Introduction to the Kontsevich Integral of Framed Tangles

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Abstract

The Kontsevich integral is the most interesting knot invariant that has ever existed. It does not only contain a lot of the previously known knot invariants, like the HOMFLY, Jones, Kauffman or Alexander polynomials which will be recovered from the Kontsevich integral in Pierre Vogel's lectures [V] or the Milnor invariants which will be recovered from the Kontsevich integral in Gregor Masbaum's lectures [H-M]; but it also organizes them in a suitable way. Whether the Kontsevich integral separates knots is still unknown, but the Kontsevich integral is known to separate braids (defined below). Furthermore, the Kontsevich integral admits a geometric construction.

Here, we will follow the original approach of Kontsevich (and [Ba, C-D, LM1, LM2]) in order to give a self-contained presentation of the Kontsevich integral and of all of its properties which are useful for the study of the universal finite type invariant of 3-manifolds of Le, Murakami and Ohtsuki [LMO, Le1]. This LMO invariant is the main subject of our Summer School.

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1 The Kontsevich Integral for braids

Fix two integers n and p, $p \geq 1$.

We will first introduce some pieces of links where the Kontsevich integral is well- and easily-defined, the $\mathbf{C}^p$*-configuration paths* which can be seen as parametrized braids.

1.1 $\mathbf{C}^p$-configuration paths

Throughout these lectures, $\mathbf{C}$ denotes the field of complex numbers. $\mathbf{\Delta}$ denotes the *big diagonal* of $\mathbf{C}^p$, that is the subset of $\mathbf{C}^p$ whose complement is the set of (ordered) *configurations* of p distinct points of $\mathbf{C}$:

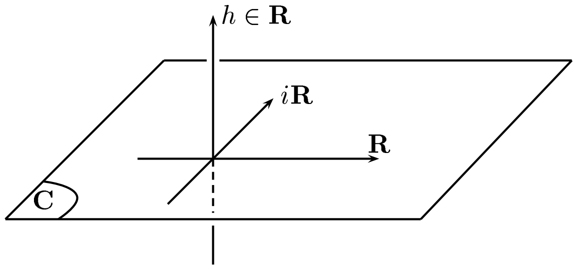
$$\mathbf{C}^p \setminus \mathbf{\Delta} = \{(z_1, z_2, \ldots, z_p) \mid i \neq j \Rightarrow z_i \neq z_j\}$$

This section is devoted to constructing the Kontsevich integral for a path

$$\begin{array}{cccc} \gamma: & [0,1] & \longrightarrow & \mathbf{C}^p \setminus \mathbf{\Delta} \\ & h & \mapsto & (z_1(h), z_2(h), \ldots, z_p(h)) \end{array}$$

of configurations. Our paths are continuous of course, and they are furthermore assumed to be piecewise C^∞. These paths γ will be called $\mathbf{C}^p$*-configuration paths*.

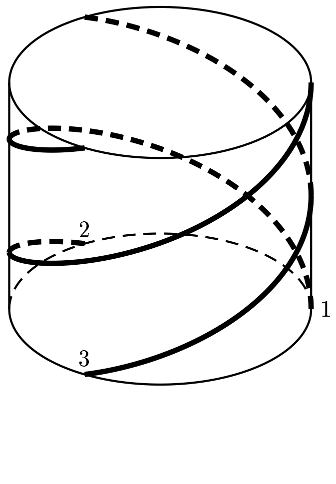
We represent them as the graphs of the p complex-valued functions $(h \mapsto z_i(h))$. These graphs sit in $\mathbf{R}^3$ which is decomposed into the product $\mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$ of the horizontal complex plane by the vertical real axis pointing upward. The vertical axis represents the axis of the *(height)* parameter $h \in [0,1]$, and, in the horizontal complex plane, the real axis runs from left to right in the blackboard (or paper sheet) plane and the imaginary axis points inward the blackboard as below.



Example 1.1 The torus braid $(3,n)$: $p = 3$, $\lambda = \frac{n}{3}2i\pi$.

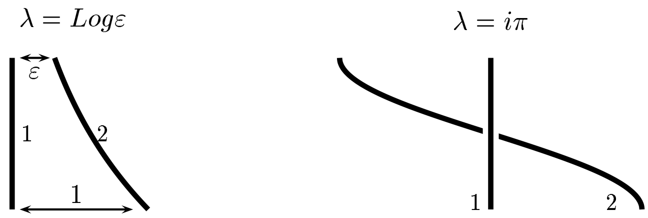
$$z_1(h) = \exp(\lambda h), z_2(h) = z_1(h)\exp(\frac{2i\pi}{3}), z_3(h) = z_1(h)\exp(\frac{4i\pi}{3})$$

Here is a picture of the torus braid $(3,2)$:



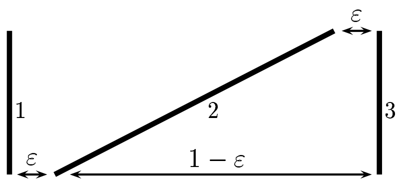
We usually only draw the projections of the graphs onto the blackboard plane that is the plane of the real horizontal axis and the vertical real axis as in the following examples.

Examples 1.2 Some $\mathbf{C}^2$-configuration paths: $p = 2$, $z_1(h) = 0, z_2(h) = \exp(\lambda h)$:



Example 1.3 The associator $\mathbf{C}^3$-configuration path: $p = 3$,

$$z_1(h) = 0, z_2(h) = \varepsilon + h(1 - 2\varepsilon), z_3(h) = 1.$$

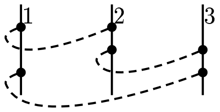


1.2 Coefficients of the Kontsevich integral of $\mathbf{C}^p$-configuration paths

Let $P = \{\{j,k\} \subset \{1,2,\ldots,p\}\}$ denote the set of pairs of elements of $\{1,2,\ldots,p\}$. We represent an n-tuple

$$x = (\{j_1,k_1\},\{j_2,k_2\},\ldots,\{j_n,k_n\}) \in P^n$$

by an *n-chord diagram* Γ_x on p parallel vertical intervals numbered from left to right as shown in the figure below, where we draw n horizontal *chords* at different heights. The r^{th} chord from bottom to top joins the j_r^{th} interval to the k_r^{th} vertical interval.

Here is Γ_x with $x = (\{1,3\},\{2,3\},\{1,2\})$: 

With such an $x \in P^n$, we associate a complex coefficient $Z(\gamma,x)$ of the Kontsevich integral. This coefficient $Z(\gamma,x)$ is defined as the integral

$$Z(\gamma,x) = \int_{\Delta^n} \omega_x$$

of a complex-valued n-form

$$\omega_x \in \Omega^n(\Delta^n,\mathbf{C})$$

defined as follows on the simplex

$$\Delta^n = \{(h_1,h_2,\ldots,h_n) \in \mathbf{R}^n |\ 0 \le h_1 \le h_2 \le \ldots \le h_n \le 1\}.$$

$$\omega_x(h_1,h_2,\ldots,h_n) = \left(\frac{1}{2i\pi}\right)^n \tilde{\omega}_{x_1}(h_1)dh_1 \wedge \tilde{\omega}_{x_2}(h_2)dh_2 \wedge \ldots \tilde{\omega}_{x_n}(h_n)dh_n$$

where

$$\tilde{\omega}_{x_r} = \frac{(z_{j_r} - z_{k_r})'}{(z_{j_r} - z_{k_r})}$$

Example 1.4 In the examples 1.1 and 1.2, the function $\tilde{\omega}_{x_r}$ is always constant, with value λ. Thus, $\omega_x = (\frac{\lambda}{2i\pi})^n dh_1 \wedge dh_2 \wedge \ldots dh_n$ and $Z(\gamma, x) = (\frac{\lambda}{2i\pi})^n \mathrm{Volume}(\Delta^n)$, where the volume of Δ^n is $\frac{1}{n!}$ (for example because the cube $[0,1]^n$ is the union of $n!$ simplices isometric to Δ^n which intersect along sets of measure zero). In these cases, for any $x \in P^n$, we have:

$$Z(\gamma, x) = \frac{(\frac{\lambda}{2i\pi})^n}{n!} \tag{1.5}$$

In general, the computation is not so simple as in the following exercise.

Side-exercise 1.6 In the case of Example 1.3, compute

$$Z(\gamma, (\{1,2\},\{2,3\})) = -\frac{1}{4\pi^2}\left(\mathrm{Log}(\varepsilon)\mathrm{Log}\left(\frac{1-\varepsilon}{\varepsilon}\right) + \sum_{n=1}^{\infty} \frac{(1-\varepsilon)^n - \varepsilon^n}{n^2}\right).$$

1.3 The target of the Kontsevich integral: the space of diagrams

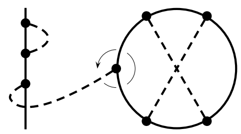
Let X be a compact one-manifold.

Definition 1.7 An *n-chord diagram* on X is the datum of

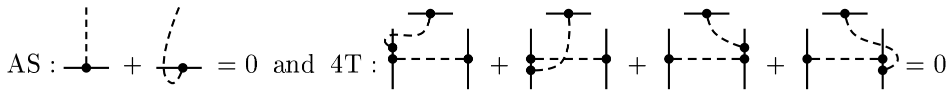
- an injection i of the set U of the 2n endpoints, called *univalent vertices*, of n closed intervals, called *chords*, into the interior of X, up to isotopy of X and isomorphism of the set of chords; (so far this is equivalent to the datum of the 0-dimensional submanifold $i(U)$ (up to isotopy of X) together with a partition of $i(U)$ into n pairs).

- for every point $i(u \in U)$, a cyclic order on the set the 3 half-edges (the chord and the two (natural) parts of a (natural) neighborhood of $i(u)$ in X) which meet at $i(u)$. This cyclic order is called a *(local) orientation* at $i(u)$. It is equivalent to the datum of a local orientation of X near $i(u)$. Namely, number the dashed half-edge by 1 and the plain ones by 2 and 3 according to the cyclic order, then the induced orientation goes from edge $\sharp 2$ to edge $\sharp 3$.

In pictures, the manifold X is represented by plain lines, the chords are represented by dashed lines, big dots represent the univalent vertices, and the local orientation is the trigonometric order of the 3 half-edges which meet at $i(u)$. See the following example of a 4-chord diagram on $X = | \bigcirc$.



Let $\mathcal{D}_n(X)$ be the (finite-dimensional !) $\mathbf{C}$-vector space freely generated by the n-chord diagrams on X. Define the antisymmetry relation (AS) and the four-term-relation (4T) as follows:



In words, (4T) reads: If four diagrams are identical except in the neighborhood of two chords, where they look as above, then their sum is zero. Similarly, AS means that the sum of two diagrams which differ only by the orientation at one vertex is zero.

Let

$$\mathcal{A}_n(X) = \frac{\mathcal{D}_n(X)}{AS, 4T}$$

denote the quotient of $\mathcal{D}_n(X)$ by these relations (that is the quotient of $\mathcal{D}_n(X)$ by the vector space generated by the elements of $\mathcal{D}_n(X)$ of the form of the left-hand side of the relations). Equip $\mathcal{A}_n(X)$ with the standard topology of a complex finite-dimensional vector space.

Note that, by definition, $\mathcal{A}_0(X)$ is the $\mathbf{C}$-vector space freely generated by the unique diagram with 0 chord, called the *empty diagram*, or the *unit of* $\mathcal{A}(X)$, and denoted by 1_X.

$$\mathcal{A}_0(X) = \mathbf{C}1_X$$

We let

$$\mathcal{A}(X) = \prod_{n \in \mathbf{N}} \mathcal{A}_n(X)$$

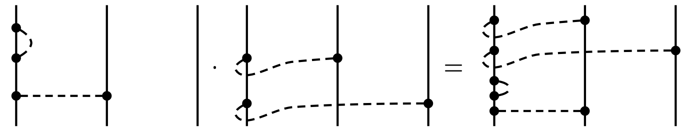
denote the product of the $\mathcal{A}_n(X)$ equipped with its natural vector space structure and with the product topology. The element $(x_n \in \mathcal{A}_n(X))_{n\in\mathbf{N}}$ is simply denoted by $\sum_{n\in\mathbf{N}} x_n$ by making use of the obvious inclusions from the $\mathcal{A}_n(X)$ to $\mathcal{A}(X)$. (Note that such a series always converges in $\mathcal{A}(X)$.)

When X is made of p vertical numbered intervals, $\mathcal{A}(X = \coprod_{i=1}^{p} I_i)$ is denoted by $\mathcal{A}(p)$. In that case, there is a unique bilinear continuous product

$$\mathcal{A}(p) \times \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$$

$$(d, e) \mapsto de$$

such that, if d and e are (the projections in $\mathcal{A}(p)$ of) two chord diagrams on the p vertical numbered intervals drawn as below, de is obtained from d and e by stacking e above d as follows.



Note that this product is well-defined (i.e. compatible with AS and 4T) and maps $\mathcal{A}_n(p) \times \mathcal{A}_m(p)$ to $\mathcal{A}_{n+m}(p)$. In particular, $\mathcal{A}(p)$ is an algebra with a well-defined and continuous exponential map:

$$\begin{array}{rccl} \exp: & \mathcal{A}(p) & \longrightarrow & \mathcal{A}(p) \\ & x & \mapsto & \exp(x) = \sum_{n\in\mathbf{N}} \frac{x^n}{n!} \end{array}$$

Let ε denote the *augmentation map* from $\mathcal{A}(X)$ to $\mathbf{C} = \mathcal{A}_0(X)$, that is the natural projection. Then, there is a continuous logarithm:

$$\begin{array}{rccl} \mathrm{Log}: & \varepsilon^{-1}(1) \subset \mathcal{A}(p) & \longrightarrow & \mathcal{A}(p) \\ & 1 + x & \mapsto & \mathrm{Log}(1+x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n} \end{array}$$

Similarly, when $r \in \mathbf{R}$, we define the r^{th} power of an element $y \in \varepsilon^{-1}(1) \subset \mathcal{A}(p)$ by the usual formula:

$$y^r = \exp(r\mathrm{Log}(y)).$$

and the $(-1)^{th}$ power of such a y is its inverse in $\mathcal{A}(p)$.

Exercise 1.8 Prove that for any three distinct elements i, j, k of $\{1, 2, \ldots, p\}$, we have

$$[\Gamma_{ij}][\Gamma_{jk}] - [\Gamma_{jk}][\Gamma_{ij}] = [\Gamma_{jk}][\Gamma_{ki}] - [\Gamma_{ki}][\Gamma_{jk}]$$

in $\mathcal{A}_2(p)$ where $\Gamma_{ij} = \Gamma_{\{\{i,j\}\}}$, and the brackets (which will be often omitted) are used to denote the projections onto the quotients $\mathcal{A}_n(p)$.

