q}$-factorial.

(ii) The cone $\mathrm{Nef}(X)$ is generated by finitely many classes of semi-ample line bundles.

(iii) There is a finite collection of SQMs $f_i : X \dashrightarrow X_i$ such that each X_i satisfies (ii) and $\bigcup_i f_i^* \mathrm{Nef}(X_i) = \mathrm{Mov}(X)$.

The following result is due to Hu and Keel:

Theorem

A normal projective $\mathbb{Q}$-factorial variety X is a Mori dream space if and only if the group $\mathrm{Pic}(X)$ is finitely generated and the section ring $R(X; L_1, \ldots, L_r)$ is a finitely generated algebra for any line bundles $L_1, \cdots, L_r$ on X.

Hu and Keel also obtained a converse to Cox's theorem:

Proposition

Let X be a smooth projective variety such that $\mathrm{Pic}(X)$ is free of finite rank and $\mathrm{Cox}(X)$ is a polynomial ring. Then X is a toric variety.

Reflexive sheaves

Definition

Let X be a variety, and $\mathcal{F}$ a coherent sheaf on X. The **dual** sheaf is $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. Then $\mathcal{F}^\vee$ is coherent and we have a natural map $j : \mathcal{F} \to \mathcal{F}^{\vee\vee}$. We say that $\mathcal{F}$ is **reflexive** if j is an isomorphism.

If X is normal (resp. smooth), then the reflexive sheaves of rank 1 on X are exactly the sheaves $\mathcal{O}_X(D)$, where D is a Weil divisor on X (resp. the invertible sheaves). Moreover, we have $\mathcal{O}_X(D)^\vee = \mathcal{O}_X(-D)$.

Assume that X is normal, and denote by $i : U \to X$ the inclusion of its smooth locus (a big open subset of X). Then the assignment $\mathcal{L} \mapsto i_*(\mathcal{L})$ yields a bijective correspondence between invertible sheaves on U and reflexive sheaves of rank 1 on X. The inverse is given by $\mathcal{F} \mapsto i^*(\mathcal{F})$.

This identifies the inverse (or dual) of invertible sheaves with the dual of reflexive sheaves. Also, the tensor product of invertible sheaves is identified with the map

$$(\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} * \mathcal{G} = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^{\vee\vee}, \quad (\mathcal{O}_X(D), \mathcal{O}_X(E)) \longmapsto \mathcal{O}_X(D+E).$$

This yields identifications $\mathrm{Pic}(U) = \mathrm{Cl}(X)$ and $\mathcal{O}(U) = \mathcal{O}(X)$.

Rigidified sheaves

We still consider a normal variety X and its smooth locus U.

Definition

Let $x \in U$, and $\mathcal{L}$ an invertible sheaf on U. A **rigidification** of $\mathcal{L}$ at x is an isomorphism

$$f : \mathcal{L}(x) \xrightarrow{\sim} \mathbb{C}.$$

There are obvious notions of morphisms and tensor product of rigidified invertible sheaves. The isomorphism classes of such sheaves form an abelian group $\mathrm{Pic}^x(U)$ equipped with a homomorphism $\mathrm{Pic}^x(U) \to \mathrm{Pic}(U)$ which forgets the rigidification.

Since $\mathrm{Aut}(\mathcal{L}) = \mathcal{O}(U)^* = \mathcal{O}(X)^*$, this yields an exact sequence

$$0 \longrightarrow \mathcal{O}(X)^*/\mathbb{C}^* \longrightarrow \mathrm{Pic}^x(U) \longrightarrow \mathrm{Pic}(U) \longrightarrow 0.$$

In particular, if $\mathcal{O}(X)^* = \mathbb{C}^*$ then $\mathrm{Pic}^x(U) = \mathrm{Pic}(U)$. This holds e.g. if X is complete.

The Cox sheaf of a normal variety

Let X be a normal variety such that $\mathcal{O}(X)^* = \mathbb{C}^*$. Choose a point x in the smooth locus U; then every divisor class $[\mathcal{F}] \in \mathrm{Cl}(X) = \mathrm{Pic}(U)$ has a unique representative by a rigidified reflexive sheaf $\mathcal{F}^x$ of rank 1.

Given two such classes $[\mathcal{F}]$, $[\mathcal{G}]$ with sum $[\mathcal{F} * \mathcal{G}]$, the multiplication map $\mathcal{F}^x \otimes_{\mathcal{O}_X} \mathcal{G}^x \to (\mathcal{F} * \mathcal{G})^x$ is well defined. Thus, we obtain an $\mathcal{O}_X$-algebra structure on the sheaf

$$\mathcal{R}_X = \bigoplus_{[\mathcal{F}] \in \mathrm{Cl}(X)} \mathcal{F}^x.$$

Definition

We say that $\mathcal{R}_X$ is the **Cox sheaf** of the pointed variety (X, x).

The quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{R}_X$ is graded by the divisor class group $\mathrm{Cl}(X)$. If this group is finitely generated, then we denote by S_X the "dual" diagonalizable group, with character group $\mathrm{Cl}(X)$.
The structure morphism

$$q : \hat{X} = \mathrm{Spec}_X(\mathcal{R}_X) \longrightarrow X$$

is then a good quotient by the action of S_X.

The Cox ring of a normal variety

Definition

With the above notation and assumptions, the **Cox ring** of (X, x) is the ring $\mathrm{Cox}(X) = H^0(X, \mathcal{R}_X)$.

One shows that the Cox sheaf and ring are independent of the choice of the base point x, up to isomorphism.

Also, we still have an "affinization" morphism

$$\iota : \hat{X} \longrightarrow \tilde{X} = \mathrm{Spec}(\mathrm{Cox}(X)),$$

which is S_X-equivariant. If the algebra $\mathrm{Cox}(X)$ is finitely generated, then ι is the inclusion of a big open subset. Moreover, $\mathrm{Cox}(X) = \mathrm{Cox}(X_0)$ for any big open subset $X_0 \subset X$.

Example

Let X be a toric variety with base point x, and assume that $\mathcal{O}(X)^* = \mathbb{C}^*$. Then $\mathrm{Cox}(X)$ is still the polynomial ring $\mathbb{C}[s_{D_1}, \ldots, s_{D_n}]$ in the canonical sections of the prime T-stable divisors. This gives back the quotient presentation of X as a good quotient of the big open subset $\hat{X} \subset \mathbb{A}^N$ by the diagonalizable group S_X with character group $\mathrm{Cl}(X)$.

Some references for Lecture 1

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