Introduction to spherical varieties: Lecture 3

Michel BRION

Institut Fourier, Université Grenoble Alpes

16 December 2021

- Lecture 1: overview; basic notions and results on actions and representations of algebraic groups; toric varieties.
- Lecture 2: further background on the structure and representations of linear algebraic groups; projective homogeneous varieties; spherical varieties: definition, finiteness properties, local structure.
- Lecture 3: embeddings of spherical homogeneous spaces; wonderful varieties; line bundles on spherical varieties; open questions.

Recap

Let G be a connected reductive algebraic group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus. We identify $X^*(B)$ with $X^*(T) = \Lambda$.

Let G/H be a spherical homogeneous space with open B-orbit $B/B \cap H$. We have an exact sequence of abelian groups

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathbb{C}(G/H)^{(B)} \longrightarrow \Lambda(G/H) \longrightarrow 1,$$

where the *weight group* $\Lambda(G/H) \subset \Lambda$ is a free abelian group of rank $r(G/H)$, the *rank* of G/H.

A (normal, equivariant) *embedding* of G/H is a normal G-variety X equipped with a point x such that the orbit $G \cdot x$ is open in X and the stabilizer G_x equals H. We identify G/H with the open G-orbit X_G^0 in X, and $\mathbb{C}(X)$ with $\mathbb{C}(G/H)$.

Also, recall that X is the union of the G-stable open subsets

$$X_{Y,G} = \{\xi \in X \mid Y \subset \overline{G \cdot \xi}\},$$

where Y runs over the finitely many G-orbit closures in X. Moreover, each pair $(X_{Y,G}, x)$ is a *simple* embedding of G/H with closed G-orbit Y_G^0. Every simple embedding is a quasi-projective variety.

Simple embeddings

Let (X, x) be a simple embedding with closed orbit Y, and

$$X_{Y,B} = \{\xi \in X \mid Y \subset \overline{B \cdot \xi}\}.$$

Recall that $X_{Y,B}$ is an open affine B-stable subset of X, with complement the union of the prime B-stable divisors which do not contain Y.

There are only finitely many prime B-stable divisors of X. Each of them is either G-stable (then its closure contains Y), or is the closure of a prime B-stable divisor in G/H (then its closure may or may not contain Y).

We denote by $\mathcal{V}_X$ the set of prime G-stable divisors in X, and by $\mathcal{D}$ the set of prime B-stable divisors in G/H. Those whose closure contains Y form the subset $\mathcal{D}_X \subset \mathcal{D}$.

For any prime divisor D of X, we denote by $\mathcal{O}_{X,D} \subset \mathbb{C}(X)$ the corresponding local ring, consisting of rational functions defined along D.

Proposition

(i) $\mathcal{O}(X_{Y,B}) = \mathcal{O}(B/B \cap H) \cap \bigcap_{D \in \mathcal{V}_X \cup \mathcal{D}_X} \mathcal{O}_{X,D}$.

(ii) *The embedding (X, x) is uniquely determined by the pair $(\mathcal{V}_X, \mathcal{D}_X)$.*

Valuations

Let X be a normal irreducible variety. A (discrete) *valuation* of the function field $\mathbb{C}(X)$ is a non-zero map $v : \mathbb{C}(X)^* \to \mathbb{Z}$ such that

(i) $v(f+g) \geq \min(v(f), v(g))$ for all f, g such that $f + g \neq 0$.
(ii) $v(fg) = v(f) + v(g)$ for all f, g.
(iii) $v(t) = 0$ for all $t \in \mathbb{C}^*$.

We extend any valuation v to $\mathbb{C}(X)$ by setting $v(0) = \infty$.
Then v defines the *valuation ring* $\mathcal{O}_v = \{f \in \mathbb{C}(X) \mid v(f) \geq 0\}$,
a local ring with maximal ideal $\mathfrak{m}_v = \{f \in \mathbb{C}(X) \mid v(f) > 0\}$.
Conversely, $\mathcal{O}_v$ determines v uniquely if v is *normalized*, i.e., $v(\mathbb{C}(X)^*) = \mathbb{Z}$.

Example

Let D be a prime divisor of X. Then D defines a normalized valuation v_D of $\mathbb{C}(X)$, such that $v_D(f)$ is the order of zero or pole of f along D.
Moreover, $\mathcal{O}_{v_D} = \mathcal{O}_{X,D}$. and hence D is uniquely determined by v_D.
More generally, given a birational map of normal varieties $X' \dashrightarrow X$ and a prime divisor D on X', we obtain a valuation v_D on $\mathbb{C}(X) \simeq \mathbb{C}(X')$.

Centers, invariant valuations

We still consider a normal irreducible variety X, and a valuation v of $\mathbb{C}(X)$.

