Rational actions of algebraic groups
Lecture 3: Automorphism groups of projective varieties

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Locally algebraic groups

As we will see in this lecture, the automorphism groups of projective algebraic varieties are generally not algebraic groups. But they have a natural structure of "locally algebraic groups", or "algebraic groups with possibly infinitely many components", in the following sense:

Definition

A topological group G is **locally algebraic** if it is the disjoint union of open and closed subspaces G_i ($i \in I$) such that:

- each G_i is an irreducible variety,
- for all $i, j \in I$, there exists $k \in I$ such that the multiplication map induces a morphism of varieties $G_i \times G_j \to G_k$, and
- for any $i \in I$, there exists $j \in J$ such that the inverse map induces a morphism $G_i \to G_j$.

Every algebraic group G is locally algebraic, with the G_i being the (finitely many) cosets gG^0 for $g \in G$.
Every discrete topological group is locally algebraic as well.

Locally algebraic groups (continued)

The construction and properties of the neutral component of an algebraic group extend readily to the setting of locally algebraic groups:

Proposition

Let G be a locally algebraic group, and $G^0 \subset G$ the connected component containing the neutral element e. Then G^0 is a connected algebraic group, normal in G. Moreover, we have an exact sequence of topological groups

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \pi_0(G) \longrightarrow 1,$$

where the group of components $\pi_0(G) = G/G^0$ is discrete.

We now reconsider two examples of subgroups of Cr_2 from Lecture 1.

- The group U of transformations of the form $(X, Y) \mapsto (X, Y + P(X))$ is not locally algebraic. Indeed, U is the increasing union of algebraic subgroups $U_n \simeq \mathbb{G}_a^{n+1}$.
- The group M of monomial transformations, of the form $(X, Y) \mapsto (aX^p Y^q, bX^r Y^s)$, is locally algebraic and not algebraic. We have $\mathrm{M}^0 = \mathbb{G}_m^2$ and $\pi_0(\mathrm{M}) = \mathrm{GL}_2(\mathbb{Z})$.

Families of automorphisms

Let X be a projective variety. Our aim is to equip the automorphism group $\mathrm{Aut}(X)$ with the structure of a locally algebraic group. For this, we introduce a notion of families of automorphisms parameterized by a variety S, which may be thought of morphisms $S \to \mathrm{Aut}(X)$.

Definition

Let X, S be varieties. A **family of automorphisms of X parameterized by S** is a morphism of varieties $f : S \times X \to X$ such that the map

$$(\mathrm{pr}_1, f) : S \times X \longrightarrow S \times X, \quad (s, x) \longmapsto (s, f(x, s))$$

is an automorphism of $S \times X$.

The families of automorphisms of X parameterized by S (viewed as automorphisms of $S \times X$) form a group, denoted by $\mathrm{Aut}(S \times X/S)$.

For example, every action a of an algebraic group G yields a family of automorphisms parameterized by G. The inverse of $(\mathrm{pr}_1, a) : (g, x) \mapsto (g, g \cdot x)$ is the map $(g, x) \mapsto (g, g^{-1} \cdot x)$.

Families of automorphisms (continued)

Given a family of automorphisms $f : S \times X \to X$ and a morphism of varieties $u : T \to S$, the **pull-back** $u^*(f)$ is the morphism

$$T \times X \longrightarrow X, \quad (t, x) \longmapsto f(u(t), x).$$

This is a family of automorphisms of X parameterized by T.
The pull-back yields a group homomorphism

$$u^* : \mathrm{Aut}(S \times X/S) \longrightarrow \mathrm{Aut}(T \times X/T).$$

We say that a family of automorphisms $f : S \times X \to X$ is **universal** if for any family of automorphisms $g : T \times X \to X$, there exists a unique morphism $u : T \to S$ such that $g = u^*(f)$.

Such a universal family is unique up to unique isomorphism, if it exists.
A natural candidate is the action map

$$a : \mathrm{Aut}(X) \times X \longrightarrow X, \quad (g, x) \longmapsto g(x).$$

The automorphism group of a projective variety

Theorem

Let X be a projective variety. Then $\mathrm{Aut}(X)$ has a unique structure of locally algebraic group such that the action map yields the universal family of automorphisms.

