# ON HOMOTOPY INVARIANTS OF COMBINGS OF THREE-MANIFOLDS

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Abstract. Combings of compact, oriented 3-dimensional manifolds $M$ are homotopy classes of nowhere vanishing vector fields. The Euler class of the normal bundle is an invariant of the combing, and it only depends on the underlying Spin$^c$-structure. A combing is called torsion if this Euler class is a torsion element of $H^2(M;\mathbb{Z})$. Gompf introduced a $\mathbb{Q}$-valued invariant $\theta_G$ of torsion combings on closed 3-manifolds, and he showed that $\theta_G$ distinguishes all torsion combings with the same Spin$^c$-structure. We give an alternative definition for $\theta_G$ and we express its variation as a linking number. We define a similar invariant $p_1$ of combings for manifolds bounded by $S^2$. We relate $p_1$ to the $\Theta$-invariant, which is the simplest configuration space integral invariant of rational homology 3-balls, by the formula $\Theta = \frac{1}{4}p_1 + 6\lambda(\hat{M})$ where $\lambda$ is the Casson-Walker invariant. The article also includes a self-contained presentation of combings for 3-manifolds.

## Contents



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## 1. Introduction

1.1. **Preamble: Conventions and notations.** Unless otherwise mentioned, all manifolds are oriented. Boundaries are oriented by the outward normal first convention. Products are oriented by the order of the factors. More generally, unless otherwise mentioned, the order of appearance of coordinates or parameters orients chains or manifolds. When $C$ is a manifold, $(-C)$ denotes the manifold obtained from $C$ by reversing its orientation. The fiber $N_x(A)$ of the normal bundle $N(A)$ to an oriented submanifold $A$ of $C$ at $x \in A$ is oriented so that $N_x(A)$ followed by the tangent bundle $T_x(A)$ to $A$ at $x$ induces the orientation of $C$. The orientation of $N_x(A)$ is a *coorientation* of $A$ at $x$. The transverse preimage of a submanifold under a map $f$ is oriented so that $f$ preserves the coorientations. The transverse intersection of two submanifolds $A$ and $B$ in a manifold $C$ is oriented so that the normal bundle $N_x(A \cap B)$ to $A \cap B$ at $x \in A \cap B$ is oriented as $(N_x(A) \oplus N_x(B))$. If the two manifolds are of complementary dimensions, then the sign of an intersection point is $+1$ if the orientation of its normal bundle coincides with the orientation of the ambient space, which is if $T_xC = N_xA \oplus N_xB$ (as oriented vector spaces), this is equivalent to $T_xC = T_xA \oplus T_xB$. Otherwise, the sign is $-1$. If $A$ and $B$ are compact and if $A$ and $B$ are of complementary dimensions in $C$, their *algebraic intersection* is the sum of the signs of the intersection points, it is denoted by $\langle A, B \rangle_C$. The *linking number* of two rationally null-homologous disjoint links in a 3-manifold is the algebraic intersection of a rational chain (i.e. a $\mathbb{Q}$-linear combination of surfaces) bounded by one of the links, and the other link.

1.2. **Introduction.** In this article, $M$ is an oriented connected compact smooth 3-manifold. The boundary $\partial M$ of $M$ is either empty or identified with the unit sphere $S^2$ of $\mathbb{R}^3$. In the latter case, a neighborhood $N(\partial M)$ of $\partial M$ in $M$ is identified with a neighborhood of $S^2$ in the unit ball of $\mathbb{R}^3$. The tangent bundle of $M$ is denoted by $TM$, and the unit tangent bundle of $M$ is denoted by $UM$. Its fiber is $U_mM = (T_mM \setminus \{0\})/\mathbb{R}^{*+}$.

It has long been known that $M$ is parallelizable. (For a proof, see [11, p. 46] or [20, Section 6.2].) All considered parallelizations $\tau\colon M \times \mathbb{R}^3 \to TM$ of $M$ are assumed to coincide with the parallelization induced by the standard parallelization $\tau_s$ of $\mathbb{R}^3$ over $N(\partial M)$, and all sections of $UM$ are assumed to be constant with respect to this parallelization over $N(\partial M)$. Homotopies of parallelizations or sections satisfy these assumptions at any time. When $\partial M = \emptyset$, the parallelizations of $M$ also induce the orientation of $M$.

A *combing* of $M$ is a homotopy class of such sections of $UM$. According to Turaev [26], a *Spin$^c$-structure* on $M$ may be seen as an equivalence class of sections of $UM$, where two sections are in the same class if and only if they are homotopic over the complement of a point that sits in the interior of $M$.

The *Euler class* of a combing $[X]$ represented by a section $X$, is the Euler class of the normal bundle $X^\perp = TM/\mathbb{R}X$. The Euler class of $[X]$ is denoted by $e(X^\perp)$. It belongs to $H^2(M;\mathbb{Z})$ (here, $H^2(M;\mathbb{Z}) = H^2(M, \partial M;\mathbb{Z})$). The Euler class is the obstruction to the existence of a nowhere zero section of $X^\perp$ that is defined in

Lemma 2.15. The Euler class $e(X^{\perp})$ only depends on the Spin$^c$-structure of the combing.

A *torsion combing* of $M$ is a combing whose Euler class is a torsion element of $H^2(M, \partial M; \mathbb{Z})$. A *torsion section* of $UM$ is a section that represents a torsion combing. A *torsion Spin$^c$-structure* is a Spin$^c$-structure represented by torsion combings.

For a section $X$ of $UM$, $-X$ denotes the opposite section. When a parallelization $\tau$ of $M$ is given, a section of $UM$ is nothing but a map from $M$ to $S^2$ that is constant on $\partial M$, and two sections $X$ and $Y$ of $UM$ induce a map $(X, Y) \colon M \to S^2 \times S^2$. Two sections $X$ and $Y$ are said to be *transverse* if the induced maps $(X, Y)$ and $(X, -Y)$ are transverse to the diagonal of $S^2 \times S^2$, that is if their graphs in $M \times S^2 \times S^2$ are transverse to the product of $M$ and the diagonal of $S^2 \times S^2$. This is generic and independent of $\tau$. (Genericity and transversality are explained in [7].) For two transverse sections $X$ and $Y$, let $L_{X=Y}$ be the preimage of the diagonal of $S^2 \times S^2$ under the map $(X, Y)$. Thus $L_{X=Y}$ is a link in the interior of $M$. It is oriented as follows with respect to the conventions of Subsection 1.1.

The sphere $S^2$ is oriented as the boundary of the unit ball of $\mathbb{R}^3$. The diagonal of $(S^2)^2$ inherits an orientation from $S^2$. It is therefore also cooriented in $(S^2)^2$, which is equipped with the product orientation. The map $(X, Y)$ pulls back the coorientation of $(S^2)^2$ to a coorientation of $L_{X=Y}$, which, in turn, orients $L_{X=Y}$. Note that

$$L_{X=Y} = L_{Y=X} = -L_{-X=-Y}.$$

In Subsection 3.2, we prove the following theorem, with elementary arguments.

**Theorem 1.1.** *Let $X$ be a fixed section of $UM$. Two sections $Y$ and $Y'$ of $UM$ transverse to $X$ represent the same Spin$^c$-structure if and only if the links $L_{Y=-X}$ and $L_{Y'=-X}$ are homologous.*

*A section $Y$ of $UM$ transverse to $X$ is a torsion section if and only if the homology class $([L_{Y=X}] + [L_{Y=-X}])$ is a torsion element of $H_1(M; \mathbb{Z})$.*

*If $X$ and $Y$ are transverse torsion sections, then the links $L_{Y=-X}$ and $L_{Y=X}$ are rationally null-homologous in $M$.*

*If $X$ is a torsion section, then two torsion sections $Y$ and $Y'$ of $UM$ transverse to $X$ represent the same combing if and only if the links $L_{Y=-X}$ and $L_{Y'=-X}$ are homologous, and $lk(L_{Y=-X}, L_{Y=X}) = lk(L_{Y'=-X}, L_{Y'=X})$.*

This theorem is a variant of a Pontrjagin theorem recalled in Subsection 2.1, which treats the case when $X$ extends to a parallelization. It might be already known. I thank Patrick Massot for pointing out to me that Dufraine proved similar results in [3].

The first Pontrjagin class induces a canonical map $p_1$ from the set of parallelization homotopy classes of $M$ to $\mathbb{Z}$. When $\partial M = \emptyset$, the map $p_1$, denoted as $\delta(M, .)$, is studied by Hirzebruch in [8, §3.1], and Kirby and Melvin study $p_1$ under the name *Hirzebruch defect* in [10], and they denote it as $h$, there. This map $p_1$ is studied in [16, 20] when $\partial M = S^2$ and $H_1(M; \mathbb{Q}) = 0$. The definition of $p_1$ and some of its properties are recalled in Subsection 4.1.

The main original result of this article is the following theorem. It is proved in Subsection 4.2.

**Theorem 1.2.** *There exists a unique map*

$$p_1 \colon \{\textit{Torsion combings of } M\} \to \mathbb{Q}$$

*such that*

- *for any section $X$ of $UM$ that extends to a parallelization $\tau$, $p_1([X]) = p_1(\tau)$, and*
- *for any two transverse torsion sections $X$ and $Y$ of $UM$,*

$$p_1([Y]) - p_1([X]) = 4lk(L_{X=Y}, L_{X=-Y}). \tag{1.3}$$

*The map $p_1$ satisfies the following properties:*

- *For any section $X$, $p_1([X]) = p_1([-X])$.*
- *For a Spin$^c$-structure $\xi$, let $\mathcal{C}(\xi)$ denote the set of combings that represent $\xi$. For any torsion Spin$^c$-structure $\xi$, the restriction of $p_1$ to $\mathcal{C}(\xi)$ is injective.*

The variation of $p_1$ under simple operations on torsion combings is presented in Subsection 4.4.

The image of $p_1$ is determined by the following theorem, which is proved in Subsection 4.2.

Let $\ell\colon \text{Torsion}(H_1(M;\mathbb{Z})) \to \mathbb{Q}/\mathbb{Z}$ denote the *self-linking number* (which is the linking number of a representative and one of its parallels). View an element $\overline{a}$ of $\mathbb{Q}/\mathbb{Z}$ as its class $(a+\mathbb{Z})$ in $\mathbb{Q}$ so that $4\ell(\text{Torsion}(H_1(M;\mathbb{Z}))$ is a subset of $\mathbb{Q}$, invariant by translation by 4.

**Theorem 1.4.** *Let $\tau$ be a parallelization of $M$ inducing a section $X$ of $UM$. For any torsion section $Y$ of $UM$ transverse to $X$,*

$$p_1([Y]) \in (p_1(\tau) - 4\ell([L_{Y=-X}])).$$

$$p_1(\{\text{Torsion combings}\}) = p_1(\tau) - 4\ell(\text{Torsion}(H_1(M;\mathbb{Z})).$$

Here $p_1(\tau)$ is an integer whose parity is determined in Theorem 4.3. Note that the image of $p_1$ is not an affine space in general. (When $H_1(M;\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$, $4\ell(\text{Torsion}(H_1(M;\mathbb{Z}))$ is either $4\mathbb{Z} \cup (\frac{4}{3} + 4\mathbb{Z})$ or $4\mathbb{Z} \cup (\frac{8}{3} + 4\mathbb{Z})$. )

In Subsection 4.3, we prove that the invariant $p_1$ coincides with an invariant $\theta_G$ defined by Gompf in [4] when $\partial M = \emptyset$. The Gompf invariant is denoted by $\theta$ in [4], and it is denoted by $\theta_G$ in this article to prevent confusion with $\Theta$.

In [24, Section 2.6], Ozsváth and Szabó associate a Spin$^c$-structure to a generator $\mathbf{x}$ of the Heegaard Floer homology $\widehat{HF}$. Gripp and Huang refine this process in [5] in order to associate a combing $\tilde{gr}(\mathbf{x})$ to such a generator $\mathbf{x}$, and they relate the Gompf invariant to the absolute $\mathbb{Q}$-grading $\overline{gr}$ of Ozsváth and Szabó for the Heegaard Floer homology of 3-manifolds equipped with torsion Spin$^c$ structures of [25]. According to [5, Corollary 4.3], $\overline{gr}(\mathbf{x}) = \frac{2+\theta_G(\tilde{gr}(\mathbf{x}))}{4}$.

An *integer homology* 3*-sphere* (resp. an *integer homology* 3*-ball*) is a smooth, compact, oriented 3-manifold with the same homology as the sphere $S^3$ (resp. as a point), with coefficients in $\mathbb{Z}$. A *rational homology* 3*-sphere* (resp. a *rational homology* 3*-ball*) is a smooth, compact, oriented 3-manifold with the same homology as the sphere $S^3$ (resp. as a point), with coefficients in $\mathbb{Q}$.

The work of Witten [28] pioneered the introduction of many rational homology 3-sphere invariants, and Witten's insight into the perturbative expansion of Chern-Simons theory led Kontsevich to outline a construction of invariants associated to graph configuration spaces in [12]. In [13], G. Kuperberg and D. Thurston applied the Kontsevich scheme to show the existence of such an invariant $Z_{KKT}$ of rational homology 3-spheres, which is equivalent to the LMO invariant $Z_{LMO}$ of Le, Murakami and Ohtsuki [15] for integer homology 3-spheres. Both $Z_{KKT}$

and $Z_{LMO}$ dominate all finite type invariants of integer homology 3-spheres. The invariant $Z_{KKT}$ is in fact constructed as a graded invariant of parallelized rational homology 3-balls $M$. Its degree one part is called the $\Theta$-invariant. Here, we denote it by $\Theta_{KKT}$. Gluing a standard 3-dimensional ball along the boundary of a rational homology 3-ball $M$ provides a well-defined rational homology 3–sphere $\hat{M}$.

According to a Kuperberg-Thurston theorem [13] generalized to rational homology 3-spheres in [17, Theorem 2.6 and Section 6.5], for a rational homology 3-ball $M$ equipped with a parallelization $\tau$,

$$\Theta_{KKT}(M,\tau) = 6\lambda(\hat{M}) + \frac{p_1(\tau)}{4},$$

where $\lambda$ is the Casson-Walker invariant, normalized as in [1,21] for integer homology 3-spheres, and as $\frac{\lambda_W}{2}$ for rational homology 3-spheres, where $\lambda_W$ is the Walker normalization in [27].

In Section 5, we define an invariant $\Theta$ of combings $[X]$ in a rational homology 3-ball $M$ from an algebraic intersection in a two-point configuration space. This invariant $\Theta$ satisfies the same variation formula as $\frac{1}{4}p_1$ (Formula 1.3) so that $\Theta(M,[X]) - \frac{p_1([X])}{4}$ only depends on the rational homology 3-ball $M$. When $X$ is the first vector of a trivialization $\tau$, the definition of $\Theta(M,[X])$ agrees with the definition of $\Theta_{KKT}(M,\tau)$ as an algebraic intersection of three chains in a two-point configuration space, which can be found in [17, Section 6.5] and in [18, Theorem 2.14] so that

$$\Theta(M,X) = 6\lambda(\hat{M}) + \frac{1}{4}p_1([X]).$$

## 2. Combings

This section is devoted to a general presentation of combings of our fixed manifold $M$. Their constructions with respect to a fixed section of $UM$ are described in Subsection 2.1. The affine structure over $H_1(M;\mathbb{Z})$ (or dually over $H^2(M,\partial M;\mathbb{Z})$) of the set $\mathcal{S}(M)$ of Spin$^c$-structures of $M$, and the properties of the Euler classes of Spin$^c$-structure are presented in Subsection 2.2. Finally, the action of $\pi_3(S^2)=\mathbb{Z}$ on the set of combings, which equips each set $\mathcal{C}(\xi)$ of combings with underlying Spin$^c$-structure $\xi$, with an affine structure, is presented in Subsection 2.3. Our presentation, which centers around links of type $L_{X=\pm Y}$, contains lemmas that will be used in the following sections.

