Introduction to spherical varieties: Lecture 2

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- Lecture 1: overview; basic notions and results on actions and representations of algebraic groups; toric varieties.
- Lecture 2: further background on the structure and representations of linear algebraic groups; projective homogeneous varieties; spherical varieties: definition, finiteness properties, local structure.
- Lecture 3: embeddings of spherical homogeneous spaces; wonderful varieties; line bundles on spherical varieties; open questions.

Some subgroups of GL_n

For any integer $n \geq 1$, we denote by B_n the subgroup of GL_n consisting of upper triangular matrices, and by T_n the subgroup of diagonal matrices. Then $\mathrm{T}_n = \mathbb{G}_m^n$ is an n-dimensional torus. Moreover, $\mathrm{B}_n = \mathrm{U}_n \rtimes \mathrm{T}_n$, where U_n denotes the subgroup of B_n consisting of matrices with diagonal coefficients 1. We have $\mathrm{U}_n \simeq \mathbb{C}^{n(n-1)/2}$ as varieties. In particular, U_n is connected, and hence so is B_n. Also, $\mathrm{U}_n = [\mathrm{B}_n, \mathrm{B}_n]$ (the derived subgroup) and U_n is nilpotent. In particular, B_n is solvable.

The homogeneous space $\mathrm{GL}_n / \mathrm{B}_n$ is isomorphic to the *variety of complete flags in* $\mathbb{C}^n$, defined as follows. A complete flag in $\mathbb{C}^n$ is a sequence of linear subspaces $V_1 \subset \cdots \subset V_{n-1}$ such that $\dim(V_i) = i$ for $i = 1, \ldots, n-1$. The complete flags form a closed subvariety $\mathrm{Fl}_n \subset \mathrm{Gr}(1,n) \times \cdots \times \mathrm{Gr}(n-1,n)$, where $\mathrm{Gr}(i,n)$ denotes the Grassmannian of i-dimensional subspaces of $\mathbb{C}^n$ for $i = 1, \ldots, n-1$.

In particular, Fl_n is a projective variety equipped with an action of GL_n. One may easily check that this action is transitive, and B_n is the stabilizer of the standard flag $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_{n-1} \rangle$, where $(e_1, \ldots, e_n)$ denotes the standard basis of $\mathbb{C}^n$.

Connected solvable algebraic groups

From now on, we only consider *linear* algebraic groups.

Let G be a connected solvable algebraic group. We will see that several properties of B_n extend to G. This is based on Borel's fixed point theorem:

Theorem

Let X be a complete G-variety. Then G has a fixed point in X.

Proof sketch: let N be a closed normal subgroup of G. Then the fixed point subset X^N is a closed subvariety of X, stable by G and on which G acts via the quotient G/N. By induction on $\dim(G)$, we may thus assume G commutative. Let $x \in X$ such that the orbit $Y = G \cdot x$ is closed in X. Then $Y \simeq G/G_x$ is complete, affine and connected, hence a point.

Corollary

Let $\rho : G \to \mathrm{GL}_n$ be a finite-dimensional representation. Then $\rho(G)$ is conjugate to a subgroup of B_n.

Proof: apply the theorem to the G-action on Fl_n.
In particular, every simple G-module is isomorphic to $\mathbb{C}_\chi$ for a unique $\chi \in \mathrm{X}^*(G)$.

Structure of connected solvable groups

Definition

An algebraic group G is *unipotent* if it is isomorphic to a closed subgroup of U_n for some n.

Proposition

Let G be a unipotent algebraic group.

(i) *G is connected and every character of G is trivial.*

(ii) *Every non-zero G-module contains a non-zero fixed point.*

Proof sketch: one shows that G is an iterated extension of copies of $\mathbb{G}_a$.

Theorem

Let G be a connected solvable algebraic group.

(i) *G has a largest closed unipotent subgroup U. Moreover, $U \triangleleft G$.*

(ii) *$G = U \rtimes T$ for some maximal torus $T \subset G$.*

(iii) *Any two maximal tori of G are conjugate in U.*

Connected solvable groups (continued)

Theorem

Let G be an algebraic group, and X a G-homogeneous space.

(i) *If G is unipotent, then $X \simeq \mathbb{C}^n$ as a variety for some integer $n \geq 0$.*

(ii) *If G is connected and solvable, then $X \simeq \mathbb{C}^m \times (\mathbb{C}^*)^n$ as a variety for some integers $m, n \geq 0$.*

A *Borel subgroup* of an algebraic group G is a maximal closed connected solvable subgroup.

For example, the Borel subgroups of GL_n are exactly the conjugates of B_n.

Theorem

Let $B \subset G$ be a Borel subgroup. Then the Borel subgroups of G are exactly the conjugates of B, and G/B is a projective variety.

Corollary

Any two maximal tori of G are conjugate.

Proof: every torus is contained in a Borel subgroup.

Radical, unipotent radical

Let G be an algebraic group. One may easily show that G has a largest closed connected solvable normal subgroup: the *radical* $\mathrm{R}(G)$. Also, G has a largest closed unipotent normal subgroup: the *unipotent radical* $\mathrm{R}_u(G)$. We have $\mathrm{R}_u(G) \subset \mathrm{R}(G)$ as the largest closed unipotent subgroup, and $\mathrm{R}(G) = \mathrm{R}_u(G) \rtimes T$ where T is a maximal torus of $\mathrm{R}(G)$.

