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A combinatorial definition of the Θ-invariant from Heegaard diagrams

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Abstract

The invariant Θ is the simplest 3-manifold invariant defined by counting graph configurations. It is actually an invariant of rational homology 3-spheres M equipped with a combing X over the complement of a point. The invariant $\Theta(M,X)$ is the sum of $6\lambda(M)$ and $\frac{p_1(X)}{4}$, where λ denotes the Casson-Walker invariant, and p_1 is an invariant of combings that is an extension of a first relative Pontrjagin class, and that is simply related to a Gompf invariant θ_G. In [Les12a], we proved a combinatorial formula for the Θ-invariant in terms of decorated Heegaard diagrams. In this article, we study the variations of the invariants p_1 or θ_G when the decorations of the Heegaard diagrams that define the combings change, independently. Then we prove that the formula of [Les12a] defines an invariant of combed once punctured rational homology 3-spheres without referring to configuration spaces. Finally, we prove that this invariant is the sum of $6\lambda(M)$ and $\frac{p_1(X)}{4}$ for integral homology spheres, by proving surgery formulae both for the combinatorial invariant and for p_1.

Keywords: Θ-invariant, Heegaard splittings, Heegaard diagrams, combings, Gompf invariant, Casson-Walker invariant, finite type invariants of 3-manifolds, homology spheres, configuration space integrals, perturbative expansion of Chern-Simons theory
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Contents

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1 Introduction

In this article, a $\mathbb{Q}$–*sphere* or *rational homology sphere* is a smooth closed oriented 3-manifold that has the same rational homology as S^3.

1.1 General introduction

The work of Witten [Wit89] pioneered the introduction of many $\mathbb{Q}$–sphere invariants, among which the Le-Murakami-Ohtsuki universal finite type invariant [LMO98] and the Kontsevich configuration space invariant [Kon94] that was proved to be equivalent to the LMO invariant for integral homology spheres by G. Kuperberg and D. Thurston [KT99]. The construction of the Kontsevich configuration space invariant for a $\mathbb{Q}$–sphere M involves a point ∞ in M, an identification of a neighborhood of ∞ with a neighborhood $S^3 \setminus B(1)$ of ∞ in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, and a parallelization τ of $(\check{M} = M \setminus \{\infty\})$ that coincides with the standard parallelization of $\mathbb{R}^3$ on $\mathbb{R}^3 \setminus B(1)$, where $B(r)$ denotes the ball centered at 0 with radius r in $\mathbb{R}^3$. The Kontsevich configuration space invariant is in fact an invariant of (M, τ). Its degree one part $\Theta(M, \tau)$ is the sum of $6\lambda(M)$ and $\frac{p_1(\tau)}{4}$, where λ is the Casson-Walker invariant and p_1 is a Pontrjagin number associated with τ, according to a theorem of G. Kuperberg and D. Thurston [KT99] generalized to rational homology spheres in [Les04b]. Here, the Casson-Walker invariant λ is normalized like in [AM90, GM92, Mar88] for integral homology spheres, and like $\frac{1}{2}\lambda_W$ for rational homology spheres where λ_W is the Walker normalisation in [Wal92].

Let B_M denote the complement in M of the neighborhood of ∞ identified with $S^3 \setminus B(1)$, B_M is a rational homology ball. An ∞-*combing* of such a rational homology sphere M is a section of the unit tangent bundle $U\check{M}$ of $\check{M}$ that is constant on $\check{M} \setminus B_M$ (via the identifications above with $\mathbb{R}^3 \setminus B(1)$ near ∞), up to homotopies through this kind of sections. As it is shown in [Les12a], $\Theta(M, .)$ is actually an invariant of rational homology spheres equipped with such an ∞-combing.

Every closed oriented 3–manifold M can be written as the union of two handlebodies $H_{\mathcal{A}}$ and $H_{\mathcal{B}}$ glued along their common boundary that is a genus g surface as

$$M = H_{\mathcal{A}} \cup_{\partial H_{\mathcal{A}}} H_{\mathcal{B}}$$

where $\partial H_{\mathcal{A}} = -\partial H_{\mathcal{B}}$. Such a decomposition is called a *Heegaard decomposition* of M. A *system of meridians* for $H_{\mathcal{A}}$ is a system of g disjoint curves α_i of $\partial H_{\mathcal{A}}$ that bound disjoint disks $D(\alpha_i)$ properly embedded in $H_{\mathcal{A}}$ such that the union of the α_i does not separate $\partial H_{\mathcal{A}}$. For a positive integer g, we will denote the set $\{1, 2, \ldots, g\}$ by $\underline{g}$. Let $(\alpha_i)_{i \in \underline{g}}$ be a system of meridians for $H_{\mathcal{A}}$ and let $(\beta_j)_{j \in \underline{g}}$ be such a system for $H_{\mathcal{B}}$. Then the surface equipped with the collections of the curves α_i and the curves $\beta_j = \partial D(\beta_j)$ determines M. When the collections $(\alpha_i)_{i \in \underline{g}}$ and $(\beta_j)_{j \in \underline{g}}$ are transverse, the data $\mathcal{D} = (\partial H_{\mathcal{A}}, (\alpha_i)_{i \in \underline{g}}, (\beta_j)_{j \in \underline{g}})$ is called a *Heegaard diagram.*

Such a Heegaard diagram may be obtained from a Morse function f_M of M that has one minimum mapped to (-3), one maximum mapped to 9, g index one points a_i and g index 2 points b_j, such that f_M maps index 1 points to 1 and index 2 points to 5, and f_M satisfies generic Morse-Smale conditions ensuring transversality of descending and ascending manifolds of critical points, with respect to a Euclidean metric $\mathfrak{g}$ of M. Thus the surface $\partial H_{\mathcal{A}}$ is $f_M^{-1}(3)$, the ascending manifolds of the a_i intersect $H_{\mathcal{A}}$ as disks $D(\alpha_i)$ bounded by the α_i and the descending manifolds of the b_j intersect $H_{\mathcal{B}}$ as disks $D(\beta_j)$ bounded by the β_i, and the flow line closures from a_i to b_j are in natural one-to-one correspondence with the crossings of $\alpha_i \cap \beta_j$. Conversely, for any Heegaard diagram, there exists a Morse function f_M with the properties above.

A *matching* in a genus g Heegaard diagram

$$\mathcal{D} = (\partial H_{\mathcal{A}}, \{\alpha_i\}_{i=1,\ldots,g}, \{\beta_j\}_{j=1,\ldots,g})$$

is a set $\mathfrak{m}$ of g crossings such that every curve of the diagram contains one crossing of $\mathfrak{m}$. An *exterior point* in such a diagram $\mathcal{D}$ is a point of $\partial H_{\mathcal{A}} \setminus \left(\coprod_{i=1}^{g} \alpha_i \cup \coprod_{j=1}^{g} \beta_j\right)$. The choice of a matching $\mathfrak{m}$ and of an exterior point w in a diagram $\mathcal{D}$ of M equips M with the following ∞-combing $X(w, \mathfrak{m}) = X(\mathcal{D}, w, \mathfrak{m})$.

