# Equivariant normalization

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# Notation and a preliminary result

Let $k$ be an algebraically closed field of characteristic $p \geq 0$.

- An **algebra** is a commutative, associative $k$-algebra with unit.
- A **grading** of an algebra $A$ by an abelian group $M$ is a decomposition $A = \bigoplus_{m \in M} A_m$, where $A_0 \supset k$ and $A_m \cdot A_{m'} \subset A_{m+m'}$ for all $m, m' \in M$. The nonzero elements of $A_m$ are called **homogeneous of degree** $m$.
- An **affine domain** is a finitely generated algebra without zero divisor.
- The **normalization** $\widetilde{A}$ of an affine domain $A$ is the integral closure of $A$ in its fraction field. This is an affine domain, finitely generated as an $A$-module.

## Proposition

*Let $A$ be an affine domain, and $\tilde{A}$ its normalization. Assume that $A$ is graded by a $p$-torsion-free abelian group $M$. Then this grading extends uniquely to a grading of the normalization $\widetilde{A}$.*

This does not hold when $M$ has $p$-torsion, as shown by the following :

# Example

Let $f \in k[x]$ (the algebra of polynomials in the variable $x$) and consider the algebra

$$A = k[x, y]/(y^p - f(x)).$$

The polynomial algebra $k[x, y]$ is graded by $\mathbb{Z}/p\mathbb{Z}$, where $\deg(x) = 0$ and $\deg(y) = 1 \pmod{p}$. Moreover, $y^p - f(x)$ is homogeneous of degree 0. Thus, $A$ has a $\mathbb{Z}/p\mathbb{Z}$-grading via

$$A = \bigoplus_{m=0}^{p-1} k[x]\, y^m.$$

The algebra $A$ is an affine domain if $f \notin k[x^p]$. Then $A$ has fraction field $K = k(x)[y]/(y^p - f(x))$. It has a $\mathbb{Z}/p\mathbb{Z}$-grading as well :

$$K = \bigoplus_{m=0}^{p-1} k(x)\, y^m.$$

# Example (continued)

## Proposition

*If $f'$ is nonconstant, then $\widetilde{A} \neq A$.*

*If $f$ has simple roots, then $A$ is the largest $\mathbb{Z}/p\mathbb{Z}$-graded subspace of $\widetilde{A}$.*

**Proof sketch** :
We view $X$ as the affine plane curve with equation $y^p = f(x)$.
Then $\widetilde{A} = A$ if and only if $X$ is nonsingular.
By the Jacobian criterion, this is equivalent to $f'$ having no zero, i.e., being constant.

Next, consider a homogeneous element $g = g(x, y) \in \widetilde{A}$ of degree $m$.
Then $g \in K_m = k(x)y^m$, i.e., $g = h(x)\, y^m$ for some $h \in k(x)$ and some integer $m$ such that $0 \leq m \leq p-1$.
Moreover, we have $g^p \in \widetilde{A}_{mp} = \widetilde{A}_0$.
But since $A_0 = k[x]$ is integrally closed in $K_0$, we obtain $\widetilde{A}_0 = A_0$ and hence $g^p \in k[x]$.
As $g^p = h(x)^p\, y^{mp} = h(x)^p\, f^m$, this forces $h(x) \in k[x]$ if $f$ has simple roots.

# Gradings and derivations

Recall that a **derivation** of an algebra $A$ is a linear map $D : A \to A$ such that $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$.

If $p = 0$ then every $\mathbb{Z}$-grading $A = \bigoplus_{n \in \mathbb{Z}} A_n$ defines a derivation $D$ such that $D(a) = na$ for any $a \in A_n$. Clearly, $D$ is diagonalizable with integer eigenvalues. Conversely, every diagonalizable derivation with integer eigenvalues arises from a $\mathbb{Z}$-grading of $A$.

If $p > 0$ then one obtains similarly a bijective correspondence between $\mathbb{Z}/p\mathbb{Z}$-gradings of $A$ and diagonalizable derivations $D$ with eigenvalues being integers mod $p$ ; equivalently, $D^p = D$.

