Introduction to spherical varieties: Lecture 1

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14 December 2021

- Lecture 1: overview; basic notions and results on actions and representations of algebraic groups; toric varieties.
- Lecture 2: further background on the structure and representations of linear algebraic groups; projective homogeneous varieties; spherical varieties: definition, finiteness properties, local structure.
- Lecture 3: embeddings of spherical homogeneous spaces; wonderful varieties; line bundles on spherical varieties; open questions.

Overview

Spherical varieties form a remarkable class of algebraic varieties equipped with an action of an algebraic group, which contains several classes of interest:

- Toric varieties
- Projective homogeneous varieties
- Symmetric spaces, complete symmetric varieties
- Wonderful varieties

Toric varieties are classified by fans, which provide a well-developed dictionary between their geometry and combinatorics. This makes toric varieties an appropriate testing ground for many algebro-geometric questions, even if they form a very special class.

The class of spherical varieties is much wider, and includes many examples from classical projective geometry. Spherical varieties also admit a combinatorial classification, whose relation to geometry is less understood.

The lectures will present some basic results on spherical varieties, together with the relevant background on the structure, actions and representations of algebraic groups. They will conclude with open questions.

Algebraic groups: basic definitions

We consider algebraic varieties over the field $\mathbb{C}$ of complex numbers. Varieties need not be irreducible. We identify each variety X with its set of complex points $X(\mathbb{C})$ equipped with the Zariski topology and with the structure sheaf $\mathcal{O}_X$.

Definition

An *algebraic group* is a group G equipped with the structure of a variety such that the multiplication map $m : G \times G \to G$, $(g, h) \mapsto gh$ and the inverse map $i : G \to G$, $g \mapsto g^{-1}$ are morphisms of varieties.

For example, the *general linear group* GL_n has a natural structure of algebraic group. Indeed, viewing the set of $n \times n$ matrices M_n as an affine space of dimension n^2, we may view GL_n as an affine open subset: the complement of the zero locus of the determinant. Moreover, the matrix multiplication is polynomial in the matrix entries, and the matrix inverse is polynomial in these entries and the inverse of the determinant.

Definition

A *linear algebraic group* is a closed subgroup of GL_n.

Examples

- Finite groups.
- The *multiplicative group* $\mathbb{G}_m = (\mathbb{C}^*, \times)$.
- The *additive group* $\mathbb{G}_a = (\mathbb{C}, +)$.
- The *special linear group* $\mathrm{SL}_n = \{A \in \mathrm{GL}_n \,|\, \det(A) = 1\}$.
- The *orthogonal group* O_n and the *special orthogonal group* SO_n.
- The *symplectic group* Sp_{2n}.
- The *projective linear group* $\mathrm{PGL}_n = \mathrm{GL}_n / \mathbb{G}_m$, where $\mathbb{G}_m$ is viewed as the subgroup of scalar matrices.

All these groups are linear: $\mathbb{G}_m = \mathrm{GL}_1$; $\mathbb{G}_a = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{C}\}$;
SL_n, O_n, SO_n are closed in GL_n and likewise for Sp_{2n};
PGL_n is the automorphism group of the algebra M_n of $n \times n$ matrices, and hence is closed in GL_{n^2}.

Elliptic curves provide examples of non-linear algebraic groups.

The neutral component of an algebraic group

Proposition

Let G be an algebraic group, e its neutral element, and G^0 the connected component of G containing e.

(i) *G^0 is a closed normal subgroup of G, and a smooth irreducible variety.*

(ii) *The connected components of G are the cosets gG^0, where $g \in G$.*

(iii) *G^0 is the largest closed subgroup of finite index of G.*

In particular, G is smooth and all its components are isomorphic as varieties. Thus, G is equidimensional.

The finite group G/G^0 is called the *group of components* of G. We denote it by $\pi_0(G)$. Also, G^0 is called the *neutral component* of G.

Examples

GL_n, SL_n, Sp_{2n}, SO_n and PGL_n are connected. We have $\pi_0(\mathrm{O}_n) \simeq \{\pm 1\}$ via the determinant.

Algebraic group actions

Let G be an algebraic group with neutral element e.

Definition

An *action* of G on a variety X is a group action $a : G \times X \to X$ which is a morphism of varieties.

Recall that a is a group action if it satisfies $a(e, x) = x$ for all $x \in X$, and $a(gh, x) = a(g, a(h, x))$ for all such x and all $g, h \in G$. We denote $a(g, x)$ by $g \cdot x$ for simplicity.

