Rational actions of algebraic groups
Lecture 1: Notions of birational geometry

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Overview

The following result is known as Weil's regularization theorem:

For any rational action of an algebraic group G on an irreducible variety X, there exists a G-variety Y which is birationally isomorphic to X.

This theorem and its refinements are key ingredients for classifying algebraic subgroups of groups of birational transformations for surfaces and threefolds, in classical work of Enriques, Fano, Mukai, Umemura and recent work of Blanc, Fanelli, Fong, Schneider, Terpereau, Zimmermann, and others.

The lectures will first discuss the notions from birational geometry and algebraic groups occuring in the regularization theorem. We will then present applications of this theorem to rational actions on surfaces.

- Lecture 1: Notions of birational geometry
- Lecture 2: Algebraic groups and their actions
- Lecture 3: Automorphism groups of projective varieties
- Lecture 4: Rational actions and their regularization
- Lecture 5: Proof sketch of the regularization theorem

Function fields

A **function field** over the field $\mathbb{C}$ of complex numbers is a finitely generated field extension $K/\mathbb{C}$.

For example, we may take $K = \mathbb{C}(T_1, \ldots, T_n)$, the field of rational functions in the variables $T_1, \ldots, T_n$.

Proposition

Let K be a function field. Then $K \simeq \mathbb{C}(x_1, \ldots, x_n, x_{n+1})$, where $x_1, \ldots, x_n$ are algebraically independent and x_{n+1} is algebraic over $\mathbb{C}(x_1, \ldots, x_n)$.

Proof.

We may choose a transcendence basis $(x_1, \ldots, x_n)$, i.e., $x_1, \ldots, x_n \in K$ are algebraically independent and K is algebraic over its subfield $\mathbb{C}(x_1, \ldots, x_n) \simeq \mathbb{C}(T_1, \ldots, T_n)$. Then n is the transcendence degree of $K/\mathbb{C}$. Since K is finitely generated, it is a finite extension of $\mathbb{C}(x_1, \ldots, x_n)$. As this extension is separable, we may choose a primitive element x_{n+1}. □

For K as above, we have $K \simeq \mathbb{C}(T_1, \ldots, T_n)[T]/(P(T))$, where $P(T)$ is a polynomial with coefficients in $\mathbb{C}(T_1, \ldots, T_n)$. We then say that K is a **function field of n variables**.

A preliminary result

Proposition

Let K be a function field, and $L \subset K$ a subfield containing $\mathbb{C}$. Then L is finitely generated over $\mathbb{C}$.

Proof.

Note that $L/\mathbb{C}$ admits a finite transcendence basis $(x_1, \ldots, x_m)$, which is part of a transcendence basis $(x_1, \ldots, x_n)$ of $K/\mathbb{C}$. Choose a basis $(y_i)_{i \in I}$ of L as a vector space over $\mathbb{C}(x_1, \ldots, x_m)$. It suffices to show that the y_i (viewed in K) are linearly independent over $\mathbb{C}(x_1, \ldots, x_n)$, since K is a finite-dimensional vector space over this subfield.

We argue by contradiction. Consider a linear relation $\sum_{i \in I} f_i(x_1, \ldots, x_n)\, y_i = 0$, where $f_i \in \mathbb{C}(x_1, \ldots, x_n)$, $f_i = 0$ except for finitely many indices i, and $f_{i_0} \neq 0$ for some i_0. Clearing denominators, we may assume that $f_i \in \mathbb{C}[x_1, \ldots, x_n]$ for all i. Then there exist $t_{m+1}, \ldots, t_n \in \mathbb{C}$ such that $f_{i_0}(x_1, \ldots, x_m, t_{m+1}, \ldots, t_n) \neq 0$. Moreover, $\sum_{i \in I} f_i(x_1, \ldots, x_m, t_{m+1}, \ldots, t_n)\, y_i = 0$. But this contradicts the linear independence of the y_i over $\mathbb{C}(x_1, \ldots, x_m)$. □

Automorphisms of function fields

Our main object of interest will be the automorphism group of the field extension $K/\mathbb{C}$, denoted by $\mathrm{Aut}_{\mathbb{C}}(K)$.