Definition

A closed irreducible subvariety Y of X is the *center* of v if $\mathcal{O}_{X,Y} \subset \mathcal{O}_v$ and $\mathfrak{m}_{X,Y} \subset \mathfrak{m}_v$, where $\mathfrak{m}_{X,Y}$ denotes the maximal ideal of the local ring $\mathcal{O}_{X,Y}$.

For example, every prime divisor D of X is the center of v_D. More generally, every closed irreducible subvariety Y of X is the center of some valuation of $\mathbb{C}(X)$ (consider the normalized blow-up of Y in X, and the valuation associated with an irreducible component of the exceptional divisor).

By the valuative criterion for separatedness, the center of v is unique if it exists. Moreover, X is complete iff every valuation of $\mathbb{C}(X)$ has a center in X, by the valuative criterion for completeness.

Next, assume that X is a G-variety. Then G acts on the set of valuations of $\mathbb{C}(X)$ via $(g \cdot v)(f) = v(g^{-1} \cdot f)$, and hence we may speak of *G-invariant valuations*. A valuation v is G-invariant iff $\mathcal{O}_v$ is G-stable.

The center of every G-invariant valuation is G-stable, and every irreducible G-stable subvariety of X is obtained in this way. Moreover, X is complete iff every G-invariant valuation has a center.

Invariant valuations (continued)

Every valuation v of $\mathbb{C}(G/H)$ yields a homomorphism

$$\rho(v) : \Lambda(G/H) = \mathbb{C}(G/H)^{(B)}/\mathbb{C}^* \longrightarrow \mathbb{Z},$$

i.e., a point of the dual lattice N of the weight lattice $M = \Lambda(G/H)$.

Proposition

The map $v \mapsto \rho(v)$ identifies the set of G-invariant valuations with $N \cap \mathcal{V}$, where $\mathcal{V}$ is a closed convex cone in $N_{\mathbb{R}}$ with non-empty interior.

We say that $\mathcal{V}$ is the *valuation cone* of G/H.

For any simple embedding (X, x), we may view $\mathcal{V}_X$ as a finite subset of $N \cap \mathcal{V}$. We denote by $\mathcal{C}_X$ the (rational, polyhedral) convex cone of $N_{\mathbb{R}}$ generated by $\mathcal{V}_X$ and $\rho(\mathcal{D}_X)$.

Proposition

(i) *The rays of $\mathcal{C}_X$ which do not meet $\rho(\mathcal{D}_X)$ are exactly the rays $\mathbb{R}_{\geq 0} v$, where $v \in \mathcal{V}_X$.*

(ii) *The embedding (X, x) is uniquely determined by the pair $(\mathcal{C}_X, \mathcal{D}_X)$.*

Classification of simple embeddings

Definition

A *colored cone* is a pair $(\mathcal{C}, \mathcal{F})$, where $\mathcal{C} \subset N_{\mathbb{R}}$ and $\mathcal{F} \subset \mathcal{D}$ satisfy the following conditions:

(i) $\mathcal{C}$ is a rational polyhedral convex cone generated by $\rho(\mathcal{F})$ and finitely many elements of the valuation cone $\mathcal{V}$.

(ii) The relative interior $\mathcal{C}^0$ meets $\mathcal{V}$.

The elements of $\mathcal{F}$ are called *colors*.

The colored cone $(\mathcal{C}, \mathcal{F})$ is *strictly convex* if it satisfies in addition

(iii) $\mathcal{C}$ is strictly convex and $0 \notin \rho(\mathcal{F})$.

With this definition, we may obtain a combinatorial classification of simple embeddings:

Theorem

The map $(X, x) \mapsto (\mathcal{C}_X, \mathcal{D}_X)$ yields a bijection between the isomorphism classes of simple embeddings and the strictly convex colored cones.

Example: torus embeddings

Take for G a torus T, and for H the trivial subgroup. Then $M = X^*(T)$ and $N = X_*(T)$. Also, $B = T$ and hence $\mathcal{D}$ is empty.

Every $\theta \in N$ defines an invariant valuation v_θ of $\mathbb{C}(T)$ as follows.
Let $f \in \mathcal{O}(T) \setminus \{0\}$, then we have $f = \sum_{i \in I} a_i m_i$ where I is a finite set, $a_i \in \mathbb{C}^*$ and $m_i \in M$. Let

$$v_\theta(f) = \min_{i \in I} \langle m_i, \theta \rangle.$$

One may check that v_θ satisfies the properties of a valuation, and hence extends uniquely to a valuation of $\mathbb{C}(T)$, also denoted by v_θ. Since $t \cdot f = \sum_{i \in I} a_i m_i(t) m_i$ for any $t \in T$, we see that v_θ is invariant. Also, $v_\theta(m) = \langle m, \theta \rangle$ for any $m \in M$, and hence $\rho(v_\theta) = \theta$.