### 2.1. Generalization of a Pontrjagin construction in dimension 3.

A *framing* of a link $L$ of $M$ is a homotopy class of sections of the unit normal bundle to $L$. Pushing $L$ in the direction of such a section yields a *parallel* $L_{\parallel}$ of $L$ up to isotopy in $N(L)\setminus L$, where $N(L)$ is a tubular neighborhood of $L$. Since this isotopy class of $L_{\parallel}$ determines the framing, a *framing* of $L$ can equivalently be defined as an isotopy class of parallels of $L$.

A *framed cobordism* from $(L, L_{\parallel})$ to another framed link $(L', L'_{\parallel})$ is a cobordism $\Sigma$ from $\{0\}\times L$ to $\{1\}\times L'$, properly embedded in $[0,1]\times M$, and equipped with a homotopy class of unit normal sections to $T\Sigma$ in $T([0,1]\times M)$ that induce the given framings of $L$ and $L'$. Two framed links are *framed cobordant* if and only if their exists a framed cobordism from one to the other one.

When a parallelization of $M$ is fixed, the Pontrjagin construction, which is recalled in Theorem 2.9 below, identifies combings with framed links up to framed cobordism. In this subsection, we present a generalized version of this construction referring to a fixed section of $UM$ rather than to a fixed parallelization.

Let $[X]^c$ denote the Spin$^c$-structure of $M$ represented by a section $X$ of $UM$.

**Lemma 2.1.** *Combings are generically transverse. For two transverse sections $X$ and $Y$ of $UM$, the homology classes of $L_{Y=X}$ and $L_{Y=-X}$ only depend on the Spin$^c$-structures $[X]^c$ and $[Y]^c$.*

PROOF: When $X$ extends as a parallelization, this parallelization identifies $UM$ with $M \times S^2$, so that $Y$ may be seen as a map from $M$ to $S^2$, and a homotopy of $Y$ is a map from $[0,1] \times M$ to $S^2$, for which $X$ is a regular value, for a generic homotopy. In particular, the preimage of $X$ under such a homotopy $h$ yields a cobordism from $L_{Y_0=X}$ and $L_{Y_1=X}$, and the homology class of $L_{Y=X}$ only depends on the homotopy class of $Y$, when $X$ is fixed. Since any $X$ locally extends as a parallelization, the local transversality arguments hold for any $X$ so that the above proof may be adapted to any $X$ by using a homotopy $(Y_t, X)$ valued in $S^2 \times S^2$ (with respect to some reference trivialization) and the preimage of the diagonal under this homotopy. Similarly, the homology class of $L_{Y=-X}$ only depends on the homotopy classes of $X$ and $Y$. When $X$ (resp. $Y$) is changed to some section $X'$ (resp. $Y'$) that coincides with $X$ (resp. $Y$) outside a ball $B^3$ embedded in $M$, the homology classes of $L_{Y=-X}$ and $L_{Y=X}$ are unchanged. Thus the homology classes of $L_{Y=-X}$ and $L_{Y=X}$ only depend on $[X]^c$ and $[Y]^c$. $\diamond$

**Definition 2.2.** Let $X$ be a section of $UM$. Let $NL$ be the normal bundle to a link $L$ in $M$. Let $S(NL, (-X)^\perp)$ denote the space of homotopy classes of sections of the bundle $\mathrm{Isom}^+(NL, (-X)^\perp)$ over $L$ whose fiber over $x$ is the space of orientation-preserving linear isomorphisms from the fiber $N_xL \cong T_xM/T_xL$ of $NL$ to $(-X(x))^\perp = T_xM/\mathbb{R}(-X(x))$. An *$X$–framing* of $L$ is an element of $S(NL, (-X)^\perp)$.

Any section $Y$ of $UM$ transverse to $X$ yields an $X$–framing

$$\sigma(Y, X) \in S(NL_{Y=-X}, (-X)^\perp)$$

of $L_{Y=-X}$, which is naturally induced by the restriction to $L_{Y=-X}$ of the tangent map to $Y \colon M \to UM$. (The tangent map to $Y$ at $x \in L_{Y=-X}$ maps $T_xM$ into $T_xM \oplus U_xM$. Since its composition with the projection onto $U_xM$ maps $T_xL_{Y=-X}$ to $\mathbb{R}X(x)$, this composition induces a map from $N_xL_{Y=-X}$ to $T_xM/\mathbb{R}(-X(x))$. Our transversality assumptions and our orientation conventions imply that this map is an orientation-preserving isomorphism.)

Equip $M$ with a Riemannian structure (all of them are homotopic). Note that the normal bundle $X^\perp$ is canonically isomorphic to the bundle of planes orthogonal to $X$ so that our notation $X^\perp$ for the normal bundle is not too abusive.

**Remark 2.3.** Let $L$ be a link of $M$ and let $\sigma_N$ be a unit section of $NL$. The section $\sigma_N$ and the Gram-Schmidt process induce a bundle retraction by deformation from $\mathrm{Isom}^+(NL_{Y=-X}, (-X)^\perp)$ to the $S^1$-bundle of orientation-preserving isometries from $N_xL_{Y=-X}$ to $T_xM/\mathbb{R}(-X(x))$. (These quotients are equipped with their metrics induced by the canonical isomorphisms that identify them with orthogonal complements).

Let $[\sigma] \in S(NL, (-X)^\perp)$ be an $X$–framing of $L$ represented by an isometry $\sigma \colon NL \to (-X)^\perp$. Set $Z(\sigma, \sigma_N)(x) = \sigma(x)(\sigma_N(x))$. Then $Z(\sigma, \sigma_N)$ is a section of

$(-X)^{\perp}$. Note that $[\sigma]$ is determined by the homotopy classes of $\sigma_N$ and $Z(\sigma,\sigma_N)$, where the homotopy class of $\sigma_N$ may be replaced with the isotopy class of the parallel $L_{\parallel}$ of $L$ induced by $\sigma_N$. Therefore, elements of $S(NL,(-X)^{\perp})$ can be thought of as pairs $(L_{\parallel}, Z(\sigma,\sigma_N))$, up to simultaneous twists of $L_{\parallel}$ and $Z(\sigma,\sigma_N)$.

**Definition 2.4.** A section $X$ of $UM$ and a link $L$ of $M$ equipped with an $X$–framing $[\sigma]$ represented by an isometry $\sigma\colon NL \to (-X)^{\perp}$ induce the following section $C(X,L,\sigma)$ of $UM$ (up to homotopy). Let $N(L)$ be a tubular neighborhood of $L$. The fiber $N_x(L)$ of $N(L)$ at $x$ is seen as $\{uv; u\in[0,1], v\in N_xL, \parallel v\parallel = 1\}$. Regard $X$ as a map to $S^2$ with respect to some trivialization such that $X$ is constant on $N_x(L)$ for all $x\in L$. Let $[-X(x),X(x)]_{\sigma(v)}$ denote the geodesic arc of $U_xM\cong S^2$ from $(-X(x))$ to $X(x)$ through $\sigma(v)\in(-X(x))^{\perp}$. Then $[-X(x), C(X,L,\sigma)(uv)]$ is the subarc of $[-X(x),X(x)]_{\sigma(v)}$ of length $u\pi$ starting at $-X(x)$. This defines $C(X,L,\sigma)$ on $N(L)$, and $C(X,L,\sigma)$ coincides with $X$ outside $N(L)$.

**Lemma 2.5.** *Let $X$ and $Y$ be two transverse sections of $UM$. Then $Y$ is homotopic to $C(X,L_{Y=-X},\sigma(Y,X))$. Furthermore, the Spin$^c$-structure of $Y$ is determined by $[X]^c$ and $L_{Y=-X}$.*

Proof: Outside $L_{Y=-X}$, there is a homotopy from $Y$ to $X$. When $Y(m)\neq -X(m)$, there is a unique geodesic arc $[Y(m),X(m)]$ with length ($\ell\in[0,\pi[$) from $Y(m)$ to $X(m)$. For $t\in[0,1]$, let $Y_t(m)\in[Y(m),X(m)]$ be such that the length of $[Y(m)=Y_0(m),Y_t(m)]$ is $t\ell$. Let $D^2$ be the unit disk of $\mathbb{C}$. Write $N(L_{Y=-X})$ as $D^2\times L_{Y=-X}$, and let $\chi$ be a smooth increasing bijective function from $[0,1]$ to $[0,1]$ whose derivatives vanish at 0 and 1. Set $\tilde{Y}_t(m)=Y_t(m)$ if $m\notin N(L_{Y=-X})$ and $\tilde{Y}_t(v\in D^2,\ell\in L_{Y=-X}) = \begin{cases} Y_{\chi(|v|)t}(v,\ell) & \text{if } v\neq 0\\ Y(0,\ell) & \text{if } v=0.\end{cases}$
Then $\tilde{Y}_1$ is homotopic to $Y$, and when $N(L)$ is small enough, it is easy to see that $\tilde{Y}_1$ is homotopic to $C(X,L_{Y=-X},\sigma(Y,X))$, too.

Let us prove that $[Y]^c=[C(X,L_{Y=-X},\sigma(Y,X))]^c$ does not depend on the $X$-framing $\sigma(Y,X)$ of $L_{Y=-X}$. Two representatives $\sigma_1$ and $\sigma_2$ of any two $X$–framings of a link may be assumed to coincide over the link except over one little interval for each link component. Thus, the associated $C(X,L_{Y=-X},\sigma_1)$ and $C(X,L_{Y=-X},\sigma_2)$ coincide outside a finite union of balls, which embeds in a larger ball. Then $[Y]^c$ is determined by $X$ and $L_{Y=-X}$. Now, for a fixed $L$, changing $X$ inside its homotopy class or changing $X$ over a ball does not affect $[C(X,L,.)]^c$. ⋄

Let $(-X)^{\perp}$ also denote the pull-back of $(-X)^{\perp}$ under the natural projection from $[0,1]\times M$ to $M$. Let $\Sigma$ be a properly embedded surface in $[0,1]\times M$, which is equipped with the product Riemannian metric. Let $S(N\Sigma,(-X)^{\perp})$ denote the space of homotopy classes of sections of the bundle over $\Sigma$ whose fiber over $x$ is the space of orientation-preserving isomorphisms from the fiber $N_x\Sigma = T_x([0,1]\times M)/T_x\Sigma$ of $N\Sigma$ to $(-X(x))^{\perp}$. Again, this bundle deformation retracts onto the $S^1$–bundle over $\Sigma$ whose fiber over $x$ is the space of orientation-preserving linear isometries from $N_x\Sigma$ to $(-X(x))^{\perp}$, so that $S(N\Sigma,(-X)^{\perp})$ is canonically isomorphic to the space of homotopy classes of sections of the latter bundle, which is an $S^1$–bundle. An *$X$–framing* of $\Sigma$ is an element of $S(N\Sigma,(-X)^{\perp})$.

Two $X$–framed links $L$ and $L'$ are *$X$–framed cobordant* if and only if there exists an $X$–framed cobordism $\Sigma$ (which is a cobordism equipped with an $X$-framing) properly embedded in $[0,1]\times M$, from $\{0\}\times L$ to $\{1\}\times L'$ that induces the $X$–framings of $L$ and $L'$.

**Theorem 2.6.** *Let $X$ be a section of $UM$. Two sections $Y$ and $Z$ of $UM$ transverse to $X$ are homotopic if and only if $(L_{Y=-X}, \sigma(Y,X))$ and $(L_{Z=-X}, \sigma(Z,X))$ are $X$-framed cobordant.*

PROOF: View a homotopy $Y_t$ from $Y = Y_0$ to $Z = Y_1$ as a section $Y_t$ of the pull-back of $UM$ under the natural projection from $[0,1] \times M$ to $M$, and assume without loss that $(Y_t, -X)$ is transverse to the diagonal of $S^2 \times S^2$ (with respect to some parallelization). Then the preimage $\Sigma$ of the diagonal is a cobordism from $L_{Y=-X}$ and $L_{Z=-X}$, which is canonically $X$–framed by an $X$-framing that induces those of $L_{Y=-X}$ and $L_{Z=-X}$. Conversely, an $X$–framed cobordism $\Sigma$ from $(L_{Y=-X}, \sigma(Y,X))$ to $(L_{Z=-X}, \sigma(Z,X))$ induces a section $C(X,\Sigma)$ of the pull-back of $UM$ under the natural projection from $[0,1] \times M$ to $M$, which is defined as $C(X,L,\sigma)$ in Definition 2.4, so that the restriction $C_t$ of $C(X,\Sigma)$ on $\{t\} \times M$ defines a homotopy from $D_0 = C(X, L_{Y=-X}, \sigma(Y,X))$ to $D_1 = C(X, L_{Z=-X}, \sigma(Z,X))$, and, according to Lemma 2.5, $Y$ and $Z$ are homotopic. ◇

**Remark 2.7.** In [14, 1.4], François Laudenbach proves a similar result for nowhere zero sections of a cotangent bundle of a manifold of arbitrary dimension. This result can easily be adapted to any other real bundle over a manifold of the same dimension. Again, I thank Patrick Massot for pointing out this reference to me.

**Corollary 2.8.** *Let $X$ be a section of $UM$. The Spin$^c$–structure of a section $Y$ of $UM$ transverse to $X$ is determined by $[X]^c$ and by the homology class $[L_{Y=-X}]$ of $L_{Y=-X}$ in $H_1(M;\mathbb{Z})$.*

◇

A parallelization $\tau$ with $X$ as first vector identifies $X$-framings of links with framings of links as follows: The second vector $X_2$ of $\tau$ is a section of $(-X)^{\perp}$, and $\tau$ identifies an $X$–framing $[\sigma] \in S(NL, (-X)^{\perp})$ represented by $\sigma$ with the isotopy class of parallels $L_{\parallel}$ of $L$ induced by the section $\sigma^{-1}(X_2)$. Set

$$C(\tau, L, L_{\parallel}) = C(X, L, \sigma).$$

Similarly, a parallelization $\tau$ with $X$ as first vector identifies $X$-framings of cobordisms with framings of cobordisms.

This allows us to state the following Pontrjagin theorem [22, Section 7, Theorem B] as a corollary of Lemma 2.5 and Theorem 2.6.

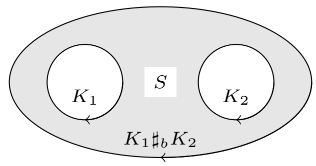
**Theorem 2.9** (Pontrjagin construction)**.** *Let $\tau$ be a parallelization of $M$. Any section of $UM$ is homotopic to $C(\tau, L, L_{\parallel})$ for a framed link $(L, L_{\parallel})$ of the interior of $M$. Two sections $C(\tau, L, L_{\parallel})$ and $C(\tau, L', L'_{\parallel})$ are homotopic if and only if $(L, L_{\parallel})$ and $(L', L'_{\parallel})$ are framed cobordant.*

◇

Pontrjagin proved generalizations of this theorem to every dimension. See [22, Section 7].

Let $\Sigma_M$ be an embedded cobordism from a link $L$ to a link $L_1$ in $M$. The graph of a Morse function $f$ from $\Sigma_M$ to $[0,1]$ such that $f^{-1}(0) = L$ and $f^{-1}(1) = L_1$ yields a proper embedding $\Sigma$ of $\Sigma_M$ into $[0,1] \times M$. The positive normal to $\Sigma_M$ in $M$ at $m$ seen in $T_{(f(m),m)}\{f(m)\} \times M$ frames $\Sigma$. This framing of $\Sigma$ identifies the $X$-framings of $\Sigma$ with homotopy classes of unit sections of $(-X)^{\perp}$ over $\Sigma$. When $\Sigma_M$ is connected, and when $K$ is a boundary component of $\Sigma$, any $X$-framing defined on $\partial\Sigma \setminus K$ extends as an $X$-framing of $\Sigma$, and the extension of the $X$-framing over $K$ is determined by the restriction of the $X$-framing to $\partial\Sigma \setminus K$.