If $K = \mathbb{C}(T_1, \dots, T_n)$ then $\mathrm{Aut}_{\mathbb{C}}(K)$ is the **Cremona group of rank n**, denoted by Cr_n. It may be viewed as the set of maps

$$(T_1, \dots, T_n) \longmapsto (f_1(T_1, \dots, T_n), \dots, f_n(T_1, \dots, T_n)),$$

where $f_1, \dots, f_n \in \mathbb{C}(T_1, \dots, T_n)$ generate this field over $\mathbb{C}$ (then they are algebraically independent). The group law is the composition of maps.

Example

The group Cr_1 consists of the rational functions $f(T) \in \mathbb{C}(T)$ such that $\mathbb{C}(f(T)) = \mathbb{C}(T)$. Such rational functions are exactly the maps $T \mapsto \frac{aT+b}{cT+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Thus,

$$\mathrm{Cr}_1 \simeq \mathrm{PGL}_2(\mathbb{C})$$

(the quotient of the group $\mathrm{GL}_2(\mathbb{C})$ of invertible 2×2 matrices by the subgroup of matrices $t\,\mathrm{I}_2$, where $t \in \mathbb{C}^*$).

Some subgroups of Cr_2

Recall that the group $\mathrm{Cr}_2 = \mathrm{Aut}_{\mathbb{C}}\,\mathbb{C}(X,Y)$ consists of the maps $(X,Y) \longmapsto (f(X,Y), g(X,Y))$ such that we have $\mathbb{C}(f(X,Y), g(X,Y)) = \mathbb{C}(X,Y)$.

The maps $(X,Y) \mapsto (X, Y + P(X))$, where $P(X) \in \mathbb{C}[X]$, form a subgroup $\mathrm{U} \subset \mathrm{Cr}_2$ isomorphic to the additive group of polynomials. Note that U is the increasing union of its subgroups U_n, consisting of the above maps with $\deg(P) \leq n$.

We also have the **monomial subgroup** $\mathrm{M} \subset \mathrm{Cr}_2$ consisting of the maps $(X,Y) \mapsto (aX^pY^q, bX^rY^s)$, where $a, b \in \mathbb{C}^*$ and $p, q, r, s \in \mathbb{Z}$ satisfy $ps - qr = \pm 1$. Thus, $M \simeq (\mathbb{C}^*)^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$, where $\mathrm{GL}_2(\mathbb{Z})$ acts on $(\mathbb{C}^*)^2$ via $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot (a,b) = (a^pb^q, a^rb^s)$.

The **projective linear group** $\mathrm{PGL}_3(\mathbb{C})$ is also a subgroup of Cr_2, as we will see later in this lecture.

Rational maps

Let X, Y be algebraic varieties over $\mathbb{C}$. In loose words, a **rational map** $f : X \dashrightarrow Y$ is a morphism $U \to Y$, where $U \subset X$ is some (unspecified) dense open subset.

More specifically, we consider pairs (U, f), where U is as above and $f : U \to Y$ is a morphism. We say that two pairs (U, f), (V, g) are **equivalent** if $f|_{U\cap V} = g|_{U\cap V}$ (recall that $U \cap V$ is open and dense in X). This is indeed an equivalence relation, and we define the rational maps as the equivalence classes.

Given a rational map $f : X \dashrightarrow Y$ and a representative $f_U : U \to Y$, we say that f is **defined on** U. There exists a largest such subset U: the **domain of definition**, $\mathrm{Dom}(f)$. The complement $X \setminus \mathrm{Dom}(f)$ is called the **indeterminacy locus**.

