SOME AUTOMORPHISM GROUPS ARE LINEAR ALGEBRAIC

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Abstract. Consider a normal projective variety X, a linear algebraic subgroup G of $\mathrm{Aut}(X)$, and the field K of G-invariant rational functions on X. We show that the subgroup of $\mathrm{Aut}(X)$ that fixes K pointwise is linear algebraic. If K has transcendence degree 1 over k, then $\mathrm{Aut}(X)$ is an algebraic group.

1. Introduction

Let X be a projective variety over an algebraically closed field k. The automorphism group functor of X, which assigns to any k-scheme S the group of relative automorphisms $\mathrm{Aut}_S(X \times S)$, is represented by a k-group scheme Aut_X, locally of finite type (see [Gro61, p. 268]). Thus, the reduced subscheme of Aut_X is equipped with a structure of a smooth k-group scheme $\mathrm{Aut}(X)$. But $\mathrm{Aut}(X)$ is not necessarily an algebraic group; equivalently, it may have infinitely many components, e.g. when X is a product of two isogenous elliptic curves. Still, the automorphism group is known to be a linear algebraic group for some interesting classes of varieties including smooth Fano varieties, complex almost homogeneous manifolds (this is due to Fu and Zhang, see [FZ13, Thm. 1.2]) and normal almost homogeneous varieties in arbitrary characteristics (see [Bri19, Thm. 1]; we say that X is almost homogeneous if it admits an action of a smooth connected algebraic group with an open dense orbit). In this note, we obtain a relative version of the latter result:

Theorem 1. *Let X be a normal projective variety, $G \subset \mathrm{Aut}(X)$ a linear algebraic subgroup, $K := k(X)^G$ the field of G-invariant rational functions on X, and $\mathrm{Aut}_K(X)$ the subgroup of $\mathrm{Aut}(X)$ fixing K pointwise. Then $\mathrm{Aut}_K(X)$ is a linear algebraic group.*

Here and later, closed subgroups of $\mathrm{Aut}(X)$ are equipped with their reduced subscheme structure unless otherwise specified. Thus, they may be identified with their group of k-rational points.

By a theorem of Rosenlicht (see [Ros63], and [BGR17] for a modern proof), the rational functions in K separate the G-orbit closures of general points of X. Thus, $\mathrm{Aut}_K(X)$ is the largest subgroup of $\mathrm{Aut}(X)$ having the same general orbit closures as G. Also, note that $K = k$ if and only if X is almost homogeneous; then $\mathrm{Aut}_K(X)$ is just the full automorphism group.

The proof of Theorem 1 is presented in Section 2. As in [FZ13, Bri19], the idea is to construct a big line bundle on X, invariant under $\mathrm{Aut}_K(X)$. To handle our relative situation, we use some tools from algebraic geometry over an arbitrary field, which seem to be unusual in this setting.

As an application of Theorem 1 and its proof, we show that $\mathrm{Aut}(X)$ is a linear algebraic group under additional assumptions:

Theorem 2. *Let X be a normal projective variety, $G \subset \mathrm{Aut}(X)$ a linear algebraic subgroup, and $K := k(X)^G$. If K has transcendence degree 1 over k, then $\mathrm{Aut}(X)$ is an algebraic group. If in addition G is connected and K is not the function field of an elliptic curve, then $\mathrm{Aut}(X)$ is linear.*

This is again due (in essence) to Fu and Zhang when X is a complex manifold and $K \simeq \mathbb{C}(t)$, see [FZ13, Appl. 1.4]. Another instance where this result was known occurs when G is a torus, say T. Then X is said to be a T-variety of complexity one; its automorphism group is explicitly described in [AHHL14], when $K \simeq \mathbb{C}(t)$ again.

Theorem 2 is proved in Section 3, as well as the following consequence:

Corollary 3. *Let X be a normal projective surface having a non-trivial action of a connected linear algebraic group. Then $\mathrm{Aut}(X)$ is an algebraic group. Moreover, $\mathrm{Aut}(X)$ is linear unless X is birationally equivalent to $Y \times \mathbb{P}^1$ for some elliptic curve Y.*

For smooth rational surfaces, this result is due to Harbourne (see [Har87, Cor. 1.4]); for ruled surfaces over an elliptic curve, it also follows from the explicit description of automorphism groups obtained by Maruyama (see [Mar71, Thm. 3]). Note that the linearity assumption for the acting group cannot be suppressed, as shown again by the example of a product of two isogenous elliptic curves. Also, there exist (much more elaborate) examples of smooth projective surfaces having a discrete, non-finitely generated automorphism group (see [DO19, Ogu19]).

Theorem 1 may be viewed as a first step towards "Galois theory" for linear algebraic subgroups of $\mathrm{Aut}(X)$. With this in mind, it would be interesting to characterize the fields of invariants of linear algebraic groups among the subfields $K \subset k(X)$, and the linear algebraic groups which occur as $\mathrm{Aut}_K(X)$. In characteristic 0, it is easy to see that a subfield $K \subset k(X)$ is the field of invariants of a connected linear algebraic group if and only if K is algebraically closed in $k(X)$, the $k(X)$-vector space $\mathrm{Der}_K(k(X))$ is spanned by global vector fields, and X is unirational over $\bar{K}$.

Notation and conventions. The ground field k is algebraically closed, of arbitrary characteristic. A *variety* is an integral separated scheme of finite type. An *algebraic group* G is a group scheme of finite type. The *neutral component* G^0 is the connected component of G containing the neutral element. We say that G is *linear* if it is smooth and affine.

2. Proof of Theorem 1

It proceeds via a sequence of reduction steps and lemmas. We begin with two easy and useful observations:

Lemma 4. (i) $\mathrm{Aut}_K(X)$ *is a closed subgroup of* $\mathrm{Aut}(X)$.
(ii) *If* G *is connected, then* K *is algebraically closed in* $k(X)$.