Remove an open ball around a flow line from the minimum to the maximum that goes through w, so that we are left with a rational homology ball

$$B_M(2) = B_M \cup_{\partial B(1) = \partial B_M} B(2) \setminus \mathring{B}(1)$$

where the gradient field of f_M is vertical near the boundary. Reversing the gradient field along the flow lines $\gamma(c)$ associated with the crossings c of $\mathfrak{m}$ as in Subsection 3.1 below produces the ∞-combing $X(w, \mathfrak{m})$ of M.

Let θ_G denote the invariant of combings of rational homology spheres introduced by Gompf in [Gom98, Section 4]. A choice of a standard modification described in Subsection 4.2 of $X(w, \mathfrak{m})$ in the fixed neighborhood of ∞ identified with $S^3 \setminus B(2)$ transforms $X(w, \mathfrak{m})$ into a combing $X(M, w, \mathfrak{m})$ such that $p_1(X(w, \mathfrak{m})) - \theta_G(X(M, w, \mathfrak{m}))$ is independent of $(M, w, \mathfrak{m})$.

In [Les12a, Theorem 1.5], we express $\Theta(M, X(w, \mathfrak{m}))$ as a combination

$$\tilde{\Theta}(\mathcal{D}, w, \mathfrak{m}) = \ell_2(\mathcal{D}) + s_\ell(\mathcal{D}, \mathfrak{m}) - e(\mathcal{D}, w, \mathfrak{m})$$

of invariants of Heegaard diagrams $\mathcal{D}$ equipped with a matching $\mathfrak{m}$ and an exterior point w. First combinatorial expressions of the ingredients $\ell_2(\mathcal{D})$, $s_\ell(\mathcal{D}, \mathfrak{m})$, and $e(\mathcal{D}, w, \mathfrak{m})$ are given in the end of this introduction section whereas Section 2 provides alternative expressions and properties of these quantities.

In this article, we give several expressions of the variations of $p_1(X(w, \mathfrak{m}))$, or, equivalently of $\theta_G(X(M, w, \mathfrak{m}))$, when w and $\mathfrak{m}$ vary, for a fixed Heegaard diagram. Expressions in terms of linking numbers are given in Subsection 3.2 and derived combinatorial expressions can be found in Section 4.

The latter ones allow us to give combinatorial proofs that $\left(4\tilde{\Theta}(\mathcal{D}, w, \mathfrak{m}) - p_1(X(w, \mathfrak{m}))\right)$ is independent of $(w, \mathfrak{m})$ in Section 5. We prove that

$$\tilde{\lambda}(\mathcal{D}) = \frac{1}{24}\left(4\tilde{\Theta}(\mathcal{D}, w, \mathfrak{m}) - p_1(X(w, \mathfrak{m}))\right)$$

only depends on the presented rational homology sphere M, combinatorially, in Section 6. We set $\tilde{\lambda}(M) = \tilde{\lambda}(\mathcal{D})$ so that $\tilde{\lambda}$ is a topological invariant of rational homology 3-spheres.

Then we give a direct combinatorial proof that $\tilde{\lambda}$ satisfies the Casson surgery formula for $\frac{1}{n}$–Dehn surgeries along null-homologous knots in Section 7. This implies that $\tilde{\lambda}$ coincides with the Casson invariant for integral homology 3-spheres. Our proof also yields a surgery formula for p_1 that is stated in Theorem 7.2.

Thus this article contains an independent construction of the Casson invariant that includes a direct proof of the Casson surgery formula, and an independent combinatorial proof of the formula of [Les12a, Theorem 3.8] for the Θ-invariant in terms of Heegaard diagrams in the case of $\mathbb{Z}$-spheres. It also describes the behaviour of the four quantities $\ell_2(\mathcal{D})$, $s_\ell(\mathcal{D},\mathfrak{m})$, $e(\mathcal{D},w,\mathfrak{m})$ and $p_1(X(\mathcal{D},w,\mathfrak{m}))$ (or equivalently $\theta_G(X(M,w,\mathfrak{m})))$ associated with Heegaard diagrams $\mathcal{D}$ decorated with $(w,\mathfrak{m})$ under standard modifications of Heegaard diagrams, and Dehn surgeries. These quantities might show up in combinatorial definitions of other invariants from Heegaard diagrams, as θ_G that Gripp and Huang use to define the Heegaard Floer homology $\widehat{HF}$ grading in [GH11].

The definitions introduced in [Les12a, Theorem 3.8] are also given here, causing overlaps. I thank Jean-Mathieu Magot for useful conversations.

1.2 Conventions and notations

Unless otherwise mentioned, all manifolds are oriented. Boundaries are oriented by the outward normal first convention. Products are oriented by the order of the factors. More generally, unless otherwise mentioned, the order of appearance of coordinates or parameters orients chains or manifolds. The normal bundle $\mathfrak{N}(A)$ of an oriented submanifold A is oriented so that the normal bundle followed by the tangent bundle of the submanifold induce the orientation of the ambient manifold (fiberwise). The transverse intersection of two submanifolds A and B of a manifold C is oriented so that the normal bundle $\mathfrak{N}_x(A\cap B)$ of $A\cap B$ at x is oriented as $(\mathfrak{N}_x(A)\oplus\mathfrak{N}_x(B))$. When the dimensions of two such submanifolds add up to the dimension of C, each intersection point is equipped with a sign ±1 that is 1 if and only if $(\mathfrak{N}_x(A)\oplus\mathfrak{N}_x(B))$ (or equivalently $(T_x(A)\oplus T_x(B)))$ induces the orientation of C. When A is compact, the sum of the signs of the intersection points is the *algebraic intersection number* $\langle A,B\rangle_C$. The *linking number* $lk(L_1,L_2) = lk_C(L_1,L_2)$ of two disjoint null-homologous cycles L_1 and L_2 of respective dimensions d_1 and d_2 in an oriented (d_1+d_2+1)-manifold C is the algebraic intersection $\langle L_1,W_2\rangle_C$ of L_1 with a chain W_2 bounded by L_2 in C. This definition extends to rationally null-homologous cycles by bilinearity.

1.3 Introduction to the combinatorial definition of $\tilde{\Theta}$

In the end of this section, we give explicit formulas for the ingredients $\ell_2(\mathcal{D})$, $s_\ell(\mathcal{D},\mathfrak{m})$ and $e(\mathcal{D},w,\mathfrak{m})$ in the formula

$$\tilde{\Theta}(\mathcal{D},w,\mathfrak{m}) = \ell_2(\mathcal{D}) + s_\ell(\mathcal{D},\mathfrak{m}) - e(\mathcal{D},w,\mathfrak{m})$$

for a Heegaard diagram $\mathcal{D}$ equipped with a matching $\mathfrak{m}$ and an exterior point w. These ingredients will be studied in more details in Section 2.