A rational map $f : X \dashrightarrow Y$ is **dominant** if $f(\mathrm{Dom}(f))$ is dense in Y. Equivalently, there exists dense open subsets $U \subset X$, $V \subset Y$ such that f is defined on U and $f(U) = V$. If in addition $f_U : U \to V$ is an isomorphism, we say that f is **birational**. Then X, Y are said to be **birationally equivalent**.

Composition of rational maps

Given two rational maps

$$f : X \dashrightarrow Y, \quad g : Y \dashrightarrow Z$$

where X, Y, Z are varieties, the composition

$$g \circ f : X \dashrightarrow Z$$

is not always defined (as a rational map): for example, if f is constant with image $y \in Y$, and g is not defined at y (that is, $y \notin \mathrm{Dom}(g)$).

The composition $g \circ f$ is defined if and only if $f(\mathrm{Dom}(f))$ contains a dense open subset of $\mathrm{Dom}(g)$. This holds for example if f is dominant, or if g is a morphism.

In particular, the composition of any two birational maps is always defined. Also, one may check that a rational self-map $f : X \dashrightarrow X$ admits a rational inverse if and only if f is birational.

As a consequence, the birational self-maps $f : X \dashrightarrow X$ form a group for the composition: the **group of birational transformations**, denoted by $\mathrm{Bir}(X)$. It contains the automorphism group $\mathrm{Aut}(X)$.

The graph of a rational map

Let $f : X \dashrightarrow Y$ be a rational map between varieties, and $f_U : U \to Y$ a representative. Recall that the graph $\Gamma(f_U)$ is the subset of $U \times Y$ consisting of the pairs $(x, f_U(x))$. This is a closed subvariety of $U \times Y$, isomorphic to U via the first projection.

We define the **graph** $\Gamma = \Gamma(f)$ as the closure of $\Gamma(f_U)$ in $X \times Y$. Then Γ is a closed subvariety of $X \times Y$, and the projections $X \times Y \to X, Y$ yield two morphisms $p : \Gamma \to X$, $q : \Gamma \to Y$ such that $q = f \circ p$ as rational maps.

The following may be easily checked:

- p restricts to an isomorphism $p^{-1}(U) = \Gamma(f_U) \to U$ for any dense open subset $U \subset \mathrm{Dom}(f)$.
- Γ is independent of the choice of U, and p is birational.
- If Y is complete, then q is proper.
- f is dominant (resp. birational) if and only if q is dominant (resp. birational).

Rational functions

Let X be a variety. A **rational function** on X is a rational map $f : X \dashrightarrow \mathbb{A}^1$, where $\mathbb{A}^1$ denotes the affine line.

Thus, a rational map on X is an equivalence class of pairs (U, f), where $U \subset X$ is a dense open subset and $f : X \to \mathbb{A}^1$ is a morphism; equivalently, $f \in \mathcal{O}(U)$ (the algebra of regular functions on U).

Given two pairs (U, f), (V, g), we define their sum as $(U \cap V, f + g)$ and their product as $(U \cap V, fg)$. In particular, we have the product tf, where $t \in \mathbb{C}$ is viewed as a constant function. One may check that this defines an algebra structure on the equivalence classes. Thus, the rational functions on X form a $\mathbb{C}$-algebra, denoted by $\mathbb{C}(X)$.

Proposition

If X is irreducible, then the algebra of rational functions $\mathbb{C}(X)$ is the field of fractions of $\mathcal{O}(U)$ for any dense open affine subset $U \subset X$.

For an arbitrary variety X with irreducible components $X_1, \ldots, X_n$, we have $\mathbb{C}(X) \simeq \mathbb{C}(X_1) \times \cdots \times \mathbb{C}(X_n)$.

Irreducible varieties and function fields

As a consequence of the above proposition, $\mathbb{C}(X)$ is a function field for any irreducible variety X. Indeed, the algebra $\mathcal{O}(U)$ is finitely generated for any affine variety U.

Conversely, every function field K of n variables is the field of rational functions of some irreducible variety X of dimension n. We then say that X is a **model** of K.