Definition

A *G-variety* is a variety X equipped with an action of G.
Given two G-varieties X, Y and a morphism of varieties $f : X \to Y$, we say that f is *equivariant* (or a *G-morphism*) if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$.

Definition

Let X be a variety, and $Y \subset X$ a (locally closed) subvariety. We say that Y is *G-stable* (resp. *G-fixed*) if for all $g \in G$ and $y \in Y$, we have $g \cdot y \in Y$ (resp. $g \cdot y = y$). The G-fixed points in X form a closed subvariety X^G.

Orbits

Definition

Let G be an algebraic group, X a G-variety, and $x \in X$.
The *orbit map* of x is the morphism $a_x : G \to X$, $g \mapsto g \cdot x$.
The *orbit* $G \cdot x$ is the image of a_x.
The *stabilizer* (or *isotropy group*) of x is $G_x = \{g \in G \mid g \cdot x = x\}$.

Thus, G_x is the fiber of a_x at e, and hence is a closed subgroup of G. Also, for any $g \in G$, we have $G_{g \cdot x} = gG_xg^{-1}$.

Proposition

(i) *$G \cdot x$ is a smooth (locally closed) subvariety of X.*

(ii) *$G \cdot x$ is equidimensional of dimension* $\dim(G) - \dim(G_x)$.

(iii) *The boundary $\overline{G \cdot x} \setminus G \cdot x$ is a union of orbits of strictly smaller dimension.*

(iv) *Every orbit of minimal dimension is closed.*

In particular, every closed G-stable subvariety of X contains a closed orbit.

Homomorphisms

Definition

Let G, H be algebraic groups. A *homomorphism* is a morphism of varieties $f : G \to H$ which is also a group homomorphism.

Proposition

Let $f : G \to H$ be a homomorphism of algebraic groups.

(i) *The kernel* $\mathrm{Ker}(f) = f^{-1}(e)$ *is a closed normal subgroup of G.*

(ii) *The image* $\mathrm{Im}(f)$ *is a closed subgroup of H.*

(iii) *We have* $\dim(G) = \dim \mathrm{Ker}(f) + \dim \mathrm{Im}(f)$.

Proof sketch: (i) is clear.
(ii) Consider the action of G on H via $g \cdot h = f(g)h$. Then the orbits are the cosets $\mathrm{Im}(f)h$, where $h \in H$, and the isotropy group of any point is $\mathrm{Ker}(f)$. Since closed orbits exist, all orbits are closed. In particular, $\mathrm{Im}(f)$ is closed.
(iii) follows similarly from the formula for the dimension of orbits.

Characters and one-parameter subgroups

Definition

Let G be an algebraic group.
A *character* of G is a homomorphism $\chi : G \to \mathbb{G}_m$.
A *one-parameter subgroup* of G is a homomorphism $\theta : \mathbb{G}_m \to G$.

The characters of G form a group under pointwise multiplication: the *character group* $\mathrm{X}^*(G)$. For example, $\mathrm{X}^*(\mathrm{GL}_n) \simeq \mathbb{Z}$ with generator the determinant.

If G is commutative, then the one-parameter subgroups also form a group $\mathrm{X}_*(G)$ under pointwise multiplication. Moreover, we have a pairing

$$\langle -, - \rangle : \mathrm{X}^*(G) \times \mathrm{X}_*(G) \longrightarrow \mathbb{Z}$$

defined by $\chi(\theta(t)) = t^{\langle \chi, \theta \rangle}$ for all $\chi \in \mathrm{X}^*(G)$, $\theta \in \mathrm{X}_*(G)$ and $t \in \mathbb{G}_m$.

Proposition

Let G be a connected algebraic group. Then $\mathrm{X}^*(G)$ is a free abelian group of finite rank.

Representations

Let G be an algebraic group, and V a finite-dimensional vector space. A *representation of G in V* is a homomorphism of algebraic groups $\rho : G \to \mathrm{GL}(V)$. We then say that V is a *G-module*.

If V is identified with $\mathbb{C}^n$, then ρ is a group homomorphism $G \to \mathrm{GL}_n$ such that the matrix coefficients are regular functions on G.

For example, the representations of dimension 1 correspond bijectively to the characters of G via $\chi \in \mathrm{X}^*(G) \mapsto \mathbb{C}_\chi$.