Proof. (i) It suffices to show that the stabilizer $\mathrm{Aut}_f(X)$ is closed in $\mathrm{Aut}(X)$ for any non-zero $f \in k(X)$. Let

$$\mathrm{Aut}_{(f)}(X) := \{g \in \mathrm{Aut}(X) \mid g^*(f) = \lambda(g)f \text{ for some } \lambda(g) \in k^*\}.$$

Then $\mathrm{Aut}_f(X) \subset \mathrm{Aut}_{(f)}(X) \subset \mathrm{Aut}(X)$.

Denote by D_0 (resp. D_∞) the scheme of zeroes (resp. poles) of f, and by $\mathrm{Aut}(X, D_0, D_\infty) \subset \mathrm{Aut}(X)$ the common stabilizer of these subschemes of X. We claim that $\mathrm{Aut}_{(f)}(X) = \mathrm{Aut}(X, D_0, D_\infty)$. Indeed, the inclusion $\mathrm{Aut}_{(f)}(X) \subset \mathrm{Aut}(X, D_0, D_\infty)$ is obvious, and the opposite inclusion follows from the fact that every $g \in \mathrm{Aut}(X, D_0, D_\infty)$ satisfies $\mathrm{div}(g^*(f)) = \mathrm{div}(f)$.

By the claim, $\mathrm{Aut}_{(f)}(X)$ is closed in $\mathrm{Aut}(X)$. Moreover, $\mathrm{Aut}_{(f)}(X)$ stabilizes the open subset $U := X \setminus (D_0 \cup D_\infty) \subset X$, and $\mathrm{Aut}_f(X)$ is the stabilizer of $f \in \mathcal{O}(U)$. So $\mathrm{Aut}_f(X)$ is closed in $\mathrm{Aut}_{(f)}(X)$.

(ii) Let $f \in k(X)$ be algebraic over K. Then the stabilizer of f in G is a closed subgroup of finite index, and hence is the whole G. So $f \in K$.

□

Step 1. We may assume that G is connected.

Proof. Denote by $\pi_0(G) := G/G^0$ the group of components. Then the invariant field $L := k(X)^{G^0}$ is equipped with an action of the finite group $\pi_0(G)$, and $L^{\pi_0(G)} = K$. Thus, L/K is a Galois extension with Galois group a quotient of $\pi_0(G)$. As G^0 is connected, L is algebraically closed in $k(X)$ (Lemma 4). Therefore, L is the algebraic closure of K in $k(X)$, and hence is stable under $\mathrm{Aut}_K(X)$. Since the composition $G \to \mathrm{Aut}_K(X) \to \mathrm{Aut}_K(L)$ is surjective and sends G^0 to the identity, this yields an exact sequence of algebraic groups

$$1 \longrightarrow \mathrm{Aut}_L(X) \longrightarrow \mathrm{Aut}_K(X) \longrightarrow \mathrm{Aut}_K(L) \longrightarrow 1,$$

and in turn the assertion in view of Lemma 4 again. □

Lemma 5. $\mathrm{Aut}_K(X)^0$ *is a linear algebraic group.*

Proof. Let $\widetilde{G} := \mathrm{Aut}_K(X)^0$. Then $G \subset \widetilde{G}$ and $k(X)^{\widetilde{G}} = K$. In view of Rosenlicht's theorem mentioned in the introduction, there exists a dense open G-stable subset $U \subset X$ such that the orbit $G \cdot x$ is open in $\widetilde{G} \cdot x$ for all $x \in U$. Since G is connected and linear, it follows that the variety $\widetilde{G} \cdot x$ is unirational, and hence its Albanese variety $\mathrm{Alb}(\widetilde{G} \cdot x)$ is trivial.

We now recall a group-theoretic description of $\mathrm{Alb}(\widetilde{G} \cdot x)$. By Chevalley's structure theorem, $\widetilde{G}$ has a largest connected linear normal subgroup $\widetilde{G}_{\mathrm{aff}}$ and the quotient $\widetilde{G}/\widetilde{G}_{\mathrm{aff}}$ is an abelian variety (see e.g. [BSU13, Thm. 1.1.1]). Denote by $H = \widetilde{G}_x$ the stabilizer of x; then $\widetilde{G}/(\widetilde{G}_{\mathrm{aff}}\, H)$ is an abelian variety as well. As the Albanese morphism of $\widetilde{G}$ (resp. of $\widetilde{G} \cdot x \simeq \widetilde{G}/H$) is invariant by $\widetilde{G}_{\mathrm{aff}}$, it follows readily that $\mathrm{Alb}(\widetilde{G}) = \widetilde{G}/\widetilde{G}_{\mathrm{aff}}$ and $\mathrm{Alb}(\widetilde{G} \cdot x) = \widetilde{G}/(\widetilde{G}_{\mathrm{aff}}\, H)$. Thus, $\widetilde{G} = \widetilde{G}_{\mathrm{aff}}\, H$. But since H is affine (see e.g. [BSU13, Lem. 2.3.2]), the natural map $\widetilde{G}/\widetilde{G}_{\mathrm{aff}} \to \widetilde{G}/\widetilde{G}_{\mathrm{aff}}\, H$ is an isogeny. Therefore, $\widetilde{G} = \widetilde{G}_{\mathrm{aff}}$. □

Step 2. In view of Step 1 and Lemma 5, we may assume that $G = \mathrm{Aut}_K(X)^0$. Then K is algebraically closed in $k(X)$ by Lemma 4.

We may further assume that there exists a morphism $f : X \to Y$, where Y is a normal projective variety satisfying the following conditions: f induces an isomorphism $k(Y) \simeq K$, we have $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, and $\mathrm{Aut}_K(X)$ is isomorphic to a closed subgroup of the relative automorphism group $\mathrm{Aut}_Y(X)$, containing $\mathrm{Aut}_Y(X)^0$.

