Rational actions of algebraic groups
Lecture 2: Algebraic groups and their actions

Michel BRION

Institut Fourier, Université Grenoble Alpes

30 October 2024

Lectures at the School and Workshop

Selected topics in Arithmetic Algebraic Geometry

Hanoi, 28 October - 8 November 2024

http://www-fourier.univ-grenoble-alpes.fr/~mbrion/RAAG2.pdf

Algebraic groups

Definition

An **algebraic group** is a variety G equipped with morphisms $m : G \times G \to G$ (the multiplication map) and $i : G \to G$ (the inverse map), and with a point $e \in G$ (the neutral element) which satisfy the group axioms.

Examples

- Every finite group is algebraic.
- The **additive group** $\mathbb{G}_a$ is the affine line $\mathbb{A}^1$ equipped with the addition. The inverse map is $x \mapsto -x$ and the neutral element is 0.
- The **multiplicative group** $\mathbb{G}_m$ is the punctured affine line $\mathbb{A}^1 \setminus \{0\}$ equipped with the multiplication. The inverse map is $x \mapsto x^{-1}$ and the neutral element is 1.
- Every **elliptic curve** E (equipped with a point 0) has a unique structure of algebraic group with neutral element 0.

Examples of algebraic groups (continued)

- The **general linear group** GL_n is the group of invertible $n \times n$ matrices. This is the open subset of M_n (the affine space of $n \times n$ matrices) defined by the nonvanishing of the determinant. Thus, GL_n is an irreducible affine variety. It is an algebraic group, since the product of matrices is a polynomial in the matrix coefficients, and the inverse is a polynomial in these coefficients and the inverse of the determinant.
- The **special linear group** SL_n is the closed subgroup of GL_n consisting of matrices of determinant 1.
- The **projective linear group** PGL_n is the quotient of GL_n by its central subgroup consisting of the scalar matrices $t\,\mathrm{I}_n$, where $t \in \mathbb{G}_m$. This is the automorphism group of the algebra of matrices M_n, and hence a closed subgroup of GL_{n^2}.

An algebraic group G is **linear** if it is isomorphic to a closed subgroup of some general linear group GL_n.

For example, $\mathbb{G}_a$ and $\mathbb{G}_m = \mathrm{GL}_1$ are linear, as well as SL_n and PGL_n. But elliptic curves are not linear, since they are projective varieties.

The neutral component

Proposition

Let G be an algebraic group, and $G^0 \subset G$ the connected component containing the neutral element e. Then G^0 is a normal subgroup of G, and a smooth irreducible variety. Moreover, the connected components of G are exactly the cosets gG^0, where $g \in G$.

Proof.

The inverse map i is an automorphism of the variety G that fixes e, and hence stabilizes G^0. Thus, for any $g \in G^0$, the translate $g^{-1}G^0$ is a connected component of G containing e, and hence equals G^0. Therefore, $g^{-1}h \in G^0$ for any $g, h \in G^0$. So G^0 is a subgroup of G. It is normal in G, since the conjugation by any $g \in G$ is an automorphism fixing e.
The smooth locus of G^0 is a non-empty open subset, stable by left multiplication by any $g \in G^0$. Thus, G^0 is smooth everywhere.
Since G^0 is connected, it is irreducible.
Let C be a connected component of G, and let $g \in C$. Then $g^{-1}C$ is a connected component containing e. Thus, $g^{-1}C = G^0$, and $C = gG^0$. □

The neutral component (continued)

We still consider an algebraic group G. As a consequence of the above proposition, G is smooth and equidimensional, i.e., all its connected components have the same dimension. Moreover, we have an exact sequence of algebraic groups

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \pi_0(G) \longrightarrow 1,$$

where the **neutral component** G^0 is connected, and the **group of components** $\pi_0(G)$ is finite.

Examples

- The groups $\mathbb{G}_a$, $\mathbb{G}_m$ and the elliptic curves are connected.
  One can show that they yield all the connected algebraic groups of dimension 1.
- The classical groups GL_n, SL_n, PGL_n are connected.
- The orthogonal group O_n is not connected. We have $\mathrm{O}_n^0 = \mathrm{SO}_n$ (the special orthogonal group), and $\pi_0(\mathrm{O}_n) \simeq \{\pm 1\}$ via the determinant.

