Invariants of links and 3-manifolds that count graph configurations.

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Abstract

We present ways of counting configurations of uni-trivalent Feynman graphs in 3-manifolds in order to produce invariants of these 3-manifolds and of their links, following Gauss, Witten, Bar-Natan, Kontsevich and others. We first review the construction of the simplest invariants that can be obtained in our setting. These invariants are the linking number and the Casson invariant of integer homology 3-spheres. Next we see how the involved ingredients, which may be explicitly described using gradient flows of Morse functions, allow us to define a functor on the category of framed tangles in rational homology cylinders. Finally, we describe some properties of our functor, which generalizes both a universal Vassiliev invariant for links in the ambient space and a universal finite type invariant of rational homology 3-spheres.

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0.1 Introduction

These notes are the notes of a series of lectures given in Pisa in February 2020 for Winter Braids. They contain all what has been said during the lectures, and more.

They present ways of counting configurations of uni-trivalent Feynman graphs in 3-manifolds in order to produce invariants of these 3-manifolds and of their links, following Gauss, Witten, Bar-Natan, Kontsevich and others. We first review the construction of the simplest invariants that can be obtained in our setting, in Section 1. These invariants are the linking number and the Casson invariant of integer homology 3-spheres. Next we see how the involved ingredients, which may be explicitly described using gradient flows of Morse functions, allow us to define an invariant Z of framed tangles in rational homology cylinders in Section 2. Finally, in Section 3, we describe some properties of our functorial invariant Z, which generalizes both a universal Vassiliev invariant for links in the ambient space and a universal finite type invariant of rational homology 3-spheres.

For more details about the presented material, we refer the reader to the book [Les20], where the above functor Z has been constructed, and where all its mentioned properties are carefully proved. These notes may also be used as an introduction or as a reading guide to [Les20].

I warmly thank the organisers of the great session 2020 of Winter Braids, and the referee for her/his careful reading and her/his helpful comments.

1 On the linking number and the Theta invariant

The modern powerful invariants of links and 3–manifolds that are studied in these series of lectures can be thought of as generalizations of the linking number. In this section, we warm up by defining this classical basic invariant in several ways. This allows us to introduce conventions and methods that will be useful throughout these notes.

1.1 The linking number as a degree

Let S^1 denote the unit circle of the complex plane $\mathbb{C}$.

$$S^1 = \{z; z \in \mathbb{C}, |z| = 1\}.$$

Consider a C^∞ embedding

$$J \sqcup K \colon S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$$

of the disjoint union $S^1 \sqcup S^1$ of two circles into the ambient space $\mathbb{R}^3$ as the one pictured in Figure 1. Such an embedding represents a 2–*component link*. Each of the embeddings $J \colon S^1 \hookrightarrow \mathbb{R}^3$ and $K \colon S^1 \hookrightarrow \mathbb{R}^3$ represents a *knot*.

The link embedding $J \sqcup K$ induces the *Gauss map*

$$\begin{array}{llll} p_{JK}\colon & S^1 \times S^1 & \to & S^2 \\ & (w,z) & \mapsto & \frac{1}{\|K(z)-J(w)\|}(K(z)-J(w)) \end{array}$$

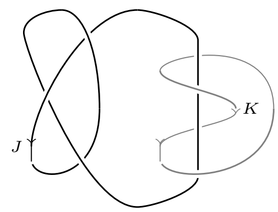
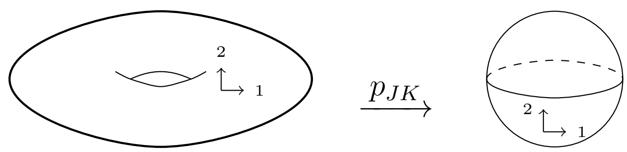


Figure 1: A 2–component link in $\mathbb{R}^3$



Definition 1.1 The *Gauss linking number* $lk_G(J, K)$ of the disjoint *knots* $J(S^1)$ and $K(S^1)$, which are simply denoted by J and K, is the degree of the Gauss map p_{JK}.

There are several (fortunately equivalent) definitions of the degree for a continuous map between two *closed* (i.e. connected, compact, without boundary) oriented manifolds of the same dimension. Let us quickly recall our favorite one for these lectures, where we work with smooth manifolds.

Definition 1.2 A point y is a *regular value* of a smooth map $p: M \to N$ between two smooth manifolds M and N, if $y \in N$ and, for any $x \in p^{-1}(y)$, the tangent map T_xp at x is surjective[1], and, when the boundary ∂M of M is non-empty, and possibly stratified[2], the restriction of the tangent map T_xp to the tangent space of ∂M or to the stratum of x is also surjective for any $x \in \partial M \cap p^{-1}(y)$.

An *orientation* of a real vector space V of positive dimension is a basis of V up to a change of basis with positive determinant. When $V = \{0\}$, an orientation of V is an element of $\{-1, 1\}$. An *orientation* of a smooth n–manifold is an orientation of its tangent space at each point, defined in a continuous way. A local diffeomorphism h of $\mathbb{R}^n$ is orientation-preserving at x if and only if the Jacobian determinant of its derivative T_xh is positive. If the transition maps $\phi_j \circ \phi_i^{-1}$ of an *atlas* $(\phi_i)_{i \in I}$ of a manifold M are orientation-preserving (at every point) for $\{i, j\} \subset I$, then the manifold M is *oriented* by this atlas. Unless otherwise mentioned, all manifolds are oriented in these notes.

[1] According to the Morse-Sard theorem [Hir94, Chapter 3, Theorem 1.3, p. 69], the set of regular values of such a map is dense. It is even *residual*, i.e. it contains the intersection of a countable family of dense open sets. Furthermore it is open if M is compact.

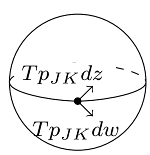
[2] In these notes, manifolds are smoothly modelled on open subspaces of $[0, 1]^n$, and covered by countably many such spaces. In particular their boundaries have strata, which correspond to the open faces of $[0, 1]^n$. They have *corners*, which correspond to the points of $\{0, 1\}^n$ and *ridges*, which correspond to the open faces of $[0, 1]^n$ of dimension in $\{1, \dots, n-2\}$. For example, in dimension 3, the ridges correspond to the edges of the cube.

When M and N are oriented, M is compact and the dimension of M coincides with the dimension of N, the *differential degree* $\deg_y(p)$ of p at a regular value y of N is the (finite) sum running over the $x \in p^{-1}(y)$ of the signs of the determinants of $T_x p$. In this case, this differential degree can be extended to a continuous function $\deg(p)$ from the complement $N \setminus p(\partial M)$ of the image of the boundary ∂M of M to $\mathbb{Z}$. See [Les20, Lemma 2.3]. In particular, when the boundary of M is empty and N is connected, the function $\deg(p)$ is constant, and its value is the *degree* of p. See [Mil97, Chapter 5].

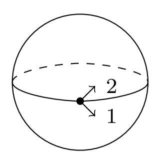
The Gauss linking number $lk_G(J,K)$ can be computed from a link diagram as in Figure 1 as follows. It is the differential degree of p_{JK} at the vector Y that points towards us. The set $p_{JK}^{-1}(Y)$ consists of the pairs of points (w,z) where the projections of $J(w)$ and $K(z)$ coincide, and $J(w)$ is under $K(z)$. They correspond to the *crossings* ${}^{J}\nwarrow\!\!\!\!\!\!\nearrow{}^{K}$ and ${}^{K}\nearrow\!\!\!\!\!\!\nwarrow{}^{J}$ of the diagram.

In a diagram, a crossing is *positive* if we turn counterclockwise from the arrow at the end of the upper strand towards the arrow of the end of the lower strand like $\nwarrow\!\!\!\!\!\!\nearrow$. Otherwise, it is *negative* like $\nearrow\!\!\!\!\!\!\nwarrow$.

For the positive crossing ${}^{J}\nwarrow\!\!\!\!\!\!\nearrow{}^{K}$, moving $J(w)$ along J following the orientation of J, moves $p_{JK}(w,z)$ towards the south-east direction $Tp_{JK}dw$, while moving $K(z)$ along K following the orientation of K, moves $p_{JK}(w,z)$ towards the north-east direction $Tp_{JK}dz$, so that the local orientation induced by the image of p_{JK} around $Y \in S^2$ is



, which is



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Therefore, the contribution of a positive crossing to the degree is 1. It is easy to deduce that the contribution of a negative crossing is (-1).

We have proved the following formula

$$\deg_Y(p_{JK}) = \sharp\, {}^{J}\nwarrow\!\!\!\!\!\!\nearrow{}^{K} - \sharp\, {}^{K}\nearrow\!\!\!\!\!\!\nwarrow{}^{J}$$

where $\sharp$ stands for the cardinality –here $\sharp\, {}^{J}\nwarrow\!\!\!\!\!\!\nearrow{}^{K}$ is the number of occurences of ${}^{J}\nwarrow\!\!\!\!\!\!\nearrow{}^{K}$ in the diagram– so that

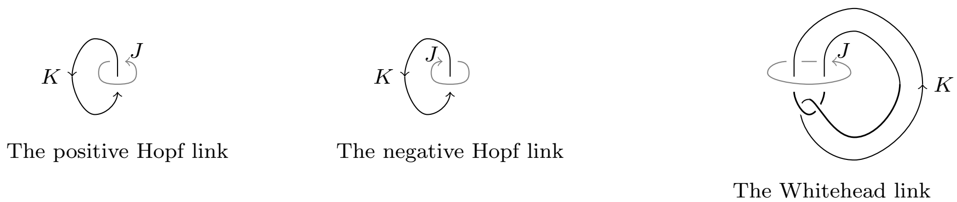
$$lk_G(J,K) = \sharp\, {}^{J}\nwarrow\!\!\!\!\!\!\nearrow{}^{K} - \sharp\, {}^{K}\nearrow\!\!\!\!\!\!\nwarrow{}^{J}.$$

Similarly, $\deg_{-Y}(p_{JK}) = \sharp\, {}^{K}\nwarrow\!\!\!\!\!\!\nearrow{}^{J} - \sharp\, {}^{J}\nearrow\!\!\!\!\!\!\nwarrow{}^{K}$ so that

$$lk_G(J,K) = \sharp\, {}^{K}\nwarrow\!\!\!\!\!\!\nearrow{}^{J} - \sharp\, {}^{J}\nearrow\!\!\!\!\!\!\nwarrow{}^{K} = \frac{1}{2}\left(\sharp\, {}^{J}\nwarrow\!\!\!\!\!\!\nearrow{}^{K} + \sharp\, {}^{K}\nwarrow\!\!\!\!\!\!\nearrow{}^{J} - \sharp\, {}^{K}\nearrow\!\!\!\!\!\!\nwarrow{}^{J} - \sharp\, {}^{J}\nearrow\!\!\!\!\!\!\nwarrow{}^{K}\right),$$

and thus $lk_G(J,K) = lk_G(K,J)$.

In the example of Figure 1, $lk_G(J,K) = 2$. Let us give some further examples. For the *positive Hopf link* of Figure 2, $lk_G(J,K) = 1$. For the *negative Hopf link*, $lk_G(J,K) = -1$, and, for the *Whitehead link*, $lk_G(J,K) = 0$.



The positive Hopf link The negative Hopf link The Whitehead link

Figure 2: The Hopf links and the Whitehead link

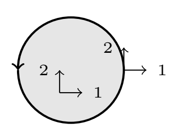
Since the differential degree of the Gauss map p_{JK} is constant on the set of regular values of p_{JK}, $lk_G(J,K) = \int_{S^1\times S^1} p_{JK}^*(\omega_S)$ for any 2-form ω_S on S^2 such that $\int_{S^2}\omega_S = 1$.

Denote the standard area form of S^2 by $4\pi\omega_{S^2}$ so that ω_{S^2} is the homogeneous volume form of S^2 such that $\int_{S^2}\omega_{S^2} = 1$. In 1833, Gauss defined the linking number of J and K, as an integral [Gau77]. In modern notation, his definition may be expressed as

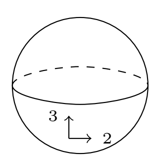
$$lk_G(J,K) = \int_{S^1\times S^1} p_{JK}^*(\omega_{S^2}).$$

1.2 The linking number as an algebraic intersection

The boundary ∂M of an oriented manifold M is oriented by the *outward normal first* convention. If $x \in \partial M$ is in a smooth part of ∂M, the outward normal to M at x followed by an oriented basis of $T_x\partial M$ induce the given orientation of M. For example, the standard orientation of the disk in the plane induces the standard orientation of the circle, counterclockwise, as the following picture shows.



As another example, the sphere S^2 is oriented as the boundary of the ball B^3, which has the standard orientation induced by the right hand rule: (Thumb, index finger (2), middle finger (3)) of the right hand.



The tangent bundle to an oriented submanifold A in a manifold M at a point x is denoted by T_xA. Two submanifolds A and B in a manifold M are *transverse*[3] if at each intersection point x, $T_xM = T_xA + T_xB$. If two transverse submanifolds A and B in a manifold M are of

[3] As shown in [Hir94, Chapter 3 (Theorem 2.4 in particular)], transversality is a generic condition.

complementary dimensions (i.e. if the sum of their dimensions is the dimension of M), then the *sign of an intersection point* is $+1$ if $T_xM = T_xA \oplus T_xB$ as oriented vector spaces. Otherwise, the sign is -1. If A and B are compact and if A and B are of complementary dimensions in M, their *algebraic intersection* is the sum of the signs of the intersection points, and is denoted by $\langle A, B\rangle_M$.

For us, a *rational chain* (resp. *integral chain*) is a linear combination of (oriented) smooth manifolds with boundary, with coefficients in $\mathbb{Q}$ (resp.in $\mathbb{Z}$). Algebraic intersection bilinearly extends to pairs of transverse chains.

When $\mathbb{K}$ is $\mathbb{Z}$ or $\mathbb{Q}$, a $\mathbb{K}$–S^3 or $\mathbb{K}$–*sphere* is a compact oriented 3-dimensional manifold[4] R with the same homology with coefficients in $\mathbb{K}$ as the standard unit sphere S^3 of $\mathbb{R}^4$. $\mathbb{Q}$-spheres (resp. $\mathbb{Z}$-spheres) are also called rational (resp. integer) homology 3-spheres. In these notes, we omit the 3 since the dimension of our homology spheres is always 3.

According to an easy case of a Thom theorem [Les20, Theorem 11.9], any knot K in a $\mathbb{Q}$–sphere R bounds[5] an oriented *rational chain* in R. If R is a $\mathbb{Z}$-sphere, K bounds an embedded surface[6], called a *Seifert surface of the knot.*

The simplest definition of the linking number of two disjoint knot embeddings in such a manifold is the following one.

Definition 1.3 The *linking number* $lk(J, K)$ of two disjoint knot embeddings J and K in a $\mathbb{Q}$-sphere R is the algebraic intersection $\langle J, \Sigma_K\rangle_R$ of J and a rational chain Σ_K bounded by K.

We will see that $lk_G(J, K) = lk(J, K)$ for 2-component links in $\mathbb{R}^3 \subset S^3$ in Lemma 1.15. See also [Les20, Proposition 2.9].

In order to generalize the Gauss definition of the linking number to 2–component links in a rational homology sphere R, let us rephrase it.

As in Subsection 1.1, consider a two-component link $J \sqcup K : S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$. This embedding induces an embedding

$$\begin{array}{rccc} J\times K\colon & S^1\times S^1 & \hookrightarrow & (\mathbb{R}^3)^2\setminus \mathrm{diag} \\ & (z_1,z_2) & \mapsto & (J(z_1),K(z_2)) \end{array}$$

into the 2–*point configuration space*

$$\check{C}_2(S^3) = (\mathbb{R}^3)^2 \setminus \mathrm{diag}.$$

Consider the map

$$\begin{array}{rccc} p_{S^2}\colon & \check{C}_2(S^3) & \to & S^2 \\ & (x,y) & \mapsto & \frac{1}{\|y-x\|}(y-x). \end{array}$$

The Gauss map p_{JK} of Section 1.1 is equal to $p_{S^2} \circ (J \times K)$.

[4] Here, all manifolds are supposed to be smooth. Since any topological 3-manifold has a unique smooth structure (see [Kui99]), we do not specify "smooth" and we often only describe 3-manifolds up to homeomorphism.

[5] The Poincaré duality ensures that this property characterizes $\mathbb{Q}$–spheres among closed oriented 3-manifolds.

[6] This property similarly characterizes $\mathbb{Z}$–spheres among closed oriented 3-manifolds.

In particular, we can rewrite $lk_G(J,K)$ as another algebraic intersection, which will generalize to 2–component links in a rational homology sphere R. For a regular value $a \in S^2$ of p_{JK},

$$lk_G(J,K) = \deg_a p_{JK} = \langle (J \times K)(S^1 \times S^1), p_{S^2}^{-1}(a) \rangle_{\check{C}_2(S^3)}$$

where the preimages are oriented as follows. The *normal bundle* T_xM/T_xA to A in M at x is denoted by N_xA. It is oriented so that (a lift of an oriented basis of) N_xA followed by (an oriented basis of) T_xA induce the orientation of T_xM. The orientation of $N_x(A)$ is a *coorientation* of A at x. The regular preimage of a submanifold under a map f is oriented so that f preserves the coorientations.

For any 2-form ω_S on S^2 such that $\int_{S^2} \omega_S = 1$, we can also use the closed 2-form $p_{S^2}^*(\omega_S)$ of $(\mathbb{R}^3)^2 \setminus \mathrm{diag}$ to write

$$lk_G(J,K) = \int_{S^1 \times S^1} p_{JK}^*(\omega_S) = \int_{(J \times K)(S^1 \times S^1)} p_{S^2}^*(\omega_S).$$

The closure of $p_{S^2}^{-1}(a)$ in a compactification $C_2(S^3)$ (defined in Section 1.3 below) of $\check{C}_2(S^3)$ is our first example of *propagating chain* or *propagator*. The closed 2-form $p_{S^2}^*(\omega_S)$ extends to $C_2(S^3)$ as an example of *propagating form* or *propagator*. Propagators are central ingredients in the construction of more general invariants of tangles in $\mathbb{Q}$–spheres that is presented in these notes.

1.3 Propagators

Let us first introduce the compact 2–point configuration spaces where propagators live. Their constructions use the following differential blow-ups.

Definition 1.4 Recall that the *unit normal bundle* of a submanifold C in a smooth manifold A is the fiber bundle over C whose fiber over $x \in C$ is $SN_x(C) = (N_x(C) \setminus \{0\})/\mathbb{R}^{+*}$, where $\mathbb{R}^{+*}$ acts by scalar multiplication. A smooth *submanifold transverse to the ridges* of a smooth manifold A is a subset C of A such that for any point $x \in C$ there exists a smooth open embedding ϕ from $\mathbb{R}^c \times \mathbb{R}^e \times [0,1[^d$ into A such that $\phi(0) = x$ and the image of ϕ intersects C exactly along $\phi(0 \times \mathbb{R}^e \times [0,1[^d)$. Here c is the *codimension* of C, d and e are integers, which depend on x, and $[0,1[$ denotes the interval $[0,1] \setminus \{1\}$.

For us, *blowing up* such a compact submanifold C in A replaces C with its unit normal bundle in order to produce the smooth manifold $B\ell(A,C)$ (with possible ridges) so that a chart $\phi\colon \mathbb{R}^c \times \mathbb{R}^e \times [0,1[^d \hookrightarrow A$ as above induces a chart $\overline{\phi}\colon ([0,\infty[\times S^{c-1}) \times \mathbb{R}^e \times [0,1[^d \hookrightarrow B\ell(A,C)$. (The origin 0 of $\mathbb{R}^c$ is replaced with the sphere $\{0\} \times S^{c-1}$ of directions around it.)