We say that G is *reductive* (resp. *semi-simple*) if $\mathrm{R}_u(G)$ is trivial (resp. G is connected and $\mathrm{R}(G)$ is trivial).

Examples

GL_n is reductive with radical the (central) subgroup of scalar matrices.
SL_n, Sp_{2n}, PGL_n are semi-simple, as well as SO_n for $n \geq 3$.
O_n is reductive, with trivial radical if $n \geq 3$. Also, $\mathrm{R}(\mathrm{O}_2) = \mathrm{SO}_2 \simeq \mathbb{G}_m$.

Proposition

Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation. Then ρ factorizes through an irreducible representation of $G/\,\mathrm{R}_u(G)$.

Proof: the fixed point subspace $V^{\mathrm{R}_u(G)}$ is non-zero and G-stable, hence is the whole V.

Reductive algebraic groups

A representation of an algebraic group is called *faithful* if its kernel is trivial, and *semi-simple* (or *completely reducible*) if it is a direct sum of irreducible representations.

Theorem

The following conditions are equivalent for an algebraic group G:

(i) *G is reductive.*

(ii) *Every representation of G is semi-simple.*

(iii) *G has a faithful semi-simple representation of finite dimension.*

Theorem

Let G be an algebraic group. Then there exists a closed subgroup $L \subset G$ such that $G = \mathrm{R}_u(G) \rtimes L$. Any two such subgroups are conjugate in $\mathrm{R}_u(G)$.

A subgroup L as above is reductive (since $L \simeq G/\mathrm{R}_u(G)$) and called a *Levi subgroup* of G; the equality $G = \mathrm{R}_u(G)L$ is the *Levi decomposition*.

Structure of connected reductive groups

Let G be a connected reductive group, $Z = Z(G)$ its center, and $[G, G]$ the commutator (or derived) subgroup. Both Z and $[G, G]$ are closed normal subgroups of G.

Theorem

(i) *Z^0 is the largest central torus of G. In particular, $Z^0 = R(G)$.*

(ii) *$[G, G]$ is semi-simple.*

(iii) *$G = Z^0[G, G]$ and $Z^0 \cap [G, G]$ is finite.*

Thus, we have $G = (T \times S)/F$, where T is a torus, S a semi-simple group, and F a finite central subgroup of $T \times S$.

Example

Let $G = \mathrm{GL}_n$. Then Z consists of the scalar matrices; in particular, $Z \simeq \mathbb{G}_m$ is connected. Moreover, $[G, G] = \mathrm{SL}_n$. Thus, $Z \cap [G, G]$ is isomorphic to the group of n-th roots of unity via $\zeta \mapsto \zeta I$, where I denotes the identity matrix.

Irreducible representations of connected reductive groups

From now on, G denotes a connected reductive algebraic group, B a Borel subgroup, U the unipotent radical of B, and T a maximal torus of B. Recall that $B = U \rtimes T$ and $\mathrm{X}^*(U)$ is trivial; thus, $\mathrm{X}^*(B) \simeq \mathrm{X}^*(T)$.

Theorem

Every simple G-module V contains a unique B-stable line. Moreover, the corresponding character of B determines V uniquely up to isomorphism.

The B-stable line $V_\lambda^{(B)}$ is called the *highest weight line*, and λ the *highest weight* of V. Let $\Lambda = \mathrm{X}^*(B)$ and $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. We denote by Λ^+ the subset of Λ consisting of highest weights of simple G-modules, and by $V(\lambda)$ the simple G-module with highest weight $\lambda \in \Lambda^+$.

Proposition

There exists a unique rational polyhedral convex cone $C \subset \Lambda_{\mathbb{R}}$ such that $\Lambda^+ = C \cap \Lambda$. Moreover, C spans $\Lambda_{\mathbb{R}}$.
Also, C is strictly convex iff G is semi-simple.

The cone C is called the *positive Weyl chamber* and Λ^+ is the set of *dominant weights*. In particular, Λ^+ is a finitely generated monoid.

Examples

Let $G = \mathrm{GL}_n$. Take $B = \mathrm{B}_n$ and $T = \mathrm{T}_n$, so that $U = \mathrm{U}_n$. Then $\Lambda \simeq \mathbb{Z}^n$ with basis the diagonal coefficients $\varepsilon_1, \ldots, \varepsilon_n$.

One may show that the monoid Λ^+ is generated by the characters $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$ where $i = 1, \ldots, n$, and by $-\varpi_n$.

The corresponding simple G-modules are the exterior powers $\wedge^i \mathbb{C}^n$ of the standard module $\mathbb{C}^n$, where $i = 1, \ldots, n$ (so that $\wedge^n \mathbb{C}^n = \mathbb{C}$ on which G acts via the determinant), and the inverse of the determinant. The highest weight line in $\wedge^i \mathbb{C}^n$ is generated by $e_1 \wedge \cdots \wedge e_i$, where $(e_1, \ldots, e_n)$ denotes the standard basis of $\mathbb{C}^n$.

If we replace G with SL_n and B (resp. T) with $\mathrm{B}_n \cap \mathrm{SL}_n$ (resp. $\mathrm{T}_n \cap \mathrm{SL}_n$), then $\Lambda = \mathbb{Z}^n / \mathbb{Z}\varpi_n$ and C is freely generated by the images of $\varpi_1, \ldots, \varpi_{n-1}$ in this quotient.