Every representation of G in V defines a linear action of G on V, and hence an action of G on the projective space $\mathbb{P}(V)$ of lines in V. If $V = \mathbb{C}^n$ then $\mathbb{P}(V) = \mathbb{P}^{n-1}$.

Theorem

Let G be a linear algebraic group, and $H \subset G$ a closed subgroup. Then there exists a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ and $x \in \mathbb{P}(V)$ such that $H = G_x$.

This identifies the quotient G/H with the orbit $G \cdot x$ in $\mathbb{P}(V)$. In particular, G/H is a smooth quasi-projective variety. If H is a normal subgroup of G, then one can show that G/H is a linear algebraic group.

Representations (continued)

Definition

Let G be an algebraic group, and V a vector space (not necessarily finite-dimensional).

A *representation* of G in V is a homomorphism of abstract groups $\rho : G \to \mathrm{GL}(V)$ such that V is a union of finite-dimensional G-stable submodules. We then say that V is a *G-module*.

Proposition

Let X be a G-variety. Then the algebra of regular functions $\mathcal{O}(X)$ is a G-module via $\rho(g)(f)(x) = f(g^{-1} \cdot x)$ for all $g \in G$, $f \in \mathcal{O}(X)$ and $x \in X$.

Corollary

Let X be an affine G-variety. Then X is equivariantly isomorphic to a closed G-stable subvariety of a finite-dimensional G-module.

Proof sketch: we may choose finitely many generators $f_1, \dots, f_n$ of the algebra $\mathcal{O}(X)$. Then there exists a finite-dimensional G-submodule $V \subset \mathcal{O}(X)$ containing $f_1, \dots, f_n$. Let $W = V^*$ be the dual G-module, and $\varphi : X \to W$, $x \mapsto (f \mapsto f(x))$. Then φ is a closed equivariant immersion.

Homogeneous spaces

Definition

Let G be an algebraic group. A G-variety X is *homogeneous* if the G-action on X is transitive.

Definition

A *G-homogeneous space* is a pair (X, x), where X is a homogeneous G-variety and $x \in X$. We then say that x is the *base point* of X.

Let (X, x) be a G-homogeneous space. Then the fibers of the orbit map $a_x : G \to X$ are exactly the cosets gG_x, where $g \in G$. So we may view X as the quotient G/G_x; this identifies the base point with the neutral coset.

Examples

The punctured affine space $\mathbb{A}^n \setminus \{0\}$ is homogeneous under the natural action of GL_n.
The projective space $\mathbb{P}^n$ is homogeneous under the action of PGL_{n+1}.
If n is odd, then $\mathbb{P}^n$ is also homogeneous under PSp_{n+1}.

Equivariant embeddings

Definition

Let G be an algebraic group, and H a closed subgroup. An *equivariant embedding* of the homogeneous space G/H is a G-variety X equipped with a point x such that $G_x = H$ and the orbit $G \cdot x$ is open dense in X.

We then say that X is an *almost homogeneous* G-variety, and identify its open orbit with G/H.

Definition

Given two equivariant embeddings (X, x) and (Y, y) of G/H, a *morphism of embeddings* is a G-equivariant morphism $f : X \to Y$ such that $f(x) = y$.

Then f restricts to the identity on the open orbits. In particular, f is unique if it exists.

We would like to classify all (equivariant) embeddings of a given homogeneous space, up to isomorphism of embeddings.

This is a very hard problem in general, as illustrated by the next examples.

The Luna–Vust theory solves this problem for normal embeddings of spherical homogeneous spaces.

Two examples of equivariant embeddings

Example

Take $G = \mathbb{G}_m$ and H the trivial subgroup. Then $G/H = \mathbb{G}_m$ has a unique smooth projective embedding, the projective line $\mathbb{P}^1$ on which G acts by multiplication: $t \cdot [x : y] = [tx : y]$.
Also, G/H has many singular projective embeddings, for example the image C_n of the morphism $\mathbb{P}^1 \to \mathbb{P}^2$, $[x : y] \mapsto [x^n : x^{n-1}y : y^n]$, where $n \geq 2$. The curves C_n are pairwise non-isomorphic.