Unlike blow-ups in algebraic geometry, this differential geometric blow-up creates boundaries. More precisely, we have the following proposition.

Proposition 1.5 *Under the assumptions of the above definition, we have the following properties.*

- *$B\ell(A,C)$ is diffeomorphic to the complement of an open tubular neighborhood of C (thought of as infinitely small).*

- *There is a canonical projection $p_b\colon B\ell(A,C) \to A$, which restricts to a diffeomorphism from the preimage of $A \setminus C$ to $A \setminus C$.*

- *If A is compact, $B\ell(A,C)$ is a compactification of $A \setminus C$.*

- *If the boundary ∂A of A is empty, then the boundary of $B\ell(A,C)$ is the unit normal bundle of C in A, and the interior $B\ell(A,C) \setminus \partial B\ell(A,C)$ of $B\ell(A,C)$ is $A \setminus C$.*

Examples 1.6 Local models are given by the following elementary blow-ups $B\ell(\mathbb{R}^c,0) \cong [0,\infty[\times S^{c-1}$, and $B\ell(\mathbb{R}^c \times A, 0 \times A) \cong [0,\infty[\times S^{c-1} \times A$.

In Figure 3, we see the result of first blowing up $(0,0)$ in $\mathbb{R}^2$, and next blowing up the closures in $B\ell(\mathbb{R}^2,(0,0))$ of $\{0\} \times \mathbb{R}^*$, $\mathbb{R}^* \times \{0\}$ and the diagonal of $(\mathbb{R}^*)^2$.

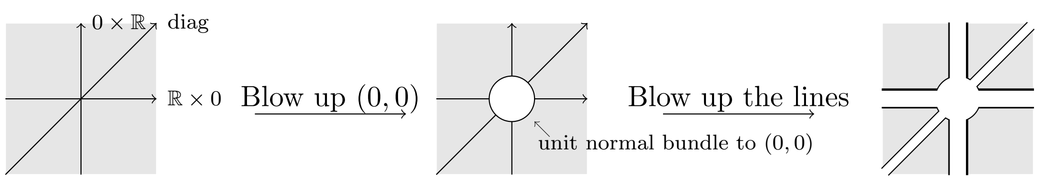


Figure 3: Iterated blow-ups of $\mathbb{R}^2$

We regard S^3 as $\mathbb{R}^3 \cup \{\infty\}$ or as two copies of $\mathbb{R}^3$ identified along $\mathbb{R}^3 \setminus \{0\}$ by the (exceptionally orientation-reversing) diffeomorphism $x \mapsto x/ \parallel x \parallel^2$. The blow-up $B\ell(S^3,\infty)$ is diffeomorphic to the compact unit ball of $\mathbb{R}^3$. As a set, $B\ell(S^3,\infty) = \mathbb{R}^3 \cup S^2_\infty$ where $(-S^2_\infty)$ denotes[7] the unit normal bundle to ∞ in S^3 and $\partial B\ell(S^3,\infty) = S^2_\infty$. There is a canonical orientation-preserving diffeomorphism $p_\infty\colon S^2_\infty \to S^2$, such that $x \in S^2_\infty$ is the limit in $B\ell(S^3,\infty)$ of a sequence of points of $\mathbb{R}^3$ approaching ∞ along a line directed by $p_\infty(x) \in S^2$.

Let R be a $\mathbb{Q}$–S^3 equipped with a point $\infty \in R$. Identify a neighborhood of ∞ in R with a neighborhood of ∞ in S^3. Let $\check{R} = R \setminus \{\infty\}$. Define the *configuration space* $C_2(R)$ to be the compact 6–manifold with boundary and ridges obtained from R^2 by first blowing up (∞,∞) in R^2, and, by next blowing up the closures of $\{\infty\} \times \check{R}$, $\check{R} \times \{\infty\}$ and the diagonal of $\check{R}^2$ in $B\ell(R^2,(\infty,\infty))$.

In particular, $\partial C_2(R)$ contains the unit normal bundle $(\frac{T\check{R}^2}{\mathrm{diag}(T\check{R}^2)} \setminus \{0\})/\mathbb{R}^{+*}$ to the diagonal of $\check{R}^2$. This bundle is canonically isomorphic to the unit tangent bundle $U\check{R}$ to $\check{R}$ via the map $([(x,y)] \mapsto [y-x])$. We have

$$\partial C_2(R) = p_b^{-1}(\infty,\infty) \cup (S^2_\infty \times \check{R}) \cup (-\check{R} \times S^2_\infty) \cup U\check{R}$$

[7]The minus sign in $(-S^2_\infty)$ reflects an orientation reversal.

and
$$\left(\check{C}_2(R) \stackrel{\text{def}}{=} C_2(R) \setminus \partial C_2(R)\right) = \check{R}^2 \setminus \text{diag}\left(\check{R}^2\right).$$

The following proposition is [Les20, Lemma 3.5].

Proposition 1.7 *Let ι_{S^2} denote the antipodal map of S^2. The S^2–valued map $p_{S^2}\colon (x,y) \mapsto \frac{1}{\|y-x\|}(y-x)$ extends smoothly from $\check{C}_2(\mathbb{R}^3)$ to $C_2(S^3)$, and its extension p_{S^2} satisfies:*

$$p_{S^2} = \begin{cases} \iota_{S^2} \circ p_\infty \circ p_1 & \text{on } S^2_\infty \times \mathbb{R}^3 \\ p_\infty \circ p_2 & \text{on } \mathbb{R}^3 \times S^2_\infty \\ p_2 & \text{on } U\mathbb{R}^3 = \mathbb{R}^3 \times S^2 \end{cases}$$

where p_1 and p_2 respectively denote the projections on the first and second factor, with respect to the above expressions.

Also note the following lemma[8].

Lemma 1.8 *$C_2(S^3)$ is homotopy equivalent to S^2.*

PROOF: $C_2(S^3)$ is homotopy equivalent to its interior $((\mathbb{R}^3)^2 \setminus \text{diag})$, which is homeomorphic to $\mathbb{R}^3 \times]0,\infty[\times S^2$ via the map

$$(x,y) \mapsto (x, \| y - x \|, p_{S^2}(x,y)).$$

□

We regard $\mathbb{R}^3$ as $\mathbb{C} \times \mathbb{R}$, where $\mathbb{C}$ is thought of as horizontal. Let $\mathcal{C}_0 = D^2 \times [0,1]$ be the *standard cylinder* of $\mathbb{R}^3$, where D^2 is the unit disk of $\mathbb{C}$. Let $\mathcal{C}_0^c$ (resp. $\check{\mathcal{C}}_0^c$) denote the closure of the complement of $\mathcal{C}_0$ in S^3 (resp. in $\mathbb{R}^3$). Here, a *rational homology cylinder* (or $\mathbb{Q}$*–cylinder*) is a compact oriented 3-manifold whose boundary neighborhood is identified with a boundary neighborhood $N(\partial\mathcal{C}_0)$ of $\mathcal{C}_0$, and that has the same rational homology as a point. Any $\mathbb{Q}$–sphere R (may and) will be viewed as the union $R(\mathcal{C})$ of $\mathcal{C}_0^c$ and of a rational homology cylinder $\mathcal{C}$ glued along $\partial\mathcal{C}_0$. It suffices to choose a point ∞ and a diffeomorphism that identifies a neighborhood of this point in R with $\mathcal{C}_0^c$ to obtain such a decomposition.

Definition 1.9 Let τ_s denote the standard parallelization of $\mathbb{R}^3$. We say that a parallelization

$$\tau\colon \check{R} \times \mathbb{R}^3 \to T\check{R}$$

of $\check{R}$ that coincides with τ_s on $\check{\mathcal{C}}_0^c$ is *asymptotically standard.* According to [Les20, Proposition 5.5], asymptotically standard parallelizations exist for any R. Such a parallelization identifies $U\check{R}$ with $\check{R} \times S^2$.

[8] More information about the homotopy groups and the homology of spaces of injective configurations of points in $\mathbb{R}^d$ or in S^d can be found in the book [FH01] by Fadell and Husseini.

An *asymptotic rational homology* $\mathbb{R}^3$ is a pair $(\check{R},\tau)$ where $\check{R}$ is a punctured rational homology sphere with a decomposition $\check{R}=\mathcal{C}\cup_{\partial\mathcal{C}_0}\check{\mathcal{C}}_0^c$ as above, equipped with an asymptotically standard parallelization τ.

In what follows, we fix such an asymptotic rational homology $\mathbb{R}^3$ $(\check{R}=\check{R}(\mathcal{C})=\mathcal{C}\cup_{\partial\mathcal{C}_0}\check{\mathcal{C}}_0^c,\tau)$ with its decomposition.

Lemma 1.10 *The parallelization τ of $\check{R}$ induces the continuous map $p_\tau\colon \partial C_2(R)\to S^2$ such that*

$$p_\tau=\begin{cases}\iota_{S^2}\circ p_\infty\circ p_1 & \text{on } S^2_\infty\times\check{R}\\ p_\infty\circ p_2 & \text{on } \check{R}\times S^2_\infty\\ p_2 & \text{on } U\check{R}\stackrel{\tau}{=}\check{R}\times S^2\\ p_{S^2} & \text{on } p_b^{-1}(\infty,\infty)\end{cases}$$

where p_1 and p_2 denote the projections on the first and second factor, respectively, with respect to the above expressions.

Proof: This is a corollary of Proposition 1.7. □

Lemma 1.11 *$H_*(C_2(R);\mathbb{Q})\cong H_*(S^2;\mathbb{Q})$ and $H_2(C_2(R);\mathbb{Q})$ is generated by the class $[S]$ of a fiber $U_x\check{R}$ of the bundle $U\check{R}$, oriented as the boundary of a ball of $T_x\check{R}$.*

Proof: The space $C_2(R)$ is homotopy equivalent to its interior $((\check{R})^2\setminus\text{diag})$, where $\check{R}$ has the rational homology of a point. The rational homology of $((\check{R})^2\setminus\text{diag})$ can be computed like the rational homology of $((\mathbb{R}^3)^2\setminus\text{diag})$, which is isomorphic to the rational homology of S^2 thanks to Lemma 1.8. □

Definition 1.12 A *volume-one form of* S^2 is a 2-form ω_S of S^2 such that $\int_{S^2}\omega_S=1$. (See [Les20, Appendix B] for a short survey of differential forms and de Rham cohomology.) Let $(\check{R},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Recall the map $p_\tau\colon\partial C_2(R)\to S^2$ of Lemma 1.10. A *propagating form* of $(C_2(R),\tau)$ is a closed 2-form ω on $C_2(R)$ whose restriction to $\partial C_2(R)$ is equal to $p_\tau^*(\omega_S)$ for some volume-one form ω_S of S^2. A *propagating chain* of $C_2(R)$ is a rational 4–chain P of $C_2(R)$ such that $\partial P\subset\partial C_2(R)$ and $\partial P\cap\left(\partial C_2(R)\setminus U\check{R}\right)=p^{-1}_{\tau|\partial C_2(R)\setminus U\check{R}}(a)$ for some $a\in S^2$. This definition does not depend on τ. A *propagating chain* of $(C_2(R),\tau)$ is a propagating chain of $C_2(R)$ such that $\partial P=p_\tau^{-1}(a)$ for some $a\in S^2$. Propagating chains and propagating forms are simply called *propagators* when their nature is clear from the context.

Example 1.13 Recall the map $p_{S^2}\colon C_2(S^3)\to S^2$ of Proposition 1.7. As already announced, for any $a\in S^2$, $p_{S^2}^{-1}(a)$ is a propagating chain of $(C_2(S^3),\tau_s)$, and for any 2-form ω_S of S^2 such that $\int_{S^2}\omega_S=1$, $p_{S^2}^*(\omega_S)$ is a propagating form of $(C_2(S^3),\tau_s)$.

For our general $\mathbb{Q}$–sphere R, propagating chains exist because the 3-cycle $p_\tau^{-1}(a)$ of $\partial C_2(R)$ bounds in $C_2(R)$ since $H_3(C_2(R);\mathbb{Q}) = 0$, according to Lemma 1.11. Dually, propagating forms exist because the restriction induces a surjective map $H^2(C_2(R);\mathbb{R}) \to H^2(\partial C_2(R);\mathbb{R})$ since $H^3(C_2(R), \partial C_2(R);\mathbb{R}) = 0$.

When R is a $\mathbb{Z}$-sphere, there exist propagating chains that are smooth 4-manifolds properly embedded in $C_2(R)$. See [Les20, Corollary 11.10]. Explicit propagating chains associated with Heegaard splittings, which were constructed with Greg Kuperberg in [Les15a], are described in Section 1.5 below. They are integral chains multiplied by $\frac{1}{|H_1(R;\mathbb{Z})|}$, where $|H_1(R;\mathbb{Z})|$ is the cardinality of $H_1(R;\mathbb{Z})$.

Lemma 1.14 *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let C be a two-cycle[9] of $C_2(R)$. For any propagating chain P of $C_2(R)$ transverse to C and for any propagating form ω of $(C_2(R), \tau)$,*

$$[C] = \int_C \omega [S] = \langle C, P\rangle_{C_2(R)}[S]$$

in $H_2(C_2(R);\mathbb{Q}) = \mathbb{Q}[S]$.

PROOF: Fix a propagating chain P. The algebraic intersection $\langle C, P\rangle_{C_2(R)}$ depends only on the homology class $[C]$ of C in $C_2(R)$. Similarly, since ω is closed, $\int_C \omega$ only depends on $[C]$. (Indeed, if C and C' cobound a chain D transverse to P, $C \cap P$ and $C' \cap P$ cobound $\pm(D \cap P)$, and $\int_{\partial D = C' - C} \omega = \int_D d\omega$ according to Stokes' theorem.) Furthermore, the dependence on $[C]$ is linear. Therefore it suffices to check the lemma for a chain that represents the canonical generator $[S]$ of $H_2(C_2(R);\mathbb{Q})$. Any fiber of $U\check{R}$ is such a chain. □

A *meridian* of a knot embedding K is the (oriented) boundary of a disk that intersects K once with a positive sign, as in Figure 4.

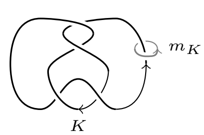


Figure 4: A meridian m_K of a knot K

Lemma 1.15 *Let $J \sqcup K$ be a two-component link embedding of $\check{R}$. The torus $J \times K = (J \times K)(S^1 \times S^1)$ is homologous to $lk(J, K)[S]$ in $H_2(C_2(R);\mathbb{Q})$. For any propagating chain P of $C_2(R)$ transverse to $J \times K$ and for any propagating form ω of $(C_2(R), \tau)$,*

$$lk(J, K) = \int_{J \times K} \omega = \langle J \times K, P\rangle_{C_2(R)}.$$

[9]A *d–cycle* is a chain of dimension d whose algebraic boundary is equal to zero. In other words, it is a d–chain such that the integral of any form of degree $d - 1$ along its boundary is zero.

If $\check{R} = \mathbb{R}^3$, then the linking number $lk(J,K)$ of Definition 1.3 is the degree $lk_G(J,K)$ of the Gauss map p_{JK}.

Proof: When $\check{R} = \mathbb{R}^3$,

$$lk_G(J,K) = \deg_a(p_{JK}) = \langle J \times K, p_{S^2}^{-1}(a)\rangle_{C_2(S^3)}$$

so that $J \times K$ is homologous to $lk_G(J,K)[S]$ in $H_2(C_2(S^3);\mathbb{Q})$ according to Lemma 1.14, with the propagator $p_{S^2}^{-1}(a)$ of Example 1.13. For an arbitrary $\check{R}$, define $lk_G(J,K)$ so that $J \times K$ is homologous to $lk_G(J,K)[S]$ in $H_2(C_2(R);\mathbb{Q})$. Recall from Definition 1.3 that $lk(J,K)$ is the algebraic intersection $\langle J, \Sigma_K\rangle_R$ of J and a rational chain Σ_K bounded by K. Lemma 1.14 reduces the proof of Lemma 1.15 to the proof that $lk(J,K)$ and $lk_G(J,K)$ coincide for any two-component link $J \sqcup K$ of $\check{R}$. Note that the definitions of $lk(J,K)$ and $lk_G(J,K)$ make sense when J and K are disjoint links. If J has several components J_i, for $i = 1,\dots,n$, then $lk_G(\sqcup_{i=1}^n J_i, K) = \sum_{i=1}^n lk_G(J_i,K)$ and $lk(\sqcup_{i=1}^n J_i, K) = \sum_{i=1}^n lk(J_i,K)$. There is no loss of generality in assuming that J is a knot for the proof, which we do. The chain Σ_K provides a rational cobordism C in $\check{R} \setminus J$ between K and a combination of meridians of J, which is homologous to $lk(J,K)[m_J]$. The product rational cobordism $J \times C$ in $\check{R}^2 \setminus \mathrm{diag}\left(\check{R}^2\right)$ allows us to see that $[J \times K] = lk(J,K)[J \times m_J]$ in $H_2(\check{R}^2 \setminus \mathrm{diag}\left(\check{R}^2\right);\mathbb{Q})$. Similarly, a chain Σ_J bounded by J provides a rational cobordism between J and a meridian m_{m_J} of m_J so that $[J \times m_J] = [m_{m_J} \times m_J]$ in $H_2(\check{R}^2 \setminus \mathrm{diag}\left(\check{R}^2\right);\mathbb{Q})$, and $lk_G(J,K) = lk(J,K)lk_G(m_{m_J}, m_J)$. Thus it suffices to prove that $lk_G(m_{m_J}, m_J) = 1$ for a positive Hopf link (m_{m_J}, m_J) in a standard ball embedded in $\check{R}$. Now, there is no loss of generality in assuming that our link is a Hopf link in $\mathbb{R}^3$. So the equality follows from that for the positive Hopf link in $\mathbb{R}^3$. □

Lemma 1.15 shows in what sense *propagators represent the linking number*. In what follows, we will use these propagators to define invariants of $\mathbb{Q}$–spheres.

1.4 On the Theta invariant

More on algebraic intersections The intersection of two transverse submanifolds A and B in a manifold M is a manifold, which is oriented so that the normal bundle to $A \cap B$ is $(N(A) \oplus N(B))$, fiberwise. In order to give a meaning to the sum $(N_x(A) \oplus N_x(B))$ at $x \in A \cap B$, pick a Riemannian metric on M, which canonically identifies $N_x(A)$ with $T_x(A)^\perp$, $N_x(B)$ with $T_x(B)^\perp$ and $N_x(A \cap B)$ with $T_x(A \cap B)^\perp = T_x(A)^\perp \oplus T_x(B)^\perp$. Since the space of Riemannian metrics on M is convex, and therefore connected, the induced orientation of $T_x(A \cap B)$ does not depend on the choice of Riemannian metric.

Let A, B, C be three pairwise transverse submanifolds in a manifold M such that $A \cap B$ is transverse to C. The oriented intersection $(A \cap B) \cap C$ is a well-defined oriented manifold. Our assumptions imply that at any $x \in A \cap B \cap C$, the sum $(T_xA)^\perp + (T_xB)^\perp + (T_xC)^\perp$ is a direct sum $(T_xA)^\perp \oplus (T_xB)^\perp \oplus (T_xC)^\perp$ for any Riemannian metric on M, so that A is also transverse to $B \cap C$, and $(A \cap B) \cap C = A \cap (B \cap C)$. Thus, the intersection of transverse, oriented submanifolds is a well defined associative operation, where *transverse submanifolds* are manifolds such that the elementary pairwise intermediate possible intersections are well defined as above. This *oriented* intersection is also commutative when the codimensions of the submanifolds are even.

