On configuration space integrals for links

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Abstract

In this article, we shall begin with an elementary introduction to the universal Vassiliev invariant called *the perturbative series expansion of the Chern-Simons theory of links in euclidean space* defined by means of configuration space integrals. Whether this universal Vassiliev invariant coincides with the Kontsevich integral is still open to mathematicians despite the substantial progress of Sylvain Poirier [22] who in particular reduced this question to the computation of the *anomaly* of Bott, Taubes, Altschuler, Freidel and D. Thurston, which is an element α of the space $\mathcal{A}(S^1)$ of Jacobi diagrams. We shall give a short survey of the Poirier work that allowed the author to define the isomorphism of $\mathcal{A}$ which transforms the Kontsevich integral into the Poirier limit of the Chern-Simons invariant of framed links, as an explicit function of α, and to prove the algebraic property of the anomaly : *The anomaly has two legs.* We end up the article by proving an additional geometric property of the anomaly that allowed Poirier to compute the anomaly up to degree 6.

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1 Introduction

There are essentially two universal Vassiliev invariants of links, the *Kontsevich integral,* and the *perturbative series expansion of the Chern-Simons theory* studied by Guadagnini Martellini and Mintchev [10], Bar-Natan [5], Axelrod and Singer [2, 3], Kontsevich [11], Polyak and Viro [23], Bott and Taubes [8], Altschuler and Freidel [1], D. Thurston [24], Yang, Poirier [22]... The question that was first raised by Kontsevich in [11] whether the two invariants coincide or not is still open to mathematicians despite the substantial progress of Sylvain Poirier who in particular reduced this question to the computation of the *anomaly* of Bott, Taubes, Altschuler, Freidel and D. Thurston, which is an element α of the space of Jacobi diagrams $\mathcal{A}(S^1)$.

In this article, we shall begin with an elementary introduction to *the perturbative series expansion of the Chern-Simons theory of links in euclidean space* defined by means of configuration space integrals in a natural and beautiful way. We shall call this series the Chern-Simons series, and we shall denote it by Z_{CS}. Its physical interpretation will not be treated here and we refer the reader to the survey [16] of Labastida for the interpretation of Z_{CS} in the context of the Chern-Simons gauge theory.

Then we shall give an introductory survey of the Poirier limit extension of the configuration space integrals to a monoidal functor on the category of combinatorial framed q-tangles. In particular, we shall review all the main properties that are shared by the Kontsevich integral and the Poirier limit of the Chern-Simons series. We shall call an invariant of framed links satisfying all these properties *a good monoidal functor from the category of framed q-tangles to* $\mathcal{A}$. Then, we shall review the author's refinement [20] of a theorem of Le and Murakami [17, Theorem 8] inspired by Drinfeld and Kontsevich that could be stated as follows: *A good monoidal functor that varies like the Kontsevich integral* Z_K *under a framing change must coincide with* Z_K *on framed links.* Our refinement is stated in Theorem 9.3. Roughly speaking, we list the possible variations of good monoidal functors under framing changes and we define some special type of isomorphisms of $\mathcal{A}$, *the* $\Psi(\beta)$, so that, when restricted to framed links, any good monoidal functor is of the form $\Psi(\beta) \circ Z_K$.

This allows us to define the isomorphism of $\mathcal{A}$ which transforms the Kontsevich integral into the Poirier limit [22] of the Chern-Simons series Z_{CS}, as an explicit function of the anomaly. This explicit relation and the Poirier estimates on the denominators of Z_{CS} allowed us to show that the denominators of the degree n part of the Kontsevich integral of framed links divide into $(2!3!\dots(n-$

5)!)$(n-5)!3^2(3n-4)!2^{2n+2}$ for $n \geq 5$ in [20]. These denominators have more prime factors than the denominators $(2!3!\dots n!)^4(n+1)!$ of Le [13] but they are smaller. Another corollary is the algebraic property of the anomaly : *The anomaly has two legs.* We end up the article by reviewing the known properties of the anomaly and by proving an additional geometric property of the anomaly that allowed Poirier to compute the anomaly up to degree 6.

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2 Introduction to configuration space integrals: The Gauss integrals.

In 1833, Carl Friedrich Gauss defined the first example of a *configuration space integral* for an oriented two-component link. Let us formulate his definition in a modern language. Consider a smooth (C^∞) embedding

$$L : S^1_1 \coprod S^1_2 \hookrightarrow \mathbb{R}^3$$

of the disjoint union of two circles $S^1 = \{z \in \mathbb{C} \text{ s.t. } |z| = 1\}$ into $\mathbb{R}^3$. With an element (z_1, z_2) of $S^1_1 \times S^1_2$ that will be called a *configuration*, we associate the oriented direction

$$\Psi((z_1, z_2)) = \frac{1}{\parallel \overrightarrow{L(z_1)L(z_2)} \parallel} \overrightarrow{L(z_1)L(z_2)} \in S^2$$

of the vector $\overrightarrow{L(z_1)L(z_2)}$. Thus, we have associated a map

$$\Psi : S^1_1 \times S^1_2 \longrightarrow S^2$$

from a compact oriented 2-manifold to another one with our embedding. This map has an integral degree $\deg(\Psi)$ that can be defined in several equivalent ways. For example, it is the *differential degree* $\deg(\Psi, y)$ of any regular value y of Ψ, that is the sum of the ± 1 signs of the Jacobians of Ψ at the points of the preimage of y [21, §5]. Thus, $\deg(\Psi)$ can easily be computed from a regular diagram of our two-component link as the differential degree of a unit vector $\overrightarrow{v}$ pointing to the reader or as the differential degree of $(-\overrightarrow{v})$.

$$\deg(\Psi) = \deg(\Psi, \overrightarrow{v}) = \sharp \,{}_2\!\times_1 - \sharp \,{}_1\!\times_2 = \deg(\Psi, -\overrightarrow{v}) = \sharp \,{}_1\!\times_2 - \sharp \,{}_2\!\times_1$$

It can also be defined as the following *configuration space integral*

$$\deg(\Psi) = \int_{S^1 \times S^1} \Psi^*(\omega)$$

where ω is the homogeneous volume form on S^2 such that $\int_{S^2} \omega = 1$. Of course, this integral degree is an isotopy invariant of L, and the reader has recognized that $\deg(\Psi)$ is nothing but the *linking number* of the two components of L.

We can again follow Gauss and associate the following similar *Gauss integral* $I(K;\theta)$ to a C^∞ embedding $K : S^1 \hookrightarrow \mathbb{R}^3$. Here, we consider the *configuration space* $C(K;\theta) = S^1 \times]0, 2\pi[$, and the map

$$\Psi : C(K;\theta) \longrightarrow S^2$$

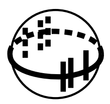
that maps (z_1, η) to the oriented direction of $\overrightarrow{K(z_1)K(z_1 e^{i\eta})}$, and we set

$$I(K;\theta) = \int_{C(K;\theta)} \Psi^*(\omega).$$

Let us compute $I(K;\theta)$ in some cases. First notice that Ψ may be extended to the closed annulus

$$\overline{C}(K;\theta) = S^1 \times [0, 2\pi]$$

by the tangent map K' of K along $S^1 \times \{0\}$ and by $(-K')$ along $S^1 \times \{2\pi\}$. Then by definition, $I(K;\theta)$ is the algebraic area (the integral of the differential degree with respect to the measure associated with ω) of the image of the annulus in S^2. Now, assume that K is contained in a horizontal plane except in a neighborhood of crossings where it entirely lies in vertical planes. Such a knot embedding will be called *almost horizontal.* In that case, the image of the annulus boundary has the shape of the following bold line in S^2.



In particular, the differential degree that extends to a constant function over S^2 outside the image of the annulus boundary, extends to a map that is constant on each hemisphere, and $I(K;\theta)$ is the average of the differential degree of two regular points located at the two hemispheres. Thus, in this case, it can be computed from a regular projection on the horizontal plane as

$$I(K;\theta) = \frac{\deg(\Psi, \overrightarrow{v}) + \deg(\Psi, -\overrightarrow{v})}{2} = \sharp \; - \sharp \; .$$

This number, that is called the *writhe* of the projection, can be changed without changing the isotopy class of the knot by local modifications where —— becomes  or . In particular, $I(K;\theta)$ can reach any integral value on a given isotopy class of knots, and since it varies continuously on such a class, it can reach any real value on any given isotopy class of knots. Thus, this Gauss integral is NOT an isotopy invariant.

However, we can follow Guadagnini, Martellini, Mintchev [10] and Bar-Natan [5] and associate configuration space integrals to any embedding L of an oriented one-manifold M and to any Jacobi diagram Γ on M without small loop like .

Let us first recall what a Jacobi diagram on a one-manifold is.

3 Definitions of the spaces of Jacobi diagrams

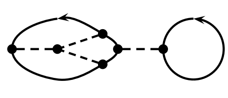
Definition 3.1 Let M be an oriented one-manifold. A *Jacobi diagram* Γ with support M is a finite uni-trivalent graph Γ such that every connected component of Γ has at least one univalent vertex, equipped with:

(1) an isotopy class of injections i of the set U of univalent vertices of Γ also called *legs* of Γ into the interior of M,

(2) an *orientation* of every trivalent vertex, that is a cyclic order on the set of the three half-edges which meet at this vertex.

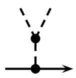
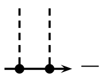
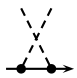
Such a diagram Γ is represented by a planar immersion of $\Gamma \cup M$ where the univalent vertices of Γ are identified with their images under i, the one-manifold M is represented by solid lines, whereas the diagram Γ is dashed. The vertices are represented by big points. The orientation of a vertex is represented by the counterclockwise order of the three dashed half-edges that meet at that vertex.

Here is an example of a diagram Γ on the disjoint union $M = S^1 \coprod S^1$ of two circles:



The *degree* of such a diagram is half the number of all its vertices.

Let $\mathcal{A}_n^{\mathbb{Q}}(M)$ denote the rational vector space generated by the degree n diagrams on M, quotiented out by the following relations AS and STU:

AS :  +  $= 0$ and STU :  =  − 

Each of these relations relate diagrams which can be represented by immersions that are identical outside the part of them represented in the pictures. For example, AS identifies the sum of two diagrams which only differ by the orientation at one vertex to zero.