In particular, if $G = \mathrm{SL}_2$ then the highest weights of the simple G-modules are just the non-negative integers. The simple G-module with highest weight d is the symmetric power $S^d \mathbb{C}^2$, with highest weight line $\mathbb{C} e_1^d$. It may also be viewed as the space $\mathbb{C}[x, y]_d$ of homogeneous polynomials of degree d, with highest weight line $\mathbb{C} y^d$.

Representations of connected reductive groups

Let V be a G-module. Then V is semi-simple, and hence decomposes uniquely as a direct sum of simple G-modules with multiplicities. We will obtain an explicit form of this decomposition.

For any $\lambda \in \Lambda^+$, denote by $\mathrm{Hom}^G(V(\lambda), V)$ the space of morphisms of G-modules $f : V(\lambda) \to V$. Choosing a highest weight vector $v_\lambda \in V(\lambda)$ defines a linear map $\mathrm{Hom}^G(V(\lambda), V) \to V$, $f \mapsto f(v_\lambda)$ with image in $V_\lambda^{(B)}$.

Theorem

(i) *The evaluation map*

$$\bigoplus_{\lambda \in \Lambda^+} \mathrm{Hom}^G(V(\lambda), V) \otimes V(\lambda) \longrightarrow V, \quad f \otimes v \longmapsto f(v)$$

is an isomorphism of G-modules, where G acts on the left-hand side via $g \cdot (f \otimes v) = f \otimes g \cdot v$.

(ii) *We have* $\mathrm{Hom}^G(V(\lambda), V) \xrightarrow{\sim} V_\lambda^{(B)}$ *for all* $\lambda \in \Lambda^+$.

In particular, the multiplicity of $V(\lambda)$ in V is $\dim V_\lambda^{(B)}$.

Projective homogeneous varieties

Let X be a projective homogeneous variety under G. By Borel's fixed point theorem, we may choose a B-fixed point $x \in X$. Equivalently, the isotropy group G_x contains B. Such a subgroup P of G is called a *standard parabolic subgroup*.

For any $\lambda \in \Lambda^+$, the stabilizer of the highest weight line in $V(\lambda)$ is a standard parabolic subgroup. By the structure of G-modules, every standard parabolic subgroup can be obtained in this way. This yields:

Proposition

The projective homogeneous G-varieties are exactly the orbits of highest weight lines in the projectivizations of simple G-modules.

Definition

A *parabolic subgroup* of G is a subgroup containing a Borel subgroup.

Theorem

Let $P \subset G$ be a parabolic subgroup. Then P is conjugate to a unique standard parabolic subgroup. Moreover, P is closed, connected, and equal to its normalizer.

Projective homogeneous varieties (continued)

Let P be a parabolic subgroup, and choose a Levi decomposition $P = \mathrm{R}_u(P) \rtimes L$. Then L is a closed connected reductive subgroup of G.

Proposition

There exists a unique parabolic subgroup $Q \subset G$ such that $P \cap Q = L$. Then $Q = \mathrm{R}_u(Q) \rtimes L$ and the map

$$\mathrm{R}_u(P) \times L \times \mathrm{R}_u(Q) \longrightarrow G, \quad (u, l, v) \longmapsto ulv$$

is an open immersion.

Let $X = G/Q$ with base point x. As a consequence of the proposition, the orbit $R_u(P) \cdot x$ is open in X and the isotropy group $\mathrm{R}_u(P)_x$ is trivial. This yields an open neighborhood of x in X, isomorphic to an affine space: the *open cell*.

If $P \supset B$ is a standard parabolic subgroup, then $B = \mathrm{R}_u(P) \rtimes (B \cap L)$, where $B \cap L$ is a Borel subgroup of L. Thus, $B \cdot x = \mathrm{R}_u(P) \cdot x$ and $B_x = B \cap L$. In particular, X contains an open dense B-orbit.

Examples of projective homogeneous varieties

Let $G = \mathrm{GL}_n$ and $B = \mathrm{B}_n$. Then the standard parabolic subgroups are exactly the groups of upper triangular block matrices, with Levi decompositions as in the following example:

$$P = \left(\begin{array}{c|c|c} * & * & * \\ \hline 0 & * & * \\ \hline 0 & 0 & * \end{array}\right), \quad \mathrm{R}_u(P) = \left(\begin{array}{c|c|c} I & * & * \\ \hline 0 & I & * \\ \hline 0 & 0 & I \end{array}\right), \quad L = \left(\begin{array}{c|c|c} * & 0 & 0 \\ \hline 0 & * & 0 \\ \hline 0 & 0 & * \end{array}\right).$$

The corresponding projective homogeneous varieties G/P are exactly the *varieties of partial flags*, consisting of sequences of subspaces

$$\{0\} \subsetneq V_1 \subsetneq \cdots \subsetneq V_m \subsetneq \mathbb{C}^n$$

where $\dim(V_1), \ldots, \dim(V_m)$ are prescribed (recall that G/B is the variety of complete flags Fl_n).

We may index the maximal parabolic subgroups $P_1, \ldots, P_{n-1}$ so that G/P_i is the Grassmannian $\mathrm{Gr}(i, n)$ for $i = 1, \ldots, n-1$.

Spherical varieties: definition and first examples

A *spherical G-variety* is a normal G-variety X which contains an open B-orbit.
We denote this orbit by X_B^0. Then X also contains an open G-orbit, namely $X_G^0 = G \cdot X_B^0$.
Choosing $x \in X_B^0$, we may identify X_G^0 with the homogeneous space G/H with $H = G_x$, and X_B^0 with $B/B \cap H$. Then (X, x) is a normal equivariant embedding of the *spherical homogeneous space* G/H.