If $A_1, \dots, A_k$ of M are transverse compact submanifolds whose codimension sum is the dimension of M, their *algebraic intersection* is defined to be $\langle A_1, \dots, A_k \rangle_M = \langle \cap_{i=1}^{k-1} A_i, A_k \rangle_M$. If M is a connected manifold, which contains a point x, the class of a 0-cycle in $H_0(M;\mathbb{Q}) = \mathbb{Q}[x] = \mathbb{Q}$ is a well-defined number, and $\langle A_1, \dots, A_k \rangle_M$ can equivalently be defined as the homology class of the (oriented) intersection $\cap_{i=1}^{k} A_i$. This algebraic intersection extends multilinearly to rational chains.

Theorem 1.16 *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let P_a, P_b and P_c be three transverse propagating chains of $(C_2(R), \tau)$ with respective boundaries $p_\tau^{-1}(a)$, $p_\tau^{-1}(b)$ and $p_\tau^{-1}(c)$ for three distinct points a, b and c of S^2. Then*

$$\Theta(R, \tau) = \langle P_a, P_b, P_c \rangle_{C_2(R)}$$

does not depend on the chosen propagators P_a, P_b and P_c. It is a topological invariant of (R, τ).

Proof: Since $H_4(C_2(R); \mathbb{Q}) = 0$, if the propagator P_a is replaced by a propagator P'_a with the same boundary, $(P'_a - P_a)$ bounds a 5-dimensional rational chain W transverse to $P_b \cap P_c$. The 1-dimensional chain $W \cap P_b \cap P_c$ does not meet $\partial C_2(R)$ since $P_b \cap P_c$ does not meet $\partial C_2(R)$. Therefore, up to a well-determined sign, the boundary of $W \cap P_b \cap P_c$ is $P'_a \cap P_b \cap P_c - P_a \cap P_b \cap P_c$. This shows that $\langle P_a, P_b, P_c \rangle_{C_2(R)}$ is independent of P_a when a is fixed. Similarly, it is independent of P_b and P_c when b and c are fixed. Thus, $\langle P_a, P_b, P_c \rangle_{C_2(R)}$ is a rational function of the connected set of triples (a, b, c) of distinct point of S^2. It is easy to see that this function is continuous, and so, it is constant. □

Lemma 1.17 *Let ω_a and ω'_a be two propagating forms of $(C_2(R), \tau)$, whose restrictions to $\partial C_2(R)$ are $p_\tau^*(\omega_A)$ and $p_\tau^*(\omega'_A)$, respectively, for two volume-one forms ω_A and ω'_A of S^2. There exists a one-form η_A on S^2 such that $\omega'_A = \omega_A + d\eta_A$. For any such η_A, there exists a one-form η on $C_2(R)$ such that $\omega'_a - \omega_a = d\eta$, and the restriction of η to $\partial C_2(R)$ is $p_\tau^*(\eta_A)$.*

Proof: Since ω_a and ω'_a are cohomologous, there exists a one-form η on $C_2(R)$ such that $\omega'_a = \omega_a + d\eta$. Similarly, since $\int_{S^2} \omega'_A = \int_{S^2} \omega_A$, there exists a one-form η_A on S^2 such that $\omega'_A = \omega_A + d\eta_A$. On $\partial C_2(R)$, $d(\eta - p_\tau^*(\eta_A)) = 0$. Thanks to the exact sequence with real coefficients

$$0 = H^1(C_2(R)) \longrightarrow H^1(\partial C_2(R)) \longrightarrow H^2(C_2(R), \partial C_2(R)) \cong H_4(C_2(R)) = 0,$$

we obtain $H^1(\partial C_2(R); \mathbb{R}) = 0$. Therefore, there exists a function f from $\partial C_2(R)$ to $\mathbb{R}$ such that

$$df = \eta - p_\tau^*(\eta_A)$$

on $\partial C_2(R)$. To obtain the result, we extend f to a C^∞ map on $C_2(R)$ and replace η by $(\eta - df)$.
□

Theorem 1.18 *Let $(\check{R},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. For any three propagating forms ω_a, ω_b and ω_c of $(C_2(R),\tau)$,*

$$\Theta(R,\tau)=\int_{C_2(R)}\omega_a\wedge\omega_b\wedge\omega_c.$$

PROOF: Let us first prove that $\int_{C_2(R)}\omega_a\wedge\omega_b\wedge\omega_c$ is independent of the propagating forms ω_a, ω_b and ω_c. Using Lemma 1.17 and its notation

$$\begin{aligned}\int_{C_2(R)}\omega'_a\wedge\omega_b\wedge\omega_c-\int_{C_2(R)}\omega_a\wedge\omega_b\wedge\omega_c &= \int_{C_2(R)}d(\eta\wedge\omega_b\wedge\omega_c)\\ &= \int_{\partial C_2(R)}\eta\wedge\omega_b\wedge\omega_c\\ &= \int_{\partial C_2(R)}p_\tau^*(\eta_A\wedge\omega_B\wedge\omega_C)=0\end{aligned}$$

since any 5-form on S^2 vanishes. Thus, $\int_{C_2(R)}\omega_a\wedge\omega_b\wedge\omega_c$ is independent of the propagating forms ω_a, ω_b and ω_c. Now, we can choose the propagating forms ω_a, ω_b and ω_c supported in very small neighborhoods of P_a, P_b and P_c and Poincaré dual to P_a, P_b and P_c, respectively, so that the intersection of the three supports is a very small neighborhood of $P_a\cap P_b\cap P_c$, from which it can easily be seen that $\int_{C_2(R)}\omega_a\wedge\omega_b\wedge\omega_c=\langle P_a,P_b,P_c\rangle_{C_2(R)}$. See [Les20, Section 11.4, Section B.2 and Lemma B.4 in particular] for more details. □

In particular, $\Theta(R,\tau)$ is equal to $\int_{C_2(R)}\omega^3$ for any propagating form ω of $(C_2(R),\tau)$. Since such a propagating form represents the linking number, $\Theta(R,\tau)$ can be thought of as the *cube of the linking number with respect to τ*. When τ varies continuously, $\Theta(R,\tau)$ varies continuously in $\mathbb{Q}$ so that $\Theta(R,\tau)$ is an invariant of the homotopy class of τ.

Example 1.19 Using (disjoint!) propagators $p_{S^2}^{-1}(a)$, $p_{S^2}^{-1}(b)$, $p_{S^2}^{-1}(c)$ associated to three distinct points a, b and c of $\mathbb{R}^3$, as in Example 1.13, it is clear that

$$\Theta(S^3,\tau_s)=\langle p_{S^2}^{-1}(a),p_{S^2}^{-1}(b),p_{S^2}^{-1}(c)\rangle_{C_2(S^3)}=0.$$

Parallelizations of 3-manifolds and Pontrjagin classes

Definition 1.20 Let $SO(3)$ be the group of orientation-preserving linear isometries of $\mathbb{R}^3$. In this paragraph, we regard S^3 as $B^3/\partial B^3$ where B^3 is the standard unit ball of $\mathbb{R}^3$ viewed as $([0,1]\times S^2)/(0\sim\{0\}\times S^2)$. Let $\chi_\pi\colon[0,1]\to[0,2\pi]$ be an increasing smooth bijection whose derivatives vanish at 0 and 1 such that $\chi_\pi(1-\theta)=2\pi-\chi_\pi(\theta)$ for any $\theta\in[0,1]$. Let $\rho\colon B^3\to SO(3)$ be the map that sends $(\theta\in[0,1],v\in S^2)$ to the rotation $\rho(\chi_\pi(\theta);v)$ with axis directed by v and with angle $\chi_\pi(\theta)$.

This map[10] induces the double covering $\tilde{\rho}\colon S^3\to SO(3)$, which identifies $SO(3)$ with the real projective space $\mathbb{R}P^3$, and which orients $SO(3)$.

[10]This double covering map allows one to deduce the first three homotopy groups of $SO(3)$ from those of S^3. The first three homotopy groups of $SO(3)$ are $\pi_1(SO(3))=\mathbb{Z}/2\mathbb{Z}$, $\pi_2(SO(3))=0$ and $\pi_3(SO(3))=\mathbb{Z}[\tilde{\rho}]$. For $v\in S^2$, $\pi_1(SO(3))$ is generated by the class of the loop that maps $\exp(i\theta)\in S^1$ to the rotation $\rho(\theta;v)$. See [Les20, Section A.2 and Theorem A.14, in particular].

For any map g from $\check{R}$ to $SO(3)$ that sends $\check{\mathcal{C}}_0^c$ to the identity element $1_{SO(3)}$ of the group $SO(3)$, define

$$\begin{array}{rccc} \psi_{\mathbb{R}}(g): & \check{R} \times \mathbb{R}^3 & \longrightarrow & \check{R} \times \mathbb{R}^3 \\ & (x,y) & \mapsto & (x, g(x)(y)). \end{array}$$

Since $GL^+(\mathbb{R}^3)$ deformation retracts onto $SO(3)$, any asymptotically standard parallelization of $\check{R}$ is homotopic to $\tau \circ \psi_{\mathbb{R}}(g)$ for some g as above.

The following classical theorem is proved in [Les20, Chapter 5]. See Theorem 4.6 and Proposition 5.22 in particular.

Theorem 1.21 *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. There exists a canonical map p_1 from the set of homotopy classes of asymptotically standard parallelizations of $\check{R}$ to $\mathbb{Z}$ such that $p_1(\tau_s) = 0$, and, for any map g from R to $SO(3)$ that sends $\mathcal{C}_0^c$ to the identity element $1_{SO(3)}$ of $SO(3)$, we have*

$$p_1(\tau \circ \psi_{\mathbb{R}}(g_{|\check{R}})) - p_1(\tau) = 2deg(g).$$

The definition of the map p_1 is given in [Les20, Section 5.5] and involves relative Pontrjagin classes. See [Les20, Definition 5.13]. It is similar to the map h studied by Hirzebruch in [Hir73, §3.1], and by Kirby and Melvin in [KM99] under the name of *Hirzebruch defect.*

The following proposition is proved in [Les20, Section 4.3]. See Proposition 4.8.

Proposition 1.22 *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. For any map g from R to $SO(3)$ that sends $\mathcal{C}_0^c$ to $1_{SO(3)}$,*

$$\Theta(R, \tau \circ \psi_{\mathbb{R}}(g_{|\check{R}})) - \Theta(R, \tau) = \frac{1}{2}deg(g).$$

Theorem 1.21 allows us to derive the following corollary from Proposition 1.22.

Corollary 1.23 $\Theta(R) = \Theta(R, \tau) - \frac{1}{4}p_1(\tau)$ *is an invariant of $\mathbb{Q}$-spheres.*

The invariant Θ coincides with $6\lambda_{CW}$ where λ_{CW} denotes the Casson-Walker invariant. The Walker invariant generalizes the Casson invariant of $\mathbb{Z}$-spheres, which counts the conjugacy classes of irreducible representations of their fundamental groups using Heegaard splittings. See [AM90, GM92, Mar88]. It is normalized as in [AM90, GM92, Mar88] for integer homology 3-spheres, and as $\frac{1}{2}\lambda_W$ for rational homology 3-spheres where λ_W is the Walker normalisation in [Wal92]. The equality $\Theta = 6\lambda_{CW}$ was proved by Kuperberg and Thurston in [KT99] for $\mathbb{Z}$–spheres, and it was generalized to $\mathbb{Q}$-spheres in [Les04, Section 6]. See [Les04, Theorem 2.6] or [Les20, Theorem 18.30].

The main part of the proof consists in comparing second derivatives or (variations of variations) of Θ and λ_{CW} under the following *Lagrangian-preserving surgeries.*

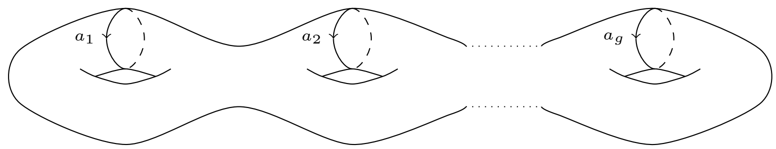


Figure 5: The standard handlebody H_g

Lagrangian-preserving surgeries

Definition 1.24 An *integer (resp. rational) homology handlebody* of genus g is a compact oriented 3-manifold A that has the same integral (resp. rational) homology as the usual solid handlebody H_g of Figure 5.

Exercise 1.25 Show that if A is a rational homology handlebody of genus g, then ∂A is a genus g surface.

The *Lagrangian* $\mathcal{L}_A$ of a compact 3-manifold A is the kernel of the map induced by the inclusion from $H_1(\partial A;\mathbb{Q})$ to $H_1(A;\mathbb{Q})$.

In Figure 5, the Lagrangian of H_g is freely generated by the classes of the curves a_i.

Definition 1.26 An *integral (resp. rational) Lagrangian-Preserving –or LP– surgery* (A'/A) is the replacement of an integral (resp. rational) homology handlebody A embedded in the interior of a 3-manifold M with another such A' whose boundary is identified with ∂A by an orientation-preserving diffeomorphism that sends $\mathcal{L}_A$ to $\mathcal{L}_{A'}$. The manifold $M(A'/A)$ obtained by such an LP-surgery is given[11] by

$$M(A'/A) = (M \setminus \mathrm{Int}(A)) \cup_{\partial A} A'.$$

Lemma 1.27 *If (A'/A) is an integral (resp. rational) LP-surgery in a 3-manifold M, then the homology of $M(A'/A)$ with $\mathbb{Z}$-coefficients (resp. with $\mathbb{Q}$-coefficients) is canonically isomorphic to $H_*(M;\mathbb{Z})$ (resp. to $H_*(M;\mathbb{Q})$). If M is a $\mathbb{Q}$-sphere, if (A'/A) is a rational LP-surgery, and if (J,K) is a two-component link of $M \setminus A$, then the linking number of J and K in M and the linking number of J and K in $M(A'/A)$ coincide.*

Proof: Exercise. □

In [Les04], I computed

$$\Theta\left(R\left(A'/A, B'/B\right)\right) - \Theta\left(R\left(A'/A\right)\right) - \Theta\left(R\left(B'/B\right)\right) + \Theta(R)$$

[11] This description defines only the topological structure of $M(A'/A)$, but we equip $M(A'/A)$ with its unique smooth structure.

and proved that it coincides with

$$6\lambda_{CW}\left(R\left(A'/A,B'/B\right)\right)-6\lambda_{CW}\left(R\left(A'/A\right)\right)-6\lambda_{CW}\left(R\left(B'/B\right)\right)+6\lambda_{CW}(R)$$

for any two rational LP-surgeries (A'/A) and (B'/B) in a $\mathbb{Q}$–sphere R such that A and B are disjoint rational homology handlebodies in $\check{R}$. Together with the property that $\Theta(-R)=-\Theta(R)$, this implies that $\Theta=6\lambda_{CW}$. See [Les20, Theorem 18.28]. In order to perform the computation of the above discrete "second derivative"

$$\left(\Theta\left(R\left(A'/A,B'/B\right)\right)-\Theta\left(R\left(B'/B\right)\right)\right)-\left(\Theta\left(R\left(A'/A\right)\right)-\Theta(R)\right)$$

of Θ, I built propagators for the four involved $\mathbb{Q}$–spheres, which coincide in the identical parts of the configuration spaces, like $(M\setminus(A\sqcup B))^2\setminus\mathrm{diag}$, for example.

1.5 A propagator associated to a Heegaard diagram

In this section, we give an example of a propagating chain associated to a Heegaard diagram or to a self-indexed Morse function of an asymptotic rational homology $\mathbb{R}^3$. I constructed such a *Morse propagator* with Greg Kuperberg in [Les15a]. Similar propagators associated to more general Morse functions have been constructed independently by Watanabe in [Wat18].

First note that the propagator $p_{S^2}^{-1}(\vec{N})$ of $C_2(S^3)$ associated to the upward vertical vector $\vec{N}$ intersects $\check{C}_2(S^3)$ as $\{(x,x+t\vec{N})\mid x\in\mathbb{R}^3,t\in]0,+\infty]\}$. The explicit propagator that we are about to construct for an asymptotic rational homology $\check{R}$ is built from the closure P_ϕ in $C_2(R)$ of $\{(x,\phi_t(x))\mid x\in\check{R},t\in]0,+\infty]\}$, where (ϕ_t) is the flow associated to a Morse function without minima and maxima of $\check{R}$, and to a metric $\mathfrak{g}$ on $\check{R}$.

Start with $\mathbb{R}^3$ equipped with its standard height function f_0 and replace the cube $[-\frac{1}{2},\frac{1}{2}]^2\times[0,1]$ with a rational homology cube C_R (which has the rational homology of a point) equipped with a Morse function f, which coincides with f_0 on $\partial\left([-\frac{1}{2},\frac{1}{2}]^2\times[0,1]\right)$, and which has $2g$ critical points: g points $a_1,\dots,a_g$ of index 1, mapped to $1/3$ by f, and g points $b_1,\dots,b_g$ of index 2, mapped to $2/3$ by f (so that $3f$ is self-indexed). Let $\check{R}$ be the associated open manifold, and let R be its one-point compactification. Equip $\check{R}$ with a Riemannian metric $\mathfrak{g}$ that coincides with the standard one outside $[-\frac{1}{2},\frac{1}{2}]^2\times[0,1]$.

The preimage H_a of $]-\infty,\frac{1}{2}]$ under f in C_R has the standard representation of the bottom part of Figure 6. Our standard representation of the preimage H_b of $[\frac{1}{2},+\infty[$ under f in C_R is shown in the upper part of Figure 6. These two pieces are equipped with standard Morse functions and metrics, a few corresponding flow lines are drawn in Figure 7. They are glued to each other by a diffeomorphism from ∂H_a to $(-\partial H_b)$.

The closure of the two-dimensional ascending manifold of a_i is denoted by $\mathcal{A}_i$. Its intersection with H_a is denoted by $D(\alpha_i)$. The disk $D(\alpha_i)$ and $\mathcal{A}_i$ are consistently oriented so that the boundary of the disk $D(\alpha_i)$ is the curve α_i of Figures 6 and 7. The descending manifold of a_i consists of two half-lines $\mathcal{L}_+(a_i)$ and $\mathcal{L}_-(a_i)$ starting as vertical lines and ending at a_i. The half-line with the orientation of the positive normal to $\mathcal{A}_i$ is called $\mathcal{L}_+(a_i)$. Thus $\mathcal{L}(a_i)=\mathcal{L}_+(a_i)\cup(-\mathcal{L}_-(a_i))$ is the descending manifold of a_i.

