A rational structure of generating functions for Vassiliev invariants

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Abstract

We formulate a conjecture about the structure of Kontsevich integral for a knot. We describe its value in terms of the generating functions for the numbers of external edges attached to the closed 3-valent diagrams. We conjecture that these functions are rational functions of the exponentials of their arguments, their denominators being the powers of the Alexander-Conway polynomial. The evidence in favor of this conjecture comes from the calculation of the colored Jones polynomial.

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1 Introduction

The quantum invariants of knots, links and 3-manifolds, such as the Jones polynominal and the Witten-Reshetikhin-Turaev invariant, were discovered about 10 years ago. However, their interpretation in terms of classical 3-dimensional topology still remains a mystery.

Let us compare the skein relation definition of the Jones polynomial to that of a much older Alexander-Conway polynomial. The single-variable Alexander-Conway polynomial $\Delta_{\mathrm{A}}(\mathcal{L};t) \in \mathbb{Z}[t^{\pm 1}]$ is a unique invariant of links in S^3 which satisfies the following two properties. First, the normalization condition:

$$\Delta_{\mathrm{A}}(\mathrm{unknot};t) = 1. \tag{1.1}$$

Second, if $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_0$ are three links whose regular projection on a plane is the same except at one spot (see Fig. **??**), then

Figure 1: The links $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_0$

$$\Delta_{\mathrm{A}}(\mathcal{L}_+;t) - \Delta_{\mathrm{A}}(\mathcal{L}_-;t) = (t^{1/2} - t^{-1/2})\,\Delta_{\mathrm{A}}(\mathcal{L}_0;t). \tag{1.2}$$

This definition is purely combinatorial and it is a bit unnatural from the 3-dimensional point of view, since it requires a projection of a link. However, there exist alternative definitions of the Alexander-Conway polynomial which are purely topological. For example, let G_0 be the fundamental group of a knot complement $G_0 = \pi_1(S^3 \setminus \mathcal{K})$. Consider the commutators $G_1 = [G_0, G_0]$, $G_2 = [G_1, G_1]$ and abelian quotients $G_0' = G_0/G_1$, $G_1' = G_1/G_2$. Obviously, $G_0' = H_1(S^3 \setminus \mathcal{K}, \mathbb{Z}) = \mathbb{Z}$. Denote by $\mathbf{t}$ the generator of G_0', it represents the meridian of $\mathcal{K}$.

The group G_0' acts on G_1' by conjugation: for $\mathbf{x} \in G_0'$, $\mathbf{y} \in G_1'$, $\mathbf{x}: \ \mathbf{y} \mapsto \mathbf{x}^{-1}\mathbf{y}\mathbf{x}$. Now the Alexander-Conway polynomial of $\mathcal{K}$ is defined (up to a factor $\pm t^n$, $n \in \mathbb{Z}$) as the simplest polynomial such that $\Delta_{\mathrm{A}}(\mathcal{K}; \mathbf{t})$ maps G_1' into 0. Another definition relates the Alexander polynomial to the Reidemeister torsion of a local system in the knot complement, the variable t being the twist acquired by that system along the meridian of $\mathcal{K}$. From both definitions of $\Delta_{\mathrm{A}}(\mathcal{K}; t)$ it is clear that t is intimately related to the meridian of $\mathcal{K}$.

The Jones polynomial of links $J_2(\mathcal{L}; q) \in \mathbb{Z}[q^{\pm 1/2}]$ can also be defined by skein relations. It is the unique invariant which satisfies the following two properties: the normalization condition

$$J_2(\text{unknot}; q) = q^{1/2} + q^{-1/2} \tag{1.3}$$

and the skein relation

$$q\, J_2(\mathcal{L}_+; q) - q^{-1}\, J_2(\mathcal{L}_-; q) = (q^{1/2} - q^{-1/2})\, J_2(\mathcal{L}_0; q), \tag{1.4}$$

where the links $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_0$ are the same as those in eq.(**??**). Despite an obvious similarity between eqs.(**??**), (**??**) and eqs.(**??**), (**??**), there exists no intrepretation of $J_2(\mathcal{L}; q)$ in terms of the "classical" objects of 3-dimensional topology, such as the fundamental group of the knot complement. In particular, there is no indication that the variable q has any connection to the meridian of $\mathcal{K}$.

A new hope for a topological interpretation of $J_2(\mathcal{L}; q)$ emerged when J. Birman, X.-S. Lin and D. Bar-Natan discovered that both the Alexander-Conway and Jones polynomials are packed with Vassiliev invariants. Consider the expansions

$$\Delta_{\mathrm{A}}(\mathcal{K}; t) = \sum_{n=0}^{\infty} a_n(\mathcal{K})\, (t-1)^n, \tag{1.5}$$

$$J_2(\mathcal{K}; q) = \sum_{n=0}^{\infty} b_n(\mathcal{K})\, (q-1)^n. \tag{1.6}$$

It is not hard to see from the skein relations (**??**) and (**??**) that the coefficients $\alpha_n(\mathcal{K})$ and $\beta_n(\mathcal{K})$ are Vassiliev invariants of degree n. However, Vassiliev invariants by definition are related to the topology of "the space of all maps $S^1 \longrightarrow S^3$, rather than to the topology of knots themselves. The latter relation is still missing although some bits of it are known, such as the relation between the tree Vassiliev invariants and Milnor's linking numbers (see [?] and references therein).

By looking at eq. (??) we may say that the Alexander-Conway polynomial presents a way of assembling some Vassiliev invariants of knots into a polynomial which has a clear interpretation in terms of the classical 3-dimensional topology. At the same time, the Jones polynomial assembles some other Vassiliev invariants into another polynomial whose topological origin is rather obscure. Therefore one may wonder if there is a way of reassembling all Vassiliev invariants into the polynomials which would be similar to the Alexander-Conway polynomial rather than to the Jones polynomial in terms of their topological interpretation.

We will present an algorithm of assembling Vassiliev invariants of knots into a sequence of functions of a variable t. We conjecture that these functions are rational: their denominators are powers of the Alexander-Conway polynomial of t while their numerators are new polynomial invariants of knots. Since these new polynomials depend on the same variable t, we expect them to have a topological interpretation in which, similarly to the case of the Alexander-Conway polynomial, t will also be related to the meridian of a knot.

2 Removing legs

Let us recall a diagrammatic description of the space of Vassiliev invariants of a knot. We will be dealing with the so-called "chinese characters", which are graphs with (1,3)-valent vertices. We will always assume that each 3-valent vertex is endowed with a cyclic ordering of 3 egdes attached to it. When we draw a picture of a graph, we assume that this ordering is counterclockwise.

The graphs have 2 degrees. They are defined as

$$\deg_1(D) = \#\text{1-vertices}, \tag{2.1}$$

$$\deg_2(D) = \#\text{chords} - \#\text{3-vertices} = \chi(D) + \deg_1(D), \tag{2.2}$$

where $\chi(D)$ is the Euler characteristic of D.

