Rational actions of algebraic groups
Lecture 4: Rational actions and their regularization

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Rational actions

Let G be an algebraic group, and X a variety.

Definition

A **rational action** of G on X is a rational map

$$a : G \times X \dashrightarrow X, \quad (g, x) \longmapsto g \cdot x$$

that satisfies the following two properties:

(i) The rational map

$$(\mathrm{pr}_1, a) : G \times X \dashrightarrow G \times X, \quad (g, x) \longmapsto (g, g \cdot x)$$

is dominant.

(ii) The two rational maps

$$G \times X \times X \dashrightarrow X, \qquad (g, h, x) \longmapsto g \cdot (h \cdot x), \quad gh \cdot x$$

are equal.

Rational actions (continued)

Note that a is dominant, since it is the composition $\mathrm{pr}_2 \circ (\mathrm{pr}_1, a)$.
Also, the rational map

$$G \times X \times X \dashrightarrow X, \quad (g, h, x) \longmapsto gh \cdot x$$

is the composition

$$G \times G \times X \xrightarrow{m \times \mathrm{id}_X} G \times X \overset{a}{\dashrightarrow} X$$

(where $m : G \times G \to G$ denotes the multiplication), and hence makes sense as $m \times \mathrm{id}_X$ is dominant.

Likewise, the rational map

$$G \times X \times X \dashrightarrow X, \quad (g, h, x) \longmapsto g \cdot (h \cdot x)$$

is the composition

$$G \times G \times X \overset{\mathrm{id}_G \times a}{\dashrightarrow} G \times X \overset{a}{\dashrightarrow} X,$$

and hence makes sense as well, since a and $\mathrm{id}_G \times a$ are dominant.

First properties

Given a rational action $a : G \times X \dashrightarrow X$, we denote its domain of definition by $V = \mathrm{Dom}(a) \subset G \times X$.

For any $g \in G$ and $x \in X$, we say that $g \cdot x$ **is defined** if $(g, x) \in V$.

The condition (ii) in the definition of rational action is equivalent to:

(iii) Let $g, h \in G$ and $x \in X$. If $h \cdot x$ and $g \cdot (h \cdot x)$ are defined, then $gh \cdot x$ is defined and equals $g \cdot (h \cdot x)$.

The intersection $V \cap (\{g\} \times X)$ is identified with an open subset $V_g \subset X$, and the morphism $a : V \to X$ yields a morphism

$$a_g : V_g \longrightarrow X, \quad x \longmapsto g \cdot x.$$

Lemma

- *V_g is dense in X and a_g is dominant.*
- *a_e is the inclusion map $V_e \to X$.*

First properties (continued)

Proof.

Consider the rational map

$$b_g : G \times X \dashrightarrow X, \quad (h, x) \longmapsto gh^{-1} \cdot (h \cdot x).$$

This is the composition $a \circ (u \times \mathrm{id}_X) \circ (\mathrm{pr}_1, a)$, where u denotes the automorphism $h \mapsto gh^{-1}$ of G. Since a and (pr_1, a) are dominant and $u \times \mathrm{id}_X$ is an isomorphism, this composition is defined, on some dense open subset $W \subset G \times X$. For any $(h, x) \in V \times W$, we have by (iii) that $g \cdot x$ is defined; moreover, $b_g = a_g$ as rational maps. This yields the first assertion.

By (iii) again, we have $e \cdot x = e \cdot (e \cdot x)$ for any $x \in V_e \cap e^{-1}(V_e)$. So $y = e \cdot y$ for all y in a dense open subset of V_e. This implies the second assertion. □

Proposition

- *The rational map* $(\mathrm{id}_G, a) : G \times X \dashrightarrow G \times X$ *is birational.*
- *For any* $g \in G$*, the rational map* $a_g : X \dashrightarrow X$ *is birational.*

First properties (continued)

Proof sketch. Let $\varphi = (\mathrm{id}_G, a) : (g, x) \mapsto (g, g \cdot x)$ and consider the rational map

$$\psi : G \times X \dashrightarrow G \times X, \quad (g, x) \longmapsto (g, g^{-1} \cdot x).$$

Then ψ is the composition

$$G \times X \xrightarrow{(i, \mathrm{id}_X)} G \times X \overset{(\mathrm{id}_G, a)}{\dashrightarrow} G \times X \xrightarrow{(i, \mathrm{id}_X)} G \times X,$$

where $i : G \to G$ denotes the inverse map. Since (i, id_X) is an isomorphism and (id_G, a) is dominant, ψ is dominant as well. Moreover, one may check by using (iii) that the rational maps φ and ψ are mutually inverse.