Example

Take $G = \mathbb{G}_a^n$ and H the trivial subgroup. Then a smooth projective embedding of $G/H = \mathbb{G}_a^n$ is $X = \mathbb{P}^n$ on which G acts by translations:
$(t_1, \dots, t_n) \cdot [x_1 : \cdots : x_{n+1}] = [x_1 + t_1 x_{n+1} : \cdots : x_n + t_n x_{n+1} : x_{n+1}]$.
The hyperplane at infinity $(x_{n+1} = 0) \simeq \mathbb{P}^{n-1}$ consists of G-fixed points. Thus, for any closed (smooth) subvariety Y of this hyperplane, the blow-up $\mathrm{Bl}_Y(X)$ is a (smooth) projective embedding of G/H. Moreover, distinct subvarieties yield non-isomorphic embeddings. So the classification problem for embeddings of $\mathbb{C}^n$ contains that of subvarieties of $\mathbb{P}^{n-1}$.

Linearization of line bundles

Definition

Let G be an algebraic group, X a G-variety, and $\pi : L \to X$ a line bundle. A *G-linearization* of L is a G-action on the variety L which lifts the G-action on X and commutes with the $\mathbb{G}_m$-action on L by multiplication on fibers.

The first assumption means that π is G-equivariant. The second assumption is equivalent to the map $g : L_x \to L_{g \cdot x}$ being linear for all $g \in G$ and $x \in X$.

Lemma

If L is G-linearized, then so are its tensor powers $L^{\otimes n}$ for all $n \in \mathbb{Z}$. Moreover, the space of global sections $\Gamma(X, L)$ is a G-module via $(g \cdot s)(x) = s(g^{-1} \cdot x)$.

For example, let $G = \mathrm{GL}(V)$ where V is a finite-dimensional vector space, and $X = \mathbb{P}(V)$. Then the line bundle $L = \mathcal{O}_{\mathbb{P}(V)}(1)$ is G-linearized, and $\Gamma(X, L)$ is the dual G-module V^*.

If we replace G with $\mathrm{PGL}(V)$, then L admits no linearization. But $L^{\otimes n}$ has a canonical linearization where $n = \dim(V)$, since $L^{\otimes n} = \det(T_{\mathbb{P}(V)})$.

Linearization of line bundles (continued)

We still consider an algebraic group G and a G-variety X.

Lemma

There is a bijective correspondence between the G-morphisms $f : X \to \mathbb{P}(V)$, where V is a G-module, and the pairs (L, φ) where L is a G-linearized line bundle on X and $\varphi : V^* \to \Gamma(X, L)$ is a morphism of G-modules whose image generates L.

The following fundamental result on the existence and uniqueness of linearizations is due to Sumihiro.

Theorem

(i) *If G is connected and X is irreducible, then any two linearizations of a line bundle L on X differ by a character of G.*

(ii) *If in addition G is linear and X is normal, then there exists a positive integer $n = n(G)$ such that $L^{\otimes n}$ is linearizable for any such L.*

In fact, the Picard group of G is finite and one can take for $n(G)$ the order of this group.

A local linearization theorem

A further result of Sumihiro will allow us to use methods from representation theory when studying equivariant embeddings.

Theorem

Let G be a connected linear algebraic group, and X a normal G-variety.

(i) *X is a union of G-stable quasi-projective open subsets.*

(ii) *Every such subset is equivariantly isomorphic to a G-stable subvariety of $\mathbb{P}(V)$ for some finite-dimensional G-module V.*

Note that X is not necessarily a union of G-stable open affine subsets (e.g., take $G = \mathrm{PGL}_{n+1}$ and $X = \mathbb{P}^n$).

The assertion (ii) is obtained by linearizing some positive tensor power of an ample line bundle. It fails for non-linear algebraic groups, e.g. for elliptic curves as they only have trivial representations.

More elaborate examples show that the connectedness and normality assumptions cannot be omitted.

Tori

Definition

A *torus* is an algebraic group isomorphic to $\mathbb{G}_m^n$ for some integer $n \geq 1$.

Lemma

Let T be a torus of dimension n.

(i) $\mathrm{X}^*(T) \simeq \mathbb{Z}^n \simeq \mathrm{X}_*(T)$. *Moreover, the pairing* $\langle -, - \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ *is non-degenerate.*

(ii) *The map* $T \to \mathrm{Hom}_{\mathrm{gp}}(\mathrm{X}^*(T), \mathbb{G}_m)$, $g \mapsto (\chi \mapsto \chi(g))$ *is an isomorphism of algebraic groups.*

(iii) *The map* $\mathbb{G}_m \otimes_{\mathbb{Z}} \mathrm{X}_*(T) \to T$, $t \otimes \theta \mapsto \theta(t)$ *is an isomorphism of algebraic groups.*

Thus, $T \mapsto \mathrm{X}^*(T)$ (resp. $T \mapsto \mathrm{X}_*(T)$) defines a contravariant (resp. covariant) bijective correspondence between tori and free abelian groups of finite rank.