Let $\tilde{\mathcal{B}}_{m,n}$ be a formal vector space (over $\mathbb{C}$) whose basis elements are in a one-to-one correspondence with (1,3)-valent graphs of degrees m and n. Together all such spaces form a graded space $\tilde{\mathcal{B}}$

$$\tilde{\mathcal{B}} = \bigoplus_{n=0}^{\infty} \tilde{\mathcal{B}}_n, \qquad \text{where } \tilde{\mathcal{B}}_n = \bigoplus_{m=0}^{\infty} \tilde{\mathcal{B}}_{m,n}. \tag{2.3}$$

The space $\tilde{\mathcal{B}}$ has two important subspaces. Let D_1 and D_2 be two graphs which are identical except that they differ in the cyclic order at one 3-valent vertex. Then the first subspace $\tilde{\mathcal{B}}_{\mathrm{AS}} \subset \tilde{\mathcal{B}}_{\mathrm{AS}}$ is spanned by the sums $D_1 + D_2$ coming from all such pairs of diagrams

$$\tilde{\mathcal{B}}_{\mathrm{AS}} = \mathrm{span}(D_1 + D_2 \quad \text{for all pairs } D_1, D_2). \tag{2.4}$$

To define the second subspace, consider a space $\tilde{\mathcal{B}}'$ whose basis vectors are graphs with 1-valent and 3-valent vertices and exactly one 4-valent vertex. We define a linear map $\partial_{\mathrm{IHX}} : \tilde{\mathcal{B}}' \longrightarrow \tilde{\mathcal{B}}$ by its action on the individual graphs of $D \in \tilde{\mathcal{B}}'$

$$\partial_{\mathrm{IHX}} : \; D \mapsto D_1 - D_2 + D_3, \tag{2.5}$$

where all four graphs D, D_1, D_2, D_3 are the same except at one spot, where they differ according to Fig. **??**.

Figure 2: The graphs D, D_1, D_2 and D_3

Then we define the second subspace $\tilde{\mathcal{B}}_{\mathrm{IHX}} \subset \tilde{\mathcal{B}}$ as the image of ∂_{IHX}. Now we introduce a space

$$\mathcal{B} = \tilde{\mathcal{B}}/(\tilde{\mathcal{B}}_{\mathrm{AS}} \cup \tilde{\mathcal{B}}_{\mathrm{IHX}}). \tag{2.6}$$

Since the graphs D_1, D_2 of (??) and the graphs D_1, D_2, D_3 of (??) have the same degree among themselves, then both subspaces $\tilde{\mathcal{B}}_{\mathrm{AS}}$ and $\tilde{\mathcal{B}}_{\mathrm{IHX}}$ respect the gradings (??) and as a result the space $\mathcal{B}$ is also graded

$$\mathcal{B} = \bigoplus_{n=0}^{\infty} \mathcal{B}_n, \qquad \text{where } \mathcal{B}_n = \bigoplus_{m=0}^{\infty} \mathcal{B}_{m,n}. \tag{2.7}$$

It is well-known that the dual space $\mathcal{B}^*$ is isomorphic to the space of all Vassiliev invariants, and the grading $\mathcal{B}^* = \bigoplus_{n=0}^{\infty} \mathcal{B}_n^*$ corresponds to the grading of Vassiliev invariants.

We are going to introduce another space $\mathcal{D}$ which is isomorphic to $\mathcal{B}$. This construction has been known to some people [?]. It appeared as an attempt to better understand the structure of $\mathcal{B}$ and, in particular, to evaluate the dimension of the spaces $\mathcal{B}_n$. I am especially indebted to A. Vaintrob for illuminating discussions on the structure of $\mathcal{D}$. In my work I had to introduce the space $\mathcal{D}$ in order to formulate a conjecture about the structure of Kontsevich integral, which was motivated by the study of the Melvin-Morton expansion of the colored Jones polynomial as it comes of R-matrix expression.

We begin by defining a bigger space $\tilde{\mathcal{D}}$. Let D be a graph with 3-valent vertices and no 1-valent vertices. We think of this graph as a CW-complex and consider a space of its rational cohomologies $H^1(D, \mathbb{Q})$. Let G_D be a group of symmetry of D (it maps 3-vertices to 3-vertices and edges to edges, preserving the cyclic order of edges at the vertices). This action extends to $H^1(D, \mathbb{Q})$ and to its symmetric algebra $S^* H^1(D, \mathbb{Q})$. We denote by $H(D)$ the G_D-invariant part of the latter space

$$H(D) = \bigoplus_{m=0}^{\infty} H_m(D), \qquad \text{where } H_m(D) = \mathrm{Inv}_{G_D}(S^m H^1(D, \mathbb{Q})), \tag{2.8}$$

while P_D is the corresponding projector

$$P_D : \, S^* H^1(D, \mathbb{Q}) \longrightarrow H(D), \qquad P_D(x) = \frac{1}{|G_D|} \sum_{g \in G_D} g(x), \tag{2.9}$$

where $|G_D|$ denotes the number of elements in G_D. Now we define a linear space $\tilde{\mathcal{D}}$ as

$$\tilde{\mathcal{D}} = \bigoplus_{m,n=0}^{\infty} \tilde{\mathcal{D}}_{m,n}, \qquad \text{where } \tilde{\mathcal{D}}_{m,n} = \bigoplus_{D:\, \chi(D)=n} H_m(D). \tag{2.10}$$

Next, we define the subspace $\tilde{\mathcal{D}}_{\rm AS} \subset \tilde{\mathcal{D}}$ which comes from the change of orientation at 3-valent vertices. Let D_1 be a 3-valent graph. Let us change the cyclic order of edges at one of the vertices of D_1. This way we get another graph D_2. The way in which we constructed D_2 provides for a natural map $f_{\rm AS}$ which maps the vertices and edges of D_1 to those of D_2. This map generates a map of cohomologies

$$\hat{f}_{\rm AS}:\ H^1(D_1,\mathbb{Q}) \longrightarrow H^1(D_2,\mathbb{Q}), \tag{2.11}$$

which can be extended as an algebra homomorphism

$$\hat{f}_{\rm AS}:\ S^*H^1(D_1,\mathbb{Q}) \longrightarrow S^*H^1(D_2,\mathbb{Q}). \tag{2.12}$$

Let $\tilde{V}_{\rm AS}$ be the graph of the latter map

$$\tilde{V}_{\rm AS} = \{(x,y)|\ y = \hat{f}_{\rm AS}(x)\} \subset S^*H^1(D_1,\mathbb{Q}) \oplus S^*H^1(D_2,\mathbb{Q}). \tag{2.13}$$

We denote by $V_{\rm AS}$ its projection onto $H(D_1) \oplus H(D_2)$

$$V_{\rm AS} = P_{D_1}P_{D_2}(\tilde{V}_{\rm AS}) \subset H(D_1) \oplus H(D_2). \tag{2.14}$$

The subspace $\tilde{\mathcal{D}}_{\rm AS} \subset \tilde{\mathcal{D}}$ is the sum of all the spaces $V_{\rm AS}$ for all 3-valent diagrams D_1 and all choices of vertices where we change the orientation.