Likewise, a_g and $a_{g^{-1}}$ are mutually inverse. □

Thus, a rational action of G on X may be viewed as a rational map $a : G \times X \dashrightarrow X$ such that the rational map $a_g : G \dashrightarrow X$, $x \mapsto a(g, x)$ is defined and birational for any $g \in G$.

The regularization theorem

Definition

Let G be an algebraic group acting rationally on two varieties X, Y. We say that a dominant rational map $f : X \dashrightarrow Y$ is G-**equivariant** if the rational maps $a : G \times X \dashrightarrow X$, $b : G \times Y \dashrightarrow Y$ satisfy $f \circ a = b \circ (\mathrm{id}_G \times f)$.

The above compositions of rational maps make sense, since a and $\mathrm{id}_G \times f$ are dominant. Also, note that f is equivariant if and only if $f(g \cdot x) = g \cdot f(x)$ whenever both sides are defined.

Theorem

Let X be an irreducible variety equipped with a rational action of G. Then there exists a projective G-variety Y and a birational G-equivariant map $f : X \dashrightarrow Y$.

This is Weil's regularization theorem (1955), with the additional assertion that Y may be taken projective. This assertion was proved in 2022 after many partial results were obtained.

The regularization theorem may be reformulated in terms of algebraic subgroups of groups of birational transformations, in the following sense:

Definition

- A rational action a of G on X is **faithful** if $a_g \neq \mathrm{id}$ for any $g \in G \setminus \{e\}$.
- An **algebraic subgroup of** $\mathrm{Bir}(X)$ is the image of the group homomorphism $H \to \mathrm{Bir}(X)$, $h \mapsto b_h$, for some faithful rational action b of some algebraic group H on X.

Theorem

Let X be an irreducible variety, and $G \subset \mathrm{Bir}(X)$ an algebraic subgroup. Then G is conjugate to a closed subgroup of $\mathrm{Aut}(Y)$ for some projective model Y of X.

If X is a curve, then it has a unique smooth projective model Y. Moreover, the equality $\mathrm{Bir}(X) = \mathrm{Aut}(Y)$ holds in view of the anti-equivalence of categories between smooth projective curves and function fields of one variable. The corollary provides a slightly stronger version of this equality.

Connected algebraic subgroups of groups of birational transformations of surfaces

Let X be a projective surface, $\varphi : Y \to X$ the equivariant resolution of singularities constructed in Lecture 3, and $\psi : Y \to Z$ a contraction to a relatively minimal model.

Recall from Lecture 3 that φ induces an injective homomorphism of locally algebraic groups

$$\varphi^* : \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(Y),$$

and ψ induces an injective homomorphism of algebraic groups

$$\psi_* : \mathrm{Aut}^0(Y) \longrightarrow \mathrm{Aut}^0(Z).$$

Thus, we may identify $\mathrm{Aut}^0(X)$ to a closed subgroup of $\mathrm{Aut}^0(Z)$. Together with the regularization theorem, this shows:

Proposition

Every connected algebraic subgroup of $\mathrm{Bir}(X)$ *is conjugate to a subgroup of* $\mathrm{Aut}^0(Z)$ *for some relatively minimal model* Z *as above.*

Combining the above proposition with the description of the relatively minimal rational surfaces (see Lecture 3), we recover a result due to Enriques (1893).

Theorem

- *Every connected algebraic subgroup of* Cr_2 *is contained in a maximal such subgroup.*
- *Every maximal connected algebraic subgroup of* Cr_2 *is conjugate to* PGL_3, $\mathrm{PGL}_2 \times \mathrm{PGL}_2$, *or* $\mathrm{Aut}(\mathbb{F}_n)$ *for some* $n \geq 2$.

From now on, we assume that X is a non-rational projective surface. We will present part of the results of a paper by Fong (2024) about the connected algebraic subgroups of $\mathrm{Bir}(X)$. We will need the following:

Definition

We say that X is **birationally ruled** if it is birationally isomorphic to $C \times \mathbb{P}^1$ for some curve C.

We may of course assume that C is smooth and projective. One then shows that C is uniquely determined by X, and has genus $g \geq 1$.

We still consider a non-rational projective surface X, and recall a result from the theory of surfaces.

Theorem

- *If X is not birationally ruled, then it has a unique relatively minimal model Z. Then Z is called a* **minimal model**.
- *If X is birationally ruled, then its relatively minimal models are exactly the ruled surfaces $\mathbb{P}(E)$, where E is a vector bundle of rank 2 over C.*

Also, recall the following:

Definition

An **abelian variety** is a complete connected algebraic group.

One shows that every abelian variety is a commutative group and a projective variety.

Theorem

If X is not birationally ruled, then every non-trivial connected algebraic subgroup $G \subset \mathrm{Bir}(X)$ is either an elliptic curve or an abelian surface. In the latter case, we have $X \simeq G$.