Embed a sphere $S$ with three holes in $M$, the 3 boundary components of $S$ are 3 knots $K_1$, $K_2$ and $-K_1\sharp_b K_2$ of $M$, which are framed by the embedding of $S$.




Then $K_1\sharp_b K_2$ is a *framed band sum* of $K_1$ and $K_2$, it is framed cobordant to the union of $K_1$ and $K_2$. Note that any $X$–framed link is $X$–framed cobordant to an $X$–framed knot by such band sums. Similarly, any framed link is framed cobordant to a framed knot.

**Lemma 2.10.** *Two framed links $(L, L_\parallel)$ and $(L', L'_\parallel)$ in a rational homology 3-sphere or in a rational homology 3-ball are framed cobordant if and only if the homology classes of $L$ and $L'$ in $H_1(M;\mathbb{Z})$ coincide and $lk(L, L_\parallel) = lk(L', L'_\parallel)$.*

Proof: When the framed links are framed cobordant, they are homologous and $lk(L, L_\parallel) = lk(L', L'_\parallel)$, since $lk(L, L_\parallel)$ is the algebraic intersection of two 2-chains bounded by $L\times\{0\}$ and $L_\parallel \times \{0\}$ in $[-1, 0]\times M$. Conversely, let $(L, L_\parallel)$ and $(L', L'_\parallel)$ be two framed links such that $L$ and $L'$ are homologous and $lk(L, L_\parallel) = lk(L', L'_\parallel)$. It is well-known that homologous links in a 3-manifold are cobordant. (A proof using triangulations of 3-manifolds, which exist according to [2], can be found in [6, p.65, Corollaire]). The links $L$ and $L'$ are respectively framed cobordant to framed knots $(K, K_\parallel)$ and $(K', K'_\parallel)$ such that $lk(K, K_\parallel) = lk(L, L_\parallel)$ and $lk(K', K'_\parallel) = lk(L', L'_\parallel)$, so that $lk(K, K_\parallel) = lk(K', K'_\parallel)$. There is a connected cobordism from $K$ to $K'$ that may be equipped with a framing that extends the framing induced by $K_\parallel$, and which therefore induces a framing of $K'$ corresponding to a parallel $K'_1$ of $K'$ such that $lk(K, K_\parallel) = lk(K', K'_1)$. Thus $lk(K', K'_1) = lk(K', K'_\parallel)$ and $K'_1$ is isotopic to $K'_\parallel$ in $N(K') \setminus K'$, so that $(K', K'_\parallel)$ is framed cobordant to $(K, K_\parallel)$. $\diamond$

2.2. **Spin$^c$-structures and Euler classes.** Let $P\colon H^2(M, \partial M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ denote the Poincaré duality isomorphism. In this subsection, we show that the Spin$^c$-structures of $M$ form an affine space $\mathcal{S}(M)$ with translation group $H_1(M;\mathbb{Z})$, such that, for any two transverse sections $X$ and $Y$ of $UM$, the difference $([X]^c - [Y]^c) \in H_1(M;\mathbb{Z})$ is the class of $L_{X=-Y}$ in $H_1(M;\mathbb{Z})$.

We also study the Euler class, and prove that it satisfies

$$P(e(X^\perp)) = [X]^c - [-X]^c$$

so that

- for two transverse homotopic sections $X$ and $Y$ of $UM$, $P(e(X^\perp))$ is the class of $L_{X=Y}$ in $H_1(M;\mathbb{Z})$, and
- for two combings $[X]$ and $[Y]$,

$$P(e(X^\perp) - e(Y^\perp)) = 2([X]^c - [Y]^c).$$

**Lemma 2.11.** *For three pairwise transverse sections $X$, $Y$ and $Z$ of $UM$,*

$$[L_{Z=-X}] = [L_{Z=-Y}] + [L_{Y=-X}]$$

*in* $H_1(M;\mathbb{Z})$.

Proof: For two sections $X$ and $Z$ of $UM$, transverse to $Y$, up to homotopy, we can assume that $L_{X=-Y}$ and $L_{Z=-Y}$ are disjoint, and pick disjoint tubular neighborhoods $N(L_{X=-Y})$ and $N(L_{Z=-Y})$ of $L_{X=-Y}$ and $L_{Z=-Y}$, respectively. Then, according to Lemmas 2.1 and 2.5 we can assume that $Z = C(Y, L_{Z=-Y}, \sigma(Z,Y))$ and that $X = C(Y, L_{X=-Y}, \sigma(X,Y))$ so that $Z = Y$ outside $N(L_{Z=-Y})$ and $X = Y$ outside $N(L_{X=-Y})$. Then $L_{Z=-X} = L_{Z=-Y} \coprod L_{Y=-X}$. $\diamond$

**Lemma 2.12.** *There is a canonical free transitive action of* $H_1(M;\mathbb{Z})$ *on the set* $\mathcal{S}(M)$ *of* Spin$^c$*-structures of* $M$ *such that for any two transverse sections* $Y$ *and* $Z$ *of* $UM$,

$$[L_{Z=-Y}][Y]^c = [Z]^c.$$

Proof: Let $Y$ be a section of $UM$ and let $[K] \in H_1(M;\mathbb{Z})$. Represent $[K]$ by a knot $K$ and equip $K$ with an arbitrary $Y$-framing $\sigma$.

Define $[K][Y]^c$ as $[Z]^c$ with $Z = C(Y, K, \sigma)$. According to Definition 2.4, $K = L_{Z=-Y}$ and, according to Corollary 2.8, $[Z]^c$ is determined by $[Y]^c$ and $[K]$. According to Lemma 2.1, if $[K][Y]^c = [Z]^c$, then $K$ is homologous to $L_{Z=-Y}$. Lemma 2.11 ensures that this defines an action of $H_1(M;\mathbb{Z})$. This action is obviously transitive since $[Z]^c = [L_{Z=-Y}][Y]^c$ and it is free. $\diamond$

**Corollary 2.13.** *This action equips* $\mathcal{S}(M)$ *with an affine structure with translation group* $H_1(M;\mathbb{Z})$. *With respect to this structure, for any two transverse sections* $X$ *and* $Y$ *of* $UM$,

$$[Y]^c - [X]^c = [L_{Y=-X}].$$

**Remark 2.14.** Classically, $\mathcal{S}(M)$ is rather equipped with an affine structure with translation group $H^2(M, \partial M; \pi_2(S^2) = \mathbb{Z})$. For two Spin$^c$-structures $[X]^c$ and $[Y]^c$, the element $([Y]^c - [X]^c)_2$ of $H^2(M, \partial M;\mathbb{Z})$ such that $[Y]^c = ([Y]^c - [X]^c)_2.[X]^c$ with respect to the $H^2(M, \partial M;\mathbb{Z})$-affine structure is the following obstruction to homotoping a section $Y$ of $UM$ that represents $[Y]^c$ to a section $X$ that represents $[X]^c$ over a two-skeleton of $M$. Fix a triangulation of $M$ and homotop $Y$ to a section $Y'$ that coincides with $X$ over the one-skeleton of $M$. Then $([Y]^c - [X]^c)_2$ is represented by a 2-cochain $G$ that maps each 2-cell $\Delta$ of $M$ to the degree $G(\Delta)$ of $Y'$ regarded as a map from $(\Delta, \partial\Delta)$ to $(S^2, X)$ with respect to a trivialization of $TM$ over $\Delta$ with $X$ as first vector.

Below, we confirm that the two structures are naturally related by Poincaré's duality, by proving that, for two transverse sections $X$ and $Y$ of $UM$,

$$P(([Y]^c - [X]^c)_2) = [Y]^c - [X]^c = [L_{Y=-X}].$$

Indeed, up to homotopy, there is no loss in assuming $Y = C(X, L_{Y=-X}, \sigma(Y,X))$ as in Lemma 2.5, for an $L_{Y=-X}$ that is transverse to the triangulation of $M$. Choose a tubular neighborhood $N(L_{Y=-X})$ that does not meet the one-skeleton of $M$. Then the cochain $G$ maps a 2-cell $\Delta$ to its algebraic intersection with $L_{Y=-X}$ so that $[L_{Y=-X}]$ is Poincaré dual to $([Y]^c - [X]^c)_2$.

The *Euler class* $e(X^\perp)$ is an obstruction to the existence of a nowhere zero section of $X^\perp$. It lives in $H^2(M;\mathbb{Z})$ and $X$ extends as a parallelization if and only if $e(X^\perp) = 0$. We will not give a more precise definition for the standard Euler class, since Lemma 2.15 below can be used as a definition in our case.

**Lemma 2.15.** *Let $X$ and $Y$ be two homotopic transverse sections of $UM$, then $[L_{Y=X}]$ is Poincaré dual to $e(X^\perp)$. Therefore, $P(e(X^\perp)) = [X]^c - [-X]^c$.*

PROOF: For a section of $X^\perp$, $X$ may be pushed slightly in the direction of the section. If $Y$ denotes the obtained combing, then $L_{Y=X}$ is the vanishing locus of the section, which is Poincaré dual to $e(X^\perp)$. ⋄

**Lemma 2.16.** *Let $X$ and $Y$ be two transverse sections of $UM$,*

$$2[L_{X=Y}] = P(e(X^\perp) + e(Y^\perp))$$

*and $2[L_{X=-Y}] = P(e(X^\perp) - e(Y^\perp))$. In particular, for two transverse torsion sections $X$ and $Y$ of $UM$, $L_{X=Y}$ and $L_{X=-Y}$ represent torsion elements in $H_1(M;\mathbb{Z})$.*

PROOF: $[L_{X=Y}] = [X]^c - [-Y]^c = [Y]^c - [-X]^c$ so that $2([L_{X=Y}]) = [X]^c - [-X]^c + [Y]^c - [-Y]^c = P(e(X^\perp) + e(Y^\perp))$. ⋄

**Lemma 2.17.** *Let $X$ and $Y$ be two transverse torsion sections of $UM$, then $lk(L_{X=Y}, L_{X=-Y})$ only depends on the homotopy classes of $X$ and $Y$.*

PROOF: Fix a trivialization of $UM$ so that sections become functions from $M$ to $S^2$. Let us prove that $lk(L_{X=Y}, L_{X=-Y})$ does not vary under a generic homotopy of $X$. Such a homotopy induces two homotopies $h_+$ and $h_-$ from $[0,1] \times M$ to $S^2 \times S^2$ where $h_\pm(t,m) = (X_t(m), \pm Y(m))$. Without loss, assume that $h_+$ and $h_-$ are transverse to the diagonal. There exists a finite sequence $0 = t_0 < t_1 < t_2 < \cdots < t_k = 1$ of times such that the projections on $M$ of the preimages of the diagonal under $h_{+|[t_i,t_{i+1}]\times M}$ and $h_{-|[t_i,t_{i+1}]\times M}$ are disjoint so that they yield two disjoint cobordisms in $M$, one from $L_{X_{t_i}=Y}$ to $L_{X_{t_{i+1}}=Y}$, and the other one from $L_{X_{t_i}=-Y}$ to $L_{X_{t_{i+1}}=-Y}$ showing that $lk(L_{X_{t_i}=Y}, L_{X_{t_i}=-Y}) = lk(L_{X_{t_{i+1}}=Y}, L_{X_{t_{i+1}}=-Y})$. ⋄

**Lemma 2.18.** *Let $X$ be a section of $UM$ that extends to a parallelization $\tau$. The homotopy class of a torsion section $Y$ of $UM$ transverse to $X$ is determined by $X$, by the homology class $[L_{Y=-X}]$ of $L_{Y=-X}$ in $H_1(M;\mathbb{Z})$, and by the linking number $lk(L_{Y=-X}, L_{Y=X})$.*

PROOF: After a homotopy, $Y$ reads $C(\tau, L_{Y=-X}, L_{Y=X_2})$ where $X_2$ is the second vector of $\tau$, and, $L_{Y=X}$ and $L_{Y=X_2}$ are parallel knots as in Theorem 2.9. According to Theorem 2.9, the combing $[Y]$ is determined by the framed cobordism class of $L_{Y=-X}$, which is determined by $[L_{Y=-X}]$ and by $lk(L_{Y=-X}, L_{Y=X_2})$, by Lemma 2.10, since $L_{Y=-X}$ is rationally null-homologous. After another homotopy that makes $Y$ transverse to $X_2$ and $X$, $lk(L_{Y=-X}, L_{Y=X_2}) = lk(L_{Y=-X}, L_{Y=X})$. ⋄

### 2.3. Action of $\pi_3(S^2)$ on combings.

**Notation 2.19.** Regard $B^3$ as the quotient of $[0,2\pi] \times S^2$ where the quotient map identifies $\{0\} \times S^2$ with a point. Then the map from $B^3$ to the group $SO(3)$ of orientation-preserving linear isometries of $\mathbb{R}^3$ that maps $(\theta \in [0,2\pi], x \in S^2)$ to the rotation $\rho(\theta,x)$ with axis directed by $x$ and with angle $\theta$ is denoted by $\rho$. It induces the standard double covering map $\tilde{\rho}$ from $S^3 = B^3/\partial B^3$ to $SO(3)$, which orients $SO(3)$, and identifies $\pi_3(S^3)$ with $\pi_3(SO(3)) = \mathbb{Z}[\tilde{\rho}]$.

The map $p_{S^2}\colon SO(3) \to S^2$ that maps an element $\phi$ of $SO(3)$ to the image of the first basis vector under $\phi$ is a fibration of fiber $SO(2) \cong S^1$. It is easy to prove that $p_{S^2}$ induces an isomorphism from $\pi_3(SO(3))$ to $\pi_3(S^2)$, with the associated long exact sequence. Let $\gamma$ be the image of $[\tilde{\rho}]$ under this isomorphism.

$$\pi_3(S^2) = \mathbb{Z}\gamma.$$

**Remark 2.20.** The group $\pi_3(S^2)$ can also be computed from the long exact sequence of the Hopf fibration $\Phi_H\colon S^3 \to S^2$, which maps an element $(z_1, z_2)$ of the unit sphere $S^3$ of $\mathbb{C}^2$ to its class $z_1/z_2$ in the complex projective line $\mathbb{C}P^1 = S^2$. For a generic smooth map $g$ from $S^3$ to $S^2$, let $lk(g^{-1}(a), g^{-1}(b))$ be the linking number of the preimages of two regular points $a$ and $b$ of $S^2$ with respect to $g$. Then $[g] = lk(g^{-1}(a), g^{-1}(b))[\Phi_H]$ in $\pi_3(S^2)$. Lemma 2.21 below shows that $[\gamma] = -[\Phi_H]$.

Let $X$ be a combing. Extend $X$ to a parallelization $(X, Y, Z)$ on a 3-ball $B$ identified with $B^3$, and see $\rho$ as a map $\rho\colon (B, \partial B) \to (SO(X, Y, Z), \mathrm{Id})$. Define $\gamma^k X$ as the section that coincides with $X$ outside $B$ and such that, for any $m \in B$,

$$\gamma^k X(m) = (\rho(m))^k(X).$$

Note that $[\gamma^k X]$ is independent of the chosen parallelization. Since $M$ is connected, any two small enough balls may be put inside a bigger one and $[\gamma^k X]$ is independent of $B$. Set $\gamma^k[X] = [\gamma^k X]$. Note that $\gamma^{k+k'}[X] = \gamma^k(\gamma^{k'}[X])$. Let $X$ and $Y$ be two sections of $UM$ that are homotopic except over a 3-ball $B^3$. Up to homotopy, we may assume that they are identical outside $B^3$. On $B^3$, $X$ extends to a parallelization and $Y$ reads as a map from $(B^3, \partial B^3)$ to $(S^2, X)$. It therefore defines an element $\gamma^k$ of $\pi_3(S^2)$, and $[Y] = \gamma^k[X]$. Thus, $\pi_3(S^2)$ acts transitively on the combings that represent a given Spin$^c$-structure. In particular, it acts transitively on the combings of an integer homology 3-sphere.