Let $\mathcal{D} = (\partial H_{\mathcal{A}}, (\alpha_i)_{i\in\underline{g}}, (\beta_j)_{j\in\underline{g}})$ be a Heegaard diagram of a rational homology 3-sphere. A *crossing* c of $\mathcal{D}$ is an intersection point of a curve $\alpha_{i(c)} = \alpha(c)$ and a curve $\beta_{j(c)} = \beta(c)$. Its sign $\sigma(c)$ is 1 if $\partial H_{\mathcal{A}}$ is oriented by the oriented tangent vector of $\alpha(c)$ followed by the oriented tangent vector of $\beta(c)$ at c as above. It is (-1) otherwise. The set of crossings of $\mathcal{D}$ is denoted by $\mathcal{C}$.

Let

$$[\mathcal{J}_{ji}]_{(j,i)\in\underline{g}^2} = [\langle\alpha_i, \beta_j\rangle_{\partial H_{\mathcal{A}}}]^{-1}$$

denote the inverse matrix of the intersection matrix.

$$\sum_{i=1}^{g} \mathcal{J}_{ji}\langle\alpha_i, \beta_k\rangle_{\partial H_{\mathcal{A}}} = \delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

When d and e are two crossings of α_i, $[d, e]_{\alpha_i} = [d, e]_\alpha$ denotes the set of crossings from d to e (including them) along α_i, or the closed arc from d to e in α_i depending on the context. Then $[d, e[_\alpha = [d, e]_\alpha \setminus \{e\}$, $]d, e]_\alpha = [d, e]_\alpha \setminus \{d\}$ and $]d, e[_\alpha = [d, e[_\alpha \setminus \{d\}$.

Now, for such a part I of α_i,

$$\langle I, \beta_j\rangle = \langle I, \beta_j\rangle_{\partial H_{\mathcal{A}}} = \sum_{c\in I\cap\beta_j} \sigma(c).$$

We shall also use the notation $|$ for ends of arcs to say that an end is half-contained in an arc, and that it must be counted with coefficient 1/2. ("$[d, e|_\alpha = [d, e]_\alpha \setminus \{e\}/2$"). We agree that $|d, d|_\alpha = \emptyset$.

We use the same notation for arcs $[d, e|_{\beta_j} = [d, e|_\beta$ of β_j. For example, if d is a crossing of $\alpha_i \cap \beta_j$, then

$$\langle [d, d|_\alpha, \beta_j\rangle = \frac{\sigma(d)}{2}$$

and

$$\langle [c, d|_\alpha, [e, d|_\beta\rangle = \frac{\sigma(d)}{4} + \sum_{c\in[c,d[_\alpha\cap[e,d[_\beta} \sigma(c).$$

Example 1.1. In the Heegaard diagrams of $\mathbb{RP}^3$ in Figure 1, $\langle [c, c|_\alpha, [c, c|_\beta\rangle = \frac{1}{4}$, $\langle [c, c|_\alpha, [c, d|_\beta\rangle = \langle [c, d|_\alpha, [c, c|_\beta\rangle = \frac{1}{2}$, $\langle [c, d|_\alpha, [c, d|_\beta\rangle = \frac{5}{4}$, $\langle [c, c|_\alpha, \beta_1\rangle = \frac{1}{2}$ and $\langle [c, d|_\alpha, \beta_1\rangle = \frac{3}{2}$.

1.4 First combinatorial definitions of ℓ_2 and $s_\ell(\mathcal{D}, \mathfrak{m})$

Choose a matching $\mathfrak{m} = \{m_i; i \in \underline{g}\}$ where $m_i \in \alpha_{\rho^{-1}(i)} \cap \beta_i$, for a permutation ρ of $\underline{g}$. For two crossings c and d of $\mathcal{C}$, set

$$\tilde{\ell}_{\mathfrak{m}}(c, d) = \langle |m_{\rho(i(c))}, c|_\alpha, |m_{j(d)}, d|_\beta\rangle - \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle |m_{\rho(i(c))}, c|_\alpha, \beta_j\rangle\langle\alpha_i, |m_{j(d)}, d|_\beta\rangle.$$

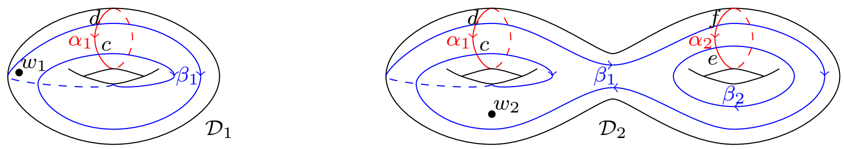

Figure 1: Two Heegaard diagrams of $\mathbb{RP}^3$

Then

$$\ell_2(\mathcal{D}) = \sum_{(c,d)\in\mathcal{C}^2} \mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)\tilde{\ell}_{\mathfrak{m}}(c,d) - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\tilde{\ell}_{\mathfrak{m}}(c,c)$$

and

$$s_\ell(\mathcal{D},\mathfrak{m}) = \sum_{(c,d)\in\mathcal{C}^2} \mathcal{J}_{j(c)i(c)}\mathcal{J}_{j(d)i(d)}\sigma(c)\sigma(d)\tilde{\ell}_{\mathfrak{m}}(c,d).$$

Example 1.2. For the genus one Heegaard diagram $\mathcal{D}_1$ of Figure 1, $\sigma(c) = 1$, $\langle\alpha_1,\beta_1\rangle_{\partial H_{\mathcal{A}}} = 2$, $\mathcal{J}_{11} = \frac{1}{2}$, choose $\{c\}$ as a matching, $\tilde{\ell}_{\{c\}}(c,c) = \tilde{\ell}_{\{c\}}(c,d) = \tilde{\ell}_{\{c\}}(d,c) = 0$, $\tilde{\ell}_{\{c\}}(d,d) = \frac{1}{2} - \mathcal{J}_{11} = 0$ so that $\ell_2(\mathcal{D}_1) = s_\ell(\mathcal{D}_1,\{c\}) = 0$.

For the genus two Heegaard diagram $\mathcal{D}_2$ of Figure 1, $\mathcal{J}_{11} = \frac{1}{2} = -\mathcal{J}_{21}$, $\mathcal{J}_{22} = 1$ and $\mathcal{J}_{12} = 0$, choose $\{c,e\}$ as a matching. For any crossing x of $\mathcal{D}_2$,

$$0 = \tilde{\ell}_{\{c,e\}}(c,x) = \tilde{\ell}_{\{c,e\}}(x,c) = \tilde{\ell}_{\{c,e\}}(e,x) = \tilde{\ell}_{\{c,e\}}(x,e) = \tilde{\ell}_{\{c,e\}}(d,d),$$

$$\begin{array}{rcl} \tilde{\ell}_{\{c,e\}}(f,f) & = & \frac{1}{4} - \frac{3}{4}\mathcal{J}_{11} - \frac{1}{4}\mathcal{J}_{12} - \frac{3}{4}\mathcal{J}_{21} - \frac{1}{4}\mathcal{J}_{22} = 0 \\ \tilde{\ell}_{\{c,e\}}(d,f) & = & \frac{3}{4} - \frac{3}{2}\mathcal{J}_{11} - \frac{1}{2}\mathcal{J}_{12} = 0 \\ \tilde{\ell}_{\{c,e\}}(f,d) & = & -\frac{1}{2}\mathcal{J}_{11} - \frac{1}{2}\mathcal{J}_{21} = 0 \end{array}$$

so that $\ell_2(\mathcal{D}_2) = s_\ell(\mathcal{D}_2,\{c,e\}) = 0$.