Let $\mathcal{A}_n(M) = \mathcal{A}_n^{\mathbb{R}}(M) = \mathcal{A}_n^{\mathbb{Q}}(M) \otimes_{\mathbb{Q}} \mathbb{R}$ and let

$$\mathcal{A}(M) = \prod_{n \in \mathbb{N}} \mathcal{A}_n(M)$$

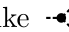
denote the product of the $\mathcal{A}_n(M)$ as a topological vector space. $\mathcal{A}_0(M)$ is equal to $\mathbb{R}$ generated by the empty diagram.

4 The Chern-Simons Vassiliev invariant

Let M be an oriented one-manifold and let

$$L : M \longrightarrow \mathbb{R}^3$$

denote a C^∞ embedding from M to $\mathbb{R}^3$.

Let Γ be a Jacobi diagram on M without small loop like . Let U denote the set of univalent vertices of Γ, and let T denote the set of trivalent vertices of Γ.

A *configuration* of Γ is an embedding

$$c : U \cup T \hookrightarrow \mathbb{R}^3$$

whose restriction $c_{|U}$ to U may be written as $L \circ j$ for some injection

$$j : U \hookrightarrow M$$

in the given isotopy class $[i]$ of embeddings of U into the interior of M. Denote the set of these configurations by $C(L;\Gamma)$,

$$C(L;\Gamma) = \left\{ c : U \cup T \hookrightarrow \mathbb{R}^3 \, ; \exists j \in [i], c_{|U} = L \circ j \right\}.$$

In $C(L;\Gamma)$, the univalent vertices move along $L(M)$ while the trivalent vertices move in the ambient space, and $C(L;\Gamma)$ is naturally an open submanifold of $M^U \times (\mathbb{R}^3)^T$.

Denote the set of (dashed) edges of Γ by $E = E(\Gamma)$, and fix an orientation for these edges. Define the map $\Psi : C(L;\Gamma) \longrightarrow \left(S^2\right)^E$ whose projection to the S^2 factor indexed by an edge from a vertex v_1 to a vertex v_2 is the direction of $\overrightarrow{c(v_1)c(v_2)}$.

This map Ψ is again a map between two orientable manifolds that have the same dimension, namely the number of (dashed) half-edges of Γ, and we can write the *configuration space integral:*

$$I(L;\Gamma) = \int_{C(L;\Gamma)} \Psi^*(\Lambda^E \omega).$$

Bott and Taubes have proved that this integral is convergent [8]. See also Sections 6, 7 below. Thus, this integral is well-defined up to sign. In fact, the orientation of the trivalent vertices of Γ provides $I(L;\Gamma)$ with a well-defined sign. Indeed, since S^2 is equipped with its standard orientation, it is enough to orient $C(L;\Gamma) \subset M^U \times (\mathbb{R}^3)^T$ in order to define this sign. This will be done by providing the set of the natural coordinates of $M^U \times (\mathbb{R}^3)^T$ with some order up to an even permutation. This set is in one-to-one correspondence with the set of (dashed) half-edges of Γ, and the vertex-orientation of the trivalent vertices provides a natural preferred such one-to-one correspondence up to some (even!) cyclic permutations of three half-edges meeting at a trivalent vertex. Fix an order on E, then the set of half-edges becomes ordered by (origin of the first edge, endpoint of the first edge, origin of the second edge, ..., endpoint of the last edge), and this order orients $C(L;\Gamma)$. The property of this sign is that the product $I(L;\Gamma)[\Gamma] \in \mathcal{A}(M)$ depends neither on our various choices nor on the vertex orientation of Γ.

Now, the *perturbative series expansion of the Chern-Simons theory for one-manifold embeddings in* $\mathbb{R}^3$ is the following sum running over all the Jacobi diagrams Γ without small loops and without vertex orientation[1]:

$$Z_{\mathrm{Chern-Simons}}(L) = \sum_{\Gamma} \frac{I(L;\Gamma)}{\sharp\mathrm{Aut}\Gamma}[\Gamma] \quad \in \mathcal{A}(M)$$

where $\sharp\mathrm{Aut}\Gamma$ is the number of automorphisms of Γ as a uni-trivalent graph with a given isotopy class of injections of U into M, but without vertex-orientation for the trivalent vertices.

[1]This sum runs over equivalence classes of Jacobi diagrams without small loops, where two diagrams are equivalent if and only if they coincide except possibly for their vertex orientation.

When L is a knot K, the degree one part of Z_{CS} is $\frac{I(K;\theta)}{2}[\theta]$ and therefore Z_{CS} is not invariant under isotopy. However, the evaluation Z^0_{CS} at representatives of knots with null Gauss integral is an isotopy invariant that is a universal Vassiliev invariant of knots. (All the finite type knot invariants in the Vassiliev sense (see [4]) factor through it.) This is the content of the following theorem, due independently to Altschuler and Freidel [1], and to D. Thurston [24], after the work of many people including Guadagnini, Martellini and Mintchev [10], Bar-Natan [5], Axelrod and Singer [2, 3], Kontsevich [11], Bott and Taubes [8]...

Theorem 4.1 (Altschuler-Freidel, D. Thurston, 1995) *If $L = K_1 \coprod \cdots \coprod K_k$ is a link, then $Z_{CS}(L)$ only depends on the isotopy class of L and on the Gauss integrals $I(K_i;\theta)$ of its components. In particular, the evaluation*

$$Z^0_{CS}(L) \in \prod_{n\in\mathbb{N}} \mathcal{A}_n(\coprod_{i=1}^{k} S^1_i)$$

at representatives of L whose components have zero Gauss integrals is an isotopy invariant of L. Furthermore, Z^0_{CS} is a universal Vassiliev invariant of links.

Recall that the normalized Kontsevich integral is also a universal Vassiliev knot invariant that is valued in the same target $\mathcal{A}$. (See [4, 19].) Thus, the still open natural question raised by Kontsevich in [11] is:

Is the Kontsevich integral of a zero framed representative of a knot K equal to $Z^0_{CS}(K)$?

On one hand, this Chern-Simons series has a beautiful, very natural and completely symmetric definition. Furthermore, it has been shown to be *rational*[2] by nature by Dylan Thurston [24]. See the end of Section 7.

On the other hand, the Kontsevich integral fits in with the framework of quantum link invariants and it can be defined in this setting [17, 15]. Therefore, it is explicitly known how to recover quantum link invariants from the Kontsevich integral [17]. Furthermore, the computation of the Kontsevich integral for links can be reduced to the computation of small link pieces called *elementary q-tangles.* In [22], Sylvain Poirier proved that the same can be done for the Chern-Simons series. Let us begin a review of his results.

[2] For a link L, the degree n part $Z^0_{CSn}(L)$ of $Z^0_{CS}(L)$ belongs to $\mathcal{A}^{\mathbb{Q}}_n(\coprod_{i=1}^k S^1_i)$.

5 The Poirier extension of Z_{CS} to tangles

A *planar configuration* is an embedding of a finite set X into the plane $\mathbb{R}^2$.

In the *ambient space* $\mathbb{R}^3 = \{(x,y,z)\}$, the *horizontal plane* is the plane $(z=0)$, whereas the *blackboard plane* is the plane $(y=0)$. The z-coordinate of a point $(x,y,z) \in \mathbb{R}^3$ is called its *vertical projection*.

A *tangle* is the intersection of the image of a link representative transverse to $\mathbb{R}^2 \times \{\beta, \tau\}$ with a horizontal slice $\mathbb{R}^2 \times [\beta, \tau]$. In particular, it is an embedded cobordism between two planar configurations.

Let $\lambda \in]0,1]$ and let h_λ be the homeomorphism of $\mathbb{R}^3$ that shrinks the horizontal plane with respect to the formula

$$h_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, x_3).$$

Let $L(M)$ be a tangle. Set

$$Z_{\text{Poirier}}(L(M)) = \lim_{\lambda \to 0} Z_{CS}(h_\lambda \circ L) \in \mathcal{A}(M)$$

Sylvain Poirier proved that this limit exists [22].

Let K be an almost horizontal knot embedding, rotated by 90 degrees around a horizontal axis, (so that it is almost contained in some vertical plane) such that $I(K;\theta) = 0$. Then $I(h_\lambda \circ K; \theta) = 0$, for any $\lambda > 0$, and the Poirier limit Z_P is the limit of a constant map. Therefore, Z_P is equal to the Chern-Simons series we started with for these representatives.

In general, we can see that the limit of $I(h_\lambda \circ K; \theta)$ depends on the differential degree of Ψ near the equator of S^2. Assume that the height function (the third coordinate) of K is a Morse function, (its second derivative does not vanish when the first one does). Then all the horizontal tangent vectors correspond to extrema of the height function. Identify the horizontal plane to $\mathbb{C}$ so that the unit horizontal vector corresponding to an extremum e is of the form $\exp(i\theta_e)$. When λ approaches 0, all the non-horizontal tangent vectors approach the poles, and the image of our annulus boundary becomes a family of straight meridians intersecting the equator at directions $\exp(i\theta_e)$, $\exp(i(\theta_e + \pi))$ corresponding to extrema e. These meridians cut our sphere like an orange, and the differential degree becomes constant on the boundaries of orange quarters and makes integral jumps at meridians. Thus, it can be seen that

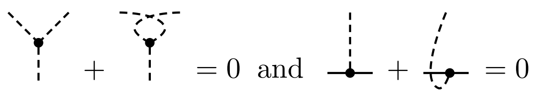
$$\lim_{\lambda \to 0} I(h_\lambda \circ K; \theta) \equiv \frac{1}{\pi} \left(\sum_{\text{e minimum}} \theta_e - \sum_{\text{e maximum}} \theta_e \right) \bmod \mathbb{Z}$$

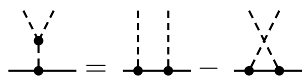
In particular, this limit is an integer when the horizontal vectors are in the blackboard plane, and does not vary under isotopies that keep the directions of the horizontal vectors fixed. This motivates our next definition of framed tangles.