A *positive (or oriented) meridian* of some knot $K$ in $M$ is the boundary of a disk that intersects $K$ once with positive sign.

**Lemma 2.21.** *Let $\tau$ be a parallelization of $M$ and let $[X(\tau)]$ denote the induced combing. Let $(U, U_-)$ be the negative Hopf link*  *($lk(U, U_-) = -1$). Then, with the notation before Theorem 2.9, $[\gamma X(\tau)] = [C(\tau, U, U_-)]$.*

Proof: First note that $[C(\tau, U, U_-)]$ reads $[\gamma^k X(\tau)]$ for an integer $k$ that does not depend on $(M, \tau)$. We prove $k = 1$ when $M = B^3$, when $\tau$ is the standard parallelization, and when $X = X(\tau)$ is the constant upward vector field, with the help of Lemma 2.18, by showing that

$$lk(L_{\gamma X(\tau) = X'}, L_{\gamma X(\tau) = -X'}) = lk(U, U_-) = -1.$$

for some constant field $X'$ near $X$. Let $N$ be the North pole of $S^2$, $(p_{S^2} \circ \rho)^{-1}(N)$ intersects the interior of $B^3$ as the vertical axis oriented from South to North while $(p_{S^2} \circ \rho)^{-1}(-N)$ intersects $B^3$ as $\pi \times (-E)$, where $E$ is the equator oriented as a positive meridian of $(p_{S^2} \circ \rho)^{-1}(N)$. Then for $N'$ near $N$, $lk((p_{S^2} \circ \rho)^{-1}(N'), (p_{S^2} \circ \rho)^{-1}(-N')) = -1$. $\diamond$

**Corollary 2.22.** *Let $\tau$ be a parallelization of $M$, let $(L, L_\|)$ be a framed link of $L$, let $(U, U_-)$ be the negative Hopf link in a ball of $M$ disjoint from $L$, and let $(U, U^+)$ be the positive Hopf link*  *in a ball of $M$ disjoint from $L$. Then*

$[\gamma C(\tau, L, L_{\|})] = [C(\tau, L \cup U, L_{\|} \cup U_-)]$ *and*

$$[\gamma^{-1} C(\tau, L, L_{\|})] = [C(\tau, L \cup U, L_{\|} \cup U_+)]. \tag{2.23}$$

*If $L$ is non-empty, for an integer $r$, let $L_{\|,r}$ be a parallel of $L$ obtained from $L_{\|}$ by adding $r$ meridians of $L$, homologically in $N(L) \setminus L$, then*

$$[\gamma^{-r} C(\tau, L, L_{\|})] = [C(\tau, L, L_{\|,r})].$$

PROOF: Note that $(L, L_{\|,\pm 1})$ is framed cobordant to $(L \cup U, L_{\|} \cup U_{\pm})$ by band sum. Thus, Formula 2.23 can be deduced from the fact that the disjoint union of two oppositely framed unknots is framed cobordant to the empty link. ◇

**Corollary 2.24.** *Let $X$ be a torsion section of $UM$, let $k \in \mathbb{Z}$ and let $Y$ be a section of $UM$ that represents $[\gamma^k X]$. Then $lk(L_{Y=X}, L_{Y=-X}) = -k$.*

PROOF: We already know that the linking number $lk(L_{Y=X}, L_{Y=-X})$ does not depend on the transverse representatives of $[X]$ and $[Y]$. Furthermore, by Theorem 2.9, $[X]$ can be represented as $C(\tau, L, L_{\|})$ as in Corollary 2.22. Assume $k \neq 0$. Let $(\cup_{i=1}^{|k|} U^{(i)}, \cup_{i=1}^{|k|} U_{\varepsilon}^{(i)})$ denote the union of $|k|$ Hopf links with sign $\varepsilon = -k/|k|$ contained in disjoint balls $B_i$ disjoint from $N(L)$, for $i = 1, \dots, k$. Let $Y$ be obtained from $C(\tau, L \cup \cup_{i=1}^{|k|} U^{(i)}, L_{\|} \cup \cup_{i=1}^{|k|} U_{\varepsilon}^{(i)})$ by a small perturbation, induced by the parallelization $\tau$ outside $N(L \cup \cup_{i=1}^{|k|} U^{(i)})$ so that it is transverse to $X$, very close to $X$, and distinct from $\pm X$ outside $N(L \cup (\cup_{i=1}^{|k|} U^{(i)}))$. Then $L_{Y=-X}$ is a parallel of $\cup_{i=1}^{|k|} U^{(i)}$ and

$$lk(L_{Y=X}, L_{Y=-X}) = \sum_{i=1}^{|k|} lk(L_{Y=X} \cap B_i, L_{Y=-X} \cap B_i) = \sum_{i=1}^{|k|} lk(U^{(i)}, U_{\varepsilon}^{(i)}) = -k.$$

◇

**Proposition 2.25.** *Let $[X]^c$ be a Spin$^c$ structure. The Euler class $e(X^{\perp}) \in H^2(M, \partial M; \mathbb{Z})$ maps $H_2(M, \partial M; \mathbb{Z})$ onto $E([X]^c)\mathbb{Z}$ for some integer $E([X]^c)$. Then the set $\mathcal{C}([X]^c)$ of combings with underlying Spin$^c$-structure $[X]^c$ is an affine space over $\mathbb{Z}/E([X]^c)\mathbb{Z}$, where the translation by the class of $1$ is the action of $\gamma$.*

PROOF: Again, fix a parallelization $\tau$ of $M$, and an induced combing $Y$. This identifies the set $\mathcal{S}(M)$ of Spin$^c$ structures with $H_1(M; \mathbb{Z})$ by mapping $[X]^c$ to the homology class $[L_{X=Y}]$. Let $\xi(\tau, [K])$ be the Spin$^c$ structure corresponding to a given class $[K]$ of $H_1(M; \mathbb{Z})$. The Pontrjagin characterization of the combings (Theorem 2.9) identifies the set $\mathcal{C}(\xi(\tau, [K]))$ with the set of framed links homologous to $[K]$ up to framed cobordism. Let $K$ be a knot that represents $[K]$, and let $K_{\|}$ be a parallel of $K$ then $\mathcal{C}(\xi(\tau, [K]))$ is the set of framed links $(K, K_{\|,r})$, for all $r \in \mathbb{Z}$ modulo framed cobordism. According to Corollary 2.22, $[\gamma^{-r} C(\tau, K, K_{\|,s})] = [C(\tau, K, K_{\|,s+r})]$.

When $[K]$ is a torsion element of $H_1(M; \mathbb{Z})$, the self-linking number $lk(K_{\|,r}, K) = lk(K_{\|}, K) + r$ makes sense, and it is a complete invariant of framings of $K$, up to framed cobordism. This shows that the action of $\pi_3(S^2)$ on the set of combings that belong to a torsion Spin$^c$-structure is free, and that this set is an affine space over $\mathbb{Z}$.

In general, for two parallels $K_{\|,r}$ and $K_{\|,s}$ of $K$ in a tubular neighborhood $N(K)$ of $K$, set $lk_{N(K)}(K_{\|,s} - K_{\|,r}, K) = s - r$. The homology class of $(K_{\|,s} - K_{\|,r})$ in

$N(K)\setminus K$ reads $lk_{N(K)}(K_{\|,s}-K_{\|,r},K)[m(K)]$ where $m(K)$ is the oriented meridian of $K$. Let $B$ be a cobordism from $0\times K_{\|,r}$ to $1\times K_{\|,s}$ in $[0,1]\times N(K)$. Then $lk_{N(K)}(K_{\|,s}-K_{\|,r},K)=\langle[0,1]\times K,B\rangle_{[0,1]\times M}$.

Let $C$ be a framed cobordism from $0\times K$ to $1\times K$ in $[0,1]\times M$, and let $C'$ be obtained from $C$ by pushing $C$ in the direction of the framing. Assume that $\partial C'=1\times K_{\|,s}-0\times K_{\|,r}$ so that $C$ is a framed cobordism from $(K,K_{\|,r})$ to $(K,K_{\|,s})$ and

$$0=\langle C,C'\rangle_{[0,1]\times M}=\langle[0,1]\times K+(C-[0,1]\times K),B+(C'-B)\rangle_{[0,1]\times M}.$$

Since $(C-[0,1]\times K)$ and $(C'-B)$ are 2-cycles in $[0,1]\times M$, $\langle(C-[0,1]\times K),(C'-B)\rangle_{[0,1]\times M}=0$, and since they are homologous $\langle[0,1]\times K,(C'-B)\rangle_{[0,1]\times M}=\langle(C-[0,1]\times K),B\rangle_{[0,1]\times M}$, so that

$$lk_{N(K)}(K_{\|,s}-K_{\|,r},K)=-2\langle[0,1]\times K,(C'-B)\rangle_{[0,1]\times M}.$$

In particular, the framing difference $(s-r)$ induced by $C$ only depends on the homology class of the projection $S$ of $C$ in $M$, and it is $-2\langle K,S\rangle_M$. Thus if the framings induced by $K_{\|,r}$ and $K_{\|,s}$ are framed cobordant, then $(s-r)$ is in $\langle 2K,H_2(M;\mathbb{Z})\rangle_M$. Conversely, for any class $S$ of $H_2(M;\mathbb{Z})$, there exists an embedded connected cobordism $C$ that projects on $S$. Any framing on $0\times K$ can be extended to $C$, and it induces a framing on $1\times K$, such that the framing difference is $-2\langle K,S\rangle_M$. Since the Euler class of $\xi(\tau,[K])$ is Poincaré dual to $2[K]$, according to Lemma 2.16, the conclusion follows. $\diamond$

## 3. Towards the variation formula 1.3

3.1. **The key proposition.** In this subsection, which will be useful in our study of the invariant $\Theta$ in Section 5, we prove the following proposition, which is the key to the extension of the map $p_1$ in Section 4.

**Proposition 3.1.** *Let $X$, $Y$ and $Z$ be three pairwise transverse torsion sections of $UM$,*

$$lk(L_{X=Y},L_{X=-Y})+lk(L_{Y=Z},L_{Y=-Z})=lk(L_{X=Z},L_{X=-Z}).$$

The *first Betti number* of $M$, which is the dimension of $H_1(M;\mathbb{Q})$, is denoted by $\beta_1(M)$. Let $(S_i)_{i=1,\dots,\beta_1(M)}$ be $\beta_1(M)$ surfaces in the interior of $M$ that represent a basis of $H_2(M;\mathbb{Q})$. Consider the 6-manifold $[0,1]\times UM$. Recall that $UM$ is homeomorphic to $M\times S^2$. For a section $Z$ of $UM$, let $Z(S_i)$ denote the graph of the restriction of $Z$ to $S_i$ in $UM$. Let $[S]$ denote the homology class of the fiber of $UM$ in $H_2(UM;\mathbb{Q})$, oriented as the boundary of a unit ball of $T_xM$.

$$H_2(UM;\mathbb{Q})=\mathbb{Q}[S]\oplus\bigoplus_{i=1}^{\beta_1(M)}\mathbb{Q}[Z(S_i)].$$

**Lemma 3.2.** *If $Y$ and $Z$ are two transverse sections of $UM$, then*

$$[Z(S_i)]-[Y(S_i)]=\langle L_{Z=-Y},S_i\rangle_M[S]$$

*in $H_2(UM;\mathbb{Q})$ (and in $H_2([0,1]\times UM;\mathbb{Q})$).*

Proof: Fix a trivialization of $UM$ so that both $Y$ and $Z$ become functions from $M$ to $S^2$, then $[Z(S_i)]-[Y(S_i)]=(\deg(Z_{|S_i})-\deg(Y_{|S_i}))[S]$. If $X$ is a section of

$UM$ induced by the trivialization, then $\deg(Z_{|S_i}) = \langle L_{Z=-X}, S_i\rangle_M$. Conclude with Lemma 2.11. ⋄

In particular, according to Lemma 2.16, the subspace $H_T$ of $H_2([0,1]\times UM;\mathbb{Q})$ generated by the $[Z(S_i)]$ for torsion combings $Z$ is canonical. Set

$$H(M) = H_2([0,1]\times UM;\mathbb{Q})/H_T.$$

Then $H(M) = \mathbb{Q}[S]$.

Let $X$ and $Y$ be two sections of $UM$. Let $X(M)$ abusively denote the graph of $X$ in $UM$. Let $\partial(X,Y)$ be the following codimension 2 submanifold of $\partial([0,1]\times UM)$. If $\partial M=\emptyset$, $\partial(X,Y) = \{1\}\times Y(M) - \{0\}\times X(M)$. If $\partial M = S^2$, recall that $\tau_s$ identifies $UM_{|\partial M}$ with $S^2\times\partial M$; let $V(X)$ and $V(Y)$ be the elements of $S^2$ such that $X=V(X)$ and $Y=V(Y)$ on $\partial M$. Let $P=P(X,Y)$ be a 1–chain in $[0,1]\times S^2$ such that $\partial P = \{1\}\times V(Y) - \{0\}\times V(X)$. Then $\partial(X,Y) = \partial(X,Y,P) = \{1\}\times Y(M) - \{0\}\times X(M) - P\times\partial M$.

**Lemma 3.3.** *For two transverse sections $X$ and $Y$ of $UM$ such that $[L_{Y=-X}]$ vanishes in $H_1(M;\mathbb{Q})$, $\partial(X,Y)$ is rationally null-homologous in $[0,1]\times UM$. It bounds a rational chain $F(X,Y)$ transverse to $\partial([0,1]\times UM)$, which is well-determined, up to the addition of a chain $\Sigma\times\partial M$ for a 2–chain $\Sigma$ of $[0,1]\times S^2$, up to the addition of a combination of $\{t_i\}\times UM_{|S_i}$ for distinct $t_i$, and up to cobordism.*

Proof: $H_3([0,1]\times UM;\mathbb{Q}) \cong H_1(M;\mathbb{Q})\otimes H_2(S^2;\mathbb{Q})$ when $\partial M = S^2$. The direct factor $\mathbb{Q}[X(M)]$ should be added when $\partial M=\emptyset$. When $\partial M = S^2$, the class of a 3–submanifold of $[0,1]\times UM$ vanishes in $H_3([0,1]\times UM;\mathbb{Q})$ if and only if its algebraic intersection with the $[0,1]\times Z(S_i)$ vanishes, for all $i$, for some section $Z$ of $UM$. For $\partial(X,Y)$, this algebraic intersection reads

$$\begin{aligned}\langle [0,1]\times Z(S_i), \partial(X,Y)\rangle_{[0,1]\times UM} &= \langle S_i, L_{Z=Y} - L_{Z=X}\rangle_M\\ &= \langle S_i, [Z]^c - [-Y]^c - ([Z]^c - [-X]^c)\rangle_M\\ &= \langle S_i, L_{Y=-X}\rangle_M = 0.\end{aligned}$$

When $\partial M=\emptyset$, the algebraic intersection with $[0,1]\times UM_{|\{x\}}$ must vanish for some $x\in M$, too. This is easily verified. Thus, $\partial(X,Y)$ bounds a rational chain $F(X,Y)$, and since $H_4(UM;\mathbb{Q}) = \bigoplus_{i=1}^{\beta_1(M)}\mathbb{Q}[UM_{|S_i}]$, the second assertion follows. ⋄

**Lemma 3.4.** *For any two transverse torsion sections $X$ and $Y$ of $UM$, for any two-cycle $C$ of $[0,1]\times UM$, the class of $C$ in $H(M)$ is $\langle C, F(X,Y)\rangle_{[0,1]\times UM}[S]$ for a $F(X,Y)$ as in Lemma 3.3.*

Proof: First note that $\langle C, F(X,Y)\rangle_{[0,1]\times UM}[S]$ only depends on the homology class of $C$, for a given $F(X,Y)$, and that $\langle [S], F(X,Y)\rangle = 1$. Now, it suffices to prove that $\langle [Z(S_i)], F(X,Y)\rangle = 0$ for any torsion combing $Z$, and for any $i$. Since $\langle [Z(S_i)], F(X,Y)\rangle = \langle [Z(S_i)], X(M)\rangle_{UM} = \langle [Z(S_i)], Y(M)\rangle_{UM}$, $\langle [Z(S_i)], F(X,Y)\rangle$ does not depend on the torsion combings $X$ and $Y$. In particular,

$$\langle [Z(S_i)], F(X,Y)\rangle = \langle [Z(S_i)], F(-Z,-Z)\rangle = 0.$$

⋄

**Definition 3.5.** In this article, *blowing up* a submanifold $A$ means replacing it with its unit normal bundle. Let $c$ be the codimension of $A$. The total space of the normal bundle to $A$ locally reads $\mathbb{R}^c\times U$ for an open subspace $U$ of $A$. It embeds into the ambient manifold as a tubular neighborhood of $A$. Its fiber $\mathbb{R}^c$ reads

$\{0\}\cup(]0,\infty[\times S^{c-1})$ where the unit sphere $S^{c-1}$ of $\mathbb{R}^c$ is the fiber of the unit normal bundle to $A$. Then the blow-up replaces $(0 \in \mathbb{R}^c)$ with $S^{c-1}$ so that the blown-up manifold locally reads $([0,\infty[\times S^{c-1} \times U)$. (In particular, unlike the blow-ups in algebraic geometry, our differential blow-ups create boundaries.) Topologically, this blow-up amounts to removing an open tubular neighborhood of $A$ (thought of as infinitely small), but the process is canonical, so that the created boundary is the unit normal bundle to $A$ and there is a canonical projection from the blown-up manifold to the initial manifold.