1.5 Combinatorial definition of $e(\mathcal{D},w,\mathfrak{m})$

Let w be an exterior point of $\mathcal{D}$. The choice of $\mathfrak{m}$ being fixed, represent the Heegaard diagrams in a plane by removing from $\partial H_{\mathcal{A}}$ a disk around w that does not intersect the diagram curves, and by cutting the surface $\partial H_{\mathcal{A}}$ along the α_i. Each α_i gives rise to two copies α'_i and α'_i of α_i that bound disjoint disks with opposite orientations in the plane. Locate the crossing m_i at the points with upward tangent vectors of α'_i and α''_i, and locate the other crossings near the points with downward tangent vectors as in Figure 2. Draw the arcs of the curves β_j so that they have horizontal tangent vectors near the crossings.

The rectangle has the standard parallelization of the plane. Then there is a map "unit tangent vector" from each partial projection of a beta curve β_j in the plane to S^1. The total

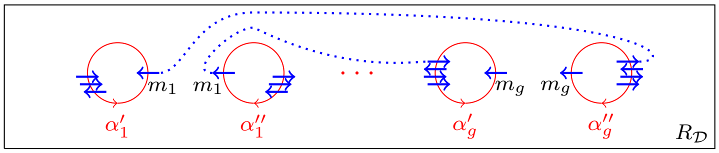

Figure 2: The Heegaard surface cut along the α_i and deprived of a neighborhood of w

degree of this map for the curve β_j is denoted by $d_e(\beta_j)$. For a crossing $c \in \beta_j$, $d_e(|m_j, c|_\beta) \in \frac{1}{2}\mathbb{Z}$ denotes the degree of the restriction of this map to the arc $|m_j, c|_\beta$. For any $c \in \mathcal{C}$, define

$$d_e(c) = d_e(|m_{j(c)}, c|_\beta) - \sum_{(r,s) \in \underline{g}^2} \mathcal{J}_{sr} \langle \alpha_r, |m_{j(c)}, c|_\beta \rangle d_e(\beta_s).$$

Set

$$e(\mathcal{D}, w, \mathfrak{m}) = \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)} \sigma(c) d_e(c)$$

and

$$\tilde{\Theta}(\mathcal{D}, w, \mathfrak{m}) = \ell_2(\mathcal{D}) + s_\ell(\mathcal{D}, \mathfrak{m}) - e(\mathcal{D}, w, \mathfrak{m}).$$

Example 1.3. For the rectangular diagram of $(\mathcal{D}_1, \{c\}, w_1)$ of Figure 3, $d_e(|c, c|_\beta) = 0$ and $d_e(c) = 0$, $d_e(|c, d|_\beta) = \frac{1}{2}$, $d_e(\beta_1) = 2$ so that $d_e(d) = -\frac{1}{2}$, $e(\mathcal{D}_1, w_1, \{c\}) = -\frac{1}{4}$ and $\tilde{\Theta}(\mathcal{D}_1, w_1, \{c\}) = \frac{1}{4}$.

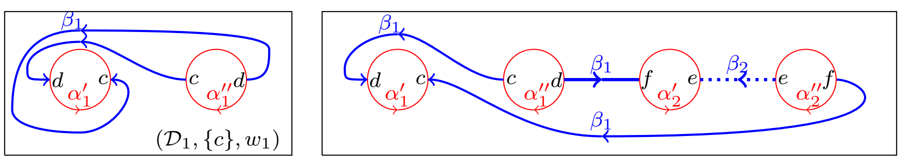

Figure 3: Rectangular diagrams of $(\mathcal{D}_1, \{c\}, w_1)$ and $(\mathcal{D}_2, \{c, e\}, w_2)$

For the rectangular diagram of $(\mathcal{D}_2, \{c, e\}, w_2)$ of Figure 3, $d_e(c) = d_e(e) = d_e(\beta_1) = d_e(\beta_2) = 0$, $d_e(d) = d_e(f) = \frac{1}{2}$, $e(\mathcal{D}_2, w_2, \{c\}) = \frac{1}{4}$ and $\tilde{\Theta}(\mathcal{D}_2, w_2, \{c\}) = -\frac{1}{4}$.

2 More on the combinatorial definition of $\tilde{\Theta}$

In this section, we show that the quantities $\ell_2(\mathcal{D})$, $s_\ell(\mathcal{D}, \mathfrak{m})$ and $e(\mathcal{D}, w, \mathfrak{m})$ defined in the previous section for a Heegaard diagram $\mathcal{D}$ equipped with a matching $\mathfrak{m}$ and an exterior point w only depend on their arguments (e.g. on $\mathcal{D}$, for $\ell_2(\mathcal{D})$ ...) and not on extra data used to define them like numberings or orientations of the diagram curves. We also give alternative definitions of $\ell_2(\mathcal{D})$ and $s_\ell(\mathcal{D}, \mathfrak{m})$.

2.1 More on $e(\mathcal{D},w,\mathfrak{m})$

Recall the notation and definitions of Subsection 1.5 with respect to a fixed matching $\mathfrak{m} = \{m_i; i \in \underline{g}\}$ where $m_i \in \alpha_{\rho^{-1}(i)} \cap \beta_i$, for a permutation ρ of $\underline{g}$.

Lemma 2.1.

$$e(\mathcal{D},w,\mathfrak{m}) = \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)d_e(c)$$

does not depend on our specific way of drawing the diagram with our conventions. It only depends on $\mathcal{D}$, w and $\mathfrak{m}$.

The topological interpretation of $e(\mathcal{D},w,\mathfrak{m})$ as an Euler class given in Corollary 4.3 yields a conceptual proof of this lemma. We nevertheless give a purely combinatorial proof below.

We use the Kronecker symbol δ_{cd} that is 1 if $c = d$ and 0 otherwise. We first prove the following lemma.