Finally, we define a subspace $\tilde{\mathcal{D}}_{\rm IHX} \subset \tilde{\mathcal{D}}$. Let D be a graph with 3-valent vertices and with a single 4-valent vertex. By adding an extra edge to D, we "resolve" the 4-valent vertex in 3 different ways, thus converting D into one of the 3-valent graphs D_1, D_2, D_3 of Fig. **??**. A removal of the extra edge generates 3 natural maps of rational homologies

$$\hat{f}_i:\ H_1(D_i,\mathbb{Q}) \longrightarrow H_1(D,\mathbb{Q}), \qquad i=1,2,3. \tag{2.15}$$

We extend the dual maps $\hat{f}_i^*:\ H^1(D,\mathbb{Q}) \longrightarrow H^1(D_i,\mathbb{Q})$ as algebra homomorphisms

$$\hat{f}_i^*:\ S^*H^1(D,\mathbb{Q}) \longrightarrow S^*H^1(D_i,\mathbb{Q}), \qquad i=1,2,3. \tag{2.16}$$

We define the map $\hat{\partial}_{\rm IHX}:\ S^*H^1(D,\mathbb{Q}) \longrightarrow \bigoplus_{i=1}^3 H(D_i)$ by the formula (*cf.* eq.(**??**))

$$\hat{\partial}_{\rm IHX} = \sum_{i=1}^3 (-1)^{1+i} P_{D_i}\hat{f}_i^*. \tag{2.17}$$

The subspace $\tilde{\mathcal{D}}_{\mathrm{IHX}}$ is the sum of the images of all the maps $\hat{\partial}_{\mathrm{IHX}}$ for all the diagrams D.

Now we define the quotient space

$$\mathcal{D} = \tilde{\mathcal{D}}/(\tilde{\mathcal{D}}_{\mathrm{AS}} \cup \tilde{\mathcal{D}}_{\mathrm{IHX}}). \tag{2.18}$$

It is easy to see that this space is still graded

$$\mathcal{D} = \bigoplus_{m,n=0}^{\infty} \mathcal{D}_{m,n}, \tag{2.19}$$

where the spaces $\mathcal{D}_{m,n}$ are the quotients of the spaces $\tilde{\mathcal{D}}_{m,n}$.

$$\mathcal{D}_{m,n} = \tilde{\mathcal{D}}_{m,n}/(\tilde{\mathcal{D}}_{\mathrm{AS}} \cup \tilde{\mathcal{D}}_{\mathrm{IHX}}). \tag{2.20}$$

3 Isomorphism

Theorem 3.1 *There exists a canonical isomorphism*

$$\hat{A}:\ \mathcal{B} \longrightarrow \mathcal{D}, \tag{3.1}$$

which respects the grading

$$\hat{A}:\ \mathcal{B}_{m,n} \longrightarrow \mathcal{D}_{m,n-m}. \tag{3.2}$$

Corollary 3.2 *If $m > n$, then $\mathcal{B}_{m,n} = \varnothing$.*

Before we prove this theorem, we have to establish some facts concerning the structure of the space $\mathcal{B}$. We call an edge of a (1,3)-valent graph *a leg* if this edge is connected to a 1-valent vertex. All other edges are called *internal*.

Lemma 3.3 *If a (1,3)-valent graph D contains a 3-valent vertex attached to two legs, then $D \in \tilde{\mathcal{B}}_{\mathrm{AS}}$.*

Proof. Suppose that a (1,3)-valent graph D contains such a 3-valent vertex. Since the 1-valent vertices of our graphs are not ordered in any way, then changing the cyclic order at that 3-valent vertex does not change the graph. Therefore $2D \in \tilde{\mathcal{B}}_{\mathrm{AS}}$ and this proves the lemma. □

Let us call a (1,3)-valent graph *restricted* if each of its 3-valent vertices contains at most one leg. Let $\tilde{\mathcal{B}}^{(\mathrm{r})}$ be a formal space whose basis vectors are restricted graphs. We introduce familiar subspaces. The subspaces $\tilde{\mathcal{B}}^{(i)}_{\mathrm{AS}} \subset \tilde{\mathcal{B}}^{(\mathrm{r})}$, $i = 0, 1$ are spanned by the sums of restricted diagrams D_1, D_2 which differ in the ordering at a 3-valent vertex which is attached to i legs. The subspaces $\tilde{\mathcal{B}}^{(i)}_{\mathrm{IHX}} \subset \tilde{\mathcal{B}}^{(\mathrm{r})}$, $i = 0, 1$ are spanned by the images of the map (**??**) acting on the (3,4)-valent diagrams whose single 4-valent vertex contains i legs. Then Lemma **??** has a simple corollary:

$$\mathcal{B}_{m,n} = \tilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n} / (\tilde{\mathcal{B}}^{(0)}_{\mathrm{AS}} \cup \tilde{\mathcal{B}}^{(0)}_{\mathrm{IHX}}), \qquad \text{where } \tilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n} = \tilde{\mathcal{B}}^{(\mathrm{r})}_{m,n} / (\tilde{\mathcal{B}}^{(1)}_{\mathrm{AS}} \cup \tilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}). \tag{3.3}$$

Indeed, this relation follows from the fact that if the 4-valent vertex of a (3,4)-valent graph D has at least two legs, then the intersection of the image of the corresponding operator (**??**) with the space $\tilde{\mathcal{B}}^{(\mathrm{r})}$ is trivial.

Now we begin to construct the isomorphism. Let D be a 3-valent graph with N edges and let C_1 be an N-dimensional vector space whose basis vectors are the oriented edges e_j, $1 \le j \le N$ of D. We assume that an edge with the reverse orientation is equal to the opposite of the original edge as a vector of C_1. We will also need the dual space C_1^* with the dual basis f_j, $1 \le j \le N$. The symmetry group of the graphs G_D acts on both spaces C_1 and C_1^*.