The quotients of a torus T are the tori corresponding to the summands of $\mathrm{X}^*(T)$. Thus, every orbit of a torus is isomorphic to $(\mathbb{C}^*)^m$ as a variety.

Representations and actions of tori

Definition

Let G be an algebraic group, $\chi : G \to \mathbb{G}_m$ a character, and V a G-module. The χ-*weight space* of V is
$V_\chi^{(G)} = V_\chi = \{v \in V \mid g \cdot v = \chi(g)v \text{ for all } g \in G\}$.

Proposition

Let T be a torus, and V as above. Then $V = \bigoplus_{\chi \in \mathrm{X}^*(T)} V_\chi$.

Corollary

Let T be a torus, and X a normal T-variety. Then X is a union of T-stable open affine subsets.

Proof sketch: By Sumihiro's theorem, we may assume that X is a (locally closed) T-stable subvariety of $\mathbb{P}(V)$, where V is a T-module. Then the closure $\bar{X}$ is T-stable, as well as $Y = \bar{X} \setminus X$. Let $x \in X$. Using the above proposition, one shows that there exists a homogeneous polynomial $P \in \mathcal{O}(V)$, which is a T-eigenvector and vanishes identically on Y but not at x. Then $\bar{X} \cap (P \neq 0) = X \cap (P \neq 0)$ is an open affine T-stable subset of X containing x.

Toric varieties

Definition

A *torus embedding* (X, x) is a normal equivariant embedding of a torus T viewed as the homogeneous space $T/\{e\}$. We then say that X is a *toric variety*.

Every toric variety X is a union of T-stable open affine subvarieties, and these are toric as well. This reduces the classification of toric varieties to those of affine toric varieties and their open equivariant immersions.

Let X be an affine toric variety with torus T. Then we may view the algebra of regular functions $\mathcal{O}(X)$ as a subalgebra of $\mathcal{O}(T)$, stable under the linear action of T via multiplication on itself. For this action, we have $\mathcal{O}(T) = \bigoplus_{\chi \in M} \mathbb{C}_\chi$ where $M = \mathrm{X}^*(T)$, and hence $\mathcal{O}(X) = \bigoplus_{\chi \in S_X} \mathbb{C}_\chi$ for a unique subset $S_X \subset M$. Thus, S_X determines X uniquely.

Lemma

A subset $S \subset M$ arises from an affine toric variety if and only if $S = C \cap M$ for a convex cone $C \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ generated by finitely many elements of M which span $M_{\mathbb{R}}$.

Orbits in affine toric varieties

Thus, the affine toric varieties are classified by *rational polyhedral convex cones* $C \subset M_{\mathbb{R}}$ with non-empty interior.

We denote by X_C the affine toric variety associated with cone C, i.e.,

$$\mathcal{O}(X_C) = \bigoplus_{\chi \in C \cap M} \mathbb{C}_\chi.$$

Let F be a *face* of C, i.e., $F = C \cap (f = 0)$ where f is a linear form on $M_{\mathbb{R}}$ such that $f(x) \geq 0$ for any $x \in C$. Then $I_F = \bigoplus_{\chi \in (C \setminus F) \cap M} \mathbb{C}_\chi$ is a prime T-stable ideal of $\mathcal{O}(X_C)$, and $\mathcal{O}(X_C)/I_F = \bigoplus_{\chi \in F \cap M} \mathbb{C}_\chi$ is the algebra of regular functions on a toric variety with torus a quotient of T. This yields:

Proposition

There is a bijective correspondence $F \mapsto Y_F$ between the faces of C and the closed irreducible T-stable subvarieties of X_C. This correspondence preserves dimensions and partial orders by inclusion.

As a consequence, T has only finitely many orbits in X_C and there is a unique closed orbit (corresponding to the smallest face of C).