Proof. Choose generators $f_1, \ldots, f_n$ of the field extension K/k. This defines a rational map $(f_1, \ldots, f_n) : X \dashrightarrow \mathbb{P}^{n-1}$ and hence a rational map $f : X \dashrightarrow Y$, where Y is a normal projective variety and f induces an isomorphism $k(Y) \simeq K$. The normalization of the graph of f yields a normal projective variety X' equipped with morphisms

$$f' : X' \longrightarrow Y, \quad g : X' \longrightarrow X$$

such that g is birational and $f' = f \circ g$. Thus, f' also induces an isomorphism $k(Y) \simeq K$. Consider the Stein factorization of f' as $X' \xrightarrow{\varphi} Z \xrightarrow{\psi} Y$. Then ψ is finite, and hence so is the extension $k(Z)/k(Y)$. But $k(Z) \subset k(X') \simeq k(X)$, and $k(Y) \simeq K$ is algebraically closed in $k(X)$. Therefore, ψ is birational. As Y is normal, ψ is an isomorphism by Zariski's Main Theorem. Thus, $f_*(\mathcal{O}_X) = \mathcal{O}_Y$.

By construction, $\mathrm{Aut}_K(X)$ acts on X' and f' is invariant under this action. This yields a homomorphism

$$u : \mathrm{Aut}_K(X) \longrightarrow \mathrm{Aut}_Y(X').$$

On the other hand, Blanchard's lemma (see e.g. [BSU13, Prop. 4.2.1]) yields a homomorphism $\mathrm{Aut}(X')^0 \to \mathrm{Aut}(X)^0$, which restricts to a homomorphism

$$g_* : \mathrm{Aut}_Y(X')^0 \longrightarrow \mathrm{Aut}_K(X)^0.$$

Since g is birational, g_* is the inverse of the restriction to neutral components $u^0 : \mathrm{Aut}_K(X)^0 \to \mathrm{Aut}_Y(X')^0$. In particular, the image of u contains $\mathrm{Aut}_Y(X')^0$, and hence is closed in $\mathrm{Aut}_Y(X')$. □

Step 3. In view of Step 2, we now consider a contraction $f : X \to Y$, i.e., a morphism of normal projective varieties such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. We assume that the algebraic group $G := \mathrm{Aut}_Y(X)^0$ is linear, and f induces an isomorphism $k(Y) \simeq K$. To prove Theorem 1, it suffices to show that $\mathrm{Aut}_Y(X)$ is a linear algebraic group.

It suffices in turn to construct a big line bundle on X which admits an $\mathrm{Aut}_Y(X)$-linearization (recall that a line bundle L on a projective variety Z is big if there exists $c > 0$ such that $\dim H^0(Z, L^{\otimes n}) > c\, n^{\dim(Z)}$ for $n \gg 0$; see e.g. [KM98, Lem. 2.60] for further characterizations of bigness). Indeed, $\mathrm{Aut}_Y(X)$ is closed in $\mathrm{Aut}(X)$ (as follows from Lemma 4), and hence in the stabilizer $\mathrm{Aut}(X, [L])$ of the isomorphism class of L for any $\mathrm{Aut}_Y(X)$-linearized line bundle L. If in addition L is big, then $\mathrm{Aut}(X, [L])$ is a linear algebraic group in view of [Bri19, Lem. 2.3]. The desired line bundle will be constructed in Step 6, after further preparations.

Denote by η the generic point of Y and by X_η the generic fiber of f. Then X_η is a projective scheme over $k(\eta) = K$, equipped with an action of the K-group scheme G_K (since G acts on X by relative automorphisms). Likewise, the geometric generic fiber $X_{\bar\eta}$ is a projective scheme over $k(\bar\eta) = \bar K$, equipped with an action of $G_{\bar K}$ (a linear algebraic group over $\bar K$).

Lemma 6. *With the above assumptions, X_η is normal and geometrically integral. Moreover, $X_{\bar\eta}$ is almost homogeneous under $G_{\bar\eta}$ and we have $k(X_{\bar\eta}) = k(X) \otimes_K \bar K$.*

Proof. By [Spr89, Lem. IV.1.5], the extension $k(X)/K$ is separable. As a consequence, the ring $k(X) \otimes_K \bar K$ is reduced (see e.g. [SP19, 10.42.5]). So X_η is geometrically reduced. As K is algebraically closed in $k(X)$, the spectrum of $k(X) \otimes_K \bar K$ is irreducible (see e.g. [SP19, 10.46.8]). Hence X_η is geometrically irreducible and the field of rational functions $\bar K(X_{\bar\eta})$ equals $k(X) \otimes_K \bar K$. Thus,

$$\bar K(X_{\bar\eta})^{G_{\bar\eta}} = (k(X) \otimes_K \bar K)^{G_{\bar\eta}} \subset (k(X) \otimes_K \bar K)^G = \bar K,$$

where G is identified with its group of k-rational points. So $\bar K(X_{\bar\eta})^{G_{\bar\eta}} = k(\bar\eta)$. By Rosenlicht's theorem, it follows that $X_{\bar\eta}$ is almost homogeneous under $G_{\bar\eta}$.

It remains to show that X_η is normal. Consider an open affine subset V of Y and an open affine cover (U_i) of $f^{-1}(V)$. Then the $(U_i)_\eta$ form an open affine cover of X_η and $\mathcal{O}((U_i)_\eta) = \mathcal{O}(U_i) \otimes_{\mathcal{O}(V)} k(\eta)$ is a localization of $\mathcal{O}(U_i)$, hence a normal domain. □

Remark 7. The generic fiber X_η is geometrically normal if $\mathrm{char}(k) = 0$. But this fails if $\mathrm{char}(k) = p > 0$, as shown by the following example: let $G = \mathbb{G}_a$ act on $X = \mathbb{P}^2$ via $t \cdot [x : y : z] = [x + ty + t^{2p} z : y : z]$. Then $K = k(\frac{y}{z})$ and the G-orbit closure of $[x : y : z]$ is singular at ∞ whenever $z \neq 0$. Likewise, the geometric generic fiber $X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\bar{K})$ is a singular curve.