Next, consider a vector space whose basis is formed by m-legged (1,3)-valent restricted graphs such that if we remove their legs, then we get the 3-valent graph D. We denote the quotient of this space by its intersection with $\tilde{\mathcal{B}}^{(1)}_{\mathrm{AS}}$ as $\tilde{\mathcal{B}}_m(D)$. We also want to consider a bigger space. Suppose that we index the edges of D and then attach m legs in order to produce restricted graphs. These graphs still carry the indexing of the edges of D. If we factor this space by its intersection with the obvious analog of $\tilde{\mathcal{B}}^{(1)}_{\mathrm{AS}}$, then we get the space

$\check{\mathcal{B}}_m(D)$. The symmetry group G_D of the graph D acts on $\check{\mathcal{B}}_m(D)$ and the invariant subspace of this action is canonically isomorphic to $\tilde{\mathcal{B}}_m(D)$:

$$\tilde{\mathcal{B}}_m(D) = \mathrm{Inv}_{G_D}(\check{\mathcal{B}}_m(D)). \tag{3.4}$$

Let us introduce a multi-index notation

$$\underline{m} = (m_1, \ldots, m_N), \qquad |\underline{m}| = \sum_{j=1}^{N} m_j. \tag{3.5}$$

For N non-negative numbers $\underline{m}$ construct a diagram $D_{\underline{m}}$ in the following way: for every j, $1 \le j \le N$ attach m_j legs to D on the left side of the edge e_j (the notion of the left side is well-defined since e_j is oriented). It is easy to see that all graphs $D_{\underline{m}}$, $|\underline{m}| = m$ form a basis of the space $\check{\mathcal{B}}_m(D)$, because after we took the quotient over the analog of the space $\tilde{\mathcal{B}}^{(1)}_{\mathrm{AS}}$, we can flip the legs of the graphs of $\check{\mathcal{B}}_m(D)$ to a particular side of each edge of D (at the cost of changing the signs of the corresponding vectors of $\check{\mathcal{B}}_m(D)$).

There is a natural isomorphism $A : \check{\mathcal{B}}_m(D) \longrightarrow S^m C_1^*$ which acts on the basis vectors as

$$\hat{A} : D_{\underline{m}} \mapsto \prod_{j=1}^{N} f_j^{m_j}. \tag{3.6}$$

Suppose that the 3-valent graph D has N_0 vertices v_j, $1 \le j \le N_0$. Consider the N_0-dimensional space C_0 whose basis vectors are in a one-to-one correspondence with these vertices. Then there is a natural boundary map $\partial : C_1 \longrightarrow C_0$. Let $\check{C}_1^*$ be the subspace of C_1^* whose elements annihilate the kernel of ∂. Apparently,

$$H^1(D, \mathbb{Q}) = C_1^* / \check{C}_1^*. \tag{3.7}$$

Let $\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D, m) = \check{\mathcal{B}}_m(D) \cap \check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}$, where the space $\check{\mathcal{B}}^{(1)}_{\mathrm{AS}}$ is the analog of the space $\tilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}$ for the graphs which come from 3-valent graphs with indexed edges.

Lemma 3.4 *The map $\hat{A}$ establishes an isomorphism between the spaces $\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D, 1)$ and $\check{C}_1^*$.*

Proof. For $1 \le j \le N_0$, denote as V_j the image in $\check{\mathcal{B}}_1(D)$ of the operator (**??**) associated with the vertex v_j of D (that is, one of the two 3-valent vertices in each of the graphs of Fig. **??**

is v_j, while the other vertex is attached to a leg). Then the space $\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,1)$ is spanned by all the spaces V_j.

For $1 \leq j \leq N_0$ and for $x \in C_1$ let $\partial_j(x)$ be the coefficient in front of $v_j \in C_0$ in the expansion of $\partial(x)$ with respect to the basis v. Then $\ker(\partial) = \bigcap_{j=1}^{N_0} \ker(\partial_j)$ and, as a result, the space $\check{C}_1^*$ is spanned by the spaces $V'_j \subset C_1^*$ which annihilate the spaces $\ker(\partial_j) \subset C_1$. It is very easy to see that for every j, $\hat{A}$ establishes an isomorphism between the corresponding spaces V_j and V'_j. This proves the lemma. $\square$

Lemma 3.5 *$\hat{A}$ establishes the isomorphism between the spaces $\check{\mathcal{B}}_m(D)/\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m)$ and $S^m H^1(D,\mathbb{Q})$.*

To prove this lemma we need a simple fact from linear algebra.

Lemma 3.6 *Let V be a finite dimensional vector space and W be its subspace. Denote by P_{S} a symmetrizing projector $P_{\mathrm{S}}: V^{\otimes m} \longrightarrow S^m V$. Then*

$$S^m V / P_{\mathrm{S}}(S^{m-1}V \otimes W) = S^m(V/W). \tag{3.8}$$

Proof. We leave the proof to the reader.

Proof of Lemma **??**. It is easy to see that $\hat{A}$ maps the space $\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m)$ onto $P_{\mathrm{S}}(S^{m-1}C_1^* \otimes \check{C}_1^*)$. Then the claim of the lemma follows from eqs. (**??**) and (**??**) if we set $V = C_1^*$ and $W = \check{C}_1^*$ in the latter equation. $\square$

Consider a space $\tilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m) = \tilde{\mathcal{B}}_m(D) \cap \tilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}$.

Lemma 3.7 *There is a natural isomorphism between the quotient spaces*

$$\tilde{\mathcal{B}}_m(D)/\tilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m) = \mathrm{Inv}_G(\check{\mathcal{B}}_m(D)/\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m)). \tag{3.9}$$

In order to prove this isomorphism we need another linear algebra lemma.

Lemma 3.8 *Let V be a finite-dimensional representation of a finite group G. Let $W \subset V$ be a subspace, which is invariant under the action of G. Then there is a natural isomorphism*

$$\mathrm{Inv}_G(V)/\,\mathrm{Inv}_G(W) = \mathrm{Inv}_G(V/W). \tag{3.10}$$

Proof. For example, one could use the fact that a finite-dimensional representation of G is a sum of irreducible representations. We leave the details to the reader. □

Proof of lemma **??**. The symmetry group D_G of the 3-valent graph D acts on the space $\check{\mathcal{B}}_m(D)$. Obviously, the symmetrization over this action projects $\check{\mathcal{B}}_m(D)$ onto $\tilde{\mathcal{B}}_m(D)$. Thus

$$\tilde{\mathcal{B}}_m(D) = \mathrm{Inv}_{G_D}(\check{\mathcal{B}}_m(D)). \tag{3.11}$$

At the same time, the subspace $\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m)$ is invariant under the action of G_D and

$$\tilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m) = \mathrm{Inv}_{G_D}(\check{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m)). \tag{3.12}$$

Then eq. (**??**) follows from eq. (**??**) in view of the relations (**??**) and (**??**). □

Let us introduce a notation $\mathcal{B}_m(D) = \tilde{\mathcal{B}}_m(D)/\tilde{\mathcal{B}}^{(1)}_{\mathrm{IHX}}(D,m)$.

Corollary 3.9 *The map $\hat{A}$ establishes the isomorphism between the spaces $\mathcal{B}_m(D)$ and $H_m(D)$ (see eq. (**??**) for the definition of the latter space).*

Proof. This isomorphism follows from the combination of Lemmas **??** and **??**. □

Now we can prove the theorem.

