Introduction to finite type invariants of links

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Ces notes de cours sont conçues pour être lisibles de manière autonome par des étudiants de DEA. Elles contiennent des références précises à des cours de base et à des cours plus avancés où le lecteur pourra compléter ses connaissances à sa guise, ainsi que des références à des articles de recherche utilisés pour la préparation du cours.

La liste de références donnée à la fin des notes évoluera au cours du semestre.

NB: Ces notes contiennent un peu plus que le cours du 28 février. Elles empiètent un peu sur le cours du 7 mars (8h30) salle 435.

Merci d'avance de me signaler les fautes même minimes que vous trouver(i)ez dans ces notes.

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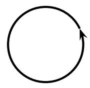
1 First steps in the folklore of knots, links and knot invariants.

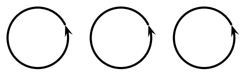
1.1 Knots and links

Intuitively, a *knot* is a circle embedded in the ambient space up to elastic deformation; a *link* is a finite family of disjoint knots.

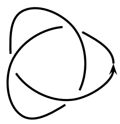
Examples 1.1 Here are some pictures of simple knots and links. More examples can be found in [Ro].



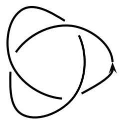
The *trivial knot*



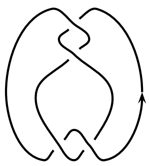
The *trivial 3-component link*



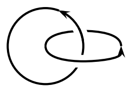
The *right-handed trefoil knot*



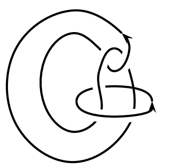
The *left-handed trefoil knot*



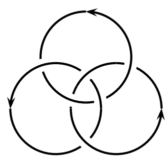
The *figure-eight knot*



The *Hopf link*



The *Whitehead link*



The *Borromean rings*

Let $\coprod^k S^1$ denote the disjoint union of k **oriented circles**. We will represent a *knot* (resp. a k-component link) by a C^∞ embedding[1] of the circle S^1 (resp. of $\coprod^k S^1$) into the ambient space $\mathbf{R}^3$.

Definition 1.2 An *isotopy* of $\mathbf{R}^3$ is a C^∞ map

$$h\ : \mathbf{R}^3 \times I \longrightarrow \mathbf{R}^3$$

such that $h_t = h(.,t)$ is a diffeomorphism for all $t \in [0,1]$. Two embeddings f and g as above are said to be *isotopic* if there is an isotopy h of $\mathbf{R}^3$ such that $h_0 =$ Identity and $g = h_1 \circ f$. Link isotopy is an equivalence relation. (Checking transitivity requires smoothing... See [Hi].)

Remark 1.3 The above definition of isotopic embeddings is equivalent to the following one:
A *link isotopy* is a C^∞ map

$$h\ : \coprod^k S^1 \times I \longrightarrow \mathbf{R}^3$$

[1]The reader is referred to [Hi] for the basic concepts of differential topology as well as for more sophisticated ones.

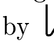
such that $h_t = h(.,t)$ is an embedding for all $t \in [0,1]$. Two embeddings f and g as above are said to be *isotopic* if there is an isotopy h such that $h_0 = f$ and $h_1 = g$.
The non-obvious implication of the equivalence comes from the isotopy extension theorem [Hi, Theorem 1.3, p.180].

Exercise 1.4 (**) Prove that for any C^∞ embedding $f : S^1 \longrightarrow \mathbf{R}^3$, there exists a continuous map $h : S^1 \times I \longrightarrow \mathbf{R}^3$ such that $h_t = h(.,t)$ is a C^∞ embedding for all $t \in [0,1]$, h_0 is a representative of the trivial knot, and $h_1 = f$. (Hint : Put the complicated part of f in a box, and shrink it.)

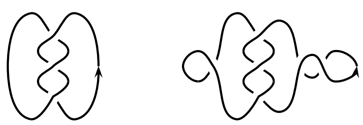
Definition 1.5 A *knot* is an isotopy class of embeddings of S^1 into $\mathbf{R}^3$. A *k-component link* is an isotopy class of embeddings of $\coprod^k S^1$ into $\mathbf{R}^3$.