For example, the affine space $\mathbb{A}^n$ and the projective space $\mathbb{P}^n$ are models of the field of rational functions $\mathbb{C}(T_1, \ldots, T_n)$.

Next, let $f : X \dashrightarrow Y$ be a dominant rational map between varieties. Then the composition $\varphi \circ f$ is defined for any rational function φ on Y, and yields a rational map on X. Moreover, the assignement $\varphi \mapsto \varphi \circ f$ is a homomorphism $f^* : \mathbb{C}(Y) \longrightarrow \mathbb{C}(X)$.
Given another dominant rational map of varieties $g : Y \dashrightarrow Z$, the composition $g \circ f$ is defined and satisfies $(g \circ f)^* = f^* \circ g^*$.

Irreducible varieties and function fields (continued)

Theorem

The category of irreducible varieties, with morphisms being dominant rational maps, is anti-equivalent to the category of function fields, with morphisms being field homomorphisms, via the assignements $X \longmapsto \mathbb{C}(X)$ *and* $(f : X \to Y) \longmapsto (f^* : \mathbb{C}(Y) \to \mathbb{C}(X))$.

More concretely, the assignement $X \mapsto \mathbb{C}(X)$ yields a bijection between irreducible varieties up to birational equivalence and function fields up to isomorphism. Given two irreducible varieties X, Y, the assignement $f \mapsto f^*$ yields a bijection between dominant rational maps $X \dashrightarrow Y$ and homomorphisms $\mathbb{C}(Y) \to \mathbb{C}(X)$.

In particular, for any irreducible variety X, we obtain an anti-isomorphism of groups

$$\mathrm{Bir}(X) \simeq \mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}(X).$$

For example, the Cremona group $\mathrm{Cr}_n = \mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}(T_1, \ldots, T_n)$ is anti-isomorphic to $\mathrm{Bir}(\mathbb{A}^n) \simeq \mathrm{Bir}(\mathbb{P}^n)$.

Rational maps from normal varieties

We consider a rational map $f : X \dashrightarrow Y$ between irreducible varieties, and its graph $\Gamma \subset X \times Y$ with projection $p : \Gamma \to X$. Recall that p is birational.

Proposition

Assume that X is normal and Y is complete. Then p is an isomorphism in codimension 1.

This means that the largest open subset of X over which p is an isomorphism (that is, the domain of definition of the inverse rational map p^{-1}) intersects every prime divisor $D \subset X$ (i.e., every closed irreducible subvariety of codimension 1).

To see this, recall that p is proper, and hence surjective. So the preimage $p^{-1}(D)$ is a finite union of prime divisors of Γ. For any such divisor E, the local ring $\mathcal{O}_{\Gamma,E} \subset \mathbb{C}(X)$ contains $p^*(\mathcal{O}_{X,D})$, where $p^* : \mathbb{C}(X) \to \mathbb{C}(\Gamma)$ is the isomorphism of function fields induced by p.
But since X is normal, $\mathcal{O}_{X,D}$ is a discrete valuation ring, and hence a maximal subring of $\mathbb{C}(X)$. Thus, p^* restricts to an isomorphism $\mathcal{O}_{X,D} \to \mathcal{O}_{\Gamma,E}$, and hence $\mathrm{Dom}(p^{-1})$ meets D.

Rational maps from normal varieties (continued)

Corollary

Let X be a normal irreducible variety, Y a complete irreducible variety, and $f : X \dashrightarrow Y$ a rational map. Then f is defined in codimension 1.

Again, this means that the domain of definition of f intersects every prime divisor.

Recall that an (irreducible) curve X is normal if and only if it is smooth; also, X is complete if and only if it is projective. So we obtain:

Corollary

Every birational map from a smooth curve to a projective curve is a morphism.

As a further consequence, the category of smooth projective curves, with morphisms being the non-constant morphisms, is equivalent to the category of function fields of one variable, with morphisms being the field homomorphisms.

In particular, $\mathrm{Aut}(X) = \mathrm{Bir}(X)$ for any smooth projective curve X.