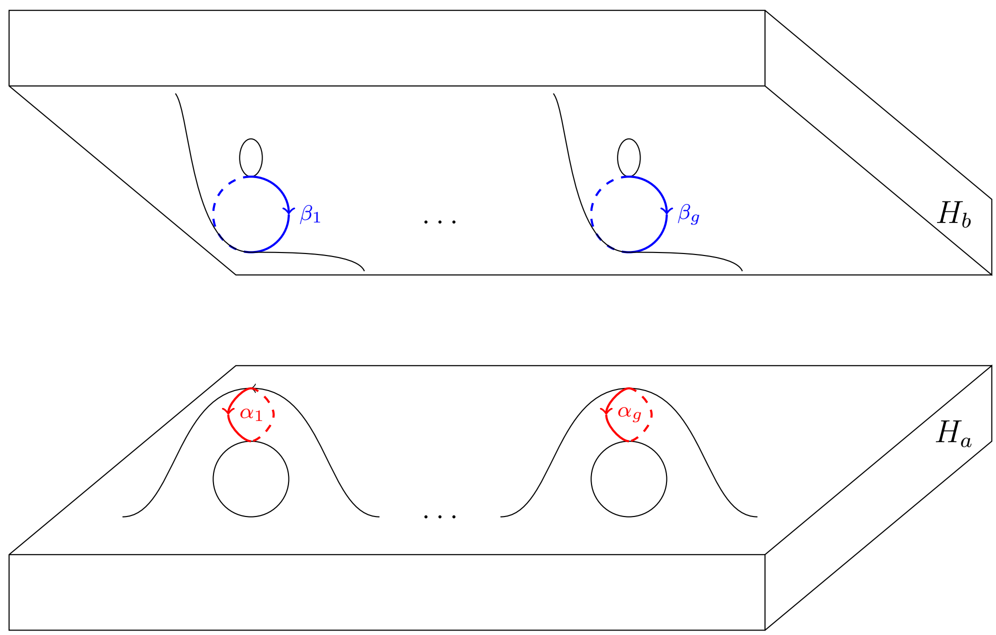


Figure 6: H_a and H_b

Symmetrically, the closure of the two-dimensional descending manifold of b_j is denoted by $\mathcal{B}_j$. The $\mathcal{B}_j$ are assumed to be transverse to the $\mathcal{A}_i$ outside the critical points. The intersection $H_b \cap \mathcal{B}_j$ is denoted by $D(\beta_j)$. The disk $D(\beta_j)$ and $\mathcal{B}_j$ are consistently oriented so that the boundary of the disk $D(\beta_j)$ is the curve β_j of Figures 6 and 7. The ascending manifold of b_j consists of two half-lines $\mathcal{L}_+(b_j)$ and $\mathcal{L}_-(b_j)$ starting at b_j and ending as vertical lines, the first $\mathcal{L}_+(b_j)$ being that whose orientation matches the orientation of the positive normal to $\mathcal{B}_j$. Thus $\mathcal{L}(b_j) = \mathcal{L}_+(b_j) - \mathcal{L}_-(b_j)$ is the ascending manifold of b_j. See Figure 7. Let

$$[\mathcal{J}_{ji}]_{(j,i)\in\{1,\dots,g\}^2} = [\langle \alpha_i, \beta_j \rangle_{\partial H_a}]^{-1}$$

be the inverse matrix of the matrix of the algebraic intersection numbers $\langle \alpha_i, \beta_j \rangle_{\partial H_a}$.

Let ϕ be the flow associated to the gradient of f and to $\mathfrak{g}$. Let P_ϕ be the closure in $C_2(R)$ of the image of

$$\begin{array}{lll} \left(\check{R} \setminus \{a_i, b_i; i \in \{1,\dots,g\}\}\right) \times]0,+\infty[& \to & C_2(R) \\ (x,t) & \mapsto & (x, \phi_t(x)), \end{array}$$

let $((\mathcal{B}_j \times \mathcal{A}_i) \cap C_2(R))$ denote the closure of $\left((\mathcal{B}_j \times \mathcal{A}_i) \cap (\check{R}^2 \setminus \mathrm{diag})\right)$ in $C_2(R)$, set

$$P_{\mathcal{I}} = \sum_{(i,j)\in\{1,\dots,g\}^2} \mathcal{J}_{ji}\left((\mathcal{B}_j \times \mathcal{A}_i) \cap C_2(R)\right) \qquad \text{and} \qquad P(f,\mathfrak{g}) = P_\phi + P_{\mathcal{I}}$$

The following proposition is proved in [Les15a]. See Theorem 4.2.

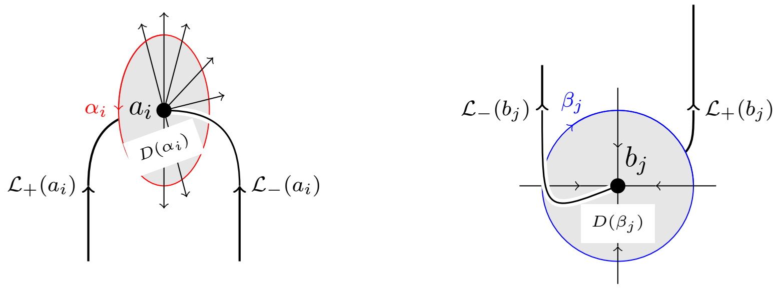


Figure 7: $\mathcal{L}_+(a_i)$, $\mathcal{L}_-(a_i)$, $\mathcal{L}_+(b_j)$, $\mathcal{L}_-(b_j)$

Proposition 1.28 (Kuperberg–Lescop) *The chain $P(f,\mathfrak{g})$ is a propagating chain of $C_2(R)$.*

In particular, $P(f,\mathfrak{g})$ can be used to compute linking numbers as in Lemma 1.15. It suffices[12] to correct the boundary of $P(f,\mathfrak{g})$ near the boundary of $C_2(R)$ to transform $P(f,\mathfrak{g})$ into a propagator of $(C_2(R),\tau)$ as in Definition 1.12.

Define a *combing* of $\check{R}$ to be a section of $U\check{R}$ which is constant on $\check{\mathcal{C}}_0^c$. For such a combing X, a *propagating chain* of $(C_2(R),X)$ is a propagating chain P of $C_2(R)$ such that $P \cap U\check{R} = X(\check{R})$. Define $\tilde{\Theta}(R,X)$ to be the algebraic intersection of a propagating chain of $(C_2(R),X)$, a propagating chain of $(C_2(R),-X)$ and any other propagating chain. It is easy to see that $\tilde{\Theta}(R,.)$ is a homotopy invariant of combings (see [Les15a, Theorem 2.1]) and that $\Theta(R,\tau) = \tilde{\Theta}(R,\tau(.,v))$, for any unit vector v of $\mathbb{R}^3$. Further properties of the invariant $\tilde{\Theta}(R,.)$ of combings are studied in [Les15b]. An explicit formula for the invariant $\tilde{\Theta}(R,.)$ from a Heegaard diagram of R was discovered by the author in [Les15a]. See [Les15a, Theorem 3.8]. It was computed directly using the above definition of $\tilde{\Theta}(R,.)$ together with the above Morse propagators, corrected near the boundary as in [Les15a, Section 5].

[12]This requires some work performed in [Les15a, Section 5].

2 Configuration space integrals

2.1 Jacobi diagrams and associated configuration space integrals

Definition 2.1 A *uni-trivalent graph* Γ is a 6-tuple

$$(H(\Gamma), E(\Gamma), U(\Gamma), T(\Gamma), p_E, p_V)$$

where $H(\Gamma)$, $E(\Gamma)$, $U(\Gamma)$ and $T(\Gamma)$ are finite sets, which are called the set of half-edges of Γ, the set of edges of Γ, the set of univalent vertices of Γ and the set of trivalent vertices of Γ, respectively, $p_E\colon H(\Gamma) \to E(\Gamma)$ is a two-to-one map (every element of $E(\Gamma)$ has two preimages under p_E) and $p_V\colon H(\Gamma) \to U(\Gamma) \sqcup T(\Gamma)$ is a map such that every element of $U(\Gamma)$ has one preimage under p_V and every element of $T(\Gamma)$ has three preimages under p_V, up to isomorphism. In other words, Γ is a set $H(\Gamma)$ equipped with two partitions, a partition into pairs (induced by p_E), and a partition into singletons and triples (induced by p_V), up to the bijections that preserve the partitions. These bijections are the *automorphisms* of the uni-trivalent graph Γ.

Such a uni-trivalent graph is pictured as and identified with the topological quotient of the disjoint union $\sqcup_{h\in H(\Gamma)}\psi_h([0,1])$ of copies $\psi_h([0,1])$ of $[0,1]$ by the relations

$$\psi_h(1) = \psi_k(1) \text{ if } p_E(h) = p_E(k), \text{ and, } \psi_h(0) = \psi_k(0) \text{ if } p_V(h) = p_V(k),$$

up to homeomorphism.

Definition 2.2 Let $\mathcal{L}$ be a one-manifold, oriented or not. A *Jacobi diagram* Γ *with support* $\mathcal{L}$, also called *Jacobi diagram on* $\mathcal{L}$, is a finite uni-trivalent graph Γ equipped with an isotopy class $[i_\Gamma]$ of injections i_Γ from the set $U(\Gamma)$ of univalent vertices of Γ into the interior of $\mathcal{L}$. For such a Γ, a Γ-*compatible injection* is an injection in the class $[i_\Gamma]$.

A Jacobi diagram Γ is represented by a planar immersion of $\Gamma \cup \mathcal{L} = \Gamma \cup_{U(\Gamma)} \mathcal{L}$ where the univalent vertices of $U(\Gamma)$ are located at their images under a Γ-compatible injection i_Γ, the one-manifold $\mathcal{L}$ is represented by dashed lines, whereas the edges of the diagram Γ are represented by plain segments. (The one-manifold $\mathcal{L}$ may be oriented in order to fix the isotopy class $[i_\Gamma]$.)

Figure 8 shows an example of a picture of a Jacobi diagram.

Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $\mathcal{L}$ be a one-manifold and let

$$L\colon \mathcal{L} \longrightarrow \check{R}$$

denote a C^∞ embedding from $\mathcal{L}$ to $\check{R}$. Let Γ be a Jacobi diagram with support $\mathcal{L}$ as in Definition 2.2. Let $U = U(\Gamma)$ denote the set of univalent vertices of Γ, and let $T = T(\Gamma)$ denote the set of trivalent vertices of Γ. A *configuration* of Γ is an injection

$$c\colon U \cup T \hookrightarrow \check{R}$$

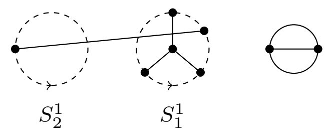


Figure 8: A Jacobi diagram Γ on the disjoint union $\mathcal{L} = S^1_1 \sqcup S^1_2$ of two (oriented) circles

whose restriction $c_{|U}$ to U may be written as $L \circ j$ for some Γ-compatible injection

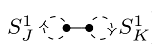
$$j\colon U \hookrightarrow \mathcal{L}.$$

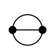
Denote the set of these configurations by $\check{C}(R, L; \Gamma)$ (or $\check{C}(L; \Gamma)$, when R is known or is part of the data).

$$\check{C}(R, L; \Gamma) = \left\{ c\colon U \cup T \hookrightarrow \check{R} \mid \exists j \in [i_\Gamma], c_{|U} = L \circ j \right\}.$$

In $\check{C}(R, L; \Gamma)$, the univalent vertices move along $L(\mathcal{L})$, while the trivalent vertices move in the ambient space $\check{R}$, and $\check{C}(R, L; \Gamma)$ is naturally an open submanifold of $\mathcal{L}^U \times \check{R}^T$. When the ambient asymptotic rational homology $\mathbb{R}^3$ is $\mathbb{R}^3$, we write $\check{C}(L; \Gamma) = \check{C}(S^3, L; \Gamma)$.

Examples 2.3 For a two-component link $J \sqcup K\colon S^1 \sqcup S^1 \to \check{R}$,

$\check{C}\,(R, J \sqcup K;$  $) = J \times K.$

$\check{C}\,(R, \emptyset;$  $) = \check{R}^2 \setminus \mathrm{diag}\left(\check{R}^2\right) = \check{C}_2(R).$

Recall that R is seen as the union $R(\mathcal{C})$ of $\mathcal{C}_0^c$ and of a rational homology cylinder $\mathcal{C}$ glued along $\partial\mathcal{C}_0$ as before Definition 1.9.

Definition 2.4 A *long tangle representative* in $\check{R} = \check{R}(\mathcal{C})$ is an embedding $L\colon \mathcal{L} \hookrightarrow \check{R}$ of a one-manifold $\mathcal{L}$, as in Figure 9, such that

- $$L(\mathcal{L}) \cap \check{\mathcal{C}}_0^c = (c^-(B^-) \times]-\infty, 0]) \cup (c^+(B^+) \times [1, \infty[)$$
  for two finite sets B^- and B^+ and two injective maps $c^-\colon B^- \hookrightarrow \mathrm{Int}(D^2)$ $c^+\colon B^+ \hookrightarrow \mathrm{Int}(D^2)$, which are called the *bottom configuration* and the *top configuration* of L, respectively, and
- $L(\mathcal{L}) \cap \mathcal{C}$ is a compact one-manifold whose unoriented boundary is $(c^-(B^-) \times \{0\}) \cup (c^+(B^+) \times \{1\})$.

Figure 10 shows an example of a Jacobi diagram Γ on its source $\mathcal{L}$ together with a configuration of $\check{C}(R, L; \Gamma)$ (where the edges are drawn just to identify the vertices, the configuration is determined by the images of the vertices).

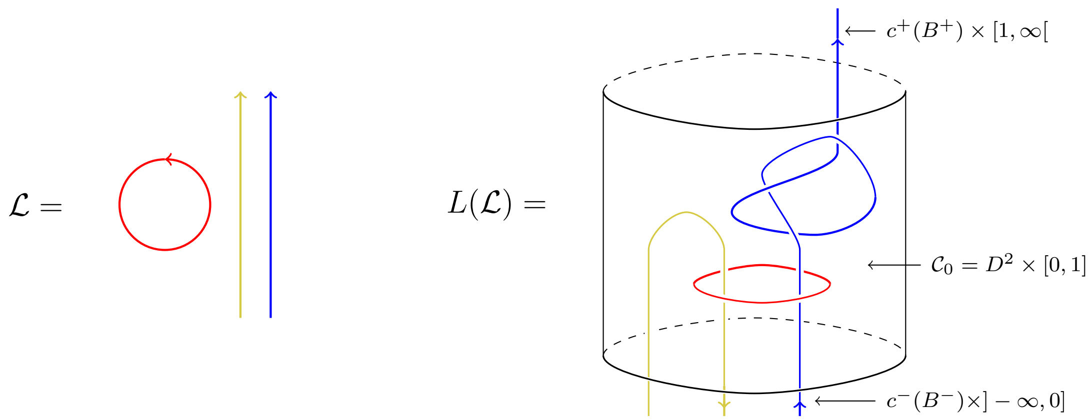
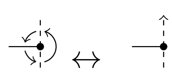
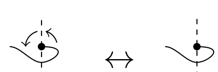


Figure 9: A long tangle representative (LTR) in $\mathbb{R}^3$

Definition 2.5 An *orientation* of a trivalent vertex of Γ is a cyclic order on the set of the three half-edges that meet at this vertex. An *orientation* of a univalent vertex u of Γ is an orientation of the connected component $\mathcal{L}(u)$ of $i_\Gamma(u)$ in $\mathcal{L}$, for a choice of Γ-compatible i_Γ, associated to u. This orientation is also called (and thought[13] of as) a *local orientation of $\mathcal{L}$ at u*.

A *vertex-orientation* of a Jacobi diagram Γ is an *orientation* of every vertex of Γ. A Jacobi diagram is *oriented* if it is equipped with a vertex-orientation[14].

In the figures, the orientation of a trivalent vertex is represented by the counterclockwise order of the three half-edges that meet at the vertex. The orientation of a univalent vertex u of a Jacobi diagram on a (non-oriented) one-manifold $\mathcal{L}$ is represented by the counterclockwise cyclic order of the three half-edges that meet at u in a planar immersion of $\Gamma \cup_{U(\Gamma)} \mathcal{L}$, where the half-edge of u in Γ is attached to the left-hand side of $\mathcal{L}$, with respect to the local orientation of $\mathcal{L}$ at u, as in the following pictures.

 and 

An *orientation* of a set X of cardinality at least 2 is a total order of the elements of X up to an even permutation.

Cut each edge of Γ into two half-edges. When an edge is oriented, define its *first* half-edge and its *second* one, so that following the orientation of the edge, the first half-edge is met first. Recall that $H(\Gamma)$ denotes the set of half-edges of Γ. When the edges of Γ are oriented, the orientations of the edges of Γ induce the following orientation of the set $H(\Gamma)$ of half-edges of

[13] A *local orientation* of $\mathcal{L}$ is simply an orientation of $\mathcal{L}(u)$, but since different vertices are allowed to induce different orientations, we think of these orientations as being *local*, i.e. defined in a neighborhood of $i_\Gamma(u)$ for a choice of Γ-compatible i_Γ.

[14] When $\mathcal{L}$ is oriented, it suffices to specify the orientations of the trivalent vertices since the univalent vertices are oriented by $\mathcal{L}$.

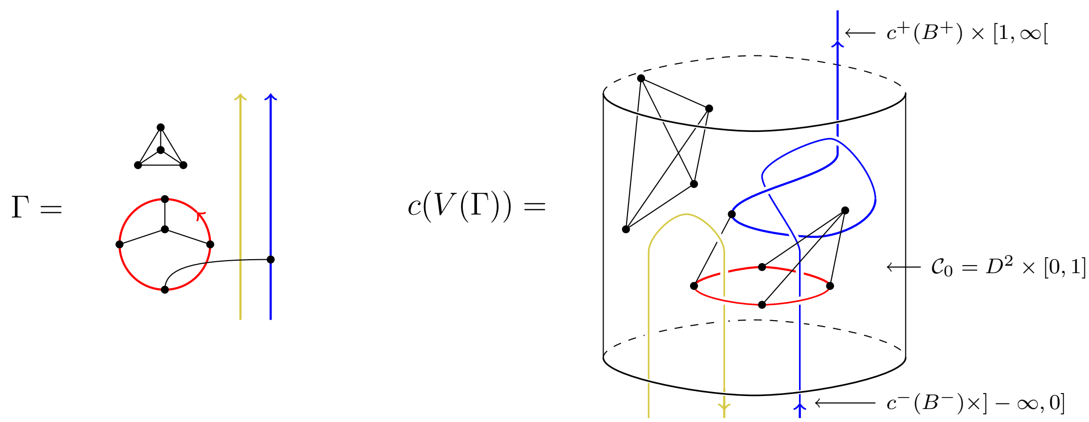


Figure 10: A (black) Jacobi diagram Γ on the source of an LTR L, and a configuration c of $\check{C}(L;\Gamma)$

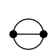
Γ: order $E(\Gamma)$ arbitrarily, and order the half-edges as (first half-edge of the first edge, second half-edge of the first edge, ..., second half-edge of the last edge). The induced orientation of $H(\Gamma)$ is called the *edge-orientation* of $H(\Gamma)$. Note that it does not depend on the order of $E(\Gamma)$.

Lemma 2.6 *When Γ is equipped with a vertex-orientation, orientations of the manifold $\check{C}(L;\Gamma)$ are in canonical one-to-one correspondence with orientations of the set $H(\Gamma)$.*

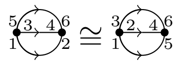
PROOF: Since $\check{C}(L;\Gamma)$ is naturally an open submanifold of $\mathcal{L}^U \times \check{R}^T$, it inherits $\mathbb{R}^{\sharp U+3\sharp T}$-valued charts from $\mathbb{R}$-valued charts of $\mathcal{L}$ and $\mathbb{R}^3$-valued orientation-preserving charts of $\check{R}$. The $\mathbb{R}$-valued charts of $\mathcal{L}$ respect the local orientations of $\mathcal{L}$ induced by the corresponding oriented univalent vertices. In order to define the orientation of $\mathbb{R}^{\sharp U+3\sharp T}$, it suffices to identify its factors and order them (up to even permutation). Each of the factors may be labeled by an element of $H(\Gamma)$: the $\mathbb{R}$-valued local coordinate of an element of $\mathcal{L}$ corresponding to the image under j of an element u of U sits in the factor labeled by the half-edge that contains u; the three ordered $\mathbb{R}$-valued coordinates of the image under a configuration c of an element t of T, with respect to an arbitrary oriented local chart, belong to the factors labeled by the three half-edges that contain t, which are cyclically ordered by the vertex-orientation of Γ, so that the cyclic orders match. $\square$

We use Lemma 2.6 to orient $\check{C}(R, L;\Gamma)$ as summarized in the following immediate corollary.