1.4 The Kontsevich integral for braids

Under the hypotheses and notation above, the *degree n* part $Z_n(\gamma)$ of the Kontsevich integral $Z(\gamma)$ of γ is

$$Z_n(\gamma) = \sum_{x \in P^n} Z(\gamma, x)[\Gamma_x] \ \in \mathcal{A}_n(p)$$

and

$$Z(\gamma) = \sum_{n\in\mathbf{N}} Z_n(\gamma) \ \in \mathcal{A}(p)$$

Examples 1.9 In Example 1.2, we have

$$Z(\gamma) = \exp\left(\frac{\lambda}{2i\pi}\Gamma_{12}\right)$$

In Example 1.1, we have

$$Z(\gamma) = \exp\left(\frac{n}{3}\left(\Gamma_{12} + \Gamma_{13} + \Gamma_{23}\right)\right)$$

If γ' is parametrized by $h \in [a,b]$ where $a, b \in \mathbf{R}, a < b$, then $Z(\gamma')$ is defined as above except that Δ^n is replaced anywhere by

$$\Delta^n_{[a,b]} \stackrel{\text{def}}{=} \{(h_1, h_2, \ldots, h_n) \in \mathbf{R}^n |\, a \leq h_1 \leq h_2 \leq \ldots \leq h_n \leq b\}.$$

Z satisfies the following easy properties:

Property 1.10 (Independence of the parameter) *Let $\phi : [a,b] \longrightarrow [0,1]$ be an increasing diffeomorphism, then*

$$Z(\gamma \circ \phi) = Z(\gamma)$$

Property 1.11 (Invariance under transvection) *Let f be a continuous function from $[0,1]$ to $\mathbf{C}$, then*

$$Z(\gamma + f : h \mapsto ((z_1 + f)(h), \ldots, (z_p + f)(h))) = Z(\gamma)$$

Property 1.12 (Invariance under global homothety) *Let $A \in \mathbf{C}$, then*

$$Z(A\gamma : h \mapsto (Az_1(h), \ldots, Az_p(h))) = Z(\gamma)$$

Property 1.13 (Multiplicativity) *Let $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbf{C}^p \setminus \mathbf{\Delta}$ be two $\mathbf{C}^p$-configuration paths such that $\gamma_1(1) = \gamma_2(0)$, and let $\gamma_1\gamma_2$ be the* composed path*:*

$$\begin{array}{cccl} \gamma_1\gamma_2 : & [0,2] & \longrightarrow & \mathbf{C}^p \setminus \mathbf{\Delta} \\ & h & \mapsto & \begin{cases} \gamma_1(h) & \text{if } h \leq 1 \\ \gamma_2(h-1) & \text{if } h \geq 1 \end{cases} \end{array}$$

Then

$$Z(\gamma_1\gamma_2) = Z(\gamma_1)Z(\gamma_2)$$

Proof of the properties: It is easy to check that all the coefficients $Z(\gamma, x)$ are unchanged by the operations of the 3 first properties. For the fourth one, just decompose

$$\begin{array}{ll} \Delta^n_{[0,2]} & = \cup_{k=0}^n \Delta^k_{[0,1]} \times \Delta^{n-k}_{[1,2]} \\ & (= \cup_{k=0}^n \{(h_1, h_2, \ldots, h_n) |\, 0 \leq h_1 \leq \ldots \leq h_k \leq 1 \leq h_{k+1} \leq \ldots \leq h_n \leq 2\}) \end{array}$$

Denote by $x_{\rightarrow k}$ the k-tuple made of the k first coordinates of x and $x_{k+1\rightarrow}$ the (n-k)-tuple made of the following ones so that $\Gamma_x = \Gamma_{x_{\rightarrow k}}\Gamma_{x_{k+1\rightarrow}}$ and observe that

$$\int_{\Delta^k_{[0,1]} \times \Delta^{n-k}_{[1,2]}} \omega_x(\gamma_1\gamma_2) = \int_{\Delta^k_{[0,1]}} \omega_{x_{\rightarrow k}}(\gamma_1) \times \int_{\Delta^{n-k}_{[1,2]}} \omega_{x_{k+1\rightarrow}}(\gamma_2(.-1))$$

This shows how

$$Z(\gamma_1\gamma_2,x)\Gamma_x=\sum_{k=0}^{n}Z(\gamma_1,x_{\to k})\Gamma_{x_{\to k}}Z(\gamma_2,x_{k+1\to})\Gamma_{x_{k+1\to}}$$

and allows us to easily conclude the proof. ◇

We now give a more compact definition of Z:
The *Knizhnik-Zamolodchikov connection* $\Omega_{KZ}\in\Omega^1(\mathbf{C}^p\setminus\mathbf{\Delta},\mathcal{A}(p))$ is the one-form valued in $\mathcal{A}(p)$ defined by:

$$\Omega_{KZ}=\frac{1}{2i\pi}\sum_{\{j,k\}\subset\{1,2,\dots,p\}}\frac{dz_j-dz_k}{z_j-z_k}\Gamma_{jk}$$

Then the Kontsevich integral Z of γ is

$$Z(\gamma)=\sum_{n\in\mathbf{N}}\int_{\Delta^n}(\gamma\circ p_1)^*(\Omega_{KZ})\wedge(\gamma\circ p_2)^*(\Omega_{KZ})\wedge\ldots\wedge(\gamma\circ p_n)^*(\Omega_{KZ})$$

where $p_i:\Delta^n\longrightarrow[0,1]$ is the i^{th} projection, $p_i(h_1,\dots,h_n)=h_i$, and where, if ω_1 and ω_2 are two complex-valued forms, and if d_1 and d_2 are two elements of $\mathcal{A}(p)$, then $\omega_1 d_1\wedge\omega_2 d_2=\omega_1\wedge\omega_2(d_1d_2)$.

Side-remark 1.14 Therefore, the Kontsevich integral is called the *holonomy of the Knizhnik-Zamolodchikov connection.* For us, it is enough to know that Z is multiplicative with respect to the path composition.

Side-exercise 1.15 Check that Z satisfies the other holonomy property:

$$\frac{\mathrm{d}}{\mathrm{dh}}Z(\gamma_{|[0,h]})(u)=Z(\gamma_{|[0,u]})\Omega_{KZ\,\gamma(u)}(\gamma'(u))$$

1.5 The isotopy invariance of the Kontsevich integral of braids

Theorem 1.16 (Kontsevich, 1992) *Let γ be a C^∞ map from $[0,1]\times[0,1]$ to $\mathbf{C}^p\setminus\mathbf{\Delta}$. For $u\in[0,1]$, let γ_u denote the configuration path:*

$$\begin{array}{llll}\gamma_u: & [0,1] & \longrightarrow & \mathbf{C}^p\setminus\mathbf{\Delta}\\ & h & \mapsto & \gamma_u(h)=\gamma(u,h)\end{array}$$

Assume $\gamma_u(0)=\gamma_0(0)$ and $\gamma_u(1)=\gamma_0(1)$, for any $u\in[0,1]$. Then

$$Z(\gamma_0)=Z(\gamma_1)$$

In other words, $Z(\gamma)$ only depends on the homotopy class[1] *with fixed extremities $\gamma(0)$, $\gamma(1)$ of γ. (This is also equivalent to say that Ω_{KZ} is flat.)*

To prove the theorem, we will use the following lemma which will be proved later:

Lemma 1.17

$$\Omega_{KZ}\wedge\Omega_{KZ}=0$$

PROOF OF THE THEOREM: Observe that $\mathrm{d}\Omega_{KZ}=0$. More precisely, for any $\{j,k\}\in P$, $\mathrm{d}\left(\frac{dz_j-dz_k}{z_j-z_k}\right)=0$ because $\frac{dz_j-dz_k}{z_j-z_k}$ can be written locally as $\mathrm{dLog}(z_j-z_k)$). Thus, the usual Stokes theorem (applied to the computation of all coefficients of the Γ_x) allows us to write:

$$\int_{\partial([0,1]\times\Delta^n)}(\gamma\circ(1\times p_1))^*(\Omega_{KZ})\wedge(\gamma\circ(1\times p_2))^*(\Omega_{KZ})\wedge\ldots\wedge(\gamma\circ(1\times p_n))^*(\Omega_{KZ})=0$$

where $1\times p_i:[0,1]\times\Delta^n\longrightarrow[0,1]\times[0,1]$ maps $(u,H\in\Delta^n)$ to $(u,p_i(H))$.

The boundary $\partial([0,1]\times\Delta^n)$ of $[0,1]\times\Delta^n$ decomposes into the following parts:

[1] Any homotopy between two C^∞ paths in $(\mathbf{C}^p\setminus\mathbf{\Delta})$ can be replaced by a C^∞ one. See [Hi, Chapter 2, Section 2].

- $\partial[0,1] \times \Delta^n$ whose contribution to the above integral is $Z_n(\gamma_1) - Z_n(\gamma_0)$,

- the face $[0,1] \times \{(h_1,\dots,h_n) \in \Delta^n | \, h_1 = 0\}$ where $(\gamma \circ (1 \times p_1))^*(\Omega_{KZ}) = 0$ because $\gamma_u(0)$ is constant and which therefore does not contribute to the integral, and the face $[0,1] \times \{(h_1,\dots,h_n) \in \Delta^n | \, h_n = 1\}$ which does not contribute either for the same reason,

- for $k = 1, 2, \dots, n-1$, the faces of $[0,1] \times \Delta^n$: $\{(h_1,\dots,h_n) \in \Delta^n | \, h_k = h_{k+1}\}$ where $1 \times p_k = 1 \times p_{k+1}$ and which therefore will not contribute because of Lemma 1.17.

Except for the proof of Lemma 1.17, the theorem is proved. $\diamond$

Proof of Lemma 1.17:

$$\Omega_{KZ} = \frac{1}{2i\pi} \sum_{\{j,k\} \in P} \omega_{jk} \Gamma_{jk}$$

where ω_{jk} is the complex-valued 1-form $\omega_{jk} = \frac{dz_j - dz_k}{z_j - z_k}$. Therefore,

$$\Omega_{KZ} \wedge \Omega_{KZ} = \left(\frac{1}{2i\pi}\right)^2 \sum_{(\{a,b\},\{c,d\}) \in P^2} (\omega_{ab} \wedge \omega_{cd}) \Gamma_{ab} \Gamma_{cd}$$

Since $\omega_{ab} \wedge \omega_{ab} = 0$, the pairs $(\{a,b\},\{a,b\})$ do not contribute to the sum. When the pairs $\{a,b\}$ and $\{c,d\}$ are disjoint, Γ_{ab} and Γ_{cd} commute and therefore, the contributions of $(\{a,b\},\{c,d\})$ and $(\{c,d\},\{a,b\})$ cancel each other. Thus, the only remaining contributions to the sum come from the pairs where $\{c,d\} \cap \{a,b\}$ contains one element, that are the pairs where the cardinality of $\{c,d\} \cup \{a,b\}$ is 3, and we have:

$$\Omega_{KZ} \wedge \Omega_{KZ} = \left(\frac{1}{2i\pi}\right)^2 \sum_{\{i,j,k\} \subset \{1,2,\dots,p\}} \begin{pmatrix} & \omega_{ij} \wedge \omega_{jk} (\Gamma_{ij}\Gamma_{jk} - \Gamma_{jk}\Gamma_{ij}) \\ + & \omega_{jk} \wedge \omega_{ki} (\Gamma_{jk}\Gamma_{ki} - \Gamma_{ki}\Gamma_{jk}) \\ + & \omega_{ki} \wedge \omega_{ij} (\Gamma_{ki}\Gamma_{ij} - \Gamma_{ij}\Gamma_{ki}) \end{pmatrix}$$

We know from Exercise 1.8 that $(\Gamma_{ij}\Gamma_{jk} - \Gamma_{jk}\Gamma_{ij})$ is invariant under any circular permutation of i, j and k. Thus, $\Omega_{KZ} \wedge \Omega_{KZ}$ can be rewritten as

$$\Omega_{KZ} \wedge \Omega_{KZ} = \left(\frac{1}{2i\pi}\right)^2 \sum_{\{i,j,k\} \subset \{1,2,\dots,p\}} (\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij})(\Gamma_{ij}\Gamma_{jk} - \Gamma_{jk}\Gamma_{ij})$$

And it suffices to prove that $(\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij})$ is zero. To do it, we express the sum as a combination of $dz_i \wedge dz_j$, $dz_j \wedge dz_k$ and $dz_k \wedge dz_i$, and compute the coefficient of $dz_i \wedge dz_j$ since the other ones can be deduced by circular permutation. This coefficient is zero. Thus, the lemma, and hence the isotopy invariance of Z are proved. $\diamond$

From now on, we will consider the configuration paths γ up to homotopy with fixed extremities and translation by a function f since these operations do not affect $Z(\gamma)$. If γ is a path from $b = \gamma(0)$ to $t = \gamma(1)$ such that b and t are in $\mathbf{R}^p$, then it (or its class modulo these operations) is determined by a regular[2] planar projection of γ with the under/over-crossing datum, and by b and t which are considered up to translation. The class of such a path will be denoted by $(D; b, t)$ or (D_b^t) where D is a picture projection of γ. The letters b and t stand for bottom and top, respectively. For b, $t \in \mathbf{R}^p \cap (\mathbf{C}^p \setminus \mathbf{\Delta})$, we will denote the (class of a) configuration path γ from b to t whose planar projection has no double point by $\mathbf{1}_b^t$ or by $(\mathbf{1}; b, t)$.

Examples 1.18 Let α, α', β and $\tau \in \mathbf{R}$.

$Z(\mathbf{1}; (\alpha, \alpha+\beta), (\alpha', \alpha'+\tau)) = Z(\mathbf{1}; (0,\beta), (0,\tau)) = Z(\mathbf{1}_{(0,\beta)}^{(0,\tau)}) = Z(\mathbf{1}_{(0,1)}^{(0,\frac{\tau}{\beta})}) \exp\left(\frac{Log(\frac{\tau}{\beta})}{2i\pi}\Gamma_{12}\right)$

$Z($$; (0,1), (0,1)) = exp(\frac{1}{2}\Gamma_{12})$

$Z($$; (0,1), (0,1)) = exp(\frac{-1}{2}\Gamma_{12})$

[2]A *regular* projection is a projection which is an immersion whose only *multiple points* (points with several preimages) are *transverse* (the two tangent vectors are independent) double points.

Recall that the inverse of a path $\gamma : [0,1] \longrightarrow X$ in a topological space X is the path $\overline{\gamma} : [0,1] \longrightarrow X$ which maps h to $\gamma(1-h)$ so that there is a homotopy with fixed extremities from the composed path $\gamma\overline{\gamma}$ to the constant path to $\gamma(0)$. Applying this remark to $X = \mathbf{C}^p \setminus \mathbf{\Delta}$ yields

$$Z(\overline{\gamma}) = Z(\gamma)^{-1}.$$

Also note that the numerotation of the strands (order of the coordinates z_i) is immaterial. The only thing that matters is to number the components of the braids and the intervals used to define $\mathcal{A}(p)$, accordingly.

1.6 Some unpleasant features of the Kontsevich integral

Let X and Y be two 1-manifolds. Let $X \coprod Y$ or $X \otimes Y$ denote their disjoint union. In pictures, we draw Y on the right-hand-side of X. There is a natural bilinear continuous operation called the *tensor product* from $\mathcal{A}(X) \times \mathcal{A}(Y)$ to $\mathcal{A}(X \coprod Y)$ which maps (d,e) to the disjoint union $d \otimes e$ of d and e, for any two diagrams $d \in \mathcal{A}(X)$ and $e \in \mathcal{A}(Y)$.

When applied to the case where X and Y are unions of intervals, this construction yields, for any two integers p and q:

$$\otimes : \mathcal{A}(p) \times \mathcal{A}(q) \longrightarrow \mathcal{A}(p+q)$$

Assume that we have a (class of a) $\mathbf{C}^p$-configuration path γ_L from $b_L \in]-\infty,0]^p$ to $t_L \in]-\infty,0]^p$, denoted by $(D_L; b_L, t_L)$, and a (class of a) $\mathbf{C}^q$-configuration path γ_R from $b_R \in [1,+\infty[^q$ to $t_R \in [1,+\infty[^q$, denoted by $(D_R; b_R, t_R)$. Assume furthermore that the images of γ_L and γ_R are contained in $(]-\infty,0] + i\mathbf{R})^p$ and $([1,+\infty[+i\mathbf{R})^q$, respectively. Denote the $\mathbf{C}^{p+q}$-configuration path diagram obtained by putting D_R on the right-hand-side of D_L by $D_L \otimes D_R$.

First bad feature:

$$Z(D_L \otimes D_R; (b_L, b_R), (t_L, t_R)) \neq Z(D_L; b_L, t_L) \otimes Z(D_R; b_R, t_R)$$

because there are unwanted contributions of diagrams with chords involving both the left-hand side part and the right-hand side part.

Exercise 1.19 Prove that $Z(D_L; b_L, t_L) \otimes Z(D_R; b_R, t_R)$ is exactly the contribution of the other diagrams (whose chords are either on the LHS part or in the RHS part) to $Z(D_L \otimes D_R; (b_L, b_R), (t_L, t_R))$. (See the proof of Property 1.13.)

Nevertheless, we will be able to diminish the unwanted contributions. In order to estimate their behaviour, we will use the following notation.

Notation 1.20 Let $d = (d_n(\varepsilon > 0))_{n \in \mathbf{N}} \in \mathcal{A}(X)$. Let $(f_n :]0,+\infty[\longrightarrow \mathbf{R})_{n \in \mathbf{N}}$ be a family of continuous maps. We use the notation $d = O(f_*(\varepsilon))$ to say: $\forall n \in \mathbf{N}$, $\exists M_n \in]0,+\infty[$ such that: $\|d_n(\varepsilon)\| \leq M_n f_n(\varepsilon)$ where $\|.\|$ denotes one of the (equivalent) norms on the finite-dimensional $\mathbf{C}$-vector space $\mathcal{A}_n(X)$. We omit the subscripts $*$ and n when f_n does not depend on n.

Let us simultaneously perform the following two homotheties $\varepsilon(.)$ and $\varepsilon(.-1)+1$ of the complex plane with ratio ε. The first one has center 0 and is performed on the left-hand-side path γ_L while the second one has center 1 and is performed on γ_R. These homotheties do not change $Z(D_L; b_L, t_L) \otimes Z(D_R; b_R, t_R)$. Then, a quick look at the behaviour of the coefficients of the boring diagrams shows:

Lemma 1.21

$$Z(D_L \otimes D_R; (\varepsilon b_L, \varepsilon(b_R - 1) + 1), (\varepsilon t_L, \varepsilon(t_R - 1) + 1)) = Z(D_L; b_L, t_L) \otimes Z(D_R; b_R, t_R) + O(\varepsilon)$$

◇

This is our first motivation to study the behaviour of $Z(\gamma)$ for configuration paths which reach some limit configurations. Another motivation for that is that we want to define a link invariant, and we would like to define $Z(\bigcap)$ from $\lim_{\varepsilon\to 0} Z(\mathbf{1};(0,1),(0,\varepsilon))$.

Second bad feature: $Z(\mathbf{1};(0,1),(0,\varepsilon)) = exp(\frac{Log(\varepsilon)}{2i\pi}\Gamma_{12})$ does not converge when ε approaches 0.

Another wish that we have is to get rid of the heavy-to-carry data of the coordinates of the bottom and top configurations. A possible way to do it would be to restrict ourselves to the case when these configurations are $(1,2,3,\dots,p)$ but the study above suggests that it may not be a very good idea. So let us begin to study some limit behaviours of Z.