As a consequence, we have an exact sequence

$$1 \longrightarrow \mathrm{Aut}^0(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow \pi_0 \,\mathrm{Aut}(X) \longrightarrow 1.$$

where $\mathrm{Aut}^0(X)$ is a connected algebraic group, and $\pi_0(X)$ is discrete.

It is easy to show that $\pi_0 \,\mathrm{Aut}(X)$ is countable. We will see in this lecture that $\pi_0(X)$ is finite if X is a smooth curve (then $\mathrm{Aut}(X)$ is an algebraic group), but is infinite for certain smooth surfaces.

Further examples (much more elaborate and quite recent) show that the group $\pi_0 \,\mathrm{Aut}(X)$ is not finitely generated for some classes of smooth surfaces X.

Outline of proof of the theorem

The idea is to encode automorphisms by their graphs, which are closed subvarieties of $X \times X$, and to use a parameter space for these subvarieties such as the Chow variety or the Hilbert scheme.

The **Chow variety** parameterizes the effective cycles of prescribed dimension n and degree d of a subvariety $Y \subset \mathbb{P}^N$. More specifically, there is a projective variety $\mathrm{Chow}_{n,d}(Y)$ whose points are the effective cycles $\sum_i a_i Z_i$, where the a_i are non-negative integers and the Z_i are irreducible subvarieties of Y such that $\dim(Z_i) = n$ for all i, and $\sum_i a_i \deg(Z_i) = d$.

In addition, there is an open subset $\mathrm{Chow}^{\mathrm{irr}}_{n,d}(Y) \subset \mathrm{Chow}_{n,d}(Y)$ whose points are the irreducible subvarieties $Z \subset Y$ of dimension n and degree d. It has a universal family $\mathrm{Univ}_{n,d}(Y) \subset \mathrm{Chow}^{\mathrm{irr}}_{n,d}(Y) \times Y$ consisting of the pairs (Z, y) such that $y \in Z$.

Taking the disjoint union of the $\mathrm{Chow}_{n,d}(Y)$ over all n, d, we obtain the Chow variety $\mathrm{Chow}(Y)$, a "variety with infinitely many components". It comes with an open subset $\mathrm{Chow}^{\mathrm{irr}}(Y)$ and with a universal family $\mathrm{Univ}(Y) \subset \mathrm{Chow}^{\mathrm{irr}}(Y) \times Y$.

Outline of proof (continued)

We may view $\mathrm{Aut}(X)$ as a subset of $\mathrm{Chow}^{\mathrm{irr}}(X \times X)$ by assigning to each $g \in \mathrm{Aut}(X)$ its graph $\Gamma(g)$, a closed subvariety of $X \times X$. For example, the graph of the identify is the diagonal $\mathrm{diag}(X)$.

The two projections $\mathrm{pr}_1, \mathrm{pr}_2 : X \times X \to X$ restrict to isomorphisms $\Gamma(g) \xrightarrow{\simeq} X$, and this condition characterizes the graphs of automorphisms among closed subvarieties of $X \times X$.

One may show that this condition is open. Thus, $\mathrm{Aut}(X)$ is identified with an open subset of $\mathrm{Chow}(X \times X)$. Moreover, the subset of $\mathrm{Aut}(X) \times X \times X$ consisting of the triples $(g, x, g(x))$ (where $g \in \mathrm{Aut}(X)$ and $x \in X$) is the pullback of the universal family, and hence is closed. It follows that the action map $a : \mathrm{Aut}(X) \times X \to X$, $(g, x) \mapsto g(x)$ is a morphism.

To show that this map is the universal family of automorphisms, one extends the graph construction

$$g \in \mathrm{Aut}(X) \longmapsto \Gamma(g) \subset X \times X$$

to families of automorphisms over a variety S, to reduce to a statement about the universal family over $\mathrm{Chow}^{\mathrm{irr}}(X \times X)$.