Thus, every $n \in N$ is the restriction of a T-invariant valuation, and hence $\mathcal{V} = N_\mathbb{R}$. So the above theorem gives back the classification of simple toric varieties in terms of strictly convex rational polyhedral convex cones in $N_\mathbb{R}$ ("simple" is equivalent to "affine" in this setting).

G-orbit structure of simple embeddings

Definition

A *face* of a colored cone $(\mathcal{C}, \mathcal{F})$ is a pair $(\mathcal{C}', \mathcal{F}')$ satisfying the following conditions:

(i) $\mathcal{C}'$ is a face of the convex cone $\mathcal{C}$.

(ii) $\mathcal{F}' = \mathcal{F} \cap \rho^{-1}(\mathcal{C}')$.

(iii) $\mathcal{C}'^0$ meets $\mathcal{V}$.

Let (X, x) be a simple embedding of G/H with closed G-orbit Y and colored cone $(\mathcal{C}_Y, \mathcal{D}_Y)$.

Proposition

The map $Z \mapsto (\mathcal{C}_Z, \mathcal{D}_Z)$ defines a decreasing bijection between the G-orbit closures in X (ordered by inclusion) and the faces of the colored cone $(\mathcal{C}_Y, \mathcal{D}_Y)$.

Under this bijection, X corresponds to the face $(0, \emptyset)$, and Y to $(\mathcal{C}_Y, \mathcal{D}_Y)$.

Classification of embeddings

Definition

A *colored fan* $\mathbb{F}$ is a finite non-empty set of colored cones which satisfies the following conditions:

(i) Every face of a colored cone in $\mathbb{F}$ is in $\mathbb{F}$.

(ii) For any $v \in \mathcal{V}$, there exists at most one $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}$ such that $v \in \mathcal{C}^0$.

The colored fan $\mathbb{F}$ is *strictly convex* if so are all its colored cones. Equivalently, $(0, \emptyset) \in \mathbb{F}$.

Next, let (X, x) be an embedding. By the previous proposition and the valuative criterion for separatedness, the colored cones $(\mathcal{C}_Y, \mathcal{D}_Y)$, where Y runs over the G-orbits in X, form a colored fan $\mathbb{F}(X)$.

Theorem

The map $(X, x) \mapsto \mathbb{F}(X)$ yields a bijection between isomorphism classes of embeddings and strictly convex colored fans.

The toroidal embeddings correspond to the *colorless* fans, i.e., those consisting of pairs $(\mathcal{C}, \emptyset)$.

Complete embeddings

Definition

Let $\mathbb{F}$ be a colored fan. The *support* of $\mathbb{F}$ is
$\mathrm{Supp}(\mathbb{F}) = \{v \in \mathcal{V} \mid v \in \mathcal{C} \text{ for some } (\mathcal{C}, \mathcal{F}) \in \mathbb{F}\}$.

Thus, the support of $\mathbb{F}$ is the disjoint union of the $\mathcal{C}^0 \cap \mathcal{V}$, where $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}$.

The valuative criterion for completeness yields readily:

Proposition

Let (X, x) be an embedding. Then X is complete iff $\mathrm{Supp}\,\mathbb{F}(X) = \mathcal{V}$.

Corollary

$\mathcal{V}$ is a rational polyhedral convex cone in $N_{\mathbb{R}}$ with non-empty interior.

Proof sketch. We already saw that $\mathcal{V}$ is a closed convex cone with non-empty interior. Also, G/H admits a complete embedding (X, x): indeed, there exists a G-equivariant immersion $G/H \to \mathbb{P}(V)$ where V is a finite-dimensional G-module, and we may take for X the normalization of the closure of the image. Thus, $\mathcal{V}$ is a finite union of rational polyhedral convex cones, and hence is rational polyhedral as well.

Morphisms of embeddings

Let H' be a closed subgroup of G containing H. Then we have a G-equivariant morphism $\varphi : G/H \to G/H'$. Thus, G/H' is spherical with open orbit $B/B \cap H'$. We denote by N', $\mathcal{V}'$, $\mathcal{D}'$ the associated data.

Also, φ induces an injective homomorphism $\varphi^* : \Lambda(G/H') \to \Lambda(G/H)$ and hence a surjective linear map $\varphi_* : N_{\mathbb{R}} \to N'_{\mathbb{R}}$. One may show that $\varphi_*(\mathcal{V}) = \mathcal{V}'$.

Let $\mathcal{D}_\varphi = \{D \in \mathcal{D} \mid \varphi(D) = G/H'\}$. Then $\varphi(D) \in \mathcal{D}'$ for any $D \in \mathcal{D} \setminus \mathcal{D}_\varphi$. This defines $\varphi_* : \mathcal{D} \setminus \mathcal{D}_\varphi \to \mathcal{D}'$.