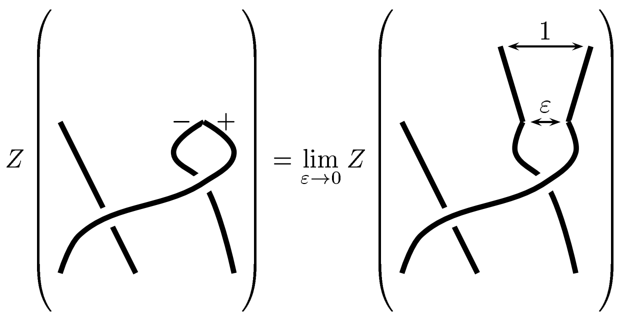
1.7 A two-point collision

Definition 1.22 Let $\mathbf{C}^p(2)$ denote the subset of $\mathbf{C}^p$ made of the elements $(z_1,z_2,\dots,z_p)$ of $\mathbf{C}^p$ which exactly have two equal coordinates $z_j = z_k$, $j \neq k$ ($z_i = z_l \Longrightarrow i = l$ or $\{i,l\} = \{j,k\}$).

A *two-point collision* of $\mathbf{C}^p$ is an element of $\mathbf{C}^p(2)$, with the additional datum of the superscript $-$ on one of the two equal coordinates, say z_j in the above example, and the superscript $+$ on the other one. Intuitively, these superscripts mean: z_j is infinitely close to z_k, and it is on the left-hand side of z_k. Let $\mathbf{C}^p_2$ be the set of two-point collisions of $\mathbf{C}^p$.

By definition, a path $\gamma : [0,1] \longrightarrow (\mathbf{C}^p \setminus \mathbf{\Delta}) \cup \mathbf{C}^p_2$ is a continuous (piecewise C^∞) path from $[0,1]$ to $(\mathbf{C}^p \setminus \mathbf{\Delta}) \cup \mathbf{C}^p(2)$ such that:
for any $h_d \in \gamma^{-1}(\mathbf{C}^p(2))$, one of the two equal coordinates of $\gamma(h_d)$, $z_{j(h_d)}$, is equipped with $-$ and the other one $z_{k(h_d)}$ is equipped with $+$, and there exists a neighborhood $N(h_d)$ of h_d in $[0,1]$ such that, for any $h \in N(h_d) \setminus \{h_d\}$, $\mathrm{Re}(z_{k(h_d)}(h) - z_{j(h_d)}(h)) > 0$.

This section is devoted to extending Z for such paths $\gamma : [0,1] \longrightarrow (\mathbf{C}^p \setminus \mathbf{\Delta}) \cup \mathbf{C}^p_2$ as suggested by the following graphic formula.



Notation 1.23 If $z \in \mathbf{C}^p_2$ is as in Definition 1.22, for $\varepsilon \in]0,\infty[$, we define

$$z(+\varepsilon) = (z_1(+\varepsilon), z_2(+\varepsilon), \dots, z_p(+\varepsilon)) \in \mathbf{C}^p$$

by $z_k(+\varepsilon) = z_k + \varepsilon$ and $z_i(+\varepsilon) = z_i$ if $i \neq k$.

In general for two points $b = (b_1,b_2,\dots,b_p)$ and $t = (t_1,t_2,\dots,t_p)$ in $\mathbf{C}^p$, let $\mathcal{B}^t_b$ denote the *barycentric path* from b to t that is

$$\begin{array}{cccl} \mathcal{B}^t_b: & [0,1] & \longrightarrow & \mathbf{C}^p \\ & h & \mapsto & (1-h)b + ht = ((1-h)b_1 + ht_1, \dots, (1-h)b_p + ht_p)) \end{array}$$

Definition 1.24 Z is defined as follows for paths $\gamma : [0,1] \longrightarrow (\mathbf{C}^p \setminus \mathbf{\Delta}) \cup \mathbf{C}^p_2$.

1. If $\gamma : [0,1] \longrightarrow (\mathbf{C}^p \setminus \mathbf{\Delta}) \cup \mathbf{C}^p_2$ is a path such that $\gamma^{-1}(\mathbf{C}^p(2)) = \{0\}$, then

$$Z(\gamma) = \lim_{\substack{\varepsilon \to 0, \varepsilon > 0 \\ h \longrightarrow 0, h > 0}} \exp(\frac{\mathrm{Log}(\varepsilon)}{2i\pi}\Gamma_{j(0)k(0)}) Z(\mathcal{B}^{\gamma(h)}_{\gamma(0)(+\varepsilon)}\gamma_{|[h,1]})$$

2. For any path $\gamma : [0,1] \longrightarrow (\mathbf{C}^p \setminus \mathbf{\Delta}) \cup \mathbf{C}_2^p$

$$Z(\overline{\gamma}) = Z(\gamma)^{-1}$$

3. Z is still multiplicative under path composition.

To prove that the definition makes sense, we need the following lemma which is proved at the end of this subsection:

Lemma 1.25 *Let $b = (b_1^-, b_2^+, b_3, b_4, \ldots, b_{p-1}, b_p) \in \mathbf{C}_2^p$, $(b_1 = b_2)$, let $\varepsilon_0 > 0$. Assume that $b(+\varepsilon) \in \mathbf{C}^p \setminus \mathbf{\Delta}$ for any ε such that $0 < \varepsilon \leq \varepsilon_0$. Then the limit*

$$\lim_{\varepsilon \to 0} \exp(\frac{\mathrm{Log}(\varepsilon)}{2i\pi}\Gamma_{12}) Z(\mathcal{B}_{b(+\varepsilon)}^{b(+\varepsilon_0)})$$

exists.

Proof of consistency of the definition: Let us first prove that the definition of Z makes sense in the first case. Let $b = \gamma(0)$. The hypotheses on γ guarantee the existence of $\varepsilon_0 > 0$ and $h_0 > 0$ such that:

1. for any (ε, h) such that $0 < \varepsilon \leq \varepsilon_0$ and $0 < h \leq h_0$, the barycentric path $\mathcal{B}_{b(+\varepsilon)}^{\gamma(h)}$ avoids $\mathbf{\Delta}$ and $\mathrm{Re}(z_2(h) - z_1(h)) > 0$
2. the homotopy class of $\mathcal{B}_{b(+\varepsilon)}^{\gamma(h)} \gamma_{|[h,1]}$ does not depend on the choice of h and coincides with the homotopy class of $\mathcal{B}_{b(+\varepsilon)}^{b(+\varepsilon_0)} \mathcal{B}_{b(+\varepsilon_0)}^{\gamma(h)} \gamma_{|[h,1]}$.

Thus, by the former properties of Z, $Z(\mathcal{B}_{b(+\varepsilon)}^{\gamma(h)} \gamma_{|[h,1]})$ is independent of h such $0 < h < h_0$ and is equal to $Z(\mathcal{B}_{b(+\varepsilon)}^{b(+\varepsilon_0)}) Z(\mathcal{B}_{b(+\varepsilon_0)}^{\gamma(h)} \gamma_{|[h,1]})$, and, according to Lemma 1.25, the definition of Z makes sense in the first case.

Now, define $Z(\gamma)$ for a path $\gamma : [0,1] \longrightarrow (\mathbf{C}^p \setminus \mathbf{\Delta}) \cup \mathbf{C}_2^p$ such that $\gamma^{-1}(\mathbf{C}^p(2)) = \{1\}$ by

$$Z(\overline{\gamma}) = Z(\gamma)^{-1}$$

In other words, if $\gamma(1) = t$,

$$Z(\gamma) = \lim_{\substack{\varepsilon \to 0, \varepsilon > 0 \\ h \longrightarrow 1, h < 1}} Z(\gamma_{|[0,h]} \mathcal{B}_{\gamma(h)}^{t(+\varepsilon)}) \exp(-\frac{Log(\varepsilon)}{2i\pi}\Gamma_{j(1)k(1)})$$

Now any path $\gamma : [0,1] \longrightarrow (\mathbf{C}^p \setminus \mathbf{\Delta}) \cup \mathbf{C}_2^p$ is a composition of paths for which Z is defined, and we define Z of a such a composition as the product of the values on elementary paths. The form of the first definition of Z makes clear that $Z(\gamma)$ does not depend on the decomposition of Z as a product. Thus, Z is compatible with products and Z is easily seen to be compatible with the passage to the inverse, too.

$\diamond$

Note that Z is still independent of the parametrization, and invariant by transvection. But we lose the invariance under homothety.

Exercise 1.26 Let $\gamma : [0,1] \longrightarrow (\mathbf{C}^p \setminus \mathbf{\Delta}) \cup \mathbf{C}_2^p$. Let $A \in]0, +\infty[$. If $\gamma(h) = (z_1(h), \ldots, z_p(h))$, then $A\gamma(h) = (Az_1(h), \ldots, Az_p(h))$. Show that:
If $\gamma^{-1}(\mathbf{C}^p(2)) = \{1\}$, then

$$Z(A\gamma) = Z(\gamma) \exp\left(-\frac{\mathrm{Log}(A)}{2i\pi}\Gamma_{j(1)k(1)}\right)$$

If $\gamma^{-1}(\mathbf{C}^p(2)) = \{0\}$, then

$$Z(A\gamma) = \exp\left(\frac{\mathrm{Log}(A)}{2i\pi}\Gamma_{j(0)k(0)}\right) Z(\gamma)$$

Proof of Lemma 1.25: Let $\gamma : h \in [0,\varepsilon_0] \mapsto (z_1(h),\ldots,z_p(h))$ be defined so that $\gamma_{|[\varepsilon,\varepsilon_0]}$ represents $\mathcal{B}_{b(+\varepsilon)}^{b(+\varepsilon_0)}$ by

$$z_j(h) = \begin{cases} b_j & \text{if } j \neq 2 \\ b_1 + h & \text{if } j = 2 \end{cases}$$

Let $x = (x_1,\ldots,x_n) \in P^n$ and let $r \in \{0,\ldots,n\}$ be such that $x_j = \{1,2\}$ for any $j \leq r$ and $x_{r+1} \neq \{1,2\}$ unless $r = n$. Let $c_x(\varepsilon)$ be the complex coefficient of Γ_x in $\exp(\frac{Log(\varepsilon)}{2i\pi}\Gamma_{12})Z(\gamma_{|[\varepsilon,\varepsilon_0]})$ that is

$$c_x(\varepsilon) = \left(\frac{1}{2i\pi}\right)^n \sum_{j=0}^{r} \frac{(\mathrm{Log}\varepsilon)^j}{j!} \int\int\cdots\int_{\varepsilon \leq h_{j+1} \leq h_{j+2} \leq \ldots \leq h_n \leq \varepsilon_0} \prod_{k=j+1}^{n} \tilde{\omega}_{x_k}(h_k)\mathrm{d}h_k$$

It suffices to prove the existence of $\lim_{\varepsilon\to 0} c_x(\varepsilon)$ for any x, that is for ours.

If $r = n$, then the multiple integral is equal to the coefficient of Γ_{12}^{n-j} in $(2i\pi)^{n-j}Z(\mathbf{1}_{(0,\varepsilon)}^{(0,\varepsilon_0)})$ that is $\frac{(\mathrm{Log}(\frac{\varepsilon_0}{\varepsilon}))^{n-j}}{(n-j)!}$ (see Example 1.4). Thus,

$$c_x(\varepsilon) = \left(\frac{1}{2i\pi}\right)^n \frac{(\mathrm{Log}\varepsilon + \mathrm{Log}(\frac{\varepsilon_0}{\varepsilon}))^n}{n!} = \left(\frac{1}{2i\pi}\right)^n \frac{(\mathrm{Log}\varepsilon_0)^n}{n!}.$$

and the same argument shows that if $r < n$, then

$$c_x(\varepsilon) = \left(\frac{1}{2i\pi}\right)^n \int_{\varepsilon}^{\varepsilon_0} f(h)dh$$

where ($h = h_{r+1}$ and)

$$f(h) = \frac{(\mathrm{Log}h)^r}{r!}\tilde{\omega}_{x_{r+1}}(h) \int\int\cdots\int_{h \leq h_{r+2} \leq h_{r+3} \leq \ldots \leq h_n \leq \varepsilon_0} \prod_{k=r+2}^{n} \tilde{\omega}_{x_k}(h_k)\mathrm{d}h_k$$

Now, remember that $\tilde{\omega}_{\{1,2\}}(h) = 1/h$, and note that there exists a $C > 0$, such that, for all $h \in]0,1]$, and for all $i > r+1$,

$$|\tilde{\omega}_{x_i}(h)| \leq C\tilde{\omega}_{\{1,2\}}(h) \text{ and } |\tilde{\omega}_{x_{r+1}}(h)| \leq C$$

Thus,

$$|f(h)| \leq C^{n-r}\frac{|\mathrm{Log}h|^r}{r!}\frac{(\mathrm{Log}\frac{\varepsilon_0}{h})^{n-r-1}}{(n-r-1)!} \leq C'|Log(h)|^{n-1}$$

for some $C' > 0$. This proves that $\int_{\varepsilon}^{\varepsilon_0} f(h)dh$ converges and concludes the proof of the lemma. ◇

Remark 1.27 Thus, the definition of Z for a path which reaches some two-point collision b is explained in the graphic formula just after Definition 1.22. Indeed, we just slightly modify the given path so that it reaches $b(+\varepsilon)$ instead of reaching b. Under our hypotheses, the choice of the modified path is canonical up to homotopy. Then, we correct the evaluation of Z on the modified path by a multiplication by $\varepsilon^{-\frac{\Gamma_{12}}{2i\pi}} = Z(\mathbf{1}_{(0,\varepsilon)}^{(0,1)})$ which involves the same ε acting on the two strands of the collision; and we take the limit.

Remark 1.28 In Definition 1.24, we choose to modify a two-point collision $b = \gamma(0) \in \mathbf{C}_2^p$ into $b(+\varepsilon)$ so that the two equal coordinate b_j^- and b_k^+ become b_j and $b_k + \varepsilon$, respectively. Let $\eta \in [0,1]$, we can define $b(\eta,+\varepsilon)$ so that $b_j(\eta,+\varepsilon) = b_j - \eta\varepsilon$ and $b_k(\eta,+\varepsilon) = b_k + (1-\eta)\varepsilon$ and $b_i(\eta,+\varepsilon) = b_i$, if $i \neq j,k$. Note that for a two-point collision b, $Z(\mathcal{B}_{b(\varepsilon)}^{b(\eta,\varepsilon)}) = 1 + O(\varepsilon)$. Therefore changing $\gamma(0)(+\varepsilon)$ into $\gamma(0)(\eta,+\varepsilon)$ in Definition 1.24 would not change the definition.

1.8 Duplication of a strand in a braid

Let X be a one-manifold, and let X_0 be a connected component of X. Let

$$X(r \times X_0) = (X \setminus X_0) \coprod \left(\coprod_{i=1}^{r} X_0^{(i)} \right)$$

be the manifold obtained from X by *duplicating* X_0 (r-1) times that is by replacing X_0 by r copies of X_0 and let

$$\pi(r \times X_0) : X(r \times X_0) \longrightarrow X$$

be the associated trivial covering, that is the identity on $(X \setminus X_0)$, and the trivial r-fold covering from $\coprod_{i=1}^{r} X_0^{(i)}$ to X_0.

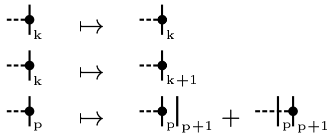
Recall from [V] that if $\pi : Y \longrightarrow X$ is a smooth map between two one-manifolds X and Y such that $\pi(\partial Y) \subset \partial X$, then one can unambiguously define the linear degree-preserving map : $\pi^* : \mathcal{A}(X) \longrightarrow \mathcal{A}(Y)$ which only depends on the homotopy class of π such that: If d is (the class of) a diagram on X whose univalent vertices avoid the critical values of π, then $\pi^*(d)$ is the sum of all diagrams on Y obtained from d by lifting each vertex to one of its preimages under π. (These diagrams have the same set of chords as d and the local orientations at vertices are naturally induced by the local orientations of the corresponding vertices of d.)

In particular, we have a natural linear *duplication map:*

$$\pi(r \times X_0)^* : \mathcal{A}(X) \longrightarrow \mathcal{A}(X(r \times X_0)).$$

When $X = \coprod_{i=1}^{p} I_i$ and $X_0 = I_j$, this map is simply denoted by $\pi(r \times j)^* : \mathcal{A}(p) \longrightarrow \mathcal{A}(p+r-1)$. Note that $\pi(r \times j)^*$ is a morphism of algebras. The local definition of $\pi(2 \times p)^*$ reads:

$\pi(2 \times p)^*$:

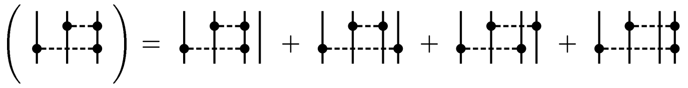


if $k < p$

if $k > p$

For example:

$\pi(2 \times 3)^*$



On the other hand, we may duplicate the p^{th} coordinate of an element $z = (z_1, \ldots, z_p)$ of $\mathbf{C}^p \setminus \mathbf{\Delta}$ to get an element $z(2 \times p) = (z_1, \ldots, z_{p-1}, z_p^-, z_p^+) \in \mathbf{C}_2^{p+1}$ Let $\gamma : [0,1] \longrightarrow \mathbf{C}^p \setminus \mathbf{\Delta}$ be our usual $\mathbf{C}^p$-configuration path which maps h to $(z_1(h), \ldots, z_p(h))$. Let

$$\begin{array}{rccl} \gamma(2 \times p)(+\varepsilon) : & [0,1] & \longrightarrow & \mathbf{C}^{p+1} \setminus \mathbf{\Delta} \\ & h & \mapsto & (z_1(h), \ldots, z_p(h), z_p(h) + \varepsilon) \end{array}$$

be obtained from γ by replacing the p^{th} strand by two parallel copies of this strand.