Examples

(i) If $G = T$ is a torus, then $B = T$ and we get back the toric varieties.

(ii) Every projective homogeneous G-variety is spherical.

(iii) Let H be a *horospherical subgroup* of G, i.e., a closed subgroup containing a conjugate of $U = \mathrm{R}_u(B)$. Then G/H is spherical.

To check (iii), let B^- be the Borel subgroup of G such that $B^- \cap B = T$. Then the multiplication map $B^- \times U \to G$ is an open immersion. Thus, the orbit of the base point $B^- \cdot x$ is open in G/H.

Symmetric spaces

Let σ be an involutive automorphism of the algebraic group G, with fixed point subgroup G^σ. Let H be a subgroup of G such that $(G^\sigma)^0 \subset H \subset G^\sigma$. Then the homogeneous space G/H is called a *symmetric space*.

A parabolic subgroup $P \subset G$ is called *σ-split* if $\sigma(P)$ is opposite to P. Then $L = P \cap \sigma(P)$ is a σ-stable Levi subgroup of both.

A subtorus $S \subset G$ is *σ-split* if $\sigma(s) = s^{-1}$ for all $s \in S$. Then $S \cap H$ is a finite group of exponent 2.

Proposition

Let P be a minimal σ-split parabolic subgroup, and $L = P \cap \sigma(P)$.
Then $[L, L] \subset H$ and the center $Z(L)$ contains a unique maximal σ-split torus S of G.
Moreover, the map $\mathrm{R}_u(P) \times S/S \cap H \to G/H$, $(u, y) \mapsto u \cdot y$ is an open immersion.

Next, choose a Borel subgroup B of P. Then $B = \mathrm{R}_u(P)(B \cap L)$ and $B \cap L \supset Z(L)^0 \supset S$. Thus, $B/B \cap H$ is open in G/H.

In particular, *every symmetric space is spherical*.

Examples of symmetric spaces

1) Consider a connected reductive group H, and the group $G = H \times H$ with involution $\sigma : (x, y) \mapsto (y, x)$. Then the fixed point subgroup is the diagonal, $\mathrm{diag}(H)$. The homogeneous space G/H may be identified with H on which $H \times H$ acts by left and right multiplication, via the map $H \times H \to H$, $(x, y) \mapsto xy^{-1}$.
The minimal σ-split parabolic subgroups are the products $B \times B^-$, where B, B^- are opposite Borel subgroups of H. The maximal σ-split tori are the images of maximal tori T of H under the map $x \mapsto (x, x^{-1})$.

2) Let $G = \mathrm{GL}_n$ with involution $A \mapsto (A^t)^{-1}$. Then $G^\sigma = \mathrm{O}_n$ and the symmetric space G/G^σ is the space of non-degenerate quadratic forms in n variables. The Borel subgroup B_n is minimal σ-split and the torus T_n is maximal σ-split.
Consider the G-module $V = S^2(\mathbb{C}^n)^*$ of all quadratic forms in n variables. Then V yields a smooth affine equivariant embedding of G/G^σ, with base point $x_1^2 + \cdots + x_n^2$. The hypersurface of degenerate forms is a closed irreducible G-stable subvariety (the orbit closure of forms of rank $n - 1$) which is singular along the forms of rank $\leq n - 2$.

The weight group and rank of a spherical variety

Let X be a spherical variety, and $\mathbb{C}(X)$ its field of rational functions. The group B acts on $\mathbb{C}(X)$ and every B-invariant rational function is constant, since X has an open B-orbit.

The set $\mathbb{C}(X)^{(B)}$ of B-eigenvectors is a subgroup of the multiplicative group $\mathbb{C}(X)^* = \mathbb{C}(X) \setminus \{0\}$, equipped with a homomorphism to $X^*(B) = \Lambda$ which sends each eigenvector to its weight. The kernel of this homomorphism is the subgroup $\mathbb{C}^*$ of non-zero constant functions. The image is the *weight group* $\Lambda(X)$. This is a free abelian group; its rank $r(X)$ is the *rank of* X.

For an embedding (X, x) of a spherical homogeneous space G/H with open B-orbit $B/B \cap H$, we have

$$\Lambda(X) = \Lambda(G/H) = \{\lambda \in X^*(B) \mid \lambda|_{B\cap H} = 1\}.$$

Examples

1) If X is projective homogeneous, then $r(X) = 0$.

2) The rank of a symmetric space is the dimension of a maximal split torus.

Affine spherical varieties

We say that a G-module M is *multiplicity-free* if the multiplicity of $V(\lambda)$ in M is 0 or 1 for any $\lambda \in \Lambda^+$. Equivalently, $\dim M_\lambda^{(B)} \leq 1$ for any such λ.

Proposition

A normal affine G-variety X is spherical iff the G-module $\mathcal{O}(X)$ is multiplicity-free. Under these conditions, we have an isomorphism of G-modules

$$\mathcal{O}(X) \simeq \bigoplus_{\lambda \in C(X) \cap \Lambda(X)} V(\lambda),$$

where $C(X)$ is a rational polyhedral convex cone in $\Lambda(X)_{\mathbb{R}}$.

Proof: Assume that $\mathcal{O}(X)$ is multiplicity-free. By a theorem of Rosenlicht, to show that X has an open B-orbit, it suffices to check that every rational B-invariant function on X is constant. Given such a function f, the set of all $\varphi \in \mathcal{O}(X)$ such that $f\varphi \in \mathcal{O}(X)$ is a non-zero B-module, and hence contains a B-eigenvector ψ. Then $\psi, f\psi \in \mathcal{O}(X)^{(B)}$ have the same weight, and hence are proportional. So $f = (f\psi)/\psi$ is constant.

