Notes on the Geometry and Topology of 3-Manifolds

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Chapter 1

Introduction

The main subject of these notes is the Geometry and Topology of 3-dimensional orientable manifolds. The point of view adopted is to emphasize the geometric properties of 3-manifolds.

A major result of the late 19th century is the uniformization theorem for Riemann surfaces:

Theorem 1.1 (Riemann Uniformization Theorem). *Any compact (orientable) surface admits a Riemannian metric of constant curvature.*

In other words, any compact surface is either spheric, Euclidean or hyperbolic: that is to say that it is the quotient of the standard 2-sphere $\mathbb{S}^2$, or the Euclidean plan $\mathbb{E}^2$ or of the hyperbolic plan $\mathbb{H}^2$ by a discrete subgroup of isometries. Moreover a compact surface belongs to a unique type, according to the Gauss-Bonnet formula that determines the Euler characteristic of the surface.

The situation for 3-manifolds is much more complicated. It is only "recently", due mainely to the seminal works of Thurston, that an analogous, but much more invoved, theory has emerged in dimension3. This theory can be very well summarized by the following Geometrization conjecture, due to Thurston:

Thurston's Geometrization Conjecture. *The interior of any compact orientable 3-manifold M can be split along a canonical family of essential spheres and tori into submanifolds whose interiors admit complete homogeneous geometric structure.*

A particular case of this conjecture is the wellknown Poincaré conjecture:

Poincaré Conjecture. *A closed simply connected 3-manifold is homeomorphic to S^3.*

Thurston's approach to the study of 3-manifolds put forwards a geometric point of view, in which the Poincaré conjecture becomes a uniformization problem.

Thurston has classified the homogeneous 3-dimensional geometries that may endow the interior of a compact 3-manifold. There are only eight possible geometries: three geometries with constant curvature, corresponding to the standard 3-sphere $\mathbb{S}^3$, the Euclidean space $\mathbb{E}^3$ and the hyperbolic space $\mathbb{H}^3$; two product geometries modeled on $\mathbb{S}^2 \times \mathbb{E}^1$ and $\mathbb{H}^2 \times \mathbb{E}^1$; three fibered geometries modeled on the Lie groups Nil, $\widetilde{SL_2}(\mathbb{R})$ and Sol.

Among these eight 3-dimensional geometries the hyperbolic geometry is the only one that does not collapse to a 2-dimensional or a 1-dimensional geometry. Thurston's work has shown that this robust hyperbolic geometry is the most common one among geometric 3-manifolds.

The topological background for Thurston's geometrization conjecture is given by the following splitting theorem for an orientable compact 3-manifold:

Theorem 1.2 (Canonical Decomposition Theorem). *Let M be a compact orientable 3-manifold different from S^3.*
a) There is a finite (perhaps empty) family of disjoint essential embedded spheres in M which splits M into prime manifolds different from S^3. Moreover, these prime factors M_i are unique up to homeomorphism.
b) For each irreducible factor M_i, there is a finite (perhaps empty) family of essential disjoint and non-parallel tori $\mathfrak{T}_i$ which splits M_i into either strongly atoroidal pieces or geometric pieces modelled on $\mathbb{H}^2 \times \mathbb{E}$, $\widetilde{SL_2(\mathbb{R})}$, $\mathbb{E}^3$, Nil or Sol. Moreover, a minimal such family $\mathfrak{T}_i$ is unique up to isotopy in M_i.

The first stage a) of the decomposition expresses the 3-manifold M as the connected sum of prime factors. These prime 3 -manifolds are either homeomorphic to $S^1 \times S^2$ or irreducible.

An orientable 3-manifold M is irreducible if any embedding of the 2-sphere into M extends to an embedding of the 3-ball into M. This notion is crucial for the study of topological properties of 3-manifold.

A compact orientable 3-manifold is atoroidal if it does not contain any essential embedded torus. It is strongly atoroidal if it is not homeomorphic to $S^1 \times D^2$, $T \times [0,1]$, $K \tilde{\times} [0,1]$ (where K is the Klein bottle) and if any finite covering of M is atoroidal.

The main and fundamental contribution of Thurston to his conjecture is the following hyperbolization theorem:

Theorem 1.3 (Thurston's Hyperbolization Theorem). *Let M be a compact orientable irreducible 3-manifold. Assume that M contains a π_1-injected properly embedded surface of strictly negative Euler characteristic. Then the interior of M admits a complete hyperbolic structure iff M is strongly atoroidal.*

A direct consequence of this hyperbolization theorem and the canonical decomposition theorem is:

Corollary 1.4. *Thurston's Geometrization Conjecture is true provided that M contains a π_1-injected properly embedded surface of strictly negative Euler characteristic.*

These results reduce Thurston's geometrization conjecture to the following uniformization conjecture:

Uniformization Conjecture. *Let M be a compact orientable irreducible strongly atoroidal 3-manifold. Then*
a) M is hyperbolic if and only if $\pi_1 M$ is infinite.
b) M is spherical if and only if $\pi_1 M$ is finite.

Theses conjectures are still widly open. They may a priori seem to be independant. It is deep idea of Thurston' programm to link these two conjectures, by showing that the spherical geometry may appear as limite of a collapsed sequence of singular cone hyperbolic structures on a given 3-manifold.

Theorem 1.5 (Thurston's Orbifold Theorem). *Thurston's Geometrization Conjecture is true, provided that there is an orientation preserving homeomorphism $\phi : M \to M$ with $\phi \neq id$, $\phi^n = id$ for some $n \geq 2$ and $Fix(\phi) \neq \emptyset$.*

These notes (which are still incomplete) are intended to be an introduction to the topology and geometry of 3-manifolds, with an emphasize on the geometric methods. Our choice is not to give detailled proofs of all the results presented in these notes, but to do it only for some results, when we think that the proofs will help to understand some basic ideas or methods in 3-dimensional geometry. So, many proofs will be only sketched and left in exercises.

Chapter 2

Thurston's Eight Geometries

A **geometric structure** on a n-manifold M is a Riemannian metric on the interior of M which is locally homogeneous.
Let X be a simply connected orientable n-manifold with a Riemannian metric such that $G = Isom_0^+(X)$ acts transitively on X. (G is the identity component of the group of orientation preserving isometries of X.)
M has a (X, G)**-structure** if it has an atlas of local coordinates each into X such that the overlap maps lie in G.
If M has a *complete* (X, G)-structure then $M = X/\Gamma$, where $\Gamma \subset G$ is a discrete subgroup acting freely and properly discontinuously on X.

2.1 2-dimensional geometries

We are looking for X with constant Gauss curvature (=sectional curvature). The classification is up to equivalence, where $(X, G) \sim (X', G')$ if and only if there is a diffeomorphism $\phi : X \to X'$ conjugating the action of G and G'. There are only three possible models (after rescaling):

$$(S^2, SO(3)), (\mathbb{E}^2, Isom^+(\mathbb{E}^2)), (\mathbb{H}^2, PSL_2(\mathbb{R}))$$

with corresponding constant sectional curvature $K \equiv 1,\ 0,\ -1$.

Theorem 2.1 (Riemann Uniformization Theorem). *Any compact orientable surface admits one of these three geometries. Any surface belongs to only one class.*

The fact that a surface belongs to a unique geometric type follows from the Gauss-Bonnet formula $\chi(M) = \int_M K\, ds$.

2.2 3-dimensional geometries

Theorem 2.2 (Thurston's classification of 3-dimensional geometries). *Up to equivalence there are exactly eight models for (X, G)-structures which admit a complete finite volume quotient. Here is a list according to the dimension of G_x, the stabilizer of $x \in X$:*
1) $G_x = SO(3)$
The sectional curvature is constant and the geometries are

$$(S^3, SO(4)), (\mathbb{E}^3, Isom^+(\mathbb{E}^3)), (\mathbb{H}^3, PSL_2(\mathbb{C}))$$

2) $G_x = SO(2)$
X is a Riemannian line bundle over a 2-dimensional homogeneous space and the geometries are either product geometries

$$(S^2 \times \mathbb{E}^1, Isom^+(S^2 \times \mathbb{E}^1)), (\mathbb{H}^2 \times \mathbb{E}^1, Isom^+(\mathbb{H}^2 \times \mathbb{E}^1))$$

or twisted product geometries:

$$(Nil, \mathbb{R} \to G \to Isom^+(\mathbb{E}^2)), (\widetilde{SL_2(\mathbb{R})}, \mathbb{R} \times \widetilde{PSL_2(\mathbb{R})})$$

Nil is the nilpotent Lie group of dimension three (Heisenberg Matrix group)

$$\{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R}\}$$

3) $G_x = \{id\}$
$X = G_x \backslash G$, $X = Sol$, a solvable Lie group given by the split extension $\mathbb{R}^2 \to Sol \to \mathbb{R}$, where $t \in \mathbb{R}$ acts on $\mathbb{R}^2$ by

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Typical closed 3-dimensional representatives for these eight geometries are:

- $(S^3, SO(4))$, spherical geometry: S^3, Lens spaces $S^3/\mathbb{Z}_n$

- $(\mathbb{E}^3, Isom^+(\mathbb{E}^3))$, Euclidean geometry: $T^3 = S^1 \times S^1 \times S^1$
  By Bieberbach's Theorem (1911), any Euclidean closed 3-manifold is finitely covered by T^3 (there are only seven orientable examples)

- $(\mathbb{H}^3, PSL_2(\mathbb{C}))$, hyperbolic geometry: There are no typical examples but the following theorem:

Theorem 2.3 (Thurston). *Let F be a closed orientable surface of genus $g(F) > 1$ and let $\phi : F \to F$ be a pseudo-Anosov diffeomorphism (i.e. there is no $\gamma \in \pi_1 F \setminus \{1\}$ such that $\phi_*^n(\gamma)$ is conjugated to γ, where ϕ_* is the induced map in π_1). Then the mapping torus*

$$(F, \phi) := F \times [0,1]/\{(x,0) \sim (\phi(x),1)\}$$

has a complete hyperbolic structure of finite volume.

The following conjecture of Thurston asserts that these mapping tori are typical representatives of closed hyperbolic 3-manifolds:

Conjecture 2.4 (Thurston). *Any closed hyperbolic 3-manifold is finitely covered by a pseudo-Anosov mapping torus.*

- Nil geometry: $\mathbb{R} \to Nil \to \mathbb{R}^2$
  S^1-bundle over T^2 with non-zero Euler class.
- $\mathbb{H}^2 \times \mathbb{E}^1$ geometry: $F_g \times S^1$, $g > 1$
- $S^2 \times \mathbb{E}^1$ geometry: There are only two orientable examples: $S^2 \times S^1$ and $\mathbb{R}P^3 \sharp \mathbb{R}P^3$
- $\widetilde{SL_2(\mathbb{R})}$ geometry: S^1-bundle over F_g, $g > 1$, with a non-trivial Euler class.
  Consider $PSL_2(\mathbb{R}) \cong T_1\mathbb{H}^2$ and $\Gamma \subset PSL_2(\mathbb{R})$ a discrete cocompact subgroup. Γ can act on $T_1\mathbb{H}^2$ as follows:
  $\gamma : (x, \vec{u}) \mapsto (\gamma(x), d\gamma_x(\vec{u}))$. Then $T_1\mathbb{H}^2/\Gamma$ has a $\widetilde{SL_2(\mathbb{R})}$ geometry.
- Sol geometry: $\mathbb{R}^2 \to Sol \to \mathbb{R}$, where the extension is given by:

$$t.(x,y) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

  Torus bundles $T^2 \to Sol/\Gamma \to S^1$, whith Anosov monodromies $\phi : T^2 \to T^2$, that is conjugated to

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \lambda > 1$$

It follows from the examples above that except for hyperbolic geometry, all other compact geometric orientable 3-manifolds admit a foliation by circles or by tori.

Proof of Theorem 2.2. Consider the pair (X, G) where X is a simply connected 3-dimensional homogeneous Riemannian manifold and $G = Isom_0^+(X)$ (component of the identity) acts transitively on X. There exists a quotient X/Γ, Γ a discrete subgroup of G, which is complete and of finite volume. $(X, G) \sim (X', G')$ if there is a diffeomorphism $\phi : X \to X'$ such that $G' = \phi G \phi^{-1}$. Let G_x be the stabilizer of $x \in X$. There are only three possibilities: $G_x = SO(3)$, $G_x = SO(2)$ or $G_x = SO(1) = \{id\}$.

1) $G_x = SO(3)$
It follows that the sectional curvature is constant. After rescaling, we have
$K \equiv 1 : (S^3, SO(4))$, spherical geometry
$K \equiv 0 : (\mathbb{E}^3, Isom^+(\mathbb{E}^3))$, Euclidean geometry
$K \equiv -1 : (\mathbb{H}^3, PSL_2(\mathbb{C}))$, hyperbolic geometry

2) $G_x = SO(2)$
Then $T_xX = \Lambda_x \perp P_x$, where Λ_x is a line invariant by G_x and P_x is an invariant plane orthogonal to Λ_x.
Because X is simply connected, we can choose a coherent orientation on Λ_x. We obtain a unit vector field $\vec{v}$ on X which is G-invariant, i.e. $\vec{v}_{g(x)} = Dg_x(\vec{v}_x)$, $\forall g \in G$. Moreover $\{P_x\}_{x \in X}$ is a G-invariant plane field.

Claim. *$\vec{v}$ is an isometric vector field (i.e. the induced flow Φ_t is isometric).*

Proof. To prove the claim, we use the fact that Φ_t commutes with the action of G. Let $L_x :=$ denote the trajectory of the flow Φ_t through x. Then $\forall y \in L_x, G_y \cong G_x \cong G_L$.
Given $x \in X$ and $\Phi_t(x) \in L_x$, let $g \in G$ be such that $g(\Phi_t(x)) = x$. We want to show that $D(g\Phi_t)_x : T_xX \to T_xX$ is an isometry. $D(g\Phi_t)_x$ is the identity on Λ_x and commute with the action of G_x, so it is a composition of a rotation around Λ_x and a homothety on P_x. We have only to show that the homothety on P_x is the identity. This is a consequence of the existence of a complete quotient $M = X/\Gamma$ of finite volume. The flow Φ_t induces a flow $\overline{\Phi_t}$ on M which preserves the volume and so $\overline{\Phi_t}$ must transversally preserve the area (on the induced plane field $\overline{P_x}$). Therefore, the flow Φ_t cannot expand or contract a direction on the plane field $\overline{P_x}$. $\square$

It follows that the leaf space $Y := X/\{x \sim \Phi_t(x)\}$ is Hausdorff; in fact one can find a saturated tubular neighborhood of each trajectory. Y is a 2-dimensional Riemannian manifold for the induced metric, which is homogeneous. Therefore it has constant sectional curvature. Y is simply connected and $X \to Y$ is a Riemannian line bundle. The plane field $\{P_x\}_{x \in X}$ gives a G-invariant connection for this line bundle. Its curvature is constant since G acts transitively on X. Either the curvature is zero or non-zero, or after rescaling 0 or +1.

	$\{P_x\}$ integrable, curv. $=0$	$\{P_x\}$ non-int., curvature $=+1$
$K \equiv 1, Y = S^2$	$S^2 \times \mathbb{E}^1$	$\widetilde{T_1 S^2} = \widetilde{SO(3)} = S^3$
$K \equiv 1, Y = \mathbb{E}^2$	$\mathbb{E}^2 \times \mathbb{E}^1 = \mathbb{E}^3$	$Nil \quad (\neq \widetilde{T_1 \mathbb{E}^2})$
$K \equiv -1, Y = \mathbb{H}^2$	$\mathbb{H}^2 \times \mathbb{E}^1$	$\widetilde{SL_2(\mathbb{R})} = \widetilde{T_1 \mathbb{H}^2}$

Exercise. Redo this classification by using Lie group theory for G.

3) $G_x = \{id\}$

Since G acts transitively, $X \cong G/G_x = G$ and X is a Lie group.

Claim. *X is a unimodular Lie group.*

Proof. By hypothesis there exists a discrete subgroup $\Gamma \subset X$ such that $\Gamma \backslash X$ is complete and of finite volume. Let D be a fundamental domain for the action of Γ and μ the (left-invariant) Haar measure on X, then $\mu(\gamma D \cap D) = 0$, for every $\gamma \neq id$. It follows that $\mu(D)$ does not depend on the fundamental domain D. Since for all $g \in G$, Dg is a fundamental domain, $\mu(Dg) = \mu(D)$ and μ is G-right invariant. □

We are looking for 3-dimensional unimodular Lie groups. Let $\mathfrak{y}$ be the 3-dimensional associated Lie algebra, $[x, y]$ the Lie bracket and $x \times y$ the cross product. There is a unique linear map $L : \mathfrak{y} \to \mathfrak{y}$ such that $L(x \times y) = [x, y]$.

Exercise. The Lie group is unimodular if and only if L is selfadjoint with respect to the left invariant metric.

Choose an orthonormal positively oriented basis $\{e_1, e_2, e_3\}$ consisting of eigenvectors for L, i.e. $Le_i = \lambda_i e_i$. We get $[e_1, e_2] = L(e_1 \times e_2) = L(e_3) = \lambda_3 e_3$ and analogously $[e_2, e_3] = \lambda_1 e_1$, $[e_3, e_1] = \lambda_2 e_2$. After rescaling and normalization: $\lambda_i \in \{-1, 0, 1\}$, $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

$(\lambda_1, \lambda_2, \lambda_3)$	$X = G$
$(+1, +1, +1)$	$SU(2)$
$(-1, +1, +1)$	$\widetilde{SL_2(\mathbb{R})}$
$(0, +1, +1)$	$\widetilde{Isom(\mathbb{E}^2)} = \mathbb{E}^2$
$(-1, 0, +1)$	$Sol = E(1, 1)$
$(0, 0, +1)$	Nil
$(0, 0, 0)$	$\mathbb{R}^3$

□

Exercise. Classify the 3-dimensional geometries up to quasi-isometries

Chapter 3

Canonical Decomposition of a 3-Manifold

In this chapter all 3-manifolds M are assumed to be orientable. A surface F is an orientable connected 2-manifold. We will work in the (PL)-category. This is not a restriction due to the fundamental theorem of Moise [27] , which asserts that every 3-dimensionamanifolds can be triangulated.

The aim of this chapter is to present the canonical topolgical decomposition of a 3-manifold along spheres and tori, involved in Thurston's geometrization conjecture.

3.1 Splitting along essential spheres and tori

We start with some classical and basic definitions

Definition. A surface $(F, \partial F) \hookrightarrow (M, \partial M)$ is **properly embedded** if $F \cap \partial M = \partial F$.

Definition. Let M be compact and orientable. A properly embedded surface $(F, \partial F) \hookrightarrow (M, \partial M)$ is **compressible** if either
i) F is a sphere which bounds a ball in M or
ii) there is a disk $D \subset M$ such that $D \cap F = \partial D$ and ∂D does not bound a disk on F (i.e. ∂D is an essential curve on F). Such a disk is called a **compressing disk**. $(F, \partial F) \hookrightarrow (M, \partial M)$ is **incompressible** if it is not compressible.

By the definition above, incompressiblity for a properly embedded surface $F \subset M$ means that F cannot be surgered along embedded disks in M to produce a simpler surface (i.e. with connected components with strictly bigger Euler characteristic).

Definition. Two properly embedded surfaces $(F_0, \partial F_0), (F_1, \partial F_1) \hookrightarrow (M, \partial M)$ are **parallel** if they cobound in M a product region $F \times [0,1]$ with $F \times \{0\} = F_0$, $F \times \{1\} = F_1$ and $\partial F \times [0,1] \subset \partial M$. A surface $(F, \partial F) \hookrightarrow (M, \partial M)$ is **∂-parallel** (boundary-parallel) if F is parallel to a subsurface of ∂M.

Definition. $(F, \partial F) \hookrightarrow (M, \partial M)$ is **∂-compressible** if either
i) $(F, \partial F)$ is a disk $(D^2, \partial D^2)$ which is ∂-parallel or
ii) there is a disk $\Delta \subset M$ such that $\partial\Delta \cap F$ is a simple arc α, $\Delta \cap \partial M$ is a simple arc β with $\partial\Delta = \alpha \cup \beta$ and $\alpha \cap \beta = \partial\alpha = \partial\beta$.