Definition 5.1 A *framed tangle* is represented by a C^∞ embedding T of a compact one-manifold M into a horizontal slice $\mathbb{R}^2 \times [\beta, \tau]$ of $\mathbb{R}^3$ such that the only horizontal tangent vectors of T occur for interior points of M and are parallel to the blackboard plane, and $T(\partial M) = T(M) \cap \mathbb{R}^2 \times \{\beta, \tau\}$. Such a T is considered up to the isotopies which fix ∂M and which satisfy the above hypotheses at any time, and up to a *rescaling of the height parameter,* that is a composition by $1_{\mathbb{R}^2} \times h$ where h is an increasing diffeomorphism from $[\beta, \tau]$ to another interval of $\mathbb{R}$. The projection onto $\mathbb{R}^2$ of $T(M) \cap \mathbb{R}^2 \times \{\beta\}$ is called the *bottom configuration* of the tangle whereas the projection onto $\mathbb{R}^2$ of $T(M) \cap \mathbb{R}^2 \times \{\tau\}$ is called the *top configuration* of the tangle.

It is more convenient to work with unoriented tangles. Therefore, we shall discuss tangle orientations in order to forget them forever afterwards.

When assigning a sign to a configuration space integral $I(L(M); \Gamma)$, we can replace the datum of a global orientation of M by the datum of a local orientation of M near the image under i of every given univalent vertex (that will orient the coordinate in M corresponding to this vertex in $C(L; \Gamma)$). Then $I(L(M); \Gamma)$ will be multiplied by -1 if the local orientation of one vertex is modified. The datum of such a local orientation at $(v \in U)$ is equivalent to the datum of a cyclic order of the three half-edges of $M \cup_U \Gamma$ meeting at v. Namely, we agree that the cyclic order v (without any specified orientation on the solid line) represents the local orientation v. A *diagram on a non-necessarily oriented one-manifold M* is now a diagram as before equipped with a cyclic order called *orientation* at all its univalent vertices, and we define $\mathcal{A}_n^{\mathbb{Q}}(M)$ as the rational vector space generated by these degree n diagrams quotiented by AS and STU, with

AS : 

STU : 

The linear map that maps a diagram (modulo relations) on an oriented manifold M as before to the diagram obtained by giving each univalent vertex the orientation matching the global orientation of M is a natural isomorphism from the former $\mathcal{A}_n^{\mathbb{Q}}(M)$ to the new one. With this definition, the product $I(L(M);\Gamma)[\Gamma]$ is well-defined for an unoriented M.

When the top configuration of a tangle T_1 coincides with the bottom configuration of a tangle T_2, these tangles can be composed by *stacking* T_2 *above* T_1. The *product* of two such framed tangles $T_1 = T_1(M_1)$ and $T_2 = T_2(M_2)$ is defined as the tangle $T = T_1 T_2 = T(M)$ such that T, T_1 and T_2 are represented by embeddings T, T_1 and T_2 such that there exists a regular value γ of the vertical projection of the embedding $T : M \longrightarrow \mathbb{R}^2 \times [\beta, \tau]$ where $M_1 = T^{-1}(\mathbb{R}^2 \times [\beta, \gamma])$, $M_2 = T^{-1}(\mathbb{R}^2 \times [\gamma, \tau])$, and where T_1 and T_2 are the restrictions of T to M_1 and M_2, respectively.

Now, let us define a product of diagrams corresponding to this composition of tangles. Assume that a one-manifold M is decomposed as a union $M = M_1 \cup M_2$ of two one-manifolds with disjoint interiors. Then, we define the *product associated to this decomposition*:

$$\mathcal{A}(M_1) \times \mathcal{A}(M_2) \longrightarrow \mathcal{A}(M)$$

as the continuous bilinear map which maps $([\Gamma_1], [\Gamma_2])$ to $[\Gamma_1 \coprod \Gamma_2]$, if Γ_1 is a diagram with support M_1 and if Γ_2 is a diagram with support M_2, where $\Gamma_1 \coprod \Gamma_2$ denotes their disjoint union. (Of course, the needed isotopy class of injections is naturally induced by the two former ones.) In the particular case where M_1 and M_2 are disjoint, this product is sometimes denoted by $\otimes$.

Let $I = [0, 1]$ be the compact oriented interval. Another particular case is the case where M is an ordered union of p intervals which are seen as vertical

$$M = | \, | \cdots | = \{1, 2, \ldots, p\} \times [0, 1]$$

Then if we naturally identify M to $M_1 = \{1, 2, \ldots, p\} \times [0, 1/2]$, and to $M_2 = \{1, 2, \ldots, p\} \times [1/2, 1]$, the above process turns $\mathcal{A}(M)$ into an algebra where the elements with degree zero part 1 admit an inverse and a unique square root whose degree zero part is 1.

With each choice of a connected component C of M and of an orientation of C, we associate an $\mathcal{A}(I)$*-module structure on* $\mathcal{A}(M)$, that is given by the continuous bilinear map:

$$\mathcal{A}(I) \times \mathcal{A}(M) \longrightarrow \mathcal{A}(M)$$

such that if Γ is a diagram with support I and if Γ' is a diagram with support M, then $([\Gamma], [\Gamma'])$ is mapped to the diagram obtained by inserting $(I \cup \Gamma)$ along C outside the vertices of Γ', according to the given orientation. The obtained class is independent of the choice of the insertion locus. (See [4, Proof of Lemma 3.1].) In particular, $\mathcal{A}(I)$ is a commutative algebra. The morphism from $\mathcal{A}(I)$ to $\mathcal{A}(S^1)$ induced by the identification of the two endpoints of I is an isomorphism from $\mathcal{A}(I)$ to $\mathcal{A}(S^1)$. (See [4, Lemma 3.1].)

In [22], Sylvain Poirier proved that his limit Z_P is a functorial invariant of framed tangles. More precisely, he proved that his limit is invariant under the framed isotopies that fix $T(\partial M)$ up to translations and homotheties with a positive ratio. The homotheties with a positive ratio will be called *dilatations*. Furthermore, he proved that Z_P is multiplicative with respect to the product of tangles. Graphically, this reads

$$Z_P\left(\begin{array}{|c|}\hline T_2 \\ \hline T_1 \\ \hline\end{array}\right) = \begin{array}{|c|}\hline Z_P(T_2) \\ \hline Z_P(T_1) \\ \hline\end{array}$$

In other words, he proved that his limit process kills the configurations where two univalent vertices of some dashed connected component of a diagram Γ sit in different horizontal slices. The idea of considering this limit is natural and was considered before, but the proof that it is a well-defined and well-behaved isotopy invariant of tangles requires a deep understanding of configuration spaces and of their compactifications that are similar to those presented in [14, 8, 24]. Therefore, we shall make a short trip inside these spaces following their presentation by Poirier [22, Subsection 10.1].

6 Compactifications of configuration spaces

Let $X = \{\xi_1, \xi_2, \ldots, \xi_p\}$ be a finite set of cardinality $p \geq 2$, let k denote a positive integer. Let $C_0(X; \mathbb{R}^k)$ (resp. $C(X; \mathbb{R}^k)$) denote the set of injections (resp. of non-constant maps) f from X to $\mathbb{R}^k$, up to translations and dilatations. $C_0(X; \mathbb{R}^k)$ is the quotient of

$$\{(x_1 = f(\xi_1), x_2 = f(\xi_2), \ldots, x_p = f(\xi_p)) \in (\mathbb{R}^k)^p; x_i \neq x_j \text{ if } i \neq j\}$$

by the translations which identify $(x_1, x_2, \ldots, x_p)$ to $(x_1 + T, x_2 + T, \ldots, x_p + T)$ for all $T \in \mathbb{R}^k$ and by the dilatations which identify $(x_1, x_2, \ldots, x_p)$ to $(\lambda x_1, \lambda x_2, \ldots, \lambda x_p)$ for all $\lambda > 0$.

Examples 6.1 1. For example, $C_0(X; \mathbb{R})$ has $p!$ connected components corresponding to the possible orders of the set X. Each of its components can be identified with the interior $\{(x_2, x_3, \ldots, x_{p-1}) \in \mathbb{R}^{p-2}; 0 < x_2 < x_3 < \cdots < x_{p-1} < 1\}$ of a $(p-2)$ simplex.

2. As another example, $C(\{1,2\};\mathbb{R}^k) = C_0(\{1,2\};\mathbb{R}^k)$ is homeomorphic to the sphere S^{k-1}.

In general, the choice of a point $\xi \in X$ provides a homeomorphism

$$\begin{array}{rlcl} \phi_\xi : & C(X,\mathbb{R}^k) & \longrightarrow & S^{kp-k-1} \\ & f & \mapsto & \left(x \mapsto \frac{f(x)-f(\xi)}{\|\sum_{i=1}^p (f(\xi_i)-f(\xi))\|} \right) \end{array}$$

where S^{kp-k-1} is the unit sphere of $\left(\mathbb{R}^k\right)^{p-1}$. These homeomorphisms equip $C(X,\mathbb{R}^k)$ with an analytic (C^ω) structure and make clear that $C(X,\mathbb{R}^k)$ is compact. There is a natural embedding

$$\begin{array}{rlcl} i : & C_0(X;\mathbb{R}^k) & \hookrightarrow & \prod_{A\subset X;\sharp A\geq 2} C(A;\mathbb{R}^k) \\ & c_X & \mapsto & (c_{X|A})_{A\subset X;\sharp A\geq 2} \end{array}$$

where $c_{X|A}$ denotes the restriction of c_X to A. Define the compactification $C(X;k)$ of $C_0(X;\mathbb{R}^k)$ as

$$C(X;k) = \overline{i(C_0(X;\mathbb{R}^k))} \subset \prod_{A\subset X;\sharp A\geq 2} C(A;\mathbb{R}^k)$$

In words, in $C(X;k)$, some points of X are allowed to collide with each other, or to become infinitely closer to each other than they are to other points, but the compactification provides us with the magnifying glasses $C(A;\mathbb{R}^k)$ that allow us to see the infinitely small configurations at the scales of the collisions.

Observe that the elements $(c_A)_{A\subset X;\sharp A\geq 2}$ of $C(X;k)$ satisfy the following condition $(\star)$.

$(\star)$: If $B \subset A$, then the restriction $c_{A|B}$ of c_A to B is either constant or equal to c_B.