Let $\mathbb{F}$ (resp. $\mathbb{F}'$) be a colored fan for G/H (resp. G/H'). We say that $\mathbb{F}$ *dominates* $\mathbb{F}'$ if for any $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}$, there exists $(\mathcal{C}', \mathcal{F}') \in \mathbb{F}'$ such that $\varphi_*(\mathcal{C}) \subset \mathcal{C}'$ and $\varphi_*(\mathcal{F} \setminus \mathcal{D}_\varphi) \subset \mathcal{F}'$.

Theorem

Let (X, x) (resp. (X', x')) be an embedding of G/H (resp. G/H'). Then φ extends to a morphism $\psi : (X, x) \to (X', x')$ iff $\mathbb{F}(X)$ dominates $\mathbb{F}(X')$. Moreover, ψ is proper iff $\operatorname{Supp} \mathbb{F}(X) = \varphi_*^{-1} \operatorname{Supp} \mathbb{F}(X')$.

This gives back the existence of the discoloration $\tilde{X}$ of a spherical variety X: take $\mathbb{F}(\tilde{X}) = \{(\mathcal{C} \cap \mathcal{V}, \emptyset) \mid (\mathcal{C}, \mathcal{F}) \in \mathbb{F}(X)\}$.

The normalizer of a spherical subgroup

Denote by $\mathrm{N}_G(H) = \{g \in G \mid gHg^{-1} = H\}$ the normalizer of H in G. This is a closed subgroup of G containing the center $Z(G)$.
The action of $N_G(H)$ on G via right multiplication yields an action on G/H, which commutes with the G-action and has kernel H. This identifies $\mathrm{N}_G(H)/H$ with the group of G-equivariant automorphisms of G/H.

Theorem

(i) *The group $\mathrm{N}_G(H)/H$ is diagonalizable, of the same dimension as $\mathrm{lin}(\mathcal{V}) = \mathcal{V} \cap (-\mathcal{V})$ (the largest linear subspace of $\mathcal{V}$).*

(ii) *The valuation cone of $G/\mathrm{N}_G(H)$ is the image of $\mathcal{V}$ under the quotient map $N_{\mathbb{R}} \to N_{\mathbb{R}}/\mathrm{lin}(\mathcal{V})$.*

(iii) *We have $\mathrm{N}_G(H) = \mathrm{N}_G(H^0)$ and $\mathrm{N}_G(\mathrm{N}_G(H)) = \mathrm{N}_G(H)$.*

Corollary

$\mathcal{V} = N_{\mathbb{R}}$ iff H is a horospherical subgroup of G.

Proof sketch: H is horospherical in G iff $\mathrm{N}_G(H)$ is a parabolic subgroup of G iff the valuation cone of $G/\mathrm{N}_G(H)$ is trivial.

Sober spherical homogeneous spaces

A further direct consequence of the above theorem is:

Corollary

The valuation cone $\mathcal{V}$ is strictly convex iff $\mathrm{N}_G(H)/H$ is finite.

We say that the spherical homogeneous space G/H is *sober* if it satisfies the above equivalent conditions. Then G/H has a *canonical embedding* $(\mathbf{X}, \mathbf{x})$: the simple projective embedding with colored cone $(\mathcal{V}, \emptyset)$.

By the classification of embeddings and their morphisms, an embedding is toroidal iff it admits a morphism to $(\mathbf{X}, \mathbf{x})$. Also, $(\mathbf{X}, \mathbf{x})$ admits a morphism to any simple projective embedding.

Theorem

If G/H is sober, then $\mathcal{V}$ is simplicial. If in addition $H = \mathrm{N}_G(H)$, then $\mathcal{V}$ is generated by a basis of N; equivalently, $\mathbf{X}$ is smooth.

For a sober homogeneous space G/H, we have $H \supset Z(G)^0$ and hence $G/H = (G/Z(G)^0)/(H/Z(G)^0)$ is homogeneous under the semi-simple group $G/Z(G)^0$. If $H = \mathrm{N}_G(H)$ then $H \supset Z(G)$ and hence G/H is homogeneous under the adjoint semi-simple group $G/Z(G)$.

Example: group embeddings

Let H be a connected reductive group, and $G = H \times H$. Recall that $G/\mathrm{diag}(H) \simeq H$ on which G acts by left and right multiplication.

As a Borel subgroup of G, we take $B \times B^-$, where B, B^- are opposite Borel subgroups of H with common torus T. Its open orbit in H is $BB^- \simeq U \times T \times U^-$, where U (resp. U^-) is the unipotent part of B (resp. B^-). Thus, the weight group is $\Lambda = X^*(T)$, and we have $N = X_*(T)$.