Step 4. We may further assume that $X(k(\eta))$ contains a point x_0 such that the orbit $G_{\bar\eta} \cdot x_0$ is open in $X_{\bar\eta}$.

Proof. Denote by K^s the separable closure of K in $\bar{K}$. Then $X(K^s)$ is dense in $X_{\bar\eta} = X_{\bar K}$, since the latter is integral (Lemma 6). So we may find $x_0 \in X(K^s)$ such that $G_{\bar\eta} \cdot x_0$ is open in $X_{\bar\eta}$. Then $x_0 \in X(K')$ for some finite Galois extension K'/K.

Denote by Γ the Galois group of K'/K, by Y' the normalization of Y in K', and by X' the normalization of X in $k(X) \otimes_K K'$ (the latter is a field as a consequence of Lemma 6). Then we have a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f} & Y, \end{array} \tag{2.1}$$

where g, h are the quotients by Γ. Denoting by η' the generic point of Y', this yields a commutative square

$$\begin{array}{ccc} X'_{\eta'} & \longrightarrow & \eta' \\ \downarrow & & \downarrow \\ X_\eta & \longrightarrow & \eta, \end{array} \tag{2.2}$$

where the vertical arrows are quotients by Γ again, and x_0 is a section of the top horizontal arrow. Since the right vertical arrow is a Γ-torsor, so is the left vertical arrow and the square (2.2) is cartesian. Therefore,

$$X'_{\overline{\eta'}} = X'_{\eta'} \times_{\eta'} \overline{\eta'} = (X_\eta \times_\eta \eta') \times_{\eta'} \overline{\eta'} = X_\eta \times_\eta \overline{\eta'} = X_\eta \times_\eta \bar\eta = X_{\bar\eta}.$$

In particular, $X'_{\eta'}$ is normal and geometrically integral (Lemma 6 again).

It follows that $f'_*(\mathcal{O}_{X'}) = \mathcal{O}_{Y'}$ by an argument of Stein factorization as in Step 2. More specifically, f' is the composition

$$X' \longrightarrow \mathrm{Spec}\, f'_*(\mathcal{O}_{X'}) =: Z' \xrightarrow{g'} Y',$$

where Z' is a variety and g' is finite. Moreover, $Z'_{\eta'}$ is geometrically integral (since so is $X'_{\eta'}$) and the finite morphism $Z'_{\eta'} \to \eta'$ has a section (since so does

$X'_{\eta'} \to \eta'$). Thus, g' is an isomorphism at η'. In view of the normality of Y' and Zariski's Main Theorem, we conclude that g' is an isomorphism.

Next, we construct an isomorphism

$$\mathrm{Aut}^{\Gamma}(X') \xrightarrow{\simeq} \mathrm{Aut}(X), \tag{2.3}$$

where the right-hand side denotes the subgroup of Γ-invariants in $\mathrm{Aut}(X')$. For any scheme S, (2.1) yields a commutative square

$$\begin{array}{ccc} X' \times S & \longrightarrow & Y' \times S \\ \downarrow & & \downarrow \\ X \times S & \longrightarrow & Y \times S, \end{array}$$

where the vertical arrows are again quotients by Γ. Since these quotients are categorical (see e.g. [Mum08, §12, Thm. 1]), every Γ-equivariant automorphism of $X' \times S$ over S induces an automorphism of $X \times S$ over S. This yields a morphism of (abstract) groups

$$\mathrm{Aut}^{\Gamma}_S(X' \times S) \longrightarrow \mathrm{Aut}_S(X \times S)$$

which is clearly functorial in S. Thus, we obtain a morphism of automorphism group schemes $u : \mathrm{Aut}^{\Gamma}_{X'} \to \mathrm{Aut}_X$ with an obvious notation. The induced morphism of Lie algebras is the natural map $\mathrm{Der}^{\Gamma}(\mathcal{O}_{X'}) \to \mathrm{Der}(\mathcal{O}_X)$ with an obvious notation again. The kernel of this map is contained in $\mathrm{Der}_{k(X)}(k(X'))$, and hence is zero since $k(X')$ is separable algebraic over $k(X)$. Restricting u to the reduced subscheme of $\mathrm{Aut}^{\Gamma}_{X'}$ yields a homomorphism

$$v : \mathrm{Aut}^{\Gamma}(X') \longrightarrow \mathrm{Aut}(X)$$

such that the induced morphism of Lie algebras is injective; thus, v is étale. Its set-theoretic kernel is contained in $\mathrm{Aut}^{\Gamma}_{k(X)}(k(X'))$, and hence is trivial by Galois theory. So v is a closed immersion. Moreover, for any $g \in \mathrm{Aut}(X)$, we have a Γ-equivariant automorphism $g \otimes \mathrm{id}$ of $k(X) \otimes_K K' = k(X')$, which stabilizes the normalization of $\mathcal{O}_X$ and hence yields a lift of g in $\mathrm{Aut}^{\Gamma}(X')$. So v is surjective on k-rational points. This yields the desired isomorphism (2.3).

By construction, v restricts to an isomorphism

$$\mathrm{Aut}^{\Gamma}_{Y'}(X') \xrightarrow{\simeq} \mathrm{Aut}_Y(X). \tag{2.4}$$

In particular, if $\mathrm{Aut}_{Y'}(X')$ is linear algebraic, then so is $\mathrm{Aut}_Y(X)$.