Proof of the theorem

We will need two preliminary results:

Lemma

Let Z be a minimal model. Then $\mathrm{Aut}^0(Z)$ is an abelian variety.

Proof of the lemma. Let $G = \mathrm{Aut}^0(Z)$. By Chevalley's structure theorem, there exists a unique exact sequence of algebraic groups

$$1 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where L is connected and linear, and A is an abelian variety. If L is non-trivial, then it contains a non-trivial Borel subgroup, i.e., a maximal connected solvable subgroup. Using the structure of connected solvable linear algebraic groups, it follows that L contains a copy H of $\mathbb{G}_a$ or $\mathbb{G}_m$. Moreover, by results of Rosenlicht, Z contains a dense open H-stable subset, isomorphic to $H \times Y$ for some curve Y. So X is birationally ruled, a contradiction. □

Proposition

Let G be an algebraic group acting faithfully on an irreducible variety X. Then the isotropy group G_x is linear for any $x \in X$.

Proof sketch of the proposition.

The group G acts on the function field $\mathbb{C}(X)$ via $(g \cdot f)(x) = f(g^{-1} \cdot x)$. The local ring $\mathcal{O}_{X,x}$ is stable by the action of G_x as well as the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ and every positive power $\mathfrak{m}_x^n$. This yields a linear representation of G_x in the quotient $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$, a finite-dimensional vector space over $\mathbb{C}$. One may check that the resulting map $G_x \to \mathrm{GL}(\mathcal{O}_{X,x}/\mathfrak{m}_x^n)$ is a homorphism of algebraic groups. Thus, its kernel K_n is a closed subgroup of G_x. So the decreasing sequence (K_n) stabilizes, say at K. Then K acts trivially on the local ring $\mathcal{O}_{X,x}$, and hence on its fraction field $\mathbb{C}(X)$. Since G acts faithfully on X, it follows that K is trivial. Thus, G_x is isomorphic to a closed subgroup of $\mathrm{GL}(\mathcal{O}_{X,x}/\mathfrak{m}_x^n)$ for $n \gg 0$. □

Proof of the theorem.

By the above results, G is an abelian variety. Let $x \in X$, then the isotropy group G_x is linear, and hence an affine variety. Also, G_x is a projective variety, since it is closed in G. Thus, G_x is finite, and hence $\dim(G) = \dim(G \cdot x) \leq \dim(X) = 2$. If equality holds, then the orbit $G \cdot x$ is open in X. But $G \cdot x$ is also closed in X as G is complete. Since $G \cdot x \simeq G/G_x$ and G is commutative and acts faithfully on X, this forces $G_x = \{e\}$ and $G \simeq X$. □

Ruled surfaces

We recall some structure results on ruled surfaces

$$f : X = \mathbb{P}(E) \longrightarrow C,$$

where C is a smooth projective curve of genus g, and E is a vector bundle of rank 2 on C. Here $\mathbb{P}(E) = (E \setminus X)/\mathbb{G}_m$ where X is identified with the zero section of E, and $\mathbb{G}_m$ acts by scalar multiplication on fibers.

Given another vector bundle F of rank 2 over C, the surfaces $\mathbb{P}(E)$ and $\mathbb{P}(F)$ are isomorphic as ruled surfaces over C if and only if $F \simeq E \otimes L$ for some line bundle L on C.

In addition, we may normalize E so that $H^0(X, E^*) \neq 0$ but $H^0(X, E^* \otimes M) = 0$ for any line bundle M on C such that $\deg(M) < 0$. Then one of the following cases occurs:

- $E = \mathcal{O}_X \oplus L$, where L is a line bundle such that $\deg(L) \geq 0$.
- E is indecomposable.

The latter case does not occur if and only if $g = 0$, i.e., $C \simeq \mathbb{P}^1$; then we get back the Hirzebruch surfaces. From now on, we assume that $g \geq 1$.

Connected automorphism groups of ruled surfaces

We still consider a ruled surface $f : X = \mathbb{P}(E) \to C$.
A detailed description of the full automorphism group $\mathrm{Aut}(X)$ is due to Maruyama (1971). Here we will obtain some qualitative information on the neutral component $\mathrm{Aut}^0(X)$.

The contraction f yields a homomorphism of algebraic groups

$$f_* : \mathrm{Aut}^0(X) \longrightarrow \mathrm{Aut}^0(C).$$

Also, recall that $\mathrm{Aut}^0(C)$ is trivial if $g \geq 2$. If $g = 1$, then C is an elliptic curve, and hence $\mathrm{Aut}^0(C) = C$.