Actions of algebraic groups

Definition

An **action** of an algebraic group G on a variety X is a morphism $a : G \times X \to X$ which satisfies the axioms of a group action.

Specifically, one requires that $a(g, a(h, x)) = a(gh, x)$ for all $g, h \in G$ and $x \in X$, and $a(e, x) = x$ for all such x. We will then denote $a(g, x)$ by $g \cdot x$.

Examples

- Every algebraic group G acts on itself by left multiplication ($g \cdot h = gh$) and by right multiplication ($g \cdot h = hg^{-1}$).
- The general linear group GL_n acts linearly on the affine space $\mathbb{A}^n$ by matrix multiplication: $A \cdot v = Av$. This yields an action of GL_n on the projective space $\mathbb{P}^{n-1}$ via $A \cdot [v] = [Av]$, for which the scalar matrices act trivially. So we arrive at an action of PGL_n on $\mathbb{P}^{n-1}$.
- The group $\mathbb{G}_a^{n+1} = \mathbb{G}_a \times \cdots \times \mathbb{G}_a$ ($n+1$ times) acts on the affine space $\mathbb{A}^2$ via $(t_0, \ldots, t_n) \cdot (x, y) = (x, y + t_0 + t_1 x + \cdots + t_n x^n)$. This gives back the subgroup $\mathrm{U}_n \subset \mathrm{Aut}(\mathbb{A}^2) \subset \mathrm{Bir}(\mathbb{A}^2) = \mathrm{Cr}_2$ of Lecture 1.

G-varieties, orbits

Let G be an algebraic group.
A G-**variety** is a variety X equipped with a G-action

$$a : G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x.$$

A (locally closed) subvariety $Y \subset X$ is G-**stable** if $g \cdot y \in Y$ for all $g \in G$ and $y \in Y$.
Given two G-varieties X, Y, a morphism $f : X \to Y$ is **equivariant** if it satisfies $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$.
For any G-variety X and $x \in X$, we denote by

$$G_x = \{g \in G \mid g \cdot x = x\}$$

the **isotropy group** at x (this is a closed subgroup of G), and by

$$G \cdot x = \{g \cdot x \mid g \in G\}$$

the **orbit** of x. Then $G \cdot x$ is the image of the **orbit map**

$$a_x : G \longrightarrow X, \quad g \longmapsto g \cdot x.$$

The fibers of a_x are exactly the cosets gG_x, where $g \in G$.

Orbits (continued)

Proposition

Let G be an algebraic group, X a G-variety, and $x \in X$.

(i) *The orbit $G \cdot x$ is a smooth, locally closed subvariety of X, and is equidimensional of dimension $\dim(G) - \dim(G_x)$.*

(ii) *The boundary $\overline{G \cdot x} \setminus G \cdot x$ is a union of G-orbits of strictly smaller dimension.*

(iii) *If $G \cdot x$ has minimal dimension among all G-orbits, then it is closed.*

In particular, every G-variety contains a closed G-orbit.

Proof sketch. We assume that G is connected, and hence irreducible. The general case follows by arguing with the neutral component.

(i) Let $\mathcal{O} = G \cdot x$. Recall that $\mathcal{O}$ is the image of the orbit map $a_x : G \to X$. By a theorem of Chevalley, it follows that $\mathcal{O}$ is a constructible subset of X, and contains a dense open subset U of its closure $\overline{\mathcal{O}}$. Since the translates $g \cdot U$ ($g \in G$) cover $\mathcal{O}$, we see that $\mathcal{O}$ is open in $\overline{\mathcal{O}}$.
The smoothness of $\mathcal{O}$ is proved by a similar argument.

Orbits (continued)

The assertion on the dimension is a consequence of a general result: For any dominant morphism of irreducible varieties $f : X \to Y$, there exists a dense open subset $V \subset Y$ such that the fiber $f^{-1}(y)$ is non-empty and equidimensional of dimension $\dim(X) - \dim(Y)$ for any $y \in V$.

(ii) This follows from the fact that $\mathcal{O}$ is a dense open G-stable subset of the irreducible G-variety $\overline{\mathcal{O}}$.