Since $G = \mathrm{Aut}_Y(X)^0$ has an open orbit in the general fibers of f, it follows that the connected algebraic group $G' := \mathrm{Aut}_{Y'}(X')^0$ has an open orbit in the general fibers of f'. As a consequence, $K' = k(X')^{G'}$. Likewise, the orbit $G'_{\overline{\eta'}} \cdot x_0$ is open in $X'_{\overline{\eta'}}$.

To complete the proof, it remains to show that if X' admits a big line bundle L' which is $\mathrm{Aut}_{Y'}(X')$-linearized, then X admits a big line bundle L which is

$\mathrm{Aut}_Y(X)$-linearized. This will follow from a norm argument. More specifically, let $M := \otimes_{\gamma\in\Gamma} \gamma^*(L')$. Then M is a big line bundle on X' and is $\mathrm{Aut}_{Y'}(X')^\Gamma$-linearized. Moreover, $M = g^*(L)$ for a line bundle L on X (the norm of M, see [EGA, II.6.5]); we have $L = g_*(M)^\Gamma$. Thus, L is $\mathrm{Aut}_Y(X)$-linearized in view of the isomorphism (2.4). Furthermore, we have

$$H^0(X', M^{\otimes n}) = H^0(X', g^*(L)^{\otimes n}) = H^0(X, L^{\otimes n} \otimes g_*(\mathcal{O}_{X'}))$$

for any integer n. Since $g_*(\mathcal{O}_{X'})^\Gamma = \mathcal{O}_X$, it follows that

$$H^0(X', M^{\otimes n})^\Gamma = H^0(X, L^{\otimes n}).$$

Thus, the section ring

$$R(X,L) := \bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})$$

satisfies $R(X,L) = R(X',M)^\Gamma$. As a consequence, the fraction fields of the domains $R(X,L)$ and $R(X',M)$ have the same transcendence degree over k. Since M is big, it follows that L is big as well. □

Step 5. Let $d := \dim(X) - \dim(Y)$; this is the maximal dimension of the G-orbits in X. Let X_0 denote the set of $x \in X$ such that the orbit $G \cdot x$ has dimension d; then X_0 is an open G-stable subset of X. Since $\mathrm{Aut}_Y(X)$ normalizes G, it stabilizes X_0 as well.

We first consider the case where $\mathrm{char}(k) = 0$. Denote by $\mathfrak{g}$ the Lie algebra of G; then $\mathrm{Aut}_Y(X)$ acts on $\mathfrak{g}$ via its conjugation action on G. Consider the Tits morphism (see [Hab74, HO84])

$$\tau_0 : X_0 \longrightarrow \mathrm{Grass}^d(\mathfrak{g}), \quad x \longmapsto \mathfrak{g}_x,$$

where $\mathfrak{g}_x \subset \mathfrak{g}$ denotes the Lie algebra of the stabilizer of x, and $\mathrm{Grass}^d(\mathfrak{g})$ stands for the Grassmannian of subspaces of $\mathfrak{g}$ of codimension d. Then τ_0 is equivariant for the natural actions of $\mathrm{Aut}_Y(X)$. We view τ_0 as a rational map $X \dashrightarrow \mathrm{Grass}^d(\mathfrak{g})$, and denote by X' the normalization of its graph. Then X' is equipped with morphisms

$$\varphi : X' \longrightarrow X, \quad \tau' : X' \longrightarrow \mathrm{Grass}^d(\mathfrak{g})$$

such that φ restricts to an isomorphism above X_0, and $\tau' = \tau_0 \circ \varphi$. Moreover, the action of $\mathrm{Aut}_Y(X)$ on X lifts to an action on X' such that φ and τ' are equivariant. Arguing as at the end of Step 2, one may check that $G \simeq \mathrm{Aut}_Y(X')^0$ and the image of $\mathrm{Aut}_Y(X)$ in $\mathrm{Aut}_Y(X')$ is closed. Since $f \circ \varphi$ is a contraction, this yields a reduction to the case where τ_0 extends to a morphism

$$\tau : X \longrightarrow \mathrm{Grass}^d(\mathfrak{g}). \qquad (2.5)$$

Next, we consider the case where $\mathrm{char}(k) = p > 0$. For any integer $n > 0$, we then have the nth iterated Frobenius homomorphism

$$F^n_{G/k} : G \longrightarrow G^{(p^n)}$$

and its kernel G_n. Recall that G_n is an infinitesimal group scheme, characteristic in G; moreover, its formation commutes with base change, see e.g. [SGA3, VIIA.4.1]. (Note that G_1 is the infinitesimal group scheme of height 1 corresponding to the Lie algebra of G). For the G_n-action on X, the stabilizer Stab_{G_n} is a subgroup scheme of $G_{n,X} = G_n \times X$, stable by the diagonal action of $\mathrm{Aut}_Y(X)$. Since X is integral, it has a largest open subset $U = U_n$ over which Stab_{G_n} is flat, or equivalently locally free. Then U is stable by $\mathrm{Aut}_Y(X)$; moreover, since $\mathcal{O}_{\mathrm{Stab}_{G_n}}$ is a quotient of $\mathcal{O}(G_n) \otimes \mathcal{O}_X$, we obtain a morphism

$$\tau_n : U \to \mathrm{Grass}^m(\mathcal{O}(G_n)), \quad x \longmapsto \mathrm{Stab}_{G_n}(x)$$

for some $m \geq 1$. By construction, τ_n is $\mathrm{Aut}_Y(X)$-equivariant. As above, we may reduce to the case where τ_n extends to a morphism

$$\tau : X \longrightarrow \mathrm{Grass}^m(\mathcal{O}(G_n)). \tag{2.6}$$

Step 6. We first record a general observation that will be used repeatedly. Given a morphism of projective varieties $f : X \to Y$, we say that a line bundle on X is *f-big* if its pull-back to the generic fiber X_η is big (this notion turns out to be equivalent to that of [KM98, Def. 3.22]).