The automorphism group of the projective space

Theorem

For any positive integer n, we have $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}$.

Proof sketch. One may readily check that the action map $\mathrm{PGL}_{n+1} \times \mathbb{P}^n \to \mathbb{P}^n$ is a morphism of varieties. To complete the proof, it suffices to show that the natural map $\mathrm{PGL}_{n+1}(\mathbb{C}) \to \mathrm{Aut}(\mathbb{P}^n)$ is bijective.

The injectivity is easy. To show the surjectivity, let $g \in \mathrm{Aut}(\mathbb{P}^n)$. Then g acts on the Picard group $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}\,[\mathcal{O}(1)]$ via pull-back, and hence sends the generator $[\mathcal{O}(1)]$ to $\pm[\mathcal{O}(1)]$. Since $-[\mathcal{O}(1)] = [\mathcal{O}(-1)]$ has no nonzero global section, we must have $g^*[\mathcal{O}(-1)] = [\mathcal{O}(-1)]$.
As a consequence, $g : \mathbb{P}^n \to \mathbb{P}^n$ lifts to an automorphism $\hat{g} : \mathcal{O}(-1) \to \mathcal{O}(-1)$ which commutes with the action of $\mathbb{G}_m$ by multiplication on fibers of the projection $\mathcal{O}(-1) \to \mathbb{P}^n$.

In turn, $\hat{g}$ induces an automorphism γ of the algebra of regular functions on the variety $\mathcal{O}(-1)$, i.e., the homogeneous coordinate ring $\mathbb{C}[x_0, \ldots, x_n]$. Moreover, γ preserves the grading, and hence acts on $\mathbb{C}x_0 + \cdots + \mathbb{C}x_n$ via some $A \in \mathrm{GL}_{n+1}(\mathbb{C})$. One may then check that $g = [A] \in \mathrm{PGL}_{n+1}(\mathbb{C})$. □

Further examples of automorphism groups

1) Let X be an elliptic curve. Choosing a point $0 \in X$ defines a commutative algebraic group law $+$ on X with neutral element 0. In particular, for any $x \in X$, we have the translation

$$\tau_x : X \longrightarrow X, \quad y \longmapsto x + y.$$

This yields a transitive action of X on itself via the homomorphism $\tau : X \to \mathrm{Aut}(X)$. Thus, every $g \in \mathrm{Aut}(X)$ can be written uniquely as $\tau_x \circ h$, where $x \in X$ and $h \in \mathrm{Aut}(X)$ satisfies $h(0) = 0$. We write $h \in \mathrm{Aut}(X, 0)$.
One then shows that h is a homomorphism, and hence $h\tau_x h^{-1} = \tau_{h(x)}$. It follows that

$$X \rtimes \mathrm{Aut}(X, 0) \xrightarrow{\sim} \mathrm{Aut}(X), \quad (x, h) \longmapsto \tau_x \circ h.$$

Moreover, $\mathrm{Aut}(X, 0)$ is a finite cyclic group of order 2, 4 or 6.

In particular, $\mathrm{Aut}^0(X) \simeq X$ is not linear. Also, $\pi_0 \, \mathrm{Aut}(X) \simeq \mathrm{Aut}(X, 0)$ is non-trivial, and hence $\mathrm{Aut}(X)$ is a non-connected algebraic group.

Further examples of automorphism groups (continued)

2) Let X be a smooth projective curve of genus g.

- If $g = 0$ then $X \simeq \mathbb{P}^1$ and $\mathrm{Aut}(X) \simeq \mathrm{PGL}_2$.
- If $g = 1$ then X is an elliptic curve and $\mathrm{Aut}(X) \simeq X \rtimes \mathrm{Aut}(X, 0)$ (Example 1).
- If $g \geq 2$ then $\mathrm{Aut}(X)$ is a finite group of order $\leq 84(g-1)$ (Hurwitz).

3) Let $X = Y \times Y$, where Y is an elliptic curve with origin 0. Then X is a commutative algebraic group with neutral element $(0,0)$.