(iii) This is a direct consequence of (ii). □

Corollary

Let $f : G \to H$ be a homomorphism of algebraic groups. Then the kernel $\mathrm{Ker}(f) \subset G$ *and the image* $\mathrm{Im}(f) \subset H$ *are closed subgroups, and we have* $\dim(G) = \dim \mathrm{Ker}(f) + \dim \mathrm{Im}(f)$.

Proof sketch. Apply the proposition to the action of G on H via $g \cdot h = f(g)h$. The orbits are exactly the cosets $\mathrm{Im}(f)h$, where $h \in H$. □

For example, the homomorphism $\mathrm{GL}_n \to \mathrm{GL}_{n^2}$ given by conjugation of $n \times n$ matrices, has kernel $\mathbb{G}_m$ and image PGL_n.

Homogeneous varieties

A G-variety X is **homogeneous** if the G-action is transitive.

This means that for any $x, y \in X$, there exists $g \in G$ such that $y = g \cdot x$. Equivalently, X is a unique G-orbit.

Every homogeneous G-variety X is smooth, and isomorphic to the **homogeneous space** G/G_x for any $x \in X$.

Examples

- The affine space $\mathbb{A}^n$ is homogeneous under the group $\mathbb{G}_a^n$ acting by translations: $(g_1, \ldots, g_n) \cdot (x_1, \ldots, x_n) = (g_1 + x_1, \ldots, g_n + x_n)$. Each isotropy group is trivial.
- $\mathbb{A}^n$ is also homogeneous under the affine group $\mathrm{Aff}_n \simeq \mathbb{G}_a^n \rtimes \mathrm{GL}_n$, where GL_n acts on $\mathbb{G}_a^n$ and on $\mathbb{A}^n$ by matrix multiplication. The isotropy group at the origin 0 is GL_n.
- The punctured affine space $\mathbb{A}^n \setminus \{0\}$ is homogeneous under the action of GL_n.
- The projective space $\mathbb{P}^{n-1}$ is homogeneous under the action of PGL_n.

The normalization

Proposition

Let G be an algebraic group, X an irreducible G-variety, and $f : X' \to X$ the normalization map. Then there is a unique action of G on X' such that f is equivariant.

Proof sketch. Consider the morphism

$$\varphi : G \times X' \longrightarrow X, \quad (g, x') \longmapsto g \cdot f(x').$$

If G is connected, then it is smooth and irreducible, so that $G \times X'$ is a normal irreducible variety. By the universal property of the normalization, it follows that φ lifts to a unique morphism

$$\psi : G \times X' \longrightarrow X'.$$

One may readily check that ψ is an action by using the fact that f is birational.

In the general case, where G is not necessarily connected, one argues as above for any connected component. $\square$

Contractions

Definition

Let $f : X \to Y$ be a proper morphism of irreducible varieties. Then f is a **contraction** if it satisfies the following condition:
For any open subset $V \subset Y$, the homomorphism

$$\mathcal{O}(V) \longrightarrow \mathcal{O}(f^{-1}(V)), \quad \varphi \longmapsto \varphi \circ f$$

is bijective.

In terms of sheaves, this means that the homomorphism

$$f^{\#} : \mathcal{O}_Y \longrightarrow f_*(\mathcal{O}_X)$$

is bijective. As a consequence, it suffices to check the above condition on an open affine covering (V_i) of Y.

Example

Given two irreducible varieties Y, Z, where Z is complete, the projection $\mathrm{pr}_1 : Y \times Z \to Y$ is a contraction. Indeed, for any open subset $V \subset Y$, we have $\mathcal{O}(\mathrm{pr}_1^{-1}(V)) = \mathcal{O}(V \times Z) = \mathcal{O}(V) \otimes_{\mathbb{C}} \mathcal{O}(Z) = \mathcal{O}(V)$ as $\mathcal{O}(Z) = \mathbb{C}$.

Contractions (continued)

Proposition

Let $f : X \to Y$ be a proper morphism of irreducible varieties.

(i) *If f is a contraction, then f is surjective and its fibers are connected.*

(ii) *Conversely, if f is surjective with connected fibers and Y is normal, then f is a contraction.*

(iii) *If f is birational and Y is normal, then f is a contraction.*

Proof. (i) This follows from the theorem on formal functions.
(ii) By the Stein factorization, f factors as $h \circ g$, where $g : X \to Z$ is a contraction, and $h : Z \to Y$ is finite and surjective. Since the fibers of f are connected, h is an isomorphism.
(iii) This follows again from the Stein factorization, since every finite birational morphism to a normal variety is an isomorphism. □

In particular, every birational morphism of smooth projective irreducible varieties is a contraction.