Proposition

We have an isomorphism of algebraic groups

$$\mathrm{Ker}(f_*)^0 \simeq \mathrm{Aut}_C(E)/\mathbb{G}_m,$$

where $\mathrm{Aut}_C(E)$ *denotes the automorphism group of the vector bundle* E*, and* $\mathbb{G}_m$ *the central subgroup of scalar multiplications on fibers. Moreover,* $\mathrm{Aut}_C(E)$ *is a connected linear algebraic group.*

Proof. Note that $\mathrm{End}_C(E) = H^0(C, E^* \otimes E)$ is a finite-dimensional algebra over $\mathbb{C}$. Moreover, $\mathrm{Aut}_C(E)$ is the group of invertible elements of this algebra, and hence is a principal open subset of the corresponding affine space. As a consequence, $\mathrm{Aut}_C(E)$ is a connected linear algebraic group.

By a result of Grothendieck, we have an exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Aut}_C(E) \longrightarrow \mathrm{Aut}_C(\mathbb{P}(E)) \longrightarrow \Gamma \longrightarrow 1,$$

where $\Gamma \subset \mathrm{Pic}(C)$ denotes the group of isomorphism classes of line bundles L such that $E \otimes L \simeq E$. Taking determinants, it follows that $L^{\otimes 2} \simeq \mathcal{O}_C$, i.e., $[L] \in \mathrm{Pic}(C)$ is 2-torsion. In particular, $[L] \in J(C)$, where the Jacobian $J(C) = \mathrm{Pic}^0(C)$ is an abelian variety. Since the n-torsion subgroup of an abelian variety is finite for any $n \geq 2$, we get an exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Aut}_C(E) \longrightarrow \mathrm{Aut}_C(\mathbb{P}(E))^0 \longrightarrow 1.$$

This yields our statement, since $\mathrm{Ker}(f_*)^0 = \mathrm{Aut}_C(\mathbb{P}(E))^0$. □

Corollary

If $g \geq 2$ then $\mathrm{Aut}^0(X)$ is linear.

We still consider a ruled surface $f : X = \mathbb{P}(E) \to C$, and assume that $g = 1$, i.e., C is an elliptic curve. We first treat the case where $E = \mathcal{O}_C \oplus L$ is decomposable.

Proposition

- *If* $\deg(L) \neq 0$, *then* $\mathrm{Aut}^0(X) = \mathbb{G}_m$.
- *If* $\deg(L) = 0$ *and* L *is non-trivial, then* $\mathrm{Aut}^0(X)$ *is commutative and lies in a non-split exact sequence of algebraic groups* $0 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Aut}^0(X) \xrightarrow{f_*} C \longrightarrow 0.$
- *If* $L = \mathcal{O}_X$ *then* $\mathrm{Aut}^0(X) = C \times \mathrm{PGL}_2$.

Next, we treat the case where E is indecomposable and normalized. By a result of Atiyah (1957), we then have either $E \simeq E_0$ or $E \simeq E_1$, where E_0, E_1 lie in non-split extensions

$$0 \to \mathcal{O}_C \to E_0 \to \mathcal{O}_C \to 0, \quad 0 \to \mathcal{O}_C \to E_1 \to \mathcal{O}_C(c) \to 0$$

with $c \in C$.

Proposition

- *The group* $\mathrm{Aut}^0(\mathbb{P}(E_0))$ *is commutative and lies in a non-split exact sequence of algebraic groups*
  $0 \longrightarrow \mathbb{G}_a \longrightarrow \mathrm{Aut}^0(\mathbb{P}(E_0)) \longrightarrow C \longrightarrow 0.$
- *We have an isomorphism* $C \simeq \mathrm{Aut}^0(\mathbb{P}(E_1))$ *which identifies* $f_* : \mathrm{Aut}^0(\mathbb{P}(E_1)) \to C$ *with the multiplication by* 2 *in* C.

These two propositions are due to Maruyama again. The next theorem was obtained by Fong:

Theorem

Let C *be a smooth projective curve of genus* $g \geq 1$. *Then the maximal connected algebraic subgroups of* $\mathrm{Bir}(C \times \mathbb{P}^1)$ *are exactly the conjugates of the groups* $\mathrm{Aut}^0(\mathbb{P}(E))$, *where*

- $E = E_0, E_1$, *or* $\mathcal{O}_C \oplus L$ *for a line bundle* L *of degree* 0, *if* $g = 1$,
- $E = \mathcal{O}_C \oplus \mathcal{O}_C$ *if* $g \geq 2$ *(then* $\mathrm{Aut}^0(\mathbb{P}(E)) = \mathrm{PGL}_2$*).*

Moreover, every connected algebraic subgroup of $\mathrm{Bir}(C \times \mathbb{P}^1)$ *is contained in a maximal one if* $g = 1$, *but not if* $g \geq 2$.