Proof of the proposition (continued)

For the converse, assume that X is spherical. Since X_B^0 is an open affine B-stable subset of X, its complement is a union of prime B-stable divisors $D_1, \ldots, D_m$. Since $\mathcal{O}(X)$ is integrally closed, we have

$$\mathcal{O}(X) = \mathcal{O}(X_B^0) \cap \bigcap_{i=1}^{m} \mathcal{O}_{X,D_i}$$

as subrings of $\mathbb{C}(X)$, where

$$\mathcal{O}_{X,D_i} = \{f \in \mathbb{C}(X) \mid \mathrm{ord}_{D_i}(f) \geq 0\}$$

denotes the local ring of X along D_i for $i = 1, \ldots, m$.

Moreover, $\mathcal{O}(X_B^0)^{(B)}/\mathbb{C}^* \simeq \Lambda(X)$ and each ord_{D_i} yields a homomorphism $\nu_i : \Lambda(X) \to \mathbb{Z}$. For any $f \in \mathcal{O}(X_B^0)^{(B)}$ of weight $\lambda \in \Lambda(X)$, the condition that $\mathrm{ord}_{D_i}(f) \geq 0$ translates into the linear inequality $\nu_i(\lambda) \geq 0$.

These inequalities define a rational polyhedral cone $C(X) \subset \Lambda(X)_{\mathbb{R}}$ such that the set of weights of $\mathcal{O}(X)^{(B)}$ is $C(X) \cap \Lambda(X)$.

Affine spherical varieties (continued)

With the notation of the proposition, $C(X)$ is called the *weight cone* of the affine spherical variety X. The intersection $C(X) \cap \Lambda(X)$ is a finitely generated submonoid of Λ, called the *weight monoid*.

The datum of the weight monoid is equivalent to that of the pair $(\Lambda(X), C(X))$. Unlike for affine toric varieties, this does not determine X up to equivariant isomorphism, as shown by the following:

Example

Let $G = \mathrm{SL}_2$ and consider the irreducible representation $V_2 = \mathbb{C}[x, y]_2$. The orbits of non-degenerate forms are closed and isomorphic to $X = \mathrm{SL}_2 / \mathrm{SO}_2$. The variety of degenerate forms is $X_0 = G \cdot x^2 \cup \{0\}$ (the affine cone over $\mathbb{P}^1$ embedded via $\mathcal{O}(2)$).
Both X and X_0 are spherical with $\Lambda(X) = 2\mathbb{Z} \subset \mathbb{Z} = \Lambda$ and $C(X) = \mathbb{R}_{\geq 0}$.

In the positive direction, we have a result of Losev:

Theorem

Any two smooth affine spherical G-varieties having the same weight monoid are equivariantly isomorphic.

Global sections of line bundles

Proposition

Let X be a spherical G-variety, and L a G-linearized line bundle on X. Then the G-module $\Gamma(X, L)$ is multiplicity-free.
Conversely, let X be a normal quasi-projective variety such that the G-module $\Gamma(X, L)$ is multiplicity-free for any G-linearized line bundle L on X. Then X is spherical.

This is proved by adapting the arguments of the preceding proposition. Moreover, the construction of convex polytopes in toric geometry extends to spherical varieties as follows.

Proposition

Let X be a complete spherical G-variety, L a G-linearized line bundle on X having non-zero sections, and λ_0 a highest weight of the G-module $\Gamma(X, L)$. Then there exists a unique rational convex polytope $P \subset \Lambda_{\mathbb{R}}$ such that

$$\Gamma(X, L^{\otimes k}) \simeq \bigoplus_{\lambda \in kP \cap (k\lambda_0 + \Lambda(X))} V(\lambda) \quad \textit{for any integer } k \geq 1.$$

Global sections of line bundles (continued)

The above polytope $P = P(X, L)$ is contained in the affine space $\lambda_0 + \Lambda(X)_{\mathbb{R}}$. If L is ample, then P has a non-empty interior.

Examples

1) Let X be a projective homogeneous variety. Recall that X is the orbit of the highest weight line in $\mathbb{P}(V(\lambda))$ for some $\lambda \in \Lambda^+$. This defines a very ample G-linearized line bundle $L = \mathcal{O}_X(1)$.
For any $k \geq 1$, we have $\Gamma(X, L^{\otimes k}) = V(k\lambda_0)$, where λ_0 denotes the highest weight of the dual G-module $V(\lambda)^*$. Thus, $P(X, L) = \{\lambda_0\}$.

2) Let $G = \mathrm{SL}_n$ and $X = \mathbb{P}(S^2(\mathbb{C}^n)^*)$. Then X is the space of quadratic hypersurfaces in $\mathbb{P}^{n-1}$ and we have $\Lambda(X) = 2\Lambda$. Let $L = \mathcal{O}_X(1)$, so that $\Gamma(X, L) \simeq S^2\mathbb{C}^n = V(2\varpi_1)$. One may show that the vertices of $P(X, L)$ are exactly 0 and the rational weights $\frac{2\varpi_i}{i}$, where $i = 1, \ldots, n-1$. In particular, $P(X, L)$ is not a lattice polytope when $n \geq 4$.