Lemma 2.2. *A full positive twist of a curve α'_i or a curve α''_i in Figure 2 changes $d_e(c)$ to $d_e(c) + \frac{1}{2}\delta_{i(c)i} - \frac{1}{2}\delta_{\rho(i)j(c)}$.*

Proof: When a crossing is moved counterclockwise along a curve α, (like along α''_i in Figure 14) the degree increases (by 1 for a full loop) when the crossing enters (the disk bounded by) α in Figure 2 and decreases when the crossing goes out. Furthermore the positive crossings enter α'_i and the negative ones enter α''_i. Then letting all the crossings make a full positive loop around α''_i (resp. around α'_i) changes $d_e(\beta_s)$ to $d_e(\beta_s) - \langle\alpha_i,\beta_s\rangle$ (resp. to $d_e(\beta_s) + \langle\alpha_i,\beta_s\rangle$). Now, for a full positive loop around α''_i, $d_e(|m_{j(c)},c|_\beta)$ is changed to

$$d_e(|m_{j(c)},c|_\beta) - \langle\alpha_i,]m_{j(c)},c[_\beta\rangle - \delta_{i(c)i}\delta_{(-1)\sigma(c)}\sigma(c) - \delta_{\rho(i)j(c)}\delta_{1\sigma(m_{j(c)})}\sigma(m_{j(c)}).$$

Indeed, right before c, $\beta_{j(c)}$ hits α''_i iff $\sigma(c) = -1$ and $i(c) = i$. Similarly, after $m_{j(c)}$, $\beta_{j(c)}$ exits α''_i iff $\sigma(c) = 1$ and $\rho^{-1}(j(c)) = i$. This expression can be rewritten as

$$d_e(|m_{j(c)},c|_\beta) - \langle\alpha_i, |m_{j(c)},c|_\beta\rangle + \frac{1}{2}\delta_{i(c)i} - \frac{1}{2}\delta_{\rho(i)j(c)}.$$

Similarly, for a full positive loop around α'_i, $d_e(|m_{j(c)},c|_\beta)$ is changed to

$$d_e(|m_{j(c)},c|_\beta) + \langle\alpha_i, |m_{j(c)},c|_\beta\rangle + \frac{1}{2}\delta_{i(c)i} - \frac{1}{2}\delta_{\rho(i)j(c)}.$$

Now, since

$$\sum_{(r,s)\in\underline{g}^2} \mathcal{J}_{sr}\langle\alpha_r, |m_{j(c)},c|_\beta\rangle\langle\alpha_i,\beta_s\rangle = \langle\alpha_i, |m_{j(c)},c|_\beta\rangle,$$

$d_e(c)$ is changed to $d_e(c) + \frac{1}{2}\delta_{i(c)i} - \frac{1}{2}\delta_{\rho(i)j(c)}$ in both cases. $\square$

Proof of Lemma 2.1: Note that $e(\mathcal{D},w,\mathfrak{m})$ does not depend on the numberings of the diagram curves. We prove that $e(\mathcal{D},w,\mathfrak{m})$ does not depend on our specific way of drawing the diagram with our conventions when the orientations of the diagram curves are fixed. When the curves α'_i and α''_i move in the plane without being twisted, the $d_e(c)$ stay in $\frac{1}{2}\mathbb{Z}$ and are therefore invariant. Therefore it suffices to prove that $e(\mathcal{D}, w, \mathfrak{m})$ is invariant under a full twist of a curve α'_i or a curve α''_i. Since

$$\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)(\delta_{i(c)i} - \delta_{\rho(i)j(c)}) = \sum_{c\in\alpha_i} \mathcal{J}_{j(c)i}\sigma(c) - \sum_{c\in\beta_{\rho(i)}} \mathcal{J}_{\rho(i)i(c)}\sigma(c) = 1 - 1 = 0,$$

according to Lemma 2.2, $e(\mathcal{D}, w, \mathfrak{m})$ does not vary under these moves. It is not hard to prove that $e(\mathcal{D}, w, \mathfrak{m})$ does not depend on the orientations of the curves β. Changing the orientation of a curve α permutes α'_i and α''_i and does not modify $e(\mathcal{D}, w, \mathfrak{m})$ either so that the lemma is proved. □

2.2 More on $s_\ell(\mathcal{D}, \mathfrak{m})$

Fix a point a_i inside each disk $D(\alpha_i)$ and a point b_j inside each disk $D(\beta_j)$. Then join a_i to each crossing c of α_i by a segment $[a_i, c]_{D(\alpha_i)}$ oriented from a_i to c in $D(\alpha_i)$, so that these segments only meet at a_i for different c. Similarly define segments $[c, b_{j(c)}]_{D(\beta_{j(c)})}$ from c to $b_{j(c)}$ in $D(\beta_{j(c)})$. Then for each c, define the *flow line* $\gamma(c) = [a_{i(c)}, c]_{D(\alpha_{i(c)})} \cup [c, b_{j(c)}]_{D(\beta_{j(c)})}$. When $\gamma(c)$ is smooth, $\gamma(c)$ is a flow line closure of a Morse function f_M giving birth to $\mathcal{D}$ discussed in the introduction.

For each point a_i in the disk $D(\alpha_i)$ as in Subsection 1.1, choose a point a_i^+ and a point a_i^- close to a_i outside $D(\alpha_i)$ so that a_i^+ is on the positive side of $D(\alpha_i)$ (the side of the positive normal) and a_i^- is on the negative side of $D(\alpha_i)$. Similarly fix points b_j^+ and b_j^- close to the b_j and outside the $D(\beta_j)$.

Then for a crossing $c \in \alpha_{i(c)} \cap \beta_{j(c)}$, $\gamma(c)_\parallel$ will denote the following chain. Consider a small meridian curve $m(c)$ of $\gamma(c)$ on $\partial H_{\mathcal{A}}$, it intersects $\beta_{j(c)}$ at two points: $c_{\mathcal{A}}^+$ on the positive side of $D(\alpha_{i(c)})$ and $c_{\mathcal{A}}^-$ on the negative side of $D(\alpha_{i(c)})$. The meridian $m(c)$ also intersects $\alpha_{i(c)}$ at $c_{\mathcal{B}}^+$ on the positive side of $D(\beta_{j(c)})$ and $c_{\mathcal{B}}^-$ on the negative side of $D(\beta_{j(c)})$. Let $[c_{\mathcal{A}}^+, c_{\mathcal{B}}^+]$, $[c_{\mathcal{A}}^+, c_{\mathcal{B}}^-]$, $[c_{\mathcal{A}}^-, c_{\mathcal{B}}^+]$ and $[c_{\mathcal{A}}^-, c_{\mathcal{B}}^-]$ denote the four quarters of $m(c)$ with the natural ends and orientations associated with the notation, as in Figure 4.