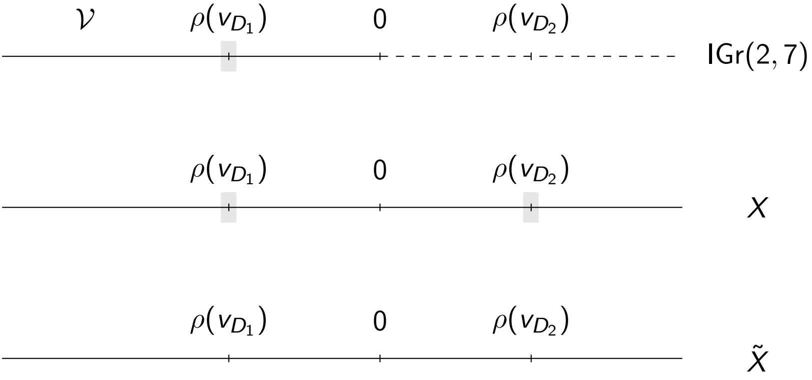
Proposition

The map $\rho : \mathcal{D} \to N$ is injective, and its image consists of the simple coroots. The valuation cone $\mathcal{V}$ is the negative Weyl chamber.

Also, we have $\mathrm{N}_G(H) = (Z(G) \times Z(G))H$ and hence $\mathrm{N}_G(H)/H \simeq Z(G)$. In particular, G/H is sober iff G is semi-simple, and $H = \mathrm{N}_G(H)$ iff G is adjoint semi-simple. In the latter case, Λ is freely generated by the simple roots, and these generate the dual cone of $\mathcal{V}$.

Part of this picture extends to any symmetric space G/H; e.g., G/H is sober if G is semi-simple, and $H = \mathrm{N}_G(H)$ if G is adjoint semi-simple.

Further examples



Here IGr(2, 7) is the isotropic Grassmannian of planes in $\mathbb{C}^7$, viewed as a spherical variety under the simple group of type G_2. Moreover, X is the horospherical degeneration of IGr(2, 7) constructed by Pasquier and Perrin, and $\tilde{X}$ is its discoloration.

Wonderful varieties

Definition

A G-variety X is *wonderful* if it satisfies the following conditions:

(i) X is smooth, projective, and contains an open orbit X^0_G.

(ii) The boundary $\partial X = X \setminus X^0_G$ is a union of prime divisors $D_1, \ldots, D_r$ intersecting transversally.

(iii) The G-orbit closures are exactly the partial intersections $D_{i_1} \cap \cdots \cap D_{i_s}$, where $1 \leq i_1 < \cdots < i_s \leq r$.

In particular, X contains a unique closed G-orbit $Y = D_1 \cap \cdots \cap D_r$. By a result of Luna, X is spherical of rank r. Together with the above theorem and the local structure of spherical varieties, this yields:

Corollary

Every wonderful G-variety is the canonical embedding of a sober spherical homogeneous space G/H.
As a partial converse, if G/H is spherical and $\mathrm{N}_G(H) = H$ then the canonical embedding of G/H is a wonderful G-variety.

The Demazure embedding

Assume that $\mathrm{N}_G(H) = H$. Then $H = \mathrm{N}_G(H^0) = \mathrm{N}_G(\mathfrak{h})$, where $\mathfrak{h} = \mathrm{Lie}(H) \subset \mathrm{Lie}(G) = \mathfrak{g}$. View $\mathfrak{h}$ as a point of the Grassmannian $\mathrm{Gr}(\mathfrak{g})$ of subspaces of $\mathfrak{g}$. Then G acts on $\mathrm{Gr}(\mathfrak{g})$ via its adjoint action on $\mathfrak{g}$, and the orbit $G \cdot \mathfrak{h}$ is isomorphic to G/H.

The following result is due to Losev, with earlier contributions by Demazure, De Concini and Procesi,...

Theorem

The pair $(\overline{G \cdot \mathfrak{h}}, \mathfrak{h})$ is the wonderful embedding of G/H.

This yields a canonical construction of the wonderful embedding $(\mathbf{X}, \mathbf{x})$.

Note that $\overline{G \cdot \mathfrak{h}}$ is contained in the variety of Lie subalgebras of $\mathfrak{g}$. If H is reductive, then $\overline{G \cdot \mathfrak{h}}$ is an irreducible component of this variety.

Proposition

The logarithmic tangent bundle $T_{\mathbf{X}}(-\log \partial \mathbf{X})$ is the pull-back of the tautological (quotient) vector bundle on $\mathrm{Gr}(\mathfrak{g})$.

The divisor class group of a spherical variety

We return to an embedding (X, x) of a spherical homogeneous space G/H. Denote by $\mathcal{V}_X \subset \mathcal{V}$ the subset of valuations associated with prime G-stable divisors in X. Then $\mathcal{V}_X$ consists of the primitive generators of the rays of $\mathbb{F}(X)$ which are not generated by colors. For any $v \in \mathcal{V}_X$, we denote by D_v the corresponding divisor.