Also, for $n = 2$ we get $P(X, L) = [0, 2]$. Let $X_0 \subset \mathbb{P}(V_2 \oplus V_0)$ be the G-orbit closure of $[x^2 \oplus 1]$, and $L_0 = \mathcal{O}_{X_0}(1)$. Then $\Lambda(X_0) = 2\Lambda$ and $P(X_0, L_0) = [0, 2]$. So $(\Lambda(X), P(X, L))$ does not determine (X, L) uniquely.

A finiteness property of spherical varieties

Theorem

Let X be a spherical G-variety. Then G has only finitely many orbits in X, and their closures are spherical G-varieties as well. Moreover, for any such orbit closure Y, we have $\Lambda(Y) \subset \Lambda(X)$ with equality iff $Y = X$.

Proof sketch: Assume that X is affine. Then one shows that every prime G-stable ideal in $\mathcal{O}(X)$ is of the form

$$\bigoplus_{\lambda \in (C(X) \setminus F) \cap \Lambda(X)} V(\lambda)$$

for a unique face F of $C(X)$. This implies the assertions in this case.

For an arbitrary X, one reduces to the case where $X \subset \mathbb{P}(V)$ for some G-module V by using Sumihiro's theorem. Then one reduces further to the affine case, by considering the affine cone over X with its natural action of the connected reductive group $G \times \mathbb{G}_m$.

Corollary

With the above notation, we have $r(Y) \leq r(X)$ with equality iff $Y = X$.

Further finiteness properties

The above finiteness theorem has a remarkable consequence:

Corollary

Let X be a spherical variety of rank 0. Then X is projective homogeneous.

Indeed, X_G^0 admits a projective embedding with no additional orbit.

Also, the theorem has a converse:

Proposition

Let G/H be a homogeneous space such that every embedding has only finitely many orbits. Then G/H is spherical.

Actually, there is a stronger finiteness result:

Theorem

Every spherical G-variety contains only finitely many B-orbits.

Yet the combinatorics of B-orbits and the geometry of their closures are much less understood than those of G-orbits. For instance, B-orbit closures are not necessarily normal, as shown by the next example.

Example: a non-normal B-orbit closure in a spherical homogeneous space

Let X be the set of triples $(\ell, \{x, y\})$, where ℓ is a line in $\mathbb{P}^2$ and $\{x, y\}$ is an unordered pair of distinct points of ℓ. Then X is a variety of dimension 4, homogeneous under the natural action of $G = \mathrm{PGL}(3)$.

Fix a line ℓ_0 in $\mathbb{P}^2$ and a point $x_0 \in \ell_0$. Let B be the Borel subgroup of G that stabilizes x_0 and ℓ_0. One may check that B has a dense open orbit in X: the locus where $x_0 \notin \ell$ and $x, y \notin \ell_0$. In particular, X is spherical.

Let $Y \subset X$ be the locus where $x \in \ell_0$ or $y \in \ell_0$. Then Y is a closed irreducible B-stable subvariety of dimension 3.

Also, consider the set Z of ordered triples (ℓ, x, y), where ℓ is a line in $\mathbb{P}^2$, $x \in \ell \cap \ell_0$, $y \in \ell$ and $x \neq y$. Then Z is a smooth threefold, since the projections $(\ell, x, y) \mapsto (\ell, x) \mapsto x$ are $\mathbb{P}^1$-bundles. Moreover, the natural map $f : Z \to Y$ is a finite morphism that restricts to an isomorphism over the open subset where $\ell \neq \ell_0$, and has fibers of order 2 at all points of the complement.

So Y is non-normal with normalization Z.

Local structure of normal G-varieties

Let X be a normal G-variety, and Y a closed G-stable subvariety.

Proposition

There exists an open affine B-stable subset $X^0 \subset X$ which meets Y.

This is proved by using representation theory, as for the existence of an open covering of X by affine T-stable subsets.

Choose such a subset X^0 and let $P = P(X^0) = \{g \in G \mid g \cdot X^0 = X^0\}$. Then $P \supset B$ is a standard parabolic subgroup of G with Levi decomposition $P = \mathrm{R}_u(P) \rtimes L$, where $L \supset T$.

Theorem

There exists an L-stable subvariety Z of X^0 such that the map $\mathrm{R}_u(P) \times Z \to X^0$, $(u, z) \mapsto u \cdot z$ is an isomorphism.

Proof sketch: we may assume that $X = G \cdot X^0$. By a linearization argument, one constructs a G-morphism $f : X \to \mathbb{P}(V)$ where V is a simple G-module, and $X \setminus X^0 = f^{-1}(H)$ where $H \subset V$ is the unique B-stable hyperplane. Then one constructs an L-stable subvariety $S \subset \mathbb{P}(V) \setminus H$ such that $\mathrm{R}_u(P) \times S \xrightarrow{\sim} \mathbb{P}(V) \setminus H$ and one takes $Z = f^{-1}(S)$.

Local structure of spherical varieties

Let X be a spherical G-variety, and Y a G-orbit closure in X. Recall that Y is a spherical G-variety as well.

Lemma

Let $X_{Y,G} = \{x \in X \mid \overline{G \cdot x} \supset Y\}$. Then $X_{Y,G}$ is a G-stable open subset containing Y_G^0 as its unique closed G-orbit, and such a subset is unique.

This follows from the finiteness of G-orbits in X.
We then say that $X_{Y,G}$ is a *simple* spherical variety.

Next, let $X_{Y,B} = \{x \in X \mid \overline{B \cdot x} \supset Y\}$. Then $X_{Y,B}$ is a B-stable open subset of $X_{Y,G}$ containing Y_B^0 as its unique closed B-orbit.