Rational varieties

An irreducible variety X is **rational** if it is birationally equivalent to the projective space $\mathbb{P}^n$. This means that there exist dense open subsets $U \subset X$ and $V \subset \mathbb{P}^n$ such that $U \simeq V$. Equivalently, $\mathbb{C}(X) \simeq \mathbb{C}(T_1, \ldots, T_n)$.

We have $\mathrm{Bir}(X) \simeq \mathrm{Cr}_n$ for any rational variety X of dimension n. Taking $X = \mathbb{P}^n$ yields a more concrete description of the Cremona group Cr_n.

For example, Cr_2 consists of the rational transformations of $\mathbb{P}^2$ of the form

$$f : (x : y : z) \longmapsto (P(x, y, z) : Q(x, y, z) : R(x, y, z)),$$

where P, Q, R are homogeneous polynomials of the same degree without common factor. The points of indeterminacy of f are exactly the common zeroes of P, Q, R in $\mathbb{P}^2$ (the **base points**).

In particular, taking P, Q, R of degree 1, we see that Cr_2 contains $\mathrm{PGL}_3(\mathbb{C})$ as a subgroup. In fact, $\mathrm{PGL}_3(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^2)$ as we will see in Lecture 3.

An example of degree 2 is the **standard Cremona involution**

$$(x : y : z) \longmapsto (yz : xz : xy) = (\frac{1}{x} : \frac{1}{y} : \frac{1}{z}).$$

Finite subgroups of rational automorphism groups

Theorem

Let X be an irreducible variety, and G a finite subgroup of $\mathrm{Bir}(X)$. Then there exists a projective model Y of X such that $G \subset \mathrm{Aut}(Y)$.

Proof sketch. We may assume that X is affine. We first show that there exists a dense open subset $U \subset X$ such that $G \subset \mathrm{Aut}(U)$.

Recall that every $g \in G$ has a domain of definition $\mathrm{Dom}(g) \subset X$. The intersection $V = \cap_{g \in G} \mathrm{Dom}(g)$ is a dense open subset of X, and every $g \in G$ yields a morphism $g_V : V \to X$. Let $U = \cap_{g \in G} g_V^{-1}(V)$; this is a dense open subset of V. One may check that every $g \in G$ yields an automorphism of U.

Next, we show that the above open subset U may be taken affine. Let $x \in U$ and consider the orbit $G \cdot x = \{g(x), g \in G\}$. This is a finite subset of U, and hence is contained in an open affine subset $U' \subset U$. Then $\cap_{g \in G} g(U')$ is a dense open affine subset of U (the intersection of finitely many dense open affine subsets) which contains x and is G-stable.

Proof of the theorem (continued)

Finally, we show that the above affine variety U on which G acts by automorphisms, is an open subset of a projective variety Y on which G acts by automorphisms extending its action on U.

Consider the coordinate ring $\mathcal{O}(U)$: this is a finitely generated algebra on which G acts by algebra automorphisms via $(g \cdot f)(x) = f(g^{-1}(x))$. Thus, the algebra $\mathcal{O}(U)$ is generated by a finite-dimensional G-stable subspace V (spanned by finitely many generators of $\mathcal{O}(U)$ and their G-translates). This defines a morphism $U \longmapsto V^*$, $x \longmapsto (v \mapsto v(x))$ which identifies U to a closed subvariety of the affine space V^*, stable by the linear action of G. We take for Y the closure of U in the projective completion $\mathbb{P}(V^* \oplus \mathbb{C})$ of V^*. □

For example, the standard Cremona involution f is not an automorphism of $\mathbb{P}^2$, since it has base points $(1:0:0), (0:1:0), (0:0:1)$. But f is an automorphism of the model $\mathbb{P}^1 \times \mathbb{P}^1$ of $\mathbb{P}^2$, since $f(x:y:1) = (\frac{1}{x} : \frac{1}{y} : 1)$.

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