One shows as in Example 1 that $\mathrm{Aut}(X) \simeq X \rtimes \mathrm{Aut}(X, (0,0))$, where $\mathrm{Aut}(X, (0,0))$ is a discrete group. Thus, we have again $\mathrm{Aut}^0(X) \simeq X$ and $\pi_0 \, \mathrm{Aut}(X) \simeq \mathrm{Aut}(X, (0,0))$.

But now $\mathrm{Aut}(X, (0,0))$ is infinite, as it contains the group $\mathbb{Z}$ acting via $n \cdot (y_1, y_2) = (y_1 + n y_2, y_2)$. Thus, $\mathrm{Aut}(X)$ is not an algebraic group.

In fact, $\pi_0 \, \mathrm{Aut}(X) \supset \mathrm{GL}_2(\mathbb{Z})$ acting via linear combinations of y_1, y_2. Moreover, equality holds if and only if $\mathrm{End}(Y) = \mathbb{Z}$, i.e., Y has no complex multiplication.

Functorial properties of automorphism groups

Proposition

Let X be an irreducible projective variety, and $f : X' \to X$ the normalization map. Then there is a unique homomorphism of locally algebraic groups

$$f^* : \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(X')$$

such that $g(f(x')) = f(f^*(g)(x'))$ for all $g \in \mathrm{Aut}(X)$ and $x' \in X'$.

This follows from the lifting property of (locally) algebraic groups under the normalization, see Lecture 2. Likewise, Blanchard's lemma yields:

Theorem

Let $f : X \to Y$ be a contraction of projective varieties. Then there is a unique homomorphism of algebraic groups

$$f_* : \mathrm{Aut}^0(X) \longrightarrow \mathrm{Aut}^0(Y)$$

such that $f(g(x)) = f_*(g)(f(x))$ for any $g \in \mathrm{Aut}^0(X)$ and $x \in X$.

Functorial properties of automorphism groups (continued)

Corollary

Let X, Y be irreducible projective varieties. Then the map

$$f : \mathrm{Aut}^0(X) \times \mathrm{Aut}^0(Y) \longrightarrow \mathrm{Aut}^0(X \times Y), \quad (g, h) \longmapsto g \times h$$

is an isomorphism.

Indeed, one may readily check that $(\mathrm{pr}_{1,*}, \mathrm{pr}_{2,*})$ is inverse to f.

Corollary

Let $f : X \to Y$ be a birational morphism of irreducible projective varieties, where Y is normal. Then $f_* : \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(Y)$ is an isomorphism to its image, which is a closed subgroup of $\mathrm{Aut}^0(Y)$.

Proof.

Since f is birational, f_* is injective. Thus, its image is a closed subgroup. The corresponding homomorphism $\mathrm{Aut}^0(X) \to \mathrm{Im}(f)$ is a bijective morphism of smooth varieties, and hence is an isomorphism. □

Automorphism groups of projective surfaces

Let X be a surface. We may first take the normalization $X' \to X$ and then blow up the finitely many singular points of X', to obtain a projective birational morphism $X_1 \to X$. Iterating this process yields a sequence of such morphisms $X_n \to \cdots \to X_1 \to X$.

By a theorem of Zariski, the surface $Y = X_n$ is smooth for $n \gg 0$. This yields a **resolution of singularities** $f : Y \to X$.

If X is projective, then X' and X_1 are projective as well. Moreover, we have a pullback homomorphism of locally algebraic groups $\mathrm{Aut}(X) \to \mathrm{Aut}(X_1)$. Iterating, we obtain a homomorphism

$$f^* : \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(Y)$$

such that $g(f(y)) = f(f^*(g)(y))$ for all $g \in \mathrm{Aut}(X)$ and $y \in Y$. We say that f is an **equivariant resolution of singularities**.

Since f is birational, f^* is injective. In particular, it identifies $\mathrm{Aut}^0(X)$ with a closed subgroup of $\mathrm{Aut}^0(Y)$.

Connected automorphism groups of smooth projective surfaces

From now on, surfaces are assumed to be smooth and projective.