Indeed, the above condition holds for elements of $i(C_0(X;\mathbb{R}^k))$, and it can be rewritten as the following condition that is obviously closed. *For any two sets A and B such that $B \subset A$, if $x \in B$, the two vectors of $(\mathbb{R}^k)^{B\setminus\{x\}}$, $(c_{A|B}(y) - c_{A|B}(x))_{y\in B\setminus\{x\}}$ and $(c_B(y) - c_B(x))_{y\in B\setminus\{x\}}$, are colinear, and their scalar product is non negative.*

Lemma 6.2 *The set $C(X;k)$ has a natural structure of an analytic manifold with corners*[3] *and*

$$C(X;k) = \{(c_A) \in \prod_{A\subset X;\sharp A\geq 2} C(A;\mathbb{R}^k);\ (c_A) \text{ satisfies } (\star)\}$$

[3]Every point c of $C(X;k)$ has a neighborhood diffeomorphic to $[0,\infty[^r \times \mathbb{R}^{n-r}$, and the transition maps are analytic.

Proof Set

$$\tilde{C}(X;k) = \{(c_A) \in \prod_{A \subset X; \sharp A \geq 2} C(A;\mathbb{R}^k);\ (c_A) \text{ satisfies } (\star)\}$$

We have already proved that $C(X;k) \subset \tilde{C}(X;k)$. In order to prove the reverse inclusion, we first study the structure of $\tilde{C}(X;k)$. Let $(c^0_A)_{A \subset X; \sharp A \geq 2} \in \tilde{C}(X;k)$. Given this point (c^0_A), we construct the rooted tree $\tau((c^0_A))$, with oriented edges, whose vertices are some subsets of X with cardinality greater than 1, in the following way. The root is X. The edges starting at a vertex $A \subset X$ are in one-to-one correspondence with the maximal subsets B of A with $\sharp B \geq 2$, such that $c^0_{A|B}$ is constant. The edge corresponding to a subset B goes from A to B. Note that the tree structure can be recovered from the set of vertices. Therefore, we identify the tree $\tau^0 = \tau((c^0_A))$ with its set of vertices. τ^0 is a set of subsets of X.

Now, we construct a chart of $\tilde{C}(X;k)$ near (c^0_A). Assign a parameter $\lambda_B \in [0,\varepsilon]$ to the edge arriving at B. This parameter λ_B will measure the ratio of the scale in B by the scale in the smallest set in τ^0 that contains it.

Let $C_{\tau^0}(B;\mathbb{R}^k)$ be the subspace of $C(B;\mathbb{R}^k)$ made of maps from B to $\mathbb{R}^k$ such that two elements of B have the same image in $\mathbb{R}^k$ if and only if they belong to a common endpoint (subset of X) of an edge starting at B. (Note that $C_{\tau^0}(B;\mathbb{R}^k)$ is naturally homeomorphic to $C_0(B_{\tau^0};\mathbb{R}^k)$ where B_{τ^0} is the set obtained from B by identifying two elements of B that belong to a common strict subset of B in τ^0.) Let $V \subset \prod_{B \in \tau^0} C_{\tau^0}(B;\mathbb{R}^k)$ be an open neighborhood of $(c^0_B)_{B \in \tau^0}$ in $\prod_{B \in \tau^0} C_{\tau^0}(B;\mathbb{R}^k)$. Let $\varepsilon > 0$. When V and ε are small enough, we define the map

$$\begin{array}{rccc} F: & [0,\varepsilon[^{\tau^0 \setminus \{X\}} \times V & \longrightarrow & \prod_{D \subset X; \sharp D \geq 2} C(D;\mathbb{R}^k) \\ & \left(P = (\lambda_B)_{B \in \tau^0 \setminus \{X\}}, (c_B)_{B \in \tau^0}\right) & \mapsto & (F(P)_D)_{D \subset X; \sharp D \geq 2} \end{array}$$

where $F(P)_D$ is equal to $F(P)_{A|D}$ if $A \in \tau^0$ is the smallest element of τ^0 that contains D, and $F(P)_A$ is represented by the map

$$\tilde{F}(P)_A : A \longrightarrow \mathbb{R}^k$$

that maps an element $(x \in A)$ to the vector that admits the following recursive definition. Let $B_1 \subset B_2 \subset \cdots \subset B_m \subset B_{m+1} = A$ be the sequence of vertices of τ^0 such that B_1 is the smallest element of τ^0 that contains x, and B_{r+1} is the smallest element of τ^0 that contains B_r. Fix a point $\xi(B)$ in any subset $B \in \tau^0$ so that if $B' \in \tau^0$ and if $\xi(B) \in B' \subset B$, then $\xi(B') = \xi(B)$ (the $\xi(B)$ depend on τ^0 that is fixed). Then $\tilde{F}(P)_{B_1}(x) = \phi_{\xi(B_1)}(c_{B_1})(x)$ and

$$\tilde{F}(P)_{B_{k+1}}(x) = \phi_{\xi(B_{k+1})}(c_{B_{k+1}})(x) + \lambda_k \tilde{F}(P)_{B_k}(x)$$

When V and ε are small enough $F(P)_D$ is never constant and F is well-defined. Furthermore, F is then a homeomorphism onto its image that is an open subset of $\tilde{C}(X;k)$. Also notice that the tree (or its vertices set) corresponding to a point in the image of $F((\lambda_B)_{B\in\tau^0\setminus\{X\}},(c_B)_{B\in\tau^0})$ is obtained from τ^0 by removing the subsets B such that $\lambda_B > 0$. The points of $i\left(C_0(X;\mathbb{R}^k)\right)$ are the points whose tree is reduced to its root X. In particular, the point we started with is in the closure of $F(]0,\varepsilon[^{B\in\tau^0\setminus\{X\}}\times V) \subset i(C_0(X;\mathbb{R}^k))$. This finishes proving that $C(X;k) \subset \tilde{C}(X;k)$. Furthermore, F and its inverse, that can be defined on an open subset of $\prod_{D\subset X;\sharp D\geq 2} C(D;\mathbb{R}^k)$, are analytic. Therefore the above local homeomorphisms provide $C(X;k)$ with the structure of a C^ω manifold with corners. $\square$

$C(X;k)$ is also provided with a partition by the associated trees of the above proof. Note that the part $F(\tau)$ corresponding to a given tree τ is an open submanifold of dimension $\left(\dim(C_0(X;\mathbb{R}^k)) - (\sharp\tau - 1)\right)$ that is homeomorphic to $\prod_{B\in\tau} C_\tau(B;\mathbb{R}^k)$. In particular, the boundary of $C(X;k)$ has a partition into open *faces,* corresponding to trees τ with $\sharp\tau > 1$, of codimension $(\sharp\tau - 1)$.

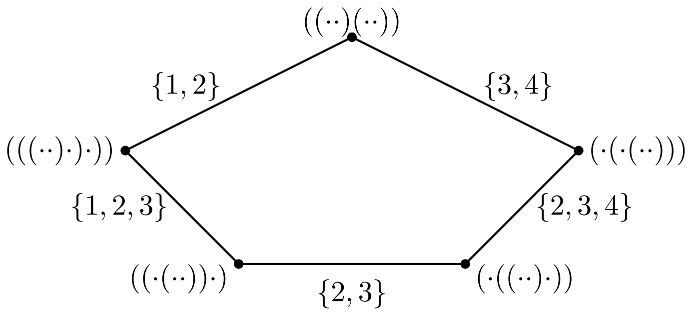
Example 6.3 Let us again consider the case where $k = 1$. Let us fix one order on X and let us study the corresponding component $C_<(X;1)$ of $C(X;1)$. Here the trees that provide non-empty faces are those which are made of subsets containing consecutive elements, and all the faces are homeomorphic to open balls. When $X = \{1,2,3\}$, $C_<(X;1)$ is an interval whose endpoints correspond to the two trees $\{X,\{1,2\}\}$ and $\{X,\{2,3\}\}$. For $X = \{1,2,\dots,p\}$, we find a *Stasheff polyhedron* that is a polyhedron whose maximal codimension faces are points that can be described as *non-associative words* as in the following definition.

Definition 6.4 A *non-associative word* or *n.a. word* w in the letter $\cdot$ is an element of the free non-associative monoid generated by $\cdot$. The *length* of such a w is the number of letters of w. Equivalently, we can define a *non-associative word* by saying that each such word has an integral *length* $\ell(w) \in \mathbb{N}$, the only word of length 0 is the *empty word*, the only word of length 1 is $\cdot$, the product $w'w''$ of two n.a. words w' and w'' is a n.a. word of length $(\ell(w') + \ell(w''))$, and every word w of length $\ell(w) \geq 2$ can be decomposed in a unique way as the product $w'w''$ of two n.a. words w' and w'' of nonzero length.

Example 6.5 The unique n.a. word of length 2 is $(\cdot\cdot)$. The two n.a. words of length 3 are $((\cdot\cdot)\cdot)$ and $(\cdot(\cdot\cdot))$. There are five n.a. words of length 4 drawn in the following picture of $C_<(\{1,2,3,4\};1)$.

A n.a. word corresponds to the binary tree of subsets of points between matching parentheses.

In particular, $C_<(\{1,2,3,4\};1)$ is the following well-known pentagon, where the edges are labeled by the element of the corresponding $\tau \setminus \{\{1,2,3,4\}\}$.



As another example, the reader can recognize that $C(\{1,2,3\};2)$ is diffeomorphic to the exterior of a 3-component Hopf link in S^3 (made of three Hopf fibers).