Definition. $(F, \partial F) \hookrightarrow (M, \partial M)$ is an **essential surface** if it is incompressible, ∂-incompressible and not ∂-parallel.

Our goal is to study essential surfaces with non-negative Euler characteristic. First we give now some examples of essential surfaces:

Example. i) A meridian disk in a handlebody is an essential surface.
ii) Surface Bundles over S^1: consider the mapping torus $F \times [0,1]/\{(x,0) \sim (\phi(x),1)\}$, where $\phi : F \to F$ is an orientation preserving homeomorphism, then any fiber $F \times \{\star\}$ is essential.
iii) Let M be a compact orientable 3-manifold and α a non-trivial class $\in H_2(M, \partial M; \mathbb{Z}) \setminus \{0\}$. Then there is an essential surface $(F, \partial F) \hookrightarrow (M, \partial M)$ that represents α. In particular, any properly embedded non separating surface of greatest Euler characteristic in its homology class is essential.
iv) Let M be a compact orientable 3-manifold with $\partial M \neq \emptyset$ such that ∂M is incompressible in M, then $\partial M \subset DM := M^+ \cup_{id_{\partial M}} M^-$ is essential.

Here is a fundamental result about closed surfaces in S^3.

Theorem 3.1 (Alexander). *Any (PL-)embedded closed orientable surface in S^3 (or $\mathbb{R}^3$) is compressible.*

Proof of Theorem 3.1. Take any Morse function h on S^3 and $F \subset S^3$ a closed orientable surface. By isotopy on F we can assume that $h|_F$ is generic. This means that F avoids the critical points of h and the critical points of $h|_F$ belong to distinct levels. If $c_1 < c_2 < \ldots < c_n$ are the critical levels of $h|_F$, define the **thinness** of F (with respect to h) as follows

$$T(F) := inf \sum_{c_i < a_i < c_{i+1}} card\, \pi_0(F \pitchfork h^{-1}(a_i))$$

where the infimum is taken over the isotopy class of F.

Lemma 3.2 (Gabai). *If F is incompressible and in thin position (i.e. F realizes the thinness $T(F)$) with respect to h, then there is a level surface $h^{-1}(t)$ intersecting F transversally such that $h^{-1}(t) \pitchfork F$ is essential in F.*

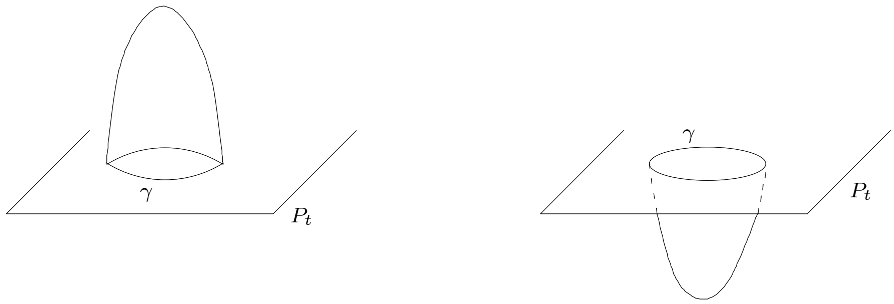

Figure 3.1: low/high level surface

First we show how Gabai's Lemma implies Alexander's Theorem:
Take h to be the standard Morse function on S^3 with 2-spheres as levels. If $T(F) = 1$, we have only two critical levels and therefore by assumption two critical points. We conclude that F is S^2 built up by 2-disks, so F is compressible. In the remaining case $T(F) > 1$, we assume that F is incompressible. Gabai's claim implies that there is a t such that $S^2_t \pitchfork F$ is essential in F. Therefore $S^2_t \pitchfork F$ bounds a disk on $S^2_t \subset S^3$ and is compressible contradicting our assumption. $\square$

We give now the proof of the claim.

Proof of Gabai's Lemma. Let m be the level of the highest local minimum for $h|_F$ and μ the level of the lowest local maximum of $h|_F$ which is higher than m. Define for $t \in [m, \mu]$ the level surface $P_t = h^{-1}(t)$ such that P_t intersects transversally F. Let $\gamma \in P_t \pitchfork F$ be an inessential curve on F which is **innermost** on F. It means that the disk $\Delta \subset F$ bounded by γ does not meet any other intersection curve of $P_t \pitchfork F$, and thus Δ lies on one side of the splitting of $\mathbb{S}^3$ by the level surface P_t. It is either **above** P_t (if $h(\Delta) \geq t$) or **under** P_t (if $h(\Delta) \leq t$). In the first case we say that P_t is a textbflow level surface, while in the second case we say that P_t is a textbfhigh level surface.

Let Define the following two sets:
$L := \{t \in [m, \mu] : P_t \pitchfork F$ and is a low level surface$\}$
$H := \{t \in [m, \mu] : P_t \pitchfork F$ and is a high level surface$\}$

For the definition of a low (respectively high) level surface, see Figure 3.1.

Exercise. Since $T(F)$ is minimal, show that $H \cap L = \emptyset$.

Exercise. Show that:
1) $L \neq \emptyset$ $(\mu - \epsilon \in L)$ and $H \neq \emptyset$ $(m + \epsilon \in H)$
2) L and H are open.

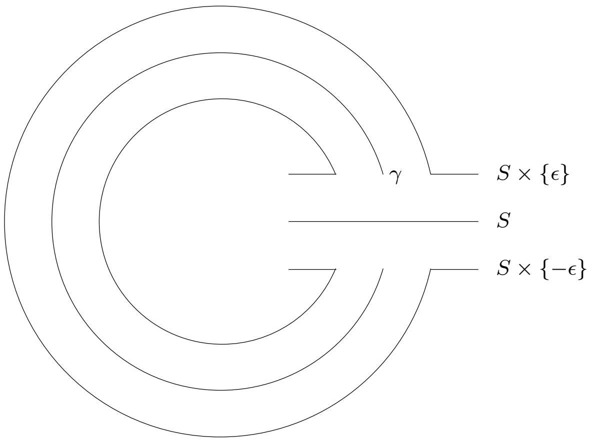

Figure 3.2: Proposition 3.3

Now the connectivity of $[m, \mu]$ implies that $L \cup H \neq [m, \mu]$, hence there exists an essential level surface. $\square$

Definition. An orientable 3-manifold is **prime** if it does not contain any essential separating 2-sphere.

Example. S^3, $\mathbb{R}^3$ and $S^1 \times S^2$ are prime. (In $S^1 \times S^2$, $\{\star\} \times S^2$ is an essential *non*-separating sphere.)

Exercise. Show that $S^1 \times S^2$ is prime. (Hint: use Alexander's theorem.)

Definition. An orientable 3-manifold M is **irreducible** if any embedded 2-sphere bounds a ball in M.

Example. S^3, $\mathbb{R}^3$ are irreducible, $S^1 \times S^2$ is not irreducible.

An irreducible 3-manifold is always prime. The following proposition describes the only compact orientable prime 3-manifold which is prime but not irreducible.

Proposition 3.3. *Let M be a compact orientable 3-manifold. If M is prime but not irreducible then $M \cong S^1 \times S^2$.*

Proof. Assume that there is an essential sphere $S \subset M$, it must be separating. Cut M along S. $M \setminus (S \times (-\epsilon, \epsilon))$ is connected. Take a simple arc γ properly embedded in $M \setminus (S \times (-\epsilon, \epsilon))$ joining $S \times \{\epsilon\}$ to $S \times \{-\epsilon\}$ (see Figure 3.2)

Take $\Sigma = (S \times \{\epsilon\}) \cup (S \times \{-\epsilon\}) \cup \partial \mathcal{N}(\gamma)$. Σ is a connected sum of two 2-spheres, therefore $\Sigma \cong S^2$. Being separating, Σ must bound a ball. $M \setminus$ ball $= (S \times [-\epsilon, \epsilon]) \cup \mathcal{N}(\gamma) = S^1 \times S^2 \setminus$ ball. $M = N \sharp (S^1 \times S^2)$, M is prime $\Rightarrow N = S^2 \Rightarrow M = S^1 \times S^2$. $\square$

A useful criterion for irreducibility is the following proposition:

Proposition 3.4. *Let M be an orientable 3-manifold and $p : \tilde{M} \to M$ a covering. If $\tilde{M}$ is irreducible then M is irreducible.*

Proof. Let $S \subset M$ be a sphere. $\tilde{S} := p^{-1}(S)$ is a union of copies of spheres each bounding a ball in $\tilde{M}$. Take $\Sigma \subset \tilde{S}$ a sphere bounding an innermost ball $\tilde{B}$.

Exercise. Show that the restriction $p : \tilde{B} \to p(\tilde{B})$ is a covering.

It follows that $p(\tilde{B})$ is a ball bounding S. □

Remark. The converse of Proposition 3.4 is also true, see the Equivariant sphere Theorem 3.22.

The following Corollary explains why one need to split 3-manifolds along 2-spheres before trying to uniformize them.

Corollary 3.5. *Any 3-manifold covered by $\S^3$ or $\mathbb{R}^3$ is irreducible. A geometric 3-manifold is irreducible, except if it is modelled on the product geometry $S^2 \times \mathbb{E}^1$.*

Remark. The same statement is false with primeness instead of irreducibility, e.g. the geometric prime manifold $S^1 \times S^2$ covers the non-prime manifold $\mathbb{R}P^3 \sharp \mathbb{R}P^3$.

Exercise. Describe the involution $\tau : S^1 \times S^2 \to S^1 \times S^2$ correspondingto the deck transformation of the 2-fold covering $p : S^1 \times S^2 \to \mathbb{R}P^3 \sharp \mathbb{R}P^3$.

We are now considering the existence of essential 2-tori in irreducible 3-manifolds

Definition. A compact orientable 3-manifold is **atoroidal** if it does not contain any essential embedded torus.

Proposition 3.6. *Let M be a compact orientable irreducible 3-manifold. Then any embedded compressible torus T in M either bounds a solid torus or is embedded in a ball.*

Proof. Let $T \subset M$ be an embedded compressible torus. There is a disk $D \subset M$ with $D \cap T = \partial D$ essential on T. Cut T along D. By this surgery we get a 2-sphere which bounds a ball B^3 in M (using the irreducibility of M). There are two possibilities. Either $D \cap B^3 = \emptyset$, then by reversing the process, T bounds a solid torus, or $D \subset B^3$, but then $T \subset B^3$. □

There are atoroidal 3-manifolds which admit finite regular coverings which contain essential tori.

Example. Let $\Gamma \subset PSL_2(\mathbb{R})$ be a triangle group, i.e. $\Gamma := T(p,q,r) := \langle \gamma_1, \gamma_2, \gamma_3 \mid \gamma_1^p = \gamma_2^q = (\gamma_1\gamma_2)^r = 1\rangle$. The quotient $T_1\mathbb{H}^2/\Gamma$ is a geometric $(\widetilde{SL_2(\mathbb{R})})$ atoroidal 3-manifold finitely covered by a quotient $T_1\mathbb{H}^2/\Gamma'$, where $\Gamma' \subset \Gamma$ is a torsionfree subgroup of finite index. This last quotient is a S^1-bundle over a surface of genus ≥ 2, which always contains essential tori.

Exercise. a)Show that a compact orientable S^1-bundle over an orientable surface of Euler characteristic ≤ -2 contains an essential torus.
b) Show that an orientable S^1-bundle over a sphere with three holes is topologically atoroidal.

Since topological atoroidality is not preserved by finite covering , the following definition makes sense.

Definition. A compact orientable 3-manifold is **strongly atoroidal** if it is not homeomorphic to $S^1 \times D^2$, $T \times [0,1]$, $K \tilde{\times} [0,1]$ (where K is the Klein bottle) and if any finite covering of M is atoroidal.

We can now state the main Theorem, whose proof will occupy the end of this chapter. The part of this theorem dealing with the decomposition along spheres is due to H. Kneser . The part dealing with the decomposition along tori is due to W. Jaco and P. Shalen and independantly to K. Johannson, but only for the case of **Haken** 3-manifolds (these are irreducible 3-manifolds that contains an essential surface, see Chapter 4). The work of Jaco-Shalen and Johannson has been completed by the works of P. Scott, P. Tukia, G. Mess, D. Gabai and A. Casson and D. Jungreis, to get the general version of the torus decomposition of an irreducible 3-manifold.

Here, using the notion of srong atoroidality, we give an easier, but weaker, version of this torus decompositon. However, it has a more geometric flavour.

Theorem 3.7 (Canonical Decomposition Theorem). *Let M be a compact orientable 3-manifold different from S^3.*
a) There is a finite (perhaps empty) family $\mathfrak{S}$ of disjoint essential embedded spheres in M which splits M into prime manifolds different from S^3. Moreover, these prime factors M_i are unique up to homeomorphism.
b) For each irreducible factor M_i, there is a finite (perhaps empty) family of essential disjoint and non-parallel tori $\mathfrak{T}_i$ which splits M_i into either strongly atoroidal pieces or geometric pieces modelled on $\mathbb{H}^2 \times \mathbb{E}$, $\widetilde{SL_2(\mathbb{R})}$, $\mathbb{E}^3$, Nil or Sol. Moreover, a minimal such family $\mathfrak{T}_i$ is unique up to isotopy in M_i.

Remark. If M is a torus bundle over S^1, it is either an Euclidean or a *Sol* geometric manifold. Hence, in this case the minimal family of splitting tori is empty.

The strategy of the proof is as follows:
1) We prove first a weaker version of the theorem, where the closure of each

component of $M_i \setminus \mathfrak{T}_i$ is either Seifert fibered or *Sol* or topologically atoroidal.
2) Then we prove the following theorem:

Theorem 3.8. *Let M be a compact orientable irreducible atoroidal 3-manifold. Then one of the following exclusive cases occurs: either M is geometric of type $(\mathbb{H}^2 \times \mathbb{E}^{1\!\!K})$, $\widetilde{SL_2(\mathbb{R})}$, $\mathbb{E}^3$, Nil or M is strongly atoroidal.*

Remark. Theorem 3.7 and 3.17 reduce Thurston's Geometrization Conjecture to the following uniformization conjecture:

Uniformization Conjecture. *Let M be a compact orientable irreducible strongly atoroidal 3-manifold. Then*
a) M is hyperbolic if and only if $\pi_1 M$ is infinite.
b) M is spherical if and only if $\pi_1 M$ is finite.

In the next section we describe a crucial step in the canonical decomposition Theorem, that is the Haken-Kneser finiteness theorem. Then in section 3.3 we deduce from it the weaker version of the canonical decomposition Theorem. Lastly in section 3.4 we prove Theorem 3.7

3.2 Haken-Kneser finiteness Theorem

The purpose of this section is to prove a fundamental result for all finiteness properties of 3-manifolds.

Theorem 3.9 (Haken-Kneser Finiteness Theorem). *Let M be a compact orientable 3-manifold. There is an integer $h(M)$ such that for any family of n disjoint embedded essential surfaces in M either $n < h(M)$ or at least two surfaces are parallel.*

Remark. For the canonical decomposition theorem we need only to work with closed essential surfaces. Haken-Kneser finiteness theorem implies the existence of a finite family of non-separating 2-spheres splitting a 3-manifold into prime factors. It gives also the existence of a finite family of essential tori splitting an irreducible 3-manifold into topologically atoroidal 3 sumanifolds.

In the following, M will be endowed with a fixed (finite) triangulation Δ. A key tool for the proof of Haken-Kneser finiteness Theorem is the notion of (pseudo)-normal surface with respect to the given triangulation Δ.

Definition. A **pseudo-normal surface** in M (with respect to Δ) is a properly embedded surface $(F, \partial F) \hookrightarrow (M, \partial M)$ such that
1) F is transversal to the 1-skeleton $\Delta^{(1)}$ and $F \cap \Delta^{(0)} = \emptyset$.
2) F intersects each 2-simplex in simple arcs with ends on different edges.
3) F intersects each 3-simplex in a disjoint union of 2-disks.

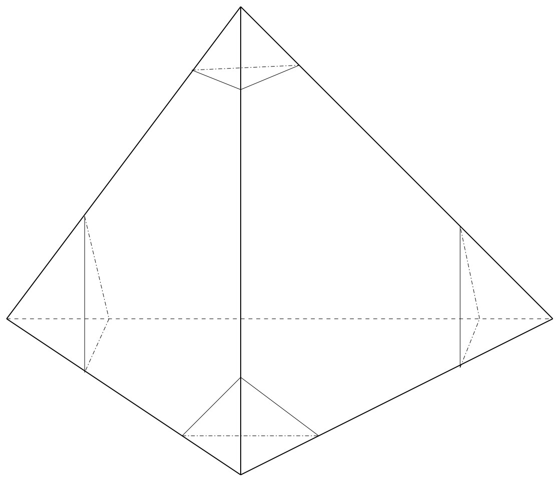

Figure 3.3: Triangular disks

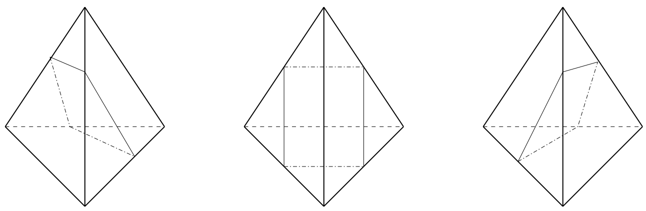

Figure 3.4: Quadrilateral disks

F is said **normal** if moreover we require that:
4) Each disk of ($F \cap$ 3-simplex) intersects each edge in at most one point.

Remark. For a normal surface there are only seven types of intersection disks with a 3-simplex:four types of triangular disks (Figure 3.3) and three types of quadrilateral disks (Figure 3.4).

Moreover, since quadrilateral disks of different types must intersect in a given 3-simplex, a normal surface can meet a given 3-simplex in at most one type of quadrilateral disks.

These restrictions make normal surfaces much more "rigid" than pseudo-normal surfaces. In particular, the arcs of intersections with the 2-skeleton $\Delta^{(2)}$ determine simple closed curves which bound disjoint disks of restricted types in the 3-simplices. Moreover, the data of this collection of simple closed curves in $\Delta^{(2)}$ together with the types of the disks they bound in the 3-simplices is called the **patern** of the normal surface. This patern determines the normal surface up to a normal isotopy in M (i.e. an isotopy that leaves invariant the triangulation Δ).

Let now attach to a (pseudo)-normal surface a complexity, which is an integer, called its **weight**.

Definition. Let $F \subset M$ be a properly embedded (pseudo-)normal surface. Define its **weight** of F as $w(F) := card(F \pitchfork \Delta^{(1)})$.

Exercise. Let $F \subset M$ be a compact normal surface. Show that:

$$w(F) = \chi(F) + \#(\text{quadrilateral disks}) + \frac{1}{2}\#(\text{triangular disks}).$$

(Hint: use the fact that the triangulation Δ of M induces a cell-structure on F)

The following results are the key points of the proof of Haken-Kneser finiteness Theorem. In fact, we need only to get pseudo-normal surfaces, which is easier to achieve.

Proposition 3.10 (H. Kneser). *Let M be a compact orientable triangulated 3-manifold. Let $\mathfrak{S} = (S_1, \dots, S_n)$ be a system of $n \geq 1$ disjoint embedded essential spheres such that no component of $M \setminus \mathfrak{S}$ is a punctured S^3 (i.e. a S^3 with open 3-balls removed). Then there exists a system $\mathfrak{S}' = (S'_1, \dots, S'_n)$ of n disjoint embedded essential (pseudo-)normal spheres with the same property (i.e. no component of $M \setminus \mathfrak{S}'$ is a punctured S^3).*

Proposition 3.11 (W. Haken). *Let M be a compact orientable triangulated irreducible 3-manifold and $\mathfrak{F} = (F_1, \dots, F_n)$ be a system of $n \geq 1$ disjoint pairwise non-parallel essential surfaces. Then there is a system $\mathfrak{F}' = (F'_1, \dots, F'_n)$ of n disjoint pairwise non-parallel essential surfaces which are homeomorphic to $(F'_1, \dots, F'_n)$ and (pseudo-)normal.*

Remark. 1) In Proposition 3.10 one cannot assume only that the 2-spheres are pairwise non-parallel because this condition is not always preserved by surgery on the 2-spheres. A typical example is given by three 2-spheres cobounding a punctured S^3 : some surgery on one 2-sphere of the system produces two new 2-spheres that are parallel to some other spheres of the original system.
However, Proposition 3.10 is sufficient to prove Haken-Kneser finiteness Theorem because there are only finitely many disjoint and pairwise non-parallel essential 2-spheres in a punctured S^3.