Denote by P the stabilizer of $X_{Y,B}$. This is a standard parabolic subgroup of G with Levi decomposition $P = \mathrm{R}_u(P) \rtimes L$.

Theorem

There exists an affine L-stable subvariety Z of $X_{Y,B}$ such that the natural map $\mathrm{R}_u(P) \times Z \to X_{Y,B}$ is an isomorphism. Moreover, $Z \cap Y$ is the unique closed L-orbit in Z and is fixed pointwise by $[L, L]$.

Local structure of spherical varieties (continued)

The above theorem applies to the open orbit $Y = X_G^0 = G/H$ and yields an isomorphism $\mathrm{R}_u(P) \times S/S \cap H \xrightarrow{\sim} B/B \cap H = P/P \cap H$, where P is the stabilizer of $X_B^0 = B/B \cap H$, and S is a central subtorus of L; moreover, $[L, L] \subset H$.

Here are further consequences of the theorem:

Corollary

(i) *$X_{Y,B}$ is affine.*

(ii) *The irreducible components of $X \setminus X_{Y,B}$ are exactly the prime B-stable divisors of X which do not contain Y.*

(iii) *There exists an L-equivariant fibration $f : Z \to Z \cap Y$ with fiber an affine spherical variety under a connected reductive subgroup of L, having a fixed point.*

All of this reduces the local structure of a spherical variety along a G-orbit to an affine spherical variety having a fixed point, as in toric geometry.

Example: quadratic forms

Consider again the symmetric space $\mathrm{GL}_n / \mathrm{O}_n$ and its embedding in the G-module X of quadratic forms in n variables. Every G-orbit closure Y in X consists of the forms of rank at most m for some $m \in \{0, \ldots, n\}$. Thus, $Y_G^0 = G \cdot y$ and $Y_B^0 = B \cdot y$, where $y = x_1^2 + \cdots + x_m^2$.

Moreover, $X_{Y,B}$ consists of the forms whose restrictions to $V_1, \ldots, V_m$ are non-degenerate, where $V_i = \langle e_1, \ldots, e_i \rangle$ for $i = 1, \ldots, n$. The stabilizer of $X_{Y,B}$ is the standard parabolic subgroup P which stabilizes $V_1, \ldots, V_m$, and $L = \mathrm{T}_m \times \mathrm{GL}_{n-m}$.

Let Z be the subset of X consisting of the forms

$$q(x_1, \ldots, x_n) = a_1 x_1^2 + \cdots + a_m x_m^2 + q'(x_{m+1}, \ldots, x_n),$$

where $a_1, \ldots, a_m \in \mathbb{C}^*$. Then Z is an L-stable subvariety of $X_{Y,B}$ and the natural map $\mathrm{R}_u(P) \times Z \to X_{Y,B}$ is an isomorphism.
Moreover, $Y \cap Z = \{a_1 x_1^2 + \cdots + a_m x_m^2 \mid a_1, \ldots, a_m \in \mathbb{C}^*\} = L \cdot y$ is fixed pointwise by $\mathrm{SL}_{n-m} = [L, L]$.

The map $q \mapsto a_1 x_1^2 + \cdots + a_m x_m^2$ is an L-equivariant fibration. Its fiber at y is isomorphic to the L_y-variety of quadratic forms in $n - m$ variables, where $L_y = \{\pm 1\}^m \times \mathrm{GL}_{n-m}$ acts via the natural action of GL_{n-m}.

Toroidal varieties

Keep the above assumptions and let $\Delta = X_G^0 \setminus X_B^0$. Then Δ is the union of the prime B-stable divisors in the open G-orbit X_G^0.

Proposition

The following conditions are equivalent:

(i) *The closure $\overline{\Delta}$ contains no G-orbit in X.*

(ii) *There exists a closed L-stable subvariety $Z \subset X \setminus \overline{\Delta}$ such that the natural map $\mathrm{R}_u(P) \times Z \to X \setminus \overline{\Delta}$ is an isomorphism. Moreover, $[L, L]$ fixes Z pointwise, and Z is a toric variety under a quotient of $Z(L)^0$. Every G-orbit in X meets Z along a unique L-orbit.*

A spherical variety satisfying the above conditions is called *toroidal*. Its G-orbit structure is that of a toric variety.

Proposition

For any spherical G-variety X, there exists a unique toroidal G-variety $\tilde{X}$ equipped with a proper birational equivariant morphism $f : \tilde{X} \to X$, and minimal for this property. We say that $\tilde{X}$ is the discoloration of X.

Toroidal varieties (continued)

Let X be a smooth toroidal G-variety, and $\partial X = X \setminus X^0_G$. Then ∂X is a divisor with strict normal crossings; its irreducible components are the prime G-stable divisors $D_1, \ldots, D_m$.

Proposition

The map $\mathrm{op}_{X,D} : \mathcal{O}_X \otimes \mathrm{Lie}(G) \to T_X(-\log \partial X)$ *is surjective. Moreover,* $-K_X = \partial X + E$*, where* E *is an effective* B*-stable divisor with support* Δ*. In particular,* $-K_X - \partial X$ *is generated by its global sections.*

Proof sketch: use the local structure to reduce to the toric case.

This formula for $-K_X$ extends to any spherical G-variety X. Indeed, the union of X^0_G and the G-orbits of codimension 1 is a smooth toroidal G-stable open subset, and its complement has codimension ≥ 2.