Proof of Theorem **??**. According the definition (**??**) of the space $\tilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n}$,

$$\tilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n+m} = \bigoplus_{D:\ \chi(D)=n} \mathcal{B}_m(D), \tag{3.13}$$

while by its definition

$$\tilde{\mathcal{D}}_{m,n} = \bigoplus_{D:\ \chi(D)=n} H_m(D). \tag{3.14}$$

It is easy to see that $\hat{A}$ establishes the isomorphisms

$$\hat{A}:\ \tilde{\mathcal{B}}^{(0)}_{\mathrm{AS}} \cap \tilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n+m} \longrightarrow \tilde{\mathcal{D}}_{\mathrm{AS}} \cap \tilde{\mathcal{D}}_{m,n}, \qquad \tilde{\mathcal{B}}^{(0)}_{\mathrm{IHX}} \cap \tilde{\mathcal{B}}^{(\mathrm{r},0)}_{m,n+m} \longrightarrow \tilde{\mathcal{D}}_{\mathrm{IHX}} \cap \tilde{\mathcal{D}}_{m,n}. \tag{3.15}$$

Then eq. (**??**) follows from eqs. (**??**) and (**??**) together with the isomorphism of Corollary **??**. □

4 Conjecture

Recall that Kontsevich integral of a knot $\mathcal{K} \in S^3$ is a sequence of vectors $I^{\mathcal{B}}_{m,n}(\mathcal{K}) \in \mathcal{B}_{m,n}$, $m \geq 0$, $n \geq m$ depending on the topological class of $\mathcal{K}$. The space $\mathcal{B}_{0,0}$ is 1-dimensional, its basis vector is the empty graph, so it can be naturally identified with $\mathbb{C}$. It is known that $I^{\mathcal{B}}_{0,0}(\mathcal{K}) = 1$.

We combine the vectors $I^{\mathcal{B}}_{n}(\mathcal{K})$ into a formal power series of a formal variable $\hbar$

$$I^{\mathcal{B}}(\mathcal{K};\hbar) = 1 + \sum_{\substack{m\geq 0\\ n\geq m\\ m+n\geq 1}} I^{\mathcal{B}}_{m,n}(\mathcal{K})\,\hbar^n \in \mathcal{B}. \tag{4.1}$$

Prior to formulating a conjecture about the structure of $I^{\mathcal{B}}(\mathcal{K};\hbar)$ we have to apply to it some transformations. First, we apply the wheeling map $\hat{\Omega}:\ \mathcal{B} \longrightarrow \mathcal{B}$, described in [?], in order to produce

$$I^{\Omega}(\mathcal{K};\hbar) = \hat{\Omega}(\,I^{\mathcal{B}}(\mathcal{K};\hbar)) = 1 + \sum_{\substack{m,n\geq 0\\ m+n\geq 1}} I^{\Omega}_{m,n}(\mathcal{K})\,\hbar^{m+n} \in \mathcal{D}, \qquad I^{\Omega}_{m,n} \in \mathcal{B}_{m,n}. \tag{4.2}$$

Then we apply the isomorphism $\hat{A}$, which maps Kontsevich integral from $\mathcal{B}$ to $\mathcal{D}$. Thus we get

$$I^{\mathcal{D}}(\mathcal{K};\hbar) = \hat{A}(\,I^{\mathcal{B}}(\mathcal{K};\hbar)) = 1 + \sum_{\substack{m,n\geq 0\\ m+n\geq 1}} I^{\mathcal{D}}_{m,n}(\mathcal{K})\,\hbar^{m+n} \in \mathcal{D}, \qquad I^{\mathcal{D}}_{m,n}(\mathcal{K}) \in \mathcal{D}_{m,n}. \tag{4.3}$$

Next, recall that the space $\mathcal{B}$ has a multiplication. A product of two (1,3)-valent graphs D_1 and D_2 is the graph which is a disjoint union of D_1 and D_2. The isomorphism $\hat{A}$ transfers this multiplication to the space $\mathcal{D}$. If a graph D is a disjoint union of two different 3-valent graphs D_1 and D_2, then $H(D) = H(D_1) \otimes H(D_2)$ (if $D_1 = D_2$, then the tensor product has to be symmetrized). Thus the product of two elements $x \in H(D_1)$ and $y \in H(D_2)$ is $x \otimes y \in H(D)$.

By using this product and manipulating formal power series in $\hbar$ we can define the logarithm of Kontsevich integral:

$$I^{(\log)}(\mathcal{K};\hbar) = \log I^{\mathcal{D}}(\mathcal{K};\hbar) = \sum_{\substack{m,n\geq 0\\ m+n\geq 1}} I^{(\log)}_{m,n}(\mathcal{K})\,\hbar^{m+n} \in \mathcal{D}, \qquad I^{(\log)}_{m,n}(\mathcal{K}) \in \mathcal{D}_{m,n}. \tag{4.4}$$

Kontsevich integral $I^{(\log)}(\mathcal{K};\hbar)$ belongs to the quotient space (**??**). Let $\tilde{I}^{(\log)}(\mathcal{K};\hbar)$ be a representative of $I^{(\log)}(\mathcal{K};\hbar)$ in the space $\tilde{\mathcal{D}}$. Of course, $\tilde{I}^{(\log)}(\mathcal{K};\hbar)$ is defined only up to an element of $\tilde{\mathcal{D}}_{\mathrm{AS}}+\tilde{\mathcal{D}}_{\mathrm{IHX}}$. Let $\mathbf{D}$ be a set of 3-valent graphs D such that each type of the graph (without distinguishing them by cyclic order at vertices) is represented there exactly once. Then we can pick $\tilde{I}^{(\log)}(\mathcal{K};\hbar)$ in such a way that

$$\tilde{I}^{(\log)}(\mathcal{K};\hbar)=\sum_{D\in\mathbf{D}}\sum_{m=0}^{\infty}x_m(\mathcal{K},D)\,\hbar^{\chi(D)+m}, \tag{4.5}$$

where $x_m(\mathcal{K},D)\in H_m(D)$.

Now we are almost ready to formulate our conjecture. Recall that for a 3-valent diagram D with N vertices e_j, $1\le j\le N$ denote the oriented edges forming a basis in the space C_1, while f_j, $1\le j\le N$ form the dual basis in the dual space C_1^*. In view of eq.(**??**) we can think of f_j as elements of $H^1(D,\mathbb{Q})$. For an element $x\in H^1(D,\mathbb{Q})$ we can define $e^x\in S^*H^1(D,\mathbb{Q})$ by the standard power series. Consider the lattice $H^1(D,\mathbb{Z})\subset H^1(D,\mathbb{Q})$. The exponentials of its elements generate an algebra over $\mathbb{Q}$, which is a subalgebra of $S^*H^1(D,\mathbb{Q})$. We denote the G_D-invariant part of this subalgebra as $H^{(\exp)}(D,\mathbb{Q})$.