7 Back to configuration space integrals for links

We can use these compactifications to study the configuration space integrals defined in Section 4. Indeed, there is a natural embedding

$$i : C(L;\Gamma) \hookrightarrow M^U \times (S^3 = \mathbb{R}^3 \cup \infty)^T \times C(U \cup T;3).$$

Define the compactification $\overline{C}(L;\Gamma)$ of $C(L;\Gamma)$ as the closure of $i(C(L;\Gamma))$ in this compact space. As before, the compactification can be provided with a structure of a C^∞ manifold with corners, with a stratification that will again be given by trees recording the different relative collapses of points.[4] Furthermore, since Ψ is defined on $C(U \cup T;3)$ as the projection on $\prod_E C(E;3)$ where an

[4]There are two main differences with the already studied case, due to the one-manifold embedding L. First, the univalent vertices vary along L, and when they approach each other, their direction that makes sense in the compactification approaches the direction of the tangent vector to L at the point where they meet. Second, there is a preferred observation scale namely the scale of the ambient space where the embedding lies.

edge E is seen as the pair of its endpoints ordered by the orientation, Ψ extends to $\overline{C}(L;\Gamma)$. This extension is smooth, and we have

$$I(L;\Gamma) = \int_{\overline{C}(L;\Gamma)} \Psi^*(\Lambda^E \omega).$$

In particular, this shows the convergence of the integrals of Section 4. The variation of $I(L;\Gamma)$ under a C^∞ isotopy

$$\begin{array}{rccc} L: & M \times I & \longrightarrow & \mathbb{R}^3 \\ & (m,t) & \mapsto & L_t(M) \end{array}$$

is computed with the help of the Stokes theorem. Since $\Psi^*(\Lambda^E\omega)$ is a closed form defined on $\cup_{t\in I} C(L_t;\Gamma)$, the variation $(I(L_1;\Gamma) - I(L_0;\Gamma))$ is given by the sum over the codimension one faces $F(\tau)(L;\Gamma)$ of the

$$V(F(\tau)(L;\Gamma)) = \int_{\cup_{t\in I} F(\tau)(L_t;\Gamma)} \Psi^*(\Lambda^E\omega).$$

The Altschuler-Freidel proof and the Thurston proof that Z^0_{CS} provides a link invariant now rely on a careful analysis of the codimension one faces, and of the variations that they induce. See [1, 24, 22]. This analysis was successfully started by Bott and Taubes [8]. It shows that the faces that indeed contribute in the link case, where $M = \coprod_{i=1}^k S^1_i$, are of four possible forms.

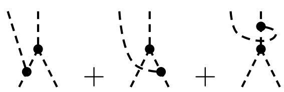
(1) Two trivalent vertices joined by an edge collide with each other.

(2) Two univalent vertices consecutive on M collide with each other.

(3) A univalent vertex and a trivalent vertex that are joined by an edge collide with each other.

(4) The *anomalous faces* where some connected component of the dashed graph Γ collapses at one point.

The algebraic STU and AS relations imply the following IHX relation in the vector spaces $\mathcal{A}_n(M)$.

IHX :  $= 0$

The STU and IHX relation make the first three kinds of variations cancel. Let us see roughly how it works for the first kind of faces. Such a face is homeomorphic to the product of the sphere S^2 by the configuration space of the graph obtained from Γ by identifying the two colliding points (which become a four-valent vertex), where S^2 is the configuration space of the two endpoints of the infinitely small edge. Let Γ_1, Γ_2 and Γ_3 be three graphs related by an

IHX relation as above so that $[\Gamma_1]+[\Gamma_2]+[\Gamma_3]=0$. Let τ_i be the tree made of $U\cup T$ and the visible edge of Γ_i. Then $V(F(\tau_i)(L;\Gamma_i))$ is independent of i. Therefore,

$$V(F(\tau_1)(L;\Gamma_1))[\Gamma_1]+V(F(\tau_2)(L;\Gamma_2))[\Gamma_2]+V(F(\tau_3)(L;\Gamma_3))[\Gamma_3]=0.$$

Thus, the sum[5] of these variations plugged into the Chern-Simons series is zero. The STU relation makes the variations of the second kind and the third kind of faces cancel each other in a similar way.

For the anomalous faces, we do not have such a cancellation. But, we are about to see that we have a formula like

$$\frac{\partial}{\partial t}(Z_{CS}(L_t))=\left(\frac{1}{2}\sum_{i=1}^{k}\frac{\partial}{\partial t}I(K_i;\theta)\alpha\sharp_i\right)Z_{CS}(L_t).$$

where α is the *anomaly* that is the constant of $\mathcal{A}(\mathbb{R})$ that is defined below, and $\sharp_i$ denotes the $\mathcal{A}(\mathbb{R})$-module structure on $\mathcal{A}(\coprod_{i=1}^k S^1_i)$ by insertion on the i^{th} component.

Let us define the anomaly. Let $v\in S^2$. Let D_v denote the linear map

$$\begin{array}{rccc} D_v: & \mathbb{R} & \longrightarrow & \mathbb{R}^3\\ & 1 & \mapsto & v\end{array}$$

Let Γ be a Jacobi-diagram on $\mathbb{R}$. Define $C(D_v;\Gamma)$ and Ψ as in Section 4. Let $\hat{C}(D_v;\Gamma)$ be the quotient of $C(D_v;\Gamma)$ by the translations parallel to D_v and by the dilatations. Then Ψ factors through $\hat{C}(D_v;\Gamma)$ that has two dimensions less. Now, allow v to run through S^2 and define $\hat{C}(\Gamma)$ as the total space of the fibration over S^2 where the fiber over v is $\hat{C}(D_v;\Gamma)$. The map Ψ becomes a map between two smooth oriented manifolds of the same dimension. Indeed, $\hat{C}(\Gamma)$ carries a natural smooth structure and can be oriented as follows. Orient $C(D_v;\Gamma)$ as before, orient $\hat{C}(D_v;\Gamma)$ so that $C(D_v;\Gamma)$ is locally homeomorphic to the oriented product (translation vector $(0,0,z)$ of the oriented line, ratio of homothety $\lambda\in]0,\infty[$) $\times\hat{C}(D_v;\Gamma)$ and orient $\hat{C}(\Gamma)$ with the (base($=S^2$)$\oplus$fiber) convention[6]. Then we can again define

$$I(\Gamma)=\int_{\hat{C}(\Gamma)}\Psi^*(\Lambda^E\omega).$$

[5]To make this sketchy proof work, to avoid thinking of the $(1/\sharp\mathrm{Aut}\Gamma)$ factor and to make sure that all the variations cancel, it is better to deal with the *labeled graphs* that are described at the end of the section.

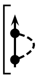
[6]This can be summarized by saying that the S^2-coordinates replace (z,λ).

Now, the *anomaly* is the following sum running over all connected Jacobi diagrams on the oriented lines (again without vertex-orientation and without small loop):

$$\alpha = \sum \frac{I(\Gamma)}{\sharp \mathrm{Aut}\Gamma}[\Gamma] \quad \in \mathcal{A}(\mathbb{R})$$

Its degree one part is

$$\alpha_1 = \left[\quad\right].$$



Then the formula

$$\frac{\partial}{\partial t}(Z_{CS}(L_t)) = \left(\sum_{i=1}^{k} \frac{\partial}{\partial t}\frac{I(K_i;\theta)}{2}\alpha\sharp_i\right) Z_{CS}(L_t)$$

expresses the following facts. Let Γ be a connected dashed graph on the circle. The set $U = \{u_1, u_2, \dots, u_k\}$ of its univalent vertices is cyclically ordered, and the anomalous faces for Γ correspond to the different total orders (which are visible at the scale of the collision) inducing the given cyclic order. Assume that $u_1 < u_2 < \dots < u_k = u_0$ is one of them. Denote the Jacobi diagram on $\mathbb{R}$ obtained by cutting the circle between u_{i-1} and u_i by Γ_i. The group of automorphisms of Γ_i is isomorphic to the subgroup $\mathrm{Aut}_0(\Gamma)$ of $\mathrm{Aut}(\Gamma)$ made of the automorphisms of Γ that fix U pointwise. The quotient $\frac{\mathrm{Aut}(\Gamma)}{\mathrm{Aut}_0(\Gamma)}$ is a subgroup of the cyclic group of the permutations of U that preserve the cyclic order of U, of order $\frac{k}{p}$, for some integer p that divides into k; and Γ_i is isomorphic to Γ_{i+p}, for any integer $i \leq (k-p)$.

The contribution of the collapse that orders U like Γ_i to the variation $(I(K_1;\Gamma) - I(K_0;\Gamma))$ during a knot isotopy $((z,t) \mapsto K_t(z))$ is proportional to the area covered by the unit derivative of K on S^2 during the isotopy, that is $\frac{I(K_1;\theta)-I(K_0;\theta)}{2}$. More precisely, it is

$$\frac{I(K_1;\theta) - I(K_0;\theta)}{2} I(\Gamma_i).$$

Therefore, the contribution to the variation $\frac{(I(K_1;\Gamma)-I(K_0;\Gamma))}{\sharp\mathrm{Aut}(\Gamma)}$ of the anomalous faces is

$$\sum_{i=1}^{p} \frac{I(K_1;\theta) - I(K_0;\theta)}{2\sharp\mathrm{Aut}(\Gamma_i)} I(\Gamma_i)$$

In general, one must multiply the infinitesimal variation due to the collapse of one connected component of the dashed graph by the contributions of the other connected components of the dashed graphs.

The integration of the above formula shows the Altschuler and Freidel formula:

$$Z_{CS}(L)=\exp(\frac{I(K_1;\theta)}{2}\alpha)\sharp_1\exp(\frac{I(K_2;\theta)}{2}\alpha)\sharp_2\dots\exp(\frac{I(K_k;\theta)}{2}\alpha)\sharp_k Z^0_{CS}(L).$$

The fact that Z^0_{CS} is a universal Vassiliev invariant is far easier to obtain than its invariance. It almost follows from the definition of Vassiliev invariants that will not be given here. See [24, p.10] for example.

As it has been first noticed by Dylan Thurston in [24], Z^0_{CS} is rational. This means that for any integer n, and for any link L whose components have zero Gauss integrals, the degree n part $Z^0_{CSn}(L)$ of $Z^0_{CS}(L)$ is in $\mathcal{A}^{\mathbb{Q}}_n(\coprod_{i=1}^k S^1_i)$. Indeed, if L is almost horizontal, $Z^0_{CSn}(L)$ may be interpreted as the following differential degree.