Proposition

The divisor class group of X lies in an exact sequence

$$0 \longrightarrow \mathrm{X}^*(G) \cap \mathcal{V}_X^{\perp} \longrightarrow \Lambda(X) \xrightarrow{\mathrm{div}} \mathbb{Z}^{\mathcal{V}_X \cup \mathcal{D}} \longrightarrow \mathrm{Cl}(X) \longrightarrow 0,$$

where $\mathrm{div}(\lambda) = \sum_{v \in \mathcal{V}_X} \langle v, \lambda \rangle D_V + \sum_{D \in \mathcal{D}} \langle \rho(v_D), \lambda \rangle D$ *for any* $\lambda \in \Lambda(X)$.

Proof sketch: the open B-orbit $B \cdot x$ is isomorphic to $\mathbb{C}^m \times (\mathbb{C}^*)^n$, and hence its divisor class group is trivial. Thus, $\mathrm{Cl}(X)$ is generated by the classes of the irreducible components of $X \setminus B \cdot x$, i.e., by $\mathcal{V}_X \cup \mathcal{D}$.

The relations come from divisors of regular invertible functions on $B \cdot x$, and we have $\mathcal{O}(B \cdot x)^*/\mathbb{C}^* = \mathcal{O}(B/B \cap H)^*/\mathbb{C}^* = \Lambda(X)$.

The polytope associated with a line bundle

Let (X, x) be a complete embedding, and L a G-linearized line bundle on X having non-zero global sections. Choose $s_0 \in \Gamma(X, L)^{(B)}$ and denote its weight by λ_0. Recall that there exists a unique rational convex polytope $P = P(X, L) \subset \Lambda_{\mathbb{R}}$ such that $\Gamma(X, L^{\otimes k}) = \bigoplus_{\lambda \in kP \cap (k\lambda_0 + \Lambda(X))} V(\lambda)$ for any integer $k \geq 1$. Moreover, $P(X, L) \subset \lambda_0 + \Lambda(X)_{\mathbb{R}}$.

Also, we have

$$\operatorname{div}(s_0) = \sum_{v \in \mathcal{V}_X} n_v D_v + \sum_{D \in \mathcal{D}} n_D D$$

for unique families of integers (n_v), (n_D).

Proposition

The polytope $-\lambda_0 + P(X, L) \subset \Lambda(X)_{\mathbb{R}}$ is defined by the linear inequalities

$$\langle v, - \rangle + n_v \geq 0 \quad (v \in \mathcal{V}_X), \qquad \langle \rho(v_D), - \rangle + n_D \geq 0 \quad (D \in \mathcal{D}).$$

Proof sketch: the B-eigenvectors in $\Gamma(X, L)$ are exactly the products fs_0, where $f \in \mathbb{C}(X)^{(B)}$ and $\operatorname{div}(f) + \operatorname{div}(s_0)$ is effective.

The Picard group of a simple spherical variety

Theorem

Let (X, x) be a simple embedding with colored cone $(\mathcal{C}_X, \mathcal{D}_X)$, and L a (globally generated, resp. ample) line bundle on X.

(i) $L \simeq \mathcal{O}_X(\sum_{D \in \mathcal{D} \setminus \mathcal{D}_X} n_D D)$ *for some (non-negative, resp. positive) integers* n_D.

(ii) *We have an exact sequence*

$$0 \longrightarrow \mathrm{X}^*(G) \cap \mathcal{V}_X^\perp \longrightarrow \Lambda(X) \cap \mathcal{C}_X^\perp \xrightarrow{\mathrm{div}} \mathbb{Z}^{\mathcal{D} \setminus \mathcal{D}_X} \longrightarrow \mathrm{Pic}(X) \longrightarrow 0,$$

where $\mathrm{div}(\lambda) = \sum_{D \in \mathcal{D} \setminus \mathcal{D}_X} \rho(v_D)(\lambda) D$ *for any* $\lambda \in \Lambda(X)$.

Proof sketch: using the local structure theorem and a linearization argument as for toric varieties, one shows that $\mathrm{Pic}(X_{Y,B}) = 0$. Thus, every Cartier divisor on X is linearly equivalent to some $\sum_{D \in \mathcal{D} \setminus \mathcal{D}_X} n_D D$.
Also, for any $D \in \mathcal{D} \setminus \mathcal{D}_X$, the translates $g \cdot D$ ($g \in G$) have no common point. By a linearization argument again, it follows that D is Cartier and generated by its global sections.