Lemma 8. *Let $f : X \to Y$ be a morphism of projective varieties, L a line bundle on X, and M a line bundle on Y. Assume that L is f-big and M is ample. Then $L \otimes f^*(M^{\otimes n})$ is big for $n \gg 0$.*

Proof. We adapt the argument of [CCP08, Lem. 2.4]. Denote by η the generic point of Y. Let A be an ample line bundle on X, and m a positive integer. Then $f_*(L^{\otimes m} \otimes A^{-1})$ is a coherent sheaf on Y, and $f_*(L^{\otimes m} \otimes A^{-1})_\eta = H^0(X_\eta, L^{\otimes m} \otimes A^{-1})$ (see e.g. [Har77, III.9.4]). Since L is big on X_η, it follows that $f_*(L^{\otimes m} \otimes A^{-1})_\eta \neq 0$ for $m \gg 0$ by arguing as in the proof of [KM98, Lem. 2.60]. As M is ample on Y, we thus have $H^0(Y, f_*(L^{\otimes m} \otimes A^{-1}) \otimes M^{\otimes n}) \neq 0$ for $n \gg m \gg 0$. Equivalently, $H^0(X, L^{\otimes m} \otimes A^{-1} \otimes f^*(M^{\otimes n})) \neq 0$. So $L^{\otimes m} \otimes f^*(M^{\otimes n}) = A \otimes E$ for some effective line bundle E on X. Thus, taking n to be a large multiple of m yields the statement in view of [KM98, Lem. 2.60] again. □

Next, we return to the setting of Step 5, and consider again the open $\mathrm{Aut}_Y(X)$-stable subset $X_0 \subset X$ consisting of G-orbits of maximal dimension d. Replacing X with the normalized blow-up of $X \setminus X_0$, we may assume that there is an effective Cartier divisor Δ on X such that $X \setminus \mathrm{Supp}(\Delta)$ consists of G-orbits of dimension d, and $\mathcal{O}_X(\Delta)$ is $\mathrm{Aut}_Y(X)$-linearized; in particular, $\mathrm{Supp}(\Delta)$ is $\mathrm{Aut}_Y(X)$-stable.

Let τ be the morphism as in (2.5) (if $\mathrm{char}(k) = 0$) or in (2.6) (if $\mathrm{char}(k) = p$). Denote by M the Plücker line bundle on the Grassmannian and let $L := \tau^*(M)$. Then L is a line bundle on X, equipped with an $\mathrm{Aut}_Y(X)$-linearization. So $L^{\otimes m}(\Delta)$ is $\mathrm{Aut}_Y(X)$-linearized as well for any integer m.

Lemma 9. *With the above notation, the line bundle $L^{\otimes m}(\Delta)_\eta$ on X_η is big for $m \gg 0$ (and for $n \gg 0$ if* $\mathrm{char}(k) > 0$*).*

Proof. Choose $x_0 \in X(k(\eta))$ such that $G_{\bar\eta} \cdot x_0$ is open in $X_{\bar\eta}$ (Step 4); then $G_{\bar\eta} \cdot x_0 = (X_0)_{\bar\eta}$. In particular, X_η is a normal projective variety, almost homogeneous under G_η. This is the setting of Theorem 1 in [Bri19] (except that $k(\eta)$ need not be algebraically closed), and the desired assertion may be obtained by arguing as in the last paragraph of the proof of that theorem. We provide an alternative argument based on Lemma 8.

Denote by H the stabilizer of x_0 in G_η and by $\mathfrak{h}$ its Lie algebra. If $\mathrm{char}(k) = 0$, then we have the equality of normalizers $N_{G_\eta}(H^0) = N_{G_\eta}(\mathfrak{h})$. If $\mathrm{char}(k) > 0$, then $N_{G_\eta}(H^0) = N_{G_\eta}(H_n)$ for $n \gg 0$, in view of [Bri19, Lem. 3.1].

Let $Z := \tau(X)$ and $z_0 := \tau(x_0)$, so that $z_0 = \mathfrak{h}$ if $\mathrm{char}(k) = 0$ and $z_0 = H_n$ if $\mathrm{char}(k) > 0$. Then Z_η is almost homogeneous under G_η and its open orbit is $G_\eta \cdot z_0 \simeq G_\eta / N_{G_\eta}(H^0)$. Moreover, τ restricts to an affine morphism $f : G_\eta/H \to G_\eta/N_{G_\eta}(H^0)$ by [Bri19, Lem. 3.1] again. Thus, $X_\eta \setminus \mathrm{Supp}(\Delta_\eta)$ is affine over $G_\eta \cdot z_0$. Denoting by ζ the generic point of Z (or equivalently of Z_η), it follows that $X_\zeta \setminus \mathrm{Supp}(\Delta_\zeta)$ is affine. As a consequence, Δ is τ-big (see e.g. [Bri19, Lem. 2.4]). Since M is ample on Z, we conclude by Lemma 8. □

By Lemmas 8 and 9, X admits a big line bundle which is $\mathrm{Aut}_Y(X)$-linearized. As seen in Step 3, this completes the proof of Theorem 1.

Remark 10. Assume that $\mathrm{char}(k) = 0$ and consider $f : X \to Y$ as in Step 2. Choose a desingularization $\psi : Y' \to Y$ and denote by X' the normalization of the irreducible component of $X \times_Y Y'$ which dominates Y. By arguing as at the end of the proof of Step 2, one shows that $\mathrm{Aut}_Y(X)$ is isomorphic to a closed subgroup of $\mathrm{Aut}_{Y'}(X')$. So we may assume that Y is smooth.