2)In Proposition 3.11, since M is irreducible, if ∂M is compressible or if $\mathfrak{F}$ is a system of closed surfaces $\mathfrak{F}'$ is isotopic to $\mathfrak{F}$.

Proof of Propositions 3.10 and 3.11. The proofs of these two propositions follow really the same schemes.

Step 1: By hypothesis, we can start with a family $\mathfrak{S}$ of 2-spheres or $\mathfrak{F}$ of surfaces such that:

1) The surfaces are essential, pairwise non-parallel and do not cobound a punctured S^3.
2) $card\,\mathfrak{S} = card\,\mathfrak{F} = n$ is fixed.
3) $\mathfrak{S}$ or $\mathfrak{F}$ is in general position with respect to the 2-skeleton $\Delta^{(2)}$.
4) $\mathfrak{S}$ or $\mathfrak{F}$ is of least total weight: it meets the 1-skeleton in the least possible number of points among all families of 2-spheres or surfaces homeomorphic to $(F_1, \dots, F_n)$ which verify 1), 2) and 3).

Step 2: Remove all intersections of $\mathfrak{S}$ or $\mathfrak{F}$ with the 2-simplices which are closed curves. This can be done by surgery without changing the weight in the following way: start with an innermost curve of intersection γ on a 2-simplex and do a surgery along it.

When one does a surgery on a sphere $S_i \subset \mathfrak{S}$, one obtains two spheres S_i', S_i'' and two new systems $\mathfrak{S}'$, $\mathfrak{S}''$ by replacing S_i by either S_i' or S_i''.

Exercise. Show that at least one of this new system $\mathfrak{S}'$ or $\mathfrak{S}''$ verifies 1) to 3). (Hint : use the fact that the 2-spheres S_i, S_i' and S_i" can be disjoined and cobound a punctured S^3.)

For $F_i \subset \mathfrak{F}$, the closed curve γmust bound a disk on F_i, because F_i is essential. After surgery on F_i along γ, there is only one component F_i' which is homeomorphic to F_i, because F_i is not a 2-sphere. So, replacing F_i by F_i' gives a new system verifying 1) to 3).

Now, the following claim finishes the proof of Propositions 3.10 and 3.11. $\square$

Claim. *The system of essential 2-spheres $\mathfrak{S}$ or surfaces $\mathfrak{F}$ obtained after steps 1 and 2 is formed by (almost)-normal surfaces with respect to the given triangulation.*

Proof of Claim. To show that $\mathfrak{S}$ or $\mathfrak{F}$ is a system of almost-normal surfaces, we have to verify two properties:

a) The arcs of intersections of $\mathfrak{S}$ or $\mathfrak{F}$ with any 2-simplex τ go to different edges of τ.
b) The intersection of $\mathfrak{S}$ or $\mathfrak{F}$ with any 3-simplex σ is a collection of properly embedded disks in σ.

This is done by contradiction, using the minimality of the total weight of the systems $\mathfrak{S}$ or $\mathfrak{F}$ verifying properties 1) to 3) of step 1.

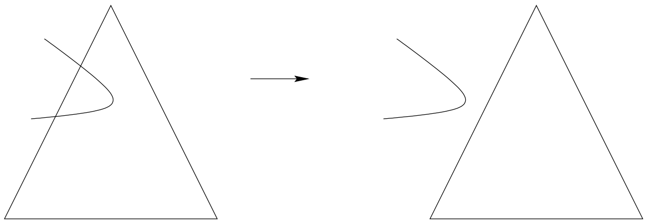

Figure 3.5: Step 3

Suppose that a) is not verified for some 2-simples τ. Then there is an arc $\alpha \subset \tau \cap \mathfrak{S}$ or $\tau \cap \mathfrak{F}$ having both ends in the same edge of τ and which is outermost in τ (cf. Figure 3.5).

Exercise. Show that in this situation, one can strictly reduce the total weight of the system $\mathfrak{S}$ or $\mathfrak{F}$.(Hint: use the disk $D \subset \tau$ bounded by α and the arc β on the edge of τ having the same ends as α. There are two cases to consider according whether $\beta \subset \int(M)$ or $\beta \subset \partial M$).

Suppose that b) is not verified for some 3-simples σ. Then some connected component G of $\sigma \cap \mathfrak{S}$ or $\sigma \cap \mathfrak{F}$ is not a disk. Then there is a simple closed curve component $\gamma \subset \partial G$ which bounds a disk $D \subset \mathfrak{S}$ or $\mathfrak{F}$ not contained in σ. Moreover γ bounds a disk $D' \subset \sigma$ disjoint from $\mathfrak{S}$ or $\mathfrak{F}$ except along γ.

Exercise. Show that by replacing the disk D by D' one obtains a new system verifying properties 1) to 3) of step 1, but with strictly smaller total weight.

So at this point, we got a system $\mathfrak{S}$ or $\mathfrak{F}$ of pseudo-normal surfaces, with least total weight, among system of pseudo-normal surfaces verifying properties 1) and 2) of step 1.
The minimality of the total weight of these systems of pseudo-normal surfaces show that in fact the surfaces are normal.

Exercise. Show that an essential pseudo-normal surface of least weight among all essential surfaces with the same topological type is normal.

□

□

We can now start the proof of Haken-Kneser finiteness Theorem.

Proof of Theorem 3.9. By the Propositions 3.10 and 3.11, we can assume that $\mathfrak{F}$ is in a pseudo-normal position with respect to the triangulation Δ of M. Then an explicit upper bound for $card\,\mathfrak{F}$ is given by the following Lemma:

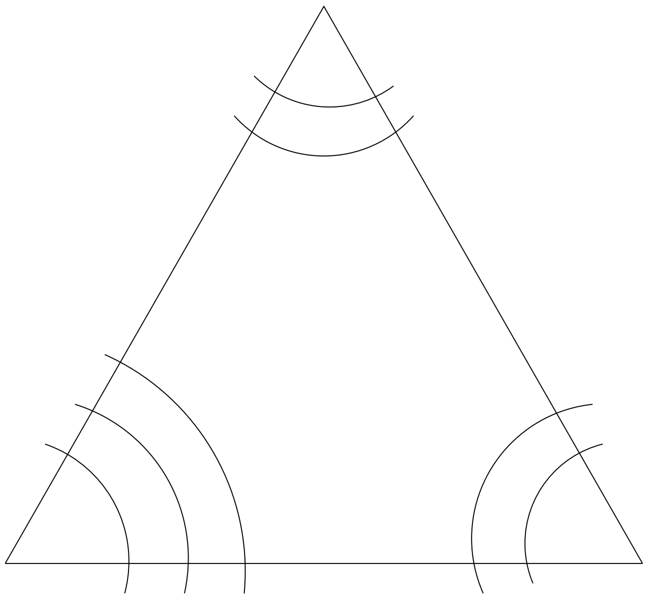

Figure 3.6: Product regions

Lemma 3.12.

$$card\,\mathfrak{F} < h(M) := 4\,card(\textit{2-simplices}) + dim_{\mathbb{Z}_2} H_2(M, \partial M; \mathbb{Z}_2).$$

Proof of Lemma. Consider the intersection of $\mathfrak{S}$ or $\mathfrak{F}$ with a 2-simplex. Except at most four regions per 2-simplex (see Figure 3.6), all the other regions cut off by $\mathfrak{F}$ on a 2-simplex are product regions (rectangles). We set $\delta := card$(2-simplices) and conclude that except at most 4δ components of $M \setminus \mathfrak{F}$, the other components will meet the 2-simplices in product regions. THen we have the following claim:

Claim. *The closure of a component R of $M \setminus \mathfrak{F}$ meeting each 2-simplex in a product region (rectangle) is an I-bundle.*

Proof of the Claim. The intersection of R with a 3-simplex, if it is not empty, is $D^2 \times I$. The I-bundle structures on $D^2 \times I$ match together to give an I-bundle structure on R. □

If such a region appears, then either R is a trivial bundle $F \times [0,1]$, $F \in \mathfrak{F}$, or a twisted bundle $G\tilde{\times}[0,1]$, where G is a non-orientable surface and $\partial(G\tilde{\times}[0,1]) = F \in \mathfrak{F}$.

In the first case, M would be a bundle over S^1 with fiber F and the Lemma follows then from the following proposition:

Proposition 3.13 (Waldhausen). *Let $F \times [0,1]$ be a trivial I-bundle over an orientable surface (perhaps with boundary). Then any essential surface S in $F \times [0,1]$, such that $\partial S \cap ((F \times \{0\} \cup F \times \{1\})) = \emptyset$ is isotopic to a level surface (of the height function).*

From this proposition it follows that if there is a trivial product region, then $\mathfrak{F} = \{F\}$ and therefore cetainely $card\, \mathfrak{F} \leq 4\delta$.

So we may assume that except 4δ components, the other components of $M \setminus \mathfrak{F}$ bound twisted I-bundles over non-orientable surfaces. Let $F_1, \dots, F_k$ be the maximal set of non-separating surfaces in $\mathfrak{F}$, such that $M \setminus \{F_1, \dots, F_k\}$ is connected. Let $G_1, \dots, G_m$ be the family of non-orientable surfaces corresponding to the twisted product regions of $M \setminus \mathfrak{F}$ and such that $F_{k+i} \in \mathfrak{F}$ is the boundary of a regular neighborhood of G_i for $1 \leq i \leq m$.

Claim. $m + k \leq dim_{\mathbb{Z}_2} H_2(M, \partial M; \mathbb{Z}_2)$.

Proof of the Claim. The proof follows from the fact that $F_1, \dots, F_k, G_1, \dots, G_m$ are linearly independent in $H_2(M, \partial M; \mathbb{Z}_2)$. Otherwise, a linear relation in $H_2(M, \partial M; \mathbb{Z}_2)$ between these elements would imply that some subcollection of them bound a compact submanifold of M, contradicting the choice of $F_1, \dots, F_k$.

□

Let $card\, \mathfrak{F} = n = m + k$. The number of components $n - k + 1$ of $M \setminus \mathfrak{F}$ is $\leq 4\delta + m$. Hence $n - k + 1 \leq 4\delta + m$ that we rewrite as $n \leq m + k + 4\delta - 1$. Then, using the last claim, we obtain: $n < m + k + 4\delta \leq dim_{\mathbb{Z}_2} H_2(M, \partial M; \mathbb{Z}_2) + 4\delta$ which completes the proof of the Lemma and of Theorem 3.9. □

□

3.3 The Kneser-Jaco-shalen-Johannson splitting

This Haken-Kneser finiteness result implies that for M compact orientable and different from S^3, there is a finite family of disjoint essential spheres which cut M into prime manifolds different from S^3. Moreover, for each irreducible factor M_i there is a finite family of tori $\mathfrak{T}_i$ which cut M_i into atoroidal pieces.
Now we want to prove some uniqueness properties of such a minimal splting.

The following Proposition is due to Kneser. Another proof has been given by Milnor.

Proposition 3.14. *Let M be a compact oriented 3-manifold, which is not homeomorphic to S^3. If $M = M_1 \sharp \dots \sharp M_k \sharp l(S^1 \times S^2) = N_1 \sharp \dots \sharp N_j \sharp m(S^1 \times S^2)$, with M_i, $N_i \not\cong S^3$, then $j = k$, $l = m$ and after a permutation $M_i \cong N_i$.*

Proof of the Proposition. Let $\mathfrak{S}_1$ be the first family of spheres and $\mathfrak{S}_2$ the second family of spheres (essential, no component of $M \setminus \mathfrak{S}_i$ is a punctured S^3). Let $\mathfrak{S}$ be any system of disjoint embedded spheres containing $\mathfrak{S}_1$. It follows that $M \setminus \mathfrak{S}$ contains $M_1, \ldots, M_k$ plus some punctured 3-spheres. The idea is to include $\mathfrak{S}_1$ and $\mathfrak{S}_2$ into the same system of disjoint embedded spheres.

Exercise. Starting with $\mathfrak{S}_1$ and $\mathfrak{S}_2$, one can find a new system of spheres $\mathfrak{S}_1'$ which cut M into $M_1, \ldots M_k$ plus some punctured spheres and such that $\mathfrak{S}_1' \cap \mathfrak{S}_2 = \emptyset$.

(Hint: Make $\mathfrak{S}_1$ meet $\mathfrak{S}_2$ transversally such that $\mathfrak{S}_1 \pitchfork \mathfrak{S}_2$ is minimal. Reduce the number of intersection curves of $\mathfrak{S}_1 \pitchfork \mathfrak{S}_2$ by surgery on $\mathfrak{S}_1$: cut $S \in \mathfrak{S}_1$ along innermost disks into S', S'' and then replace S by $S' \cup S''$. Show that the new system $\mathfrak{S}_1'$ obtained in this way cuts M into $M_1, \ldots, M_k$ plus some punctured spheres, using the fact that S, S', S'' bound together a punctured 3-sphere.)

Once we know that $k = j$ and $M_1 \sharp \ldots \sharp M_k = N_1 \sharp \ldots \sharp N_j$, an easy calculation of the dimension of the rational first homology groups shows that $l = m$; It uses the fact that $H_1(S^1 \times S^2; \mathbb{Q}) \cong \mathbb{Q})$. $\square$

The following very useful notion has been introduced by W. Neumann and G. Swarup [33].

Definition. Let M be an irreductible orientable compact 3-manifold. A textbfcanonical torus in M is an essential torus which can be isotoped to be disjoint from any other essential torus in M .

By Haken-Kneser finiteness theorem, there is in M a maximal finite collection of disjoint, pairwise non-parallel canonical tori. The following theorem gives the properties of such a family:

Theorem 3.15 (Canonical tori decomposition [33]). *Let M be a compact orientable irreducible 3-manifold and let $\mathfrak{C} \subset M$ be a maximal finite (perhaps empty) collection of disjoint and non-parallel canonical tori. Then:*
(i) The collection $\mathfrak{C}$ is unique up to isotopy in M.
(ii) $\mathfrak{C}$ splits M into either atoroidal pieces or Seifert fibered pieces.
(iii) The Seifert fibrations on two adjacent pieces do not match up.

Definition. Let M be an irreducible orientable compact 3-manifold. We call the maximal family of canonical disjoint and pairwise non-parallel tori the **J-S-J family of tori**

Before starting the proof of the canonical tori decomposition we need to introduce the notion of Seifert fibration, which is a generalization of the notion of circle bundle:

Definition. A **Seifert fibration** on a compact orientable 3-manifold M is a foliation by cicles, where each circle admits a foliated tubular neighborhood. If such a foliation exists on M, M is called a **Seifert (fibered) 3-manifold**.

Exercise. Show that a compact orientable 3-manifold is a Seifert 3-manifold iff it admits a geometry modelled on one of the following six models: $\mathbb{S}^3$, $\mathbb{E}^3$, Nil, $\mathbb{H}^2 \times \mathbb{E}^1$, $\widetilde{SL_2(\mathbb{R})}$ or $S^2 \times \mathbb{E}^1$.
(Hint: Show that one can find a Riemannian metric on M such that the leaves are closed geodesics. Then uniformize the space of leaves, using the fact that it is a 2-dimensional orbifold).

Proof of Theorem 3.15. Let M be compact orientable irreducible 3-manifold and let $\mathfrak{C} \subset M$ be a maximal collection of pairwise non-parallel canonical tori. Let $T \subset M$ be a canonical torus. From the fact that T is canonical and the maximality of the family $\mathfrak{C}$ it follows easily that:

Claim. *T can be isotoped to be parallel to a torus $\subset \mathfrak{C}$.* $\square$

This Claim implies that any another maximal collection $\mathfrak{C}'$ of pairwise non-parallel canonical tori can be isotoped to be parallel to $\mathfrak{C}$. The maximality of $\mathfrak{C}$ and $\mathfrak{C}'$ imply that theses two collectionsof tori are in fact isotopic. This proves the assertion i) of Theorem 3.15.

The proof of the assertion ii) of Theorem 3.15 follows from:

Lemma 3.16. *Let N be a compact orientable irreducible 3-manifold. If N is not atoroidal, but does not contain any canonical torus, then N is Seifert fibered.*

Proof of Lemma. . Let $\mathfrak{T} \subset M$ be a maximal collection of essential disjoint and pairwise non-parallel tori. By Haken-Kneser finiteness theorem, such a finite collection exists, and by hypothesis it is not empty. Moreover each torus $T \subset \mathfrak{T}$ is not canonical. Let W be the closure of a component of $N \setminus \mathfrak{T}$ and let $\partial_0 W = \partial W \cap \mathfrak{T}$. For each component T of $\partial_0 W$) there must exist an essential torus T' which cannot be disjoined by isotopy from T, since T is not canonical. By making the intersection $T \cap T'$ transversal and minimal, a component of $W \cap T'$ gives an essential annulus $(A, \partial A) \hookrightarrow (W, \partial_O W)$. Then the proof of Lemma follows from the following:

Claim. *W admits a Seifert fibration for wich the essential annulus A is vertical (i.e an union of fibers). More precisely the seifert fibration belongs to the following restricted list:*
1) A fibration over the disk with two exceptional fibers.
2) A fibration over an annulus with one exceptionnal fiber.
3) A product fibration over a disk with two holes.

Exercise. Show that W can admit, up to isotopy, only one Seifert fibration of type 1),2) or 3).

We first deduce the Lemma from the Claim. Let W and W' be the closures of two adjacent components of $N \setminus \mathfrak{T}$ and let T be a component of the intersection $\partial W \cap \partial W'$. Since T is not canonical, by the argument above there is an essential torus $T' \subset N$ such that $W \cap T'$ and $W' \cap T'$ are essential annuli in respectivly W and W'. By the Claim II.3.5 W and W' both admit Seifert fibrations in wich the annuli $W \cap T'$ and $W' \cap T'$ are vertical. In particular these Seifert fibrations are isotopic on T, and hence can be match up along T. The unicity, up to isotopy, of the Seifert fibrations of type 1), 2) or 3) and a finite induction shows that N is Seifer fibered. □

Proof of Claim. By maximality of the collection $\mathfrak{T} \subset N$, W is atoroidal. We distinguish two cases according whether or not ∂A belongs to only one component of ∂W.

If $\partial A \subset T$, where T is a component of ∂W, let A_1 and A_2 be the closures of $T - \partial A$. Since A is not boundary-parallel, the tori $T_1 = A \cup A_1$ and $T_2 = A \cup A_2$ cannot be both boundary-parallel.

a) If both T_1 and T_2 are not boundry-parallel, they are compressible and not contained in a ball, because A is essential in W. Hence they both bounds disjoint solid tori in V_1 and V_2 in W, and W is obtained by gluing V_1 and V_2 along an essential annulus in their boundaries.

Exercise. Show that an orientable manifold obtained by gluing two solid tori along an essential annulus in their boundaries (i.e. whose core meets each meridian in more than one point) admits a Seifert fibration with two exceptionnal fibers and basis a disk. (Hint: start foliating the gluing annulus,then extend the foliation to the boundaries of the two solid tori, and then to the solid tori with an exceptional fiber for each core, because the gluing annulus is essential.)

b) If T_2 is boundary-parallel, but not T_1, then T_1 bounds a solid torus V_1 and T_2 cobounds a product $T^2 \times [0,1]$ with a component of ∂W. Hence W is obtained by gluing to $T^2 \times [0,1]$ the solid torus V_1 along an essential annulus in ∂V_1 and $T^2 \times \{0\}$.

Exercise. Show that an orientable manifold obtained by gluing $T^2 \times [0,1]$ and a solid torus along an essential annulus in their boundaries admits a Seifert fibration with one exceptionnal fiber and basis a disk. (Hint: as in Exercise II.3.4, start foliating the gluing annulus, and then extends the foliation to W.)

c) If both T_1 and T_2 are boundary-parallel, then W is obtained by gluing two copies of $T^2 \times [0,1]$ along an essential annulus in their boundaries. Hence W is a product $S^1 \times F$, where F is a disk with two holes.

If ∂A belongs to two different components $T \cup T' \subset \partial W$, let A_1 and A_2 be the annuli obtained by cutting T and T' along ∂A. Then $T_0 = A_1 \cup \partial\mathcal{N}(A) \cup A_2$ is a torus which cannot be embedded in a ball since A isessential.

d) If T_0 is not boundary-parallel, it bounds a solid torus $V_0 \subset W$ and W is obtained by gluing two disjoint essential annuli on ∂V_0.