The Picard group of a wonderful variety

Let X be a wonderful G-variety. Then X is simple, $\mathcal{D}_X = \emptyset$, and $\mathcal{C}_X = \mathcal{V}$ has non-empty interior in $N_{\mathbb{R}}$. So the above theorem yields:

Proposition

Let X be a wonderful G-variety. Then $\mathrm{Pic}(X)$ is freely generated by the classes of the divisors in $\mathcal{D}$, and these generate the nef cone. Moreover, every nef divisor on X is generated by its global sections.

Recall that the divisors in $\mathcal{D}$ are exactly the irreducible components of $X \setminus X_{Y,B}$, where Y denotes the closed G-orbit. Also, the local structure theorem yields an isomorphism $\mathrm{R}_u(P) \times Z \simeq X_{Y,B}$ where $P = \mathrm{R}_u(P) \rtimes L$, and $Z \simeq \mathbb{C}^r$ on which L acts linearly via r linearly independent weights. In particular, $X_{Y,B} \simeq \mathbb{C}^n$ is an "open cell". So the above proposition generalizes a well-known result for projective homogeneous varieties.

Corollary

Every wonderful variety X is log Fano, i.e., $-K_X - \partial X$ is ample.

Indeed, recall that $-K_X = \partial X + E$, where $E = \sum_{D \in \mathcal{D}} a_D D$ with $a_D > 0$ for all $D \in \mathcal{D}$.

(Almost) Fano wonderful varieties

A smooth projective variety X is called *almost Fano* if $-K_X$ is big and nef.

Proposition

Let X be a smooth projective spherical variety. Then $-K_X$ is big.

Proof: $-K_X = \sum_{v \in \mathcal{V}_X} D_v + \sum_{D \in \mathcal{D}} a_D D$ has a section with zero locus the complement of an affine open subset (the open B-orbit).

Let X be a *wonderful symmetric variety*, i.e., the wonderful embedding of the symmetric space G/G^σ, where σ is an involution of the adjoint semi-simple group G. The following result is due to Ruzzi:

Theorem

With the above assumptions, X is almost Fano. Moreover, X is Fano unless $r(X) = r(G)$.

Thus, the wonderful embedding of every adjoint semi-simple group G is Fano. But its boundary divisors are not necessarily almost Fano. This happens e.g. if G is simple of type D_n or E_n, and the divisor is associated with the triple node of the Dynkin diagram.

In particular, wonderful varieties are not necessarily almost Fano.

Some topics not covered in the lectures

- Picard group of spherical varieties (not necessarily simple).
- Nef cone, cone of effective divisors, Cox ring.
- Equivariant cohomology of spherical varieties.
- Combinatorial invariants of spherical homogeneous spaces.
- Relation to symplectic geometry, multiplicity-free spaces.
- Singularities of spherical varieties.
- ($\mathbb{Q}$)-Gorenstein Fano spherical varieties.

Some open questions

1) *Determine the fixed points of a maximal torus in a spherical variety*.

These fixed points yield much topological and geometric information. Their description is known for special classes of spherical varieties: toric varieties, projective homogeneous varieties, horospherical varieties, complete symmetric varieties.

2) *Determine the faces of the convex polytope associated with an ample line bundle on a projective spherical variety*.

In the toric case, these faces correspond bijectively to the orbit closures. For any arbitrary group G, there is still a bijective correspondence between G-orbit closures and certain faces, but not all of them.

3) *Find generators and relations for the rational cohomology ring of a smooth complete spherical variety*.

Such a presentation is known for toric varieties and projective homogeneous varieties. Also, there is a description of the rational G-equivariant cohomology ring of a smooth projective spherical G-variety, as a subring of the product of equivariant cohomology rings of orbits.

Some open questions (continued)

4) *Describe the families of minimal rational curves on a smooth projective spherical variety*.

The notion of minimal rational curve generalizes that of lines in a projective embedding, and yields important local invariants of uniruled varieties.

The families of minimal rational curves are known for projective homogeneous varieties and toric varieties. They are partially understood for complete symmetric varieties.

5) *Describe the moduli spaces of rational curves on a smooth projective spherical variety*

These spaces are known to be irreducible and unirational for projective homogeneous varieties, and for certain wonderful varieties and "log curves".

Also, these spaces are irreducible for wonderful group embeddings and curves meeting the open orbit. But they may be reducible for arbitrary complete symmetric varieties.

Some general references for Lecture 3

- C. De Concini, C. Procesi: *Complete symmetric varieties*, pp. 1–44 in: Invariant theory, Lect. Notes Math. **996**, Springer, 1983.
- J. Gandini: *Embeddings of spherical homogeneous spaces*, Acta Math. Sin., Engl. Ser. **34** (2018), 299–340.
- F. Knop: *The Luna-Vust theory of spherical embeddings*, pp. 225–249 in: Proceedings of the Hyderabad conference on algebraic groups, Manoj Prakashan (1991).
- F. Knop: *Automorphisms, root systems, and compactifications of homogeneous varieties*, J. Amer. Math. Soc. **9** (1996), 153–174.
- I. Losev: *Demazure embeddings are smooth*, Int. Math. Res. Not. **14** (2009), 2588–2596.
- D. Luna: *Toute variété magnifique est sphérique*, Transform. Groups **1** (1996), 249–258.
- G. Pezzini: *Lectures on wonderful varieties*, Acta Math. Sin., Engl. Ser. **34** (2018), 417–438.