Affine toric varieties (continued)

Denote by $N = \mathrm{X}_*(T)$ the dual lattice of M, and by $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ the dual vector space of $M_{\mathbb{R}}$.
The classification of affine toric varieties in terms of cones in $M_{\mathbb{R}}$ can be reformulated in terms of cones in $N_{\mathbb{R}}$ as follows: given a closed convex cone $C \subset M_{\mathbb{R}}$, the *dual cone*

$$C^{\vee} = \{n \in N_{\mathbb{R}} \mid \langle n, m \rangle \geq 0 \text{ for all } m \in C\}$$

is a closed convex cone in $N_{\mathbb{R}}$.
We have $C = (C^{\vee})^{\vee}$; in particular, the data of C and $C^{\vee}$ are equivalent.
Moreover, C is rational polyhedral iff so is $C^{\vee}$. Also, C spans $M_{\mathbb{R}}$ iff $C^{\vee}$ is *strictly convex*, i.e., it contains no line. This yields:

Proposition

The affine toric varieties are classified by the strictly convex rational polyhedral cones in $N_{\mathbb{R}}$.

We denote by σ such a cone, and by X_{σ} the corresponding affine torus embedding with base point x_{σ}.

Local structure of toric varieties

Consider an affine toric variety $X = X_\sigma$. Then the T-orbits in X are in bijection with the faces τ of σ as follows: let $C = \sigma^\vee$, then $F = C \cap \tau^\perp$ is a face of C and hence corresponds to a closed irreducible T-stable subvariety Y_F of X. We denote by $\mathcal{O}_\tau$ the open T-orbit in Y_F. The map $\tau \mapsto \mathcal{O}_\tau$ is the desired bijection. It is decreasing with respect to the partial order on orbits by inclusion of closures, and satisfies $\dim(\mathcal{O}_\tau) = \mathrm{codim}(\tau)$.

Proposition

With the above notation, there exists a unique open T-stable subset $U_\tau \subset X$ such that $\mathcal{O}_\tau$ is closed in U_τ.
Moreover, U_τ is affine and there exists a T-equivariant fibration $f_\tau : U_\tau \to \mathcal{O}_\tau$ with fiber an affine toric variety (under a subtorus of T) having a fixed point.

This reduces the structure of a toric variety along an orbit to that of an affine toric variety with a fixed point. These correspond to cones with non-empty interior.

Classification of toric varieties

Recall that every toric variety is obtained by gluing affine toric varieties along open affine T-stable subsets. These can be described in combinatorial terms:

Lemma

Let σ, τ be strictly convex rational polyhedral cones in $N_{\mathbb{R}}$. Then there exists a morphism of embeddings $f : (X_\tau, x_\tau) \to (X_\sigma, x_\sigma)$ if and only if $\tau \subset \sigma$. Moreover, f is an open immersion if and only if τ is a face of σ.

Definition

A *fan* in $N_{\mathbb{R}}$ is a non-empty finite set Σ of strictly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying the following:

(i) Every face of a cone in Σ lies in Σ as well.

(ii) Any two cones in Σ intersect along a face of both.

Theorem

The toric T-varieties are classified by the fans in $N_{\mathbb{R}}$.

More toric geometry

Let Σ be a fan, and X_Σ the associated toric variety. The T-orbits in X_Σ are in bijective correspondence with the cones of Σ via $\tau \mapsto \mathcal{O}_\tau$, and we have $\dim(\mathcal{O}_\tau) = \mathrm{codim}(\tau)$. Thus, the orbit closures of codimension 1 are exactly the *toric divisors* $D_\tau = \bar{\mathcal{O}}_\tau$, where τ is a ray of Σ.

Many geometric properties of X_Σ can be read off Σ. Here is a sample:

Proposition

(i) *X_Σ is affine iff Σ consists of all faces of a strictly convex rational polyhedral cone σ.*

(ii) *X_Σ is complete iff* $\mathrm{Supp}(\Sigma) = N_\mathbb{R}$*, where* $\mathrm{Supp}(\Sigma)$ *denotes the union of the cones of Σ.*

(iii) *X_Σ is smooth iff every cone of Σ is generated by part of a basis of N. Then every orbit closure is smooth as well, and is a transversal intersection of toric divisors.*

For (iii), one shows that a smooth affine toric variety having a fixed point is isomorphic to $\mathbb{C}^n$ on which $T \simeq \mathbb{G}_m^n$ acts by coordinatewise multiplication.

The logarithmic tangent bundle

Let X be a smooth variety, and $D \subset X$ a *divisor with strict normal crossings*, i.e. its irreducible components $D_1, \ldots, D_m$ are smooth divisors which intersect transversally.

We identify vector bundles on X with their sheaves of local sections. Then the tangent bundle T_X is identified with the sheaf of derivations of the structure sheaf $\mathcal{O}_X$. The *logarithmic tangent bundle* $T_X(-\log D)$ is the subsheaf of T_X which preserves the ideal sheaf of D.