Figure 4: $m(c)$, $c_{\mathcal{A}}^+$, $c_{\mathcal{A}}^-$, $c_{\mathcal{B}}^+$ and $c_{\mathcal{B}}^-$

Let $\gamma^+_{\mathcal{A}}(c)$ (resp. $\gamma^-_{\mathcal{A}}(c)$) be an arc parallel to $[a_{i(c)}, c]_{D(\alpha_{i(c)})}$ from $a^+_{i(c)}$ to $c^+_{\mathcal{A}}$ (resp. from $a^-_{i(c)}$ to $c^-_{\mathcal{A}}$) that does not meet $D(\alpha_{i(c)})$. Let $\gamma^+_{\mathcal{B}}(c)$ (resp. $\gamma^-_{\mathcal{B}}(c)$) be an arc parallel to $[c, b_{j(c)}]_{D(\beta_{j(c)})}$ from $c^+_{\mathcal{B}}$ to $b^+_{j(c)}$ (resp. from $c^-_{\mathcal{B}}$ to $b^-_{j(c)}$) that does not meet $D(\beta_{j(c)})$.

$$\gamma(c)_{\|} = \frac{1}{2}(\gamma^+_{\mathcal{A}}(c) + \gamma^-_{\mathcal{A}}(c)) + \frac{1}{4}([c^+_{\mathcal{A}}, c^+_{\mathcal{B}}] + [c^+_{\mathcal{A}}, c^-_{\mathcal{B}}] + [c^-_{\mathcal{A}}, c^+_{\mathcal{B}}] + [c^-_{\mathcal{A}}, c^-_{\mathcal{B}}]) + \frac{1}{2}(\gamma^+_{\mathcal{B}}(c) + \gamma^-_{\mathcal{B}}(c)).$$

Set $a_{i\|} = \frac{1}{2}(a^+_i + a^-_i)$ and $b_{j\|} = \frac{1}{2}(b^+_j + b^-_j)$. Then $\partial\gamma(c)_{\|} = b_{j(c)\|} - a_{i(c)\|}$.

Recall our matching $\mathfrak{m} = \{m_i; i \in \underline{g}\}$ where $m_i \in \alpha_{\rho^{-1}(i)} \cap \beta_i$, for a permutation ρ of $\underline{g}$, so that $\gamma_i = \gamma(m_i)$ goes from $a_{\rho^{-1}(i)}$ to b_i.

Set

$$L(\mathcal{D}, \mathfrak{m}) = \sum_{i=1}^{g} \gamma_i - \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\gamma(c).$$

Note that $L(\mathcal{D}, \mathfrak{m})$ is a cycle since

$$\partial L(\mathcal{D}, \mathfrak{m}) = \sum_{i=1}^{g}(b_i - a_i) - \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle\alpha_i, \beta_j\rangle_{\partial H_{\mathcal{A}}}(b_j - a_i) = 0.$$

Set $L(\mathcal{D}, \mathfrak{m})_{\|} = \sum_{i=1}^{g} \gamma_i - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\gamma(c)_{\|}$.

In this subsection, we prove the following proposition.

Proposition 2.3. *For any Heegaard diagram $\mathcal{D}$ equipped with a matching $\mathfrak{m}$,*

$$s_\ell(\mathcal{D}, \mathfrak{m}) = lk(L(\mathcal{D}, \mathfrak{m}), L(\mathcal{D}, \mathfrak{m})_{\|}).$$

This proposition has the following easy corollary.

Corollary 2.4. *The real number $s_\ell(\mathcal{D}, \mathfrak{m})$ is an invariant of the Heegaard diagram $\mathcal{D}$ equipped with $\mathfrak{m}$ that does not depend on the orientations and numberings of the curves α_i and β_j, and that does not change when the roles of the α-curves or the β-curves are permuted.*

□

We first prove the following lemma that will be useful later, too.

Lemma 2.5. *For any curve α_i (resp. β_j), choose a basepoint $p(\alpha_i)$ (resp. $p(\beta_j)$). These choices being made, for any crossing c of $\mathcal{C}$, define the triangle $T_\beta(c)$ in the disk $D(\beta_{j(c)})$ such that*

$$\partial T_\beta(c) = [p(\beta(c)), c]_\beta + (\gamma(c) \cap H_{\mathcal{B}}) - (\gamma(p(\beta(c))) \cap H_{\mathcal{B}}).$$

Similarly, define the triangle $T_\alpha(c)$ in the disk $D(\alpha_{i(c)})$ such that

$$\partial T_\alpha(c) = -[p(\alpha(c)), c]_\alpha + (\gamma(c) \cap H_{\mathcal{A}}) - (\gamma(p(\alpha(c))) \cap H_{\mathcal{A}}).$$

Let $K=\sum_{c\in\mathcal{C}}k_c\gamma(c)$ *be a cycle of* M.

Let $\Sigma_T(K)=\sum_{c\in\mathcal{C}}k_c(T_\alpha(c)+T_\beta(c))$ *and*

$$\Sigma_D(K)=\sum_{(i,j,c)\in\underline{g}^2\times\mathcal{C}}\mathcal{J}_{ji}k_c\left(\langle|p(\alpha(c)),c|_\alpha,\beta_j\rangle D(\alpha_i)-\langle\alpha_i,|p(\beta(c)),c|_\beta\rangle D(\beta_j)\right).$$

There exists a 2-chain $\Sigma_\Sigma(K)$ *in* $\partial H_{\mathcal{A}}$ *whose boundary* $\partial\Sigma_\Sigma(K)$ *is*

$$\sum_{(i,j,c)\in\underline{g}^2\times\mathcal{C}}\mathcal{J}_{ji}k_c\left(\langle\alpha_i,|p(\beta(c)),c|_\beta\rangle\beta_j-\langle|p(\alpha(c)),c|_\alpha,\beta_j\rangle\alpha_i\right)+\sum_{c\in\mathcal{C}}k_c([p(\alpha(c)),c]_\alpha-[p(\beta(c)),c]_\beta)$$

so that the boundary of

$$\Sigma(K)=\Sigma_\Sigma(K)+\Sigma_D(K)+\Sigma_T(K)$$

is K.

Though it is not visible from the notation, the surfaces depend on the basepoints.

Proof of Lemma 2.5:

$$\partial\Sigma_T(K)-K=\sum_{c\in\mathcal{C}}k_c([p(\beta(c)),c]_\beta-[p(\alpha(c)),c]_\alpha)$$

is a cycle. Any 1-cycle σ of $\partial H_{\mathcal{A}}$ is homologous to $\sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}(\langle\sigma,\beta_j\rangle\alpha_i+\langle\alpha_i,\sigma\rangle\beta_j)$. Therefore by pushing $(\partial\Sigma_T(K)-K)$ in the directions of the positive and negative normals to the α and the β in $\partial H_{\mathcal{A}}$, and by averaging, we see that $(K-\partial\Sigma_T(K))$ is homologous in $\partial H_{\mathcal{A}}$ to

$$\sum_{(i,j,c)\in\underline{g}^2\times\mathcal{C}}\mathcal{J}_{ji}k_c\left(\langle|p(\alpha(c)),c|_\alpha,\beta_j\rangle\alpha_i-\langle\alpha_i,|p(\beta(c)),c|_\beta\rangle\beta_j\right)$$

that bounds

$$\Sigma_D(K)=\sum_{(i,j,c)\in\underline{g}^2\times\mathcal{C}}\mathcal{J}_{ji}k_c\left(\langle|p(\alpha(c)),c|_\alpha,\beta_j\rangle D(\alpha_i)-\langle\alpha_i,|p(\beta(c)),c|_\beta\rangle D(\beta_j)\right).$$