We may further assume that X is smooth, in view of the existence of a canonical desingularization (see [Kol07, Thm. 3.36]; such a desingularization is $\mathrm{Aut}(X)$-equivariant by the argument of [Kol07, Prop. 3.9.1]). Then the generic fiber X_η is smooth (by generic smoothness, see e.g. [Har77, III.10.7]); also, $X_{\bar\eta}$ is almost homogeneous under $G_{\bar\eta}$ in view of Lemma 6. By [FZ13, Thm. 1.2], it follows that the anticanonical class $-K_{X_{\bar\eta}}$ is big, hence so is $-K_{X_\eta}$. Also, $\mathcal{O}_X(-K_X)$ is $\mathrm{Aut}(X)$-linearized. Combining this with Lemmas 9 and 8, this yields a shorter proof of Theorem 1, in characteristic zero again.

3. Proofs of Theorem 2 and Corollary 3

Like that of Theorem 1, the proof of Theorem 2 goes through a succession of reduction steps.

Step 1. Let Y be the smooth projective curve such that $K = k(Y)$. Then we may assume that the inclusion $k(Y) \subset k(X)$ comes from a contraction $f : X \to Y$, and $G = \mathrm{Aut}_Y(X)^0$.

Proof. The invariant field $k(X)^{G^0}$ is algebraic over K, and therefore has transcendence degree 1 over k. Thus, we may replace G with G^0, and hence assume that G is connected.

Denote by $f : X \dashrightarrow Y$ the rational map associated with the inclusion $k(Y) \subset k(X)$. Arguing as in Step 2 of Section 2, we may further assume that f is a morphism; then it is a contraction.

By Lemma 5, the group $\mathrm{Aut}_Y(X)^0 = \mathrm{Aut}_K(X)^0$ is linear; thus, we may also assume that $G = \mathrm{Aut}_Y(X)^0$. □

Step 2. We may further assume that X is not almost homogeneous under $\widetilde{G} := \mathrm{Aut}(X)^0$.

Proof. By Blanchard's lemma again (see e.g. [BSU13, Prop. 4.2.1]), the action of $\widetilde{G}$ on X induces a unique action on Y such that f is equivariant. This yields an exact sequence of algebraic groups

$$1 \longrightarrow \widetilde{G} \cap \mathrm{Aut}_Y(X) \longrightarrow \widetilde{G} \longrightarrow \mathrm{Aut}(Y)^0,$$

where $\widetilde{G} \cap \mathrm{Aut}_Y(X)$ has reduced neutral component G, and hence is affine.

If X is almost homogeneous under $\widetilde{G}$, then so is Y. Thus, either $Y \simeq \mathbb{P}^1$ or Y is an elliptic curve. In the former case, we claim that every rational map from X to an abelian variety is constant. Indeed, any such map $\varphi : X \dashrightarrow A$ is G-invariant. Since $k(X)^G = k(\mathbb{P}^1)$, it follows that φ factors through a rational map $\mathbb{P}^1 \dashrightarrow A$, implying the claim.

By this claim, the open orbit of $\widetilde{G}$ in X has a trivial Albanese variety. Arguing as in the proof of Lemma 5, it follows that $\widetilde{G}$ is linear. Thus, so is $\mathrm{Aut}(X)$ in view of [Bri19, Thm. 1].

On the other hand, if Y is an elliptic curve, then f is the Albanese morphism of X, since the latter morphism is G-invariant. Thus, there is a unique action of $\mathrm{Aut}(X)$ on Y such that f is equivariant. This yields an exact sequence of group schemes, locally of finite type

$$1 \longrightarrow N \longrightarrow \mathrm{Aut}(X) \xrightarrow{f_*} \mathrm{Aut}(Y), \tag{3.1}$$

where the reduced subscheme of N is $\mathrm{Aut}_Y(X)$. In view of Theorem 1, it follows that N is an affine algebraic group. Also, the image of f_* contains

$Y = \mathrm{Aut}(Y)^0$, since X is almost homogeneous under $\widetilde{G}$. As $\mathrm{Aut}(Y)$ is a non-linear algebraic group, we conclude that so is $\mathrm{Aut}(X)$. □

Step 3. We may further assume that $G = \widetilde{G} = \mathrm{Aut}(X)^0$, and $\mathrm{Aut}(X)$ acts on Y so that we still have the exact sequence (3.1); in addition, $Y \simeq \mathbb{P}^1$ or Y is an elliptic curve.

Proof. By Rosenlicht's theorem again, the subfield $k(X)^{\widetilde{G}} \subset K$ has transcendence degree 1 over K. As $k(X)^{\widetilde{G}}$ is algebraically closed in $k(X)$ (Lemma 4), it follows that $k(X)^{\widetilde{G}} = K$. So $\widetilde{G}$ is linear in view of Lemma 5.

Therefore, we may assume that $G = \mathrm{Aut}(X)^0$. Then $\mathrm{Aut}(X)$ stabilizes K, and hence acts on Y; this yields (3.1).

Denote by g the genus of Y. If $g \geq 2$ then $\mathrm{Aut}(Y)$ is finite, and hence $\mathrm{Aut}(X)$ is a linear algebraic group. So we may further assume that $g \leq 1$. □

Step 4. We may further assume that X has an $\mathrm{Aut}(X)$-linearized line bundle L which is f-big and f-globally generated, i.e., the adjunction map

$$u : f^* f_*(L) \longrightarrow L$$

is surjective.

Proof. Arguing as in Steps 4, 5 and 6 of Section 2, we may assume that X has an f-big line bundle L which is $\mathrm{Aut}(X)$-linearized; moreover, $H^0(X, L) \neq 0$. Then $f_*(L)$ is a coherent, torsion-free sheaf on Y, and hence is locally free. Thus, so is $f^* f_*(L)$; moreover, u is a morphism of $\mathrm{Aut}(X)$-linearized sheaves. Also, $u \neq 0$ as L has non-zero global sections. Therefore, u yields a non-zero morphism $L^{-1} \otimes f^* f_*(L) \to \mathcal{O}_X$. Its image is the ideal sheaf of a closed subscheme $Z \subsetneq X$, stable by $\mathrm{Aut}(X)$.