The following is a special case of a result of Seidenberg :

## Proposition

*If $p = 0$ then every derivation of an affine domain extends uniquely to its normalization.*

This fails if $p > 0$ in view of the above example (basically due to Seidenberg).

# Affine (group) schemes

## Definition

- A **(group) functor** is a covariant functor from the category of algebras to that of sets (resp. groups).
- An **affine (group) scheme** is a representable (group) functor.

Equivalently, a functor $X$ is an affine scheme if there exists an algebra $A$ such that $X(R) = \mathrm{Hom}_{\mathrm{alg}}(A, R)$ for any algebra $R$. Then $A$ is unique ; we write $X = \mathrm{Spec}(A)$ and $A = \mathcal{O}(X)$. We say that $X$ is an (affine) **variety** if $A$ is an affine domain.
For a group scheme $G$, the functorial group structure on each $G(R)$ corresponds to a **Hopf algebra** structure on $\mathcal{O}(G)$.

## Examples

- The assignement $R \mapsto (R, +)$ yields a group functor $\mathbb{G}_a$ : the **additive group**. It is represented by the algebra $k[T]$.
- Likewise, $R \mapsto (R^\times, \times)$ yields a group functor $\mathbb{G}_m$ : the **multiplicative group**. It is represented by the algebra $k[T, T^{-1}]$.

# Finite group schemes

### Definition

An affine group scheme $G$ is **finite** if the algebra $\mathcal{O}(G)$ is finite-dimensional as a $k$-vector space.
The dimension of $\mathcal{O}(G)$ is then called the **order** of $G$, and denoted by $|G|$.

### Example

Let $G$ be a finite group of order $N$, and $\mathcal{O}(G)$ the algebra of $k$-valued functions on $G$ equipped with pointwise multiplication.
(This is the dual of the group algebra $k[G]$).
Then $G$ is a finite group scheme with Hopf algebra $\mathcal{O}(G)$ and order $N$.
Moreover, $G(k) \simeq G$ via evaluation of functions.

The following is a special case of a theorem of Cartier :

### Proposition

*If $p = 0$ then every finite group scheme is a (genuine) finite group.*

# Finite group schemes (continued)

In characteristic $p > 0$, there are additional examples :

- The group functor $R \mapsto \{t \in R \mid t^p = 0\}$ is represented by a finite group scheme $\alpha_p$ such that $\mathcal{O}(\alpha_p) = k[t]/(t^p)$.
  Moreover, $\alpha_p$ is the (schematic) kernel of the Frobenius endomorphism
  $$F : \mathbb{G}_a \longrightarrow \mathbb{G}_a, \quad t \longmapsto t^p.$$
- The group functor $R \mapsto \{t \in R^\times \mid t^p = 1\}$ is represented by a finite group scheme $\mu_p$ such that
  $\mathcal{O}(\mu_p) = k[t, t^{-1}]/(t^p - 1) \simeq k[t]/(t^p - 1)$.
  Moreover, $\mu_p$ is the kernel of the Frobenius endomorphism of $\mathbb{G}_m$.

Both $\alpha_p$ and $\mu_p$ have order $p$. One can show that every finite group scheme of order $p$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, $\alpha_p$ or $\mu_p$.

Also, $\alpha_p(k) = \{0\}$ and $\mu_p(k) = \{1\}$. We say that $\alpha_p$ and $\mu_p$ are **infinitesimal**.

# Actions of affine group schemes

Let $G$ be an affine group scheme, and $X$ an affine scheme. An **action** of $G$ on $X$ is an action of the group $G(R)$ on the set $X(R)$ for any algebra $R$, which is functorial in $R$.

The action is **free** if $G(R)$ acts freely on $X(R)$ for all $R$.

A morphism of affine schemes $f : X \to Y$ is $G$-**equivariant** if $f(R) : X(R) \to Y(R)$ is $G(R)$-equivariant for all $R$.