Lemma 1.29 $Z(\gamma(2 \times p)(+\varepsilon)) = \pi(2 \times p)^*(Z(\gamma)) + O(\varepsilon)$

PROOF: Replacing $Z(\gamma(2 \times p)(+\varepsilon))$ by $\pi(2 \times p)^*(Z(\gamma))$ amounts to replace $\frac{\mathrm{d}z_{p+1} - \mathrm{d}z_j}{z_{p+1} - z_j}$ by $\frac{\mathrm{d}z_p - \mathrm{d}z_j}{z_p - z_j}$ in the iterated integrals, where $z_{p+1} = z_p + \varepsilon$. $\diamond$

Lemma 1.30 *Let*

$$\begin{array}{rccl} \gamma(2 \times p)(+\varepsilon) : & [0,1] & \longrightarrow & (\mathbf{C}^{p+1} \setminus \mathbf{\Delta}) \cup \mathbf{C}_2^{p+1} \\ & h & \mapsto & (z_1(h), \ldots, z_p(h), z_p(h) + \varepsilon) \end{array}$$

be a path from a two-point collision

$$\gamma(2\times p)(+\varepsilon)(0) = (z_1^-(0), z_2^+(0), z_3(0), \dots, z_p(0), z_p(0)+\varepsilon)$$

such that $\gamma(2\times p)(+\varepsilon)(]0,1]) \subset \mathbf{C}^{p+1} \setminus \mathbf{\Delta}$. *Then*

$$Z(\gamma(2\times p)(+\varepsilon)) = \pi(2\times p)^*(Z(\gamma)) + O(\varepsilon).$$

PROOF: The proof is the same as above once it has been noticed that the majorations of the proof of Lemma 1.25 can be made uniform. ◇

Now, we know enough about the Kontsevich integral of braids, which may be seen as embeddings of trivial cobordisms between two $\mathbf{C}^p$-configurations, to be able to extend it on more general cobordisms between limit configurations (including the links themselves) called *tangles*.

2 The Kontsevich Integral for general tangles

This section is devoted to constructing the Kontsevich integral for *framed tangles* using what we know of the Kontsevich integral of braids by the preceeding section. We first define framed tangles.

2.1 Some limit configurations

Definition 2.1 A *non-associative word* or *n.a. word* w in the letter $\cdot$ is an element of the free non-associative monoid generated by $\cdot$. The *length* of such a w is the number of letters of w. Equivalently, we can define a *non-associative word* by saying that each such word has an integral *length* $\ell(w) \in \mathbf{N}$, the only word of length 0 is the *empty word*, the only word of length 1 is $\cdot$, the product $w'w''$ of two n.a. words w' and w'' is a n.a. word of length $(\ell(w') + \ell(w''))$, and every word w of length $\ell(w) \geq 2$ can be decomposed in a unique way as the product $w'w''$ of two n.a. words w' and w'' of nonzero length.

Example 2.2 The unique n.a. word of length 2 is $(\cdot\cdot)$. The two n.a. words of length 3 are $((\cdot\cdot)\cdot)$ and $(\cdot(\cdot\cdot))$. There are five n.a. words of length 4.

Definition 2.3 Let $\mathcal{O}_{p,k}$ be the set of equivalence classes $[(a_1; w_1), (a_2; w_2), \ldots, (a_k; w_k)]$ of the k-tuples $((a_1; w_1), (a_2; w_2), \ldots, (a_k; w_k))$ such that:

- for $i = 1, \ldots, k$, a_i is a real number and w_i is a non-empty n.a. word,
- $a_1 < a_2 < \ldots < a_k$ and $\sum_{i=1}^{k} \ell(w_i) = p$

where two such k-tuples are equivalent if and only if they are obtained from one another by a real translation. (For any $T \in \mathbf{R}$, $[(a_1; w_1), \ldots, (a_k; w_k)] = [(a_1 + T; w_1), \ldots, (a_k + T; w_k)]$.)

An *object of length p* is an element of the set $\mathcal{O}_p$ defined by:

$$\mathcal{O}_p = \cup_{k=1}^{p} \mathcal{O}_{p,k}$$

The objects of $\mathcal{O}_{p,p}$ are the sequences $[(a_i; \cdot)_{i=1,\ldots,p}]$ which will be simply denoted by $(a_1, a_2, \ldots, a_p)$ and represent *actual*[3] configurations of $(\mathbf{R}^p \setminus \mathbf{\Delta})$ /translation). Among these, $(1, 2, \ldots, p)$ will be simply denoted by $[p]$.

The objects of $\mathcal{O}_{p,p-1}$ represent two point-collisions: in these objects, there is one word of the form (..) whose left-hand-side letter represents the left-hand side point of the collision and whose RHS letter represents the RHS point of the collision.

In general, an object of length p is a *limit configuration* of p points (that is an element of a suitable[4] compactification of $(\mathbf{R}^p \setminus \mathbf{\Delta})$ /translation). In the object $[(a_1; w_1), (a_2; w_2), \ldots, (a_k; w_k)]$, there is a collision of $\ell(w_i)$ points at a_i, and the word w_i will tell us the way of seeing this collision as a composition of two-point collisions and duplications in Subsection 3.3.

The objects of $\mathcal{O}_{p,1}$ are of the form $[(0; w)]$ for a n.a. word w of length p and will be simply denoted by w. They are simply n.a. words.

An object $[(1; w_1), (2; w_2), \ldots, (k; w_k)]$ is simply denoted by $(w_1, w_2, \ldots, w_k)$ and is called a *word sequence*.

2.2 Framed tangles

Definition 2.4 Let T be an embedding[5] of a one-manifold X into a compact submanifold M of $\mathbf{R}^3 = \mathbf{C} \times \mathbf{R}$. It is understood in the word embedding that

$$T(X) \cap \partial M = T(\partial X)$$

[3]For us, an *actual* configuration is an element of $\mathbf{C}^p \setminus \mathbf{\Delta}$, i. e. it is not a limit configuration.

[4]Suitable compactifications can be found in [Po]. For us, it is enough to keep this interpretation in mind at an intuitive level.

[5]See [Hi] for the basic concepts of differential topology.

and that the tangent vectors of X at the boundary points of X are transverse to the boundary ∂M of M (i. e. are not contained in the tangent space to ∂M). For us, an *extremum* of $T(X)$ is a point $T(x)$ where the tangent vector $T'(x)$ to $T(X)$ is *horizontal* ($\in \mathbf{C} = \mathbf{C} \times \{0\} \subset \mathbf{C} \times \mathbf{R}$), and where, if $p_V : \mathbf{C} \times \mathbf{R} \longrightarrow: \mathbf{R}$ denotes the vertical projection, $(p_V \circ T)''(x) \neq 0$. We will say that an embedding as above is *almost admissible* if all of its horizontal tangent vectors are *real* ($\in \mathbf{R} \subset \mathbf{C}$) and correspond to extrema. Such an embedding is said to be *admissible* if furthermore, the extrema occur at distinct heights.

Definition 2.5 A *tangle* is a triple $(T; b, t)$ where

- T is an admissible embedding of a one-manifold X into $\mathbf{C} \times [0,1]$ such that:

$$T(\partial X) \subset \mathbf{R} \times \{0,1\}$$

  considered up to almost admissible isotopy[6] and rescaling[7] $[0,1]$. In particular, T provides a partition of the boundary ∂X of X into two subsets, namely the *bottom* $b(X) = T^{-1}(\mathbf{R} \times \{0\})$ and the *top* $t(X) = T^{-1}(\mathbf{R} \times \{1\})$ of X, each of which inherits the standard order of $\mathbf{R}$; the triple $(X, b(X), t(X))$ is called the *support* of T.

- b is an object whose length is the cardinality of $b(X)$. Similarly, $t \in \mathcal{O}_{\sharp t(X)}$.

The triple $(T; b, t)$ will also be denoted by T_b^t or simply by T.

Note that such a tangle is completely determined by b, t and the orthogonal projection of one of its representatives on the blackboard plane together with the under and over-crossing datum as soon as the chosen representative projection is regular. Again, $(T; b, t)$ will often be denoted by $(D; b, t)$ where D is a regular projection of T. Again, $\mathbf{1}$ will be used to denote p vertical parallel intervals.

Because of the condition on horizontal tangencies, the normal bundle of each component $T(X_0)$ of $T(X)$ admits a natural trivialization given by the horizontal imaginary direction, and therefore one or two (up to boundary-fixing homotopy) other ones for which the vectors are always in the blackboard plane (one if X_0 is a circle, two otherwise). In other words, $T(X = X \times \{0\})$ admits a *parallel* that is another embedding $T(X \times \{1\})$ satisfying the conditions of the definition such that $T_{|X \times \{0,1\}}$ extends to an embedding of $X \times [0,1]$ (satisfying the conditions for any $X \times \{u\}$) such that, for any $x \in X$, $T(\{x\} \times [0,1])$ is contained in a plane parallel to the blackboard plane. The isotopy class of $T(X \times \{0,1\})$ is well-defined provided that we allow exchanges of $X_0 \times \{0\}$ and $X_0 \times \{1\}$ for each component X_0 with non-empty boundary. In pictures, the two copies of $T(X_0)$ are parallel.

In particular, we have a well-defined operation of *duplication of a set of connected components* $T(X_0)$ *of* $(T(X), b, t)$ which consists in replacing:

1. X by $X(2 \times X_0) = X \cup X_0 \times \{1\}$,
2. $T = T_{|X \times \{0\}}$ by its above extension on $X \times \{0\} \cup X_0 \times \{1\}$
3. every letter $\cdot$ corresponding to a boundary point of X_0 by the word $(\cdot\cdot)$.

The obtained *duplicated tangle* will be denoted by $T(2 \times X_0)$.

A tangle whose bottom and top objects are n.a. words is called a *q-tangle*. When $T_1 = (T_1; b_1, t_1)$ and $T_2 = (T_2; b_2, t_2)$ are q-tangles, we can build their *tensor product*:

$$T_1 \otimes T_2 = (T_1 \otimes T_2; b_1 b_2, t_1 t_2).$$

When the top of a tangle $T_1 = (T_1(X_1); b, c)$ coincides with the bottom of a tangle $T_2 = (T_2(X_2); c, t)$, then we can form the *product* tangle $T_1 T_2 = (T_1 T_2; b, t)$ by stacking T_2 above T_1.

[6] An almost admissible isotopy between two admissible embeddings T_0 and T_1 is a C^∞ map $T : [0,1] \times X \longrightarrow \mathbf{C} \times [0,1]$ such that, for any $u \in [0,1]$, $T_u = T_{|\{u\} \times X}$ is an almost admissible embedding such that $T_u(\partial X) \subset \mathbf{R} \times \{0,1\}$.

[7] Rescaling $[0,1]$ means composing T by the product of the identity of $\mathbf{C}$ by an increasing diffeomorphism which maps $[0,1]$ onto another compact interval of $\mathbf{R}$.

Then, X_1X_2 is defined as the support of T_1T_2, and we have a natural *product* obtained by "stacking above" from $\mathcal{A}(X_1) \times \mathcal{A}(X_2)$ to $\mathcal{A}(X_1X_2)$. In other words, X_1X_2 is defined from the disjoint union $X_1 \otimes X_2$ of X_1 and X_2 by identifying $b(X_2)$ and $t(X_1)$, and the product is the composition of the natural maps:

$$\mathcal{A}(X_1) \times \mathcal{A}(X_2) \longrightarrow \mathcal{A}(X_1 \otimes X_2) \longrightarrow \mathcal{A}(X_1X_2)$$

A component of a tangle is said to be *regular* if it has no extremum. A tangle is *regular* if all its components are regular.

2.3 From the language of braids to the language of tangles.

A configuration path $\gamma : [0,1] \longrightarrow \mathbf{C}^p \setminus \boldsymbol{\Delta}$ such that $\gamma(0)$ and $\gamma(1)$ are in $\mathbf{R}^p$ gives rise to a well-determined regular tangle

$$T(\gamma) = (T(\gamma)(X); \sigma_b(\gamma(0)), \sigma_t(\gamma(1)))$$

where the image of $T(\gamma)$ is the graph of γ, and, $\sigma_b(\gamma(0))$ and $\sigma_t(\gamma(1))$ coincide with $\gamma(0)$ and $\gamma(1)$, respectively, up to a permutation of the coordinates.

Conversely, any regular tangle T whose bottom and top objects are actual configurations, may be written as $T(\gamma)$ for some configuration path which is well-determined up to homotopy with fixed boundary (and permutation of the strands). Thus, for such a tangle $T(\gamma)$, we unambiguously set:

$$Z(T(\gamma)) = Z(\gamma).$$

Now, if $(T; b, t)$ is a regular tangle such that b is a 2-point collision and t is an usual configuration, then represent $T(X)$ as the image of a path γ such that $\gamma(0) = b(+\varepsilon)$, for some very small ε, and $\gamma(1)$ coincides with t up to a permutation of the coordinates, and set

$$Z(T; b, t) = Z(\mathcal{B}_b^{b(+\varepsilon)}\gamma)$$

where, of course, $b(+\varepsilon)$ is obtained form b by changing $[\ldots, (a_i; (\cdot\cdot)), \ldots]$ into $[\ldots, (a_i; \cdot), (a_i + \varepsilon; \cdot), \ldots]$. Proceed similarly when t is a 2-point collision.

Thus, Z is well-defined for regular tangles whose bottom and top configurations are either actual configurations or two-point collisions. Furthermore, Z is compatible with the product (and inverse) for these tangles.

Example 2.6

$$Z(\mathbf{1}_{((\cdot\cdot),\cdot)}^{(\cdot,(\cdot\cdot))}) = Z(\mathbf{1}_{[(0;(\cdot\cdot)),(1;\cdot)]}^{[(0;\cdot),(1;(\cdot\cdot))]}) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \exp(\frac{\mathrm{Log}(\varepsilon)}{2i\pi}\Gamma_{12}) Z(\mathbf{1}_{(0,\varepsilon,1)}^{(0,1-\varepsilon,1)}) \exp(-\frac{\mathrm{Log}(\varepsilon)}{2i\pi}\Gamma_{23})$$

(See Remark 1.28.)

Side-exercise 2.7 Recall that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Compute

$$Z(\mathbf{1}_{((\cdot\cdot),\cdot)}^{(\cdot,(\cdot\cdot))}) = 1_{|||} + \frac{1}{24}(\Gamma_{23}\Gamma_{12} - \Gamma_{12}\Gamma_{23}) + O(3)$$

where $O(3)$ makes up for a sum of terms of degree ≥ 3.

Examples 2.8

$$Z(\mathbf{1}_{(0,1)}^{(\cdot\cdot)}) = 1_{||}$$

$$Z(\mathbf{1}_{(\cdot\cdot)}^{(0,1)}) = 1_{||}$$

$$Z(\times_{(\cdot\cdot)}^{(\cdot\cdot)}) = exp(\frac{1}{2}\Gamma_{12})$$

$$Z(\times_{(\cdot\cdot)}^{(\cdot\cdot)}) = exp(-\frac{1}{2}\Gamma_{12})$$

As another example, we prove a little lemma which will be useful later.

Lemma 2.9 *Let $\gamma : [0,1] \longrightarrow \mathbf{C}^{p-1} \setminus \mathbf{\Delta}$ be a configuration path. Let X_0 denote a strand of γ. Then*

$$Z\left(T(\gamma)(2 \times X_0)\right) = \pi(2 \times X_0)^* Z(T(\gamma))$$

PROOF: Assume that the duplicated strand is the first one, whose copies are labeled by $\sharp 1$ and $\sharp 2$. Then by definition,

$$Z\left(T(\gamma)(2 \times X_0)\right) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \exp(\frac{\mathrm{Log}(\varepsilon)}{2i\pi}\Gamma_{12}) Z\left((\gamma)(2 \times X_0)(+\varepsilon)\right) \exp(-\frac{\mathrm{Log}(\varepsilon)}{2i\pi}\Gamma_{12})$$

Now, using Lemma 1.29 together with the fact that $\lim_{\varepsilon \to 0} \varepsilon(\mathrm{Log}\varepsilon)^n = 0$ allows us to replace $Z\left((\gamma)(2 \times X_0)(+\varepsilon)\right)$ by $\pi(2 \times X_0)^* Z(T(\gamma))$ in the above expression. In order to conclude the proof, observe that $(4T)$ implies that Γ_{12} commutes with the image $\pi(2 \times X_0)^*$. (It is also a consequence of Lemma 2.16 which will be proved in Subsection 2.6). ⋄

2.4 The theorem

Definition 2.10 An element of $\mathcal{A}(1)$ is said to be *symmetric* if it is unchanged by the involution of $\mathcal{A}(1)$ induced by the flip of the interval. This involution reverses the order of the vertices (and reverses their orientations (in even number!)). It is unknown whether there exist elements of $\mathcal{A}(1)$ that are not symmetric under this symmetry.