Definition 1.6 Let $\pi : \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ be the projection defined by $\pi(x,y,z) = (x,y)$. Let $f : \coprod^k S^1 \hookrightarrow \mathbf{R}^3$ be a representative of a link L. The *multiple* (resp. *double*) points of $\pi \circ f$ are the points of $\mathbf{R}^2$ that have several (resp. two) inverse images under $\pi \circ f$. A double point is said to be *transverse* if the two tangent vectors to $\pi \circ f$ at this point generate $\mathbf{R}^2$. $\pi \circ f : \coprod^k S^1 \hookrightarrow \mathbf{R}^2$ is a *regular projection* of L if and only if $\pi \circ f$ is an immersion whose only multiple points are transverse double points.

Proposition 1.7 *Any link L has a representative f whose projection $\pi \circ f$ is regular.*

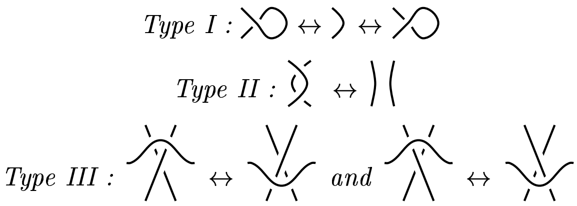
Sketch of proof: In fact, it could be justified with the help of [Hi] that when the space of representatives of a given link is equipped with a suitable topology, the representatives whose projection is regular form a dense open subspace of this space. The reader can also complete the following sketch of proof. A *PL* or *piecewise linear* link representative is an embedding of a finite family of polygons whose restrictions to the polygon edges are linear. Such a PL representative can be *smoothed* by replacing a neighbborhood of a vertex like  by  in the same plane. It is a representative of our given link if the smooth representatives obtained by smoothing close enough to the vertices are representatives of our link. A planar linear projection of such a PL representative is *regular* if there are only finitely many multiple points which are only double points without vertices in their inverse image. Observe that an orthogonal projection of a generic PL representative is regular if the direction of the kernel of the projection avoids :
I. the vector planes parallel to the planes containing one edge and one vertex outside that edge.
II. the directions of the lines that meet the interiors of 3 distinct edges.
Fix a triple of pairwise non coplanar edges. Then for every point in the third edge there is at most one line intersecting this point and the two other edges. One can even see that the set of directions of lines intersecting these three edges is a dimension one compact submanifold of the projective plane $\mathbf{R}P^2$ parametrized by subintervals of this third edge. Thus, the set of allowed oriented directions for the kernel of the projection is the complement of a finite number of one-dimensional submanifolds of the sphere S^2. Therefore it is an open dense subset in S^2 according to a weak version of the Morse-Sard theorem [Hi, Proposition 1.2, p.69] or [Mi, p.16]. Note that changing the direction of the projection amounts to composing the embedding by a rotation of $SO(3)$. Now, it is easy to smooth the projection, and to get a smooth representative whose projection is linear.
QED

Definition 1.8 A *diagram* of a link is a regular projection equipped with the additional under/over information : at a double point, the strand that crosses under is broken near the crossing. Note that a link is well-determined by one of its diagrams. The converse is not true as the following diagrams of the right-handed trefoil knot show.



Nevertheless, we have the following theorem

Theorem 1.9 (Reidemeister theorem (1926) [Re]) *Up to orientation-preserving diffeomorphism of the plane, two diagrams of a link can be related by a finite sequence of* Reidemeister moves *that are local changes of the following type:*



As it will often be the case during this course, these pictures represent local changes. They should be understood as parts of bigger diagrams that are unchanged outside their pictured parts.
LACK OF PROOF OF THE REIDEMEISTER THEOREM: It could be proved by studying the topology of the space of representatives of a given link, that a generic path between two representatives whose projections are regular in this space (i.e. a generic isotopy) only meets a finite number of times three walls made of singular representatives. The first wall that leads to the first Reidemeister move is made of the representatives that have one vertical tangent vector (and nothing else prevent their projections from being regular). The second wall that leads to the second move is made of embeddings whose projections have one non-transverse double point whereas the third wall (that leads to RIII) is made of embeddings whose projections have one triple point. *QED*

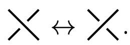
Remark 1.10 The unproved Reidemeister theorem will not be needed in the following chapters of the course. Nevertheless, the following exercise can help understanding a proof idea of the Reidemeister theorem.