If $X = \mathrm{Spec}(A)$, then the $G$-actions on $X$ correspond bijectively to the $G$-actions on $A$ by algebra automorphisms (given for any algebra $R$ by an action of $G(R)$ on the $R$-algebra $A \otimes_k R$, functorial in $R$).

## Proposition

- *The actions of $\alpha_p$ on an affine scheme $X = \mathrm{Spec}(A)$ correspond bijectively to the derivations $D$ of $A$ such that $D^p = 0$.*
- *The actions of $\mu_p$ on $X$ correspond bijectively to the derivations $D$ of $A$ such that $D^p = D$.*

Thus, the $\mu_p$-actions on $X$ correspond bijectively to the $\mathbb{Z}/p\mathbb{Z}$-gradings of $A = \mathcal{O}(X)$.

# Quotients by finite group schemes

Let $G$ be a finite group scheme acting on an affine scheme $X = \mathrm{Spec}(A)$. Consider the subalgebra of invariants

$$A^G = \{f \in A \mid f(g \cdot x) = f(x) \text{ for all } R,\ g \in G(R),\ x \in X(R)\}.$$

The inclusion $A^G \subset A$ corresponds to a morphism of affine schemes

$$\pi : X \longrightarrow Y = X/G,$$

the **(categorical) quotient** by $G$.

If $A$ is finitely generated, then $A^G$ is finitely generated as well. If in addition $G$ is a genuine finite group and $X$ is a variety, then

$$A^G = \{f \in A \mid f(g \cdot x) = f(x) \text{ for all } \ g \in G(k),\ x \in X(k)\}$$

and $X/G$ is a variety.

## $G$-normal varieties

Let $G$ be a finite group scheme. A $G$-**variety** is a variety $X$ equipped with a $G$-action.

We say that an affine $G$-variety $X = \mathrm{Spec}(A)$ is $G$-**normal**, if $A$ is the largest $G$-stable subalgebra of its normalization. This is equivalent to the following geometric definition (which makes sense for a possibly non-affine $G$-variety $X$) :

*For any $G$-variety $Y$, every $G$-equivariant finite birational morphism $Y \to X$ is an isomorphism.*

### Proposition

*Let $X$ be a $G$-variety.*

- *Then $X$ admits a $G$-normalization $X' \to X$, unique up to unique $G$-equivariant isomorphism.*
- *If $X$ is $G$-normal, then the quotient variety $X/G$ is normal.*
- *Conversely, if $X/G$ is normal and $G$ acts freely on $X$, then $X$ is $G$-normal.*

# Example (continued)

Let again $A = k[x,y]/(y^p - f(x))$, where $f \notin k[x^p]$, and $X = \mathrm{Spec}(A)$. Then $\alpha_p$ acts on $X$ via

$$t \cdot (x,y) = (x, y+t).$$

This action is free, and the corresponding derivation is the partial derivative $\partial_y$. The quotient is $x : X \to \mathbb{A}^1$.

Also, $\mu_p$ acts on $X$ via

$$t \cdot (x,y) = (x, ty).$$

This action is free away from the points $(x,0)$, where $f(x) = 0$. The corresponding derivation is $y\partial_y$. The quotient is again $x : X \to \mathbb{A}^1$.

Since $\alpha_p$ acts freely on $X$ and the quotient variety $\mathbb{A}^1$ is normal, $X$ is $\alpha_p$-normal.

Also, we have seen that $X$ is $\mu_p$-normal if $f$ has simple roots.

Yet $X$ is nonnormal if $f'$ is nonconstant.