We can now state the main theorem which gives the general definition of the Kontsevich integral Z for tangles.

Theorem 2.11 *There is a unique extension of Z to tangles which satisfies the following properties:*

Isotopy invariance: *For any tangle $T = (T(X); b, t)$, $Z(T) \in \mathcal{A}(X)$, $Z_0(T) = 1$, and $Z(T)$ only depends on b, t, and the usual*[8] *isotopy class of $T(2 \times X)$.*

Functoriality: *For any two tangles $T_1 = (T_1; b, c)$, $T_2 = (T_2; c, t)$ whose product is defined*

$$Z(T_1 T_2) = Z(T_1) Z(T_2)$$

Monoidality: *For any two q-tangles, $T_1 = (T_1; b_1, t_1)$ and $T_2 = (T_2; b_2, t_2)$*

$$Z(T_1 \otimes T_2) = Z(T_1) \otimes Z(T_2)$$

Duplication: *For any component $T(X_0)$ of a tangle $T = (T(X); b, t)$, such that either $T(X_0)$ is boundary-less or $T(X_0)$ runs from bottom to top,*

$$Z(T(2 \times X_0)) = \pi(2 \times X_0)^* (Z(T))$$

Extremum: *$Z(\bigcup^{(\cdot\cdot)})$ is a symmetric element ν of $\mathcal{A}(1)$ and:*

$$Z(\overset{(\cdot\cdot)}{\bigcup}) = Z(\underset{(\cdot\cdot)}{\bigcap}) = \nu$$

In particular, Z provides an invariant of *parallelized links*, that are links each component of which is equipped with a parallel (up to isotopy). Indeed, links can be represented by boundary-less tangles (with top and bottom objects of length 0). The theorem will be proved in Section 4. Its proof will only use the results of Subsection 2.6 which is completely independent of the rest of the notes. In the other subsections of this section, we assume the theorem, and, of course, we will not use their results in Section 4.

[8] The admissibility condition is dropped: here, a usual isotopy between two admissible embeddings T_0 and T_1 is a C^∞ map $T : [0,1] \times X \longrightarrow \mathbf{C} \times [0,1]$ such that, for any $u \in [0,1]$, $T_u = T_{|\{u\} \times X}$ is an embedding such that $T_u(\partial X) \subset \mathbf{R} \times \{0,1\}$.

2.5 First computations of Z on tangles

Example 2.12 Let us compute the *Drinfeld associator*

$$\Phi_{KZ} = Z(\mathbf{1}; ((\cdot\cdot)\cdot), (\cdot(\cdot\cdot)))$$

By Example 2.8 and by the Duplication property (D),

$$Z(\mathbf{1}^{(\cdot(\cdot\cdot))}_{(\cdot,(\cdot\cdot))}) = \pi(2 \times 2)^*(1_{||}) = 1_{|||}$$

Similarly,

$$Z(\mathbf{1}^{((\cdot\cdot),\cdot)}_{((\cdot\cdot)\cdot)}) = 1_{|||}$$

Thus, by the Functoriality property,

$$\Phi_{KZ} = Z(\mathbf{1}^{(\cdot,(\cdot\cdot))}_{((\cdot\cdot),\cdot)})$$

whose expression has been defined in Example 2.6 and whose low degree terms are given in the exercise immediately after. A complete expression of Φ_{KZ} in terms of generalized η-functions can be found in [LM2].

Example 2.13 Let w be a n.a. word of length p. Let T^t_b be a q-tangle. Let $\mathbf{1}^w_w(\frac{T^t_b}{i})$ be the tangle obtained from $\mathbf{1}^w_w$ by replacing the i^{th} strand of $\mathbf{1}^w_w$ by T (replacing in particular the i^{th} letter of the bottom w by b and the i^{th} letter of the top w by t and removing unneeded parentheses if one of b and t is empty). We will similarly denote the bottom word of $\mathbf{1}^w_w(\frac{T^t_b}{i})$ by $w(\frac{b}{i})$. Then, by the Monoidality property, and by induction on p, we have that

$$Z(\mathbf{1}^w_w(\frac{T^t_b}{i})) = 1_{\mathcal{A}(i-1)} \otimes Z(T) \otimes 1_{\mathcal{A}(p-i)}$$

Thus, for example,

$$Z(|_1|_2 \cdots |_{i-1} \times |_{i+2} \cdots |_{p+1}; w(2 \times i), w(2 \times i)) = exp(\frac{1}{2}\Gamma_{\{i,i+1\}})$$

and

$$Z(|_1|_2 \cdots |_{i-1} \times |_{i+2} \cdots |_{p+1}; w(2 \times i), w(2 \times i)) = exp(-\frac{1}{2}\Gamma_{\{i,i+1\}})$$

for any n.a. word w of length p. (Note that any n.a. word which contains the two-letter word that groups the i^{th} letter with the $(i+1)^{th}$ one, as a subword, can be written as $w(2 \times i)$.)

As another example, let w_1, w_2 and w_3 be three n.a. words, then by the duplication property

$$Z(\mathbf{1}; ((w_1 w_2) w_3), (w_1(w_2 w_3))) = \pi(\ell(w_1) \times 1, \ell(w_2) \times 2, \ell(w_3) \times 3)^*(\Phi_{KZ})$$

where we use an obvious generalisation of the notation of Subsection 1.8. Let w be our n.a. word of length p. Then

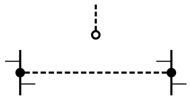
$$Z(\mathbf{1}; w(\frac{((w_1 w_2) w_3)}{i}), w(\frac{(w_1(w_2 w_3))}{i})) = 1_{\mathcal{A}(i-1)} \otimes \pi(\ell(w_1) \times 1, \ell(w_2) \times 2, \ell(w_3) \times 3)^*(\Phi_{KZ}) \otimes 1_{\mathcal{A}(p-i)}$$

Before going any further in example computations, we will need to know a little more about diagrams, namely a commutation principle.

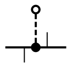
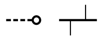
2.6 Commutation principle in $\mathcal{A}$

Let us first introduce a graphical notation. An edge with a free vertex drawn as a circle, and a set of little hooks like ($\perp$) elsewhere on a diagram, denotes the sum over the set of hooks of all (classes of) diagrams obtained by attaching the free vertex to the attaching point of a hook, where the neighborhood of the new vertex is the same as the hook neighborhood.

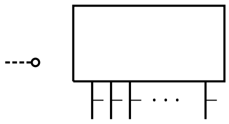
With this notation, (4T) reads

 $= 0,$

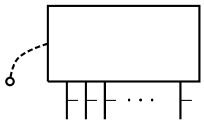
while (AS) implies:

 $= 0$ and  $= 0.$

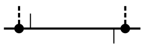
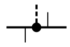
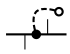
This allows us to prove the following commutation identities on classes of diagrams.
First commutation identity:

 $= 0 \qquad (2.14)$

Second commutation identity:

 $= 0 \qquad (2.15)$

In each of these identities, the sum over the hooks relates diagrams which are arbitrary but identical inside the box and outside the represented parts, such all the outputs of the box (connections between inside and outside) are represented and such that there are no boundary points of the support X of the diagrams inside the box.

We prove the identities: by AS, we may add two hooks on each plain edge in the inside diagram, one near each extremity of the plain edge, which show the left-hand side to someone located on the corresponding vertex like  without changing the sum. Now, every vertex inside the box is equipped with two hooks which turn left like . For each chord inside the box, the contribution of its two vertices vanishes by the above hook version of 4T. This proves the first identity, since the only initial hooks which are not near a vertex can be grouped in pairs which vanish by AS ( $= 0$). In the second case, we further use that  vanishes.

The first commutation identity applied to the case when the box contains nothing but an element of $\mathcal{A}(1)$ shows how the choice of a connected component X_0 of a one-manifold X and an orientation of X_0 provides a natural $\mathcal{A}(1)$-*module structure* on $\mathcal{A}(X)$, that is given by the bilinear continuous map:

$$\begin{array}{ccc} \mathcal{A}(1) \times \mathcal{A}(X) & \longrightarrow & \mathcal{A}(X) \\ (\alpha, d) & \mapsto & \alpha d \end{array}$$

where, if α and d are (classes of) diagrams, then αd is obtained by inserting α somewhere in X_0 so that the orientation of X_0 matches the orientation from bottom to top of the vertical interval. The independence on the chosen insertion locus is a consequence of the first commutation identity (and of AS).

Of course, if α is a symmetric element of $\mathcal{A}(1)$, there is no need to specify the orientation of X_0 in order to define αd.

With this module structure in mind, the extremum property of the theorem is better written as:

$$Z\Big(\overset{(\cdot\cdot)}{\bigcup}\Big) = \nu 1_{\cup}$$

$$Z\Big(\underset{(\cdot\cdot)}{\bigcap}\Big) = \nu 1_{\cap}$$

Similarly, note that

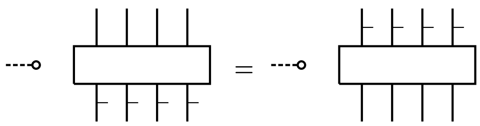
$$exp(\frac{1}{2}\Gamma_{12||})1_{\cap} = exp(-\frac{1}{2}\Gamma_{11})1_{\cap}$$

where $\Gamma_{11} \in \mathcal{A}(1)$ denotes the one-chord diagram on the vertical interval such that the local orientations at its two vertices give rise to the same orientation of the interval.

Note that the first commutation identity also implies that the algebra $\mathcal{A}(1)$ is *commutative*.

Let X be a compact one-manifold whose boundary is made of two points, and let $\overline{X}$ be the quotient of X obtained by identifying these two boundary points. The second commutation identity applied to the case when the box contains nothing but an element of $\mathcal{A}(X)$ deprived from a neighborhood of ∂X and half a dashed edge is exactly the relation of "closing the circle" that should be put on $\mathcal{A}(X)$ in order to build $\mathcal{A}(\overline{X})$ from $\mathcal{A}(X)$. Thus, since it is true in $\mathcal{A}(X)$, it shows that the natural map from $\mathcal{A}(X)$ to $\mathcal{A}(\overline{X})$ is an isomorphism. In particular, $\mathcal{A}(1)$ is isomorphic to $\mathcal{A}(S^1)$.

The first commutation identity applied to the case when the box contains nothing but an element of $\mathcal{A}(r)$ can be redrawn as below, and stated as in the following lemma:



Lemma 2.16 *An r-duplicated vertex always commutes with an element of $\mathcal{A}(r)$.*

2.7 More examples and corollaries

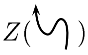
Example 2.17 By the Monoidality property (M),

$$Z(\cup \uparrow^{((\cdot\cdot)\cdot)}) = Z(\cup^{(\cdot\cdot)}) \otimes Z(\uparrow) = \nu 1_{\cup} \otimes 1_{\uparrow}$$

Similarly,

$$Z(\uparrow \cap_{(\cdot(\cdot\cdot))}) = 1_{\uparrow} \otimes \nu 1_{\cap}$$

Now, by functoriality,



$$Z(\quad) = Z(\cup \uparrow^{((\cdot\cdot)\cdot)})\Phi_{KZ}Z(\uparrow \cap_{(\cdot(\cdot\cdot))}) = \nu^2 1_{\cup\uparrow}\Phi_{KZ}1_{\uparrow\cap}$$

By isotopy invariance, this expression must be equal to $Z(\uparrow) = 1_{\uparrow}$. Thus, by the condition that the degree 0 part of ν is 1, we get the *defining formula* of ν:

$$\nu = (1_{\cup\uparrow}\Phi_{KZ}1_{\uparrow\cap})^{-\frac{1}{2}} \in \mathcal{A}(\uparrow)$$

Lemma 2.18 *Any q-tangle may be written as a product of elementary tangles of the form $\mathbf{1}^w_w(\frac{T}{i})$ (as in Example 2.13) where $T = \times^{(\cdot\cdot)}_{(\cdot\cdot)}, \times^{(\cdot\cdot)}_{(\cdot\cdot)}, \cap_{(\cdot\cdot)}, \cup^{(\cdot\cdot)}, (\mathbf{1};((w_1w_2)w_3),(w_1(w_2w_3)))$ or $(\mathbf{1};(w_1(w_2w_3)),((w_1w_2)w_3))$.*

The proof is easily reduced to the proof of the lemma for a q-tangle $(\mathbf{1};a,b)$ which is left as an exercise.

◇

By Examples 2.13 and 2.17, we know how to express Z as a function of the associator Φ_{KZ} (of Example 2.12) for all the elementary q-tangles that are mentioned in the above lemma. More precisely, these examples and the lemma show that the statement of the theorem together with the values of $Z(\times^{(\cdot\cdot)}_{(\cdot\cdot)})$ and $Z(\mathbf{1};((\cdot\cdot)\cdot),(\cdot(\cdot\cdot)))$ completely defines Z on q-tangles. (Of course, the consistency of the definition will be a consequence of the theorem.)

Example 2.19 The *Kontsevich integral of the trivial knot* is

$$Z(\bigcirc) = Z(\cup)Z(\cap) = \nu^2 1_{\circ}$$

It has been recently computed by D. Bar-Natan, T. Le and D. Thurston (see D. Thurston's lectures).

Example 2.20 Let us now compute the Kontsevich integral of twisted extrema:

$$Z(\;_{(\cdot\cdot)}) = Z(\;^{(\cdot\cdot)}_{(\cdot\cdot)})Z(\cap_{(\cdot\cdot)}) = exp(-\frac{1}{2}\Gamma_{12})\nu 1_{\cap}$$

Thus, by commutation,

$$Z(\;_{(\cdot\cdot)}) = exp(\frac{1}{2}\Gamma_{11})\nu 1_{\cap}$$

Similarly,

$$Z(\;_{(\cdot\cdot)}) = exp(-\frac{1}{2}\Gamma_{11})\nu 1_{\cap}$$

$$Z(\;^{(\cdot\cdot)}) = exp(\frac{1}{2}\Gamma_{11})\nu 1_{\cup}$$

$$Z(\;^{(\cdot\cdot)}) = exp(-\frac{1}{2}\Gamma_{11})\nu 1_{\cup}$$

Thus, the Kontsevich integral varies during these twists. However, one can observe that if we quotient $\mathcal{A}$ out by the relation $1T$ which identifies the diagrams which have an isolated chord, that is an inserted Γ_{11}, to zero, then the Kontsevich integral becomes a usual isotopy invariant of unparallelized tangles (since it is easy to observe that the above twists allow us to go from any parallelized tangle to any other isotopic tangle with another parallelization). The quotient of $\mathcal{A}_n(X)$ by $(1T)$ will be denoted by $\overline{\mathcal{A}}_n(X)$, and the image of a diagram class d in this quotient will be denoted by $\overline{d}$.

2.8 Universality of Z among finite type invariants

Given an invariant I of oriented knots, valued in an abelian group, we extend it to an invariant of knots with transverse double points[9] by making use of the following local formula:

$$I(\;) = I(\;) - I(\;)$$

Such an invariant is said to be of *finite type* or of *finite degree* less or equal to n, if it maps all the knots with (n+1) double points to zero. Let $\overline{\mathcal{A}}_{\leq n}(S^1) = \oplus_{i=0}^{n}\overline{\mathcal{A}}_i(S^1)$. For a knot K, let $\overline{Z}_{\leq n}(K)$ denote the natural projection of $\overline{Z}(K)$ onto $\overline{\mathcal{A}}_{\leq n}(S^1)$. In this section, we will prove the universality of Z which can be stated as follows:

Theorem 2.21 (Kontsevich, 1992) *If I is a complex-valued knot invariant of degree $\leq n$, then there exists a linear map*

$$\phi : \overline{\mathcal{A}}_{\leq n}(S^1) \longrightarrow \mathbf{C}$$

such that $I = \phi \circ \overline{Z}_{\leq n}$.

To every oriented singular knot $K : S^1 \longrightarrow \mathbf{R}^3$ whose only singularities are n transverse double points, we associate its *diagram*

$$D(K) \in \mathcal{D}_n(S^1)$$

which is the n-chord diagram obtained by relating the two preimages of every double point by a chord and giving the vertices the orientation induced by the local orientations of the crossing strands .

The following lemma implies in particular that $\overline{Z}_{\leq n}$ is a finite type invariant of degree less or equal than n, so that the above theorem exactly describes the degree less or equal than n complex-valued knot invariants.

Lemma 2.22 *If K is a singular knot with n double points, then $\overline{Z}_{\leq n}(K) = \overline{D(K)}$.*

[9]When we are dealing with other non-necessarily oriented 1-manifolds, we demand that, near every double point, the two strands of the 1-manifold are equipped with a local orientation.