Let e_n be a number of edges greater or equal than the number of edges of degree n diagrams that might contribute with a non zero integral to the Chern-Simons series. For $n\geq 3$, $e_n=3n-3$ works, see [22]. We wish to interpret $Z^0_{CSn}(L)$ as the differential degree of a map to $\left(S^2\right)^{e_n}$. We first modify the configuration space $\overline{C}(L;\Gamma)$ of a degree n diagram Γ whose set of edges is $E(\Gamma)$ by

$$\hat{C}(L;\Gamma)=\overline{C}(L;\Gamma)\times\left(S^2\right)^{e_n-\sharp E(\Gamma)}.$$

Next, in order to be able to map it to $\left(S^2\right)^{e_n}$, we *label* Γ, that is we orient the edges of Γ and we define a bijection from $E(\Gamma)\cup\{1,2,\dots,e_n-\sharp E(\Gamma)\}$ to $\{1,2,\dots,e_n\}$. This bijection transforms the map

$$\Psi\times\mathrm{Identity}(\left(S^2\right)^{e_n-\sharp E(\Gamma)}):\hat{C}(L;\Gamma)\longrightarrow\left(S^2\right)^{E(\Gamma)}\times\left(S^2\right)^{e_n-\sharp E(\Gamma)}$$

into a map

$$\hat{\Psi}:\hat{C}(L;\Gamma)\longrightarrow\left(S^2\right)^{e_n}$$

For a given degree n diagram Γ, there are $\frac{2^{\sharp E(\Gamma)}e_n!}{\sharp\mathrm{Aut}(\Gamma)}$ labeled diagrams. Now,

$$Z_{CSn}(L)=\sum_{\Gamma\text{ labeled diagram of degree }n}\frac{1}{2^{\sharp E(\Gamma)}e_n!}\int_{\hat{C}(L;\Gamma)}\hat{\Psi}^*(\Lambda^{e_n}\omega)[\Gamma]$$

Define the differential degree $\deg(\Psi,x)$ of Ψ over the formal union

$$\cup_{\Gamma\text{ labeled diagram of degree }n}\frac{1}{2^{\sharp E(\Gamma)}e_n!}[\Gamma]\hat{C}(L;\Gamma)$$

as follows for a regular[7] point $x\in(S^2)^{e_n}$:

$$\deg(\Psi,x)=\sum_{\Gamma\text{ labeled diagram of degree }n}\frac{1}{2^{\sharp E(\Gamma)}e_n!}\deg(\Psi_{|\hat{C}(L;\Gamma)},x)[\Gamma]$$

[7] Here, regular means regular with respect to all the $\Psi_{|\hat{C}(L;\Gamma)}$.

where $\deg(\Psi_{|\hat{C}(L;\Gamma)}, x)$ is a usual differential degree. Then D. Thurston proved that $\deg(\Psi, x)$ does not vary across the images of the codimension one faces of the $\hat{C}(L;\Gamma)$ See also [22]. In other words, the above weighted union of configuration spaces behaves as a closed $2e_n$-dimensional manifold from the point of view of the differential degree theory. In particular, ω can be replaced by any volume form of S^2 with total volume 1. Computing Z^0_{CSn} as the degree of a generic point of $\left(S^2\right)^{e_n}$ shows that Z^0_{CSn} belongs to the lattice of $\mathcal{A}^{\mathbb{Q}}_n(\coprod_{i=1}^k S^1_i)$ generated by the $\frac{(e_n-\sharp E(\Gamma))!}{2^{\sharp E(\Gamma)}e_n!}[\Gamma]$, where the Γ's are the degree n graphs that may produce a nonzero integral. This interpretation is more convenient for computational purposes.

8 Further properties of the Poirier limit

8.1 The Poirier connection

In order to prove the isotopy invariance and the functoriality of his limit, Poirier considered the $(2\sharp E+1)$ configuration space viewed as the closure of the image[8] of $\cup_{\lambda\in]0,1]}C(h_\lambda\circ L(M);\Gamma)$ inside the compact space $[0,1]\times M^U\times(S^3)^T\times C(U\cup T;3)$ where the first interval receives the parameter λ. He interpreted his limit as the integral over the intersection $C_\ell(L(M);\Gamma)$ of the above closure with $\{0\}\times M^U\times(S^3)^T\times C(U\cup T;3)$. He studied the stratification of $C_\ell(L(M);\Gamma)$; and he proved that the only non-vanishing contributions come from limit configurations of graphs Γ where any connected component Γ_c is at some height h_c (that can be recovered from the coordinates in M^U). Furthermore, the restriction to the set U_c of univalent vertices of Γ_c of such a limit configuration projects (by forgetting the height coordinate of points) to the planar configuration lying at the intersection of the image of L with the horizontal plane $\mathbb{R}^2\times\{h_c\}$.

More precisely, he described his limit as follows for braids, that will be seen as paths $\gamma:[\beta,\tau]\longrightarrow C_0(X;\mathbb{R}^2)$.

Let $f:X\hookrightarrow\mathbb{R}^2$ represent a planar configuration. Let Γ be a diagram on $X\times\mathbb{R}$. Let $C(f\times 1_{\mathbb{R}};\Gamma)$ be the configuration space associated with the embedding

$$f\times\mathrm{Identity}(\mathbb{R}):X\times\mathbb{R}\hookrightarrow\mathbb{R}^3$$

as in Section 4. The quotient $\hat{C}(f;\Gamma)$ of $C(f\times 1_{\mathbb{R}};\Gamma)$ by the vertical translations is a $(2\sharp E-1)$-manifold with corners. Furthermore, the fibered space over $C_0(X;\mathbb{R}^2)$ whose fiber over f is $\hat{C}(f;\Gamma)$ is also a smooth manifold equipped

[8] In fact, he used a less separating compactification for technical reasons.

with the $(2\sharp E)$-form $\Psi^*\Lambda^E\omega$. Integrating this form along the fiber provides a one-form[9]

$$\Omega_P(\Gamma) : T(C_0(X;\mathbb{R}^2)) \longrightarrow \mathbb{R}.$$

Then the Poirier connection is

$$\Omega_P = \sum_{\Gamma \text{ connected diagram on } X\times\mathbb{R}} \frac{1}{\sharp\mathrm{Aut}(\Gamma)}\Omega_P(\Gamma)[\Gamma] \in \Omega^1(C_0(X;\mathbb{R}^2);\mathcal{A}(X\times\mathbb{R}))$$

Sylvain Poirier proved that his limit for a braid, that is seen as a path $\gamma : [\beta,\tau] \longrightarrow C_0(X;\mathbb{R}^2)$ is the Chen holonomy of Ω_P along this path.

$$Z_P(\gamma) =$$

$$\sum_{n\in\mathbb{N}} \int_{\beta\leq h_1\leq h_2\leq\ldots\leq h_n\leq\tau} \Omega_P(\gamma'(h_1))\Omega_P(\gamma'(h_2))\ldots\Omega_P(\gamma'(h_n))dh_1dh_2\ldots dh_n$$

Sylvain Poirier proved that, unlike the complex Knizhnik-Zamolodchikov connection that provides a similar definition of the Kontsevich integral for braids, the holonomy of his real connection converges without any regularisation for paths reaching limit planar configurations. Thus, when restricting his invariant of tangles to the framed *q-tangles* introduced in [17] that are framed cobordisms between two limit configurations on the real line of $\mathbb{R}^2 = \mathbb{C}$, he got a functor from the category of framed q-tangles to $\mathcal{A}$ that satisfies the expected natural properties that are described below.

8.2 The Poirier functor on q-tangles

Definition 8.1 A framed *q-tangle* is a triple $(T(M);b,t)$ where b and t are two non-associative words and T is a C^∞ embedding of a compact one-manifold M into a horizontal slice $\mathbb{R}^2\times[\beta,\tau]$ of $\mathbb{R}^3$ such that:

$$T(M)\cap(\mathbb{R}^2\times\{\beta,\tau\}) = T(\partial M) \subset \mathbb{R}\times\{0\}\times\{\beta,\tau\},$$

the set of letters of b and t are in natural one-to-one correspondences induced by the order of $\mathbb{R}$ with $T^{-1}(\mathbb{R}\times(0,\beta))$ and $T^{-1}(\mathbb{R}\times(0,\tau))$, respectively, and the only horizontal tangent vectors of M occur for interior points of M and are parallel to the blackboard plane. T is considered up to the isotopies which satisfy these hypotheses at any time and up to a rescaling of the height

[9]The sign of this form is defined by letting the tangent vector replace the vertical translation parameter.

parameter. The letters b and t stand for the *bottom word* and the *top word*, respectively.

Here, framed q-tangles will be simply called *tangles.*

Examples 8.2 Tangles are unambiguously defined by the data of a regular projection of the involved embedding onto the blackboard plane, together with the bottom and top words. Since there is only one n. a. word of length 0, 1 or 2, these words do not need to be specified. Here is an example of a q-tangle:

$$\bigcup = (\bigcup; \emptyset, (\cdot\cdot)) \neq$$ 

The bottom and top words are sometimes shown in pictures by the relative positions of the bottom (or top) points. For example,

$$|/| = (|\,|\,|; ((\cdot\cdot)\cdot), (\cdot(\cdot\cdot))$$

 $= ($  $; ((\cdot\cdot)(\cdot\cdot)), \emptyset)$

The *product* of two q-tangles $T_1 = (T_1(M_1); b_1, t_1)$ and $T_2 = (T_2(M_2); b_2, t_2)$ is defined as soon as $t_1 = b_2$ by *stacking* T_2 *above* T_1 as before. For example,

$$(\bigcup)\,(\times) =$$ 

By definition, Z_P is *multiplicative* with respect to the product.