Exercise. Show that an orientable manifold obtained from a solid torus by identifying two disjoint essential annuli in its boundary admits a Seifert fibration with one exceptionnal fibers and basis an annulus.

e) If T_0 is boundary-parallel, then W is obtained from $T^2 \times [0,1]$ by identifying two disjoint essential annuli in $T^2 \times \{0\}$. Hence it is a product $S^1 \times F$, where F is a disk with two holes.

This finishes the proof of the claim . $\square$ $\square$

We prove now the assertion iii) of Theorem 3.15.

Let N_1 and N_2 the closures of two components of $M \setminus \mathfrak{C}$ which are adjacent : $\partial N_1 \cap \partial N_2 \not\emptyset$. Let $T \subset \mathfrak{C}$ be a component of $\partial N_1 \cap \partial N_2$. Since T is incompressible in both N_1 and N_2, there are two vertical essential annuli $(A_1, \partial A_1) \hookrightarrow (N_1, T)$ and $(A_2, \partial A_2) \hookrightarrow (N_2, T)$. If the Seifert fibrations on N_1 and N_2 match up along T, after an isotopy on T one can glue A_1 and A_2 along their boundaries to get an essential torus $T' \subset M$ which cannot be disjoined from T. This finishes the proof of the Theorem 3.15. $\square$ $\square$

3.4 Atoroidal 3-manifolds that are not strongly atoroidal

In general, Seifert 3-manifolds are not atoroidal. In fact:

Exercise. Show that a compact orientable Seifert 3-manifold with infinite fundamental group always admit a regular finite covering which is not atoroidal, and hence is never strongly atoroidal.
(Hint : think about a S^1-bundle over an orientable surface M. Choose $\rho : M \to F$, $\gamma \subset F$ an essential simple closed curve, then show that $p^{-1}(\gamma)$ is an essential torus in M.)

The main result of this section shows that it is the following:

Theorem 3.17. *Let M be a compact orientable irreducible atoroidal 3-manifold. If M is not strongly atoroidal, then M is a Seifert manifold with infinite fundamental group.*

Proof of Theorem **??**. Let $p : \overline{M} \to M$ be a regular finite covering with deck transformation group G and assume that $\overline{M}$ contains an essential torus. We want to show that M is a Seifert manifold.

Proposition 3.18 (Scott [44]). *There exists in $\overline{M}$ an essential torus $\overline{T}$ such that $p(\overline{T}) \hookrightarrow M$ is a self transversal immersion with only double points (no triple points) or there is a Klein bottle $\overline{K}$ in $\overline{M}$ with the same properties.*

To prove this proposition, start with an essential torus $\overline{T}_0 \subset \overline{M}$ in a normal form with respect to a G-invariant triangulation of $\overline{M}$ and define the complexity

$$\sum_{g_1 \neq g_2} card(g_1(\overline{T}_0) \pitchfork g_2(\overline{T}_0) \pitchfork \overline{T}_0)$$

where $g_1, g_2 \in G \setminus \{id\}$.

Look now to the essential normal torus $\overline{T}$ with least possible complexity in $\overline{M}$. Either the complexity is 0 or there is an embedded essential Klein bottle $\overline{K}$ with

$$\sum_{g_1 \neq g_2} card(g_1(\overline{K}) \pitchfork g_2(\overline{K}) \pitchfork \overline{K}) = 0.$$

The following Lemma together with Scott's Proposition completes the proof of Theorem **??**.

Lemma 3.19. *Let M be compact orientable irreducible and atoroidal. Assume that $f : T \hookrightarrow M$ (or $f : K \hookrightarrow M$) is self transversal without triple points and that the immersion is essential, i.e. $f_* : \pi_1(T) \to \pi_1(M)$ is injective (or $f_* : \pi_1(K) \to \pi_1(M)$ is injective). Then M is a Seifert manifold.*

Exercise. Replace the injectivity on π_1 by the fact that $f(T)$ is finitely covered by an essential torus.

Proof of the Lemma. :
We can get rid of the *inessential* double curves by an isotopy of f because the immersion is essential and M is irreducible. On the torus, the preimage of the double curves is now a collection of essential parallel closed curves (using the assumption that there are no triple points). Let $\gamma \in T$ be a simple closed curve which meets every double curve in only one point. $f(\gamma)$ is a closed curve with double points in M. A regular neighborhood of $f(T)$

$$N = \mathcal{N}(f(T)) = S^1 \times \mathcal{N}(f(\gamma)) = S^1 \times F$$

is a Seifert manifold. F is a surface with non-empty boundary and ∂N is a union of tori.

Claim. *$M \setminus int(N)$ is a union of solid tori.*

This Claim implies that M is Seifert fibered by extending the Seifert fibration of N. This is possible because the immersion is essential. □

Proof of the Claim. :
N is boundary-incompressible (i.e. ∂N is incompressible in N), otherwise N would be a solid torus and $f_*(\pi_1 T) \subset \pi_1 N = \mathbb{Z}$, contradicting the fact that the immersion is essential. From the atoroidality of M follows that ∂W is compressible in W for all components W of $M \setminus int(N)$. If W is irreducible then W is a solid torus. Assume now W is not irreducible and let $S^2 \hookrightarrow W$ be an embedded sphere. Since M is irreducible, $S^2 = \partial B^3$, $B^3 \hookrightarrow M$, $\partial B^3 \cap \partial W = \emptyset$. There are two cases: either $B^3 \hookrightarrow W$ and S^2 bounds a ball in W, or $\partial W \subset B^3$ and then the double curves on $f(T)$ are null homotopic in M, contradicting the injectivity of $f_* : \pi_1 T \to \pi_1 M$.

To finish the proof of the claim we have to consider the remaining case:

Step 1: By isotopy on f, make the double curves essential on K.
Step 2: Either K is a twisted S^1-bundle over S^1 ($K = S^1 \tilde{\times} S^1$) or K is a Seifert fibered 2-manifold obtained by gluing two copies of a Möbius band. This gives us two cases: either the double curves are 2-sided and non-separating (they are fibers of $S^1 \tilde{\times} S^1$) or the double curves are 1-sided or separating (they are fibers of the Seifert fibration).
Step 3: In any case $\mathcal{N}(f(K))$ is a Seifert fibration. One achieves the proof as in the torus case. □

□

The Canonical Decomposition Theorem **??** for irreducible 3-manifolds follows then from the canonical tori decomposition 3.15 and the theorem **??** above.

3.5 Some results on Seifert 3-manifolds

We start with a precise description of a saturated tubular neighborhood of a fiber of a Seifert fibration. The fibration of the tubular neighborhood of a fiber f may not be the product one. Define $T(\alpha, \beta)$ to be a solid torus with the circle foliation given by the gluing map $z \mapsto e^{2\pi i \beta/\alpha} z$. $\beta/\alpha \in \mathbb{Q}/\mathbb{Z}$ is an invariant of f. We choose the normalization $\alpha \geq 1$. α can be interpreted as $card(\partial D^2 \pitchfork \text{ fiber})$. We define the projection $T(\alpha, \beta) \to$ leaf space, where the metric induced on the leaf space by the metric of $T(\alpha, \beta)$ is a cone metric. From the geometric point of view the leaf space is an orbifold.

Definition. The fiber f is **exceptional** if its saturated tubular neighborhood is isomorphic to $T(\alpha, \beta)$ with $\alpha > 1$ (**isomorphic** means that there is a fiber preserving diffeomorphism). $\beta/\alpha \in (\mathbb{Q}/\mathbb{Z})^\star$ is called the type of the exceptional fiber.

Exercise. The solid torus $T(1, 0) = D^2 \times S^1$ with the product fibration is a finite regular covering of $T(\alpha, \beta)$ of order α induced by

$$\begin{aligned} D^2 \times S^1 &\to D^2 \times S^1 \\ (z, x) &\mapsto (e^{2\pi i \beta/\alpha} z, e^{2\pi i \alpha} x). \end{aligned}$$

Fact 1: If M is compact and Seifert fibered, there are only finitely many exceptional fibers.
Fact 2: If M is not covered by S^3, the base of the fibration (the leaf space of the foliation) is a 2-dimensional geometric orbifold whose geometry is modelled on S^2, $\mathbb{E}^2$ or $\mathbb{H}^2$.
Fact 3: Fact 2 is not true for spherical 3-manifolds. The basis of such a Seifert fibered manifold is not always a geometric 2-dimensional orbifold. The only exceptions are S^3 and the Lens spaces $L(p, q)$ which have a Seifert fibration with basis a “bad” orbifold, i.e. a sphere with exactly one cone point (teardrop) or the 2-sphere with two cone points of different order (see Scott [42, p. 425]).
Fact 4: Let M be a compact orientable Seifert fibered 3-manifold. One can associate to the Seifert fibration of M a set of invariants:
a) The genus g of the underlying space of the basis (with $g \geq 0$ if the basis is orientable , and $-g$ if the basis is non-orientable)
b) the rational Euler class $e_0 \in \mathbb{Q}$,
c) the invariant of the exceptional fibers $\beta_1/\alpha_1, \ldots, \beta_n/\alpha_n$ with $\beta_i/\alpha_i \in (\mathbb{Q}/\mathbb{Z})^\star$, $\alpha_i > 1$
Definition of e_0:
If M is not covered by S^3, from Fact 2, M admits a finite covering $\overline{M} \to F$ of order $|G|$ which is a S^1-bundle over an orientable surface B (Explain!). Define $e_0 := e/|G| \in \mathbb{Q}$, where e is the Euler class of this S^1-bundle.

$$\begin{array}{ccc} \overline{M} & \longrightarrow & F \\ \downarrow & & \downarrow \\ M & \longrightarrow & B \end{array}$$

Exercise. e_0 does not depend on the covering chosen. e_0 is the obstruction for the existence of an essential horizontal surface in M (ramified section to the projection on the basis).

Here is a general definition of e_0 which works for any Seifert fibration:
Let $f_1, \ldots, f_r$ be the singular fibers. Pull out regular saturated neighborhoods T_i of these fibers and one saturated solid torus T_0 around a regular fiber f_0.

$M \setminus int(T_0 \cup \ldots \cup T_r)$ is a S^1-bundle over a surface with boundary. It is a product fibration because the Euler class vanishes. Choose sections $s_0, s_1, \ldots, s_r$ to the Seifert fibration on each torus ∂T_i, $i = 0, \ldots r$.
$e \in \mathbb{Z}$ is the obstruction to find a horizontal surface in $F \times S^1$ (F= punctured base) with boundary the given sections. e depends on the choice of sections (depends on α and β), but e_0 is well defined:

$$e_0 := e - \sum_{i=1}^{r} \beta_i / \alpha_i$$

Theorem 3.20 (Waldhausen). *Let M be a compact oriented Seifert fibered 3-manifold. Any incompressible surface F in M is either isotopic to a **vertical** surface (which is a union of fibers) or to a **horizontal** surface (everywhere transversal to the fibers).*

Exercise. 1) Show that a horizontal surface in an orientable compact Seifert 3-manifold M is always a fiber of a fibration over S^1.
2) Show that a compact orientable Seifert 3-manifold fibers over S^1 iff $e_0 = 0$. (Hint: use Waldhausen's Theorem above)
3) Show tha a Seifert 3-manifold is modelled on one of the geometries $\mathbb{E}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$, or $S^2 \times \mathbb{E}$.

Exercise. 1)Show that the only atoroidal and not strongly atoroidal closed orientable 3-manifolds are the seifert 3-manifolds modelled on $\mathbb{H}^2 \times \mathbb{E}^1$ and $\widetilde{SL_2(\mathbb{R})}$, with three exceptionnal fibers and basis a 2-sphere with 3 cone points.// 2) Show moreover that such manifolds do not contain any incompressible orientable surfaces iff $e_0 \neq 0$.

3.6 Equivariant Theorems

Proposition 3.21 (Jaco-Rubinstein). *Let M be an orientable 3-manifold with a fixed triangulation Δ and G a group of simplicial orientation preserving homeomorphisms of M, such that $Fix(G) = \{x \in M \mid g(x) = x$ for some $g \in G\}$ is a subcomplex of Δ.*
a) If $S^2 \subset M$ is an embedded essential sphere, then there exists a G-invariant embedded essential sphere Σ^2 (i.e. $g(\Sigma^2) = \Sigma^2$ or $g(\Sigma^2) \cap \Sigma^2 = \emptyset$, $\forall g \in G$).
b) Let M be irreducible and ∂-incompressible. Assume that $F \subset M$ is an essential embedded surface such that gF is "homotopically disjoint" from F for all $g \in G$. Then F can be isotoped to a G-invariant essential surface F'.

Idea of the proof. Put the essential surface S^2 or F in normal form with respect to Δ. Define a complexity of F (can be S^2) with respect to the G-action by

$$C_G(F) := \sum_{g \in G} card(F^{(1)} \pitchfork g(F^{(1)}))$$

(By an isotopy we assume that either $F = g(F)$ or $F \cap g(F) = \emptyset$ or $F \pitchfork g(F)$; only the case $F \pitchfork g(F)$ gives a contribution in this definition). $F^{(1)}$ is the 1-skeleton of the cell structure induced by Δ on F and we have $g(F^{(1)}) = g(F)^{(1)}$.

Claim. *If $C_G(F) = 0$ then one can isotope F to a G-equivariant surface.*

Again, define a complexity for F:

$$C(F;G) := (w(F), C_G(F))$$

where $w(F) = card(F \pitchfork \Delta^{(1)})$ is the weight of F as defined earlier.

Claim (Jaco-Rubinstein). *Let S^2 or F be as in Proposition 3.21 and in normal form with the least possible complexity $C(F;G)$. Then S^2 or F are G-equivariant.*

□

Corollary 3.22 (Equivariant Sphere Theorem). *Let M be a compact orientable 3-manifold and $p : \overline{M} \to M$ any regular covering. Then M is irreducible if and only if $\overline{M}$ is irreducible.*

The if direction of this statement was already proved in Proposition 3.4.

Corollary 3.23 (Equivariant Dehn Lemma). *Let M be a compact orientable irreducible 3-manifold with $\partial M \neq \emptyset$ and $p : \overline{M} \to M$ a regular finite covering, Then $\partial\overline{M}$ is compressible if and only if ∂M is compressible.*

Proof. $p : \overline{M} \cup_{\partial\overline{M}} \overline{M} \to M \cup_{\partial M} M$. If $(\overline{D}, \partial\overline{D}) \hookrightarrow (\overline{M}, \partial\overline{M})$ is an essential disk we get an essential sphere $\overline{S}$ in $\overline{M} \cup_{\partial\overline{M}} \overline{M}$. By the Equivariant Sphere Theorem there is an equivariant sphere in $\overline{M} \cup_{\partial\overline{M}} \overline{M}$ and therefore an essential sphere S in $M \cup_{\partial M} M$ which meets ∂M because M is irreducible. An innermost disk on S gives the compression disk for ∂M. □

Chapter 4

Homotopy versus Topology

Poincaré Conjecture. *Any closed simply connected* 3*-manifold is homeomorphic to* S^3*.*

Conjecture 4.1. *The universal cover of any irreducible compact* 3*-manifold is* $\mathbb{R}^3$ *or* S^3*.*

Remark. This conjecture is stronger than the Poincaré Conjecture, because if Σ^3 is a fake 3-sphere, then there is an irreducible fake 3-sphere just by applying the Prime Decomposition Theorem to Σ^3. (A **fake** 3-sphere is a space homotopy equivalent but not homeomorphic to S^3.)

Remark. A **Whitehead manifold** is an irreducible contactible open (i.e. non-compact and without boundary) 3-manifold not homeomorphic to $\mathbb{R}^3$. The first such example was constructed by Whitehead ([55]) in 1935. It is known that there are uncountably many Whitehead manifolds, but it is still an unsolved problem if a Whitehead manifold can cover a compact 3-manifold. For certain classes of Whitehead manifolds the answer is negative (see [32], [56], [52]), e.g. the so-called genus one Whitehead manifold (there are uncountably many of them, see [21]) cannot even non-trivially cover any 3-manifold. If Conjecture 4.1 is true, then no Whitehead manifold can have a compact quotient. In dimension $n > 3$, Davis ([6]) has given examples of contractible open n-manifolds not homeomorphic to $\mathbb{R}^n$ which cover compact manifolds.

An important property of $\mathbb{R}^3$ or S^3 was given by Alexander's Theorem (Theorem 3.1). In order to extend this theorem to any simply connected 3-manifold, we need the following theorem.

Theorem 4.2 (Loop Theorem, Papakyriakopoulos, [37]). *Let M be a compact orientable* 3*-manifold with $\partial M \neq \emptyset$ and $F \subset \partial M$ a connected component. Then if $\pi_1 F \to \pi_1 M$ is not injective, F is compressible in M.*

Corollary 4.3. *Let $F \hookrightarrow M$ be an orientable embedded surface different from S^2. Then F is incompressible if and only if $\pi_1 F$ injects in $\pi_1 M$.*

Corollary 4.4. *If M is simply connected, the only orientable embedded essential surfaces are spheres.*

Corollary 4.5 (Sphere Theorem, Papakyriakopoulos, [36]). *If M is an orientable 3-manifold with $\pi_2 M \neq \{0\}$, then M contains an essential sphere.*

Corollary 4.5 follows from Corollary 4.4 and the Equivariant Sphere Theorem (Corollary 3.22).

4.1 Haken 3-manifolds

Definition. A compact orientable 3-manifold is called **Haken manifold** if it is irreducible and contains an essential surface, or if it is homeomorphic to B^3.

Example. If M is irreducible, connected and $H^1(M;\mathbb{Q}) \neq \{0\}$ (e.g. if $\partial M \neq \emptyset$), then M is Haken.

Proof. First assume that $\partial M \neq \emptyset$. If ∂M is a union of 2-spheres then $M = B^3$ (using that M is irreducible and connected) and M is Haken by definition. If ∂M is not a union of 2-spheres then there exist two essential (simple) closed curves $\gamma_1, \gamma_2 \in \partial M$ such that the intersection number $\gamma_1 \cdot \gamma_2$ is 1. The inclusion $i : \partial M \to M$ induces the homomorphism $i_* : H_1(\partial M;\mathbb{Q}) \to H_1(M;\mathbb{Q})$. If $i_*(\gamma_1) = i_*(\gamma_2) = 0$, then γ_1 and γ_2 are boundaries of singular 2-chains: $\gamma_1 = \partial c_1$, $\gamma_2 = \partial c_2$. It follows that $\gamma_1' \cdot c_2 \equiv 0 \bmod 2$ in M (where γ_1' is γ_1 pushed into $int(M)$), contradicting $\gamma_1 \cdot \gamma_2 = 1$ in ∂M. Therefore, we obtain $i_*(H_1(\partial M;\mathbb{Q})) \neq \{0\}$ which implies $H_1(M;\mathbb{Q}) \neq \{0\}$ and by Poincaré duality $H_2(M, \partial M;\mathbb{Q}) \neq \{0\}$. By the Hopf Theorem, one can realize any non-trivial element of $H_2(M, \partial M;\mathbb{Z})$ by a surface. Doing surgery along compressing disks, we see that M contains an essential surface.

Assume now $H^1(M;\mathbb{Q}) \neq \{0\}$ (which is satisfied if $\partial M \neq \emptyset$ because of $H_2(M, \partial M;\mathbb{Q}) \cong H^1(M;\mathbb{Q})$). There is an other way to get the essential surface. The assumption implies $H^1(M;\mathbb{Z}) \neq \{0\}$ which gives the existence of a surjective homomorphism $\phi : \pi_1 M \to \mathbb{Z}$. This ϕ can be realized by a continuous map $h : M \to S^1$. Approximate h by a C^∞ map and apply Sard's Theorem to get an embedded surface (take the preimage h^{-1} of a regular value). After surgery on compressing disks we get an essential non-separating surface. $\square$

Remark. Conversely, any non-separating properly embedded surface in M defines a morphism $\phi : \pi_1 M \to \mathbb{Z} \to 0$.

Example. There are Haken 3-manifolds M with trivial homology constructed as follows. Take any non-trivial knot k and its knot space $N = S^3 \setminus int(\mathcal{N}(k))$. Define $M := N \underset{f:\partial N \circlearrowleft}{\cup} N$ such that $f_* : \pi_1 \partial N \to \pi_1 \partial N$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(note that $\pi_1 \partial N \cong \mathbb{Z} \oplus \mathbb{Z}$).