PROOF: We can represent K as a product of singular tangles $K = T_0 S_1 T_1 S_2 T_2 \dots S_n T_n$ so that the T_i are usual tangles and every S_i is of the form $S_i = \mathbf{1}_w^w(\ )$ for a n.a. word $w = w_i$ (see Example 2.13 for the notation) with exactly one double point whose two strands point upward. Let $S_i^+ = \mathbf{1}_w^w(\ )$ and $S_i^- = \mathbf{1}_w^w(\ )$. Then, by functoriality and by definition,

$$\overline{Z}(K) = \overline{Z}(T_0)\left(\overline{Z}(S_1^+) - \overline{Z}(S_1^-)\right)\overline{Z}(T_1)\dots\overline{Z}(T_{n-1})\left(\overline{Z}(S_n^+) - \overline{Z}(S_n^-)\right)\overline{Z}(T_n)$$

where, since the T_i are usual tangles $Z_0(T_i) = 1$, and according to Example 2.13,

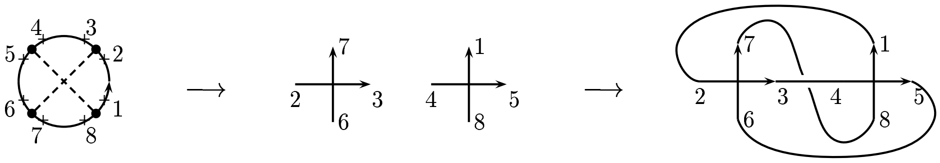
$$Z_{\leq 1}(S_i^+) - Z_{\leq 1}(S_i^-) = \Gamma_{12}$$

is exactly the one-chord diagram whose chord connects the two preimages of the double point in the prescripted way. The conclusion is now easy. ◇

Lemma 2.23 *Every chord diagram on S^1 is the diagram of a singular knot. Furthermore, two singular knots which have the same diagram are related by a finite number of crossing changes (where a* crossing change *is the modification which transforms into).*

PROOF: Though the first assertion is very easy, we prove it because its proof is the beginning of the proof of the second assertion. Let d be an n-chord diagram on S^1. Put 4n *cutting points,* numbered from 1 to 4n along S^1, on the support S^1 of d, one near each extremity of the 2n intervals separated by the vertices, so that $2i-1$ and $2i$ are on the same interval. Next embed the neighborhoods of the n double points, bounded by the 4n cutting points, into n fixed disjoint 3-balls $\mathbf{R}^3$ so that the cutting points lie on the boundary of the closure C of the complement of our n fixed balls.

Then, in order to construct our first representative K^0 of d, it is enough to notice that we have enough room to embed the *remaining 2n intervals* of S^1 (the $[2i-1, 2i]$) into C. The projections of these three steps are represented in the following example:



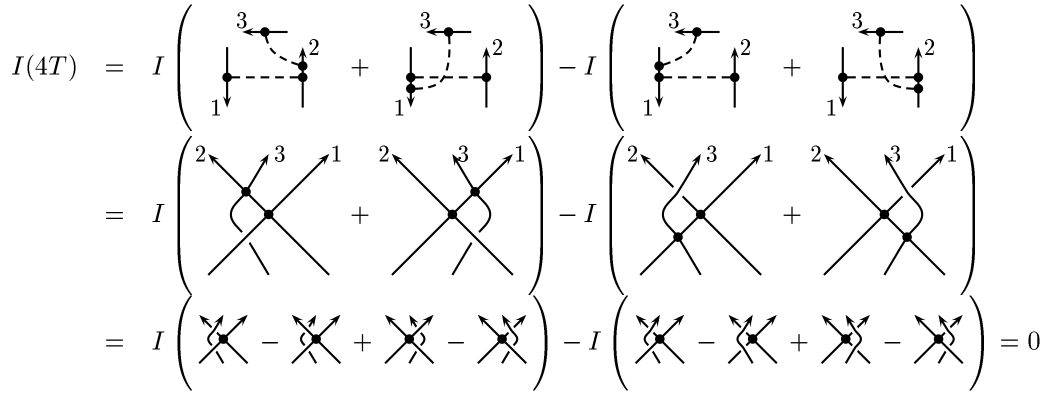
Let K be another representative of d. After an isotopy, we may assume that K intersects our n balls like K^0 does. Then, since $\pi_1(C)$ is trivial, there is a boundary-fixing homotopy in C that maps the remaining 2n intervals for K to the remaining 2n intervals for K^0. Such a homotopy may be approximated by a finite sequence of (isotopies and) crossing changes, and we are done.

Minor changes in the proof show that the result remains true for knots without double points. ◇

Lemma 2.24 *Let I be a degree less or equal than n complex-valued invariant. Let K be a singular knot with n double points. Then $I(K)$ only depends on $D(K)$ mod (1T) and (4T). In other words, I induces a map*

$$\begin{array}{rlll} \overline{I}: & \overline{\mathcal{A}}_n(S^1) & \longrightarrow & \mathbf{C} \\ & \overline{D(K)} & \mapsto & \overline{I}(D(K)) = I(K) \end{array}$$

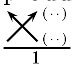
PROOF: That $I(K)$ only depends on $D(K)$ is an immediate corollary of the preceeding lemma. Set $I(D(K)) = I(K)$. Thus, it remains to show that I maps the relations (4T) and (1T) to zero. To see that (1T) is mapped to zero, it suffices to represent a diagram with an isolated chord by a diagram with a loop like so that the two desingularizations are isotopic. For (4T), locally represent the involved diagrams by embedding the three arcs of S^1 (which inherit their local orientations from the cyclic orders of the first diagram) so that the two arcs of the crossing of the fixed chord sit in the blackboard plane and the other one points to us. Then



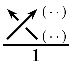
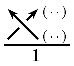
$\diamond$

PROOF OF THEOREM 2.21: The proof of the theorem is now easy by induction on n. An invariant of degree 0 is nothing but a constant map on the set of maps, according to Lemma 2.23. Now, for I as in the statement of the theorem, the above lemma and Lemma 2.22 show the existence of $\overline{I} : \overline{\mathcal{A}}_n(S^1) \longrightarrow \mathbf{C}$ such that $I = \overline{I} \circ \overline{Z}_n$ on singular knots with n double points. In particular, if we extend $\overline{I}$ to $\overline{I} : \overline{\mathcal{A}}_{\leq n}(S^1) \longrightarrow \mathbf{C}$ so that it maps all diagrams with less than n chords to zero, then $I - \overline{I} \circ \overline{Z}_{\leq n}$ is an invariant of degree at most (n-1) and we easily conclude. $\diamond$

Side-remark 2.25 Let $(X, b(X), t(X))$ be a tangle support. Let $b \in \mathcal{O}_{\sharp b(X)}$ and let $t \in \mathcal{O}_{\sharp t(X)}$. In all this subsection, we can replace S^1 by X, and, knot by tangle (without parallelisation, in this remark) from b to t with support $X = (X, b(X), t(X))$, simultaneously. Everything works exactly in the same way in this setting.

We could also have given more algebraic statements. Namely, let $\mathcal{V}(X)$ denote the complex vector space freely generated by the tangles from b to t with support X. Let $\tilde{\partial}$ denote the desingularisation map which maps a singular tangle S, which can be represented as a product $S = T_0 S_1 T_1 S_2 T_2 \dots S_n T_n$ where the T_i are usual tangles and the S_i are of the form $S_i = \mathbf{1}_w^w($$)$, to the sum

$$\tilde{\partial}(S) = \sum_{\varepsilon : \{1,2,\dots,n\} \longrightarrow \{+,-\}} (-1)^{\sharp \varepsilon^{-1}(-)} T_0 S_1^{\varepsilon(1)} T_1 S_2^{\varepsilon(2)} T_2 \dots S_n^{\varepsilon(n)} T_n \ \in \mathcal{V}(X)$$

where $S_i^+ = \mathbf{1}_w^w($$)$ and $S_i^- = \mathbf{1}_w^w($$)$. Let $\mathcal{V}_n(X)$ denote the subspace of $\mathcal{V}(X)$ generated by the images of the singular tangles from b to t with support X with n double points under the desingularisation map. Observe that:

$$\mathcal{V}_{n+1}(X) \subset \mathcal{V}_n(X) \subset \dots \subset \mathcal{V}_0(X) = \mathcal{V}(X)$$

Thus, $(\mathcal{V}_n(X))_{n \in \mathbf{N}}$ is a filtration of $\mathcal{V}(X)$. It is called the *Vassiliev filtration* (of the space of knots when $X = S^1$). What can be proved following the lines of that section is that $\overline{Z}_n$ induces an isomorphism from $\frac{\mathcal{V}_n(X)}{\mathcal{V}_{n+1}(X)}$ to $\overline{\mathcal{A}}_n(X)$ (which furthermore sends the desingularisation of a singular tangle to the diagram of the singular tangle). In other words, the graded space associated to the Vassiliev filtration is isomorphic to $\overline{\mathcal{A}}_n(X)$, and is thus finite-dimensional at each degree.

With this notation, a complex-valued invariant of degree less or equal than n is an element of the dual of $\frac{\mathcal{V}(X)}{\mathcal{V}_{n+1}(X)}$. Let $\mathcal{I}_n(X) = \mathrm{Hom}(\frac{\mathcal{V}(X)}{\mathcal{V}_{n+1}(X)}, \mathbf{C}) = \left(\frac{\mathcal{V}(X)}{\mathcal{V}_{n+1}(X)}\right)^*$ be the space of these invariants. Then $\frac{\mathcal{I}_n(X)}{\mathcal{I}_{n-1}(X)}$ is isomorphic to $\left(\frac{\mathcal{V}_n(X)}{\mathcal{V}_{n+1}(X)}\right)^* = \left(\overline{\mathcal{A}}_n(X)\right)^*$.

3 The proof of the theorem

3.1 Sketch of the proof

Let $\nu \in \mathcal{A}(\uparrow)$ be defined in Example 2.17. We do not know yet that it is symmetric. Define an *e-tangle* as a tangle where the number of extrema and their relative heights are fixed that is as an admissible embedding up to admissible isotopies[10].

There are six steps in the proof.

1. Extend Z to all the oriented e-tangles whose bottom and top configurations are actual configurations so that:

   - $Z(T(X))$ is defined as follows for a tangle $T(X)$ which only has one extremum at height c. If this extremum is a maximum, then let $\gamma_1 : [0,c] \longrightarrow \mathbf{C}^{p+2}$ denote the (natural up to a permutation of the numbering of the strands) path whose graph is $T \cap (\mathbf{C} \times [0,c])$. It is a path from an actual configuration to a 2-point collision. Define:
   $$Z(T) = Z(\gamma_1)\nu 1_X Z(T \cap (\mathbf{C} \times [c,1]) \setminus \{\text{maximum}\})$$
   where ν acts on the maximum of X according to the given orientation of X.
   If this extremum is a minimum, then define similarly the path $\gamma_2 : [c,1] \longrightarrow \mathbf{C}^{p+2}$ as the path going from a two-point collision to an actual configuration whose graph is $T\cap(\mathbf{C}\times[c,1])$ and define:
   $$Z(T) = Z(T \cap (\mathbf{C} \times [0,c]) \setminus \{\text{minimum}\})\nu 1_X Z(\gamma_2)$$
   where ν acts on the minimum of X according to the given orientation of X.
   - Z satisfies the functoriality property for these tangles.
   - Z is invariant under admissible isotopies and rescaling. Since, by the former properties of Z, the two first bullets define Z unambiguously for a given embedding of a tangle up to rescaling, the only thing which remains to be proved in this step is this invariance under admissible isotopy which will occupy Subsection 3.2.

2. Define Z for all the tangles $(\mathbf{1}; a, b)$ where a and b are two objects of the same length $p \in \mathbf{N}$ so that:

   - It coincides with our Z when a and b are actual $\mathbf{R}^p$-configurations or $\mathbf{R}^p$-two-point collisions.
   - It satisfies the duplication property for these tangles.
   - It is functorial on these tangles: $Z(\mathbf{1}; a, b)Z(\mathbf{1}; b, c) = Z(\mathbf{1}; a, c)$.

   and prove that these conditions define Z unambiguously on these tangles. In other words, prove that these conditions define Z unambiguously on these tangles and are consistent, this will be the goal of Subsection 3.3.

3. Define Z for a general e-tangle $(T; b, t)$ where b is an object of length p and t is an object of length q by
   $$Z(T; b, t) = Z(\mathbf{1}; b, [p])Z(T; [p], [q])Z(\mathbf{1}; [q], t)$$
   Thus, after the first two steps, the so-defined Z is a functorial invariant of e-tangles which takes the value ν at extrema and coincides with our Z when b and t are actual $\mathbf{R}^p$-configurations or $\mathbf{R}^p$-two-point collisions.

4. Prove that the so-defined Z is invariant under almost admissible isotopies and satisfies the monoidality property. This will be done in Subsection 3.4.

[10] An admissible isotopy between two admissible embeddings T_0 and T_1 is a C^∞ map $T : [0,1] \times X \longrightarrow \mathbf{C} \times [0,1]$ such that, for any $u \in [0,1]$, $T_u = T_{|\{u\}\times X}$ is an admissible embedding such that $T_u(\partial X) \subset \mathbf{R} \times \{0,1\}$. Note that the number of extrema and the relative heights of the extrema are fixed during such an admissible isotopy.

5. Prove that Z is invariant under any isotopy which does not change the parallelisation. Prove that ν is symmetric. This will be the goal of Subsection 3.5. Therefore the definition of Z does not depend on the chosen orientation[11].

6. Prove that the so-defined Z satisfies the duplication property. This will be done in Subsection 3.6.

The unicity comes from the following easy facts. Any tangle is isotopic to a product of elementary tangles of the form $(\mathbf{1}; a, b)$ or $(\mathbf{1}^w_w(\frac{T}{i}))$ (as in Example 2.13) where $T = \times^{(\cdot\cdot)}_{(\cdot\cdot)}, \times^{(\cdot\cdot)}_{(\cdot\cdot)}, \cap_{(\cdot\cdot)}$ or $\cup^{(\cdot\cdot)}$. For the tangles of the latter form, the definition is imposed by the statement of the theorem as shown in Examples 2.13 and 2.17. For the tangles $Z(\mathbf{1}; a, b)$, according to the second step, the definition is also imposed by the statement of the theorem.

We will close this section by the computation of Z for iterated torus knots.

3.2 The admissible isotopy invariance of Z

Let $T : [0,1] \times X \longrightarrow \mathbf{C} \times [0,1]$ be an isotopy between two admissible tangle embeddings that fixes the boundary[12]. Let $p_V : \mathbf{C} \times [0,1] \longrightarrow [0,1]$ denote the vertical projection. T is said to be *horizontal* if, for any $x \in X$, $p_V \circ T_{[0,1]\times\{x\}}$ is constant.

Note that the definition of $Z(T(X))$ for a given embedding only depends on the image of $T(X)$ and not on the parametrization. Furthermore, rescaling the vertical parameter does not change $Z(T(X))$. Thus, up to rescaling $[0,1]$, any admissible isotopy may be considered as a horizontal one.

Lemma 3.1 *Z is invariant under admissible isotopies.*

PROOF: For a C^∞ homotopy $\gamma : [0,1] \times [a,b] \longrightarrow \mathbf{C}^p \setminus \mathbf{\Delta}$ we let $\gamma_{(.,c)}$ be the path described by the homotopy at height c: $\gamma_{(.,c)}(h) = \gamma(h,c)$. With this notation, even if $\gamma_{(.,a)}$ and $\gamma_{(.,b)}$ are not constant, γ_0 is homotopic by a boundary-fixing homotopy to $\gamma_{(.,a)}\gamma_1\overline{\gamma_{(.,b)}}$, and hence

$$Z(\gamma_0) = Z(\gamma_{(.,a)})Z(\gamma_1)Z(\gamma_{(.,b)})^{-1}$$

Thus, in general the (boundary-fixing) isotopy invariance of the functorial Z is equivalent to the above equation for not boundary-fixing isotopies. And the good behaviour of the above equation under the vertical product allows us to restrict ourselves to the case of an admissible boundary-fixing horizontal isotopy T of a tangle with only one extremum, for which we are about to prove $Z(T_1) = Z(T_0)$.

Assume that this extremum is a maximum at height c.

Let $u \in [0,1]$. Let $\gamma^b_u : [a,c] \longrightarrow (\mathbf{C}^{p+2} \setminus \mathbf{\Delta}) \cup \mathbf{C}^{p+2}_2$ denote a configuration path to a two-point collision - occuring between the two last strands and corresponding to the maximum - whose graph is the image of $T_u(X) \cap (\mathbf{C} \times [a,c])$. Let $\gamma^t_u : [c,b] \longrightarrow \mathbf{C}^p \setminus \mathbf{\Delta}$ denote a configuration path whose graph is $T_u(X) \cap (\mathbf{C} \times [c,b]) \setminus \{\text{maximum}\}$.

Then, by definition

$$Z(T_u) = Z(\gamma^b_u)\nu 1_X Z(\gamma^t_u)$$

where as in the introduction to the proof,

$$Z(\gamma^t_1) = Z(\gamma^t_{(.,c)})^{-1} Z(\gamma^t_0)$$

Now, note that if the isotopy moves neither the points at the critical height c nor the tangent vector at the maximum, then the invariance comes from the properties of the Kontsevich integral for braids. Thus, we are allowed to modify our isotopy by this kind of isotopy, and, we suppose without loss that, for some $\varepsilon_0 > 0$, the restriction of γ^b_u to $[c - \varepsilon_0^2, c]$ has the following form:

$$\begin{array}{rccl} \gamma^b_u : & [c-\varepsilon_0^2, c] & \longrightarrow & \mathbf{C}^{p+2} \\ & c - h^2 & \mapsto & (z_1(u,c), z_2(u,c), \dots, z_p(u,c), z_{p+1}(u,c) - \frac{h}{2}, z_{p+1}(u,c) + \frac{h}{2}) \end{array}$$

[11] Orient S^1. Let K be an oriented knot, and let $\overleftarrow{K}$ denote the knot obtained from K by reversing the orientation. If $Z(K) \in \mathcal{A}(S^1)$ were not symmetric, then $Z(K)$ and $Z(\overleftarrow{K})$ would still be different elements of $\mathcal{A}(S^1)$.