The above proposition has a partial converse:

Theorem

Let X *be a smooth complete* G*-variety, and* $D \subset X$ *a divisor with strict normal crossings. If the map* $\mathrm{op}_{X,D}$ *is surjective, then the* G*-variety* X *is toroidal and* $D = \partial X$*.*

Local rigidity

The following is a special case of a vanishing theorem due to Knop:

Theorem

Let X be a smooth complete toroidal G-variety, and $\partial X = X \setminus X^0_G$. Then $H^i(X, T_X(-\log \partial X)) = 0$ for any $i \geq 1$. In particular, the pair $(X, \partial X)$ is locally rigid.

Corollary

With the above assumptions, if X is Fano then $H^i(X, T_X) = 0$ for any $i \geq 1$. In particular, X is locally rigid.

Proof sketch: recall the exact sequence $0 \to T_X(-\log \partial X) \to T_X \to \bigoplus_{i=1}^m N_{D_i/X} \to 0$, where $\partial X = D_1 \cup \cdots \cup D_m$. By the theorem, it suffices to show that $H^i(D_j, N_{D_j/X}) = 0$ for $i \geq 1$ and all j. Equivalently, $H^i(D_j, \mathcal{O}(-K_X + D_j)) = 0$ for all such i, j. But this follows from the Kodaira vanishing theorem, since $-K_X$ is ample.

As a consequence, *every Fano toric manifold X is locally rigid*. (The above vanishing theorem is easily proved in this setting, since $T_X(-\log \partial X)$ is trivial and $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$).

Example: a non-locally rigid spherical Fano manifold

We present an example due to Pasquier and Perrin. Consider the group SO_7 and its standard representation in $\mathbb{C}^7$. The adjoint representation is isomorphic to $\wedge^2\mathbb{C}^7$. The orbit of the highest weight line in the projective space $\mathbb{P}(\wedge^2\mathbb{C}^7)$ is the isotropic Grassmannian $\mathrm{IGr}(2,7)$. Its Picard group is $\mathbb{Z}$, generated by the pull-back of $\mathcal{O}_{\mathbb{P}(\wedge^2\mathbb{C}^7)}(1)$.

Next, consider the simple algebraic group $G = \mathrm{G}_2 \subset \mathrm{SO}_7$. Then $\mathrm{IGr}(2,7)$ is a spherical G-variety with two orbits, and the closed orbit Y has codimension 2. As a representation of G, we have $\wedge^2\mathbb{C}^7 \simeq V(\varpi_1) \oplus V(\varpi_2)$, where ϖ_1, ϖ_2 are the fundamental weights, i.e., the generators of the monoid of dominant weights. Moreover, $V(\varpi_1)$ is the natural representation of G in $\mathbb{C}^7$, and $V(\varpi_2) = \mathrm{Lie}(G)$ is the adjoint representation.

Also, $\Lambda(\mathrm{IGr}(2,7)) = \mathbb{Z}(\varpi_2 - \varpi_1)$. In particular, $\mathrm{IGr}(2,7)$ has rank 1.

We have $Y \simeq G/P_2$, where P_2 is the standard parabolic subgroup stabilizing the highest weight line in $V(\varpi_2)$. Moreover, Y is the projectivization of the minimal nilpotent orbit in $\mathrm{Lie}(G)$.

A non-locally rigid spherical Fano manifold (continued)

Pasquier and Perrin constructed a G-variety $\mathcal{X}$ equipped with a G-invariant proper morphism $f : \mathcal{X} \to \mathbb{C}$ such that $f^{-1}(t) \simeq \mathrm{IGr}(2,7)$ for any $t \in \mathbb{C}^*$. Moreover, $f^{-1}(0) = X$ is the G-orbit closure of $[v_{\varpi_1} + v_{\varpi_2}]$ in $\mathbb{P}(V(\varpi_1) \oplus V(\varpi_2))$, where v_{ϖ_i} are highest weight vectors in $V(\varpi_i)$ for $i = 1, 2$.

Thus, X is a projective horospherical variety and $\Lambda(X) = \mathbb{Z}(\varpi_2 - \varpi_1)$. Moreover, $X = X_G^0 \cup Y_1 \cup Y_2$, where $Y_i = G \cdot [v_{\varpi_i}] \simeq G/P_i$ are closed orbits for $i = 1, 2$. The G-stable open subset $X_{Y_i,G} = X_G^0 \cup Y_i$ is equipped with a G-equivariant map to Y_i, the restriction of the projection $\mathbb{P}(V(\varpi_1) \oplus V(\varpi_2)) \dashrightarrow \mathbb{P}(V(\varpi_i))$. In fact, $X_{Y_i,G}$ is a G-equivariant vector bundle of rank 2 over Y_i for $i = 1, 2$. In particular, X is smooth.

Since the closed G-orbits have codimension 2, we have $\mathrm{Pic}(X) \simeq \mathrm{Pic}(X_G^0)$. Also, $X_G^0 \simeq G/H$ where $H = U \rtimes \mathrm{Ker}(\varpi_2 - \varpi_1)$, and hence $\mathrm{Pic}(X_G^0) \simeq \mathbb{Z}$ with generator the equivariant line bundle associated with the character $\varpi_2 - \varpi_1$ of H. Thus, $\mathrm{Pic}(X) \simeq \mathbb{Z}$; in particular, X is Fano.

Pasquier showed that X is not homogeneous under its full automorphism group, and hence not isomorphic to $\mathrm{IGr}(2,7)$.

Some references for Lecture 2

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