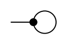
Corollary 2.7 *If Γ is equipped with a vertex-orientation $o(\Gamma)$ and if the edges of Γ are oriented, then the induced edge-orientation of $H(\Gamma)$ orients $\check{C}(L;\Gamma)$, via the canonical correspondence described in Lemma 2.6.*

Example 2.8 Equip the diagram  with its vertex-orientation induced by the picture. Orient its three edges so that they start from the same vertex. Then the orientation of $\check{C}(R, L;$ $)$

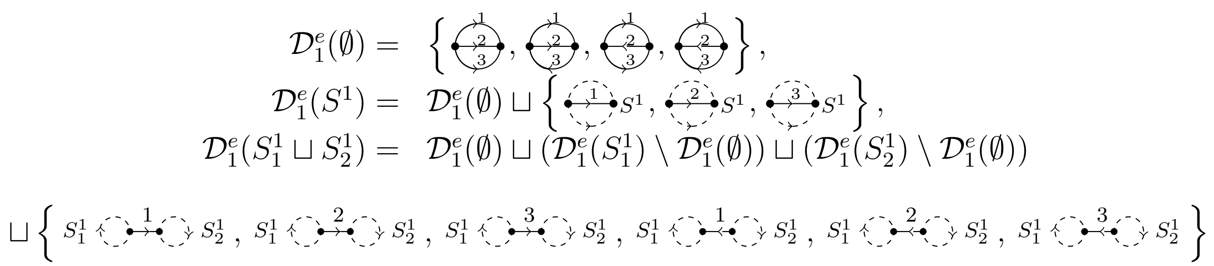
induced by this edge-orientation of  matches the orientation of $(\check{R} \times \check{R}) \setminus \mathrm{diag}$ induced by the order of the two factors, where the first factor corresponds to the position of the vertex where the three edges start, as shown in the following picture.



For an integer $k \in \mathbb{N}$, set $\underline{k} = \{1, 2, \dots, k\}$.

Definition 2.9 The *degree* of a Jacobi diagram is half the number of all its vertices. A *numbered degree* n *Jacobi diagram* is a degree n Jacobi diagram Γ whose edges are oriented, equipped with an injection $j_E \colon E(\Gamma) \hookrightarrow \underline{3n}$. Such an injection numbers the edges. Note that this injection is a bijection when $U(\Gamma)$ is empty. Let $\mathcal{D}^e_n(\mathcal{L})$ denote the set of numbered degree n Jacobi diagrams with support $\mathcal{L}$ without *looped edges* like .

Examples 2.10



Definition 2.11 Let Γ be a numbered degree n Jacobi diagram with support $\mathcal{L}$. An edge e oriented from a vertex v_1 to a vertex v_2 of Γ induces the following canonical map

$$\begin{array}{rccc} p_e \colon & \check{C}(R, L; \Gamma) & \to & C_2(R) \\ & c & \mapsto & (c(v_1), c(v_2)). \end{array}$$

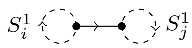
Let $o(\Gamma)$ be a vertex-orientation of Γ. For any $i \in \underline{3n}$, let $\omega(i)$ be a propagating form of $(C_2(R), \tau)$. Define the *configuration space integral*

$$I\left(R, L, \Gamma, o(\Gamma), (\omega(i))_{i \in \underline{3n}}\right) = \int_{(\check{C}(R,L;\Gamma), o(\Gamma))} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$$

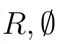
where $(\check{C}(R, L; \Gamma), o(\Gamma))$ denotes the manifold $\check{C}(R, L; \Gamma)$ equipped with the orientation induced by $o(\Gamma)$ and by the edge-orientation of Γ, as in Corollary 2.7.

Note that the dimension of the space $\check{C}(R, L; \Gamma)$ is equal to the degree of the integrated form $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$ since both coincide with the number of half-edges of Γ.

Examples 2.12 For any three propagating forms $\omega(1)$, $\omega(2)$ and $\omega(3)$ of $(C_2(R),\tau)$,

$$I(R, K_i \sqcup K_j \colon S^1_i \sqcup S^1_j \hookrightarrow \check{R},$$  $$, (\omega(i))_{i\in\underline{3}}) = lk(K_i, K_j)$$

and

$$I(R, \emptyset,$$  $$, (\omega(i))_{i\in\underline{3}}) = \Theta(R,\tau)$$

for any numbering of the (plain) diagrams.

Definition 2.13 The involution $(x,y)\mapsto(y,x)$ of $\check{R}^2 \setminus \mathrm{diag}\left(\check{R}^2\right)$ extends to an involution ι of $C_2(R)$. A propagating form ω of $(C_2(R),\tau)$ is *antisymmetric* if $\iota^*(\omega) = -\omega$.

Recall that ι_{S^2} denotes the antipodal map of S^2. Since $\iota^*_{S^2}(\omega_{S^2}) = -\omega_{S^2}$, the standard propagating form $p^*_{S^2}(\omega_{S^2})$ of $(C_2(S^3),\tau_s)$ is antisymmetric. When the $\omega(i)$ are antisymmetric, $I\left(R, L, \Gamma, o(\Gamma), (\omega(i))_{i\in\underline{3n}}\right)$ is independent of the orientation of the edges of Γ. Indeed, reversing the orientation of an edge changes the orientation of the configuration space and multiplies the integrated form by (-1). For any propagating form ω of $(C_2(R),\tau)$, $\frac{1}{2}(\omega - \iota^*(\omega))$ is an antisymmetric propagating form ω of $(C_2(R),\tau)$.

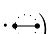
When all the $\omega(i)$ coincide with a given propagating form ω, $I\left(R, L, \Gamma, o(\Gamma), (\omega(i))_{i\in\underline{3n}}\right)$ is simply denoted by $I\left(R, L, \Gamma, o(\Gamma), \omega\right)$. When $\check{R} = \mathbb{R}^3$, and when $\omega = p^*_{S^2}(\omega_{S^2})$, we simply write $I\left(L, \Gamma, o(\Gamma)\right)$ and we also omit $o(\Gamma)$ when Γ is oriented by a picture.

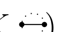
The study of these configuration space integrals was initiated by the articles of Witten [Wit89], Guadagnini, Martellini and Mintchev [GMM90], Bar-Natan [BN95b] on the perturbative expansion of the Chern-Simons theory,[15] in the case of links in $\mathbb{R}^3$, with the standard propagator $p^*_{S^2}(\omega_{S^2})$ on every edge. Let us compute some examples in this original setting.

2.2 Configuration space integrals associated to one chord

Let $K\colon S^1 \hookrightarrow \check{R}$ be a smooth embedding of the circle into $\check{R}$.

Consider the associated *configuration space*

$$\check{C}(K;$$  $$) = \{(K(z), K(z\exp(2i\pi t))) \mid z\in S^1, t\in]0,1[\},$$

which is naturally identified with an open annulus $S^1\times]0,1[$, and set $I_\theta(K) = I\left(K,\right.$  $)$.

When $\check{R} = \mathbb{R}^3$, the *direction map*

$$\begin{array}{llll} d\colon & \check{C}(K;\Gamma) & \to & S^2 \\ & (z,t) & \mapsto & \frac{1}{\|K(z\exp(2i\pi t)) - K(z)\|}(K(z\exp(2i\pi t)) - K(z)) \end{array}$$

allows us to write

$$I_\theta(K) = I\left(K,\right.$$  $$\left.\right) = \int_{\check{C}(K;\,\cdot)} d^*(\omega_{S^2}).$$ 

[15]The relation between the perturbative expansion of the Chern-Simons theory of the Witten article and the configuration space integral viewpoint is explained by Polyak in [Pol05] and by Sawon in [Saw06].

The annulus $\check{C}(K;\leftrightarrow)$ can be compactified to the closed annulus $C(K;\leftrightarrow) = S^1 \times [0,1]$, to which d extends smoothly. The extended d, also denoted by d, maps $(z,0) \in S^1 \times \{0\}$ (resp. $(z,1) \in S^1 \times \{1\}$) to the direction of the tangent vector to K at z (resp. to the opposite direction).

In particular, our integral $I_\theta(K)$ converges. It is the *algebraic area* $\int_{d(C(K;\leftrightarrow))} \omega_{S^2}$ of $d(C(K;\leftrightarrow))$ in the following sense. The degree of d is a continuous map from $S^2 \setminus d(\partial C(K;\leftrightarrow))$ to $\mathbb{Z}$, and the algebraic area of $d(C(K;\leftrightarrow))$ is $\int_{S^2} \deg(d)\omega_{S^2}$, which is the sum over the connected components C of $S^2 \setminus d(\partial C(K;\leftrightarrow))$ of the area of C multiplied by the value of the degree at C. Let us compute it for the following embeddings of the trivial knot.

Let O be an embedding of the circle in the horizontal plane. The image under d of the whole annulus lies in the horizontal great circle of S^2. Its area is zero so that $I_\theta(O) = 0$.

Let K_1 and K_{-1} be embeddings of S^1, which project to the horizontal plane as in Figure 11, which lie in the horizontal plane everywhere except when they cross over, and which lie in the union of two orthogonal planes.

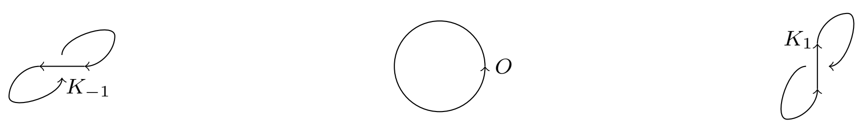
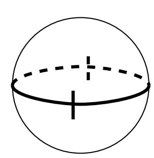


Figure 11: Diagrams of the trivial knot

The image of the boundary of $C(K_{\pm 1};\leftrightarrow) = S^1 \times [0,1]$ in S^2 lies in the union of the great circles of the two planes, or more precisely in the union of the horizontal great circle and two vertical arcs as in the following figure, where the vertical arcs are the images of the restriction to the portion of $K_{\pm 1}$ that crosses over of the direction of the tangent map to K, and of the opposite direction.



In our example with K_1, the degree is constant on each side of our horizontal equator. Computing it at the North Pole $\vec{N}$ as in Subsection 1.1, we find that the degree of d is 1 on the Northern Hemisphere. One computes the degree of d on the Southern Hemisphere similarly. It is also 1.

Therefore, $I_\theta(K_1) = 1$. Similarly, $I_\theta(K_{-1}) = -1$.

An *isotopy* between two knot embeddings K and K_1 is smooth map $\psi \colon [0,1] \times S^1 \to \mathbb{R}^3$ such that the restriction $\psi(t,.)$ of ψ to $\{t\} \times S^1$ is a knot embedding for any $t \in [0,1]$, $\psi(0,.) = K$ and $\psi(1,.) = K_1$. When there exists such an isotopy, K and K_1 are said to be *isotopic* or in the same *isotopy class*. A *knot* is an isotopy class of knot embeddings. For example, K_1, K_{-1} and

O are in the same isotopy class. They represent the same knot. Therefore, I_θ is not invariant under isotopy.

Definition 2.14 A knot embedding K that lies in the union of the horizontal plane and a finite union of vertical planes so that the unit tangent vector to K is never vertical is called *almost-horizontal*. An almost-horizontal embedding K has a natural parallel $K_\parallel$ obtained from K by pushing it down. An embedding from S^1 to $\mathbb{R}^3$ is of *constant (resp. null) I_θ-degree* if the degree of the associated direction map $(d\colon \check{C}(K;\leftrightarrow) \to S^2)$ can be extended to a constant (resp. everywhere 0) function on S^2.

Lemma 2.15 *Almost-horizontal knot embeddings have constant I_θ-degree. Any knot of $\mathbb{R}^3$ may be represented by an almost-horizontal knot embedding K. For an almost-horizontal knot embedding K, $I_\theta(K) = lk(K, K_\parallel)$.*

PROOF: The *writhe* of an almost-horizontal knot embedding is the number of positive crossings minus the number of negative crossings of its orthogonal projection onto the horizontal plane. As in the previous examples, we see that an almost-horizontal knot embedding has a constant I_θ-degree, which is its *writhe*. The parallel below $K_\parallel$ is isotopic in the complement of K to the parallel $K_{\parallel,\ell}$ on the left-hand side of K, and the formulas of Section 1.1 show that $lk(K, K_{\parallel,\ell})$ is the writhe of K. □

It is easy to construct an embedding of null I_θ-degree in every isotopy class of embeddings of S^1 into $\mathbb{R}^3$, by adding kinks such as or to a horizontal projection. Since I_θ varies continuously under an isotopy of K, for any knot K of $\mathbb{R}^3$, I_θ maps the space of embeddings of S^1 into $\mathbb{R}^3$ isotopic to K onto $\mathbb{R}$.

For a *long component* (i.e. a non-compact connected component) K of a long tangle representative in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, define

$$I_\theta(K) = 2I\left(K, \hat{\vdots}, p^*_{S^2}(\omega_{S^2})\right) = 2I\left(K, \hat{\vdots}\right).$$

Examples 2.16 Let us compute $I_\theta(K_{\ell,i}) = 2I\left(K_{\ell,i}, \hat{\vdots}, \omega\right)$ for the long tangles of Figure 12, which shows their projections onto the plane $\mathbb{R} \times \mathbb{R} \subset \mathbb{C} \times \mathbb{R}$. Assume that the images of the embeddings lie in this plane everywhere, except when they cross over, so that the image of each one-component tangle lies again in the union of two orthogonal planes.

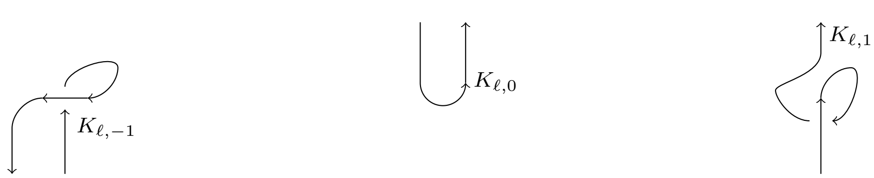


Figure 12: Long tangle representatives

The *configuration space* $\check{C}(K = K_{\ell,i}; \hat{\circlearrowleft})$ associated to $\Gamma = \hat{\circlearrowleft}$ and to $K\colon \mathbb{R} \hookrightarrow \mathbb{R}^3$ is

$$\check{C}(K; \hat{\circlearrowleft}) = \{(K(t), K(u)) \mid (t,u) \in \mathbb{R}^2, t < u\},$$

which is naturally identified with the open triangle $\{(t,u) \in \mathbb{R}^2, t < u\}$. The *direction map*

$$\begin{array}{llll} d\colon & \check{C}(K; \hat{\circlearrowleft}) & \to & S^2 \\ & (K(t), K(u)) & \mapsto & \frac{1}{\| K(u) - K(t) \|}(K(u) - K(t)) \end{array}$$

allows us to write

$$I_\theta(K) = 2I\left(K, \hat{\circlearrowleft}\right) = 2\int_{\check{C}(K; \hat{\circlearrowleft})} d^*(\omega_{S^2}).$$

Again, since $K_{\ell,0}$ is contained in $\mathbb{R} \times \mathbb{R}$, d maps $\check{C}(K_{\ell,0}; \hat{\circlearrowleft})$ to the vertical great circle $S^1_{\mathbb{R}}$ that contains the real direction of $\mathbb{C}$ and $I_\theta(K_{\ell,0}) = 0$.

The configuration space $\check{C}(K_{\ell,1}; \hat{\circlearrowleft})$ embeds in the closed triangle

$$\tilde{C}(K_{\ell,1}; \hat{\circlearrowleft}) = \{(t,u) \in [-\infty, \infty]^2 \mid t \leq u\} = ◤ ,$$

where d extends. The extended d maps $(\{-\infty\} \times [-\infty, \infty]) \cup ([-\infty, \infty] \times \{\infty\})$ to the vertical upward vector $\vec{N}$, and it maps (u,u) to the unit tangent vector to K at u directed by $\mathbb{R}$. So far, this applies to any long K that goes from bottom to top. For our $K_{\ell,1}$, d maps the boundary of the triangle to the union of $S^1_{\mathbb{R}}$ and an arc of an orthogonal great circle. Here, the degree of d is 1 on the hemisphere behind $S^1_{\mathbb{R}}$ and it is zero in front of it so that $\int_{\check{C}(K_{\ell,1}; \hat{\circlearrowleft})} d^*(\omega_{S^2}) = \frac{1}{2}$ and $I_\theta(K_{\ell,1}) = 1$.

Let us now compute $I_\theta(K_{\ell,-1}) = -1$. In this case, $\check{C}(K_{\ell,-1}; \hat{\circlearrowleft})$ still embeds in the former closed triangle, but the map d does not extend continuously at $(-\infty, \infty)$. It extends to $\{-\infty\} \times [-\infty, \infty[$ and it maps $\{-\infty\} \times [-\infty, \infty[$ to $\vec{N}$, and it extends to $]-\infty, \infty] \times \{\infty\}$ and it maps $]-\infty, \infty] \times \{\infty\}$ to $(-\vec{N})$, but we need to blow up the triangle at $(-\infty, \infty)$ so that d extends. After such a blow-up, which transforms the closed triangle into $\tilde{\tilde{C}}(K_{\ell,-1}; \hat{\circlearrowleft})$, (the extension of) d maps the boundary of $\tilde{\tilde{C}}(K_{\ell,-1}; \hat{\circlearrowleft})$ to the union of $S^1_{\mathbb{R}}$ and an arc of a great circle. Here, the degree of d is -1 on the hemisphere in front of $S^1_{\mathbb{R}}$ and it is zero behind so that $I_\theta(K_{\ell,-1}) = -1$.

Definition 2.17 A propagating form of $(C_2(R), \tau)$ is *homogeneous* if its restriction to $\partial C_2(R)$ is equal to $p_\tau^*(\omega_{S^2})$ for the homogeneous volume-one form ω_{S^2} of S^2.

Lemma 2.18 *Let $K\colon \mathbb{R} \hookrightarrow \check{R}$ be a component of a long tangle representative in an asymptotic rational homology $\mathbb{R}^3$. Let ω be a homogeneous propagating form of $(C_2(R), \tau)$. Then $I(R, K, \hat{\circlearrowleft}, \omega)$ is independent of the chosen homogeneous propagating form ω. (It depends on the embedding K and on τ.) It is denoted by $\frac{1}{2}I_\theta(K, \tau)$.*

See [Les20, Lemma 12.5 and Definition 12.6].

2.3 More examples of configuration space integrals

Examples 2.19 For any trivalent numbered degree n Jacobi diagram

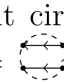
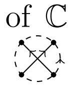
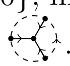
$$I(\Gamma) = I\left(S^3, \emptyset, \Gamma, o(\Gamma)\right) = 0.$$

Indeed, $I(\Gamma)$ is equal to

$$\int_{(\check{C}(S^3,\emptyset;\Gamma),o(\Gamma))} \left(\prod_{e\in E(\Gamma)} p_{S^2}\circ p_e\right)^* \left(\bigwedge_{e\in E(\Gamma)} \omega_{S^2}\right)$$

where

- $\bigwedge_{e\in E(\Gamma)} \omega_{S^2}$ is a product volume form of $(S^2)^{E(\Gamma)}$ with total volume one.
- $\check{C}(S^3,\emptyset;\Gamma)$ is the space $\check{C}_{\underline{2n}}(\mathbb{R}^3)$ of injections of $\underline{2n}$ into $\mathbb{R}^3$,
- the degree of $\wedge_{e\in E(\Gamma)}\omega_{S^2}$ is equal to the dimension of $\check{C}(S^3,\emptyset;\Gamma)$, and
- the map $\left(\prod_{e\in E(\Gamma)} p_{S^2}\circ p_e\right)$ is never a local diffeomorphism since it is invariant under the action of global translations on $\check{C}(S^3,\emptyset;\Gamma)$.