Conjecture 4.1 *The representative $\tilde{I}^{(\log)}(\mathcal{K};\hbar)\in\tilde{\mathcal{D}}$ of Kontsevich integral $I^{(\log)}(\mathcal{K};\hbar)\in\mathcal{D}$ can be chosen in such a way that for any $D\in\mathbf{D}$, $\chi(D)\ge 1$ there exists an element $y(\mathcal{K},D)\in H^{(\exp)}(D,\mathbb{Q})$ such that*

$$\sum_{m=0}^{\infty}x_m(\mathcal{K},D)=\frac{y(\mathcal{K},D)}{\prod_{j=1}^{N}\Delta_{\mathrm{A}}(\mathcal{K};\exp(f_j))}. \tag{4.6}$$

Remark 4.2 The *r.h.s.* of eq.(**??**) is a well-defined element of $H(D)$. Indeed, since

$$\Delta_{\mathrm{A}}(\mathcal{K};1/t)=\Delta_{\mathrm{A}}(\mathcal{K};t), \tag{4.7}$$

then the denominator of that expression is G_D-invariant and thus belongs to $H(D)$. On the other hand, the Alexander-Conway polynomial satisfies the property $\Delta_{\mathrm{A}}(\mathcal{K};1)=1$ which guarantees that the denominator can be inverted within $H(D)$.

Remark 4.3 The only 3-valent graph D with $\chi(D)=0$ is the circle. The value of the *l.h.s.* of eq. (**??**) has been established by D. Bar-Natan and S. Garoufalidis in [**?**]

$$\sum_{m=0}^{\infty} x_m(\mathcal{K},\mathrm{circle}) = \frac{1}{2}\left(\log\frac{\sinh(f/2)}{(f/2)} - \log\Delta_{\mathrm{A}}(\mathcal{K};\exp(f))\right), \tag{4.8}$$

where f represents the integral generator of $H^1(\mathrm{circle},\mathbb{Q})$.

Remark 4.4 D. Thurston presented the arguments which show that if Conjecture **??** is true as it is formulated, then it should also be true if one defines $I^{\mathcal{D}}(\mathcal{K},\hbar)$ directly as an image of $I^{\mathcal{B}}(\mathcal{K},\hbar)$ under the isomorphism $\hat{A}$ without appying the wheeling map $\hat{\Omega}$ of eq. (**??**).

5 Evidence

We are going to present some evidence in favor of Conjecture **??**. It is hard to calculate Kontsevich integral directly, so we have to rely on an indirect method of gaining information about its properties. We will use a well-know relation between Kontsevich integral and the colored Jones polynomial. Let $\mathfrak{g}$ be a simple Lie algebra endowed with the invariant scalar product normalized in such a way that long roots have length $\sqrt{2}$ (this scalar product allows us to identify the dual space $\mathfrak{g}^*$ with $\mathfrak{g}$ itself). Let $\vec{\alpha}\in\mathfrak{h}$ be the hightest weight of a representation of $\mathfrak{g}$, shifted by $\vec{\rho}$ (which is half the sum of positive roots of $\mathfrak{g}$). Reshetikhin and Turaev associate to this data a polynomial $J_{\vec{\alpha}}(\mathcal{K};q)\in\mathbb{Z}[q^{\pm 1/2}]$. If we substitute

$$q = e^{\hbar}, \tag{5.1}$$

then we can expand $J_{\vec{\alpha}}(\mathcal{K};q)$ in power series of $\hbar$

$$J_{\vec{\alpha}}(\mathcal{K};q) = \sum_{n=0}^{\infty} p_n(\mathcal{K};\vec{\alpha})\,\hbar^n, \tag{5.2}$$

whose coefficients $p_n(\mathcal{K};\vec{\alpha})$ are polynomials of $\vec{\alpha}$.

The same series (**??**) can be deduced from the value of Kontsevich integral.

The data $\mathfrak{g},\vec{\alpha}$ defines an element in the dual space $\mathcal{B}^*$, which is called *the weight system.* We will define it in such a way that it will be suitable for application to $I^{\Omega}(\mathcal{K};\hbar)$.

The first steps in the definition of the weight systems are fairly standard. Let $\vec{x}_a$, $1 \leq a \leq \dim \mathfrak{g}$ be a basis of $\mathfrak{g}$. Define the structure constants f_{abc} by the relation

$$[\vec{x}_a, \vec{x}_b] = \sum_{c=1}^{\dim \mathfrak{g}} f_{ab}{}^{c}\, \vec{x}_c. \tag{5.3}$$

We can raise and lower the indices of $f_{ab}{}^{c}$ with the help of the metric tensor

$$h_{ab} = \vec{x}_a \cdot \vec{x}_b \tag{5.4}$$

and its inverse h^{ab}.

Let D be a (1,3)-valent graph, $\deg_1(D) = m$, $\deg_2(D) = n + m$. Suppose that if we strip off its legs, then we get a 3-valent graph D_0. Let us pick an orientation of the edges of D_0 and assign orientation to the edges of D in such a way that it is compatible with the orientation of D_0 and legs are oriented in the direction from 1-valent vertex to 3-valent vertex. Next, we assign the tensors f to 3-valent vertices, assigning their indices to attached edges according to the cyclic ordering. We use the upper indices for the incoming edges and lower indices for the outgoing edges. Finally, we take the product of all tensors f assigned to 3-valent vertices, contract each pair of indices of f's along each internal edge, while contracting each index assigned to a leg with α_a ($\vec{\alpha} = \sum_{a=1}^{\dim \mathfrak{g}} \alpha_a\, \vec{x}_a$). Thus we get a homogeneous polynomial $w^{\Omega}(D, \vec{\alpha})$ of $\vec{\alpha}$ of degree m. It is easy to see that it does not depend on the choice of orientation of the edges of D_0. For a fixed weight $\vec{\alpha}$, w^{Ω} assignes a number to each (1,3)-valent graph, so $w^{\Omega} \in \tilde{\mathcal{B}}^*$. In fact, due to the anti-symmetry of f and to the Jacobi identity, satisfied by the commutator (**??**), w^{Ω} annihilates the subspaces $\tilde{\mathcal{B}}_{\mathrm{AS}}$ and $\tilde{\mathcal{B}}_{\mathrm{IHX}}$ and therefore it can be projected to $\mathcal{B}^*$. We explained in [**?**] that

$$J_{\vec{\alpha}}(\mathcal{K}; q) = d_{\vec{\alpha}} \left(1 + \sum_{\substack{m,n \geq 0 \\ m+n \geq 1}} w^{\Omega}(\, I^{\Omega}_{m,n}(\mathcal{K}), \vec{\alpha})\, \hbar^{m+n} \right), \tag{5.5}$$

where $d_{\vec{\alpha}}$ is the dimension of the representation of $\mathfrak{g}$ with the shifted highest weight $\vec{\alpha}$.

The inverse of the dual isomorphism map $\hat{A}^*$ maps the weight system $w^{\Omega} \in \mathcal{B}^*$ into an element of $\mathcal{D}^*$, which we will call $w^{\mathcal{D}}$. In order to see how $w^{\mathcal{D}}$ acts on $\mathcal{D}$ we come back to the calculation of $w^{\Omega}(D, \vec{\alpha})$ and modify it.