Till the 70's, there were known few (irreducible) non-Haken 3-manifolds and they all have a finite covering which is Haken.

Conjecture 4.6 (Waldhausen, Haken). *1) The fundamental group of any $K(\pi, 1)$ closed 3-manifold contains a (closed) surface group.*
2) Any compact (closed) irreducible orientable 3-manifold with infinite fundamental group is finitely covered by a Haken 3-manifold.
3) Any compact (closed) irreducible orientable 3-manifold with infinite fundamental group is finitely covered by a 3-manifold with strictly positive first Betti number.
4) Any compact (closed) irreducible orientable 3-manifold with an infinite fundamental group that does not contain a solvable subgroup of finite index is finitely covered by a 3-manifold with arbitrary large first Betti number.
5) (Thurston) Any complete hyperbolic 3-manifold of finite volume is finitely covered by a bundle over S^1.

Remark. 1) is still widely open.
2) implies Thurston's Geometrization Conjecture for $K(\pi, 1)$ spaces.
3) is true for arithmetic hyperbolic manifolds.
4) is true if $\partial M \neq \emptyset$ or if M contains an essential torus.

Here is a crucial property of a compact Haken 3-manifold.

Proposition 4.7. *Let M be a compact orientable Haken 3-manifold, then M admits a finite **hierarchy** of the following type. There is a finite sequence of pairs*

$$(M_1, F_1) \rightsquigarrow (M_2, F_2) \rightsquigarrow \ldots \rightsquigarrow (M_n, \emptyset),$$

where i) $M_1 = M$.
ii) F_i is an essential (connected orientable) surface in M which is not a disk and $\partial F_i \neq \emptyset$ if $\partial M_i \neq \emptyset$.
iii) $M_{i+1} = M_i \setminus int(\mathcal{N}(F_i))$ (cut M_i along F_i).
iv) M_n is a disjoint union of handlebodies.

Exercise. If a Haken 3-manifold admits as essential surfaces only disks then it is a handlebody.

Proposition 4.7 follows from this exercise and the Haken-Kneser Finiteness Theorem (Theorem 3.9) which we reformulate in an adapted weaker version.

Theorem 4.8 (Haken-Kneser Finiteness Theorem). *Given a compact orientable irreducible* 3*-manifold* M*, there is a number* $h(M) \in \mathbb{N}$ *such that* M *contains at most* $h(M)$ *incompressible and pairwise non-parallel closed surfaces.*

Claim. *For the length* n *of the hierarchy in Proposition 4.7 we have* $n \leq 2h(M) + 2$*.*

Proof of the Claim. We assume that a hierarchy $\{(M_i, F_i)\}$ is given as in Proposition 4.7 and construct a family $(G_1, \ldots, G_{n-1})$ of disjoint incompressible closed orientable surfaces in M. If F_i is closed, we take $G_i = F_i$ (this is only possible for $i = 1$). F_i is an essential surface in M_i, $\partial F_i \neq \emptyset$ (for $i \geq 2$). If ∂M_i is incompressible in M_i then it will be incompressible in M. Take for G_i the component of ∂M_i meeting ∂F_i. If ∂M_i is compressible in M_i, because F_i is incompressible and ∂-incompressible, one can find a complete family of compressing disks for ∂M_i which avoids F_i. Do the compressions and take for G_i the closed incompressible components obtained by compression and which meets ∂F_i.

Exercise. If G_i is incompressible in M_i, then it is incompressible in M.

In this way we construct $n - 1$ incompressible closed surfaces. If $n - 1 > 2h(M) + 1$ then there exist at least three surfaces G_i, G_j, G_k (i, j, k pairwise different) which are parallel and different from G_1. Assume G_i is in the product region between G_j and G_k. It follows that one of the surface (say G_j) must be a subset of M_i and therefore we have $M_j \subset M_i$ which implies $j > i$. F_i must be contained in the product region between G_i and G_j (by construction $F_i \cap \partial M_j = \emptyset$, $F_i \subset M_i \Rightarrow F_i \subset M_i \setminus M_j$).

Lemma 4.9 (Waldhausen). F_i *must be parallel to* G_i *(isotopic to a subsurface of* G_i*).*

Subclaim. *If* F_i *is parallel to* G_i *then* F_i *is* ∂*-reducible (not* ∂*-incompressible).*

The proof of the subclaim goes like this: F_i is isotopic to $F_i' \subset G_i$, therefore $F_i' \setminus \{\text{some disks}\} \subset \partial M_i$ and there is an essential arc on $F_i' \setminus \{\text{disks}\}$ which stays essential on F_i' because F_i' is not a disk (like F_i). This gives an essential arc on F_i which is parallel to an (essential) arc on ∂M. □

Definition. Let M be a compact orientable Haken 3-manifold. We define n to be the length of a hierarchy of M given as in Proposition 4.7. Further, for a union of handlebodies, let g be the sum of the genera of the boundary surfaces.

Define now the complexity $C(M)$ in the following way:

$$C(M) := \begin{cases} (sup\, n; 0), & \text{if } M \text{ is } \textit{not} \text{ a union of handlebodies.} \\ (0; g), & \text{if } M \text{ is a union of handlebodies.} \end{cases}$$

Take the lexicographical order, i.e. $(r, s) < (r', s')$ if either $r < r'$, or $r = r'$ and $s < s'$. It follows directly from this definition that $C(M) = (0; 0)$ if and only if M is a union of balls.

Proof of the Loop Theorem for Haken manifolds. The proof is by induction on the complexity $C(M)$. For $C(M) = (0; g)$ it is trivially true because ∂M is compressible for handlebodies. Assume $C(M) = (n; 0)$, $n > 0$. By assumption $\partial M \neq \emptyset$ and there is an essential surface $(G, \partial G) \hookrightarrow (M, F)$ which is not a disk (F denotes a connected component as in Theorem 4.2). Define $N = M \setminus int(\mathcal{N}(G))$. N is Haken with complexity $C(N) < C(M)$.

Claim. *Let $\partial_0 N$ be the component of ∂N containing the two copies of G, then $\partial_0 N$ must be compressible in N.*

First we show how this claim implies the Loop Theorem. Let $(\Delta, \partial\Delta) \hookrightarrow (N, \partial_0 N)$ be a compressing disk. We have $G^+ \amalg G^- \subset \partial_0 N$ and $\partial\Delta \not\subseteq G^+ \cup G^-$ because G is essential in M. If $\partial\Delta \cap (G^+ \cup G^-) \neq \emptyset$, make $\Delta \cap (G^+ \cup G^-)$ transversal. Look to an outermost arc of $\Delta \cap (G^+ \cup G^-)$ (there is no closed curve because G is essential). It follows that G is ∂-reducible and $\partial\Delta \cap (G^+ \cup G^-) = \emptyset$. F is compressible in M because $(\Delta, \partial\Delta) \hookrightarrow (M, F)$ is a compressing disk.

Proof of the claim:
Assume that $\partial_0 N$ is incompressible in N. This implies that $\pi_1 \partial_0 N$ injects in $\pi_1 N$ because of $C(N) < C(M)$ using the induction hypothesis. Set

$$W = M \setminus int(N) = F \times [0, 1] \underset{\partial G \times [0,1]}{\cup} G \times [0, 1]$$

$$\Rightarrow \pi_1 W = \pi_1 F \underset{\pi_1 \partial G}{*} \pi_1 G = \pi_1(F \cup G) = HNN(\pi_1 \partial_0 N)$$

$\Rightarrow \pi_1 \partial_0 N \rightarrowtail \pi_1 W$
$\Rightarrow \pi_1 \partial_0 N \rightarrowtail \pi_1 M$
$\Rightarrow \pi_1 W \rightarrowtail \pi_1 M$
$\Rightarrow \pi_1 F \rightarrowtail \pi_1 M$ which contradicts the assumption. □

Theorem 4.10 (Waldhausen, [54]). *Let M be a compact orientable Haken 3-manifold. Then the universal cover $\tilde{M}$ is homeomorphic to $B^3 \setminus \Sigma$ where Σ is a closed subset of ∂B^3.*

Proof. The proof is by induction on the complexity $C(M)$. If $C(M) = (0; 0)$ then $M = B^3$ and we are done. If $C(M) = (0; g)$ then M is a handlebody

(Exercise: do it by hand, by cutting along essential disks). Assume the theorem is true for complexity $< C(M)$. Let $(F, \partial F) \hookrightarrow (M, \partial M)$ be an essential surface and $N := M \setminus int(\mathcal{N}(F))$ with $C(N) > C(M)$. Let $p : \tilde{M} \to M$ be the universal covering. $p^{-1}(F) =: \tilde{F} = \bigcup_{i\in\mathbb{N}} \tilde{F}_i$ where each $\tilde{F}_i$ is homeomorphic to the universal cover of F because of the Loop Theorem. We have

$$\tilde{M} \setminus p^{-1}(\mathring{N}) = p^{-1}(F \times [0,1]) = \bigcup_{i\in\mathbb{N}} \tilde{F}_i \times [0,1]$$

and

$$p^{-1}(N) =: \tilde{N} = \bigcup_{i\in\mathbb{N}} \tilde{N}_i.$$

Each $\tilde{N}_i$ is homeomorphic to the universal covering of N (respectively of one component of N if F is separating). By the induction hypothesis, there are embeddings $\tilde{f}_i : \tilde{N}_i \hookrightarrow B^3$ such that $\mathring{B}^3 \subset \tilde{f}_i(\tilde{N}_i)$. We choose a renumbering of the $\tilde{N}_i$, $\tilde{F}_i$ and define $N^{(i)}$ $(i \in \mathbb{N})$ such that

$$N^{(1)} = \tilde{N}_1, \quad N^{(i)} \cap \Big(\tilde{F}_i \times [0,1]\Big) = \tilde{F}_i \times \{0\}$$

$$\tilde{N}_{i+1} \cap \Big(\tilde{F}_i \times [0,1]\Big) = \tilde{F}_i \times \{1\}$$

$$N^{(i+1)} = N^{(i)} \cup \Big(\tilde{F}_i \times [0,1]\Big) \cup \tilde{N}_{i+1}$$

Remark. There is a tree with $\tilde{N}_i$ as vertices and $\tilde{F}_i \times [0,1]$ as edges.

Assume that we have built an embedding $f_i : N^{(i)} \hookrightarrow B^3$ such that $\mathring{B}^3 \subset f_i(N^{(i)})$ (in particular $f_i(\tilde{F}_i \times \{0\}) \subset \partial B^3$). We can assume this at least for $i = 1$ because of $N^{(1)} = \tilde{N}_1$. Moreover, there exists an embedding $g_i : \tilde{F}_i \times \{0\} \hookrightarrow B^2$ with the property $\mathring{B}^2 \subset g_i(\tilde{F}_i \times \{0\})$. Identify B^2 with the unit disk in the plane $z = 0$ in $\mathbb{R}^3$ and let P, Q_i be cones over $g_i(\tilde{F}_i \times \{0\})$ with cone points respectively $p = (0,0,-1)$, $q_i = (0,0,z_i)$ where $-1/i < z_i < 0$. Further, let C_i be the cylinder $g_i(\tilde{F}_i \times \{0\}) \times [0,1/2] \subset B^3$. Define an embedding $Q_i \cup C_i \hookrightarrow Q_i$ by choosing a line l through q_i and identifying linearly $\overline{l \cap (Q_i \cup C_i)}$ with $\overline{l \cap Q_i}$.

Exercise. Using this embedding, define an embedding $h_i : N^{(i)} \cup (\tilde{F}_i \times [0,1/2]) \hookrightarrow B^3$ such that $\mathring{B}^3 \subset h_i(N^{(i)} \cup \tilde{F}_i \times [0,1/2])$.

Do the same construction for $(\tilde{F}_i \times [1/2,1]) \cup \tilde{N}_{i+1}$ using the given embedding $\tilde{N}_{i+1} \hookrightarrow B^3$ by defining $h'_i : (\tilde{F}_i \times [1/2,1]) \cup \tilde{N}_{i+1} \hookrightarrow {B'}^3$. This gives us an embedding $h_{i+1} : N^{(i+1)} \hookrightarrow B^3 \cup {B'}^3$ where the two balls are glued (to a ball) along $h_i(\tilde{F}_i \times \{1/2\}) = h'_i(\tilde{F}_i \times \{1/2\})$ whose closure is a disk. Taking the limit $i \to \infty$, we obtain the required embedding $\tilde{M} \hookrightarrow B^3$. $\square$

4.2 Topological Rigidity

Conjecture 4.11 (Borel). *Let M, N be two orientable compact $K(\pi, 1)$ n-manifolds and let $f : M \to N$ be a (proper??) homotopy equivalence such that the restriction $f_| : \partial M \to \partial N$ is a homeomorphism. Then f is homotopic (relativ to ∂M) to a homeomorphism.*

Theorem 4.12 (Waldhausen). *Borel's Conjecture is true for Haken 3-manifolds.*

Remark. 1) Borel's Conjecture is true in dimension 2 by Nielsen (just using homotopy theory).
2) Thurston's Geometrization Theorem implies Borel's Conjecture in dimension 3.
3) In dimension 3 you have not to put an assumption on $f_| : \partial M \to \partial N$ if M is acylindrical (i.e. if there is no essential annulus in M).

Proof of Theorem 4.12. The proof is again by induction on the complexity $C(M)$. If $C(M) = (0; 0)$ then $M = B^3$ and $f_| : S^2 \xrightarrow{\cong} \partial N$. We obtain $N = B^3$ because N is irreducible...
Assume now that $C(M) = (0; g)$. In this case M is a handlebody which we denote by H_g. Let $(F, \partial F) \hookrightarrow (N, \partial N)$ be a properly embedded essential surface. The preimage $f^{-1}(F)$ is a surface embedded in H_g which can be made essential by a homotopy of f, hence $f^{-1}(F)$ is a union of disks. Using the fact that $f_| : \partial H_g \to \partial N$ is a homeomorphism, we conclude that $f^{-1}(F)$ is just one disk Δ. f can be homotoped such that $f_| : \Delta \to F$ is a homeomorphism. Now cut H_g along Δ and N along F to reduce the complexity.
In the general case, assume that Theorem 4.12 is true for complexity $< C(M)$. First we consider the case $\partial M \neq \emptyset \neq \partial N$. Let $(F, \partial F) \hookrightarrow (N, \partial N)$ be an essential surface different from a disk (using the general assumption that N is Haken). By homotopy on f, we can make $G := f^{-1}(F) \subset M$ be a properly embedded essential surface (perhaps not connected). Take a connected component $G_i \subset G$, then $f_{|*} : \pi_1 G_i \to \pi_1 F$ is injective. A theorem of Nielsen implies that $f_|$ is homotopic to a covering map $G_i \to F$ (in particular $\partial G_i \neq \emptyset$ if $\partial F \neq \emptyset$). $f_| : \partial G_i \to \partial F$ has degree one, hence $f_|$ is homotopic to a homeomorphism $G_i \to F$. Because of $f^{-1}(\partial F) = \partial G_i$ and the fact that $f_| : \partial M \to \partial N$ is a homeomorphism, it follows that $G = G_i$ is connected. Therefore we see that (after the homotopy made before) $f^{-1}(F) = G$ is connected and $f_| : f^{-1}(F) = G \to F$ is a homeomorphism. By cutting M along G and N along F, the map $f_| : M \setminus int(\mathcal{N}(G)) \to N \setminus int(\mathcal{N}(F))$ is a homotopy equivalence (using the fact that G and F are essential) and a homeomorphism on the boundary. The complexity is reduced: $C(M \setminus int(\mathcal{N}(G))) < C(M)$ and by the induction hypothesis f is homotopic to a homeomorphism.

In order to prove the more difficult case $\partial M = \partial N = \emptyset$, we need a lemma.

Lemma 4.13 ([39]). *Let M be an irreducible orientable closed 3-manifold and $G \subset M$ be a closed orientable incompressible (connected) surface. If $\pi_1 G \subset \Gamma \subset \pi_1 M$ and $\Gamma \cong \pi_1 F$ for a closed orientable surface F, then $\pi_1 G \cong \pi_1 F$.*

Exercise. Instead of the assumption that Γ is a surface group, assume only that Γ is finitely presented with index $[\Gamma : \pi_1 G] < \infty$. Then the conclusion still holds.

Proof of Lemma 4.13: $d := [\Gamma : \pi_1 G] < \infty$ because G and M are closed. Let $p : \overline{M} \to M$ be the covering associated to Γ (of degree $[\pi_1 M : \Gamma]$) then $\pi_1 \overline{M} \cong \Gamma \cong \pi_1 F$. The irreducibility of M implies that $\overline{M}$ is also irreducible. M is Haken, hence both M and $\overline{M}$ are $K(\pi, 1)$. Therefore, the isomorphism $\pi_1 \overline{M} \cong \pi_1 F$ can be realized by a homotopy equivalence $g : \overline{M} \to F$. The inclusion $i : G \to M$ can be lifted to $\overline{M}$ which gives the following commutative diagram:

$$\begin{tikzcd} & \overline{M} \arrow[r, "g", "\simeq"'] \arrow[d, "p"] & F \\ G \arrow[ur, dotted, "\exists \overline{i}"] \arrow[r, "i"] & M & \end{tikzcd}$$

The composition $g \circ \overline{i} : G \to F$ induces an injective map $(g \circ \overline{i})_* : \pi_1 G \to \pi_1 F$. By the Nielsen Theorem $g \circ \overline{i}$ is homotopic to a finite covering of degree d and in the following diagram the map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by d.

$$\begin{tikzcd} H_2(G;\mathbb{Z}) \arrow[r, "\overline{i}_*"] \arrow[d, "\cong"'] & H_2(\overline{M};\mathbb{Z}) \arrow[r, "g_*", "\cong"'] & H_2(F;\mathbb{Z}) \arrow[d, "\cong"] \\ \mathbb{Z} \arrow[rr, "\cdot d"] & & \mathbb{Z} \end{tikzcd}$$

$\overline{i}_*([G]) \neq 0$ in $H_2(\overline{M};\mathbb{Z})$, we want to show that it is a primitive class. G is incompressible in $\overline{M}$ because $\pi_1 G$ injects in $\pi_1 \overline{M}$. Moreover, G must separate $\overline{M}$ (otherwise $\pi_1 \overline{M} = H \underset{\pi_1 G}{*}$ is a HNN-extension for some group H which gives the contradiction $d = [\pi_1 \overline{M} : \pi_1 G] = \infty$). Therefore we have $\overline{M} = X \cup_G Y$ with X, Y non-compact and $\pi_1 \overline{M} \cong \pi_1 X \underset{\pi_1 G}{*} \pi_1 Y$. Take a path $\gamma : \mathbb{R} \to \overline{M}$ going at infinity in X and Y and meeting G in only one point. $[\gamma] \neq 0$ in $H_1(\overline{M}, Ends(\overline{M}); \mathbb{Z})$ which implies by duality that $\overline{i}_*([G])$ is a primitive class in $H_2(\overline{M};\mathbb{Z})$ and $d = +1$. This proves the lemma.

Now we continue the main proof in the case $\partial M = \partial N = \emptyset$. Let $F \subset N$ be a closed essential (i.e. incompressible) surface. The following claim implies the theorem as in the case $\partial M \neq \emptyset \neq \partial N$.