[12] For any $x \in \partial X$, $T_{[0,1]\times\{x\}}$ is constant.

where $\{z_1(u,c), z_2(u,c), \dots, z_p(u,c), z_{p+1}(u,c)\}$ is the configuration occuring at the maximum. Let

$$\mathcal{C} : u \in [0,1] \mapsto (z_1(u,c), z_2(u,c), \dots, z_p(u,c), z_{p+1}(u,c))$$

be the path described by the homotopy at the maximum height. The above normalisation of the isotopy shows that the tangle $T(\gamma_1^b)$ is equal to the product of the tangles $T(\gamma_0^b)$ and $T(\mathcal{C})(2\times(p+1))$. Thus,

$$Z(\gamma_1^b) = Z(\gamma_0^b) Z\left(T(\mathcal{C})(2\times(p+1))\right)$$

where, by Lemma 2.9, $Z\left(T(\mathcal{C})(2\times(p+1))\right) = \pi(2\times(p+1))^*(Z(\mathcal{C}))$. Therefore, to conclude the proof it suffices to prove that

$$\pi(2\times(p+1))^*(Z(\mathcal{C}))1_X = i(Z(\gamma^t_{(.,c)}))$$

where $i : \mathcal{A}(p) \longrightarrow \mathcal{A}(X)$ is induced by the inclusion. Observe that $\pi(2\times(p+1))^*(Z(\mathcal{C}))1_X$ is nothing but $\pi^*(Z(\mathcal{C}))$ where $\pi : X \longrightarrow \coprod_{i=1}^{p+1} I_i$ identifies the two strands going up to the maximum to some bottom part of I_{p+1}. Now, since π is homotopic to the map which sends the whole component of the maximum to the bottom point of I_{p+1}, π^* amounts to forget the contribution of diagrams which have univalent vertices on the maximum component. (See the beginning of Subsection 1.8.) Thus, $\pi^*(Z(\mathcal{C})) = i(Z(\gamma^t_{(.,c)}))$ and we are done. Of course, the minimum case works exactly in the same way.
$\diamond$

3.3 From an actual real configuration to a limit one

We fix an integer p.

We first define Z for all the tangles $(\mathbf{1}; a, b)$ where a and b are two word sequences of length p. Let $W = (w_1, w_2, \dots, w_k)$ be a word sequence of length p. For $i < k$, $W(i)$ denotes the word sequence

$$W(i) = (w_1, w_2, \dots, w_i w_{i+1}, w_{i+2}, \dots, w_k).$$

In particular, $[k](i) = [(1;\cdot), (2;\cdot), \dots, (i;(\cdot\cdot)), \dots, (k-1;\cdot)]$, and there is a natural tangle from W to $W(i)$ obtained by duplicating the j^{th} strand of $(\mathbf{1}; [k], [k](i))$ $(\ell(w_j)-1)$ times for any $j \in \{1, 2, \dots, k\}$.

Set

$$Z(W \to W(i)) = \pi(\ell(w_1)\times 1, \ell(w_2)\times 2, \dots, \ell(w_k)\times k)^* Z(\mathbf{1}; [k], [k](i))$$

and

$$Z(W(i) \to W) = Z(W \to W(i))^{-1}$$

Define a *path P of word sequences* from a word sequence W^0 to another one W^r as a sequence

$$P = (W^0 \to W^1 \to W^2 \to \dots \to W^{r-1} \to W^r)$$

where all the W^j are word sequences, and, for any $j \in \{1, 2, \dots, r\}$, there exists i such that either $W^j(i) = W^{j-1}$ or $W^j = W^{j-1}(i)$. Then define:

$$Z(P) = Z(W^0 \to W^1) Z(W^1 \to W^2) \dots Z(W^{r-1} \to W^r)$$

Lemma 3.2 *Let $W = (w_1, w_2, \dots, w_k)$ be a word sequence of length p. Let i and j be two integers such that $1 \le i \le j-2 \le k-3$. Then $W(i)(j-1) = W(j)(i)$ and*

$$Z(W \to W(i) \to W(i)(j-1)) = Z(W \to W(j) \to W(j)(i))$$

PROOF: We may assume that $W = [k]$, since the general case can be deduced from this particular case by duplications. In this case, using Notation 1.23,

$$Z([k] \to [k](j)) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} Z(\mathbf{1}; [k], [k](j)(+\varepsilon)) \exp\left(-\frac{\mathrm{Log}(\varepsilon)}{2i\pi}\Gamma_{j,j+1}\right)$$

and

$$Z([k](j) \to [k](j)(i)) = \pi(2\times j)^* \left(\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} Z(\mathbf{1}; [k-1], [k-1](i)(+\varepsilon)) \exp\left(-\frac{\mathrm{Log}(\varepsilon)}{2i\pi}\Gamma_{i,i+1}\right)\right)$$

Now, the continuities of the product and of $\pi(2\times j)^*$ and the commutation lemma allow us to write:

$$Z(W \to W(j) \to W(j)(i)) =$$

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} Z(\mathbf{1}; [k], [k](j)(+\varepsilon))\pi(2\times j)^*(Z(\mathbf{1}; [k-1], [k-1](i)(+\varepsilon))) \exp\left(-\frac{\mathrm{Log}(\varepsilon)}{2i\pi}\Gamma_{j,j+1}\right) \exp\left(-\frac{\mathrm{Log}(\varepsilon)}{2i\pi}\Gamma_{i,i+1}\right)$$

where

$$\pi(2\times j)^*(Z(\mathbf{1}; [k-1], [k-1](i)(+\varepsilon))) = Z(\mathbf{1}; [k](j)(+\varepsilon), ([k](j)(+\varepsilon))\,(i)(+\varepsilon)) + O(\varepsilon|\mathrm{Log}^{*-1}(\varepsilon)|).$$

(See the proofs of Lemmas 1.25 and 1.29.) Thus, and because of the horizontal isotopy invariance and of the multiplicativity of Z for braids, we find that

$$Z(W \to W(j) \to W(j)(i)) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} Z(\mathbf{1}; [k], ([k](j)(+\varepsilon))\,(i)(+\varepsilon))\varepsilon^{-\frac{1}{2i\pi}\Gamma_{i,i+1}}\varepsilon^{-\frac{1}{2i\pi}\Gamma_{j,j+1}}$$

Since this form is symmetric in i and j, the lemma is proved. $\diamond$

Let $P = (W^0 \to \ldots \to W^{r-1} \to W^r)$ be a path of word sequences. The *inverse* of P is the path $\overline{P} = (W^r \to \ldots \to W^1 \to W^0)$. By definition, we have that $Z(\overline{P}) = Z(P)^{-1}$. Every word sequence W of length p is connected to $[p]$ by a path (i.e. there exists P as above such that $W^0 = [p]$ and $W^r = W$). If a word sequence P as above is such that $W^0 = W^r = [p]$, we say that P is a *loop based at* $[p]$.

Lemma 3.3 *Let $P = W^0 \to W^1 \to W^2 \to \ldots \to W^{r-1} \to W^r$ be a path of word sequences, then $Z(P)$ only depends on W^0 and W^r.*

PROOF: Because of the remarks before the lemma, it is sufficient to prove the lemma when P is a loop based at $[p]$, that is to prove that Z maps such a loop P to 1.

Define the *complexity* of a word sequence $W = (w_1, w_2, \ldots, w_k)$ of length p as the non-negative integer $c(W) = p - k$ so that the only word sequence of complexity 0 is $[p]$. Then define the *complexity* of P as the sum $c(P) = \sum_{i=0}^r c(W^i)$ so that the only path of complexity zero is the constant path $P_0 = ([p])$.

Then it suffices to prove that for any non-trivial loop P based at [p], there exists a loop P' based at [p] such that $c(P') < c(P)$ and $Z(P') = Z(P)$, in order to conclude by induction on $c(P)$. In such a loop P, there exists some word sequence W^i of maximal complexity. By definition of a path, $W^i = W^{i-1}(j) = W^{i+1}(k)$ for some integers j and k. If $j = k$, then $W^{i-1} = W^{i+1}$, we can remove $(\to W^i \to W^{i+1})$ from the path without changing Z and we are done. Otherwise, there exists a word sequence W such that $c(W) = c(W^i) - 2$, and

$$\begin{array}{lll} W^{i-1} = W(k+1), & W^{i+1} = W(j) & \text{if } j < k, \text{ and,} \\ W^{i-1} = W(k), & W^{i+1} = W(j+1), & \text{if } j > k \end{array}.$$

Thus, we may decrease the complexity of P by replacing W^i by W without changing $Z(P)$, thanks to Lemma 3.2. $\diamond$

For any two word sequences W and W', choose a path P of word sequences from W to W', set

$$Z(\mathbf{1}; W, W') = Z(P)$$

The above lemma guarantees that this definition is unambiguous.

Now, for any object $a = [(a_1; v_1), (a_2; v_2), \ldots, (a_j; v_j)]$ of length p, set

$$Z(\mathbf{1}; a, V) = \pi(\ell(v_1) \times 1, \ell(v_2) \times 2, \ldots, \ell(v_j) \times j)^* \left(Z(\mathbf{1}; (a_1, a_2, \ldots), [j])\right)$$

where $V = (v_1, v_2, \ldots, v_j)$ is the word sequence corresponding to a, and

$$Z(\mathbf{1}; V, a) = Z(\mathbf{1}; a, V)^{-1}$$

Now, for any object $b = [(b_1; w_1), (b_2; w_2), \ldots, (b_k; w_k)]$, set

$$Z(\mathbf{1}; a, b) = Z(\mathbf{1}; a, V) Z(\mathbf{1}; V, W) Z(\mathbf{1}; W, b)$$

where $W = (w_1, w_2, \ldots, w_k)$ is the word sequence corresponding to b. Note that this final definition of $Z(\mathbf{1}; a, b)$ was imposed by the announced properties.

By Lemma 2.9, this definition coincides with the one we knew for path from an actual configuration to a two-point collision.

Now, the definition of Z is obviously mutiplicative, and behaves like we want with respect of duplication, and we are done for the second step. According to Step 3, *Z is well-defined for e-tangles.*

3.4 Monoidality

Note the following property of the extension of Z for e-tangles.

Property 3.4 *Let $T(X_0)$ be a regular component of an e-tangle T, then*

$$Z(T(2 \times X_0)) = \pi(2 \times X_0)^*(Z(T))$$

PROOF: Decompose T as a product of

- regular tangles whose bottom and top are actual configurations,
- tangles with only one extremum (not on X_0) whose bottom and top are actual configurations,
- tangles of the form $(\mathbf{1}; a, b)$.

For all of these, the property is true. For the third ones, it comes from the definition. For the first ones, this is Lemma 2.9. The second case can be proved exactly like Lemma 2.9 using Lemma 1.30. Thus, the general property is true. $\diamond$

Lemma 3.5 *Assume that T_2 is the trivial tangle with one vertical strand.*

Let $T_1 = (T_1; a = ((a_1; v_1), (a_2; v_2), \ldots, (0 = a_j; v_j)), b = ((b_1; w_1), (b_2; w_2), \ldots, (0 = b_k; w_k)))$ be a tangle so that the real coordinates of the limit configurations (the a_i and the b_i) are in $]-\infty, 0]$. Let $\eta \in]0, +\infty[$. Set $\eta a = ((\eta a_1; v_1), (\eta a_2; v_2), \ldots, (0 = a_j; v_j))$ and $\eta T_1 = (T_1; \eta a, \eta b)$. Let $\eta T_1 \tilde{\otimes} T_2$ denote the tangle obtained by putting T_2 as one vertical strand from 1 to 1 on the right-hand side of ηT_1.

Then

$$Z(\eta T_1 \tilde{\otimes} T_2) - Z(\eta T_1) = O(\eta |\mathrm{Log}^{*-1}(\eta)|).$$

Since this property of T_1 is compatible with products and with (linear, degree-preserving) duplication of strands we can restrict ourselves to the case where T_1 is either:

1. a regular tangle whose bottom and top are actual configurations (done by Lemma 1.21),

2. a tangle with only one extremum whose bottom and top are actual configurations,
3. a tangle of the form $(\mathbf{1}; a, b)$ where one of a and b is a two-point collision, and the other one is an actual configuration.

Since the second case may be seen as a particular case of the third one, we may only deal with the third one where the coefficients are easy to analyze. $\diamond$

Applying this lemma when T_1 is a q-tangle (for which $\eta T_1 = T_1$ and $Z(\eta T_1 \tilde{\otimes} T_2) = Z(T_1 \otimes T_2)$, for any η) yields:

$$Z(T_1 \otimes |) = Z(T_1)$$

Thus, by the above duplication property (3.4), for any n.a. word w,

$$Z(T_1 \otimes (\mathbf{1}; w, w)) = Z(T_1) \otimes 1_{\ell(w)}$$

A symmetry of center $1/2$ of the proof or of the result yields: for any n.a. word w', and for any q-tangle T_2,

$$Z((\mathbf{1}; w', w') \otimes T_2) = 1_{\ell(w')} \otimes Z(T_2)$$

Thus, if $T_1 = (T_1; b_1, t_1)$ and $T_2 = (T_2; b_2, t_2)$ are two q-tangles such that $T_1 \otimes T_2$ has at most one extremum (so that $T_1 \otimes T_2$ is defined as an *e-tangle*), we have:

$$Z(T_1 \otimes T_2) = Z(T_1 \otimes (\mathbf{1}; b_2, b_2)) Z((\mathbf{1}; t_1, t_1) \otimes T_2)$$

$$= (Z(T_1) \otimes 1_{\ell(b_2)})(1_{\ell(t_1)} \otimes Z(T_2)) = Z(T_1) \otimes Z(T_2)$$

Now, for any two e-q-tangles, $T_1 = (T_1; b_1, t_1)$ and $T_2 = (T_2; b_2, t_2)$, we have several[13] e-tangles T_σ which represent the tangle $T_1 \otimes T_2$. To each of them, we associate a product of elementary tensor products as above, and the good behaviour of the above formula under product tells us that for any chosen e-representative T_σ of $T_1 \otimes T_2$, we have

$$Z(T_\sigma) = Z(T_1) \otimes Z(T_2)$$

In particular, we may use this independence property to prove that me may exchange the height of two extrema without changing Z: using the functoriality and the invariance under admissible isotopy, we may assume that the two extrema to be exchanged are in the two parts of a tensor product of two q-tangles and are the only extrema of the tensor product. We have just proved:

Lemma 3.6 *Z is invariant under almost admissible isotopies.*

$\diamond$

Now, *Z is an invariant of tangles,* the tensor product of tangles makes sense, and *the monoidality property is proved in its full generality.*

3.5 The general isotopy invariance of Z

We know that we have a monoidal invariant Z of oriented tangles.

The replacement of a trivial ascending strand $\uparrow$ by an ascending snake with two extrema from right to left is called an *ascending snake move.* The replacement of a trivial descending strand $\downarrow$ by a descending snake from right to left is called an *descending snake move.*

Lemma 3.7 *Z is invariant under ascending and descending snake moves.*

[13]corresponding to the shuffle permutations of the extrema

By monoidality, and by definition of ν (2.17), Z is invariant under an ascending snake move. Start with a maximum $\cap = \cap_{(\cdot\cdot)}$ oriented from right to left, and perform an ascending snake move, this does not change Z. Now, exchange the heights of the two extrema, this does not change Z either and the new tangle is obtained from $\cap$ by a descending snake move. Therefore

$$\nu = Z(\cap) = Z(\text{descending snake})Z(\cap)$$

and since ν is inversible, $Z(\text{descending snake}) = 1$ and we are done. $\diamond$

Lemma 3.8 *Let $T_0(X)$ and $T_1(X)$ denote the images of two isotopic admissible embeddings, then T_1 is obtained from T_0 by a finite sequence of the following operations:*

1. *almost admissible isotopies,*
2. *ascending and descending snake moves,*
3. *twists of the extrema as in Example 2.20.*

SKETCH OF PROOF: The reader may read [Hi] to complete this proof. The embeddings T of a compact one-manifold X in $\mathbf{C} \times [0,1]$ which have only non-degenerate critical points at different heights form a dense open subset in the set of embeddings (with fixed boundary) equipped with a suitable topology (C^{∞}_S). Here, a *critical point* is a point x such that $(p_V \circ T)'(x) = 0$ where p_V is the vertical projection, and it is *non-degenerate* when $(p_V \circ T)''(x) \neq 0$. In other words, this property of embeddings is generic (density) and stable under small deformations (openness).

The C^{∞} isotopies from $I \times X$ to $\mathbf{C} \times [0,1]$ whose restrictions to $\{u\} \times X$ satisfy the above conditions except for a finite number of u where one of the accidents "two critical points at the same height" or "one degenerate critical point at which $(p_V \circ T)'''(x) \neq 0$ does not vanish" occur also form a dense open subset of the set of C^{∞} isotopies from $I \times X$ to $\mathbf{C} \times [0,1]$ equipped with a suitable topology (C^{∞}_S).