# Reduction to infinitesimal group schemes

The following classical result reduces somehow the structure of finite group schemes to the infinitesimal case :

## Proposition

*Let $G$ be a finite group scheme, and $G(k)$ its group of $k$-points.*
*Then $G$ has a largest infinitesimal normal subgroup scheme $G^0$.*
*Moreover, $G = G^0 \rtimes G(k)$.*

There is a similar reduction for equivariantly normal varieties :

## Theorem

*Let $G$ be as above, and $X$ a $G$-variety. Then $X$ is $G$-normal if and only if it is $G^0$-normal.*

In particular, if $G$ is a genuine finite group then $G$-normality is equivalent to normality (this is easily checked directly).

# Equivariantly normal curves

A **function field of one variable** is a separable, finitely generated field extension $K/k$ of transcendence degree 1.

Equivalently, $K$ is the field of rational functions $k(X)$, where $X$ is a curve. Moreover, $X$ may be taken normal (or equivalently, smooth) and projective.

For instance, $k(\mathbb{A}^1) = k(\mathbb{P}^1)$ is the field of rational functions $k(t)$, where $\mathbb{P}^1$ denotes the projective line.

Classically, the assignement $X \mapsto k(X)$ yields an anti-equivalence of categories between normal projective curves and function fields of one variable. This can be generalized as follows :

### Theorem

*Let $G$ be a finite group scheme. Then the above assignement yields an anti-equivalence of categories between $G$-normal projective curves and function fields of one variable equipped with a (functorial) $G$-action.*

## Examples

Let $K$ be a function field in one variable, and $K^p \subset K$ the subfield of $p$th powers. Then there exists $t \in K$ such that

$$K = \bigoplus_{m=0}^{p-1} K^p \, t^m$$

(this is equivalent to $K$ being separable algebraic over $k(t)$).

The above decomposition is a $\mathbb{Z}/p\mathbb{Z}$-grading of $K$, and hence corresponds to a $\mu_p$-action. The corresponding derivation $D$ is uniquely determined by the condition that $D(t) = t$, i.e., $D = t \, \frac{\mathrm{d}}{\mathrm{d}t}$.

The data of $K$ and $t$ yield a unique $\mu_p$-normal projective curve $X$ such that $k(X) = K$. If the geometric genus of $X$ is at least 2, then $X$ is non-normal (since a smooth projective curve of genus $g \geq 2$ has no nonzero global vector fields).

One can show that every projective $\mu_p$-normal curve is obtained via this construction. Likewise, every projective $\alpha_p$-normal curve arises from a derivation of the form $\frac{\mathrm{d}}{\mathrm{d}t}$ for $t$ as above.

# A $G$-normality criterion for curves

**Theorem**

*Let $G$ be an infinitesimal group scheme, and $X$ a $G$-curve.
Then $X$ is $G$-normal if and only if the largest $G$-stable ideal of the local ring $\mathcal{O}_{X,x}$ is principal for any $x \in X$.*

We may then see each $\mathcal{O}_{X,x}$ as a $G$-discrete valuation ring.

**Corollary**

*Let $G$ be infinitesimal of order $p$, and $X$ a $G$-curve.
Then $X$ is $G$-normal if and only if it satisfies the following conditions : $X/G$ is smooth and $X$ is smooth at every $G$-fixed point.*

**Example**

Let $f \in k[x, y]$ be homogeneous of degree $p$, and not a $p$th power. Let $X \subset \mathbb{P}^2$ the curve with homogeneous equation $z^p - f(x, y) = 0$.
Then $\mu_p$ acts on $X$ via $t \cdot (x, y, z) = (x, y, tz)$. The quotient is the projection $(x, y) : X \to \mathbb{P}^1$ and the fixed points are exactly the $(x, y, 0)$, where $f(x, y) = 0$. So $X$ is $\mu_p$-normal if and only if $f$ has simple roots.

# A further $G$-normality criterion

Let again $G$ be a finite group scheme.
Then $G$ is a subgroup scheme of a smooth connected algebraic group $G^\#$.
For example, we may take $G^\# = \mathrm{GL}(\mathcal{O}(G))$ in which $G$ embeds via the regular representation.
Given a $G$-variety, we may form the "induced" variety $X^\# = (G^\# \times X)/G$, where $G$ acts on $G^\# \times X$ via $g \cdot (h, x) = (hg^{-1}, g \cdot x)$.
This is indeed a variety, equipped with an action of $G^\#$ via left multiplication on itself.