Claim. *After homotopy on f, $G = f^{-1}(F)$ is a closed essential (connected) surface and $f_| : G \to F$ is homotopic to a homeomorphism.*

Proof of the claim:
Let $G_i \subset G$ be a connected component. $f_{|*} : \pi_1 G_i \to \pi_1 F$ is injective, hence by the Nielsen Theorem $f_| : G_i \to F$ is homotopic to a covering map of finite degree (because F and G_i are closed), in particular $[\pi_1 F : f_*(\pi_1 G_i)] < \infty$. Applying Lemma 4.13 to $\pi_1 G_i \cong f_*(\pi_1 G_i) \subset \pi_1 F \subset \pi_1 N \cong \pi_1 M$, we conclude that $\pi_1 G_i \cong \pi_1 F$, so $f_| : G_i \to F$ is homotopic to a homeomorphism. Write $f^{-1}(F) = G_1 \cup \ldots \cup G_q$ and choose a base point e on F and base points e_j on G_j, $j = 1, \ldots q$ such that $f^{-1}(e) = e_1 \cup \ldots \cup e_q$. Choose two components (e.g. G_1, G_2) and a path α from e_1 to e_2. $f(\alpha)$ is a loop in N. Let β be a loop based at e_1 representing $f_*^{-1}([f(\alpha)]^{-1})$ and let $\lambda = \beta \cdot \alpha$ be the composition of β with α. Then $f(\lambda)$ is nullhomotopic in $\pi_1(N, e)$. We can assume that λ is immersed in M in such a way to be transverse to the G_j and meet them only at the base points e_j. $\lambda \cap f^{-1}(F)$ divides λ in a finite number of arcs $\lambda_1, \ldots, \lambda_r$ and each $f(\lambda_j)$ is a loop in N. Parametrize λ by $\lambda : S^1 \to M$, then $f \circ \lambda : S^1 \to N$ is nullhomotopic, so it extends to a map $h : D^2 \to N$. We make h transversal to F. $h^{-1}(F)$ is a union of arcs and closed curves. Take an innermost closed curve $C = \partial D$ in $h^{-1}(F)$. $h(C)$ must be nullhomotopic in F because F is incompressible (Loop Theorem). Look to $h(D) \cup \Delta$ where $\Delta \subset F$, $\partial \Delta = h(C)$. $h(D) \cup \Delta$ is a sphere which is nullhomotopic in N (because of $\pi_2(N) = 0$). After a finite number of homotopies we can assume that $h^{-1}(F)$ contains only arcs. Consider an outermost arc $a \in h^{-1}(F)$ and $b \subset \partial D^2$ such that $a \cup b = \partial D_0$ bounds a disk D_0 which does not meet any other arc in $h^{-1}(F)$. From $h(b) = (f \circ \lambda)(b) = f(\lambda_j)$ (for some j) we conclude that the loop $f(\lambda_j)$ is homotopic to the loop $h(a)$ in $\pi_1(F, e)$. If λ_j is a loop in M with base point e_i, we can homotope λ_j modulo e_i to lie in $\pi_1(G_i, e_i)$ and decrease the number of intersection points $\lambda \cap \{G_1 \cup \ldots G_q\}$. For this reason, we can assume that λ_j is a path from (G_k, e_k) to (G_l, e_l) with $k \neq l$ and $f(\lambda_j) \in \pi_1(F, e)$. Cut M along G_k and G_l and consider the compact component X which contains λ_j and with boundary $\partial X = G_k \cup G_l$. To finish the proof we need the following lemma.

Lemma 4.14. *$G_k \hookrightarrow X$ and $G_l \hookrightarrow X$ are both homotopy equivalences (in fact by Waldhausen $X \cong G_k \times [0,1]$).*

Proof of the lemma:
Let $p : \overline{X} \to X$ be the covering such that $\pi_1 \overline{X} = \pi_1(G_k, e_k)$. Since $\pi_1(G_k, e_k) \cong \pi_1(G_l, e_k)$, both inclusions $G_k \hookrightarrow X$, $G_l \hookrightarrow X$ can be lifted to $\overline{X}$:

$$\begin{tikzcd} & \overline{X} \arrow[d] \\ G_j \arrow[r] \arrow[ur, dotted, "\exists"] & X \end{tikzcd} \qquad \begin{tikzcd} & \overline{X} \arrow[d] \\ G_k \arrow[r] \arrow[ur, dotted, "\exists"] & X \end{tikzcd}$$

Moreover, $G_k \hookrightarrow \overline{X}$, $G_l \hookrightarrow \overline{X}$ are homotopy equivalences, in particular we have $H_2(\overline{X};\mathbb{Q}) \cong H_2(G_k;\mathbb{Q}) \cong H_2(F;\mathbb{Q}) \cong \mathbb{Q}$. Consider the following exact sequence:

$$H_3(\overline{X}, \partial\overline{X};\mathbb{Q}) \to H_2(\partial\overline{X};\mathbb{Q}) \to H_2(\overline{X};\mathbb{Q}) \cong \mathbb{Q}$$

$\partial\overline{X}$ contains at least two components because ∂X contains two components. This implies that the rank of $H_2(\partial\overline{X};\mathbb{Q})$ is at least two. Therefore $H_3(\overline{X}, \partial\overline{X};\mathbb{Q})$ is not trivial, hence $\overline{X}$ is compact. We conclude that $\overline{X} \to X$ is a finite covering. From the exercise after Lemma 4.13 (applied to $\pi_1 G_k \cong \pi_1\overline{X} \subset \pi_1 X \subset \pi_1 M$) we obtain $\pi_1\overline{X} \cong \pi_1 X$ which implies that the covering must have degree one, so $\overline{X}$ is homeomorphic to X. This completes the proof of Lemma 4.14. Since $G_k \hookrightarrow X$, $G_l \hookrightarrow X$ are homotopy equivalences, one can homotope $f : M \to N$ such that $f(X)$ embeds in F. A further homotopy eliminates G_k and G_l from $f^{-1}(F)$. Repeating this procedure completes the proof of the claim and Theorem 4.12. □

4.3 Strong Torus Theorem

In this section we give a homotopical characterization of strong atoroidality.

Theorem 4.15 (Strong torus theorem). *A closed orientable irreducible* 3-*manifold M is strongly atoroidal iff any $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$ is conjugated to the fundamental group of a component of ∂M.*

In the case of Haken 3-manifolds, this theorem has been discovered by Waldhausen in the late sixties. Subsequently several authors including Feustel [7, 8], Jaco and Shalen [14], Johannson [16], and Scott [43] gave proofs of various forms of this theorem for Haken 3-manifolds. All these approaches involve quite intricate topological arguments. Among these, Scott's acount is the most easily digestible and use mainely properties of the fundamental group of the manifold. In fact Scott's work ([43, 44]) is the starting point of the proof of the strong torus theorem in the general case. This proof, which is beyond the scoop of classical methods in dimension 3, is a consequence of the works of several people: Casson and Jungreis [5], Gabai [10], Mess [**?**] and Tukia [53]. In particular, Mess's paper uses a lot of ideas from geometric group theory.

Chapter 5

Thurston's Hyperbolization Theorem

5.1 Some hyperbolic geometry

We will consider two models. The **upper half space model** (Poincaré)

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

with a Riemannian metric given by $ds^2 = \sum dx_i^2/x_n^2$ and the **unit disk model** (Poincaré)

$$\mathbb{D}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 < 1\}$$

with $ds^2 = 4\sum dx_i^2/(1-\sum x_i^2)$. In the sequel, we will use the following inclusions and identifications:

$$\mathbb{H}^n \subset \mathbb{R}^n \subset \widehat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\} = S^n.$$

$$\overline{\mathbb{H}^n} \subset \widehat{\mathbb{R}}^n. \quad \partial\overline{\mathbb{H}^n} = \mathbb{R}^{n-1} \cup \{\infty\} = S^{n-1}_\infty = \partial\overline{\mathbb{D}^n}.$$

Exercise. One can define an isometry between these two models $\Phi_0 : \mathbb{D}^n \to \mathbb{H}^n$ by taking the inversion in the $(n-1)$-sphere of radius $\sqrt{2}$ and center $(0, \ldots, 0, -1) \in \mathbb{R}^n$ (see Figure 5.1). Φ_0 is not orientation preserving. An orientation preserving isometry between $\mathbb{D}^n$ and $\mathbb{H}^n$ is given by the composition of the inversion in the $(n-1)$-sphere of radius $\sqrt{2}$ and center $(0, \ldots, 0, 1) \in \mathbb{R}^n$ with the reflection in $\mathbb{R}^{n-1} \times \{0\}$.

An **inversion** ρ in a sphere $S(O, R)$ is determined by $\rho(P) = P'$ with $|\overrightarrow{OP}| \cdot |\overrightarrow{OP'}| = R^2$ (see Figure 5.2). Inversions are conformal; they send a sphere or plane to a sphere or plane and preserve the angles. We define the Möbius group $M(\widehat{\mathbb{R}}^n) = M(S^n)$ to be the group generated by inversions in spheres of dimension $n-1$ in $\widehat{\mathbb{R}}^n$. There is a natural inclusion $M(\widehat{\mathbb{R}}^{n-1}) \subset M(\widehat{\mathbb{R}}^n)$.

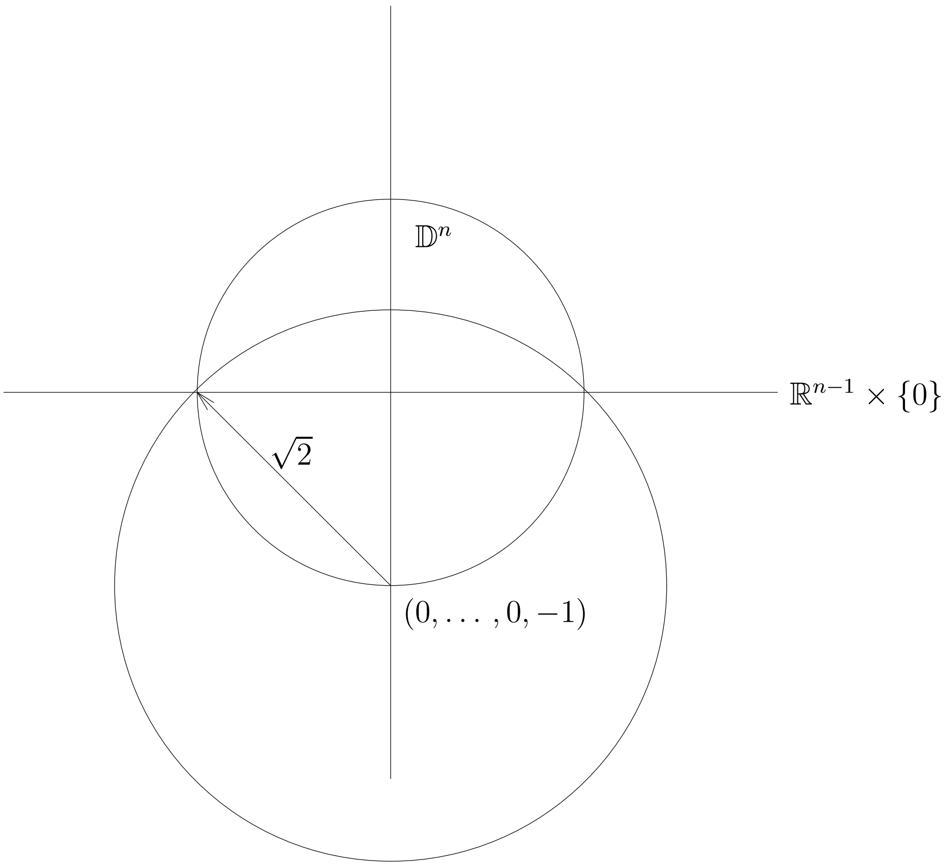

Figure 5.1: $\Phi_0 : \mathbb{D}^n \to \mathbb{H}^n$

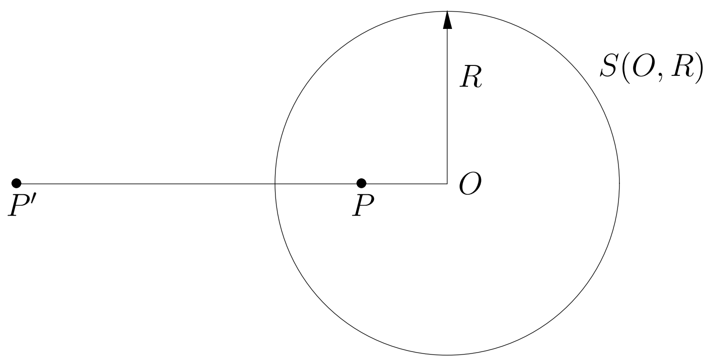

Figure 5.2: Inversion in $S(O, R)$

Theorem 5.1 (Liouville). *The Möbius group $M(\widehat{\mathbb{R}}^n)$ is isomorphic to $Conf(S^n)$, the group of conformal transformations of S^n.*

Examples of Möbius transformations are
i) $X \mapsto \lambda AX + B$, where X, $B \in \mathbb{R}^n$, $A \in O(n)$, $\lambda > 0$ and $\infty \mapsto \infty$ or
ii) $X \mapsto \lambda A\rho(X) + B$, where in addition ρ is an inversion.

Lemma 5.2 (Rigidity). *Given two sets of $n{+}2$ independent points $(P_0, P_1, \ldots, P_{n+1})$, $(Q_0, Q_1, \ldots, Q_{n+1})$ in S^n, there is a unique conformal transformation $f \in M(\widehat{\mathbb{R}}^n)$ which sends P_i to Q_i. The points $(P_0, P_1, \ldots, P_{n+1})$ are called **independent**, if the vectors $\{\overrightarrow{P_i - P_0}\}_{i=1,\ldots,n+1}$ are linearly independent in $\mathbb{R}^{n+1}$.*

We leave the proof of this lemma as an exercise.

Theorem 5.3. *a) The group of orientation preserving isometries $Isom^+(\mathbb{H}^n)$ acts simply transitively on the pairs $(P, \mathcal{R})$, where P is a point in $\mathbb{H}^n$ and $\mathcal{R}$ is a positively oriented orthonormal framing of $T_P\mathbb{H}^n$ (i.e. given $(P, \mathcal{R})$ and $(P', \mathcal{R}')$, there is a unique $f \in Isom^+(\mathbb{H}^n)$ such that $f(P) = P'$ and $df_P(\mathcal{R}) = \mathcal{R}'$).*
b) The hyperbolic lines in $\mathbb{H}^n$ are the half-circles or half-lines which are perpendicular to $\mathbb{R}^{n-1} \times \{0\}$.
c) $Isom(\mathbb{H}^n) = M(\widehat{\mathbb{R}}^{n-1}) = Conf(S^{n-1}_\infty)$ (e.g. in dimensions 2 and 3 we have the identities $Isom^+(\mathbb{H}^2) = PSL_2(\mathbb{R})$ and $Isom^+(\mathbb{H}^3) = PSL_2(\mathbb{C})$).

Proof (Sketch). a) Take $X \in \mathbb{H}^n$, $X = (Y, x_n)$, $Y \in \mathbb{R}^{n-1}$, $x_n > 0$ and define $f(X) = (\lambda A(Y) + B, \lambda x_n)$, $\lambda > 0$, $A \in O(n-1)$.

Exercise. $f \in Isom(\mathbb{H}^n)$. Given X, $X' \in \mathbb{H}^n$, there is an isometry f of the above type such that $f(X) = X'$.

Exercise. The subgroup of isometries in $Isom(\mathbb{H}^n)$ fixing a point is isomorphic to $O(n)$. To see this, go to the unit disk model and send the fixed point to 0. The isometries of $\mathbb{D}^n$ fixing 0 are the isometries of the Euclidean $\mathbb{R}^n$.

This proves a). As a corollary of assertion a) we get the following statement (using the uniqueness of f): Any isometry of $\mathbb{H}^n$ is a product of conformal transformations, in particular $Isom(\mathbb{H}^n) \subset M(\widehat{\mathbb{R}}^n) = Conf(S^n)$. Thus any isometry of $\mathbb{H}^n$ extends to a conformal transformation of $\partial\overline{\mathbb{H}^n} = S^{n-1}_\infty$ and therefore $Isom(\mathbb{H}^n) \subset M(\widehat{\mathbb{R}}^{n-1}) = Conf(S^{n-1}_\infty)$. If an isometry is the identity on S^{n-1}_∞, then it is the identity on $\overline{\mathbb{H}^n}$.

Part b) is a consequence of the following three claims:

Claim. *The vertical segment between $P = (x, x_n)$ and $Q = (x, x'_n) \in \mathbb{H}^n$ is the shortest one, hence the unique geodesic between P and Q is the vertical half-line. Moreover, the hyperbolical distance is $d(P, Q) = |\log(x_n/x'_n)|$.*

Claim. *Given a half-circle perpendicular to $\mathbb{R}^{n-1} \times \{0\}$, there is an isometry in $Isom(\mathbb{H}^n)$ which sends it to a vertical half-line, so half-circles perpendicular to $\mathbb{R}^{n-1} \times \{0\}$ are geodesic lines.*

Claim. *Between two points in $\mathbb{H}^n$ there is a unique geodesic line.*

By a) we know that $Isom(\mathbb{H}^n) \subset M(\widehat{\mathbb{R}}^{n-1})$, therefore in order to prove part c) we only have to show $M(\widehat{\mathbb{R}}^{n-1}) \subset Isom(\mathbb{H}^n)$. Take an inversion in $M(\widehat{\mathbb{R}}^{n-1})$, it sends a geodesic line to a geodesic line by b) and it preserves angles. Hence, an inversion sends a geodesic triangle to a geodesic triangle with the same angles.

Exercise. A geodesic triangle of $\mathbb{H}^n$ $(n = 2)$ is determined up to isometry by its angles. This implies that an inversion (a conformal transformation) preserves the metric.

□

Remark. Choose an arbitrary point $x \in \mathbb{H}^n$. One can define the sphere at infinity S^{n-1}_∞ as the space of directions (of semi-geodesics) through x.

Definition. A **totally geodesic submanifold** in $\mathbb{H}^n$ is a submanifold V which contains all the hyperbolic lines tangent at V to a point.

Exercise. Using the disk model, show that if $V \subset \mathbb{H}^n$ is a totally geodesic submanifold of dimension $p \leq n$, then V is isometric to $\mathbb{H}^p$.

Theorem 5.4 (Classification of isometries of $\mathbb{H}^n$). *Let $f \in Isom(\mathbb{H}^n) \setminus \{id\}$. Then either*
*i) f has at least one fixed point in $\mathbb{H}^n$ (f is called **elliptic**), or*
*ii) f has a unique fixed point in $\overline{\mathbb{H}^n}$ which belongs to $S^{n-1}_\infty = \partial\overline{\mathbb{H}^n}$ (f is called **parabolic**), or*
*iii) f has exactly two distinct fixed points in $\overline{\mathbb{H}^n}$ which belong to S^{n-1}_∞ (f is called **hyperbolic**).*

Proof. By the Brouwer fixed point theorem in $\overline{\mathbb{H}^n}$, f must have a fixed point in $\overline{\mathbb{H}^n}$. If it has no fixed point in $\mathbb{H}^n$ then f has at most two fixed points on $\partial\overline{\mathbb{H}^n}$. Otherwise, if f has at least three fixed points in $\partial\overline{\mathbb{H}^n}$, take $V^2 \subset \mathbb{H}^n$ to be a totally geodesic submanifold passing through these three points. We obtain $V^2 = \mathbb{H}^2$ and f induces an isometry of V^2 with three fixed points at infinity. Therefore $f|_{V^2} = id$, which implies $f = id$; a contradiction to the assumption. □

We give now a description of each type:

i) Elliptic isometries:
f has a fixed point in $\mathbb{H}^n$. Use the disk model and send the fixed point to 0. Then f is conjugated to an element of $O(n)$. In dimension $n \geq 3$ an orientation

preserving elliptic isometry may have fixed points on S_∞^{n-1} too, whereas for $n = 2$ this is impossible.

ii) Parabolic isometries:

Take the upper half plane model. Send the fixed point set to ∞ and assume $f(\infty) = \infty$.

Exercise. f preserves each horizontal plane (horosphere based at ∞) $\mathbb{R}^{n-1} \times \{t\}$, $t > 0$ and acts on each of these planes as an Euclidean isometry.

Definition. Let P be a point in $\partial\mathbb{D}^n = S_\infty^{n-1}$. A **horosphere** is a Euclidean sphere of dimension $n-1$ in $\mathbb{D}^n$ tangent to S_∞^{n-1} at P. It is perpendicular to all hyperbolic lines through P. A **horoball** is a ball bounded by a horosphere in $\mathbb{D}^n$. Horoballs are convex.