Examples 2.20 Let us now compute $I\left(O,\Gamma,o(\Gamma),p^*_{S^2}(\omega_{S^2})\right)$, where O denotes the representative of the unknot of S^3, that is the image of the embedding of the unit circle S^1 of $\mathbb{C}$, regarded as $\mathbb{C}\times\{0\}$, into $\mathbb{R}^3$, regarded as $\mathbb{C}\times\mathbb{R}$, for the following graphs $\Gamma_1 =$ , $\Gamma_2 =$ , $\Gamma_3 =$ , $\Gamma_4 =$ . For $i \in \underline{4}$, set $I(\Gamma_i) = I\left(S^3,O,\Gamma_i,o(\Gamma_i),p^*_{S^2}(\omega_{S^2})\right)$. Let us prove that $I(\Gamma_1) = I(\Gamma_2) = I(\Gamma_3) = 0$ and that $I(\Gamma_4) = \frac{1}{8}$.

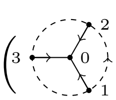
For $i \in \underline{4}$, set $\Gamma = \Gamma_i$, $I(\Gamma)$ is equal to

$$\int_{(\check{C}(S^3,O;\Gamma),o(\Gamma))} \left(\prod_{e\in E(\Gamma)} p_{S^2}\circ p_e\right)^* \left(\bigwedge_{e\in E(\Gamma)} \omega_{S^2}\right).$$

When $i \in \underline{2}$, the image of $\prod_{e\in E(\Gamma)} p_{S^2}\circ p_e$ lies in the subset of $(S^2)^2$ consisting of the pair of horizontal vectors. Since the interior of this subset is empty, $I(\Gamma_i) = 0$. When $i = 3$, the two edges that have the same endpoints must have the same direction so that the image of $\prod_{e\in E(\Gamma)} p_{S^2}\circ p_e$ lies in the subset of $(S^2)^{E(\Gamma)}$ for which two S^2-coordinates are identical (namely those in the S^2-factors corresponding to that pair of edges), and $I(\Gamma_3) = 0$ as before.

Let us finish this series of examples by proving the following lemma.

Lemma 2.21 *Let $\Gamma = \Gamma_4$. Then*

$I(\Gamma_4) = I\Big($ 

$\Big) = I\left(S^3,O,\Gamma,o(\Gamma),p^*_{S^2}(\omega_{S^2})\right) = \dfrac{1}{8}.$

Proof: Let G^+ be the set of direct triples (X_{10}, X_{20}, X_{30}) of $(S^2)^3$ where all vectors have positive heights. Recall that ι_{S^2} is the antipodal map of S^2 and let $G^- = (\iota_{S^2})^E(G^+)$. Let D be the codimension-one subspace of $(S^2)^3$ of triples of vectors such that at least one of the vectors is horizontal or the three vectors are coplanar. For any edge e, let d_e denote $p_{S^2} \circ p_e$. It is easy to see that the image of $\check{C}(K;\Gamma)$ under $\left(\prod_{e\in E} d_e\right)$ is contained in $G^+ \cup G^- \cup D$ and that the restriction of $\left(\prod_{e\in E} d_e\right)$ to the preimage of G^+ is a diffeomorphism h^+ onto G^+. Using the orientation-reversing diffeomorphism h_c of $\check{C}(K;\Gamma)$ that maps a configuration to its composition by $(-\mathrm{Id}_{\mathbb{R}^3})$, it is also clear that the restriction of $\left(\prod_{e\in E} d_e\right)$ to the preimage of G^- is the diffeomorphism $(\iota_{S^2})^E \circ h^+ \circ h_c$ onto G^-. In particular, the degree of $\left(\prod_{e\in E} d_e\right)$ is well defined on $(S^2)^E \setminus D$, it is ± 1 on $G^+ \cup G^-$, with the same sign on G^+ and G^-, and 0 elsewhere. The sign is computed in the proof of [Les20, Lemma 7.12], and the degree is 1 on G^+. This shows that $I_\Gamma(O)$ is twice the volume of G^+, so that $I_\Gamma(O) = \frac{1}{8}$. □

2.4 More compactifications of configuration spaces

Axelrod, Singer [AS94] and Kontsevich [Kon94] proved that the configuration space integrals $I\left(R, L, \Gamma, o(\Gamma), (\omega(i))_{i\in\underline{3n}}\right)$ converge, when $\mathcal{L}$ is a disjoint union of circles, using compactifications $C(R, L;\Gamma)$ “à la Fulton-MacPherson” of $\check{C}(R, L;\Gamma)$, where the maps $p_e\colon \check{C}(R, L;\Gamma) \to C_2(R)$ extend smoothly so that $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$ extends smoothly to $C(R, L;\Gamma)$, and

$$\int_{(\check{C}(R,L;\Gamma),o(\Gamma))} \bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e))) = \int_{(C(R,L;\Gamma),o(\Gamma))} \bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e))).$$

These compactifications are constructed as follows in [Les20, Chapter 8]. We first generalize the constructions of $C_2(R)$ and define a compactification $C_V(R)$ of the space $\check{C}_V(R)$ of injections of a finite set V into $\check{R}$ as in [Les20, Theorem 8.4] as follows. For a non-empty $A \subseteq V$, let Ξ_A be the set of maps from V to R that map A to ∞ and $V \setminus A$ to $\check{R}$ injectively, and let $\mathrm{diag}_A(\check{R}^V)$ be the set of maps c from V to $\check{R}$, which are constant on A and which map $V \setminus A$ to $\check{R} \setminus \{c(A)\}$ injectively.

Start with R^V. Blow up Ξ_V (which is reduced to the point $m = \infty^V$ such that $m^{-1}(\infty) = V$). Then for $k = \sharp V, \sharp V - 1, \dots, 3, 2$, in this decreasing order, successively blow up the closures of the $\mathrm{diag}_A(\check{R}^V)$ such that $\sharp A = k$ (choosing an arbitrary order among them) and, next, the closures of the Ξ_J such that $\sharp J = k - 1$ (again choosing an arbitrary order among them). Then the compactification $C(R, L;\Gamma)$ is the closure of $\check{C}(R, L;\Gamma)$ in $C_{V(\Gamma)}(R)$ as in [Les20, Proposition 8.6]. It satisfies the following properties.

Theorem 2.22 *If L is a link, then the configuration space $C(R, L;\Gamma)$ is a compact manifold with boundary and corners with the following properties.*

- *The interior of $C(R, L;\Gamma)$, which is the complement of $\partial C(R, L;\Gamma)$, is $\check{C}(R, L;\Gamma)$.*

- *For any edge e of Γ, the projection map $p_e \colon \check{C}(R,L;\Gamma) \to C_2(R)$ extends smoothly*[16] *to $C(R,L;\Gamma)$.*

- *For every non-empty subset A of $T(\Gamma)$, there is a codimension-one open face $F_\infty(A,L,\Gamma)$ of $C(R,L;\Gamma)$ which may be identified with the product of*

$$\{c \colon (V \setminus A) \hookrightarrow \check{R} \mid c_{|U} = L \circ j_\Gamma(c) \text{ for some } j_\Gamma(c) \in [i_\Gamma]\}$$

by the space $\check{\mathcal{S}}(\mathbb{R}^3, A)$ of injective *maps w from A to $(\mathbb{R}^3 \setminus 0)$ up to dilation*[17]*, so that an element $(c,[w])$ of this face is the limit in $C(R,L;\Gamma)$ when u tends to 0 of a family of injective configurations $(c, \frac{1}{u}w)_{u \in]0,\varepsilon[}$, which is defined for some small $\varepsilon > 0$, for a representative w of $[w]$.*

- *For every subset A of cardinality greater than 2 of $V(\Gamma)$ that intersects $U = U(\Gamma)$ as a (possibly empty) set of consecutive vertices on some component of $\mathcal{L}$ with respect to $[i_\Gamma]$, there is a codimension-one open face $F(A,L,\Gamma)$ which behaves as follows. Let $a \in A$ be such that $a \in A \cap U$ if $A \cap U \neq \emptyset$. Then $F(A,L,\Gamma)$ fibers over*

$$\{c \colon (V \setminus A) \cup \{a\} \hookrightarrow \check{R} \mid c_{|(U \setminus (U \cap (A \setminus \{a\}))} = L \circ j_\Gamma(c)_{|(U \setminus (U \cap (A \setminus \{a\}))} \text{ for some } j_\Gamma(c) \in [i_\Gamma]\}.$$

  - *If $A \cap U = \emptyset$, then the fiber is the space $\check{\mathcal{S}}_A(T_{c(a)}\check{R})$ made of* injective *maps w_A from A to $T_{c(a)}\check{R}$ up to translation and dilation. When $\check{R} = \mathbb{R}^3$, an element $(c,[w_A])$ of this face is the limit in $C(R,L;\Gamma)$ when u tends to 0 of a family of injective configurations $(c + uw_A)_{u \in]0,\varepsilon[}$, which is defined for some small $\varepsilon > 0$, where w_A is a representative of w_A which maps a to zero, and c and w_A are extended to V so that c is constant on A and w_A maps $V \setminus A$ to 0.*
  - *If $A \cap U \neq \emptyset$, then the fiber over c is the space of injective maps w_A from A to $T_{c(a)}\check{R}$ which map $A \cap U$ to the line $\mathbb{R}T_{c(a)}L$ through 0 directed by the tangent vector $T_{c(a)}L$ to L at $c(a)$, with respect to an order compatible with i_Γ, up to dilation and translation along the line $\mathbb{R}T_{c(a)}L$.*

- *The complement of the union of the faces described above in the boundary of $C(R,L;\Gamma)$ is a finite union of manifolds of codimension at least 2 in $C(R,L;\Gamma)$.*

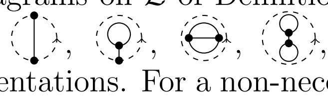
These faces are described more precisely in [Les20, Section 8.4]. Bott and Taubes analyzed the variations of the integrals $I_\Gamma(K)$ when a knot K of $\mathbb{R}^3$ varies in its isotopy class in [BT94], using such compactifications together with their codimension-one faces, described above, which correspond to the loci where one blow-up has been performed.

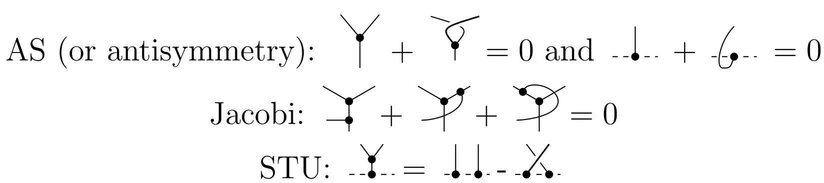
In [Poi00], Sylvain Poirier used the theory of semi-algebraic sets [BCR98] to prove the convergence of the integrals for semi-algebraic long tangle representatives in $\mathbb{R}^3$. He proved that

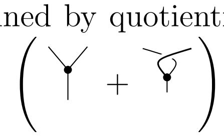
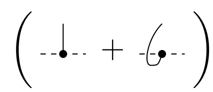
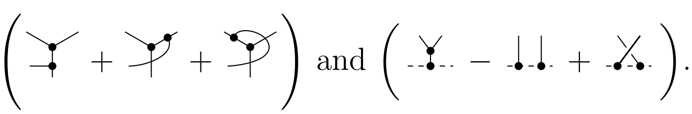
[16] See [Les20, Theorem 8.5].
[17] Dilations are homotheties with positive ratio.

the closure $C(L;\Gamma)$ of $\check{C}(L;\Gamma)$ in $C_{V(\Gamma)}(S^3)$ is a semi-algebraic set for semi-algebraic long tangle representatives L in $\mathbb{R}^3$. In [Les20, Chapter 14], I proved the convergence of the integrals for all long tangle representatives L in $\mathbb{Q}$–spheres [Les20, Theorem 12.2] by studying the structure of the closure of $\check{C}(R,L;\Gamma)$ in $C_{V(\Gamma)}(R)$. This closure is no longer a manifold. See [Les20, Theorem 14.16].

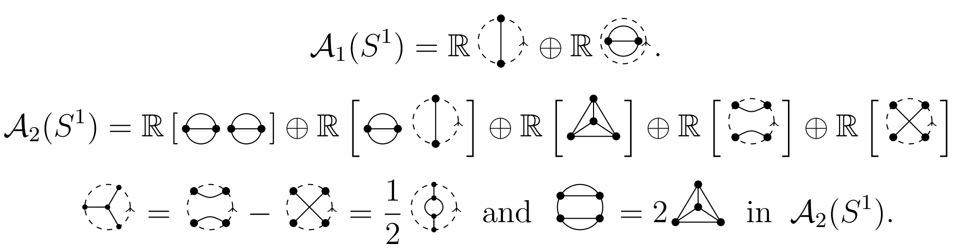
2.5 The invariant Z

For a one-manifold $\mathcal{L}$, $\mathcal{D}_n(\mathcal{L})$ denotes the real vector space generated by the degree n oriented Jacobi diagrams on $\mathcal{L}$ of Definition 2.2. For the circle S^1, these generators of $\mathcal{D}_1(S^1)$ are the diagrams , and the diagrams obtained from them by changing some vertex orientations. For a non-necessarily oriented one-manifold $\mathcal{L}$, $\mathcal{A}_n(\mathcal{L})$ denotes the quotient of $\mathcal{D}_n(\mathcal{L})$ by the following relations AS, Jacobi and STU:

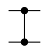
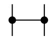


Each of these relations relate oriented Jacobi diagrams which are identical outside the pictures (or, more exactly, which can be represented by planar immersions whose images intersect a disk as in the picture and are identical outside this disk). The quotient $\mathcal{A}_n(\mathcal{L})$ is the largest quotient of $\mathcal{D}_n(\mathcal{L})$ in which these relations hold. It is obtained by quotienting $\mathcal{D}_n(\mathcal{L})$ by the vector space generated by elements of $\mathcal{D}_n(\mathcal{L})$ of the form ,  and .

Examples 2.23 Note that diagrams with looped edges vanish in $\mathcal{A}_n(\mathcal{L})$.

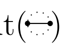
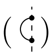
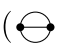
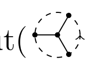


Remark 2.24 When $\partial\mathcal{L} = \emptyset$, Lie algebras provide nontrivial linear maps, called *weight systems* from $\mathcal{A}_n(\mathcal{L})$ to $\mathbb{K}$, see [BN95a], [CDM12, Chapter 6] or [Les05, Section 6]. In the weight system constructions, the Jacobi relation for the Lie bracket ensures that the maps defined for oriented

Jacobi diagrams factor through the Jacobi relation. In [Vog11], Pierre Vogel proved that the maps associated with Lie (super)algebras are sufficient to detect nontrivial elements of $\mathcal{A}_n(\mathcal{L})$ up to degree 15, and he exhibited a non-trivial element of $\mathcal{A}_{16}(\emptyset)$ that cannot be detected by such maps. The Jacobi relation was originally called IHX by Bar-Natan in [BN95a] because, up to AS, it can be written as  $=$  $-$ . Note that the four entries in this IHX relation play the same role, up to AS.

Let $\mathcal{D}^u_n(\mathcal{L})$ denote the set of unnumbered, unoriented degree n Jacobi diagrams on $\mathcal{L}$ without looped edges. Note that the product $I(R, L, \Gamma, \omega)[\Gamma]$ is independent of the orientation of Γ for an antisymmetric propagating form of $(C_2(R), \tau)$.

An *automorphism* of a graph $\Gamma \in D^u_n(\mathcal{L})$ is an automorphism of the underlying uni-trivalent graph, for which the permutation σ of $U(\Gamma)$ induced by the automorphism is such that $i_\Gamma \circ \sigma$ and i_Γ are isotopic for some (and thus any) Γ-compatible injection i_Γ. Let $\mathrm{Aut}(\Gamma)$ denote the set of these automorphisms, and let $\sharp\mathrm{Aut}(\Gamma)$ denote its cardinality.

Examples 2.25 The cardinality of Aut() is 2, $\sharp$Aut() $= 1$, $\sharp$Aut() $= 12$, $\sharp$Aut() $= 3$.

The following theorem is a consequence of [Les20, Theorem 7.20 and Proposition 7.26], when L is a link and of [Les20, Theorem 12.7, Theorem 12.13 and Lemma 13.7] in general.

Theorem 2.26 *Let L be a long tangle representative in $\check{R}$. Let L_C denote the set of connected components of L. Let ω be an antisymmetric homogeneous propagating form of $(C_2(R), \tau)$. Then*

$$Z_n(\check{R}, L, (I_\theta(K, \tau))_{K \in L_C}, p_1(\tau)) = \sum_{\Gamma \in \mathcal{D}^u_n(\mathcal{L})} \frac{1}{\sharp Aut(\Gamma)} I(R, L, \Gamma, \omega)[\Gamma] \in \mathcal{A}_n(\mathcal{L})$$

depends only on

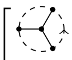
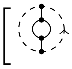
- *the pair $(\mathcal{C}, L \cap \mathcal{C})$ up to orientation-preserving diffeomorphisms[18] of $\mathcal{C}$ which preserve the bottom disk $D^2 \times \{0\}$ and the top disk $D^2 \times \{1\}$, and which preserve $c^+(B^+)$ and $c^-(B^-)$ up to (global) translation and dilation,*
- *$I_\theta(K, \tau)$ for each component K of L,*
- *$p_1(\tau)$,*

where $I(R, L, \Gamma, \omega)[\Gamma] = I(R, L, \Gamma, o(\Gamma), \omega)[\Gamma, o(\Gamma)]$ for an arbitrary orientation of Γ.

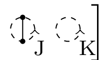
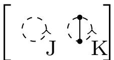
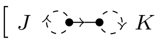
Note that the above definition of $I(R, L, \Gamma, \omega)[\Gamma]$ is consistent because the right-hand side of the above equality does not depend on $o(\Gamma)$. Also note that when L is an almost-horizontal knot K of $\mathbb{R}^3$ as in Definition 2.14, $Z_n(\mathbb{R}^3, K, I_\theta(K, \tau_s), p_1(\tau_s) = 0)$ depends only on $I_\theta(K, \tau_s) = lk(K, K_\parallel)$ (see Lemma 2.15) and on the isotopy class of K, so that Z_n induces an isotopy invariant of parallelized knots in $\mathbb{R}^3$.

[18] As often in these notes, we identify an embedding and its image.