8.3 Monoidality

The *tensor product* of two tangles $T_1 = (T_1(M_1); b_1, t_1)$ and $T_2 = (T_2(M_2); b_2, t_2)$ is defined as the tangle $T = T_1 \otimes T_2 = (T_1 \otimes T_2(M_1 \coprod M_2); b_1 b_2, t_1 t_2)$ by *putting* T_2 *on the right-hand side of* T_1. In order to construct the embedding T, choose representatives of T_1 and T_2 which embed M_1 and M_2 into $[0,1] \times \mathbb{R} \times [0,1]$ and $[2,3] \times \mathbb{R} \times [0,1]$, respectively, and define T as their disjoint union. For example,

$$\times \otimes |\,| = (\times|\,|; ((\cdot\cdot)(\cdot\cdot)), ((\cdot\cdot)(\cdot\cdot)))$$

The intuitive meaning of this tensor product is that the strands of T_1 are infinitely closer to each other than they are to strands of T_2. This allows to make the interactions of T_1 and T_2 vanish and this allowed Poirier to prove that his functor is *monoidal* (i.e. that Z_P respects the tensor product).

$$Z_P(T_1 \otimes T_2) = Z_P(T_1) \otimes Z_P(T_2).$$

Graphically, this reads

$$Z_P\left(\boxed{T_1}\boxed{T_2}\right) = \boxed{Z_P(T_1)}\boxed{Z_P(T_2)}.$$

8.4 Behaviour under duplication

The *duplication* of a component $C \subset M$ of a tangle $T(M)$ consists in replacing $T(C)$ by two closed parallel copies $T(C_1)$ and $T(C_2)$ of $T(C)$ so that, up to homotopy with fixed boundary, the section in the normal unit bundle of $T(C_1)$ induced by $T(C_2)$ coincides with one of the two sections given, thanks to the condition on horizontal tangencies, by "the intersection with the blackboard plane". Every letter of the top and bottom words corresponding to a (possible) boundary point of C is replaced by the two-letter word $(\cdot\cdot)$. The resulting tangle is denoted by $(2 \times C)(T)$. For example, duplicating the unique component of  yields . As another example, duplicating a knot $T(C)$ amounts to replace it by two parallel copies of $T(C)$ whose linking number is $w(T(C))$.

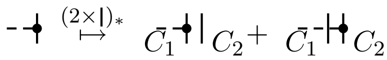
Let us describe the map corresponding to the duplication of tangles at the level of diagrams.

Define $(2 \times C)(M)$ from M by replacing C by two copies C_1 and C_2 of C. Let Γ be a diagram with support M. Let U_C denote the preimage of C under i. To a subset U_1 of U_C, we associate the diagram Γ_{U_1} on $(2 \times C)(M)$, obtained from Γ by changing i into $\tilde{i}$ so that $\tilde{i} = i$ outside U_C, $\tilde{i}_{|U_1} = \iota_1 \circ i_{|U_1}$, and $\tilde{i}_{|U_2 = U_C \setminus U_1} = \iota_2 \circ i_{|U_2}$ where ι_j is the identification morphism from C to C_j which also carries the local orientations of the vertices of U_j.

The *duplication map* $(2 \times C)_*$ from $\mathcal{A}(M)$ to $\mathcal{A}((2 \times C)(M))$ is the (well-defined!) morphism of topological vector spaces which maps $[\Gamma]$ to

$$(2 \times C)_*([\Gamma]) = \sum_{\substack{U_1 \\ \emptyset \subseteq U_1 \subseteq U_C}} [\Gamma_{U_1}].$$

Locally, this reads:



A functor is said to *be compatible with the duplication of a regular component* if for any component C of a tangle T which can be represented without horizontal tangent vector, we have

$$Z((2 \times C)(T)) = (2 \times C)_*(Z(T))$$

The functor Z_P is compatible with the duplication of a regular component. Indeed, intuitively, the two components remain parallel so that they describe a constant path in $C(\{1,2\};2)$, and they are infinitely closer to each other than they are to other components so that both of them interact in the same way with the other components.

8.5 Behaviour under deletion

The *deletion* of a component C of M consists in forgetting about C, removing the possible letters of the top and bottom words corresponding to the boundary points of C and removing the unneeded parentheses. The tangle obtained from $T = T(M; b, t)$ by deleting C will be denoted by $T \setminus C$. The corresponding operation at the level of diagrams is the natural continuous linear map $\mathcal{O}_C$ from $\mathcal{A}(M)$ to $\mathcal{A}(M \setminus C)$ such that:

$\mathcal{O}_C([\Gamma]) = [\Gamma]$ if Γ is a diagram without any leg on C and $\mathcal{O}_C([\Gamma]) = 0$ for the other diagrams.

A functor Z is said to *respect deletion* if for any component C of a tangle T, we have

$$Z(T \setminus C) = \mathcal{O}_C(Z(T))$$

It is clear that Z_P respects deletion.

The functor Z_P also respects the symmetries that have not been broken by the limit process.

9 Linking the Poirier functor to the Kontsevich integral

Set

$$\mathcal{A}_n^{\mathbb{C}}(M) = \mathcal{A}_n^{\mathbb{Q}}(M) \otimes_{\mathbb{Q}} \mathbb{C} \quad \text{and} \quad \mathcal{A}^{\mathbb{C}}(M) = \prod_{n \in \mathbb{N}} \mathcal{A}_n^{\mathbb{C}}(M).$$

The above properties of the Poirier functor can be summarized by saying that it is a real *good functor* with respect to the following definition.

Definition 9.1 A *functor* from the category of q-tangles to $\mathcal{A}^{\mathbb{C}}$ is a map Z which associates an element $Z(T(M); b, t) \in \mathcal{A}^{\mathbb{C}}(M)$ to any q-tangle $T = (T(M); b, t)$ so that Z is compatible with the products, and the degree 0 part of $Z(T(M))$ is one. Such a functor Z is said to be *good* if it is monoidal and if it satisfies the following additional properties:

(1) Z is compatible with the duplication of regular components.

(2) Z is compatible with the deletion of components.

(3) Z is invariant under the 180 degree rotation r_v around the vertical axis.

(4) Let s_h be the orthogonal symmetry with respect to the horizontal plane and let s_v be the orthogonal symmetry with respect to the blackboard plane. Let σ_v and σ_h be the two endomorphisms of the topological vector spaces $\mathcal{A}(S^1)$ such that $\sigma_v(z[\Gamma]) = (-1)^d \overline{z}[\Gamma]$ and $\sigma_h(z[\Gamma]) = (-1)^d z[\Gamma]$, where z is a complex number and $[\Gamma]$ is the image of a degree d diagram Γ in $\mathcal{A}(S^1)$. Then, for any framed knot $K(S^1)$,

$$Z \circ s_v(K) = \sigma_v \circ Z(K)$$

and

$$Z \circ s_h(K) = \sigma_h \circ Z(K)$$

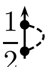
(5) The element $a^Z \in \mathcal{A}(S^1)$ such that $a_0^Z = 0$ and

Z () $= \exp(a^Z) Z$ ()

has a nonzero degree one part

$$a_1^Z \neq 0.$$

In [17], Thang Le and Jun Murakami proved that the Kontsevich integral Z_K is a good functor such that

$a^{Z_K} = \frac{1}{2}$  .

See also [19]. They also proved that any good functor such that $a^Z = a^{Z_K}$ coincides with the Kontsevich integral on framed links. The already mentioned properties of Z_P make clear that Z_P is a good functor such that

$$a^{Z_P} = \frac{1}{2}\alpha.$$

Therefore, Sylvain Poirier obtained the following corollary.

Theorem 9.2 (Poirier) *If the anomaly α vanishes in degree greater than one, then the Chern-Simons series Z^0_{CS} of links is equal to the Kontsevich integral of zero framed links.*

Say that an element $\beta = (\beta_n)_{n\in\mathbb{N}}$ in $\mathcal{A}(S^1)$ is a *two-leg element* if, for any $n \in \mathbb{N}$, β_n is a combination of diagrams with two univalent vertices.

Let β be a two-leg element. Forgetting S^1 from β gives rise to a unique series β^s of diagrams with two distinguished univalent vertices v_1 and v_2, such that β^s is symmetric with respect to the exchange of v_1 and v_2. The series β^s is well-defined thanks to the diagrammatic Bar-Natan version [4] of the Poincaré-Birkhoff-Witt theorem.

A *chord diagram* on a one-manifold M is a diagram without trivalent vertices. Its (dashed connected) components are just *chords.* The degree n chord diagrams generate $\mathcal{A}_n(M)$.

If Γ is a chord diagram, define $\Phi(\beta)([\Gamma])$ by replacing each chord by β^s. $\Phi(\beta)$ is a well-defined morphism of topological vector spaces from $\mathcal{A}(M)$ to $\mathcal{A}(M)$ for any one-manifold M, and $\Phi(\beta)$ is an isomorphism as soon as $\beta_1 \neq 0$. See [20]. The following result is proved in [20].

Theorem 9.3 *If Z is a good monoidal functor as above, then a^Z is a two-leg element of $\mathcal{A}(S^1)$, such that for any integer i, $a^Z_{2i} = 0$ and a^Z_{2i+1} is a combination of diagrams with real coefficients, and, for any framed link L,*

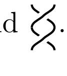
$$Z(L) = \Phi(2a^Z)(Z_K(L))$$

where Z_K denotes the Kontsevich integral of framed links (denoted by $\hat{Z}_f$ in [17] and by Z in [19]).

Corollary 9.4 *The anomaly α is a two-leg element of $\mathcal{A}(S^1)$. For any framed link L, the Poirier limit integral $Z_P(L)$ is equal to $\Phi(\alpha)(Z_K(L))$.*

Thus, we have obtained an algebraic constraint on the anomaly. Since, conversely, any functor of the form $\Phi(\alpha)(Z_K(L))$, for a real odd two-leg element α, is a good functor, nothing more can be obtained from algebra. We are now going to try to compute the low degree terms of the anomaly geometrically.

10 Some geometric properties of the anomaly

Poirier showed that the anomaly can be defined from the logarithm of the holonomy of his connexion for the two strand braid . This is restated in the following proposition.

Let Γ be a Jacobi diagram on $\{1,2\} \times \mathbb{R}$. Let $(x,y) \in \mathbb{R}^2$. Let

$$\begin{array}{llcl} f_{(x,y)}: & \{1,2\} \times \mathbb{R} & \longrightarrow & \mathbb{R}^3 \\ & (1,t) & \mapsto & (0,0,t) \\ & (2,t) & \mapsto & (x,y,t) \end{array}$$

Let $\theta \in [0,2\pi]$. Let $C_\theta(\Gamma)$ denote the quotient of $C(f_{(\cos(\theta),\sin(\theta))};\Gamma)$ by the translations by some $(0,0,z)$, with $z \in \mathbb{R}$. And let $C(\Gamma) = \cup_{\theta \in [0,2\pi]} C_\theta(\Gamma)$.