We may see these isotopies at the neighborhoods of the isotopy times u of the accidents as compositions of isotopies without accidents and isotopies where the modifications occur only near the pieces which constitute the accident. Thus, when it cannot be trivially removed, the first acident is just an exchange of the heights of two extrema that may be supposed real, and, when it cannot be trivially removed, the second accident may be transformed into an ascending snake move or a descending one. Now, we are left with isotopies without accident where the only bad thing is that the horizontal tangent vectors are not necessarily real. Nevertheless, when these vectors are real at the beginning and at the end of the isotopy, it is not hard to compose this kind of isotopy by twists of the extrema in order to see them as compositions of admissible isotopies modulo rescaling and twists of the extrema. $\diamond$

Lemma 3.9 *$\overline{Z}$ is an isotopy invariant of unframed tangles.*

PROOF: We already know that Z itself is invariant under the two first moves. The invariance of $\overline{Z}$ under the third one come from the computations of Example 2.20 which are now allowed for our Z. $\diamond$

Definition 3.10 Let K_1 and K_2 be two oriented disjoint components of a tangle T. Then the *linking number* $lk(K_1, K_2)$ of K_1 and K_2 is equal to half the sum of the signs ($+1$ or -1) of the crossings involving both K_1 and K_2 in a regular projection of the tangle $K_1 \cup K_2$, where the sign of the crossing is $+1$ and the sign of is -1. That this sum is independent of the projection can be proved from a Reidemeister-type theorem. It can also be deduced from the following exercise.

Exercise 3.11 Let $T = T(X)$ be an oriented q-tangle. Assume $X = X_1 \coprod X_2$. Let $\tilde{\mathcal{D}}_1$ be the complex vector space freely generated by one-chord diagrams such that the local orientations at the vertices match the orientations of the tangles. Let $\tilde{\mathcal{A}}_1$ denote the quotient of $\tilde{\mathcal{D}}_1$ by the relations which identify the diagrams whose two vertices are either on X_1 or on X_2 to zero and which identify all the other diagrams to some d_{12}. Let $\tilde{Z}_1(T)$ denote the projection of $\overline{Z}_1(T)$ in $\tilde{\mathcal{A}}_1$. Show that $\tilde{Z}_1(K_1 \cup K_2) = lk(K_1, K_2)d_{12}$.

(Hint: See Lemma 2.18 and note that the degree one part of the associator is zero.)

Side-remark 3.12 The author usually prefers the following definition for the linking number: For two disjoint knots K_1 and K_2, the *linking number* of K_1 and K_2 is equal to the algebraic intersection number of K_1 and an oriented compact surface embedded in $\mathbf{R}^3$ bounded by K_2.

The *framing* of a component $T(X_0)$ is equal to the linking number of the two copies of $T(X_0)$ in $T(2\times X_0)$. We denote it by $lk(T(X_0),T(X_0))$. By the above exercise, it only depends on the isotopy class of $T(2\times X_0)$ which only depends on the parallelized isotopy class of T.

Define $\hat{Z}(T(X))$ from $Z(T(X))$ by multiplying $Z(T(X))$ by $\exp(-\frac{lk(T(X_i),T(X_i))}{2}\Gamma_{11})$ acting on the component X_i for each component X_i of X.

Then as in Lemma 3.9, $\hat{Z}(T)$ is a usual unparallelized invariant of oriented tangles. This proves that *Z is an invariant of oriented framed tangles.*

Lemma 3.13 *ν is symmetric.*

PROOF: Since the framed oriented round circle is isotopic to itself with its orientation reversed $Z(\bigcirc)=\nu^2$ is symmetric. $\diamond$

Now, the proof of the isotopy invariance of Z is complete.

3.6 Duplication

By Property 3.4, we know that Z behaves like we want under duplication of regular components.

Simply denote $\cap_{(..)}(2\times\cap)$ by $(2\times\cap_{(..)})$ and set

$$A=Z(2\times\cap_{(..)})$$

Use similar notation to set

$$B=Z(2\times\cup^{(..)})$$

We do not know whether $A=\pi(2\times\cap)^*(Z(\cap_{(..)}))$. We consider A and B as elements of $\mathcal{A}(2)$. Note that they are symmetric under the simultaneous symmetry of the two vertical intervals (because the invariance of the Kontsevich integral under a homothety of ratio (-1) is preserved in the regularisations).

Let $T(X_0)$ be a component of a tangle $T(X)$. We decompose T as a product of tangles so that the extrema of $T(X_0)$ are in factors of the form $\mathbf{1}^w_w(\frac{\cap_{(..)}}{1})$ or $\mathbf{1}^w_w(\frac{\cup^{(..)}}{1})$. By the monoidality property (as in Example 2.13), the contribution of such a tangle to $Z(T(2\times X_0))$ is an insertion of A, if we have a maximum, and B otherwise, on the two strands of the involved duplicated extremum. The contribution of the same tangle to $\pi(2\times X_0)^*(Z(T))$ is the insertion of $\pi(2\times 1)^*(\nu)$ at the same place.

Lemma 3.14 *Let $T(X_0)$ be a component of a tangle $T(X)$. Assume that either $T(X_0)$ is a circle or $T(X_0)$ is an interval running from bottom to top. Let $\star$ be the bottom point of $T(X_0)$ if X_0 is an interval, and a regular point of $T(X_0)$ otherwise. Orient X_0 so that it runs from bottom to top near $\star$. Let M be the number of maxima of X_0. Then*

$$Z(T(2\times X_0))=(AB)^M\pi(2\times X_0)^*(\nu^{-2M})\pi(2\times X_0)^*(Z(T))$$

where ν acts on X_0, and $(AB)^M$ acts on $2\times X_0$ at the duplicated $\star$.

Remark 3.15 About the case of duplication of circular components. The action of $(AB)^M\in\mathcal{A}(2)$ is an insertion of $(AB)^M\in\mathcal{A}(2)$ at the duplicated $\star$. In fact, this duplicated $\star$ does not show up in diagrams of $\mathcal{A}(X_0)$ when X_0 is a circle. So, in order to give sense to that sentence, we cut X_0 at $\star$ to transform it into an interval I_0 whose boundary is made of two copies of $\star$, and we call $X(I_0/X_0)$ the new support. Now, when $\star$ is at the intersection of two tangles, then $Z(T\setminus\star)\in\mathcal{A}(X(I_0/X_0))$ has a natural definition (which depends on our product decomposition), and in this case, the result should be written as

$$Z(T(2\times X_0))=i\left((AB)^M\pi(2\times X_0)^*(\nu^{-2M})\pi(2\times I_0)^*(Z(T\setminus\{\star\}))\right)$$

where i is the natural map from $\mathcal{A}(X(I_0/X_0)(2\times I_0))$ to $\mathcal{A}(X(2\times X_0))$. (Though $Z(T\setminus\{\star\})$ is not canonical, the RHS of the above equation is well-defined. See the proof below.)

Note that when X has no boundary, by Subsection 2.6, the natural map from $\mathcal{A}(X(I_0/X_0))$ to $\mathcal{A}(X)$ is an isomorphism. Thus, in this case, $Z(T\setminus\star)$ is nothing but the preimage of $Z(T)$ under this isomorphism and is well-defined.

Proof of Lemma 3.14: $(\nu^{-2M})Z(T)$ is the Kontsevich integral of T where the contribution of the elementary tangles containing the extrema is replaced by 1. By Proposition 3.4, its duplication $\pi(2\times X_0)^*((\nu^{-2M})Z(T))$ contains the contribution of all the other tangles to $Z(T(2\times X_0))$. Thus, $Z(T(2\times X_0))$ is obtained by inserting the A and B corresponding to the extrema in this duplicated element between duplicated parts. Since, by Lemma 2.16, A and B commute with duplicated vertices, and since maxima and minima occur alternatively, starting from $\star$ with a maximum, we get the announced lemma. $\diamond$

Apply this lemma when $T(X_0)=T(X)$ is a snake. Then

$$Z(T(2\times X_0))=(AB)\pi(2\times X_0)^*(\nu^{-2})\pi(2\times X_0)^*(Z(T))$$

where T and $T(2\times X_0)$ are isotopic (by a parallelisation-preserving isotopy) to the trivial tangles $\mathbf{1}^{\cdot}_{\cdot}$ and $\mathbf{1}^{(\cdot\cdot)}_{(\cdot\cdot)}$, respectively. Thus, the above equation implies that

$$(AB)\pi(2\times X_0)^*(\nu^{-2})=1\ \in\mathcal{A}(2)$$

Now, the above lemma implies that *the duplication property is true in its full generality, and the theorem is proved.*

Also note that we have the more specific result for duplication of link components:

Lemma 3.16 *Let $T(X_0)$ be a component of a link $T(X)$. Let $\star$ be a regular point of $T(X_0)$. Use the notation of Remark 3.15 above. Then*

$$Z(T(2\times X_0))=i\circ\pi(2\times I_0)^*(Z(T\setminus\{\star\}))$$

where i is the natural map from $\mathcal{A}(X(I_0/X_0)(2\times I_0))$ to $\mathcal{A}(X(2\times X_0))$

$\diamond$

3.7 The Kontsevich integral of iterated torus knots

We see the circle S^1 as the boundary of the complex disk $D^2=\{z\in\mathbf{C}\mid |z|\leq 1\}$. Let K be a component of a framed link L. With this framed knot K, we associate an embedding

$$\Phi_K: D^2\times S^1\longrightarrow\mathbf{R}^3$$

whose image intersects L along $\Phi_K(\{0\}\times S^1)=K$ and such that $\Phi_K(\{1\}\times S^1)$ is the parallel of K given by the framing. Let U denote the unknot with the 0-framing, then Φ_U is nothing but a standard embedding of the solid torus.

Let p and q be two coprime integers. Let

$$\begin{array}{cccc}\Phi_{p,q}: & S^1 & \longrightarrow & S^1\times S^1\\ & z & \mapsto & (z^q,z^p)\end{array}$$

Then the *framed torus knot* $K_{p,q}$ is the image of $\phi_U\circ\phi_{p,q}$ equipped with its parallel which is embedded on the torus (boundary of the solid torus). Let $L(K_{p,q}/K)$ denote the link obtained from L by replacing K by $\phi_K\circ\phi_{p,q}(S^1)$ which is equipped with its parallel lying on $\phi_K(S^1\times S^1)$.

This section is devoted to proving the following propositions:

Proposition 3.17 *Let $i : \mathcal{A}(p) \longrightarrow \mathcal{A}(S^1)$ be the map induced by gluing the top of the i^{th} interval to the bottom of the $(i+1)^{th}$ interval for any integer i mod p. Recall that Γ_{11} is the one-chord diagram on the vertical interval such that the local orientations at its two vertices give rise to the same orientation of the interval. Then*

$$Z(K_{p,q}) = i \circ \pi(p \times 1)^* \left(\nu^2 \exp\left(\frac{q}{2p}\Gamma_{11}\right)\right)$$

Proposition 3.18 *Let $\star$ be a point of K. See the underlying abstract circle X_0 corresponding to K as an interval I_0 whose two endpoints are identified with $\star$. Then*

$$Z(L(K_{p,q}/K)) = i\left(\pi(p \times 1)^* \exp\left(\frac{q}{2p}\Gamma_{11}\right) \pi(p \times I_0)^*(Z(L \setminus \{\star\}))\right)$$

where $i : \mathcal{A}(L(I_0/X_0)(p \times I_0)) \longrightarrow \mathcal{A}(L)$ is the extension of the map i of the statement of the previous proposition by the identity on the other components of L.

Proof: Proposition 3.17 is clearly a corollary of Proposition 3.18, so we prove Proposition 3.18. Write L as a product of 2 q-tangles

$$L = T_1 T_2$$

so that $\star$ is a letter of the top word W of T_1. By iterating Lemma 3.16, we know that if we duplicate the component K, cut at $\star$, (p-1) times, with respect to a length p word w, then the resulting Kontsevich integral is obtained by duplication. This can be written as

$$Z((L \setminus \{\star\})(w \times I_0)) = \pi(p \times I_0)^*(Z(L \setminus \{\star\})).$$

Now, we construct a tangle $T_{p,q}$ from w to w so that

$$L(K_{p,q}/K) = T_1(w \times (I_0 \cap T_1))\mathbf{1}_W^W(\frac{T_{p,q}}{\star})T_2(w \times (I_0 \cap T_2))$$

Let

$$\begin{array}{rccl} \gamma_{p,q} : & [0,1] & \longrightarrow & \mathbf{C}^p \setminus \mathbf{\Delta} \\ & h & \mapsto & (z_1(h), z_2(h), \dots, z_p(h)) \end{array}$$

be the $\mathbf{C}^p$-configuration path defined by

$$z_k(h) = \exp(\frac{qh + k}{p} 2i\pi)$$

Let $\gamma_p : [0,1] \longrightarrow \mathbf{C}^p \setminus \mathbf{\Delta}$ be a $\mathbf{C}^p$-configuration path from $\gamma_p(0) = (1, 2, \dots, p) = [p]$ to $\gamma_p(1) = \gamma_{p,q}(0)$ and let $\sigma_{p,q}(\overline{\gamma}_p)$ be the path obtained by permuting the strands of $\overline{\gamma}_p$ in such a way that $\sigma_{p,q}(\overline{\gamma}_p)(0) = \gamma_{p,q}(1)$. Thus, $\gamma_p\gamma_{p,q}\sigma_{p,q}(\overline{\gamma}_p)$ represents a tangle from $[p]$ to $[p]$.

Now, it is easy to see that the tangle

$$\tilde{T}_{p,q} = \mathbf{1}_w^{[p]} T(\gamma_p\gamma_{p,q}\sigma_{p,q}(\overline{\gamma}_p))\mathbf{1}_{[p]}^w$$

works, up to parallelisation. To obtain a tangle $T_{p,q}$ with the right parallelisation, first observe that the difference between the parallelisations of $T_{p,q}$ and $\tilde{T}_{p,q}$ does not depend on K. Then, observe that the framing of $K_{p,q}$ is pq while the framing of the knot obtained by inserting $\tilde{T}_{p,q}$ on a duplication of the unknot with zero framing is $(p-1)q$. So $Z(T_{p,q})$ is obtained from $Z(\tilde{T}_{p,q})$ by letting $\exp(\frac{q}{2}\Gamma_{11})$ act on one strand of $Z(\tilde{T}_{p,q})$. Now,

$$Z(\tilde{T}_{p,q}) = Z(\mathbf{1}_w^{[p]})Z(\gamma_p)Z(\gamma_{p,q})Z(\sigma_{p,q}(\gamma_p))^{-1}Z(\mathbf{1}_{[p]}^w)$$

and by monoidality, $Z(L(K_{p,q}/K))$ is obtained from $Z((L \setminus \{\star\})(w \times I_0))$ by letting $Z(\tilde{T}_{p,q})$ act on $w \times I_0$ at $w \times \star$, and then by letting $\exp(\frac{q}{2}\Gamma_{11})$ act on the torus component.

Let us now describe the action of $Z(\tilde{T}_{p,q})$ on $w \times I_0$. We will let the RHS part $Z(\gamma_{p,q})Z(\sigma_{p,q}(\gamma_p))^{-1}Z(\mathbf{1}_{[p]}^w)$ of $Z(\tilde{T}_{p,q})$ act on the bottom part of $w \times I_0$, and the LHS part $Z(\mathbf{1}_w^{[p]})Z(\gamma_p)$ act on the top part of

$w \times I_0$. Thus, since $Z((L \setminus \{\star\})(w \times I_0))$ is a duplicated element, the LHS part commutes along $w \times I_0$, and therefore cancels the inverse action of $Z(\sigma_{p,q}(\gamma_p))^{-1} Z(\mathbf{1}^w_{[p]})$. The only remaining action is the action of $Z(\gamma_{p,q})$ where, as in Example 1.4,

$$Z(\gamma_{p,q}) = \exp\left(\frac{q}{p} \sum_{\{i,j\} \subset \{1,2,\dots,p\}} \Gamma_{ij}\right)$$

and

$$\sum_{\{i,j\} \subset \{1,2,\dots,p\}} \Gamma_{ij} = \frac{1}{2}\pi(p \times 1)^*(\Gamma_{11}) - \frac{1}{2}(\sum_{i=1}^{p} \Gamma_{ii})$$

Thus,

$$Z(\gamma_{p,q}) = \pi(p \times 1)^* \left(\exp\left(\frac{q}{2p}\Gamma_{11}\right)\right) \prod_{i=1}^{p} \exp\left(\frac{-q}{2p}\Gamma_{ii}\right)$$

Now, note that the contribution of the big product to the Kontsevich integral of the torus knot makes up for the framing correction. Also note that since we have a duplicated element on the copies of I_0, these copies play a symmetric role, and the rebuilding of S^1 from these copies can be made as in the statement in any case. $\diamond$

Now, if $L_{kp,kq}$ denotes the torus link obtained from $K_{p,q}$ by duplicating $(k-1)$ times the framed knot $K_{p,q}$, then by the iterated duplication property, we have:

$$Z(L_{kp,kq}) = \pi(k \times S^1)^*(Z(K_{p,q})).$$

Thus, as soon as we know the Kontsevich integral of the unknot, we will know the Kontsevich integral of all iterated torus links by the above proposition.

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