Proof of the exercise (Sketch). We choose the upper half space model $\mathbb{H}^n$. Take two horospheres $\mathbb{R}^{n-1} \times \{t\} = H \neq H' = \mathbb{R}^{n-1} \times \{t'\}$ with $t' > t > 0$ and denote by $d = \log(t'/t)$ their hyperbolic distance. Assume that $f(H) = H'$. Consider the Euclidean translation $h : H \to H'$ of distance $t' - t$ which maps $(0, \dots, 0, t)$ to $(0, \dots, 0, t')$. Denoting by $|\cdot|$ the Euclidean norm in H and H' respectively, and using the fact that f^{-1} is an isometry (with respect to the hyperbolic distance), we obtain

$$\frac{|f^{-1}(h(x))|}{t} = \frac{|h(x)|}{t'}$$

and therefore $|f^{-1}(h(x))| = (t/t')|h(x)| = e^{-d}|h(x)| = e^{-d}|x|$. For this reason, $f^{-1} \circ h$ is a (Euclidean) contraction of H (which is complete) with coefficient $e^{-d} < 1$. This implies that there is a point $P_0 \in H$ such that $f(P_0) = h(P_0)$. Therefore f preserves the hyperbolic line through 0, P_0 and ∞. This gives the contradiction that f has two fixed points at infinity. Finally, we conclude $f(H) = H$. $\square$

iii) Hyperbolic isometries:

Take the upper half space model. We can assume that $f(\infty) = \infty$, $f(0) = 0$. The hyperbolic line joining the two fixed points is called the **axis** $A(f)$ of f. f preserves the family of hyperplanes (half-spheres) perpendicular to the axis $A(f)$. f also preserves the family of Euclidean lines through 0 and perpendicular to the family of hyperplanes. f can be written as $f(Y, x_n) = (\lambda AY, \lambda x_n)$, $A \in O(n-1)$, $\lambda > 1$. For any $X_0 \in A(f)$ we have $d(f(X_0), X_0) = \inf_{x \in \mathbb{H}^n} d(f(X), X) = \log \lambda$. Fix a point $P \in \mathbb{H}^n$. Its image $f(P)$ can be constructed as follows (see Figure 5.3): First, map P to Q, where Q is the intersection of $A(f)$ with the hyperplane through P perpendicular to $A(f)$. After this, translate Q to Q' along $A(f)$, where $d(Q, Q') = \lambda$. Get Q'' by the intersection of OP with the hyperplane perpendicular to $A(f)$ at Q', finally rotate Q'' by A to get $f(P)$. If $A = id$ then f is called hyperbolic, if $A \neq id$ then it is called **loxodromic** and we have a screw motion around the axis $A(f)$.

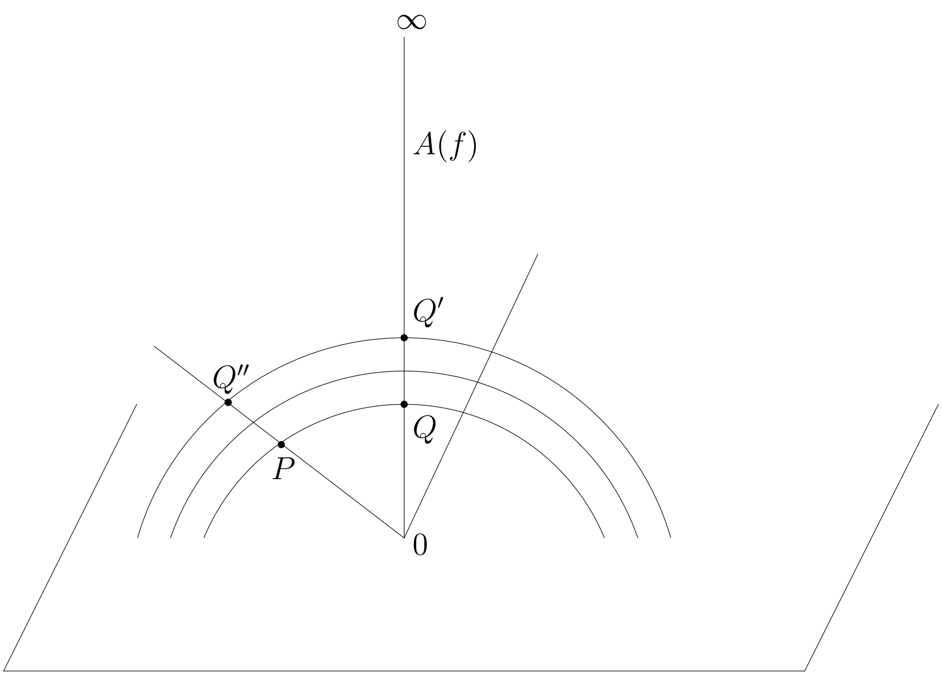

Figure 5.3: Hyperbolic isometry

Exercise. Show that for $f \in Isom(\mathbb{H}^n) \setminus \{id\}$, we have the following characterization by the **translational length** of f, defined by the expression $\inf_{x\in\mathbb{H}^n} d(x, f(x))$:
i) f is elliptic if and only if $\inf_{x\in\mathbb{H}^n} d(x, f(x)) = 0$ and the infimum is reached at $x_0 \in \mathbb{H}^n$.
ii) f is parabolic if and only if $\inf_{x\in\mathbb{H}^n} d(x, f(x)) = 0$ but the infimum is not reached.
iii) f is hyperbolic if and only if $\inf_{x\in\mathbb{H}^n} d(x, f(x)) > 0$.

5.2 Kleinian groups

From now on we will restrict to the case of $\mathbb{H}^3$ and $Isom^+(\mathbb{H}^3) = PSL_2(\mathbb{C})$.

Exercise. Let $f \in PSL_2(\mathbb{C})$:
1) f is elliptic if and only if $(tr\, f)^2 < 4$ and $tr\, f \in \mathbb{R}$.
2) f is parabolic if and only if $(tr\, f)^2 = 4$.
3) f is hyperbolic if and only if $(tr\, f)^2 > 4$ or $tr\, f \notin \mathbb{R}$.
4) Let f, g be in $PSL_2(\mathbb{C}) \setminus \{id\}$, then f and g are conjugated in $PSL_2(\mathbb{C})$ if and only if $(tr\, f)^2 = (tr\, g)^2$.

Definition. A subgroup $\Gamma \subset PSL_2(\mathbb{C})$ is **discrete** if any sequence $(\gamma_n) \in \Gamma$ with the property $\lim_{n\to\infty} \gamma_n(z) = z$, $\forall z \in S^2_\infty = \partial\overline{\mathbb{H}^3}$, is stationary. In other words, Γ is discrete if the identity $id \in \Gamma$ is isolated for the topology of pointwise convergence in S^2_∞ or if $id \in \Gamma$ is isolated for any matrix norm in $PSL_2(\mathbb{C})$.

Exercise. $\Gamma \subset PSL_2(\mathbb{C})$ is discrete if and only if Γ acts **properly discontinuously** on $\mathbb{H}^3$, i.e. if for each compact subset $K \subset \mathbb{H}^3$ the set

$$\Gamma_K := \{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$$

is finite. To prove that properly discontinuous implies discrete, use that the orbit of any point is discrete. To prove the other direction, use the fact that the stabilizer of a point $x \in \mathbb{H}^3$ in $PSL_2(\mathbb{C})$ is compact ($\cong SO(3)$).

Definition. A **Kleinian group** will be (in these lectures) a discrete, torsion free, finitely generated subgroup of $PSL_2(\mathbb{C})$.

Definition. A point $x \in \overline{\mathbb{H}^3} = \mathbb{H}^3 \cup S^2_\infty$ is a **limit point** of $\Gamma \subset PSL_2(\mathbb{C})$ if there exists a point $x' \in \overline{\mathbb{H}^3}$ and a non-trivial sequence $(\gamma_n) \in \Gamma$ such that $\lim_{n\to\infty} \gamma_n(x') = x$.

Exercise. $\Gamma \subset PSL_2(\mathbb{C})$ is discrete if and only if any limit point is in S^2_∞.

Definition. Let Γ be a Kleinian group. The **limit set** of Γ, denoted by $\Lambda(\Gamma)$, is the set of all limit points of Γ. In particular, $\Lambda(\Gamma) \subset S^2_\infty$ is a closed Γ-invariant subset.

Exercise. 1) Let $x \in \mathbb{H}^3$ be a point and let $\Gamma x := \{\gamma(x) \mid \gamma \in \Gamma\}$ be the orbit of x. Then $\Lambda(\Gamma) = \overline{\Gamma x} \setminus \Gamma x$ is the set of accumulation points of the orbit of x. Remarkably, $\Lambda(\Gamma)$ does not depend on x.
2) Show that $\Lambda(\Gamma)$ is the closure of the fixed points in S^2_∞ of elements of Γ.
3) Show that $\Lambda(\Gamma)$ is the smallest Γ-invariant closed subset of S^2_∞. In particular, the action of Γ on $\Lambda(\Gamma)$ is **minimal**, i.e. any orbit is dense in $\Lambda(\Gamma)$.

Definition. A Kleinian group is **elementary**, if it contains an abelian subgroup of finite index.

Exercise. 1) An abelian subgroup of a Kleinian group is either $\mathbb{Z}$ (generated by a hyperbolic or parabolic transformation) or $\mathbb{Z} \oplus \mathbb{Z}$ (generated by two parabolic elements).
2) A Kleinian group Γ is elementary if and only if $\Lambda(\Gamma)$ contains at most two points.
3) A Kleinian group Γ is elementary if and only if $\Lambda(\Gamma)$ contains an isolated point. (Hint: Let $f,\, g \in PSL_2(\mathbb{C})$ be hyperbolic or parabolic; if f and g have only one common fixed point, then $\langle f, g\rangle$ is not discrete.)

Remark. From this exercise follows that a non-trivial elementary Kleinian group is conjugated to one of the following:
1) A parabolic cyclic group $\langle z \mapsto z+1\rangle$.
2) A parabolic rank 2 abelian group $\langle z \mapsto z+1, z \mapsto z+\tau\rangle$, where $Im\,\tau > 0$.
3) A hyperbolic cyclic group $\langle z \mapsto \lambda z\rangle$, where $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$.

Definition. The **domain of discontinuity** of a Kleinian group Γ is the open set $\Omega(\Gamma) = S^2_\infty \setminus \Lambda(\Gamma)$, the maximal Γ-invariant open subset of S^2_∞.

Exercise. Show that a Kleinian group Γ acts properly discontinuously on $\mathbb{H}^3 \cup \Omega(\Gamma)$ (i.e. has no limit point in $\mathbb{H}^3 \cup \Omega(\Gamma)$).

Definition. Let Γ be a Kleinian group, then $M(\Gamma) := \mathbb{H}^3/\Gamma$ is an open complete hyperbolic 3-manifold. Moreover, $\overline{M(\Gamma)} := (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is a 3-manifold with boundary $\partial\overline{M(\Gamma)} = \Omega(\Gamma)/\Gamma$ and interior $M(\Gamma)$. Γ is said of the **first kind** if $\Omega(\Gamma) = \emptyset$ (i.e. $\Lambda(\Gamma) = S^2_\infty$) and of the **second kind** if $\Omega(\Gamma) \neq \emptyset$.

Remark. If Γ acts conformally on $\Omega(\Gamma)$, then the boundary $\partial\overline{M(\Gamma)} = \Omega(\Gamma)/\Gamma$ inherits a natural conformal structure; it is a countable union of Riemann surfaces lying at infinity of the complete hyperbolic 3-manifold $M(\Gamma)$.

Theorem 5.5 (Ahlfors Finiteness Theorem). *Let Γ be a (finitely generated) Kleinian group which is not elementary. If $\Omega(\Gamma) \neq \emptyset$, then $\Omega(\Gamma)/\Gamma$ is a finite union of Riemann surfaces of finite type, i.e. they are conformally equivalent to the complement of a finite number of points in a compact Riemann surface. In particular, if Γ does not contain any parabolic element, then $\Omega(\Gamma)/\Gamma$ is compact.*

We want to give a topological explanation of this theorem. The crucial hypothesis is that $\pi_1(\overline{M(\Gamma)}) = \Gamma$ is finitely generated.

Theorem 5.6 (Scott Compact Core Theorem, 1973**).** *If M is an irreducible 3-manifold with $\pi_1 M$ finitely generated, then there exists an irreducible compact submanifold $K \subset M$ (called* **core***) such that the inclusion $K \hookrightarrow M$ is a homotopy equivalence. In particular, $\pi_1 M$ is finitely presented.*

Theorem 5.7 (Relative Core Theorem, McCullough 1986**).** *Let M be an irreducible 3-manifold with $\pi_1 M$ finitely generated. If $F \subset \partial M$ is a compact (not necessarily connected) subsurface, then there exists a compact core $K \hookrightarrow M$ such that $K \cap \partial M = F$.*

Proposition 5.8. *If Γ is a Kleinian group, then the inequality $-\chi(\partial\overline{M(\Gamma)}) \leq 2(r(\Gamma) - 1)$ holds, where $r(\Gamma)$ is the minimal number of generators for Γ.*

This proposition would prove the Ahlfors Finitenesss Theorem in the case that all ends of $\partial\overline{M(\Gamma)}$ "come" from punctures.

Proof of Proposition 5.8. It is sufficient to show the inequality $-\chi(F) \leq 2(r(\Gamma)-1)$ for any compact subsurface $F \subset \partial\overline{M(\Gamma)}$ such that any component of ∂F is non-trivial in $\partial\overline{M(\Gamma)}$. From the Relative Core Theorem follows that there is a compact core $K \subset \overline{M(\Gamma)}$ such that $K \cap \partial\overline{M(\Gamma)} = F$. Moreover, K is irreducible because $\overline{M(\Gamma)}$ is irreducible. First, we show that $-\chi(F) \leq -\chi(\partial K)$. For any component S of $\partial K \setminus F$, we have $\chi(S) \leq 0$, otherwise S would be S^2 or D^2. It cannot be S^2 because of the irreducibility of K. Assume that $S = D^2$. Since $D^2 \subset \partial K$, cutting along D^2 gives a trivial decomposition of $\pi_1(\overline{M(\Gamma)}) = \pi_1 K$ which implies the contradiction that $\partial D^2 \subset \partial F$ is trivial in $\partial\overline{M(\Gamma)}$. Take the closed 3-manifold DK, the double of K along ∂K with $\chi(DK) = 0 = 2\chi(K) - \chi(\partial K)$. Therefore $-\chi(\partial K) = -2\chi(K)$. We compute $-\chi(K) = -1 + b_1 - b_2 \leq b_1 - 1 \leq r(\Gamma) - 1$, where b_i denotes the rank of $H_i(K;\mathbb{Q})$. This implies $-\chi(\partial K) = -2\chi(K) \leq 2(r(\Gamma) - 1)$. $\square$

Corollary 5.9. *Let Γ be a non-elementary Kleinian group, $\Omega_0 \subset \Omega(\Gamma)$ be a connected component and $\Gamma_0 \subset \Gamma$ be the subgroup $\Gamma_0 := \{\gamma \in \Gamma \mid \gamma(\Omega_0) = \Omega_0\}$. Then Γ_0 is finitely generated and $\Lambda(\Gamma_0) = \partial(\Omega_0) := \overline{\Omega_0} \setminus \Omega_0$ in S^2_∞.*

Proof (Sketch). Let $F_0 = \Omega_0/\Gamma_0$ be a connected component of $\partial\overline{M(\Gamma)}$. F_0 is of finite type by the Ahlfors Finiteness Theorem. Γ_0 is the image of $\pi_1(F_0)$ in Γ, hence Γ_0 is finitely generated. Let $U_0 \supset \Omega_0$ be the connected component of $S^2_\infty \setminus \Lambda(\Gamma_0)$ containing Ω_0. If $U_0 = \Omega_0$ then $\Lambda(\Gamma_0) = \partial(\Omega_0)$. Assume $\Omega_0 \subsetneq U_0$. Define $\Sigma_0 = U_0/\Gamma_0 \supset F_0$. Σ_0 is of finite type because F_0 is of finite type. Take a point $p \in \Sigma_0 \setminus F_0$ and let Δ be a disk neighborhood of P. Then $\Delta \setminus \{p\} \subset F_0$, because F_0 is of finite type. Let $\tilde{\Delta}$ be a connected component of the lift of Δ in S^2_∞. Then $\tilde{\Delta} \subset U_0$ and $\tilde{\Delta} \setminus \{\tilde{p}\} \subset \Omega_0$, where $\tilde{p}$ is a lift of $p \in U_0$. This implies that $\tilde{p}$ is an isolated point for $\Lambda(\Gamma)$, but this is impossible because Γ is non-elementary. $\square$

The following lemma explains the presence of cusps in the Riemann surface $\Omega(\Gamma)$ by the existence of a parabolic element in Γ.

Lemma 5.10 (Ahlfors). *Let Γ be a non-elementary Kleinian group such that $\Omega(\Gamma) \neq \emptyset$. Assume that the Riemann surface $\partial\overline{M(\Gamma)} = \Omega(\Gamma)/\Gamma$ has a cusp. Then there is a parabolic element $\gamma \in \Gamma$ and a simply connected domain $\tilde{U} \subset \Omega(\Gamma)$ such that $\gamma^n(\tilde{U}) = \tilde{U}$, $\forall n \in \mathbb{Z}$ and $\tilde{U}/\langle\gamma\rangle = U$ is a neighborhood of the cusp in $\partial\overline{M(\Gamma)}$.*

Proof. Look to the covering $p : \Omega(\Gamma) \to \partial\overline{M(\Gamma)}$. Let U be a neighborhood of the cusp. U is conformally equivalent to a once punctured disk $\Delta := \{\xi \in \mathbb{C} \mid 0 < |\xi| \leq 1\}$. Let $\tilde{U}$ be a connected component of $p^{-1}(U)$, then $p_| : \tilde{U} \to U$ is a holomorphic covering. Since U is homeomorphic to a 1-punctured disc, $\tilde{U}$ is homeomorphic to a 0- or 1-punctured disc. $\tilde{U}$ must be simply connected (0-punctured), otherwise $\Lambda(\Gamma)$ would have an isolated point, but this would imply that Γ is elementary. Thus $\tilde{U}$ is simply connected and invariant by $\langle\gamma\rangle \cong \mathbb{Z}$, the covering transformation group of $p_| : \tilde{U} \to U$. We have to show that γ is parabolic. The assumption that γ is hyperbolic leads to a contradiction, because if γ is hyperbolic then for all $x \in S^2_\infty$, $\gamma^n(x)$ converges to an attractive fixed point whereas $\gamma^{-n}(x)$ converges to a repulsive fixed point, therefore no simply connected region $\tilde{U}$ can be invariant by $\langle\gamma\rangle$. $\square$

Remark. One can relate the translation length of an element $\gamma \in \pi_1(\Omega(\Gamma)/\Gamma)$ for the conformal structure on the boundary with the translation length of γ for the hyperbolic structure on $M(\Gamma)$.

Lemma 5.11. *Let Γ be a Kleinian group and $F \subset \Omega(\Gamma)/\Gamma$ be a connected component such that each component of $p^{-1}(\Gamma)$ is simply connected (this means that F is incompressible in $\overline{M(\Gamma)}$). Let $l(\gamma)$ be the translational length for $\gamma \in \Gamma$ and $l_F(\gamma)$ be the translational length for $\gamma \in \pi_1(F)$ with the induced conformal structure. Then $l(\gamma) \leq 2l_F(\gamma)$.*

The explanation of this last fact is that F is not totally geodesic, so the geodesic joining two points are shorter in $M(\Gamma)$ than in $\Omega(\Gamma)/\Gamma = \partial\overline{M(\Gamma)}$.

5.3 Convex core

Definition. Let Γ be a non-elementary Kleinian group. Denote by $\widetilde{C}(\Lambda)$ the **convex hull** of $\Lambda(\Gamma)$, i.e. the smallest closed convex subset of $\mathbb{H}^3$ whose closure in $\overline{\mathbb{H}^3}$ contains $\Lambda(\Gamma)$ and by $C(\Gamma) := \widetilde{C}(\Lambda)/\Gamma$ the **convex core**, also called **Nielsen core** of Γ (or of $M(\Gamma)$).

The convex hull of $\Lambda(\Gamma)$ can be constructed as

$$\widetilde{C}(\Lambda) = \bigcap_{\overline{H_i} \supset \Lambda(\Gamma)} H_i,$$

where H_i are totally geodesic half-spaces. The convex core $C(\Gamma)$ is the smallest closed convex subset of $M(\Gamma)$ such that $i : C(\Gamma) \to M(\Gamma)$ induces a homotopy equivalence. Because of the convexity, any closed geodesic is contained in $C(\Gamma)$. In the case $n = 2$, $\partial C(\Gamma)$ is a totally geodesic 1-submanifold but in the case $n = 3$, $C(\Gamma)$ is not a differentiable submanifold of $M(\Gamma)$ and $\partial C(\Gamma)$ is not smooth in general, it is "bent" along some geodesics. To avoid this problem, we consider the closed δ-neighborhood of $C(\Gamma)$ ($\delta > 0$)

$$C_\delta(\Gamma) = \{x \in M(\Gamma) \mid d(x, C(\Gamma)) \leq \delta\}.$$

This is a submanifold and has smooth boundary.