## Proposition

*$X$ is $G$-normal if and only if $X^\#$ is normal.*
*If $X$ is a curve, this is equivalent to $X^\#$ being smooth.*

The projection $G^\# \times X \to X$ gives a morphism $\varphi : X^\# \to G^\#/G$ which is $G^\#$-equivariant, faithfully flat, and has fiber $X$ at the base point.
This yields :

## Corollary

*Every $G$-normal curve $X$ is a local complete intersection.*

# A Hurwitz formula for $G$-normal curves

A finite group scheme $G$ is **linearly reductive** if every representation of $G$ is semi-simple.
This is equivalent to the infinitesimal part $G^0$ being a product of $\mu_{p^r}$'s, and the order of $G(k)$ being prime to $p$.

### Theorem

*Let $G$ be linearly reductive, $X$ a $G$-normal curve, and $\pi : X \to Y = X/G$ the quotient. Then*

$$\omega_X = \pi^*(\omega_Y)(\sum(n_x - 1)G \cdot x),$$

*where the sum runs over the $G$-orbits $G \cdot x$, $x \in X(k)$, and $n_x$ denotes the order of the stabilizer $G_x$.*

Here $\omega_X$ denotes the dualizing sheaf of $X$ (which exists since $X$ is a local complete intersection).
If $G$ is a genuine finite group, then $X$ is smooth and hence $\omega_X = \Omega^1_X$.
We then recover the Hurwitz formula for tamely ramified Galois covers of smooth curves. But if $G$ is infinitesimal, then $\pi$ is purely inseparable.

# $G$-normal curves and smooth projective surfaces

Let $E$ be an elliptic curve, and $G \subset E$ a finite subgroup scheme. Given a projective $G$-normal curve $X$, the quotient

$$S = (E \times X)/G$$

is a smooth projective surface equipped with an action of $E$.

One can show that every smooth projective surface whose automorphism group contains an elliptic curve is obtained by this construction.

The quotient $\pi : X \to Y = X/G$ defines a morphism

$$f : S \longrightarrow Y = S/E,$$

which is an elliptic fibration with general fiber $E$.

If $G$ is linearly reductive, then the Hurwitz formula for $\omega_X$ yields a version of the canonical bundle formula for this elliptic fibration, due to Bombieri and Mumford.

The geometry of these elliptic surfaces has been recently investigated by Fong and Maccan. Their canonical ring satisfies $R(S) = R(X)^G$, where $R(X) = \bigoplus_{m=0}^{\infty} H^0(X, \omega_X^{\otimes m})$.

# Example

Consider again the curve $X \subset \mathbb{P}^2$ with homogeneous equation $z^p - f(x,y) = 0$, where $f \in k[x,y]$ is homogeneous of degree $p$ with simple roots.
Recall that $X$ is $G$-normal for the action of $G = \mu_p$ via $t \cdot (x,y,z) = (x,y,tz)$. Moreover, $X/G = \mathbb{P}^1$.
Also, $X$ is rational, and singular if $p \geq 3$. We have $\omega_X = \mathcal{O}_X(p-3)$.

Let $E$ be an ordinary elliptic curve. Then $E$ contains a unique copy of $G$.
The smooth projective surface $S = (E \times X)/G$ comes with two fibrations : the elliptic fibration $f : S \to X/G = \mathbb{P}^1$, and $\varphi : S \to E/G$ with fiber $X$.

If $p = 2$ then $\varphi$ is a ruling ; in particular, $\kappa(S) = -\infty$.

If $p \geq 3$ then $S$ is covered by rational curves (the fibers of $\varphi$) and none of them is smooth. We have $\kappa(S) = \begin{cases} 0 \text{ if } p = 3, \\ 1 \text{ if } p \geq 5. \end{cases}$

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