□

Proposition 2.6. *For any curve* α_i *(resp.* β_j*), choose a basepoint* $p(\alpha_i)$ *(resp.* $p(\beta_j)$*). These choices being fixed, set*

$$\tilde{\ell}(c,d)=\langle|p(\alpha(c)),c|_\alpha,|p(\beta(d)),d|_\beta\rangle-\sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\langle|p(\alpha(c)),c|_\alpha,\beta_j\rangle\langle\alpha_i,|p(\beta(d)),d|_\beta\rangle).$$

Let $K=\sum_{c\in\mathcal{C}}k_c\gamma(c)$ *and* $L=\sum_{d\in\mathcal{C}}g_d\gamma(d)$ *be two 1-cycles of* M. *Then*

$$lk(K,L_\|)=lk(L,K_\|)=\sum_{(c,d)\in\mathcal{C}^2}k_cg_d\tilde{\ell}(c,d).$$

PROOF: The first equality comes from the symmetry of the linking number and from the observation that $lk(K, L_{\|}) = lk(K_{\|}, L)$. Compute $lk(K, L_{\|})$ as the intersection of $L_{\|}$ with the surface bounded by K provided by Lemma 2.5. Thus $lk(K, L_{\|}) = \langle \Sigma_\Sigma(K), L_{\|}\rangle$. Now, since $L = \sum_{d\in\mathcal{C}} g_d\gamma(d)$ is a cycle,

$$L = \sum_{d\in\mathcal{C}} g_d(\gamma(d) - \gamma(p(\beta(d))))$$

and it suffices to prove the result when $L = \gamma(d) - \gamma(p(\beta(d)))$. For any path $[x, y]$ from a point x to a point y in $\partial H_{\mathcal{A}}$, when x and y are outside $\partial\Sigma_\Sigma(K)$,

$$\langle x - y, \Sigma_\Sigma(K)\rangle_\Sigma = \langle [x, y], \partial\Sigma_\Sigma(K)\rangle_{\partial H_{\mathcal{A}}}.$$

Thus by averaging,

$$\langle \gamma(d)_{\|} - \gamma(p(\beta(d)))_{\|}, \Sigma_\Sigma(K)\rangle = \langle \partial\Sigma_\Sigma(K), |p(\beta(d)), d|_\beta\rangle_{\partial H_{\mathcal{A}}}.$$

This is

$$\begin{array}{l}\sum_{c\in\mathcal{C}} k_c\langle |p(\alpha(c)), c|_\alpha, |p(\beta(d)), d|_\beta\rangle_{\partial H_{\mathcal{A}}} \\ -\sum_{(i,j,c)\in\underline{g}^2\times\mathcal{C}} k_c\mathcal{J}_{ji}\left(\langle |p(\alpha(c)), c|_\alpha, \beta_j\rangle\langle \alpha_i, |p(\beta(d)), d|_\beta\rangle_{\partial H_{\mathcal{A}}}\right)\end{array} = \sum_{c\in\mathcal{C}} k_c\left(\tilde{\ell}(c, d) - \tilde{\ell}(c, p(\beta(d)))\right).$$

□

PROOF OF PROPOSITION 2.3: Apply Proposition 2.6 with the basepoints of $\mathfrak{m}$ so that $\tilde{\ell} = \tilde{\ell}_{\mathfrak{m}}$.
□

2.3 More on $\ell_2(\mathcal{D})$

Proposition 2.7. *For any curve α_i (resp. β_j), choose a basepoint $p(\alpha_i)$ (resp. $p(\beta_j)$). These choices being made, for two crossings c and d of $\mathcal{C}$, set*

$$\ell(c, d) = \langle [p(\alpha(c)), c|_\alpha, [p(\beta(d)), d|_\beta\rangle - \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle [p(\alpha(c)), c|_\alpha, \beta_j\rangle\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle$$

and $\tilde{\ell}(c, d) = \langle |p(\alpha(c)), c|_\alpha, |p(\beta(d)), d|_\beta\rangle - \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle |p(\alpha(c)), c|_\alpha, \beta_j\rangle\langle \alpha_i, |p(\beta(d)), d|_\beta\rangle$. Then, for any 2–cycle $G = \sum_{(c,d)\in\mathcal{C}^2} g_{cd}(\gamma(c) \times \gamma(d)_{\|})$ of M^2,

$$\ell^{(2)}(G) = \sum_{(c,d)\in\mathcal{C}^2} g_{cd}\ell(c, d) = \sum_{(c,d)\in\mathcal{C}^2} g_{cd}\tilde{\ell}(c, d).$$

Furthermore, $\ell^{(2)}(G)$ is independent of the choices of the basepoints $p(\alpha_i)$ or $p(\beta_j)$, and of the numberings and orientations of the curves α_i or β_j.

Proof: Let ℓ' be defined as ℓ except that the basepoint $p_i = p(\alpha_i)$ of α_i is changed to a basepoint q_i. When $c \in \alpha_i \setminus [p_i, q_i[_\alpha$,

$$\ell'(c,d) - \ell(c,d) = -\langle [p_i, q_i[_\alpha, [p(\beta(d)), d|_\beta \rangle + \sum_{(r,j)\in \underline{g}^2} \mathcal{J}_{jr} \langle [p_i, q_i[_\alpha, \beta_j\rangle \langle \alpha_r, [p(\beta(d)), d|_\beta\rangle \tag{2.8}$$

When $c \in [p_i, q_i[_\alpha$, $[q_i, c|_\alpha \setminus [p_i, c|_\alpha = [q_i, p_i[_\alpha = \alpha_i \setminus [p_i, q_i[_\alpha$. Since

$$\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle = \sum_{(r,j)\in \underline{g}^2} \mathcal{J}_{jr} \langle \alpha_i, \beta_j\rangle \langle \alpha_r, [p(\beta(d)), d|_\beta\rangle,$$

$\ell'(c,d) - \ell(c,d)$ is given by Formula 2.8 that does not depend on $c \in \alpha_i$ in this case either. Then

$$\sum_{(c,d)\in \mathcal{C}^2} g_{cd}(\ell'(c,d) - \ell(c,d)) = \sum_{(c,d)\in \alpha_i \times \mathcal{C}} g_{cd}(\ell'(c,d) - \ell(c,d)).$$

Since

$$\partial(\gamma(c) \times \gamma(d)_\parallel) = (b_{j(c)} - a_{i(c)}) \times \gamma(d)_\parallel - \gamma(c) \times (b_{j(d)\parallel} - a_{i(d)\parallel}),$$

for any $d \in \mathcal{C}$, $\sum_{c\in\alpha_i} g_{cd} = 0$. Since the right-hand side of Formula 2.8 does not depend on $c \in \alpha_i$, this shows that $\sum_{(c,d)\in\mathcal{C}^2} g_{cd}\ell(c,d)$ does not depend on the basepoint choice on α_i. Similarly, it does not depend on the choices of the basepoints on the β_j.