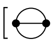
Examples 2.27 For the empty link $\emptyset$ of $\mathbb{R}^3$, $Z_n(\mathbb{R}^3, \emptyset, 0) = 0$ for all $n > 0$ and $Z_0(\mathbb{R}^3, \emptyset, 0) = [\emptyset]$. For the knot O of Example 2.20, $Z_0(\mathbb{R}^3, O, 0) = [$  $]$, $Z_1(\mathbb{R}^3, O, 0) = 0$ and

$$Z_2(\mathbb{R}^3, O, 0) = \frac{1}{24}\left[\right.$$  $$\left.\right] = \frac{1}{48}\left[\right.$$  $$\left.\right].$$

For any two-component link $J \sqcup K$ of $\mathbb{R}^3$ such that J and K are almost-horizontal,

$$Z_1(\mathbb{R}^3, J \sqcup K, 0) = \frac{1}{2} lk(J, J_\parallel)\left[\right.$$  $$\left.\right] + \frac{1}{2} lk(K, K_\parallel)\left[\right.$$  $$\left.\right] + lk(J, K)\,[$$  $$].$$

If $(\check{R}, \tau)$ is a parallelized asymptotic rational homology $\mathbb{R}^3$, then

$$Z_1(\check{R}, \emptyset, p_1(\tau)) = \frac{\Theta(R, \tau)}{12}\,[$$  $$].$$

Remark 2.28 Let ω be an antisymmetric homogeneous propagating form of $(C_2(R), \tau)$. The homogeneous definition of $Z_n(\check{R}, L, .)$ above makes clear that $Z_n(\check{R}, L, .)$ is a measure of graph configurations, where a graph configuration is an embedding of the set of vertices of a uni-trivalent graph into $\check{R}$, which maps univalent vertices to $\mathcal{L}(L)$ in a constrained way. The embedded vertices are connected by a set of abstract plain edges, which represent the measuring form. The factor $\frac{1}{\sharp \mathrm{Aut}(\Gamma)}$ ensures that every such configuration of an unnumbered, unoriented graph is measured exactly once.

Definition 2.29 A one-cycle c of S^2 is *algebraically trivial* if, for any two points x and y outside its support, the algebraic intersection of an arc from x to y transverse to c with c is zero, or, equivalently, if the integral of any one–form of S^2 along c is zero. A link embedding L is *straight* (with respect to τ) if the image $p_\tau(U^+K)$ of the direction of the tangent map to any component K of L is an algebraically trivial cycle of S^2. A straight knot embedding K can be parallelized (or framed) by pushing it in a direction $\tau(X)$ for some $X \in S^2 \setminus (p_\tau(U^+K) \cup \iota_{S^2}(p_\tau(U^+K)))$. As a consequence of [Les20, Lemma 7.35], the isotopy class of the obtained parallel $K_{\parallel,\tau}$ is independent of such an X, and we have the following lemma.

Lemma 2.30 *For any component K of a straight link embedding, $I_\theta(K, \tau) = lk(K, K_{\parallel,\tau})$.*

Note that for any link representative L in $\mathring{\mathcal{C}} \subset (\check{R} = R(\mathcal{C}))$, and for any asymptotically standard parallelization τ_0 of $\check{R}$, there is a parallelization τ homotopic to τ_0 among asymptotically standard parallelizations τ_0 of $\check{R}$ such that $p_\tau(U^+K) = \{\vec{N}\}$ for any component K of L, so that L is straight with respect to τ.

For a degree n Jacobi diagram Γ on $\mathcal{L}$, set

$$\zeta_\Gamma = \frac{(3n - \sharp E(\Gamma))!}{(3n)!\,2^{\sharp E(\Gamma)}}.$$

Theorem 2.31 *Let $L\colon \mathcal{L} \hookrightarrow \check{R}$ be a straight link embedding with respect to τ in $(\check{R},\tau)$, which is our asymptotic rational homology $\mathbb{R}^3$. For any $i \in \underline{3n}$, let $\omega(i)$ be a propagating form of $(C_2(R),\tau)$, and let P_i be a propagating chain of $(C_2(R),\tau)$. With the notation of Definition 2.9 and Theorem 2.26,*

$$\begin{aligned} Z_n(\check{R}, L, (lk(K, K_{\parallel,\tau}))_{K\in L_C}, p_1(\tau)) &= \sum_{\Gamma\in\mathcal{D}^e_n(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i\in\underline{3n}})[\Gamma] \\ &= \sum_{\Gamma\in\mathcal{D}^e_n(\mathcal{L})} \zeta_\Gamma [\bigcap_{e\in E(\Gamma)} p_e^{-1}(P_{j_E(e)})][\Gamma], \end{aligned}$$

whenever all the above intersections are transverse, and they are for generic choices of $(P_i)_{i\in\underline{3n}}$. In particular, the right-hand sides do not depend on our choices and they are rational.

Proof: The first equality is a consequence of [Les20, Theorem 7.39]. The genericity of the statement is described in [Les20, Chapter 11]. See [Les20, Definition 11.3 and Lemma 11.4], in particular. The second equality is a consequence of [Les20, Lemma 11.7]. □

In the above statement, $[\bigcap_{e\in E(\Gamma)} p_e^{-1}(P_{j_E(e)})]$ is the algebraic intersection of the codimension 2 chains $p_e^{-1}(P_{j_E(e)})$ in $C(R,L;\Gamma)$. Theorem 2.31 may be applied to compute Z with the Morse propagators of Section 1.5. In this case Z counts graph embeddings where some edges embed in the flow lines (when the pairs of points are in the part P_ϕ of $P(f,\mathfrak{g})$) and some edges $e=(v,w)$ constrain their origin vertex to belong to some descending manifold $\mathcal{B}_j$ of an index 2 critical point and their final vertex to belong to some ascending manifold $\mathcal{A}_i$ of an index 1 critical point, up to some corrections due to the behaviour of $P(f,\mathfrak{g})$ near $\partial C_2(R)$. A similar way of counting graphs was proposed by Fukaya in [Fuk96] and further studied by Watanabe [Wat18].

The following consequence of Theorem 2.31 can be deduced from independent results of Sylvain Poirier [Poi02] and Dylan Thurston [Thu99] in the case of links in $\mathbb{R}^3$, with propagating chains $p_{S^2}^{-1}(X_i)$. For any edge e, let d_e denote $p_{S^2}\circ p_e$.

Theorem 2.32 *Let $L\colon \mathcal{L} \hookrightarrow \mathbb{R}^3$ be a straight link embedding into $\mathbb{R}^3$. The subset A of $(S^2)^{3n}$ consisting of the $(X_i)_{i\in\underline{3n}}$ such that $(X_{j_E(e)})_{e\in E(\Gamma)}$ is a regular value of $\prod_{e\in E(\Gamma)} d_e\colon C(L;\Gamma) \to (S^2)^{j_E(E(\Gamma))}$ for any $\Gamma\in\mathcal{D}^e_n(\mathcal{L})$ is open and dense, and, for any $(X_i)_{i\in\underline{3n}}\in A$,*

$$Z_n(\mathbb{R}^3, L, (lk(K,K_{\parallel,\tau}))_{K\in L_C}, 0) = \sum_{\Gamma\in\mathcal{D}^e_n(\mathcal{L})} \zeta_\Gamma [\bigcap_{e\in E(\Gamma)} d_e^{-1}(X_{j_E(e)})][\Gamma].$$

This theorem tells us that $Z_n(\mathbb{R}^3, L, (I_\theta(K))_{K\in L_C}, 0)$ behaves as an $\mathcal{A}_n(\mathcal{L})$–valued degree on $(S^2)^{3n}$ and it may be proved along the following lines. Associate the map $\Pi_\Gamma = \prod_{e\in E(\Gamma)} d_e \times \mathrm{Id}_{(S^2)^{\underline{3n}\setminus j_E(E(\Gamma))}}$ from $C(L;\Gamma)\times (S^2)^{\underline{3n}\setminus j_E(E(\Gamma))}$ to $(S^2)^{3n}$ to each $\Gamma\in\mathcal{D}^e_n(\mathcal{L})$, equipped with a fixed arbitrary orientation. By definition, for any such Jacobi diagram Γ equipped with an implicit vertex-orientation,

$$I\left(L,\Gamma,p_{S^2}^*(\omega_{S^2})\right) = \int_{\check{C}(L;\Gamma)} \bigwedge_{e\in E(\Gamma)} d_e^*(\omega_{S^2})$$

is the algebraic volume of the image of Π_Γ. The degree d_Γ of Π_Γ is a continuous function on the complement of $\Pi_\Gamma\left(\partial C(L;\Gamma)\times (S^2)^{\underline{3n}\setminus j_E(E(\Gamma))}\right)$ in $(S^2)^{3n}$. The degree d_Γ changes by

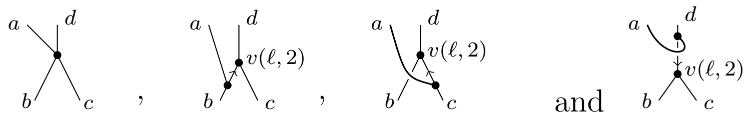


Figure 13: $\Gamma/e(\ell)$, Γ_{ab}, Γ_{ac} and Γ_{ad}, around the collapsing edge

± 1 across each wall, where a *wall* is a codimension-one image of a codimension-one face of $\Pi_\Gamma\left(C(L;\Gamma)\times(S^2)^{\underline{3n}\setminus j_E(E(\Gamma))}\right)$. Sylvain Poirier and Dylan Thurston proved independently that $D_n = \sum_{\Gamma\in\mathcal{D}^e_n(\mathcal{L})}\zeta_\Gamma d_\Gamma[\Gamma]$ can be extended to an $\mathcal{A}_n(\mathcal{L})$-valued constant function on $(S^2)^{3n}$ by gluing the above walls as in the example below.

Let $\Gamma\in\mathcal{D}^e_n(\mathcal{L})$. Let $e(\ell)$ be an edge of Γ with label ℓ, which goes from a vertex $v(\ell,1)$ to a vertex $v(\ell,2)$. Assume that no other edge of Γ contains both $v(\ell,1)$ and $v(\ell,2)$. Let $\Gamma/e(\ell)$ be the labelled edge-oriented graph obtained from Γ by contracting $e(\ell)$ to a point. The labels of the edges of $\Gamma/e(\ell)$ belong to $\underline{3n}\setminus\{\ell\}$, $\Gamma/e(\ell)$ has one four-valent vertex and its other vertices are univalent or trivalent. Let $\mathcal{E}=\mathcal{E}(\Gamma;e(\ell))$ be the set of pairs $(\tilde\Gamma,\tilde e(\ell))$ where $\tilde\Gamma\in\mathcal{D}^e_n(\mathcal{L})$ and $\tilde e(\ell)$ is an edge of $\tilde\Gamma$ with label ℓ such that $\tilde\Gamma/\tilde e(\ell)$ is equal to $\Gamma/e(\ell)$.

Let us show that there are 6 graphs in $\mathcal{E}$. Let a,b,c,d be the four half-edges of $\Gamma/e(\ell)$ that contain its four-valent vertex. In $\tilde\Gamma$, the edge $\tilde e(\ell)$ joins a vertex $v(\ell,1)$ to a vertex $v(\ell,2)$. The vertex $v(\ell,1)$ is adjacent to the first half-edge of $\tilde e(\ell)$ and to two half-edges of $\{a,b,c,d\}$. The unordered pair of $\{a,b,c,d\}$ adjacent to $v(\ell,1)$ determines $\tilde\Gamma$ as an element of $\mathcal{D}^e_n(\mathcal{L})$ and there are 6 elements in $\mathcal{E}$ labelled by the pairs of elements of $\{a,b,c,d\}$. They are $\Gamma=\Gamma_{ab}$, Γ_{ac}, Γ_{ad}, Γ_{bc}, Γ_{bd} and Γ_{cd}, equipped with the edge from $v(\ell,1)$ to $v(\ell,2)$. Three of them (Γ_{ab}, Γ_{ac} and Γ_{ad}) are drawn in Figure 13. The other ones are obtained from them by reversing the orientation of $\tilde e(\ell)$.

The face $F(\{v(\ell,1),v(\ell,2)\},L,\Gamma)$, where $e(\ell)$ collapses, is fibered over the configuration space of $\Gamma/e(\ell)$ with fiber S^2, which contains the (free) direction of the vector from $c(v(\ell,1))$ to $c(v(\ell,2))$, so that the wall determined by this space is the same for all $(\tilde\Gamma,\tilde e(\ell))$ in $\mathcal{E}$, while the variation of D_n across the wall associated to $(\tilde\Gamma,\ \tilde e(\ell))$ is $\pm\zeta_\Gamma[\tilde\Gamma]$. Checking the signs as in [Les20, Lemma 9.13] shows that the sum over the elements of $\mathcal{E}$ of the variations of D_n across the wall associated to $(\tilde\Gamma,\ \tilde e(\ell))$ vanishes, thanks to the Jacobi relation.

3 Some properties of Z

Set $\mathcal{A}(\mathcal{L}) = \prod_{n\in\mathbb{N}} \mathcal{A}_n(\mathcal{L})$. We drop the subscript n to denote the collection (or the sum) of the Z_n for $n \in \mathbb{N}$. For example,

$$Z\left(\check{R}, L, (0), p_1(\tau)\right) = \left(Z_n(\check{R}, L, (0), p_1(\tau))\right)_{n\in\mathbb{N}} = \sum_{n\in\mathbb{N}} Z_n\left(\check{R}, L, (0), p_1(\tau)\right) \in \mathcal{A}(\mathcal{L}).$$

The disjoint union of diagrams induces a commutative product on $\mathcal{A}(\emptyset)$ which maps two classes of diagrams to the class of their disjoint union. Equipped with this product, $\mathcal{A}(\emptyset)$ is a commutative algebra. The disjoint union of diagrams induces similarly an $\mathcal{A}(\emptyset)$-module structure on $\mathcal{A}(\mathcal{L})$ for any one-manifold $\mathcal{L}$.

3.1 On the invariant Z of $\mathbb{Q}$–spheres and the anomaly β

Let $\mathcal{A}^c_n(\emptyset)$ denote the subspace of $\mathcal{A}_n(\emptyset)$ generated by trivalent Jacobi diagrams with one connected component, set $\mathcal{A}^c(\emptyset) = \prod_{n\in\mathbb{N}} \mathcal{A}^c_n(\emptyset)$, and let $p^c\colon \mathcal{A}(\emptyset) \to \mathcal{A}^c(\emptyset)$ be the linear projection that maps the empty diagram and diagrams with several connected components to 0. Let $\mathcal{D}^c_n$ denote the subset of $\mathcal{D}^e_n(\emptyset)$ that contains the connected diagrams of $\mathcal{D}^e_n(\emptyset)$. For $n \in \mathbb{N}$, set

$$z_n(\check{R}, p_1(\tau)) = p^c\left(Z_n(\check{R}, p_1(\tau)) = Z_n(\check{R}, \emptyset, p_1(\tau))\right).$$

$$z_n(\check{R}, p_1(\tau)) = \sum_{\Gamma\in\mathcal{D}^c_n} \zeta_\Gamma I(R, \Gamma, \omega)[\Gamma] \in \mathcal{A}^c_n(\emptyset)$$

for some propagating form ω of $(C_2(R), \tau)$. The reader can check that

$$Z(\check{R}, p_1(\tau)) = \exp\left(z(\check{R}, p_1(\tau))\right).$$

The dependence on $p_1(\tau)$ of $z(\check{R}, p_1(\tau))$ is linear, and the following proposition is a consequence of [Les20, Corollary 10.9, Proposition 10.7 and Definition 10.5].

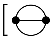
Proposition 3.1 (Kuperberg, Thurston [KT99]) *There exists an element $\beta \in \mathcal{A}(\emptyset)$ such that $\left(z(\check{R}, p_1(\tau)) - \frac{p_1(\tau)}{4}\beta\right)$ is independent of τ so that*

$$Z(R) = Z(\check{R}, p_1(\tau)) \exp\left(-\frac{p_1(\tau)}{4}\beta\right).$$

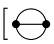
is an invariant of R. If n is even, then the degree n part β_n of $\beta = (\beta_n)_{n\in\mathbb{N}}$ is zero.

Note the following proposition.

Proposition 3.2 *Let $(\check{R},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$, then*

$$Z_1(\check{R},p_1(\tau)) = z_1(\check{R},p_1(\tau)) = \frac{\Theta(R,\tau)}{12}\,[$$  $$]$$

in $\mathcal{A}_1(\emptyset) = \mathcal{A}_1(\emptyset;\mathbb{R}) = \mathbb{R}[$$]$.

In particular, $\beta_1 = \frac{1}{12}\,[$$]$. See [Les20, Section 10.2] for more details about the anomaly β, which is unknown in odd degrees greater than 1.

In [KT99], Greg Kuperberg and Dylan Thurston proved that the restriction of Z to $\mathbb{Z}$-spheres is a universal finite type invariant of $\mathbb{Z}$-spheres, with respect to the Ohtsuki theory of finite type invariants for $\mathbb{Z}$-spheres [Oht96], see also [GGP01]. In [Les04], I generalized their result by proving that the restriction of Z to $\mathbb{Q}$-spheres is a universal finite type invariant of $\mathbb{Q}$-spheres with respect to the Moussard theory of finite type invariants of $\mathbb{Q}$-spheres based on Lagrangian-preserving surgeries [Mou12], see [Les20, Sections 18.1 and 18.5]. This implies that Z and the LMO invariant of Le, Murakami and Ohtsuki [LMO98] are equivalent in the sense that they distinguish the same $\mathbb{Q}$–spheres.

3.2 On the invariant Z of framed tangles and the anomaly α

The product $\exp\left(-\frac{p_1(\tau)}{4}\beta\right) Z(\check{R},L,(I_\theta(K,\tau))_{K\in L_C},p_1(\tau))$ is actually independent of $p_1(\tau)$, too, so that we set

$$Z(\check{R},L,(I_\theta(K,\tau))_{K\in L_C}) = \exp\left(-\frac{p_1(\tau)}{4}\beta\right) Z(\check{R},L,(I_\theta(K,\tau))_{K\in L_C},p_1(\tau)).$$

Remark 3.3 Let $\check{\mathcal{A}}(\mathcal{L})$ be the quotient of $\mathcal{A}_n(\mathcal{L})$ by the vector space generated by the diagrams that have at least one connected component without univalent vertices. Using the corresponding projection $\check{p}\colon \mathcal{A}(\mathcal{L}) \to \check{\mathcal{A}}(\mathcal{L})$ and setting $\check{Z}_n = \check{p}\circ Z_n$, we can write

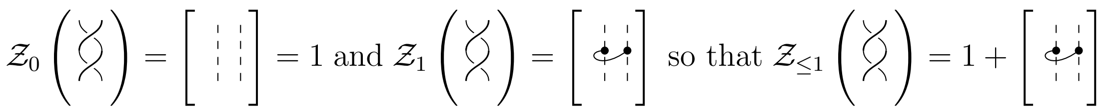
$$Z(\check{R},L,(I_\theta(K,\tau))_{K\in L_C}) = Z(R)\check{Z}(\check{R},L,(I_\theta(K,\tau))_{K\in L_C}).$$

If a one-manifold $\mathcal{L}$ is the union of two one-manifolds $\mathcal{L}_1$ and $\mathcal{L}_2$, which meet only along their boundaries, the disjoint union of diagrams again induces products from $\mathcal{A}_j(\mathcal{L}_1)\otimes\mathcal{A}_k(\mathcal{L}_2)$ to $\mathcal{A}_{j+k}(\mathcal{L})$, where the required class of injections $i_{\Gamma_1\sqcup\Gamma_2}$ for a disjoint union of a Jacobi diagram Γ_1 on $\mathcal{L}_1$ and a Jacobi diagram Γ_2 on $\mathcal{L}_2$ is naturally induced by $[i_{\Gamma_1}]$ and $[i_{\Gamma_2}]$. View $[0,1]$ as the union of $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$, together with orientation-preserving identifications of $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ with $[0,1]$. Then the above products induce an algebra structure on $\mathcal{A}([0,1])$. In [BN95a], Bar-Natan proved that the induced product of $\mathcal{A}([0,1])$ is actually commutative, and that the natural map from $\mathcal{A}([0,1])$ to $\mathcal{A}(S^1)$ obtained from the identification $S^1=[0,1]/(0\sim 1)$ is an isomorphism. See [Les20, Proposition 6.22]. In particular, the choice of an oriented connected component $\mathcal{K}$ of $\mathcal{L}$ equips $\mathcal{A}(\mathcal{L})$ with a well-defined $\mathcal{A}([0,1])$-*module structure* $\sharp_{\mathcal{K}}$, induced by an orientation-preserving inclusion from $[0,1]$ into a small part of $\mathcal{K}$ outside the vertices.