Blanchard's lemma

Let Y be an irreducible variety, $Z \subset Y$ a closed subvariety, and $f : X = \mathrm{Bl}_Z(Y) \to Y$ the blowing-up of Y along Z. Then f is a projective birational morphism (and hence a contraction if Y is normal). Given an algebraic group G acting on Y and stabilizing Z, there is a unique action of G on X such that f is equivariant.

Blanchard's lemma provides a partial converse:

Theorem

Let G be a connected algebraic group, X a G-variety, and $f : X \to Y$ a contraction. Then there is a unique action of G on Y such that f is equivariant.

Thus, if $X = \mathrm{Bl}_Y(Z)$ as above, then G stabilizes the exceptional set $E = f^{-1}(Z) \subset X$ and its image $Z \subset Y$. See Lecture 3 for applications to automorphism groups of projective varieties.

The theorem fails for non-connected algebraic groups (take $X = Y \times Y$ where Y is a complete irreducible variety, and consider the projection $\mathrm{pr}_1 : X \to Y$ and the involution $(y_1, y_2) \mapsto (y_2, y_1)$ of X).

Proof of Blanchard's lemma

Step 1. We construct an action of the abstract group G on the set Y such that f is G-equivariant. Let $y \in Y$ and consider the fiber $X_y = f^{-1}(y)$. This is a complete connected variety, and the morphism

$$\varphi : G \times X_y \longrightarrow Y, \quad (g, x) \longmapsto f(g \cdot x)$$

sends $\{e\} \times X_y$ to y. By the rigidity lemma, φ sends $\{g\} \times X_y$ to a point for any $g \in G$. Denote this point by $g \cdot y$; then for any $x \in X_y$, we have $f(g \cdot x) = g \cdot y = g \cdot f(x)$. It follows easily that the map $(g, y) \mapsto g \cdot y$ is the desired action.

Step 2. We show that the above action $b : G \times Y \to Y$ is continuous. Let $Z \subset Y$ be a closed subset; we check that $b^{-1}(Z) \subset G \times Y$ is closed. Since $\mathrm{id}_G \times f : G \times X \to G \times Y$ is proper and surjective, it suffices to show that $(\mathrm{id}_G \times f)^{-1} b^{-1}(Z) \subset G \times X$ is closed. As f is equivariant, we have $b \circ (\mathrm{id}_G \times f) = f \circ a$, and hence $(\mathrm{id}_G \times f)^{-1} b^{-1}(Z) = a^{-1} f^{-1}(Z)$ is closed indeed.

Proof of Blanchard's lemma (continued)

Step 3. We show that the continuous map $b : G \times Y \to Y$ is a morphism. For this, we check that the pullback map $b^{\#}$ (between sheaves of continuous functions) sends $\mathcal{O}_Y$ to $b_*(\mathcal{O}_{G\times Y})$.

Let $V \subset Y$ be an open subset. Then $\mathcal{O}_Y(V) \simeq \mathcal{O}_X(f^{-1}(V))$ as f is a contraction. Moreover, we have
$b_*(\mathcal{O}_{G\times Y})(V) = \mathcal{O}_{G\times Y}(b^{-1}(V) = (\mathrm{id}_G \times f)_*(\mathcal{O}_{G\times X})(b^{-1}(V))$
$= \mathcal{O}_{G\times X}(\mathrm{id}_G \times f)^{-1} b^{-1}(V)) = \mathcal{O}_{G\times X}(a^{-1} f^{-1}(V))$,
where the latter equality follows again from the equivariance of f.
One may then check that the pullback map $b^{\#}$ is identified with
$a^{\#} : \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_{G\times X}(a^{-1} f^{-1}(V))$.

Step 4. It remains to show that b is unique. But this follows again from the surjectivity and equivariance of f. □

The idea behind this proof is the "uniqueness of contractions" (EGAII, §8.11): given a contraction $f : X \to Y$ and a morphism of varieties $\varphi : X \to Z$ which is constant on the fibers of f, there exists a unique morphism $\psi : Y \to Z$ such that $\varphi = \psi \circ f$.