For a root λ of $\mathfrak{g}$ let P_λ denote the operator, projecting $\mathfrak{g}$ onto the root space $V_\lambda \subset \mathfrak{g}$. We also introduce an operator $P_\mathfrak{h}$, projecting $\mathfrak{g}$ onto $\mathfrak{h}$. Let us assign a root of $\mathfrak{g}$ or the Cartan subalgebra to each internal edge of D. Let $\tilde{\mathbf{S}}$ be a set of all such assignments. For an assignment $c \in \tilde{\mathbf{S}}$ we modify the contraction of indices of tensors f in the following way: if an internal edge carries an index a at the beginning and index b at the end, then instead of contracting them (that is, instead of setting $a = b$ and taking a sum over their values) we bring in an extra factor P^a_b, where P is the projector corresponding to the subspace assigned to that edge by c, and then contract the pairs of indices a and b independently. In other words, we project Lie algebras $\mathfrak{g}$ flowing along the internal edges of D onto root spaces and Cartan subalgebras. Let us denote the resulting number as $w^\Omega_c(D, \vec{\alpha})$. Since the sum of projectors $P_\mathfrak{h}$ and P_λ for all roots λ of $\mathfrak{g}$ is equal to the identity operator, then

$$w^\Omega(D, \vec{\alpha}) = \sum_{c \in \tilde{\mathbf{S}}} w^\Omega_c(D, \vec{\alpha}). \tag{5.6}$$

The sum in the *r.h.s.* of this equation can be simplified. Since $\vec{\alpha} \in \mathfrak{h}$, then

$$[\vec{\alpha}, \vec{y}] = (\vec{\alpha} \cdot \lambda)\, \vec{y} \qquad \text{if } \vec{y} \in V_\lambda, \qquad [\vec{\alpha}, \vec{y}] = 0 \qquad \text{if } \vec{y} \in \mathfrak{h}. \tag{5.7}$$

Therefore, $w^\Omega_c(D, \vec{\alpha}) = 0$ unless the following two conditions are met. First, c must assign the same projector to internal edges of D which correspond to the same edge of D_0. Second, there is a *compatibility requirement* at every 3-valent vertex: Cartan subalgebra can be assigned to at most one of its edges and the sum of the roots on incoming edges is equal to the sum of the roots on outgoing edges. Thus we can substitute the set $\tilde{\mathbf{S}}$ in eq. (**??**) with the set $\mathbf{S}$ whose elements assign subspaces to the edges of D_0 in such a way that the compatibility condition is satisfied at all of its vertices.

Equations (**??**) also indicate that the effect of leg contractions is easy to account for in the calculation of $w^\Omega_c(D, \vec{\alpha})$. If a leg is attached to at least one edge, to which a Cartan subalgebra is assigned, then $w^\Omega_c(D, \vec{\alpha}) = 0$. Otherwise, if m_j legs are attached on the left side of an oriented edge e_j of D_0 to which a root λ is assigned, then they contribute a factor of $(\vec{\alpha} \cdot \lambda)^{m_j}$. Let $\lambda_{c(j)}$ denote the root of $\mathfrak{g}$ assigned by $c \in \mathbf{S}$ to the edge e_j of D_0. If c

assignes $\mathfrak{h}$ to e_j, then we set $\lambda_{c(j)} = 0$. With these notations we see that

$$w_c^{\Omega}(D, \vec{\alpha}) = w_c(D_0) \prod_{j=1}^{N} (\vec{\alpha} \cdot \lambda_{c(j)})^{m_j}, \tag{5.8}$$

where $w_c(D_0) = w_c^{\Omega}(D_0, \vec{\alpha})$ (we had to introduce this new notation because the graph D_0 has no legs and as a result $w_c^{\Omega}(D_0, \vec{\alpha})$ does not depend on $\vec{\alpha}$). Note that in eq. (**??**) we adopted a convention that $0^0 = 1$.

The isomorphism (**??**) completes the translation of $w_c^{\Omega}(D, \vec{\alpha})$ into the language of 3-valent graphs. For an assignment $c \in \mathbf{S}$ consider a linear combination of edges

$$e_{c,\vec{\alpha}} = \sum_{j=1}^{N} (\vec{\alpha} \cdot \lambda_{c(j)})\, e_j \in C_1. \tag{5.9}$$

According to the compatibility condition satisfied by c, $e_{c,\vec{\alpha}} \in \ker(\partial) = H_1(D, \mathbb{Q})$. Therefore, we can evaluate an element $x \in S^* H^1(D, \mathbb{Q})$ on $e_{c,\vec{\alpha}}$ and get a number (or a formal series) $x(e_{c,\vec{\alpha}})$. Equations (**??**), (**??**) and (**??**) indicate that for an element $x \in \check{\mathcal{B}}_m(D_0)$,

$$w_c^{\Omega}(x, \vec{\alpha}) = (\hat{A}\, x)(e_{c,\vec{\alpha}})\, w_c(D_0). \tag{5.10}$$

Then, according to eq. (**??**), after taking a sum over the assignments of $\mathbf{S}$, we come to the following relation: for any $x \in \tilde{\mathcal{B}}_m(D_0)$,

$$w^{\Omega}(x, \vec{\alpha}) = w^{\mathcal{D}}(\hat{A}\, x, \vec{\alpha}), \tag{5.11}$$

where

$$w^{\mathcal{D}}(y, \vec{\alpha}) = \sum_{c \in \mathbf{S}} y(e_{c,\vec{\alpha}})\, w_c(D_0). \tag{5.12}$$

Thus eq. (**??**) defines the element $w^{\mathcal{D}} \in \mathcal{D}^*$ corresponding to $w^{\Omega} \in \mathcal{B}^*$.

Applying eq. (**??**) to eq. (**??**), we find that

$$J_{\vec{\alpha}}(\mathcal{K}; q) = d_{\vec{\alpha}} \left(1 + \sum_{\substack{m,n \geq 0 \\ m+n \geq 1}} w^{\mathcal{D}}(\, I^{\mathcal{D}}_{m,n}(\mathcal{K}), \vec{\alpha})\, \hbar^{m+n} \right). \tag{5.13}$$

It is easy to see that the weight system $w^{\mathcal{D}}$ behaves nicely under the multiplication of elements of $\mathcal{D}$: $w^{\mathcal{D}}(xy,\vec{\alpha}) = w^{\mathcal{D}}(x,\vec{\alpha})\, w^{\mathcal{D}}(y,\vec{\alpha})$ for any $x, y \in \mathcal{D}$. Therefore the analog of eq.(**??**) holds for the modified integral (**??**)

$$\log\left(J_{\vec{\alpha}}(\mathcal{K};q)/d_{\vec{\alpha}}\right) = \sum_{\substack{m,n\geq 0\\ m+n\geq 1}} w^{\mathcal{D}}\big(\, I^{(\log)}_{m,n}(\mathcal{K}),\vec{\alpha}\big)\, \hbar^{m+n}, \tag{5.14}$$

and for its representative (**??**) in the space $\tilde{\mathcal{D}}$

$$\log\left(J_{\vec{\alpha}}(\mathcal{K};q)/d_{\vec{\alpha}}\right) = \sum_{D\in\mathbf{D}} \sum_{m=0}^{\infty} w^{\mathcal{D}}\big(\, x_m(\mathcal{K},D)\big)\, \hbar^{\chi(D)+m}. \tag{5.15}$$