Denote by

$$\pi : X' \longrightarrow X$$

the normalization of the blow-up of Z in X and let

$$f' := f \circ \pi : X' \longrightarrow Y.$$

Then the action of $\mathrm{Aut}(X)$ on X lifts to a unique action on X'; moreover, $\pi^*(L)$ is $\mathrm{Aut}(X)$-linearized and we have $f'^* f'_* \pi^*(L) = \pi^* f^* f_*(L)$. The image of the adjunction map

$$u' : f'^* f'_* \pi^*(L) \longrightarrow \pi^*(L)$$

generates an invertible subsheaf $L' \subset \pi^*(L)$, as proved (in essence) in [Har77, Ex. II.7.17.3]. Note that L' is $\mathrm{Aut}(X)$-linearized and f'-globally generated.

We claim that L' is f'-big, possibly after replacing L with a positive tensor power $L^{\otimes m}$, and L' with the associated subsheaf $L'_m \subset \pi^*(L^{\otimes m})$. Indeed, denoting by η the generic point of Y, we have

$$f_*(L^{\otimes m})_\eta = H^0(X_\eta, L^{\otimes m})$$

by [Har77, III.9.4], and likewise,

$$f'_*\pi^*(L^{\otimes m})_\eta = H^0(X'_\eta, \pi^*(L^{\otimes m})) = H^0(X_\eta, L^{\otimes m}).$$

Thus, $\pi^*(L)$ is f'-big. Moreover, denoting by $i : X_\eta \to X$ and $i' : X'_\eta \to X'$ the inclusions, we obtain

$$i'^* f'^* f'_* \pi^*(L^{\otimes m}) = f'_*\pi^*(L^{\otimes m})_\eta \otimes \mathcal{O}_{X'_\eta} = H^0(X'_\eta, \pi^*(L^{\otimes m})) \otimes \mathcal{O}_{X'_\eta}.$$

Since L'_m is the subsheaf of $\pi^*(L^{\otimes m})$ generated by the image of $f'^* f'_* \pi^*(L^{\otimes m})$, we see that $i'^*(L'_m)$ is generated by the image of the natural map

$$H^0(X_\eta, L^{\otimes m}) \otimes \mathcal{O}_{X'_\eta} \longrightarrow i'^*\pi^*(L^{\otimes m}).$$

As $i'^*\pi^*(L^{\otimes m})$ is big, this proves the claim.

This claim combined with [Bri19, Lem. 2.1] yields the desired reduction. □

Step 5. Choose an ample line bundle M on Y. Then $L \otimes f^*(M^{\otimes n})$ is big for $n \gg 0$ (Lemma 8). We claim that $L \otimes f^*(M^{\otimes n})$ is also globally generated for $n \gg 0$.

Indeed, since M is ample, the sheaf $f_*(L) \otimes M^{\otimes n}$ is globally generated for $n \gg 0$. Equivalently, the evaluation map

$$H^0(Y, f_*(L) \otimes M^{\otimes n}) \otimes \mathcal{O}_Y \to f_*(L) \otimes M^{\otimes n}$$

is surjective. As f is flat, the induced map

$$H^0(Y, f_*(L) \otimes M^{\otimes n}) \otimes \mathcal{O}_X \to f^*(f_*(L) \otimes M^{\otimes n})$$

is surjective as well. Since $H^0(Y, f_*(L) \otimes M^{\otimes n}) = H^0(X, f^*(f_*(L) \otimes M^{\otimes n}))$, it follows that the sheaf $f^*(f_*(L) \otimes M^{\otimes n})$ is globally generated. Thus, so is its quotient $L \otimes f^*(M^{\otimes n})$, proving the claim.

We may now complete the proof of Theorem 2. Observe that $\mathrm{Aut}(X)$ fixes the numerical equivalence class of $f^*(M^{\otimes n})$ (since $\mathrm{Aut}(Y)$ fixes that of $M^{\otimes n}$) and hence that of $L \otimes f^*(M^{\otimes n})$. Thus, $\mathrm{Aut}(X)$ is an algebraic group in view of [Bri19, Lem. 2.2].

If $Y \simeq \mathbb{P}^1$, then $\mathrm{Aut}(X)$ fixes the isomorphism class of $f^*\mathcal{O}_{\mathbb{P}^1}(1)$, since $\mathrm{Aut}(Y) = \mathrm{PGL}_2$ fixes that of $\mathcal{O}_{\mathbb{P}^1}(1)$. So $\mathrm{Aut}(X)$ is linear algebraic by [Bri19, Lem. 2.3]. Otherwise, Y is an elliptic curve with function field K.

Proof of Corollary 3. By assumption, there exists a non-trivial connected linear algebraic subgroup $G \subset \mathrm{Aut}(X)$. Replacing G with a Borel subgroup, we may assume that it is solvable. Then $k(X)$ is a purely transcendental field extension of $K := k(X)^G$ in view of [Pop16, Thm. 1]. Also, recall that K

is algebraically closed in $k(X)$ (Lemma 4). Since $K \neq k(X)$, it follows that either $K = k$, or K has transcendence degree 1 over k. In the former case, $\mathrm{Aut}(X)$ is linear algebraic by [Bri19, Thm. 1]. In the latter case, Theorem 2 yields that $\mathrm{Aut}(X)$ is algebraic, and linear unless K is the function field of an elliptic curve Y; then $k(X) \simeq K(t) \simeq k(Y \times \mathbb{P}^1)$.

Acknowledgments. This work was prompted by a minicourse on automorphism groups of algebraic varieties that I gave at the summer school "Visions of Algebraic Groups", held at St Petersburg, 26–30 August 2019. I thank the organizers of this event for their invitation, and the participants, especially Ivan Arzhantsev, for stimulating discussions. Also, thanks to the anonymous referee for valuable comments and suggestions.

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