For example, blowing up the origin $(0,0)$ in $\mathbb{R}^2$ replaces it with the circle of half-lines of $\mathbb{R}^2$ starting at $(0,0)$. Blowing up a codimension two submanifold $L$ with a trivial tubular neighborhood $\mathbb{R}^2\times L$ replaces $\mathbb{R}^2\times L$ with the product of the previous blow-up of $\mathbb{R}^2$ at $(0,0)$ by $L$.

**Proposition 3.6.** *Let $X$ and $Y$ be two transverse torsion sections of $UM$. For any $F(X,Y)$ and $F(-X,-Y)$ as in Lemma 3.3, such that the 1–chains $P(X,Y)$ and $P(-X,-Y)$ are disjoint, the class of $F(X,Y)\cap F(-X,-Y)$ in $H(M)$ is*

$$lk(L_{X=Y},L_{X=-Y})[S].$$

Proof: Let us first prove that the class of $F(X,Y)\cap F(-X,-Y)$ is well-determined in $H(M)$. When $F(X,Y)$ is changed to $F(X,Y)+(\Sigma\times\partial M)$ for a two-chain $\Sigma$ of $[0,1]\times S^2$ transverse to $P(-X,-Y)$, $(\Sigma\times\partial M)\cap F(-X,-Y)$ is added to $F(X,Y)\cap F(-X,-Y)$. Now, $(\Sigma\times\partial M)\cap F(-X,-Y)$ is a union of $\pm(t_j,V_j)\times\partial M$, which bounds since the parallelization $\tau_s$ extends to $M$. Thus, the class of $F(X,Y)\cap F(-X,-Y)$ in $H(M)$ in unchanged. Since the class of $\{t_i\}\times UM_{|S_i}\cap F(-X,-Y)$ is in $H_T$, the class of $F(X,Y)\cap F(-X,-Y)$ is well-determined in $H(M)$.

Now, we construct an explicit $F(X,Y)$ by using the homotopy of Lemma 2.5, which we recall. Assume $M$ is Riemannian. When $X(m)\neq -Y(m)$, there is a unique geodesic arc $[X(m),Y(m)]$ with length $(\ell\in[0,\pi[)$ from $X(m)$ to $Y(m)$. For $t\in[0,1]$, let $X_t(m)\in[X(m),Y(m)]$ be such that the length of $[X_0(m)=X(m),X_t(m)]$ is $t\ell$. This defines $X_t$ on $(M\setminus L_{X=-Y})$.

Observe that this definition naturally extends to the boundary of the manifold $B\ell(M,L_{X=-Y})$ obtained from $M$ by blowing up $L_{X=-Y}$: Indeed, $X$ induces an orientation-preserving isomorphism from the normal bundle $N_xL_{X=-Y}$ to $L_{X=-Y}$ in $M$ at $x$ to $(-Y(x))^{\perp}$. Then for a unit element $n$ of $N_xL_{X=-Y}$, $X_t(n)$ describes the half great circle from $X(x)$ to $Y(x)$ through the image of $n$ under the above map. In particular, the whole sphere is covered with degree 1 by the image of $([0,1]\times(N_xL_{X=-Y}/\mathbb{R}^{*+}))$. Let $G_h$ be the closure of $\left(\cup_{t\in[0,1]}X_t\left(M\setminus L_{X=-Y}\right)\right)$.

$$G_h=\cup_{t\in[0,1]}X_t(B\ell(M,L_{X=-Y})).$$

Define the 3–cycle of $UM$

$$p(\partial(X,Y))=Y(M)-X(M)-[V(X),V(Y)]\times\partial M$$

where $[V(X),V(Y)]$ is the shortest geodesic path from $V(X)$ to $V(Y)$ in the fiber of $UM$ over $\partial M$, which is identified with $S^2$ by $\tau_s$. Then

$$\partial G_h-p(\partial(X,Y))=\cup_{t\in[0,1]}X_t(-(\partial B\ell(M,L_{X=-Y})\setminus\partial M))=UM_{|L_{X=-Y}}$$

because it is oriented like $\cup_{t\in[0,1]}X_t(\partial N(L_{X=-Y}))$. Let $\Sigma_{X=-Y}$ be a two-chain transverse to $L_{X=Y}$ and bounded by $L_{X=-Y}$ in $M$. Set $G=G_h-\left(UM_{|\Sigma_{X=-Y}}\right)$

so that $\partial G = p(\partial(X,Y))$. Let $\iota$ be the endomorphism of $UM$ over $M$ that maps a unit vector to the opposite one. Set

$$F(X,Y) = [0,1/3] \times X(M) + \{1/3\} \times G + [1/3,1] \times Y(M)$$

and

$$F(-X,-Y) = [0,2/3] \times (-X)(M) + \{2/3\} \times \iota(G) + [2/3,1] \times (-Y)(M).$$

Then $F(X,Y) \cap F(-X,-Y)$ reads

$$[1/3,2/3] \times Y(L_{Y=-X}) - \{1/3\} \times (-X)(\Sigma_{X=-Y}) + \{2/3\} \times (Y)(\Sigma_{X=-Y}).$$

Using Lemma 3.4 with $F(X,X) = [0,1] \times X(M)$ to evaluate the class of $(F(X,Y) \cap F(-X,-Y))$ in $H(M)$ finishes the proof. $\diamond$

Proof of Proposition 3.1: Compute $lk(L_{X=Z}, L_{X=-Z})$ by computing the class of $F(X,Z) \cap F(-X,-Z)$ in $H(M)$ where $F(X,Z)$ (resp. $F(-X,-Z)$) is constructed by gluing shrinked copies of $F(X,Y)$ (resp. $F(-X,-Y)$) and $F(Y,Z)$ (resp. $F(-Y,-Z)$) so that $[F(X,Z) \cap F(-X,-Z)] = [F(X,Y) \cap F(-X,-Y)] + [F(Y,Z) \cap F(-Y,-Z)]$. $\diamond$

3.2. **Proof of Theorem 1.1.** The first assertion of Theorem 1.1 follows from Lemma 2.1 and Corollary 2.8.

The second and third assertions follow from Lemma 2.16. If $X$ and $Y$ are two transverse torsion sections of $UM$, then $lk(L_{Y=X}, L_{Y=-X})$ only depends on the combings $[X]$ and $[Y]$ according to Lemma 2.17 (or to Proposition 3.6).

Now, assume that $Y'$ is another torsion section of $UM$ such that $L_{Y'=-X}$ is homologous to $L_{Y=-X}$. Then $Y$ and $Y'$ represent the same Spin$^c$-structure and there exists $k \in \mathbb{Z}$ such that $Y'$ represents $[\gamma^k Y]$. According to Corollary 2.24, $lk(L_{Y'=Y}, L_{Y'=-Y}) = -k$. According to Proposition 3.1,

$$lk(L_{Y'=X}, L_{Y'=-X}) - lk(L_{Y=X}, L_{Y=-X}) = lk(L_{Y'=Y}, L_{Y'=-Y}).$$

Thus if $lk(L_{Y'=X}, L_{Y'=-X}) = lk(L_{Y=X}, L_{Y=-X})$, $k=0$, and $Y$ and $Y'$ are homotopic. $\diamond$

## 4. On the map $p_1$

4.1. **The original map $p_1$ for parallelizations.** Let $M$ be equipped with a parallelization $\tau_M \colon M \times \mathbb{R}^3 \to TM$. Let $GL^+(\mathbb{R}^3)$ denote the group of orientation-preserving linear isomorphisms of $\mathbb{R}^3$.

Let $C^0((M,\partial M),(GL^+(\mathbb{R}^3),\mathrm{Id}))$ denote the set of maps

$$g : (M,\partial M) \longrightarrow (GL^+(\mathbb{R}^3),\mathrm{Id})$$

from $M$ to $GL^+(\mathbb{R}^3)$ that send $\partial M$ to the identity Id of $GL^+(\mathbb{R}^3)$.

Let $[(M,\partial M),(GL^+(\mathbb{R}^3),\mathrm{Id})]$ denote the group of homotopy classes of such maps, with the group structure induced by the multiplication of maps using the multiplication in $GL^+(\mathbb{R}^3)$.

For a map $g$ in $C^0((M,\partial M),(GL^+(\mathbb{R}^3),\mathrm{Id}))$, define

$$\begin{array}{rccc} \psi(g): & M \times \mathbb{R}^3 & \longrightarrow & M \times \mathbb{R}^3 \\ & (x,y) & \mapsto & (x, g(x)(y)). \end{array}$$

Then any parallelization $\tau$ of $M$ that coincides with $\tau_M$ on $\partial M$ reads

$$\tau = \tau_M \circ \psi(g)$$

for some $g \in C^0((M, \partial M), (GL^+(\mathbb{R}^3), \mathrm{Id}))$. Thus fixing $\tau_M$ identifies the set of homotopy classes of parallelizations of $M$ fixed on $\partial M$ with $[(M, \partial M), (GL^+(\mathbb{R}^3), \mathrm{Id})]$. Since $GL^+(\mathbb{R}^3)$ deformation retracts onto $SO(3)$, the group

$$[(M, \partial M), (GL^+(\mathbb{R}^3), \mathrm{Id})]$$

is isomorphic to $[(M, \partial M), (SO(3), \mathrm{Id})]$. A map $f$ from $(M, \partial M)$ to $(SO(3), \mathrm{Id})$ has an integral *degree* deg(f), which is the differential degree of $f$ at a regular point of $SO(3) \setminus \mathrm{Id}$ when $f$ is smooth.

A proof of the following long-known proposition can be found in [20, Lemmas 6.1, 6.5, 6.6].

**Proposition 4.1.** *For any compact connected oriented* 3*-manifold* $M$*, the group* $[(M, \partial M), (SO(3), Id)]$ *is abelian, and the degree*

$$deg\colon [(M, \partial M), (SO(3), Id)] \longrightarrow \mathbb{Z}$$

*is a group homomorphism, which induces an isomorphism*

$$deg\colon [(M, \partial M), (SO(3), Id)] \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}.$$

Recall that, when $n \geq 2$, $\pi_i(SU(n)) = \{1\}$ when $0 \leq i < 3$ and $\pi_3(SU(n)) = \mathbb{Z}$. Indeed, $SU(2)$ is homeomorphic to $S^3$, and, for any $n \geq 2$, and for any $i \leq 3$ the inclusion $SU(n) \hookrightarrow SU(n+1)$ induces an isomorphism from $\pi_i(SU(n))$ to $\pi_i(SU(n+1))$ thanks to the long exact sequence of the fibration $SU(n+1) \to S^{2n+1}$, which maps an element of $SU(n+1)$ to the image of the first basis vector.

Let $W$ be a connected, compact 4–dimensional manifold whose boundary is

$$\partial W = \begin{cases} M \cup_{1 \times \partial M} (-[0,1] \times S^2) \cup_{0 \times S^2} (-B^3) & \text{when } \partial M = S^2 \\ M & \text{when } \partial M = \emptyset. \end{cases}$$

When $\partial M = S^2$, $W$ has boundary and corners (that could be more accurately called ridges, here), it is identified with an open subspace of one of the products $[0,1[ \times B^3$ or $]0,1] \times M$ near $\partial W$. For any parallelization $\tau$ of $M$, the tangent vector $T_t[0,1]$ to $[0,1]$, the standard parallelization $\tau_s$ of $\mathbb{R}^3$ and $\tau$ together induce a trivialization $\tau(\partial W, \tau)$ of $TW$ over $\partial W$, this trivialization reads $T_t[0,1] \oplus \tau_s$ or $T_t[0,1] \oplus \tau$. The relative Pontrjagin class $p_1(W, \tau(\partial W, \tau))$ is the obstruction to extending the trivialization $\tau(\partial W, \tau) \otimes \mathbb{C}$ of $TW_{|\partial W} \otimes \mathbb{C}$ across $W$ (with respect to the trivialization of $det(TW)$ induced by the orientation of $W$). It lives in the module $H^4(W, \partial W; \pi_3(SU(4)))$, which is isomorphic to $\mathbb{Z}$ since $\pi_3(SU(4)) = \mathbb{Z}$. Under canonical identifications specified in [16, Section 1.5], the relative Pontrjagin class is an integral number. Recall that the *signature* of $W$ is the signature of the intersection form on $H_2(W; \mathbb{R})$, which is the number of positive entries minus the number of negative entries in a diagonal matrix of this form. The *Pontrjagin number* $p_1(\tau)$ of $\tau$ is defined by

$$p_1(\tau) = p_1(W, \tau(\partial W, \tau)) - 3\,\mathrm{signature}(W).$$

For more details, see [16, Section 1.5] or [20, Proposition 6.13] where the following classical theorem is proved.

**Theorem 4.2.** *Let $M$ be a compact connected oriented 3-manifold such that $\partial M = \emptyset$ or $S^2$. For any map $g$ in $C^0((M,\partial M),(SO(3),Id))$, for any trivialization $\tau$ of $TM$*

$$p_1(\tau \circ \psi(g)) - p_1(\tau) = 2deg(g).$$

For $n \geq 3$, a *spin structure* of a smooth $n$–manifold $A$ is a homotopy class of parallelizations over a 2-skeleton of $A$ (that is over the complement of a point when $n = 3$ and $A$ is connected).

The class of the covering map $\tilde{\rho}$ described in Notation 2.19 is the standard generator of $\pi_3(SO(3)) = \mathbb{Z}[\tilde{\rho}]$. The map $\rho$ can be used to describe the action of $\pi_3(SO(3))$ on the homotopy classes of parallelizations ($\tau\colon M \times \mathbb{R}^3 \to TM$) of $M$ as follows. Let $B$ be a 3–ball in $M$ identified with $B^3$. Let $\tau\psi(\rho)$ coincide with $\tau$ outside $B \times \mathbb{R}^3$ and read $\tau \circ \psi(\rho)$ on $B \times \mathbb{R}^3$. Set $[\tilde{\rho}][\tau] = [\tau\psi(\rho)]$. According to Theorem 4.2, $p_1([\tilde{\rho}][\tau]) = p_1(\tau) + 4$. The set of parallelizations that induce a given spin structure of our $M$ form an affine space with translation group $\pi_3(SO(3))$.