For any $x \in X$, we may choose local coordinates $z_1, \ldots, z_n$ such that $D = (z_1 \cdots z_p = 0)$ in a neighborhood U_x of x. Then $T_X(-\log D)$ is generated in U_x by $z_1\partial/\partial z_1, \ldots, z_p\partial/\partial z_p, \partial/\partial z_{p+1}, \ldots, \partial/\partial z_n$.

It follows that $T_X(-\log D)$ is a vector bundle of rank $n = \dim(X)$, and we have an exact sequence of sheaves

$$0 \longrightarrow T_X(-\log D) \longrightarrow T_X \longrightarrow \bigoplus_{i=1}^{m} N_{D_i/X} \longrightarrow 0,$$

where $N_{D_i/X} = \mathcal{O}_{D_i}(D_i)$ denotes the normal bundle. Thus, the canonical divisor K_X satisfies $\det T_X(-\log D) = \mathcal{O}_X(-K_X - D)$.

The canonical divisor of a toric variety

Keep the above assumptions and assume in addition that an algebraic group G acts on X and stabilizes D. Then the Lie algebra $\mathrm{Lie}(G)$ acts on X by global vector fields which preserve D. This yields a map

$$\mathrm{op}_{X,D} : \mathcal{O}_X \otimes \mathrm{Lie}(G) \longrightarrow T_X(-\log D).$$

Proposition

Let X be a smooth toric variety with torus T and boundary $\partial X = X \setminus T$ (the union of the toric divisors). Then $T_X(-\log \partial X) \simeq \mathcal{O}_X \otimes \mathrm{Lie}(T)$ via the above map. In particular, $K_X = -\partial X$.

The toric divisors are indexed by the rays of the fan Σ of X. Therefore, $K_X = -\sum_{\tau \in \Sigma(1)} D_\tau$, where $\Sigma(1)$ denotes the set of rays. This still holds for a possibly singular toric variety X, by considering the smooth locus.

The above proposition has a converse due to Winkelmann:

Theorem

Let X be a smooth complete rational variety, and $D \subset X$ a divisor with smooth normal crossings. If the vector bundle $T_X(-\log D)$ is trivial, then X is a toric variety with boundary D.

The divisor class group of a toric variety

Let X be a toric variety with fan Σ. Every ray $\sigma \in \Sigma(1)$ is generated by a unique primitive vector $n_\sigma \in N$.

Proposition

The divisor class group of X lies in an exact sequence

$$0 \longrightarrow M \cap \mathrm{Supp}(\Sigma)^\perp \longrightarrow M \xrightarrow{\mathrm{div}} \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X) \longrightarrow 0,$$

where $\mathrm{div}(m) = \sum_{\sigma\in\Sigma(1)} \langle m, n_\sigma \rangle D_\sigma$ for any $m \in M$.

Lemma

Let X be an affine toric variety with cone σ, and D a Cartier divisor on X with T-stable support. Then $D = \mathrm{div}(m)$ for some $m \in M$, uniquely determined modulo $M \cap \sigma^\perp$.

Proof sketch: Choose a linearization of the line bundle $L = \mathcal{O}_X(D)$. The restriction to the closed orbit $\mathcal{O}_\tau$ yields a surjective map of T-modules $\Gamma(X, L) \to \Gamma(\mathcal{O}_\tau, L)$. Choose a T-eigenvector in $\Gamma(\mathcal{O}_\tau, L)$ and lift it to a T-eigenvector $s \in \Gamma(X, L)$. Then the zero locus of s is closed, T-stable and does not meet $\mathcal{O}_\tau$, hence is empty. Thus, L is trivial as a line bundle, and s is identified with a T-eigenvector in $\mathcal{O}(T)$, i.e., a character $m \in M$.

The Picard group of a toric variety

Let X be a toric variety with fan Σ, and D a Cartier divisor on X. Replacing D with a linearly equivalent divisor, we may assume that its support is T-stable. For any $\sigma \in \Sigma$, we then have $D \cap U_\sigma = \mathrm{div}(m_\sigma)$ where $m_\sigma \in M/M \cap \sigma^\perp$. We may view m_σ as a linear map on σ.