By using the formula (**??**) for the weight system, we can rewrite eq.(**??**) as

$$\log\left(J_{\vec{\alpha}}(\mathcal{K};q)/d_{\vec{\alpha}}\right) = \sum_{D\in\mathbf{D}} \sum_{c\in\mathbf{S}} \sum_{m=0}^{\infty} x_m(\mathcal{K},D)(e_{c,\vec{\alpha}})\, w_c(D)\, \hbar^{\chi(D)+m}, \tag{5.16}$$

where $x_m(\mathcal{K},D)(e_{c,\vec{\alpha}})$ denotes the evaluation of the element $x_m(\mathcal{K},D) \in H_m(D)$ on $e_{c,\vec{\alpha}} \in H_1(D,\mathbb{Q})$. According to eq.(**??**), $e_{c,\vec{\alpha}}$ is a linear function of $\vec{\alpha}$, while $x_m(\mathcal{K},D)(e_{c,\vec{\alpha}})$ is the homogeneous polynomial of $e_{c,\vec{\alpha}}$. Therefore, eq.(**??**) can be further modified as

$$\log\left(J_{\vec{\alpha}}(\mathcal{K};q)/d_{\vec{\alpha}}\right) = \sum_{D\in\mathbf{D}} \hbar^{\chi(D)} \sum_{c\in\mathbf{S}} w_c(D) \sum_{m=0}^{\infty} x_m(\mathcal{K},D)(e_{c,\hbar\vec{\alpha}}). \tag{5.17}$$

Let us apply the conjectured formula (**??**) to $\sum_{m=0}^{\infty} x_m(\mathcal{K},D)(e_{c,\hbar\vec{\alpha}})$. Since by the definition of the dual basis $f_j(e_i) = \delta_{ij}$, then according to eq.(**??**), $f_j(e_{c,\hbar\vec{\alpha}}) = \hbar\,(\vec{\alpha}\cdot\lambda_{c(j)})$ and as a result, in view of (**??**),

$$\Delta_{\mathrm{A}}(\mathcal{K};\exp(f_j))(e_{c,\hbar\vec{\alpha}}) = \Delta_{\mathrm{A}}(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_{c(j)}}), \tag{5.18}$$

$$w_c(D)\, y(\mathcal{K},D)(e_{c,\hbar\vec{\alpha}}) = p_c(\mathcal{K},D;q^{\vec{\alpha}\cdot\lambda_1},\ldots,q^{\vec{\alpha}\cdot\lambda_k}), \tag{5.19}$$

where $\lambda_1,\ldots,\lambda_k$ are all positive roots of $\mathfrak{g}$ and

$$p_c(\mathcal{K},D;t_1,\ldots,t_k) \in \mathbb{Q}[t_1,\ldots,t_k]. \tag{5.20}$$

Applying the weight system to the contribution of the circle graph, described by eq.(**??**), is rather straightforward. We just have to take a sum over assignments of the roots of $\mathfrak{g}$ to the circle. Note that in view of eq.(**??**), an assignment of a positive root λ yields the same number as the assignment of its opposite.

Thus we proved that Conjecture **??** has the following

Corollary of Conjecture 4.1 *For a knot $\mathcal{K}$ and a simple algebra $\mathfrak{g}$ there exist the polynomials (??) such that*

$$\log\left(J_{\vec{\alpha}}(\mathcal{K};q)/d_{\vec{\alpha}}\right) = \sum_{j=1}^{k}\log\left(\frac{q^{(\vec{\alpha}\cdot\lambda_j/2)}-q^{-(\vec{\alpha}\cdot\lambda_j)/2}}{\hbar\,(\vec{\alpha}\cdot\lambda_j)}\right)-\sum_{j=1}^{k}\log\Delta_{\mathrm{A}}(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_j}) \qquad (5.21)$$
$$+\sum_{D\in\mathbf{D},\,\chi(D)\geq 1}\hbar^{\chi(D)}\sum_{c\in\mathbf{S}}\frac{p_c(\mathcal{K},D;q^{\vec{\alpha}\cdot\lambda_1},\ldots,q^{\vec{\alpha}\cdot\lambda_k})}{\prod_{j=1}^{N}\Delta_{\mathrm{A}}(\mathcal{K};q^{\vec{\alpha}\cdot\lambda_{c(j)}})}.$$

We can check this prediction for the case of $\mathfrak{g}=su(2)$. The algebra $su(2)$ has only one positive root. As a result, the elements of $\mathbf{S}$ assign the subspaces of $su(2)$ to the edges of a diagram D in such a way that for any three edges attached to the same vertex, two are assigned a root space and the third is assigned the Cartan subalgebra. A connected 3-valent graph has $3\chi(D)$ edges. In the case of $su(2)$, $\chi(D)$ of those edges carry a Cartan subalgebra, while $2\chi(D)$ edges carry a root space. In view of eq. (??), the positive and the negative root give the same contribution to the denominators of eq. (??), so denominators do not depend on c. Therefore if $\mathfrak{g}=su(2)$, then we introduce a notation

$$p_n(\mathcal{K};t)=\sum_{D\in\mathbf{D},\,\chi(D)=n}\ \sum_{c\in\mathbf{S}}p_c(\mathcal{K};t),\qquad p_n(\mathcal{K};t)\in\mathbb{Q}[t^{\pm 1}] \qquad (5.22)$$

and eq. (??) is reduced to

$$\log\left(J_\alpha(\mathcal{K};q)/\alpha\right)=-\log\Delta_{\mathrm{A}}(\mathcal{K};q^\alpha)+\sum_{n=1}^{\infty}\frac{p_n(\mathcal{K};q^\alpha)}{\Delta_{\mathrm{A}}^{2n}(\mathcal{K};q^\alpha)}\,\hbar^n, \qquad (5.23)$$

where α is the dimension of the representation attached to the knot $\mathcal{K}$.

Equation (??) can be verified directly. We proved in [?] that for a knot $\mathcal{K}$ in S^3 there exist the polynomials

$$P_n(\mathcal{K};t)\in\mathbb{Z}[t^{\pm 1}],\quad n\geq 1, \qquad (5.24)$$

such that the expansion (??) can be rewritten as

$$J_\alpha(\mathcal{K};q)=\frac{[\alpha]}{\Delta_{\mathrm{A}}(\mathcal{K};q^\alpha)}\left(1+\sum_{n=1}^{\infty}\frac{P_n(\mathcal{K};q^\alpha)}{\Delta_{\mathrm{A}}^{2n}(\mathcal{K};q^\alpha)}\,h^n\right), \qquad (5.25)$$

where

$$[\alpha]=\frac{q^{\alpha/2}-q^{-\alpha/2}}{q^{1/2}-q^{-1/2}},\qquad h=q-1. \qquad (5.26)$$

Since $J_\alpha(\text{unknot}; q) = [a]$, then eq. (**??**) follows easily from eq. (**??**).

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