The *Rohlin invariant* $\mu(M,\sigma)$ of a smooth closed 3-manifold $M$, equipped with a spin structure $\sigma$, is the mod 16 signature of a compact spin 4-manifold $W$ bounded by $M$ so that the spin structure of $W$ restricts to $M$ as a stabilization of $\sigma$.

Kirby and Melvin proved the following theorem [10, Theorem 2.6].

**Theorem 4.3.** *For any closed oriented 3-manifold $M$, for any parallelization $\tau$ of $M$,*

$$p_1(\tau) \equiv dimension(H_1(M;\mathbb{Z}/2\mathbb{Z})) + \beta_1(M) \ (mod\, 2).$$

*Let $M$ be a closed 3-manifold equipped with a given spin structure $\sigma$. Then $p_1$ is a bijection from the set of homotopy classes of parallelizations of $M$ that induce $\sigma$ to*

$$2\,(dimension(H_1(M;\mathbb{Z}/2\mathbb{Z})) + 1) + \mu(M,\sigma) + 4\mathbb{Z}$$

*When $M$ is an integer homology 3–sphere, $p_1$ is a bijection from the set of homotopy classes of parallelizations of $M$ to $(2 + 4\mathbb{Z})$.*

Extend the standard parallelization $\tau_s$ of $B^3$ as a parallelization $\widehat{\tau_s}$ of $S^3$. When $\partial M = S^2$, form $\hat{M} = (S^3 \setminus (B^3 \setminus N(\partial M))) \cup_{N(\partial M)} M$ and use $\widehat{\tau_s}$ to extend any parallelization $\tau$ of $M$ to a parallelization $\hat{\tau}$ of $\hat{M}$. Then it is easy to see that $p_1(\tau) = p_1(\hat{\tau}) - p_1(\widehat{\tau_s})$. In particular, according to Theorem 4.3,

$$p_1(\tau) \equiv \text{dimension}(H_1(M;\mathbb{Z}/2\mathbb{Z})) + \beta_1(M) \ (\text{mod } 2)$$

and, when $M$ is an integer homology 3–ball, the map $p_1$ is a bijection from the set of homotopy classes of parallelizations of $M$ to $4\mathbb{Z}$.

### 4.2. **Proofs of Theorems 1.2 and 1.4.**

**Lemma 4.4.** *Let $\tau$ be a trivialization of $TM$. Let $g \in C^0((M,\partial M),(SO(3),Id))$. Recall that $p_{S^2}\colon SO(3) \to S^2$ maps a transformation $t$ of $SO(3)$ to $t(N)$ where $N$ is the first basis vector of $\mathbb{R}^3$. Let $X$ and $Y$ be two sections of $UM$ induced by $\tau$ and $\tau\psi(g)$, respectively. Then*

$$lk(L_{Y=X}, L_{Y=-X}) = lk((p_{S^2} \circ g)^{-1}(N), (p_{S^2} \circ g)^{-1}(-N)) = -\frac{1}{2}deg(g)$$

PROOF: The first equality follows from the definition. It implies that

$$lk(L_{Y=X}, L_{Y=-X}) = lk((p_{S^2} \circ g)^{-1}(N), (p_{S^2} \circ g)^{-1}(-N)) = lk'(g)$$

only depends on $g$. Then Proposition 3.1 implies that $lk'$ is a homomorphism from $[(M,\partial M),(SO(3),\mathrm{Id})]$ to $\mathbb{Q}$. According to Proposition 4.1 it suffices to evaluate $lk'(\rho)$ for the element $\rho$ regarded as a degree 2 map of $C^0((B^3,\partial B^3),(SO(3),\mathrm{Id}))$. According to Corollary 2.24, when $g=\rho$, $lk(L_{Y=X},L_{Y=-X})=-1$. $\diamond$

Proof of Theorem 1.2: Theorem 4.2 and Lemma 4.4 show that if $X$ and $Y$ extend to parallelizations $\tau(X)$ and $\tau(Y)$, then

$$p_1(\tau(Y))-p_1(\tau(X))=-4lk(L_{Y=X},L_{Y=-X}).$$

For any torsion combing $[Y]$, define $p_1([Y])$ from a combing $[X]$ that extends to a parallelization by

$$p_1([Y])=p_1([X])+4lk(L_{X=Y},L_{X=-Y}).$$

Thanks to Proposition 3.1, since this formula is valid for combings that extend to parallelizations, this definition does not depend on the choice of $X$. Now, Proposition 3.1 implies that the above formula is valid for all pairs of torsion combings.

Since $[-X]=[X]$ for a section $X$ that extends as a trivialization, we deduce that $p_1([-Y])=p_1([Y])$, for all torsion sections $Y$ of $UM$, from the above definition.

According to the following Lemma 4.5, Proposition 2.25 ensures the injectivity of the restriction of $p_1$ to $\mathcal{C}(\xi)$ for any torsion Spin$^c$-structure $\xi$. $\diamond$

**Lemma 4.5.** *For any torsion combing* $[X]$, $p_1(\gamma[X])-p_1([X])=4$.

Recall Corollary 2.24. $\diamond$

**Proposition 4.6.** *With the notation of Theorem 2.9, if* $(L,L_\parallel)$ *is a framed rationally null-homologous link of the interior of* $M$, *then*

$$p_1([C(\tau,L,L_\parallel)])=p_1(\tau)-4lk(L,L_\parallel).$$

Proof: Set $Y=C(\tau,L,L_\parallel)$. Assume that $\tau$ reads $(X,X_2,X_3)$ so that $L=L_{Y=-X}$. Then $X$ and $X_2$ are homotopic sections of $UM$ so that $p_1(\tau)=p_1([X])=p_1([X_2])$ and, according to Theorem 1.2, $p_1([Y])=p_1(\tau)-4lk(L_{Y=X_2},L_{Y=-X_2})$. The link $(L_{Y=X_2},L_{Y=-X_2})$ is isotopic to $(L,L_\parallel)$. $\diamond$

Proof of Theorem 1.4: According to Theorem 2.9, any torsion section $Y$ of $UM$ is homotopic to $C(\tau,L,L_\parallel)$, for some $L$ and $\tau$ as in Proposition 4.6 and in its proof. In particular, since $L=L_{Y=-X}$, $p_1([Y])\in p_1(\tau)-4\ell([L_{Y=-X}]))$ and $p_1([Y])\in p_1(\tau)-4\ell(\mathrm{Torsion}(H_1(M;\mathbb{Z}))$. Conversely, any element in $\ell(\mathrm{Torsion}(H_1(M;\mathbb{Z}))$ reads $lk(L,L_\parallel)$ for some rationally null-homologous link $L$. $\diamond$

4.3. **Identifying $p_1$ with the Gompf invariant.** Let us first recall the definition of the Gompf invariant. An *almost-complex structure* on a smooth 4-dimensional manifold $W$ is an operator $J$ such that $J^2=-\mathrm{Id}$, acting smoothly on the tangent space to $W$, fiberwise, so that identifying the multiplication by $J$ with the multiplication by $i$ transforms the real bundle $TW$ into a bundle with fiber $\mathbb{C}^2$. An almost-complex structure on $W$ induces a combing of $\partial W$, which is the class of the image $[J\nu=J(\nu(\partial W))]$ under $J$ of the outward normal $\nu(\partial W)$ to $W$. Gompf showed that all the combings of a 3-manifold appear as combings $J\nu$ for some $W$ [4, Lemma 4.4]. This will be reproved below. The *first Chern class* $c_1(TW,J)$ of $(TW,J)$ lives in $H^2(W;\mathbb{Z})$. It is the obstruction to trivializing $TW$ over the two-skeleton of $W$ as an almost-complex manifold (the induced trivialization of

$TW$ must read $(X, JX, Y, JY)$). (In general, the first Chern class $c_1$ of a complex vector bundle restricts to real surfaces as the Euler class of the corresponding determinant bundle.) The restriction of $c_1(TW, J)$ to $H^2(\partial W; \mathbb{Z})$ is $e((J\nu)^\perp)$ so that the boundary of a chain $\Sigma'$ that represents the Poincaré dual $Pc_1(TW, J)$ of $c_1(TW, J)$ is Poincaré dual to $e((J\nu)^\perp)$. When $J\nu$ is a torsion combing, this boundary $\partial\Sigma'$ represents a torsion element of $H_1(\partial W; \mathbb{Z})$ so that there exists a rational 2–chain $\Sigma$ of $\partial W$ such that $(\Sigma' \cup \Sigma)$ is a closed rational 2-cycle of $W$. The algebraic self-intersection of this rational cycle is independent of $\Sigma$ and it is denoted by $(Pc_1(TW, J))^2$, and the *Gompf invariant* $\theta_G(J\nu)$, which is denoted by $\theta(J\nu)$ in [4, Section 4] is

$$\theta_G(J\nu) = (Pc_1(TW, J))^2 - 2\chi(W) - 3\,\mathrm{signature}(W)$$

where $\chi$ stands for the Euler characteristic.

In this subsection, where $M$ has no boundary, we prove that $\theta_G = p_1$.

**Lemma 4.7.** *When a combing $X$ of $M$ extends as a parallelization, $\theta_G([X]) = p_1([X])$.*

Proof: For a rank $2k$ complex bundle $\omega$ seen as a rank $4k$ real bundle $\omega_{\mathbb{R}}$, $p_1(\omega_{\mathbb{R}}) = c_1^2(\omega) - 2c_2(\omega)$, where $c_2$ denotes the second Chern class, which is the Euler class of $\omega_{\mathbb{R}}$ for a rank 2 complex bundle $\omega$. See [23, Definition p.158 § 14 and Corollary 15.5]. Let $(W, J)$ be an almost-complex connected compact manifold bounded by $M$ such that $X = J\nu$, let $Y$ be a nowhere zero section of $X^\perp \subset TM$. Consider the almost-complex parallelization $(\nu, Y)$ inducing the real parallelization $(\nu, J\nu, Y, JY)$ of $TW_{|M}$, and the complex bundle $\omega$ over $(W \cup_M (-W))$ that is trivial with fiber $\mathbb{C}\nu \oplus \mathbb{C}Y$ over $(-W)$ and that coincides with the initial one over $W$. Since the characteristic classes $p_1$, $c_1$ and $c_2$ of $\omega_{\mathbb{R}}$ or $\omega$ trivially restrict to $H^*(-W)$, they come from classes of $H^*(W \cup_M (-W), -W) \cong H^*(W, M)$. Thus $p_1(\omega_{\mathbb{R}})$ is the image of $p_1(W, (J\nu, Y, JY))[W, \partial W] \in H^4(W, \partial W)$, and $c_2(\omega)$ is the image of $c_2(TW, \nu) \in H^4(W, \partial W)$, which is $\chi(W)[W, \partial W]$ since $c_2$ is the obstruction to extending $\nu$ as a nowhere zero section of $TW$, which is the relative Euler class of $(TW, \nu)$. Similarly, $c_1(\omega)$ is the image of a lift $\tilde{c}_1$ of $c_1(TW, J)$ in $H^2(W, \partial W)$, where $P\tilde{c}_1$ is represented by a cycle of $W$ that can be constructed as in the definition of $(Pc_1(TW, J))^2$ before Lemma 4.7. The Poincaré dual $Pc_1(\omega)$ of $c_1(\omega)$ is the image of this cycle $(\Sigma' \cup \Sigma)$ in $H_2(W \cup_M (-W))$ and $p_1(W, (J\nu, Y, JY)) = (Pc_1(TW, J))^2 - 2\chi(W)$. ◇

**Lemma 4.8.** *When a combing $X$ of $M$ extends as a parallelization, $\theta_G([\gamma X]) = \theta_G([X]) + 4$.*

Proof: According to Lemma 4.5, $p_1([\gamma X]) = p_1([X]) + 4$ for any $[X]$. ◇

Any closed oriented connected 3-manifold $M$ is the boundary of a 4-manifold

$$W_L = B^4 \bigcup_{L \times D^2 \subset S^3} \coprod_{i=1,\dots,n} (D^2 \times D^2)^{(i)}$$

obtained from $B^4$ by attaching 2-handles $(D^2 \times D^2)^{(i)}_{i=1,\dots,n}$ along a tubular neighborhood $L \times D^2$ of a framed link $L = (K_i, \mu_i)_{i=1,\dots,n}$. Such a framed link $L$ is an *integral surgery presentation* of $W_L$ and $M$. The $K_i$ are the components of $L$, the $\mu_i$ are the surgery parallels $K_i \times \{1\} \subset K_i \times D^2$ which frame the $K_i$, and the handle $(D^2 \times D^2)^{(i)}$ is attached by a natural identification of $K_i \times D^2 \subset \partial B^4$ with

$((-S^1)\times D^2)^{(i)}$ that restricts to $\mu_i$ as an orientation-reversing homeomorphism onto $(S^1\times\{1\})^{(i)}$.

According to a theorem of Kaplan [9], we can furthermore demand that $lk(K_i,\mu_i)$ is even for any $i$, in the statement above. In this case, we will say that the surgery presentation is *even.*

**Lemma 4.9.** *Let $L$ be an even surgery presentation of $M$. There is an almost-complex structure $J_b$ on $W_L$ (described below) such that $e((J_b\nu)^\perp)=0$. For any Spin$^c$ structure $\xi$ on $M$, there is at least one almost complex structure $J$ on $W_L$ (described below) such that the class of $J\nu$ belongs to $\xi$ and, if $J\nu$ is a torsion combing, then $p_1(J\nu)-p_1(J_b\nu)=\theta_G(J\nu)-\theta_G(J_b\nu)$.*

Proof: We will only consider almost-complex structures $J$ that are *compatible with a* given *Riemannian metric* in the following sense: $J$ preserves the Riemannian metric and $Jx$ is orthogonal to $x$ for any $x$. Furthermore, in all the local product decompositions below, the Riemannian metric is supposed to be a product metric. Our almost-complex structures $J$ of 4-manifolds also induce the orientation via local parallelizations of the form $(X,JX,Y,JY)$. Below, $B^4$ is seen as the unit complex ball of $\mathbb{C}^2$ and equipped with the corresponding complex structure, it is equipped with its usual Riemannian structure. It is also seen as the unit ball of the quaternion field $\mathbb{H}=\mathbb{C}\oplus\mathbb{C}j$, so that $S^3$ is identified with the group of unit quaternions and $T_xS^3$ is the space of quaternions orthogonal to $x$.

A homotopy

$$J\nu\colon\quad \begin{array}{lcl}[-1,0]\times S^3 & \to & TS^3\\ (t,x) & \mapsto & J\nu(t,x)\in T_xS^3\end{array}$$

such that $J\nu(-1,x)=ix$, and $\|J\nu(t,x)\|=1$ induces a homotopic almost-complex structure on $B^4$ as follows, the complex structure is unchanged outside a collar $[-1,0]\times S^3$ of the boundary of $B^4$, and the operator $J$ of the almost-complex structure maps the unit tangent vector to $[0,1]\times\{x\in S^3\}$ at $(t,x)$ to $J\nu(t,x)$. Note that $J$ is completely determined by these conditions. If such a homotopy is such that $J\nu(0,.)$ is tangent to $K_i\times\{y\}$ on $K_i\times D^2$, then the associated almost-complex structure $J$ preserves the tangent space to $\{x\}\times D^2$ and it uniquely extends to $(D^2\times D^2)^{(i)}$ so that $J$ preserves the tangent space to $\{x\}\times D^2$ and $J$ is compatible with the product Riemannian structure on $(D^2\times D^2)^{(i)}$. In particular $J$ maps the outward normal to $(D^2\times S^1)^{(i)}\subset M$ at $(x,y\in S^1)$ to the unit tangent vector to $(\{x\}\times S^1)^{(i)}$ at $(x,y)$.