Theorem

(i) *With the above notation, the m_σ glue to a piecewise linear function f on Σ.*

(ii) *D is generated by its global sections (resp. ample) iff f is convex (resp. strictly convex).*

(iii) *The Picard group of X lies in an exact sequence*

$$0 \longrightarrow M \cap \mathrm{Supp}(\Sigma)^\perp \longrightarrow M \xrightarrow{\mathrm{div}} \mathrm{PL}(\Sigma) \xrightarrow{u} \mathrm{Pic}(X) \longrightarrow 0,$$

where $\mathrm{PL}(\Sigma)$ denotes the group of piecewise linear functions on Σ which take integral values at points of $N \cap \mathrm{Supp}(\Sigma)$, and $u(f) = (\mathrm{div}_{U_\sigma}(f|_\sigma))$ for any $f \in \mathrm{PL}(\Sigma)$.

Projective toric varieties and polytopes

Let X be a projective toric variety of dimension n with fan Σ. Then the T-fixed points in X are indexed by the subset of n-dimensional cones $\Sigma(n) \subset \Sigma$. Moreover, Σ consists of the cones in $\Sigma(n)$ and their faces.

Next, let $\pi : L \to X$ an ample T-linearized line bundle. For any T-fixed point x_σ where $\sigma \in \Sigma(n)$, the fiber L_{x_σ} is equipped with a linear action of T via a character $m_\sigma \in M$. The piecewise linear function on Σ associated with L can be identified with $(m_\sigma)_{\sigma \in \Sigma(n)}$.

Denote by $P = P(X, L)$ the convex hull in $M_{\mathbb{R}}$ of the m_σ, where $\sigma \in \Sigma(n)$. Then P is a *convex lattice polytope* in $M_{\mathbb{R}}$, i.e., a convex polytope with vertices in the lattice M.

Proposition

For any integer $k \geq 1$, we have an isomorphism of T-modules

$$\Gamma(X, L^{\otimes k}) \simeq \bigoplus_{m \in kP \cap M} \mathbb{C}_m.$$

Projective toric varieties and polytopes (continued)

The map $(X, L) \mapsto P(X, L)$ defines a bijective correspondence between projective toric varieties equipped with an ample T-linearized line bundle, and convex lattice polytopes in $M_{\mathbb{R}}$ with non-empty interior. The inverse correspondence is given by $P \mapsto (X_P, L_P)$, where X_P is the Proj of the graded T-stable subalgebra

$$\bigoplus_{m \in M, k \in \mathbb{Z}_{\geq 0},\, m \in kP} \mathbb{C}_m\, t^k \subset \mathcal{O}(T)[t] = \bigoplus_{m \in M, k \in \mathbb{Z}_{\geq 0}} \mathbb{C}_m\, t^k$$

and L_P is the twisting sheaf $\mathcal{O}(1)$.

Proposition

The map $F \mapsto X_F$ yields a bijective correspondence between faces of P and orbit closures in X, which preserves dimensions and inclusions. The cone in $N_{\mathbb{R}}$ associated with X_F is the dual of the cone in $M_{\mathbb{R}}$ generated by $-F + P$.

We say that the fan of X is the *normal fan* of the polytope P.

Fano toric varieties

Recall that every line bundle on X admits a T-linearization, and any two such linearizations differ by a character. So the above polytope $P(X, L)$ is uniquely determined by the line bundle L up to translation in M.

If $L = \mathcal{O}_X(\sum_{\tau \in \Sigma(1)} a_\tau D_\tau)$ with its natural T-linearization, then $P(X, L)$ is defined by the linear inequalities $\langle m, n_\tau \rangle + a_\tau \geq 0$, where $\tau \in \Sigma(1)$.

Taking $L = \mathcal{O}_X(-K_X) = \mathcal{O}_X(\sum_{\tau \in \Sigma(1)} D_\tau)$, this yields:

Proposition

Let X be a smooth projective toric variety with fan Σ. Then X is Fano iff the linear inequalities $\langle m, n_\tau \rangle + 1 \geq 0$ ($\tau \in \Sigma(1)$) define a convex lattice polytope $P \subset M_\mathbb{R}$ with normal fan Σ.

Example

Let $T = \mathbb{G}_m^n$ acting on $X = \mathbb{P}^n$ by pointwise multiplication. Then the rays are generated by $e_1, \ldots, e_n, -e_1 - \cdots - e_n$, where $(e_1, \ldots, e_n)$ is the standard basis of $N \simeq \mathbb{Z}^n$. So
$P = (x_1 \geq -1, \ldots, x_n \geq -1, x_1 + \cdots + x_n \leq 1) = (-1, \ldots, -1) + (n+1)S$, where S denotes the standard simplex.

Some references for Lecture 1

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