Let X be a surface. Then there is a birational morphism $f : X \to Y$, where Y is a **relatively minimal model**, i.e., a surface such that every birational morphism from Y to a surface is an isomorphism. Thus, f identifies $\mathrm{Aut}^0(X)$ to a closed subgroup of $\mathrm{Aut}^0(Y)$.

All of this reduces somehow the study of connected automorphism groups of surfaces to the case of relatively minimal models.

For a rational surface X, the relatively minimal models are exactly the projective plane $\mathbb{P}^2$ and the **rational ruled surfaces**, also known as the **Hirzebruch surfaces**

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)),$$

where $n = 0$ or $n \geq 2$. Note that $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, $\mathbb{F}_1$ is the blowing-up of $\mathbb{P}^2$ at one point, and hence is not relatively minimal.

Automorphism groups of Hirzebruch surfaces

For any integer $n \geq 1$, we have $H^0(\mathbb{P}^1, \mathcal{O}(n)) = \mathbb{C}[x, y]_n$, the space of homogeneous polynomials of degree n in the homogeneous coordinates x, y. This space is an irreducible representation of GL_2 acting by linear change of variables. The kernel of this representation is the subgroup μ_n of nth roots of unity viewed as scalar matrices.

Proposition

With this notation, we have an isomorphism of algebraic groups

$$\mathrm{Aut}(\mathbb{F}_n) \simeq \mathbb{G}_a^{n+1} \rtimes (\mathrm{GL}_2 / \mu_n),$$

where GL_2 / μ_n *acts on* $\mathbb{G}_a^{n+1}$ *via its linear action on* $\mathbb{C}[x, y]_n$.

The case where $n = 0$ is different: we then have $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and

$$\mathrm{Aut}(\mathbb{F}_0) \simeq (\mathrm{PGL}_2 \times \mathrm{PGL}_2) \rtimes \{\mathrm{id}, \sigma\},$$

where σ exchanges the two factors.

Automorphism groups of Hirzebruch surfaces (continued)

Proof sketch of the proposition. The ruling $f : \mathbb{F}_n \to \mathbb{P}^1$ induces a homomorphism of algebraic groups $f_* : \mathrm{Aut}^0(\mathbb{F}_n) \to \mathrm{Aut}^0(\mathbb{P}^1) = \mathrm{PGL}_2$. On the other hand, we have a natural action of GL_2 on $\mathcal{O}(n)$ that lifts the action of PGL_2 on $\mathbb{P}^1$. This yields a homomorphism of algebraic groups $\mathrm{GL}_2 \to \mathrm{Aut}^0(\mathbb{F}_n)$ with kernel μ_n. So we may view GL_2 / μ_n as a closed subgroup of $\mathrm{Aut}^0(\mathbb{F}_n)$. We then have $\mathrm{Aut}^0(\mathbb{F}_n) = \mathrm{Ker}(f_*) \cdot \mathrm{GL}_2 / \mu_n$ and $\mathrm{Ker}(f_*) \cap \mathrm{GL}_2 / \mu_n = \mathbb{G}_m / \mu_n \simeq \mathbb{G}_m$ (the image of the scalar matrices).

We now describe the algebraic group $\mathrm{Ker}(f_*)$, which consists of automorphisms preserving each fiber F_y, where $y \in \mathbb{P}^1$. We use the fact that f has a unique section C of self-intersection $-n$, which is thus stable by $\mathrm{Ker}(f_*)$. So $\mathrm{Ker}(f_*)$ acts on F_y by affine transformations fixing the intersection point with C. One may then check that $\mathrm{Ker}(f_*) \simeq \mathbb{G}_a^{n+1} \rtimes \mathbb{G}_m$, where $\mathbb{G}_a^{n+1}$ acts by translations, and $\mathbb{G}_m$ by multiplication.

As consequence, $\mathrm{Aut}^0(\mathbb{F}_n) \simeq \mathbb{G}_a^{n+1} \rtimes (\mathrm{GL}_2 / \mu_n)$. To complete the proof, it suffices to check that $\mathrm{Aut}(\mathbb{F}_n)$ is connected. But this follows from the uniqueness of the ruling f and its negative section C.