Exercise. $C_\delta(\Gamma) = \widetilde{C}_\delta(\Lambda)/\Gamma$ is a C^1-submanifold of $M(\Gamma)$ with strictly convex boundary, for all $\delta > 0$, where $\widetilde{C}_\delta(\Lambda)$ is the closed δ-neighborhood of $\widetilde{C}(\Lambda)$ in $\mathbb{H}^3$.

Lemma 5.12. *For all $\delta > 0$, $C_\delta(\Gamma)$ is diffeomorphic to $\overline{M(\Gamma)} = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$.*

Remark. 1) If $\Omega(\Gamma) = \emptyset$ (which is equivalent to $\Lambda(\Gamma) = S^2_\infty$) then $M(\Gamma) = \overline{M(\Gamma)} = C(\Gamma)$.
2) By the Ahlfors Finiteness Theorem, $\partial C_\delta(\Gamma)$ is a Riemann surface of finite topological type. If Γ does not contain a parabolic element, then $\partial C_\delta(\Gamma)$ is compact.

Proof of Lemma 5.12. Fix $\delta > 0$ and define the nearest point retraction $r_\delta : M(\Gamma) \setminus C_\delta(\Gamma) \to \partial C_\delta(\Gamma)$ (see Figure 5.4). This map is well defined because of the strict convexity of $\partial C_\delta(\Gamma)$. The map

$$\begin{aligned} M(\Gamma) \setminus C_\delta(\Gamma) &\to \partial C_\delta(\Gamma) \times (0, \infty) \\ x &\mapsto (r_\delta(x), d(x, r_\delta(x))) \end{aligned}$$

defines a C^1-diffeomorphism.

Exercise. Construct a C^1-diffeomorphism $C_\delta(\Gamma) \to \overline{M(\Gamma)}$ by dividing $C_\delta(\Gamma) = int(C_{\delta/2}(\Gamma)) \cup (C_\delta(\Gamma) \setminus int(C_{\delta/2}(\Gamma)))$.

It follows from the Ahlfors Finiteness Theorem that $\Omega(\Gamma)/\Gamma \cong \partial C_\delta(\Gamma)$ is of finite type. Moreover, $C_\delta(\Gamma)$ does not depend (up to diffeomorphism) of $\delta > 0$. We call it the **thickened convex core**. $\square$

Remark. Although $\partial C(\Gamma)$ is not a differentiable surface, the convexity of $C(\Gamma)$ allows to endow $\partial C(\Gamma)$ with the "path metric" induced by the metric of $M(\Gamma)$.

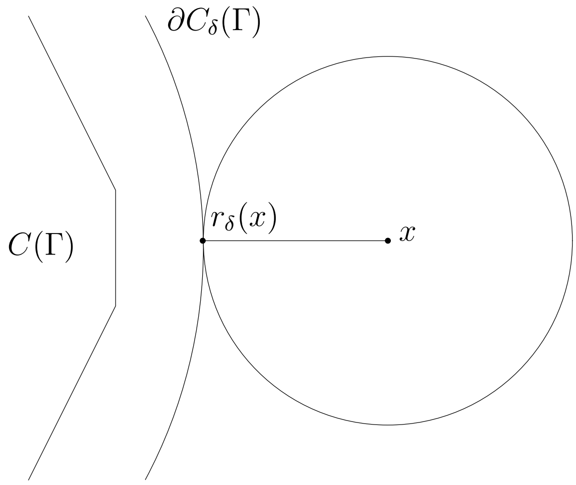

Figure 5.4: r_δ

With this induced metric, $\partial C(\Gamma)$ is locally isometric to $\mathbb{H}^2$ (i.e. the distance is hyperbolic). So to study a non-elementary Kleinian group we have:
1) The complete hyperbolic 3-manifold $M(\Gamma) = \mathbb{H}^3/\Gamma$.
2) The manifold (with boundary) $\overline{M(\Gamma)} = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$.
3) The thickened convex core $C_\delta(\Gamma) \subset M(\Gamma)$, where $C_\delta(\Gamma) \hookrightarrow M(\Gamma)$ is a homotopy equivalence.

Definition. A Kleinian group Γ is **geometrically finite** if $vol(C(\Gamma)) < \infty$.

Exercise. Show that
1) Γ is geometrically finite if and only if there is a $\delta > 0$ such that $vol(C_\delta(\Gamma)) < \infty$.
2) Γ is geometrically finite if and only if $vol(C_\delta(\Gamma)) < \infty$ for *all* $\delta > 0$.
3) Γ is geometrically finite if and only if some (any) Dirichlet fundamental domain of Γ has a finite number of sides.

5.4 Margulis decomposition of $M(\Gamma)$

We assume that Γ is a non-elementary Kleinian group.

Definition. Given $\epsilon > 0$, we define the **ϵ-thick part** of $M(\Gamma)$ to be the set

$$\begin{aligned} M(\Gamma)_{[\epsilon,\infty)} &= \{x \in M(\Gamma) \mid \text{ every geodesic loop through } x \text{ has length } \geq \epsilon\} \\ &= \{x \in M(\Gamma) \mid inj(x) \geq \frac{\epsilon}{2}\}, \end{aligned}$$

where $inj(x)$ is the injectivity radius of x. The **ϵ-thin part** of $M(\Gamma)$ is defined by

$$M(\Gamma)_{(0,\epsilon]} = M(\Gamma) \setminus int(M(\Gamma)_{[\epsilon,\infty)}) = \{x \in M(\Gamma) \mid inj(x) \leq \frac{\epsilon}{2}\}.$$

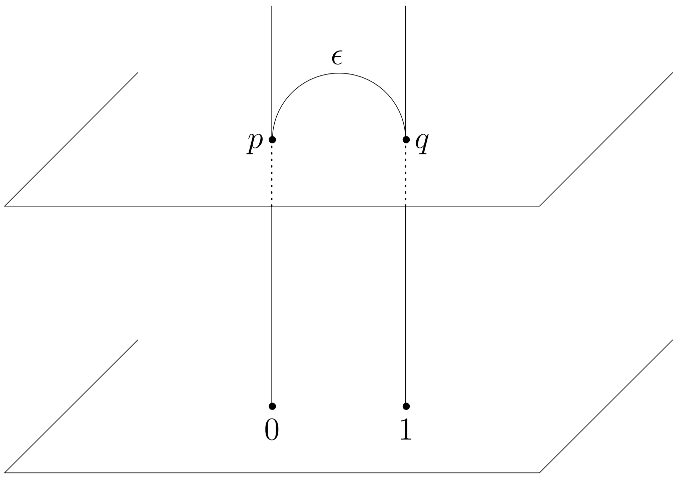

Figure 5.5: $c(\epsilon)$

Margulis Lemma. *Let Γ be a Kleinian group. There exists a constant $\epsilon_0 > 0$ such that for all $x \in \mathbb{H}^3$ the subgroup $\Gamma_{\epsilon_0,x} \subset \Gamma$ generated by the elements $\gamma \in \Gamma$ with the property $d(x, \gamma(x)) < \epsilon_0$ is elementary (i.e. $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$).*

In dimension 3, the Margulis Lemma is a direct consequence of the following proposition:

Proposition 5.13 (Jørgensen Inequality). *If $\langle \gamma_1, \gamma_2 \rangle$ generates a non-elementary discrete subgroup of $PSL_2(\mathbb{C})$ then*

$$|(tr\, \gamma_1)^2 - 4| + |tr[\gamma_1, \gamma_2] - 2| \geq 1,$$

and this inequality is sharp.

A proof of this statement can be found in **??**. An interpretation of this inequality can be given as follows: if γ_1 is near the identity, then γ_2 is far away from the identity.

Corollary 5.14. *A non-elementary group $\Gamma \subset PSL_2(\mathbb{C})$ is discrete if and only if any two-generator subgroup is discrete.*

For $\epsilon > 0$, define $c = c(\epsilon) > 0$ by the condition $d(p, q) = \epsilon$ in $\mathbb{H}^3$, where $p = (0, 0, c(\epsilon))$, $q = (1, 0, c(\epsilon))$ in the upper half-space model. Further, we define the horoball $H_c := \{(x, y, z) \in \mathbb{H}^3 \mid z > c\}$ (see Figure 5.5).

Proposition 5.15. *For any complete hyperbolic 3-manifold $M(\Gamma) = \mathbb{H}^3/\Gamma$ and for $\epsilon < \epsilon_0$ (where ϵ_0 is as in the Margulis Lemma), each connected component of $M(\Gamma)_{(0,\epsilon]}$ is isometrically equivalent to one of the following:*
1) $H_c/\langle z \mapsto z + 1 \rangle$, a ***solid cusp cylinder***

2) $H_c/\langle z \mapsto z+1, z \mapsto z+\tau\rangle$ ($Im\,\tau > 0$, $|\tau| \geq 1$), a ***solid cusp torus***
3) $V/\langle\gamma\rangle$, where γ is a hyperbolic transformation in Γ and $V \subset \mathbb{H}^3$ is a tubular neighborhood of the axis of γ. The quotient $V/\langle\gamma\rangle$ is called ***Margulis solid torus*** *or* ***Margulis tube****.*

Proof. For $\gamma \in \Gamma \setminus \{id\}$, define the (possibly empty) set

$$V_\gamma = \{x \in \mathbb{H}^3 \mid d(x, \gamma(x)) < \epsilon\}.$$

If γ is the map $z \mapsto z+1$ then $V_\gamma = H_c$ by definition of c. If γ is hyperbolic, either $V_\gamma = \emptyset$ or it is a tubular neighborhood of the axis A_γ of γ. Take the projection $p : \mathbb{H}^3 \to M(\Gamma) = \mathbb{H}^3/\Gamma$.

Exercise. Show that

$$p^{-1}(M(\Gamma)_{(0,\epsilon)}) = \bigcup_{\gamma \in \Gamma \setminus \{id\}} V_\gamma.$$

To prove this exercise, take a connected component $E \subset p^{-1}(M(\Gamma)_{(0,\epsilon)})$ and define $G_E \subset \Gamma$ to be the stabilizer of E. Then
i) since E is invariant by G_E, $\gamma \in \Gamma \setminus \{id\}$ belongs to G_E if and only if $V_\gamma \subset E$, thus

$$E = \bigcup_{\gamma \in G_E \setminus \{id\}} V_\gamma.$$

ii) G_E is elementary by the Margulis Lemma ($\epsilon < \epsilon_0$), because if $V_\gamma \cap V_{\gamma'} \neq \emptyset$, then $\langle\gamma, \gamma'\rangle \subset \Gamma_{\epsilon,x}$ (where $x \in V_\gamma \cap V_{\gamma'}$) is elementary. The connectedness of E implies that all elements in G_E have the same fixed point set, therefore G_E is elementary.
iii) G_E is either $\mathbb{Z}$, generated by a parabolic element, or $\mathbb{Z} \oplus \mathbb{Z}$, generated by two parabolic elements, or $\mathbb{Z}$, generated by a hyperbolic element. In the first two cases, we may take $E = H_c$. In the hyperbolic case, E is a tubular neighborhood of the axis A_γ. In any case, $p(E) = E/G_E$ is a connected component of the thin part. □

As ϵ goes to 0, the Margulis solid tori disappear, but not the cusps. The union of solid cusp neighborhoods is called the **cuspidal part** of $M(\Gamma)$. If Γ is geometrically finite, then $vol(C(\Gamma)) < \infty$ and all the closed geodesics are in $C(\Gamma)$. For this reason, there is no arbitrarily small closed geodesic. Moreover, if Γ is geometrically finite, there exists a number $\epsilon(\Gamma) > 0$ such that for all $\epsilon < \epsilon(\Gamma)$, $M(\Gamma)_{(0,\epsilon)}$ is the cuspidal part.

We give now a description of the solid cusp torus: Let (T^2, ds^2) be a flat torus, then $C_T = T^2 \times \mathbb{R}$ with the metric $e^{-r}ds^2 + dr^2$ is a complete open hyperbolic 3-manifold with infinite volume. A model of a solid cusp torus is $C_T^+ = T^2 \times [0, \infty)$, a finite volume hyperbolic 3-manifold. Its metric depends (up to isometry) only on the flat metric on T^2. An infinite cyclic covering $\widetilde{C}_T^+ = S^1 \times \mathbb{R} \times [0, \infty)$ of C_T^+ is a model of the solid cusp cylinder. It is of infinite volume.

Proposition 5.16. *Let Γ be a non-elementary Kleinian group. If Γ is geometrically finite, then*
1) $C_\delta(\Gamma)_{[\epsilon,\infty)} = C_\delta(\Gamma) \cap M(\Gamma)_{[\epsilon,\infty)}$ is compact for $\epsilon < \epsilon(M(\Gamma))$
2) $C_\delta(\Gamma)_{(0,\epsilon)} = C_\delta(\Gamma) \cap M(\Gamma)_{(0,\epsilon)}$ is a disjoint union of finitely many solid cusp tori and cylinders of finite volume.

Proof. In any case, up to taking a concentric neighborhood of a cusp, any cusp of $M(\Gamma)$ lies in $C_\delta(\Gamma)$ but may be of infinite volume. □

Definition. The pair

$$(N, P) := (C_\delta(\Gamma)_{[\epsilon,\infty)}, \partial C_\delta(\Gamma)_{[\epsilon,\infty)} \cap M(\Gamma)_{(0,\epsilon]})$$

is called the **topological type** of the geometrically finite group Γ. N is a compact 3-manifold and $P \subset \partial N$ is the track of the cusp neighborhoods (a union of tori and annuli).

Proposition 5.17. *Let Γ be a non-elementary Kleinian group.*
1) If V is a rank 2 cusp of $M(\Gamma)$, we can replace it by a concentric cusp V' such that $V' \subset C(\Gamma)$.
2) If V is a rank 1 cusp whose preimage in $\mathbb{H}^3$ is $H_c = \{(x, y, z) \in \mathbb{H}^3 \mid z > c\}$, then there exist constants A, B, $c' \geq c$ (A and B may be infinite) such that

$$\widetilde{C}(\Lambda) \cap H_{c'} = \{(x, y, z) \in \mathbb{H}^3 \mid A \leq y \leq B \text{ and } z > c'\}$$

A proof of this proposition can be found in [28], see also [19].

Definition. A cusp cylinder (rank 1 cusp) is of **finite type** if its intersection with $C(\Gamma)$ is of finite volume (i.e. if and only if A and B are finite).

Remark. 1) If the cusp cylinder is of finite type, its intersection with $C(\Gamma)$ is bounded by two totally geodesic parallel cylinders.
2) In the model of the horoball H_c, the parabolic element $\gamma : (x, y, z) \mapsto (x + 1, y, z)$ leaves invariant the limit set $\Lambda(\Gamma)$ which is contained in the strip $A \leq y \leq B$ of minimal size. If Γ is geometrically finite then all the cusps are of finite type and $C(\Gamma)_{[\epsilon,\infty)}$ is compact for ϵ sufficiently small.

Proposition 5.18. *Let Γ be a non-elementary geometrically finite Kleinian group. The topological type (N, P) of the* ***Kleinian manifold*** $\overline{M(\Gamma)} = (\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ *has the following topological properties:*
1) N is compact and irreducible.
2) $P \subset \partial N$ is a disjoint union of incompressible (π_1-injective) tori and annuli.

3) Any subgroup $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1 N$ is conjugated to some $\pi_1(P_i)$, where $P_i \subset P$ is a connected component (such a $\mathbb{Z} \oplus \mathbb{Z}$ must be parabolic).
4) There is no essential annulus $(A, \partial A) \hookrightarrow (N, P)$ (otherwise there would be an essential torus in N).
5) N is not homeomorphic to $T^2 \times [0,1]$ or $K^2 \tilde{\times} [0,1]$. (The only admissible pair is $(T^2 \times [0,1], T^2 \times \{1\})$.)

Definition. A pair (N, P) which satisfies properties 1) to 5) is called a **pared manifold**. The boundary of a pared manifold is $\partial_0 N = \overline{\partial N \setminus P}$. This boundary is said to be **strictly incompressible**, if there is no essential annulus $(A, \partial A) \hookrightarrow (N, \partial N)$ with one boundary component on $\partial_0 N$ and the other on P (a curve of $\partial_0 N$ homotopic to a curve of P in N must be homotopic to this curve in ∂N). We cannot avoid essential annuli for $(N, \partial_0 N)$. (If P is empty then $\partial_0 N = \partial N$.)

Here is a more precise version of Thurston's hyperbolization theorem, needed for the proof in the Haken case.

Theorem 5.19 (Thurston's hyperbolization theorem for pared manifolds). *Let N be a compact orientable Haken 3-manifold whose fundamental group $\pi_1 N$ is not virtually abelian. Then any pared 3-manifold (N, P) is the topolological type of a geometrically finite Kleinian 3-manifold.*

Remark. 1) The Haken hypothesis is essential for the proof, because the proof is by induction on the complexity of a Haken manifold, defined in Chapter 4. In particular pared manifolds appears naturally when one cuts along an essential surface to reduce the complexity.
2) Thurston's **mirror trick** (see [17], [35]) allows to reduce the gluing inductive step to the final gluing step where all the boundary $\partial_0 N$ is involved.(starting from a pared manifold (N, P), we obtain a n orbifold without boundary).

5.5 Thurston's Gluing Theorem

The main step of Thurston's hyperbolization theorem is the following theorem:

Theorem 5.20 (Thurston's gluing theorem). *Let (N, P) be a pared 3-manifold with strictly incompressible non-empty boundary $\partial_0 N$. Let $\tau : \partial_0 N \to \partial_0 N$ be an orientation reversing smooth involution which exchanges the boundary components by pairs, i.e.*

$$\tau = (f, f^{-1}) : \partial_0^+ N \amalg \partial_0^- N \to \partial_0^- N \amalg \partial_0^+ N.$$

Assume that (N,P) *is the topological type of a geometrically finite Kleinian* 3-*manifold. Then* $N/\tau := N/(x \sim \tau(x), x \in \partial_0 N)$ *admits a complete finite volume hyperbolic structure if and only if* N/τ *is atoroidal.*

Some historical comments on the proof of Thurston's hyperbolization theorem. Thurston has announced this theorem in 1977. In his notes [46], he has settled the technics and results which are fundamental for his proof. In 1980, he has presented a very detailled plan of his proof [47], in the notes about his series of talks at Bowdoin(cf. also [48], [28]). Then he has written the first part of his proof (*the bounded image theorem* in [49, 51], as well as the proof for the surface bundle case [50] (see also [34]). The remaining part of the proof had to be recovered from [47] and the chapters 8, 9 and 13 of his notes [46].

In 1985, F. Bonahon's work [4] on the *tameness* of the ends of hyperbolic 3-manifolds, allowed to simplify the part of the proof using the chapters 8 and 9 of [46].

At the same time J. Morgan and P. Shalen [29, 30, 31] gave a different and more algebraic proof of the main results of [49, 51].

Following a totally different approach from Thurston's, C. McMullen [22, 23, 24] has written in 1989 a proof of *the fixed point theorem* which isthe heart of the proof of *the gluig theorem.*

Using McMullen's work, J.P. Otal [**?**] wrote in 1996 a complete proof of Thurston's hyperbolization theorem for a Haken manifold which is not a surface bundle. Meanwhile, he has written a new proof for the surface bundle case in [**?**].

At the same time, M. Kapovich [**?**] has written a complete proof of Thurston's hyperbolization theorem for a Haken manifolds which is not a surface bundle, but by recovering the original Thurston's approach.

The following sections are still missing. Their goal is to describe more precisely the main steps of the proof of Thurston's gluing theorem. None of these steps is easy.

5.6 Thurston's Fixed Point Theorem

5.7 Bounded Image Theorem

5.8 Ends of hyperbolic 3-manifolds

5.9 Covering Theorem

5.10 Ends of hyperbolic 3-manifolds

5.11 Geometric Limit Theorem

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