Exercise 1.11 (**) Say that a PL representative of a link is *generic* if its only pairs of coplanar edges are the pairs of edges that share one vertex. Consider a generic piecewise linear link representative. Prove that for any pair $(\rho, \sigma) \in SO(3)^2$ such that $\pi \circ \rho \circ f$ and $\pi \circ \sigma \circ f$ are regular, (smoothings of) $\pi \circ \rho \circ f$ and $\pi \circ \sigma \circ f$ are related by a finite sequence of Reidemeister moves.

Exercise 1.12 (**) Prove that there are at most 4 knots that can be represented with at most 4 crossings (namely, the trivial knot, the two trefoil knots and the figure-eight knot).

Exercise 1.13 (**) Prove that the set of knots is numerable.

A *crossing change* is a local modification of the type



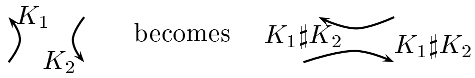
Proposition 1.14 *Any link can be unknotted by a finite number of crossing changes.*

PROOF: At a philosophical level, it comes from the fact that $\mathbf{R}^3$ is simply connected, and that a homotopy $h : S^1 \times [0,1] \longrightarrow \mathbf{R}^3$ that transforms a link into a trivial one can be replaced by a homotopy that is an isotopy except at a finite number of times where it is a crossing change. (Consider $h \times [0,1] : (x,t) \mapsto (h(x,t),t) \in \mathbf{R}^3 \times [0,1]$. h can be pertubed so that $h \times [0,1]$ is an immersion with a finite number of multiple points that are transverse double points -[Hi, Exercise 1, p.82]-...)

Here is however an elementary constructive proof of the proposition from a link diagram. Number the components of the link from 1 to m. Choose an open arc of every component K_i of the link that contains no crossing inverse image, and choose two distinct points b_i and a_i on this oriented arc so that a_i follows b_i on this arc. Then change the crossings in your link diagram if necessary so that:
1. If $i < j$, K_i crosses K_j under K_j.
2. When we follow the component K_i from a_i to b_i, we meet the lowest preimage of the crossing before meeting the corresponding highest preimage.
After these (possible) modifications, we get a diagram of a (usually different) link that is represented by an

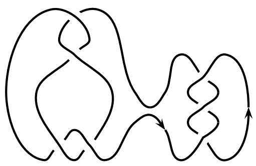
embedding whose first two coordinates can be read on the projection and whose third coordinate is given by a real-valued height function h that can be chosen so that $h(a_i) = 2i$, $h(b_i) = 2i + 1$, and h is strictly increasing from a_i to b_i and strictly decreasing from b_i to a_i. (This is consistent with the above assumptions on the crossings.) Then the obtained link is a disjoint union of components (separated by horizontal planes) that have at most two points at each height and that can therefore not be knotted. *QED*

Definition 1.15 The *disjoint union* $K_1 \coprod K_2$ of two knots K_1 and K_2 is represented by two representatives of the knots sitting in two disjoint balls of the ambient space. The disjoint union of two regular projections of K_1 and K_2 -where the two projections lie in disjoint disks of the plane $\mathbf{R}^2$- is a regular projection of $K_1 \coprod K_2$. The local change in such a projection of $K_1 \coprod K_2$:



transforms $K_1 \coprod K_2$ into the *connected sum* $K_1 \sharp K_2$ of K_1 and K_2. The connected sum of knots is a commutative well-defined operation. A knot is said to be *prime* if it cannot be written as a connected sum of two non-trivial knots. Modulo commutativity, every knot can be expressed in a unique way as the connected sum of a finite number of prime knots. (See [Li, Theorem 2.12].) Let K be a knot represented by an embedding f from S^1 into $\mathbf{R}^3$. The *reverse* $-K$ of K is the knot represented by the embedding $f \circ \mathrm{conj}$ where conj is the complex conjugation acting on the unit circle S^1 of the complex plane. The *mirror image* $\overline{K}$ of K is the knot represented by the embedding $\sigma \circ f$ where σ is the reflection of $\mathbf{R}^3$ such that $\sigma(x, y, z) = (x, y, -z)$. If K is presented by a diagram D, $\overline{K}$ is presented by the *mirror image* $\overline{D}$ of D that is obtained from D by changing all its crossings.