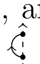
A *tangle representative* is a pair $(\mathcal{C}, \mathcal{C} \cap L)$, which is simply denoted by $(\mathcal{C}, L)$ for a long tangle representative as in Definition 2.4, where we again identify the embedding L and its image. Such a tangle representative is a cobordism in $\mathcal{C}$ from the bottom configuration of L to the top configuration of L. From now on Z is viewed as a map, which maps such a tangle representative, also denoted by L or by $(\mathcal{C}, L)$, to an element $\mathcal{Z}(\mathcal{C}, L) = Z(\check{R}(\mathcal{C}), L, (0)_{K \in L_C})$ of $\mathcal{A}(\mathcal{L})$.

Note that $\mathcal{Z}$ maps trivial braids $c(B) \times [0,1]$ of $\mathcal{C}_0$ to the class of the empty diagram, since the vertical translations act on the involved configuration spaces so that the image of $\prod_{e \in E} d_e$ in $(S^2)^{j_E(E)}$ of the configuration space is the image of the quotient, which is included in a subspace of codimension at least 1.

It is easy to compute the expansion $\mathcal{Z}_{\leq 1}$ up to degree 1 of $\mathcal{Z}$ for  and to show that



where the endpoints of the tangle representative lie on $\mathbb{R} \times \{0,1\}$. See [Les20, Lemma 12.19].

More precisely, $\mathcal{Z}$ maps the above braid $(\sigma_1)^2$ to the exponential of an element obtained by inserting a combination $2\breve{\alpha}$ of Jacobi diagrams with two free univalent vertices, which are symmetric with respect to exchanging two vertices, on the diagram with one edge between the two strands. See [Les20, Lemma 13.16]. The degree one part of $2\breve{\alpha}$ is an edge between the two vertices, and it is conjectured that $2\breve{\alpha}$ vanishes in degree greater than 1. Inserting $2\breve{\alpha}$ on the edge of  gives rise to 2α, where $\alpha \in \mathcal{A}([0,1])$ is the *Bott and Taubes anomaly,* which controls the dependence on $I_\theta(K, \tau)$ as follows.

Theorem 3.4 *Let L be a long tangle representative and let L_C denote the set of its connected components. The expression*

$$\prod_{K \in L_C} (\exp(-I_\theta(K,\tau)\alpha)\sharp_K)\, Z(\check{R} = \check{R}(\mathcal{C}), L, (I_\theta(K,\tau))_{K \in L_C})$$

is independent of the $I_\theta(K,\tau)$. It is denoted by $\mathcal{Z}(\mathcal{C}, L)$.

Here $\exp(-I_\theta(K,\tau)\alpha)$ acts on $Z(\check{R}, L, (I_\theta(K,\tau))_{K \in L_C})$, by insertion on the component of K in the source $\mathcal{L}$ of the long tangle as indicated[19] by the subscript K.

Remark 3.5 It is known that $\alpha_{2n} = 0$ for any $n \in \mathbb{N}$, and that $\alpha_3 = 0$ [Poi02, Proposition 1.4]. Sylvain Poirier also showed that $\alpha_5 = 0$ with the help of a Maple program. Furthermore, according to [Les02, Corollary 1.4], α_{2n+1} is a combination of diagrams with two univalent vertices (as mentioned above), and $\mathcal{Z}(S^3, L)$ is obtained from the Kontsevich integral Z^K by inserting d times the plain part $2\breve{\alpha}$ of 2α on some edge of each degree d connected component of a diagram. See [Les20, Section 10.3] for more about the anomaly α, which is unknown in odd degrees greater than 6.

[19] Because of the given symmetry of α, there is no need to orient K to define $(\exp(-I_\theta(K,\tau)\alpha)\sharp_K)$.

The precise natural definitions of parallels $K_\parallel$ of long tangle components K and of the corresponding linking numbers $lk(K, K_\parallel)$ are given in [Les20, Section 12.2]. With these definitions, when $L = (K)_{K \in L_C}$ is framed by some $L_\parallel = (K_\parallel)_{K \in L_C}$, we set

$$\mathcal{Z}^f(\mathcal{C}, (L, L_\parallel)) = \prod_{K \in L_C} \left(\exp(lk(K, K_\parallel)\alpha)\sharp_K\right) \mathcal{Z}(\mathcal{C}, L),$$

as in [Les20, Definition 12.12].

Let us now discuss some properties of this invariant $\mathcal{Z}^f$ of framed tangles. The first one is the following functoriality property, which is part of [Les20, Theorem 13.12], and is proved in [Les20, Section 17.2]

Theorem 3.6 $\mathcal{Z}^f$ *is* functorial: *For two framed tangles $L_1 = (\mathcal{C}_1, L_1)$ and $L_2 = (\mathcal{C}_2, L_2)$ such that the top configuration of L_1 coincides with the bottom configuration of L_2, then the naturally framed product L_1L_2 is defined by stacking L_2 on top of L_1, (and appropriately vertically rescaling) and*

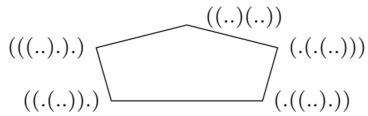
$$\mathcal{Z}^f(L_1L_2) = \mathcal{Z}^f\left(\begin{array}{|c|}\hline L_2 \\ \hline L_1 \\ \hline \end{array}\right) = \begin{array}{|c|}\hline \mathcal{Z}^f(L_2) \\ \hline \mathcal{Z}^f(L_1) \\ \hline \end{array} = \mathcal{Z}^f(L_1)\mathcal{Z}^f(L_2).$$

When applied to the case where the tangles are empty, this theorem implies that the invariant Z of $\mathbb{Q}$–spheres is multiplicative under connected sum.

3.3 Generalization to q–tangles

Here, framed tangles are framed cobordisms in $\mathbb{Q}$-cylinders between injective configurations of points in $\mathbb{C}$ up to dilations and translations. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and for a finite set B, the space $\check{\mathcal{S}}_B(\mathbb{K})$ of injective maps from B to $\mathbb{K}$ up to translation and dilation, may be compactified to a manifold $\mathcal{S}_B(\mathbb{K})$ by first embedding $\check{\mathcal{S}}_B(\mathbb{K})$ in the compact space $\overline{\mathcal{S}}_B(\mathbb{K})$ of non-constant maps from B to $\mathbb{K}$ up to translation and dilation (when $\sharp B \geq 2$), and then successively blowing up all the diagonals as in the beginning of Section 2.4. See [Les20, Section 8.3] for details.

Example 3.7 For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the configuration space $\check{\mathcal{S}}_{\underline{1}}(\mathbb{K}) = \mathcal{S}_{\underline{1}}(\mathbb{K})$ is reduced to a point. The configuration space $\check{\mathcal{S}}_{\underline{2}}(\mathbb{C}) = \mathcal{S}_{\underline{2}}(\mathbb{C})$ is a circle, while the configuration space $\check{\mathcal{S}}_{\underline{2}}(\mathbb{R}) = \mathcal{S}_{\underline{2}}(\mathbb{R})$ has two points $(0,1)$ and $(0,-1)$, where we write elements of $\check{\mathcal{S}}_{\underline{k}}(\mathbb{R})$ as elements $(c(1), \dots, c(k))$ of $\mathbb{R}^k$ such that $c(1) = 0$ and $|c(k)| = 1$, for any $k \in \mathbb{N}$ such that $k \geq 2$. In general, $\check{\mathcal{S}}_{\underline{k}}(\mathbb{R})$ and its compactification $\mathcal{S}_{\underline{k}}(\mathbb{R})$ have $k!$ components, which correspond to the orders of the $c(i)$ in $\mathbb{R}$. Denote the connected component of $\check{\mathcal{S}}_{\underline{k}}(\mathbb{R})$ where $c(1) < c(2) < \dots < c(k)$ by $\check{\mathcal{S}}_{<,\underline{k}}(\mathbb{R})$, and its closure in $\mathcal{S}_{\underline{k}}(\mathbb{R})$ by $\mathcal{S}_{<,\underline{k}}(\mathbb{R})$. Then $\check{\mathcal{S}}_{<,\underline{3}}(\mathbb{R}) = \{(0,t,1) \mid t \in]0,1[\}$, and $\mathcal{S}_{<,\underline{3}}(\mathbb{R})$ is its natural compactification $[0,1]$ where $t \in]0,1[$ represents the injective configuration $(0,t,1)$, 0 represents the limit configuration $((\bullet\bullet)\bullet) = \lim_{t \to 0}(0,t,1)$ and 1 represents the limit configuration $(\bullet(\bullet\bullet)) = \lim_{t \to 0}(0, 1-t, 1)$. The configuration space $\mathcal{S}_{<,\underline{4}}(\mathbb{R})$ is diffeomorphic to the following well-known pentagon.



In general, for $k \geq 3$, the configuration space $\mathcal{S}_{<,\underline{k}}(\mathbb{R})$ is a *Stasheff polyhedron* of dimension $(k-2)$ whose corners are labeled by *non-associative words* in the letter $\bullet$ as in the above examples. For any integer $k \geq 2$, a non-associative word w with k letters represents a limit configuration $w = \lim_{t \to 0} w(t)$, where $w(t) = (w_1(t) = 0, w_2(t), \dots, w_{k-1}(t), w_k(t) = 1)$ is an injective configuration for $t \in]0, \frac{1}{2}[$, and, if w is the product uv of a non-associative word u of length $j \geq 1$ and a non-associative word v of length $(k-j) \geq 1$, $w_i(t) = tu_i(t)$ when $1 < i \leq j$ and $w_i(t) = 1 - t + tv_{i-j}(t)$ when $k > i > j$. For example, $(((\bullet\bullet)\bullet)\bullet)(t) = (0, t^2, t, 1)$. In a limit configuration associated to such a non-associative word, points inside matching parentheses are thought of as infinitely closer to each other than they are to points outside these matching parentheses.

Definition 3.8 Define a *combinatorial q–tangle* as a framed tangle representative whose bottom and top configurations are on the real line, up to isotopies of $\mathcal{C}$ which globally preserve the intersection of the bottom disk $D^2 \times \{0\}$ with $\mathbb{R} \times \{0\}$ and the intersection of the top disk $D^2 \times \{1\}$ with $\mathbb{R} \times \{1\}$, equipped with non-associative words of the appropriate length associated to the bottom and top configurations. These non-associative words are called the *bottom and top configurations* of the combinatorial q–tangle.

Such a combinatorial q–tangle L from a bottom word w^- to a top word w^+ is thought of as the limit when t tends to 0 of the framed tangles $L(t)$ in the above isotopy class whose bottom and top configurations are $w^-(t)$ and $w^+(t)$, respectively. In [Les20, Theorem 13.8 and Remark 13.11], following Poirier [Poi00], I proved that $\lim_{t\to 0} \mathcal{Z}^f(L(t))$ exists and that it defines an isotopy invariant of these (framed) combinatorial q–tangles L. This invariant is still multiplicative under vertical composition as in Theorem 3.6, and we can now define other interesting operations.

For two combinatorial q–tangles $L_1 = (\mathcal{C}_1, L_1)$ from w_1^- to w_1^+ and $L_2 = (\mathcal{C}_2, L_2)$ from w_2^- to w_2^+, define the product $L_1 \otimes L_2$ from the bottom configuration $w_1^- w_2^-$ to the top configuration $w_1^+ w_2^+$ by shrinking $\mathcal{C}_1$ and $\mathcal{C}_2$ to make them respectively replace the products by $[0,1]$ of the horizontal disks with radius $\frac{1}{4}$ and respective centers $-\frac{1}{2}$ and $\frac{1}{2}$.

Theorem 3.9 $\mathcal{Z}^f$ *is* monoidal: *For two combinatorial q–tangles* L_1 *and* L_2*,*

$$\mathcal{Z}^f(L_1 \otimes L_2) = \mathcal{Z}^f\left(\boxed{L_1}\boxed{L_2}\right) = \boxed{\mathcal{Z}^f(L_1)}\boxed{\mathcal{Z}^f(L_2)} = \mathcal{Z}^f(L_1) \otimes \mathcal{Z}^f(L_2).$$

PROOF: This theorem can be easily deduced from the cabling property and the functoriality property of [Les20, Theorem 13.12]. $\square$

We can also *double* a component K according to its parallelization in a combinatorial q–tangle L. This operation replaces a component with two parallel components, with respect to the given framing, and, if this component has boundary points, it replaces the corresponding letters in the non-associative words with (••). The combinatorial q–tangle obtained in this way is denoted by $L(2 \times K)$.

The corresponding operation for Jacobi diagrams is the following one.

Definition 3.10 Let $\mathcal{L}$ be a one-manifold, and let $\mathcal{K}$ be a connected component of $\mathcal{L}$. Let

$$\mathcal{L}(2 \times \mathcal{K}) = (\mathcal{L} \setminus \mathcal{K}) \sqcup \left(\mathcal{K}^{(1)} \sqcup \mathcal{K}^{(2)}\right)$$

be the manifold obtained from $\mathcal{L}$ by *duplicating* $\mathcal{K}$, that is by replacing $\mathcal{K}$ with two copies $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$ of $\mathcal{K}$, and let

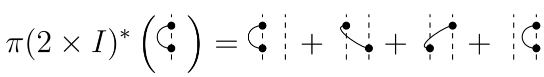
$$\pi(2 \times \mathcal{K}) \colon \mathcal{L}(2 \times \mathcal{K}) \longrightarrow \mathcal{L}$$

be the associated map, which is the identity on $(\mathcal{L} \setminus \mathcal{K})$, and the trivial 2-fold covering from $\mathcal{K}^{(1)} \sqcup \mathcal{K}^{(2)}$ to $\mathcal{K}$.

If Γ is (the class of) an oriented Jacobi diagram on $\mathcal{L}$, then $\pi(2 \times \mathcal{K})^*(\Gamma)$ is the sum of all diagrams on $\mathcal{L}(2 \times \mathcal{K})$ obtained from Γ by lifting each univalent vertex to one of its preimages under $\pi(2 \times \mathcal{K})$. These diagrams have the same vertices and edges as Γ and the local orientations at univalent vertices are naturally induced by the local orientations of the corresponding univalent vertices of Γ. This operation induces the natural linear *duplication map:*

$$\pi(2 \times \mathcal{K})^* : \mathcal{A}(\mathcal{L}) \longrightarrow \mathcal{A}(\mathcal{L}(2 \times \mathcal{K})).$$

Example 3.11



We can now state the following duplication property for $\mathcal{Z}^f$ of [Les20, Theorem 13.12], which is proved in [Les20, Section 17.4].

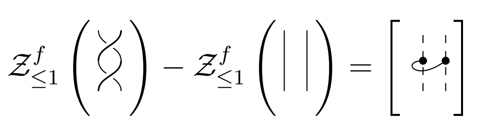
Theorem 3.12 *Let K be a component of a combinatorial q–tangle L, then*

$$\mathcal{Z}^f(L(2 \times K)) = \pi(2 \times K)^* \mathcal{Z}^f(L).$$

More properties of $\mathcal{Z}^f$ are presented in [Les20, Theorem 13.12].

3.4 Discrete derivatives of $\mathcal{Z}^f$

Since



where the endpoints of the tangles lie on $\mathbb{R} \times \{0, 1\}$, the above properties of $\mathcal{Z}^f$ allow us to completely compute n^{th} derivatives of $\mathcal{Z}_n$, where a simple derivative of $\mathcal{Z}_n$ is a difference

$\mathcal{Z}_n($  $)-\mathcal{Z}($  $)$. In particular, they imply that the restriction of $\mathcal{Z}$ to links in S^3 is a universal Vassiliev invariant of links as in [Les20, Section 17.6], without using the theorem mentioned in Remark 3.5.

The following n^{th} derivative with respect to LP-surgeries of $\mathcal{Z}^f$ is computed in [Les20, Theorem 18.5]. Let L be a q–tangle representative in a rational homology cylinder $\mathcal{C}$. Let $\sqcup_{i=1}^{x} A^{(i)}$ be a disjoint union of rational homology handlebodies embedded in $\mathcal{C} \setminus L$. Let $(A^{(i)\prime}/A^{(i)})$ be rational LP surgeries in $\mathcal{C}$ as in Definition 1.26. Set $X = [\mathcal{C}, L; (A^{(i)\prime}/A^{(i)})_{i\in\underline{x}}]$ and

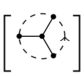
$$\mathcal{Z}_n(X) = \sum_{I\subset\underline{x}} (-1)^{x+\sharp I} \mathcal{Z}_n\left(\mathcal{C}_I, L\right),$$

where $\mathcal{C}_I = \mathcal{C}\left((A^{(i)\prime}/A^{(i)})_{i\in I}\right)$ is the rational homology cylinder obtained from $\mathcal{C}$ by performing the LP-surgeries that replace $A^{(i)}$ with $A^{(i)\prime}$ for $i \in I$. If $2n < x$, then $\mathcal{Z}_n(X)$ vanishes, and, if $2n = x$, then the expression of $\mathcal{Z}_n(X)$ is given in [Les20, Theorem 18.5].

This computation relies on constructions of propagating forms that coincide as much as possible[20] for the involved manifolds. The result of this computation implies that the restriction of Z to $\mathbb{Q}$-spheres is a universal finite type invariant of $\mathbb{Q}$-spheres with respect to the Moussard theory of finite type invariants of $\mathbb{Q}$-spheres [Mou12], as announced in Section 3.1.

This computation has also allowed the author to compute $\check{Z}_2(R, K)$ for any null-homologous knot K in a rational homology sphere R in [Les20, Theorem 18.41], and to show that

$$\check{Z}_2(R, K) = \left(\frac{1}{24} - \frac{1}{2}\Delta''_K(1)\right) [\,\cdot\,],$$



where Δ_K is the Alexander polynomial of K, normalized so that $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = 1$. This result was generalized by David Leturcq in [Let20]. See [Les20, Theorem 18.43].

3.5 Some open questions

The determination of the anomalies α and β is still open. The behaviour of $\mathcal{Z}^f$ under Dehn surgeries has not yet been investigated. Is the invariant $\mathcal{Z}$ of $\mathbb{Q}$-spheres obtained from the invariant $\mathcal{Z}^f$ of framed links in the same way as the Le-Murakami-Ohtsuki invariant [LMO98] is obtained from the Kontsevich integral?

I constructed an invariant $\tilde{\mathcal{Z}}$ of null-homologous knots in $\mathbb{Q}$-spheres from equivariant algebraic intersections in equivariant configuration spaces in [Les11, Les13]. This equivariant $\tilde{\mathcal{Z}}$ lives in a more structured space of Jacobi diagrams. It shares many properties with the Kricker lift of the Kontsevich integral of [Kri00, GK04]. Does $\tilde{\mathcal{Z}}$ lift the restriction of $\mathcal{Z}$ to null-homologous knots in $\mathbb{Q}$-spheres as the Kricker invariant lifts the Kontsevich integral?

Heegaard splittings provide propagators as in Section 1.5. How do the invariants Z, $\mathcal{Z}^f$ and $\tilde{\mathcal{Z}}$ relate to Heegaard-Floer homology?

[20] The constructed forms ω_I satisfy $\omega_I = \omega_J$ on $C_2\left((\check{R}(\mathcal{C}_I) \setminus \cup_{i\in I\cup J}\mathrm{Int}(A^{(i)})) \cup \cup_{i\in I\cap J} A^{(i)\prime}\right)$ for parts I and J of $\underline{x}$, where $C_2(X) = p_b^{-1}(X^2)$.

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