Combinatorics and Cox ring of spherical varieties

- I. Arzhantsev, U. Derenthal, J. Hausen, A. Laface: *Cox rings*, Cambridge Stud. Advanced Math. **144**, Cambridge Univ. Press, 2015.
- P. Bravi, G. Pezzini: *Wonderful subgroups of reductive groups and spherical systems*, J. Algebra **409** (2014), 101–147.
- P. Bravi, G. Pezzini: *Primitive wonderful varieties*, Math. Z. **282** (2016), 1067–1096.
- M. Brion: *The total coordinate ring of a wonderful variety*, J. Algebra **313** (2007), 61–99.
- S. Cupit-Foutou, G. Pezzini, B. Van Steirteghem: *Momentum polytopes of projective spherical varieties and related Kähler geometry*, Sel. Math., New Ser. **26**, 27 (2020).
- G. Gagliardi: *The Cox ring of a spherical embedding*, J. Algebra **397** (2014), 548–569.
- G. Gagliardi: *A combinatorial smoothness criterion for spherical varieties*, Manuscr. Math. **146** (2015), 445–461.

Fano spherical varieties

- T. Delcroix: *Kähler-Einstein metrics on group compactifications*, Geom. Funct. Anal. **27** (2017), 78–129.
- T. Delcroix: *K-stability of Fano spherical varieties*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), 615–662.
- G. Gagliardi, J. Hofscheier: *Gorenstein spherical Fano varieties*, Geom. Dedicata **178** (2015), 111–133.
- G. Gagliardi, J. Hofscheier: *The generalized Mukai conjecture for symmetric varieties*, Trans. Am. Math. Soc. **369** (2017), 2615–2649.
- B. Pasquier: *Fano horospherical varieties*, Bull. Soc. Math. France **136** (2008), 195–225.
- A. Ruzzi: *Geometrical description of smooth projective symmetric varieties with Picard number one*, Transform. Groups **15** (2010), 201–226.
- A. Ruzzi: *Smooth projective symmetric varieties with Picard number one*, Int. J. Math. **22** (2011), 145–177.
- A. Ruzzi: *Fano symmetric varieties with low rank*, Publ. Res. Inst. Math. Sci. **48** (2012), 235–278.

Cohomology and Chow rings of spherical varieties

- E. Bifet, C. De Concini, C. Procesi: *Cohomology of regular embeddings*, Adv. Math. **82** (1990), 1–34.
- M. Brion: *Equivariant Chow groups for torus actions*, Transform. Groups **2** (1997), 225–267.
- M. Brion, R. Joshua: *Equivariant Chow ring and Chern classes of wonderful symmetric varieties of minimal rank*, Transform. Groups **13** (2008), 471–493.
- V. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** (1978), 97–154.
- J. Hofscheier, A. Khovanskii, L. Monin: *Cohomology rings of toric bundles and the ring of conditions*, arXiv:2006.12043.
- E. Strickland: *The Reisner-Stanley system and equivariant cohomology for a class of wonderful varieties*, J. Algebra **356** (2012), 216–229.

Rational curves on spherical varieties

- M. Brion, B. Fu: *Minimal rational curves on wonderful group compactifications*. J. Éc. Polytech., Math. **2** (2015), 153–170.
- Y. Chen, B. Fu, J.-M. Hwang: *Minimal rational curves on complete toric manifolds and applications*, Proc. Edinburgh Math. Soc. (2) **57** (2014), 111–123.
- Q. Chen, Y. Zhu: *On the irreducibility of the space of genus zero stable log maps to wonderful compactifications*, Int. Math. Res. Not. **10** (2016), 3029–3050.
- Q. Chen, Y. Zhu: $\mathbb{A}^1$*-curves on log smooth varieties*, J. Reine Angew. Math. **756** (2019), 1–35.
- J.-M. Hwang, N. Mok: *Deformation rigidity of the rational homogeneous space associated to a long simple root*, Ann. Sci. Éc. Norm. Supér. (4) **35** (2002), 173–184.
- N. Perrin: *Courbes rationnelles sur les variétés homogènes*, Ann. Inst. Fourier **52** (2002), 105–132.
- A. Shebzukhov: *Rational curves on wonderful compactifications of symmetric varieties*, arXiv:2109.04964.