Before smoothing the ridges, $W_L$ reads $(\mathbb{R}^2\setminus\{(x,y);x<-1,y>-1\})\times(-K_i)\times S^1$ near $K_i\times S^1$. The 4-manifold $W_L$ is next smoothed around $K_i\times S^1$, the

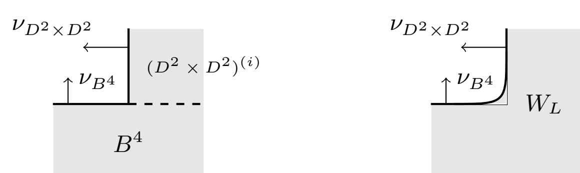



Figure 1. $W_L$ near $K_i\times S^1$ before and after smoothing.

smoothing adds the product of $K_i\times S^1$ by a triangle with two orthogonal straight sides and a smooth hypothenuse that makes null angles with the two straight sides. See Figure 1.

In the plane of the triangle, the normal $\nu$ reads $\nu = \cos(\theta)\nu_{B^4} + \sin(\theta)\nu_{D^2\times D^2}$ for some $\theta \in [0,\pi/2]$. Extend $J$ on the triangle, naturally, so that $J\nu$ reads $J\nu = \cos(\theta)J\nu_{B^4} + \sin(\theta)J\nu_{D^2\times D^2}$ and $J\nu$ goes from the tangent to $K_i \times \{y\}$ to the tangent to $(\{x\}\times S^1)^{(i)}$ on $T_{(x,y)}K_i \times S^1$ by the shortest possible way on the smooth hypothenuse.

Then $J$ and $J\nu$ are completely determined on $W_L$ by the homotopy $J\nu$ on $[-1,0]\times S^3$, and we now study them as a function of this homotopy.

We will consider homotopies induced by homotopies of orthonormal parallelizations, i.e. homotopies $J\nu$ such that there is a homotopy $V\colon [-1,0]\times S^3 \to T_xS^3$ where $V(t,x) \in T_xS^3$, $V(t,x) \perp J\nu(t,x)$, $\|V(t,x)\| = 1$ and $V(-1,x) = jx$. Furthermore, our homotopies are such that $J\nu(0,.)$ is tangent to $K_i \times \{y\}$ on $K_i \times D^2$, so that $V(0,x)$ induces a framing of $K_i$. The linking number of $K_i$ with the parallel of $K_i$ induced by this framing is denoted by $r_i$. Recall that $H_1(SO(3);\mathbb{Z}) = \pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$ is generated by a loop of rotations $(\exp(i\theta) \mapsto \rho(\theta, A))$ about a fixed arbitrary axis $A$.

**Sublemma 4.10.** *The integers $r_i$ are odd.*

Proof of Sublemma 4.10: Let $\Sigma$ be a (connected) Seifert surface of $K_i$, then $TM_{|\Sigma}$ has a trivialization $\tau_\Sigma$ whose third vector is the positive normal $\nu\Sigma$ to $\Sigma$, and whose first vector over $K_i$ is obtained from the tangent vector $v_K$ to $K_i$ by rotating it $(-\chi(\Sigma))$ times around the axis $\nu\Sigma$, along $K_i$. On the other hand, the first vector of the restriction to $K_i$ of the trivialization $\tau_{JV}$ induced by $J\nu(0,.)$ and $V(0,.)$ is $v_K$ and its third vector is obtained from $\nu\Sigma$ by rotating it $r_i$ times around $v_K$ along $K_i$. Then $\tau_\Sigma^{-1}\circ\tau_{JV}$ induces a map from $\Sigma$ to $SO(3)$ whose restriction to $K_i$ represents a trivial homology class in $H_1(SO(3))$. Since the class of this restriction is $(r_i + \chi(\Sigma))$ mod 2 and since $\chi(\Sigma)$ is odd, $r_i$ is odd, too. $\diamond$

**Sublemma 4.11.** *The integers $r_i$ may be changed to any arbitrary odd number, by perturbing the homotopy near $K_i \times D^2$.*

Proof of Sublemma 4.11: Assume without loss that $J\nu(0,.)$ is tangent to $K_i \times \{y\}$ on a bigger tubular neighborhood $K_i \times 2D^2$. Let $e_1$ denote the first basis vector of $\mathbb{R}^3$. Consider a map

$$\begin{array}{llll} F\colon & [0,1]\times S^1 & \to & SO(3) \\ & (t,\exp(i\theta)) & \mapsto & \mathrm{Id} \quad \text{if } t=1 \text{ or } \theta\in 2\pi\mathbb{Z} \\ & & & \rho(2\theta,e_1) \quad \text{if } t=0. \end{array}$$

Then $(J\nu, V, JV)(0,.)$ may be replaced on $K_i \times 2D^2$, with the homotopic map that maps $(0,(\exp(i\theta), u\exp(i\eta)))$ to

$$F(\max(0,u-1),\exp(ik_i\theta))\,((J\nu,V,JV)(0,(\exp(i\theta),u\exp(i\eta))))$$

for some integer $k_i$. Since this changes $r_i$ to $r_i + 2k_i$, this shows that $r_i$ can be changed to any odd number. $\diamond$

Now, the obstruction to extending $V$ as a unit vector tangent to the second almost-complex factor $D^2$ across $(D^2\times .)^{(i)}$ is $-(r_i - lk(\mu_i, K_i))$, and the obstruction to extending $J\nu$, which is the tangent to $K_i \times \{y\}$, as a unit vector tangent to the first almost-complex factor across $(D^2\times .)^{(i)}$ is 1. In particular, the Poincaré dual of the Chern class $c_1(TW_L, J)$ may be represented by a chain that does not intersect $B^4$ (since $H^2(B^4, S^3) = 0$) and that intersects $(D^2 \times D^2)^{(i)}$ as $(1-r_i+lk(\mu_i, K_i))(0\times D^2)^{(i)}$. Let $(J_b\nu, V_b)$ be a pair of orthogonal homotopies such that $r_i = lk(\mu_i, K_i)+1$. Then $c_1(TW_L, J_b) = 0$ and $\theta_G(J_b\nu) = -2\chi(W_L) - 3\ \mathrm{signature}(W_L)$.

Change $r_i$ to $(r_i + 2k_i)$ as in Sublemma 4.11 and denote the obtained almost-complex structure by $J$. Compare the induced vector fields and compute $L_{J\nu=V_b}$ and $L_{J\nu=-V_b}$.

The exact sequence of the fibration from $p_{S^2}\colon SO(3) \to S^2$ (defined after Notation 2.19) allows us to show that composing the map $F$ in the proof of Sublemma 4.11 by $p_{S^2}$ provides a degree $\pm 1$ map from $([0,1] \times S^1, \partial[0,1] \times S^1)$ to $(S^2, e_1)$. Thus there exists a well-determined $\varepsilon = \pm 1$ such that $L_{J\nu=V_b}$ and $L_{J\nu=-V_b}$ are homologous to $\varepsilon \sum_{i=1}^n k_i m_i$ in $(L \times D^2) \setminus L$ where $m_i$ is a meridian of $K_i$. We can furthermore assume that $L_{J\nu=V_b} \subset L \times uS^1$ and $L_{J\nu=-V_b} \subset L \times u'S^1$ for two distinct elements $u$ and $u'$ of $]1,2[$. Let $m_i$ (resp. $m_{i\|}$) denote a meridian of $K_i$ in $L \times uS^1$ (resp. in $L \times u'S^1$).

Since the meridians $m_i$ generate $H_1(M;\mathbb{Z})$, for any $Spin^c$-structure $\xi$, there exists an almost-complex structure $J$ as above such that $J\nu$ belongs to $\xi$. The combing $J\nu$ is torsion if and only if $L_{J\nu=-V_b}$ represents a torsion element in $H_1(M;\mathbb{Z})$. Assume that $J\nu$ is torsion from now on. According to Theorem 1.2

$$p_1(J\nu) - p_1(J_b\nu) = p_1(J\nu) - p_1(V_b) = -4lk(\sum_{i=1}^n k_i m_i, \sum_{i=1}^n k_i m_{i\|}).$$

On the other hand, since the boundary of a representative of $Pc_1(TW_L, J)$ is homologous to $2L_{J\nu=-V_b}$, $Pc_1(TW_L, J)$ is represented by $\Sigma' = 2\varepsilon \sum_{i=1}^n k_i (0 \times D^2)^{(i)}$. Set $\Sigma'_{\|} = 2\varepsilon \sum_{i=1}^n k_i (x \times D^2)^{(i)}$ for $x \in \mathring{D}^2 \setminus \{0\}$, and let $(-\partial\Sigma')$ and $\left(-\partial\Sigma'_{\|}\right)$ bound $\Sigma$ and $\Sigma_{\|}$ in $M$, respectively, so that

$$\begin{aligned}
&\theta_G(J\nu) - \theta_G(J_b\nu) = (Pc_1(TW, J))^2 \\
&= \langle \Sigma' \cup \Sigma, \Sigma'_{\|} \cup \Sigma_{\|} \rangle_{W_L \cup_{\partial W_L = 0\times M} [0,1]\times M} \\
&= \langle (-[0,1/2] \times \partial\Sigma) \cup (1/2 \times \Sigma), (-[0,2/3] \times \partial\Sigma_{\|}) \cup (2/3 \times \Sigma_{\|}) \rangle_{[0,1]\times M} \\
&= -\langle \Sigma, \partial\Sigma_{\|} \rangle_M \\
&= p_1(J\nu) - p_1(J_b\nu).
\end{aligned}$$

$\diamond$

The previous lemma, Lemma 4.5 and the transitivity of the action of $\pi_3(S^2)$ on the combings of a $Spin^c$-structure reduce the proof that $\theta_G = p_1$ to the proof of the following lemma.

**Lemma 4.12.** $\theta_G([\gamma X]) - \theta_G([X]) = 4$ *for any combing* $[X]$.

PROOF: We refer to the previous proof. Add a trivial knot $U$ framed by $+1$ to a surgery presentation $L$, such that $W_L$ is equipped with an almost-complex structure $J$. The structure $J$ is homotopic to a structure $J^{(1)}$ that extends on $W_{L\cup U}$ so that $Pc_1(TW, J^{(1)})$ is represented by $(0 \times D^2)^{(0)}$. Then $\theta_G(J^{(1)}\nu) - \theta_G(J\nu) = 1 - 2 - 3 = -4$. The structure $J$ is also homotopic to a structure $J^{(3)}$ that extends on $W_{L\cup U}$ so that $Pc_1(TW, J^{(3)})$ is $3(0 \times D^2)^{(0)}$, then $\theta_G(J^{(3)}\nu) - \theta_G(J^{(0)}\nu) = 9 - 2 - 3 = 4$. These two combing modifications sit in a 3-ball of $M$, so that each of them corresponds to the action of an element of $\pi_3(S^2)$ independent of $(M, J)$. According to Lemma 4.8, $[J^{(1)}\nu] = [\gamma^{-1}J\nu]$ and $[J^{(3)}\nu] = [\gamma J\nu]$. Since the above process allows us to inductively represent all the combings $[\gamma^k J\nu]$, by adding some disjoint trivial knots framed by $+1$, and to prove that $\theta_G(\gamma^k J\nu)-\theta_G(\gamma^{k-1}J\nu)=4$, for all $k\in\mathbb{Z}$, we are done. ⋄

**Remark 4.13.** For a natural integer $k$ and for a surgery presentation $L$ of $M$ in $S^3$, let $L(k)$ be the surgery presentation of $M$ obtained from $L$ by adding $k$ trivial knots framed by $+1$. On our way, we have proved that for any combing $[X]$ and for any even surgery presentation $L$ of $M$, there exist a natural integer $k$ and an almost complex structure $J$ on $W_{L(k)}$ such that $[X]=[J\nu]$.

4.4. **More variations of $p_1$.** In applications, combing modifications often arise as in Definition 4.16 or as in the statement of Proposition 4.21 below. We show how Formula 1.3 applies in these settings to yield other useful variation formulas.

**Lemma 4.14.** *Let $M$ be equipped with a torsion section $X$ of $UM$. Let $L$ be a rationally null-homologous link in the interior of $M$. Let $Z$ be a section of $UM$ orthogonal to $X$, such that $Z$ is defined on $L$ and $\partial M$. Extend $Z$ as a section $\tilde{Z}$ of the $\mathbb{R}^2$-bundle $X^\perp$, so that $\tilde{Z}$ is transverse to the zero section. Let $L(Z\subset X^\perp)$ be the zero locus of $\tilde{Z}$ cooriented by the fiber of $X^\perp$. Then $L(Z\subset X^\perp)$ is a link of $M\setminus L$ that represents the Poincaré dual of the relative Euler class of $(X^\perp,Z)$, and $L(Z\subset X^\perp)$ is homologous to the Poincaré dual of $e(X^\perp)$.*

⋄

**Remark 4.15.** Lemma 4.14 can be taken as a definition of the relative Euler class in this case. The obstruction to extending $Z$ as a section of $UM\cap X^\perp$ across a 2–cycle of $(M,L\cup\partial M)$ is the algebraic intersection of the 2–cycle with $L(Z\subset X^\perp)$.

**Definition 4.16.** Let $X$ be a section of $UM$. Let $L$ be a link in the interior of $M$ and let $Z$ be a section of $UM_{|L}$ orthogonal to $X$. Let $\eta=\pm 1$, let $L_\parallel$ be a parallel of $L$ and let $N(L)$ be a tubular neighborhood of $L$ where $Z$ is extended as a section of $UM$ orthogonal to $X$. Let $\rho(\theta,X)$ denote the rotation with axis $X$ and angle $\theta$. Let $D^2=\{u\exp(i\theta); u\in[0,1],\theta\in[0,2\pi]\}$ be the unit disk of $\mathbb{C}$. Define $D(X,L,L_\parallel,Z,\eta)$ (up to homotopy) as the section of $UM$ that coincides with $X$ outside $N(L)$ and that reads as follows in $N(L)$, which is trivialized with respect to $L_\parallel$ so that it reads $D^2\times L$.

- $D(X,L,L_\parallel,Z,\eta)(0,k\in L)=-X(0,k)$,
- when $u\in]0,1]$, $[-X,D(X,L,L_\parallel,Z,\eta)(u\exp(i\theta),k)]$ is the geodesic arc of length $u\pi$ of the half great circle $[-X,X]_{\rho(\eta\theta,X)(Z)}$ from $(-X)$ to $X$ through $\rho(\eta\theta,X)(Z)$, where $X$ and $Z$ stand for $X(u\exp(i\theta),k)$ and $Z(u\exp(i\theta),k)$, respectively,

so that $D(X,L,L_\parallel,Z,\eta)(1/2,k)=Z(1/2,k)$. Note that the homotopy class of $D(X,L,L_\parallel,Z,\eta)$ can also be defined by the following formula.

$$D(X,L,L_\parallel,Z,\eta)(u\exp(i\theta),k)=\rho(\pi(1+u),\rho(\eta\theta-\pi/2,X)(Z))(X)(u\exp(i\theta),k).$$

**Remark 4.17.** With the notation of Remark 2.3

$$C(X,L,\sigma)=D(X,L,L_\parallel,Z(\sigma,\sigma_N),-1).$$

**Proposition 4.18.** *Under the assumptions of Lemma 4.14, let $\eta=\pm 1$, let $L_\parallel$ be a parallel of $L$ and let $N(L)$ be a tubular neighborhood of $L$ where $Z$ is extended as a section of $UM$ orthogonal to $X$. For the section $D(X,L,L_\parallel,Z,\eta)$ of Definition 4.16,*

$$p_1([D(X,L,L_\parallel,Z,\eta)])-p_1([X])=4lk(L,\eta L(Z\subset X^\perp)-L_\parallel).$$

Proof: Set $Y = D(X, L, L_{\|}, Z, \eta)$. Let $\tau$ be the parallelization of $N(L)$ with first vector $X$ and second vector $Z$. Then $\tau^{-1}$ maps $Y(D^2/\partial D^2 \times k)$ to the sphere $S^2$ with degree $(-\eta)$ so that $L_{Y=-X} = -\eta L$ and $L_{X=-Y} = \eta L$. In order to use Theorem 1.2, deform $X$ to $\tilde{X}$ to make it transverse to $Y$ using $\tilde{Z}$ as follows. Let $N_{1/3}(L) = \{(u\exp(i\theta), k \in L) \in N(L); u \in [0,1/3]\}$ and $N_{2/3}(L) = \{(u\exp(i\theta), k) \in N(L); u \in [0,2/3]\}$. Consider a smooth function $\chi\colon M \to [0,1]$ that maps $\left(M \setminus N_{2/3}(L)\right)$ to 1 and $N_{1/3}(L)$ to 0. Let $\varepsilon$ be a very small positive real number, set $\tilde{X} = \frac{1}{\|X+\varepsilon\chi\tilde{Z}\|}(X+\varepsilon\chi\tilde{Z})$ so that $\tilde{X}(M)$ is now transverse to $Y(M)$. Outside $UM_{|N(L)}$, $\tilde{X}(M) \cap Y(M)$ reads $Y(L(Z \subset X^{\perp}))$, whereas on $UM_{|N(L)}$, $Y(M) \cap \tilde{X}(M)$ reads $Y(-\eta L_{\|})$ because $Y$ covers $S^2$ with degree $(-\eta)$ along a fiber of $N(L)$. $\diamond$

We have the two immediate corollaries.