Examples 1.16 The two trefoil knots are mirror images of each other. The figure-eight knot is its own mirror image. (You have surely proved it when solving Exercise 1.12!) There are knots which are not equivalent to their reverses, like the eight-crossing knot 8_{17} in [Li, Table 1.1, p.5]. Here is a picture of the connected sum of the figure-eight knot and the right-handed trefoil knot.



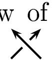
The *connected sum of the figure-eight knot and the right-handed trefoil knot*

The general open problem in knot theory is to find a satisfactory classification of knots, that is an intelligent way of producing a complete and repetition-free list of knots [2]. A less ambitious task is to be able to decide from two knot presentations whether these presentations represent the same knot. This will sometimes be possible with the help of knot invariants.

1.2 Link invariants

Definition 1.17 A link *invariant* is a function from the set of links to another set.

Such a function can be defined as a function of diagrams that is invariant under the Reidemeister moves.

Definition 1.18 A *positive crossing* in a diagram is a crossing that looks like  (up to rotation of the plane). (The "shortest" arc that goes from the arrow of the top strand to the arrow of the bottom strand turns counterclockwise.) A *negative crossing* in a diagram is a crossing that looks like  .

[2] A table of prime knots with at most 9 crossings is given in [Ro]. In this table, knots are not distinguished from their reverses and their mirror images. According to Thistlethwaithe, with the same conventions, the table of prime knots with 15 crossings contains 253293 items [Li, Table 1.2, p.6].

Definition 1.19 The *linking number* for two component links is half the number of the positive crossings that involve the two components minus half the number of negative crossings that involve the two components.

Exercise 1.20 Use Reidemeister's theorem to prove that the linking number is a link invariant. Use the linking number to distinguish the Hopf link from the Whitehead link.

Remark 1.21 Every knot bounds an oriented[3] embedded surface in $\mathbf{R}^3$. See [Ro, p.120] or [Li, Theorem 2.2]. Such a surface is called a Seifert surface of the knot. The linking number of two knots could be defined as the intersection number of a knot with a Seifert surface of the other one... and there are lots of other definitions.

Examples 1.22 There are numerical knot invariants that are easy to define but difficult to compute like:
- the *minimal number* of crossings $m(L)$ in a projection,
- the *unknotting number* of a knot that is the minimal number of crossing changes to be performed in $\mathbf{R}^3$ to unknot the knot (i.e. to make it equivalent to the trivial knot),
- the *genus* of a knot that is the minimal genus of an oriented embedded surface bounded by the knot.

An invariant is said to be *complete* if it is injective. The knot itself is a complete invariant. There are invariants coming from algebraic topology like the fundamental group of the complement of the link. A *tubular neighborhood* of a knot is a solid torus $S^1 \times D^2$ embedded in $\mathbf{R}^3$ such that its core $S^1 \times \{0\}$ is (a representative of) the knot. (See [Hi, Theorem 5.2, p.110] for the existence of tubular neighborhoods.) A *meridian* of a knot is the boundary of a small disk that intersects the knot once transversally and positively. A *longitude* of the knot is a curve on the boundary of a tubular neighborhood of the knot that is parallel to the knot. Up to isotopy, the longitudes of a knot are classified by their linking number with the knot. The *preferred longitude* of a knot is the one such that its linking number with the knot is zero. According to a theorem of Waldhausen [Wa], the fundamental group equipped with two elements that represent the oriented *meridian* of the knot and the preferred *longitude* is a complete invariant of the knot. (See also [He, Chapter 13]). According to a more recent difficult theorem of Gordon and Luecke [GL], the *knot complement*, that is the compact 3-manifold that is the closure of the complement of a knot tubular neighborhood (viewed up to orientation-preserving homeomorphism), determines the knot up to orientation. Nevertheless, these meaningful invariants are hard to manipulate.