Orient $C(\Gamma)$ by letting the θ-coordinate $\theta \in [0, 2\pi]$ replace[10] the translation parameter $z \in \mathbb{R}$.

Define the *two-strand anomaly*

$$\tilde{\alpha} = \sum_{\Gamma \text{ connected diagram on } \{1,2\}\times\mathbb{R}} \frac{1}{\sharp\mathrm{Aut}(\Gamma)} \int_{C(\Gamma)} \Psi^*(\Lambda^{E(\Gamma)}\omega)[\Gamma]$$

Let $i : \mathcal{A}(\vert\,\vert) \longrightarrow \mathcal{A}(\mathbb{R})$ be the linear continuous map induced by the inclusion from $\{1,2\} \times \mathbb{R} = \vert\,\vert$ to $\bigcap\!\downarrow\; = \lceil\!\downarrow$. The map i sends a Jacobi diagram Γ to the diagram with the same dashed graph equipped with the same orientations at trivalent vertices where the embedding of univalent vertices is composed by the above inclusion that also carries the local orientations at univalent vertices.

Proposition 10.1 (Poirier)

$$\alpha = -i(\tilde{\alpha})$$

This definition of the anomaly is easier to handle with. Instead of searching for configurations where the univalent vertices are on a common unknown line, we look for configurations where the univalent vertices have the same horizontal coordinates.

Proposition 10.2 *Let n be an integer greater than 2. If Γ is a degree n connected Jacobi diagram on $\{1,2\} \times \mathbb{R}$, that has less than three vertices on some strand, then*

$$I(\Gamma) = \int_{C(\Gamma)} \Psi^*(\Lambda^{E(\Gamma)}\omega) = 0.$$

In particular, the two-strand anomaly $\tilde{\alpha}_n$ in degree n is a combination of connected diagrams with at least 3 vertices on each strand.

Proof Let Γ be a diagram as in the above statement. Let U_2 denote the set of its univalent vertices which are on $\{2\} \times \mathbb{R}$. We shall prove that if U_2 contains less than 3 elements, then $I(\Gamma) = 0$. This will be sufficient to conclude by symmetry.

[10]This means that if the quotient $C_\theta(\Gamma)$ is oriented so that $C(f_{(\cos(\theta),\sin(\theta))}; \Gamma)$ is oriented by the (fiber $\oplus$ base) convention where the base is $C_\theta(\Gamma)$ and the fiber is the oriented vertical translation factor $\mathbb{R}$ of $\mathbb{R}^3$, then $C(\Gamma)$ is oriented with the (base$=$ $[0, 2\pi]$ $\oplus$ fiber $= C_\theta(\Gamma)$) convention.

If U_2 is empty, then all $(2\sharp E(\Gamma)-1)$-manifolds $C_\theta(\Gamma)$ are the same. Thus $\Psi(C(\Gamma))$ is the image under Ψ of one of them, and its volume in $\left(S^2\right)^{E(\Gamma)}$ is zero.

In order to study the two remaining cases where U_2 contains one or two elements, we replace $C(\Gamma)$ by a slightly larger smooth configuration space $\hat{C}(\Gamma)$ in which $C(\Gamma)$ is dense, where Ψ extends so that we shall have

$$I(\Gamma)=\int_{\hat{C}(\Gamma)}\Psi^*(\Lambda^{E(\Gamma)}\omega).$$

Fix one vertex u_0 on $\{1\}\times\mathbb{R}$, and let $C_0(f_{(x,y)};\Gamma)$ denote the subset of $C(f_{(x,y)};\Gamma)$ made of the configurations that map u_0 to the origin of $\mathbb{R}^3$. Orient it as the quotient of $C(f_{(x,y)};\Gamma)$ by the vertical translations with the ((fiber $(=\mathbb{R})$) $\oplus$ base) convention. Let $\hat{C}(\Gamma)$ denote the $(2\sharp E(\Gamma))$-dimensional quotient of

$$P(\Gamma)=\cup_{(x,y)\in\mathbb{R}^2}C_0(f_{(x,y)};\Gamma)$$

by the dilatations with a ratio $\lambda\in]0,\infty[$. The map Ψ is well-defined on this space which contains the additional configurations corresponding to $(0,0)$ that constitute a codimension 2 subspace of $\hat{C}(\Gamma)$ that will therefore not contribute to the integral.

The orientation of $\hat{C}(\Gamma)$ is obtained as follows. Orient $P(\Gamma)$ with the convention $(\text{base}=\mathbb{R}^2)\oplus(\text{fiber})$. Then the orientation of $\hat{C}(\Gamma)$ is defined by the $(\text{fiber}=\,]0,\infty[)\oplus(\text{base})$ convention.

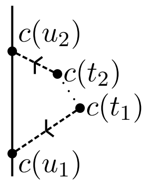
Now, let us get rid of the case where $U_2=\{u\}$, by defining a free smooth action of $]0,\infty[$ on $\hat{C}(\Gamma)$ that does not change the image of a configuration under Ψ, so that Ψ will again factor through a map from a $(2\sharp E-1)$-dimensional manifold to $(S^2)^E$ and define a zero integral. Since Γ is connected, and since its degree is greater than 1, the other end of the edge of u is a trivalent vertex t. Let $\mu\in]0,\infty[$. Let $c\in P(\Gamma)$. Define $\mu.c(x)=x$ if $x\neq u$ and $\mu.c(u)=c(t)+\mu(c(u)-c(t))$. This action is compatible with the dilatations, $\Psi(\mu.c)=\Psi(c)$, and we are finished with this case.

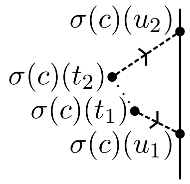
Let us study the remaining case, $U_2=\{u_1,u_2\}$. Denote the trivalent vertex connected by an edge to u_1 by t_1, and denote the trivalent vertex connected by an edge to u_2 by t_2. The vertices t_1 and t_2 may coincide. We shall use the symmetry[11] σ that maps a configuration $c\in P(\Gamma)$ to the configuration $\sigma(c)$ defined by:

[11] This symmetry resembles the Bott and Taubes symmetry that allowed them to prove that the faces corresponding to a collapsing part of a diagram that contains a trivalent vertex and exactly two of its three adjacent vertices do not contribute to the variation of the Chern-Simons series under a link isotopy.

$$
\begin{aligned}
\sigma(c)(x) &= c(x) && \text{if } x \notin U_2 \\
\sigma(c)(u_1) &= c(t_1) + c(t_2) - c(u_2) \\
\sigma(c)(u_2) &= c(t_1) + c(t_2) - c(u_1)
\end{aligned}
$$

as shown in the picture below.

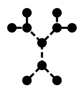
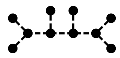
σ maps 

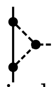
to 

.

This symmetry factors through the dilatations and reverses the orientation of $\hat{C}(\Gamma)$. Furthermore, $\Psi(\sigma(c))$ is obtained from $\Psi(c)$ by reversing the two unit vectors corresponding to the edges containing u_1 and u_2 and by exchanging them, that is, by a composition by a diffeomorphism of $\left(S^2\right)^{E(\Gamma)}$ that preserves $\Lambda^{E(\Gamma)}\omega$. This proves that $I(\Gamma)$ vanishes and this finishes the proof of the proposition. $\square$

As a corollary, α_n is a combination of Jacobi diagrams with 6 legs when $n \geq 2$. Unfortunately, unlike the previous two-leg condition, this condition is not very restrictive. Nevertheless, this provides another proof[12] of the Poirier and Yang result that $\alpha_3 = 0$. There is no connected degree 3 diagram with at least 6 univalent vertices. Recall that for any integer n, $\alpha_{2n} = 0$. Therefore, the next interesting degree is 5. The connected degree 5 diagrams with at least 6 univalent vertices are necessarily trees and their dashed parts have one of the two forms:

 or 

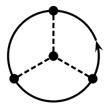
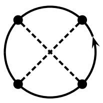
Now, observe that if Γ is a Jacobi diagram on $\{1,2\} \times \mathbb{R}$, with two univalent vertices that are connected to the same trivalent vertex, and that lie on the same vertical line, like in , then $I(\Gamma)$ vanishes. Indeed the two corresponding edges and the vertical vector are coplanar. Therefore, the image of Ψ must lie inside a codimension one subspace of $\left(S^2\right)^{E(\Gamma)}$. This additional

[12] Note that with this two-strand definition all configuration space integrals vanish whereas with the one-strand original definition, the different integrals cancel each other witout being zero individually. See [22, Section 7].

remark determines the distribution of the univalent vertices of the above graphs on the two vertical lines. Then Sylvain Poirier computed $\tilde{\alpha}_5$ with the differential degree methods outlined at the end of Section 7, and with the help of Maple, and he found that $\alpha_5 = 0$ thanks to AS and STU. As a corollary, all coefficients of the HOMFLY polynomial properly normalized that are Vassiliev invariants of degree less than seven can be explicitly written as combinations of the configuration space integrals of Section 4.

Thus, the following Bar-Natan theorem generalizes to any *canonical* [13] Vassiliev invariant of degree less than 7.

Theorem 10.3 (Bar-Natan [5], 1990) *Let Δ denote the symmetrized Alexander polynomial. For any knot K,*

$\frac{\Delta''(K)(1)}{2} = -\frac{1}{3}I(K;$  $) + \frac{1}{4}I(K;$  $) + \frac{1}{24}.$

This particular coefficient, that is the degree 2 invariant which can be extracted from the Chern-Simons series, has been further studied in [23].

11 Questions

The first question here is of course:

1. Prove or disprove the physicist conjecture:

$$\alpha_i = 0 \text{ for any } i > 1.$$

After the articles of Axelrod, Singer [2, 3], Bott and Cattaneo [6, 7, 9], Greg Kuperberg and Dylan Thurston have constructed a universal finite type invariant for homology spheres as a series of configuration space integrals similar to Z^0_{CS}, in [12]. Their construction yields two natural questions:

2. Find a surgery formula for the Kuperberg-Thurston invariant in terms of the above Chern-Simons series.

3. Compare the Kuperberg-Thurston invariant to the LMO invariant [18].

[13] Here, *canonical* can be understood as explicitly recovered from the Kontsevich integral like all the quantum invariants.

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