**Corollary 4.19.** *Under the hypotheses of Proposition 4.18, when $Z$ extends as a section of the unit bundle of $X^{\perp}$ on $M$,*

$$p_1([D(X, L, L_{\|}, Z, \eta)]) = p_1([X]) - 4lk(L, L_{\|}).$$

**Corollary 4.20.** *Under the hypotheses of Proposition 4.18, let $K = \{K(\exp(i\kappa) \in S^1)\}$ be a component of $L$, let $r \in \mathbb{Z}$, and let $Z_r = Z$ on $L \setminus K$ and $Z_r(K(k = \exp(i\kappa))) = \rho(r\kappa, X)(Z)(k)$. Then*

$$p_1([D(X, L, L_{\|}, Z_r, \eta)]) - p_1([D(X, L, L_{\|}, Z, \eta)]) = 4\eta r.$$

Note that under the hypotheses of Proposition 4.18, when $X$ is tangent to $L$, if $Z$ is induced by $L_{\|}$, then $D(X, L, L_{\|}, Z, 1)$ is independent of $Z$ and $L_{\|}$.

The following combing modification also arises in the study of combings associated to Heegaard diagrams.

**Proposition 4.21.** *Let $M$ be equipped with a torsion section $X$ of $UM$. Let $K$ be a rationally null-homologous knot in the interior of $M$. Write a tubular neighborhood $N(K)$ of $K$ as $N(K) = (D^2 \times I)/\sim$ where $I = [0,1]$, $D^2$ is the unit disk of $\mathbb{C}$ and $\sim$ identifies $(u\exp(i\theta), 1)$ with $(u\exp(i\theta + r\pi), 0)$ for $(u, \theta) \in [0,1] \times [0, 2\pi]$, for some integer $r$. Let $K^{(2)}$ be the satellite $K^{(2)} = \{(\pm\frac{1}{2}, t); t \in I\}$ of $K$ in $N(K)$, which is connected iff $r$ is odd. Let $s$ be the involution of $K^{(2)}$ such that $s(\frac{1}{2}, t) = (-\frac{1}{2}, t)$. Regard $X$ as a function from $N(K)$ to $S^2$ with respect to some trivialization of $UM_{|N(K)}$, and assume that $X$ is constant with value $X(t)$ on $\{(z,t); z \in D^2\}$, for every $t$. Let $Z$ be a section orthogonal to $X$ of the restriction of $UM$ to $K^{(2)}$, such that $Z(s(\kappa)) = -Z(\kappa)$, for all $\kappa \in K^{(2)}$. Define $D(X, K, K^{(2)}, Z, -1)$ as follows.*

- *$D(X, K, K^{(2)}, Z, -1)(0, t \in I) = -X(0,t)$,*
- *when $u \in ]0,1]$, $[-X, D(X, K, K^{(2)}, Z)(u\exp(i\theta), t)]$ is the geodesic arc of length $u\pi$ of the half great circle $[-X, X]_{\rho(-\theta, X)(Z(1/2,t))}$ from $(-X(t))$ to $X(t)$ through $\rho(-\theta, X(t))(Z(1/2,t))$,*

*so that $D(X, K, K^{(2)}, Z, -1)(1/2, t) = Z(1/2, t)$. Let $f$ be a smooth (not strictly) increasing surjective function from the interval $I$ to $[0, \pi]$, that is constant near the ends of $I$. Let $k \in \mathbb{Z}$. Define*

$$\begin{array}{rccc} T^k\colon & D^2 \times I & \longrightarrow & D^2 \times I \\ & (u\exp(i\theta), t) & \mapsto & (u\exp(i(\theta + kf(t))), t) \end{array}$$

*so that $T$ is a half-twist. Define $T^k_*(Z)$ on the satellite $\overline{T^{(k)}(K^{(2)}\cap(D^2\times]0,1[))}$ of $K$ so that for $\epsilon\in\{-1,+1\}$,*

$$T^k_*(Z)((\frac{\epsilon}{2}\exp(ikf(t)),t))=\rho(kf(t),X)(Z((\frac{\epsilon}{2},t))).$$

*Then*

$$p_1([D(X,L,T^k(L^{(2)}),T^k_*(Z),-1)])-p_1([D(X,L,L^{(2)},Z,-1)])=-4k.$$

PROOF: The variation of a section under some $T^k$ sits inside a ball $D^2\times[\varepsilon,1-\varepsilon]$. Therefore the corresponding variation of $p_1$ may be read in this ball. It does not depend on the trivialization of the ball induced by $X$ and $Z$, since all of them are homotopic. Therefore, it only depends on $k$, linearly. The coefficient is obtained by looking at the effect of the twist $T^2$ on a $D(X,K,K_{\parallel},Z,-1)$ as in Proposition 4.18. $\diamond$

## 5. The Θ-invariant of combings

In this section, we present a definition of the invariant $\Theta$ of combings of rational homology balls. This definition is deeply inspired from the definition of $\Theta_{KKT}$ that can be found in [17, Section 6.5] and in [18, Theorem 2.14]. It also appears in [19]. Finally, we prove that $\Theta=6\lambda(\hat{M})+\frac{p_1}{4}$.

5.1. **On configuration spaces.** Recall that *blowing up* a submanifold $A$ means replacing it with its unit normal bundle. See Definition 3.5.

In a closed 3-manifold $R$, we fix a point $\infty$ and define $C_1(R)$ as the compact 3-manifold obtained from $R$ by blowing up $\{\infty\}$. This space $C_1(R)$ is a compactification of $\check{R}=(R\setminus\{\infty\})$.

The *configuration space* $C_2(R)$ is the compact 6–manifold with boundary and corners obtained from $R^2$ by blowing up $(\infty,\infty)$, and the closures of $\{\infty\}\times\check{R}$, $\check{R}\times\{\infty\}$ and the diagonal of $\check{R}^2$, successively.

Then $\partial C_2(R)$ contains the unit normal bundle to the diagonal of $\check{R}^2$. This bundle is canonically isomorphic to $U\check{R}$ via the map

$$[(x,y)]\in\frac{\frac{T_r\check{R}^2}{\mathrm{diag}}\setminus\{0\}}{\mathbb{R}^{*+}}\mapsto[y-x]\in\frac{T_r\check{R}\setminus\{0\}}{\mathbb{R}^{*+}}.$$

Since $((\mathbb{R}^3)^2\setminus\mathrm{diag})$ is homeomorphic to $\mathbb{R}^3\times]0,\infty[\times S^2$ via the map

$$(x,y)\mapsto(x,\parallel y-x\parallel,\frac{1}{\parallel y-x\parallel}(y-x)),$$

$((\mathbb{R}^3)^2\setminus\mathrm{diag})$ is homotopy equivalent to $S^2$. In general, $C_2(R)$ is homotopy equivalent to $(\check{R}^2\setminus\mathrm{diag})$. When $R$ is a rational homology 3-sphere, $\check{R}$ is a rational homology $\mathbb{R}^3$ and the rational homology of $(\check{R}^2\setminus\mathrm{diag})$ is isomorphic to the rational homology of $((\mathbb{R}^3)^2\setminus\mathrm{diag})$. Thus, $C_2(R)$ has the same rational homology as $S^2$, and $H_2(C_2(R);\mathbb{Q})$ has a canonical generator $[S]$ that is the homology class of a fiber of $U\check{R}\subset C_2(R)$, oriented as the boundary of the unit ball of a fiber of $T\check{R}$. For a 2-component link $(J,K)$ of $\check{R}$, the homology class $[J\times K]$ of the image of $J\times K$ in $H_2(C_2(R);\mathbb{Q})$ reads $lk(J,K)[S]$, where $lk(J,K)$ is the linking number of $J$ and $K$, see [20, Proposition 1.6].

5.2. **On propagators.** When $R$ is a rational homology 3-sphere, a *propagator* of $C_2(R)$ is a 4–cycle $F$ of $(C_2(R), \partial C_2(R))$ that is Poincaré dual to the preferred generator of $H^2(C_2(R);\mathbb{Q})$ that maps $[S]$ to 1. For such a propagator $F$, for any 2-cycle $G$ of $C_2(R)$,

$$[G] = \langle F, G\rangle_{C_2(R)}[S]$$

in $H_2(C_2(R);\mathbb{Q})$.

Let $B$ and $\frac{1}{2}B$ be two balls in $\mathbb{R}^3$ of respective radii $\ell$ and $\frac{\ell}{2}$, centered at the origin in $\mathbb{R}^3$. Identify a neighborhood of $\infty$ in $R$ with $S^3 \setminus (\frac{1}{2}B)$ in $(S^3 = \mathbb{R}^3 \cup \{\infty\})$ so that $\check{R}$ reads $\check{R} = M \cup_{]\ell/2,\ell]\times S^2} (\mathbb{R}^3 \setminus (\frac{1}{2}B))$ for a rational homology ball $M$ whose complement in $\check{R}$ is identified with $\mathbb{R}^3 \setminus B$. There is a canonical regular map

$$p_\infty \colon (\partial C_2(R) \setminus UM) \to S^2$$

that maps the limit in $\partial C_2(R)$ of a converging sequence of ordered pairs of distinct points of $(\check{R} \setminus M)$ to the limit of the direction from the first point to the second one. See [16, Lemma 1.1]. Recall that $\tau_s \colon \mathbb{R}^3 \times \mathbb{R}^3 \to T\mathbb{R}^3$ denotes the standard parallelization of $\mathbb{R}^3$. Also recall that the sections $X$ of $UM$ that we consider are *constant* on $\partial M$, i.e. they read $\tau_{s|\partial M \times \{V(X)\}}$ for some fixed $V(X) \in S^2$ on $\partial M$. Let $X$ be such a section and let $X(M)$ abusively denote its graph in $UM$. Then the *propagator boundary* $b_X$ associated to $X$ is the following 3–cycle of $\partial C_2(R)$

$$b_X = p_\infty^{-1}(V(X)) \cup X(M)$$

and a *propagator associated to the section* $X$ is a 4–chain $F_X$ of $C_2(R)$ whose boundary reads $b_X$. Such an $F_X$ is indeed a propagator because the algebraic intersection in $UM$ of $X(M)$ and a fiber is one.

5.3. **On the $\Theta$-invariant of a combed rational homology 3-sphere.**

**Theorem 5.1.** *Let $X$ be a section of $UM$ (which is constant on $\partial M$) for a rational homology 3-ball $M$, and let $(-X)$ be the opposite section. Let $F_X$ and $F_{-X}$ be two associated transverse propagators. Then $F_X \cap F_{-X}$ is a two-dimensional cycle whose homology class is independent of the chosen propagators. It reads $\Theta(M,X)[S]$, where $\Theta(M,X)$ is a rational–valued topological invariant of $(M,[X])$.*

Proof: Recall that $C_2\left(R = M \cup_{]\ell/2,\ell]\times S^2} (S^3 \setminus (\frac{1}{2}B))\right)$ has the same rational homology as $S^2$. In particular, since $H_3(C_2(R);\mathbb{Q}) = 0$, there exist propagators $F_X$ and $F_{-X}$ with the given boundaries $b_X$ and $b_{-X}$. These propagators may be chosen transverse. Without loss, assume that $F_{\pm X} \cap \partial C_2(R) = b_{\pm X}$. Since $b_X$ and $b_{-X}$ do not intersect, $F_X \cap F_{-X}$ is a 2–cycle. Since $H_4(C_2(R);\mathbb{Q}) = 0$, the homology class of $F_X \cap F_{-X}$ in $H_2(C_2(R);\mathbb{Q})$ does not depend on the choices of $F_X$ and $F_{-X}$ with their given boundaries. It reads $\Theta(M,X)[S]$. Then it is easy to see that $\Theta(M,X) \in \mathbb{Q}$ is a locally constant function of the section $X$. $\diamond$

When $R$ is an integer homology 3-sphere, a combing $X$ is the first vector of a unique parallelization $\tau(X)$ of $\check{R}$ that coincides with $\tau_s$ outside $M$, up to homotopy. When $R$ is a rational homology 3-sphere, and when $X$ is the first vector of a such a parallelization $\tau(X)$, this parallelization is again unique. In this case, according to [17, Section 6.5] (or [18, Theorem 2.14]), the invariant $\Theta(M,X)$ is the degree 1 part of the Kontsevich invariant of $(M,\tau(X))$ [12,13,16] and

$$\Theta(M,X) = 6\lambda(\hat{M}) + \frac{p_1(\tau(X))}{4} \tag{5.2}$$

with the notation of the introduction where $\hat{M} = R$.

With our extension of the definition of $p_1$ to combings, we prove that Formula 5.2 also holds for combings.

**Theorem 5.3.** *Let $M$ be a rational homology 3-ball. Let $X$ and $Y$ be two transverse sections of $UM$. Then*

$$\Theta(M,Y) - \Theta(M,X) = lk(L_{X=Y}, L_{X=-Y}).$$

*In particular,*

$$\Theta(M,X) = 6\lambda(\hat{M}) + \frac{p_1([X])}{4}.$$

PROOF: Let us prove that $\Theta(M,Y) - \Theta(M,X) = lk(L_{X=Y}, L_{X=-Y})$. This can be done as follows. Let $F_{-1}(\pm X, \pm Y)$ be the chain $F(\pm X, \pm Y)$ of Lemma 3.3 translated by $-1$ and seen in a collar $[-1,0]\times UM$ of $UM$ in $C_2(R)$. Assume that $F_X$ and $F_{-X}$ behave as products $[-1,0]\times\partial F_{\pm X}$ in $[-1,0]\times UM$. Then replacing these parts with $F_{-1}(X,Y)$ and $F_{-1}(-X,-Y)$, respectively, and making the appropriate easy corrections in $C_2(R)\setminus C_2(M)$ transforms $F_X$ and $F_{-X}$ into chains $F_Y$ and $F_{-Y}$ so that $[F_Y\cap F_{-Y}] = [F_X\cap F_{-X}] + [F_{-1}(X,Y)\cap F_{-1}(-X,-Y)]$ where $[F_{-1}(X,Y)\cap F_{-1}(-X,-Y)] = lk(L_{X=Y}, L_{X=-Y})[S]$ according to Proposition 3.6. $\diamond$

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