Exercise 1.23 The *segments* of a link diagram are the connected components of the link diagram that are segments between two undercrossings where the diagram is broken. An *admissible 3-colouring* of a link diagram is a function from the set of segments of a diagram to the three-element set {Blue, Red, Yellow} such that, for any crossing, the image of the set of (usually three) segments that meet at the crossing contains either one or three elements (exactly). Prove that the number of admissible 3-colourings is a link invariant. Use this invariant number to distinguish the trefoil knots from the figure-eight knot, and the Borromean link from the trivial 3-component link. (In fact the admissible 3-colourings of a link are in one-to-one correspondence with the representations of the fundamental group of the link complement to the group of permutations of 3 elements that map the link meridians to transpositions.)

1.3 Finite type knot invariants

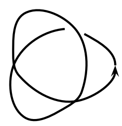
Definition 1.24 A *singular knot* with n double points is represented by an immersion with n transverse double points that is an embedding when restricted to the complement of the preimages of these double points.

Two such immersions f and g are said to be *isotopic* if there is an isotopy h of $\mathbf{R}^3$ such that $h_0 =$ Identity and $g = h_1 \circ f$.

A *singular knot* with n double points is an isotopy class of such immersions with n double points.

Example 1.25

[3]Boundaries are always oriented with the "outward normal first" convention.



A *singular knot with two double points*

Definition 1.26 Let be a double point of a singular knot. This double point can disappear in a *positive way* by changing into , or in a *negative way* by changing into . Note that the sign of such a *desingularisation,* that can be seen in a diagram as above, is defined from the orientation of the ambient space. Choose one strand involved in the double point. Call this strand the first one. Consider the tangent plane to the double point, that is the vector plane equipped with the basis (tangent vector $\vec{v_1}$ to the first strand, tangent vector $\vec{v_2}$ to the second one). This basis orients the plane, and allows us to define a positive normal vector $\vec{n}$ to the plane (that is a vector $\vec{n}$ orthogonal to $\vec{v_1}$ and $\vec{v_2}$ such that the basis $(\vec{v_1},\vec{v_2},\vec{n})$ is an oriented basis of $\mathbf{R}^3$). Then the *positive desingularisation* is obtained by pushing the first strand in the direction of $\vec{n}$. Note that this definition is independent of the choice of the *first* strand.

Notation 1.27 Let K be a singular knot with n double points numbered by 1,2 $\dots$, n. Let f be a map from $\{1, 2\dots, n\}$ to $\{+, -\}$. Then K_f denotes the genuine knot obtained by removing every crossing i by the tranformation: ${}_i$ becomes if $f(i) = +$, and ${}_i$ becomes if $f(i) = -$.
Let $\mathcal{K}$ denote the set of knots. Let $\mathbf{Z}[\mathcal{K}]$ denote the free $\mathbf{Z}$-module with basis $\mathcal{K}$. Then $[K]$ denotes the following element of $\mathbf{Z}[\mathcal{K}]$:

$$[K] = \sum_{f:\{1,2\dots,n\}\longrightarrow\{+,-\}} (-1)^{\sharp f^{-1}(-)} K_f$$

where the symbol $\sharp$ is used to denote the cardinality of a set.

Proposition 1.28 *We have the following equality in V:*

$$[\;] = [\;] - [\;]$$

that relate the brackets of three singular knots that are identical outside a ball where they look like in the above pictures.

Proof: Exercise.

Definition 1.29 Let G be an abelian group. Then any G-valued knot invariant I is extended to $\mathbf{Z}[\mathcal{K}]$, linearly. It is then extended to singular knots by the formula

$$I(K) \stackrel{\text{def}}{=} I([K])$$

Let n be an integer. A G-valued knot invariant I is said to be of *degree less or equal than* n, if it vanishes at singular knots with $(n + 1)$ double points. Of course, such an invariant if of degree n if it is of degree less or equal than n without being of degree less or equal than $(n - 1)$. A G-valued knot invariant I is said to be of *finite type* or of *finite degree* it is of degree n for some n. Note that we could have defined the extension of a G-valued knot invariant to singular knots by induction on the number of double points using the induction formula:

$$I(\;) = I(\;) - I(\;).$$

Definition 1.30 Let V be a vector space. A *filtration* is a decreasing sequence of vector spaces

$$V = V_0 \supseteq V_1 \supseteq \dots \supseteq V_n \supseteq V_{n+1} \supseteq \dots$$

that begins with $V_0 = V$. The *graded space* associated to such a filtration is the vector space

$$\mathcal{G} = \oplus_{k=1}^{\infty} \mathcal{G}_k((V_n)) \text{ where } \mathcal{G}_k((V_n)) = \frac{V_k}{V_{k+1}}$$

Let $\mathbf{K} = \mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{V}$ denote the $\mathbf{K}$-vector space freely generated by the knots. Let $\mathcal{V}_n$ denote the subspace of $\mathcal{V}$ generated by the brackets of the singular knots with n double points. The *Vassiliev filtration of* $\mathcal{V}$ is the sequence of the $\mathcal{V}_n$. Let $\mathcal{I}_n$ denote the set of $\mathbf{K}$-valued invariants of degree less or equal than n. $\mathcal{I}_n$ is nothing but the dual vector space of $\frac{\mathcal{V}}{\mathcal{V}_{n+1}}$ that is the space of linear forms on $\frac{\mathcal{V}}{\mathcal{V}_{n+1}}$.

$$\mathcal{I}_n = \mathrm{Hom}(\frac{\mathcal{V}}{\mathcal{V}_{n+1}}; \mathbf{K}) = \left(\frac{\mathcal{V}}{\mathcal{V}_{n+1}}\right)^*$$

Proposition 1.31

$$\frac{\mathcal{I}_n}{\mathcal{I}_{n-1}} = \left(\frac{\mathcal{V}_n}{\mathcal{V}_{n+1}}\right)^*$$

PROOF: Exercise.

Exercise 1.32 (*) Let $\lambda \in \mathcal{I}_n$, let $\mu \in \mathcal{I}_m$. Define the invariant $\lambda\mu$ at genuine knots K by $\lambda\mu(K) = \lambda(K)\mu(K)$. Prove that $\lambda\mu \in \mathcal{I}_{n+m}$.

Here are some natural questions, and the way we are going to try to anwer them.

Question 1: *Are there finite type invariants and how many are they?* In other words, what is the dimension of $\frac{\mathcal{V}_n}{\mathcal{V}_{n+1}}$?

In the next subsection, we are going to bound this dimension from above. Examples of easily-computable finite type invariants will be given at the end of this first chapter in Subsection 1.5.

Question 2: *How can finite type invariants be constructed?*

We will answer this question by constructing the *Kontsevich integral*. The Kontsevich integral is a knot invariant $\overline{Z}$ valued in a vector space $\overline{\mathcal{A}}$ such that any $\mathbf{K}$-valued finite type invariant is obtained as a composition of $\overline{Z}$ by a linear form. Unfortunately, the space $\overline{\mathcal{A}}$ that will be presented in the next subsection is not yet well-understood. Nevertheless, we will show that Lie algebras provide linear forms on $\overline{\mathcal{A}}$. The so-called *quantum invariants* are obtained by composing $\overline{Z}$ by such linear forms.

Question 3: *How is Z related to classical knot invariants?*

We will relate the Kontsevich integral to the *Milnor invariants* and the *Alexander polynomial* that can be defined from the π_1 of link complements.

Question 4: *What algebraic structure does the space of finite type invariants carry?*

We can see from Exercise 1.32 that this space is an algebra over $\mathbf{K}$. In fact $\oplus_{n=1}^{\infty} \frac{\mathcal{I}_n}{\mathcal{I}_{n-1}}$ is even a *Hopf algebra.* We are going to get acquainted with the Hopf algebra structures.

Question 5: *How can one relate two knots that are not distinguished by invariants of degree less than n?*

We shall prove the theorem of Stanford [St] that describes elementary topological modifications of type n such that two knots are not distinguished by invariants of degree less than n if and only if they are related by a finite sequence of these modifications.

Question 6: *Do finite type invariants separate knots?*

This question is still open...

References

Basic references

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