Similarly, $\sum_{(c,d)\in\mathcal{C}^2} g_{cd}\ell(c,d) = \sum_{(c,d)\in\mathcal{C}^2} g_{cd}\tilde{\ell}(c,d)$.

Using $\tilde{\ell}$, changing the orientation of α_i changes $|p(\alpha(c)), c|_\alpha$ to $-\alpha_i + |p(\alpha(c)), c|_\alpha$ for $c \in \alpha_i$, and does not change $\sum_{(c,d)\in\mathcal{C}^2} g_{cd}\tilde{\ell}(c,d)$. □

Remark 2.9. Let $[S]$ be the homology class of $\{x\} \times \partial B_x$ in $M^2 \setminus \text{diagonal}$, where B_x is a ball of M and x is a point inside B_x. Then $H_2(M^2 \setminus \text{diagonal}; \mathbb{Q}) = \mathbb{Q}[S]$, and it is proved in [Les12a, Proposition 3.4] that the class of a 2–cycle

$$G = \sum_{(c,d)\in\mathcal{C}^2} g_{cd}(\gamma(c) \times \gamma(d)_\parallel)$$

in $H_2(M^2 \setminus \text{diagonal}; \mathbb{Q})$ is $\ell^{(2)}(G)[S]$. Furthermore, for two disjoint one-cycles K and L of M, the class of $K \times L$ in $H_2(M^2 \setminus \text{diagonal}; \mathbb{Q})$ is $lk(K,L)[S]$ so that Proposition 2.6 provides an alternative proof of [Les12a, Proposition 3.4] when G is the product of two one-cycles. This is the needed case to produce combinatorial expressions of linking numbers involved in the variations of p_1 that we are going to study later.

Proposition 2.10. *Set*

$$G(\mathcal{D}) = \sum_{(c,d)\in\mathcal{C}^2} \mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)(\gamma(c) \times \gamma(d)_\parallel) - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)(\gamma(c) \times \gamma(c)_\parallel).$$

Then $G(\mathcal{D})$ is a 2–cycle of M^2. Let $\ell_2(\mathcal{D}) = \ell^{(2)}(G(\mathcal{D}))$. Then $\ell_2(\mathcal{D})$ is an invariant of the Heegaard diagram that does not depend on the orientations and numberings of the curves α_i and β_j. It not change when the roles of the α-curves or the β-curves are permuted either.

Proof: Let us first prove that $G(\mathcal{D})$ is a 2-cycle. Note that, for any j,

$$\sum_{c\in\beta_j}\mathcal{J}_{j(d)i(c)}\sigma(c)=\sum_{i=1}^{g}\mathcal{J}_{j(d)i}\langle\alpha_i,\beta_j\rangle=\delta_{jj(d)}=\begin{cases}1 & \text{if } j=j(d)\\ 0 & \text{if } j\neq j(d)\end{cases}$$

and, for any i, $\sum_{c\in\alpha_i}\mathcal{J}_{j(c)i(d)}\sigma(c)=\sum_{j=1}^{g}\mathcal{J}_{ji(d)}\langle\alpha_i,\beta_j\rangle=\delta_{ii(d)}$. Therefore, for any $d\in\mathcal{C}$,

$$\partial\left(\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\gamma(c)\right)=\mathcal{J}_{j(d)i(d)}(b_{j(d)}-a_{i(d)})=\mathcal{J}_{j(d)i(d)}\partial\gamma(d)$$

so that

$$\partial\left(\sum_{(c,d)\in\mathcal{C}^2}\mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)(\gamma(c)\times\gamma(d)_{\parallel})\right)=$$

$$\sum_{d\in\mathcal{C}}\sigma(d)\mathcal{J}_{j(d)i(d)}(\partial\gamma(d))\times\gamma(d)_{\parallel}-\sum_{c\in\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)\gamma(c)\times\partial\gamma(c)_{\parallel}$$

and $\partial G(\mathcal{D})=0$.

Permuting the roles of the α_i and the β_j reverses the orientation of $\partial H_{\mathcal{A}}$ and changes $\mathcal{J}$ to the transposed matrix. It does not change $\ell_2(\mathcal{D})$ because of the symmetry in the definition of $\ell^{(2)}$. □

3 The ∞-combings $X(w,\mathfrak{m})$ and their p_1

3.1 On the ∞-combing $X(w,\mathfrak{m})$

In order to finish our description of $X(w,\mathfrak{m})$ started in the introduction, we need to describe the vector field that replaces the gradient field X_{f_M} in regular neighborhoods $N(\gamma_i=\gamma(m_i))$ of the flow lines γ_i associated with a matching $\mathfrak{m}$ of $\mathcal{D}$. Up to renumbering and reorienting the β_j, assume that $m_i\in\alpha_i\cap\beta_i$ to simplify notation.

Choose a natural trivialization (X_1,X_2,X_3) of $T\check{M}$ on a regular neighborhood $N(\gamma_i)$ of γ_i, such that:

- γ_i is directed by X_1,
- the other flow lines never have X_1 as an oriented tangent vector,
- (X_1,X_2) is tangent to the ascending manifold $\mathcal{A}_i$ of a_i (except on the parts of $\mathcal{A}_i$ near b_i that come from other crossings of $\alpha_i\cap\beta_i$), and (X_1,X_3) is tangent to the descending manifold $\mathcal{B}_i$ of b_i (except on the parts of $\mathcal{B}_i$ near a_i that come from other crossings of $\alpha_i\cap\beta_i$).

This parallelization identifies the unit tangent bundle $UN(\gamma_i)$ of $N(\gamma_i)$ with $S^2 \times N(\gamma_i)$.

There is a homotopy $h\colon [0,1] \times (N(\gamma_i) \setminus \gamma_i) \to S^2$, such that

- $h(0,.)$ is the unit vector with the same direction as the gradient vector of the underlying Morse function f_M,
- $h(1,.)$ is the constant map to $(-X_1)$ and
- $h(t,y)$ goes from $h(0,y)$ to $(-X_1)$ along the shortest geodesic arc $[h(0,y), -X_1]$ of S^2 from $h(0,y)$ to $(-X_1)$.

Let 2η be the distance between γ_i and $\partial N(\gamma_i)$ and let $X(y) = h(\max(0, 1 - d(y,\gamma_i)/\eta), y)$ on $N(\gamma_i) \setminus \gamma_i$, and $X = -X_1$ along γ_i.

Note that X is tangent to $\mathcal{A}_i$ on $N(\gamma_i)$ (except on the parts of $\mathcal{A}_i$ near b_i that come from other crossings of $\alpha_i \cap \beta_i$), and that X is tangent to $\mathcal{B}_i$ on $N(\gamma_i)$ (except on the parts of $\mathcal{B}_i$ near a_i that come from other crossings of $\alpha_i \cap \beta_i$). More generally, project the normal bundle of γ_i to $\mathbb{R}^2$ in the X_1–direction by sending γ_i to 0, $\mathcal{A}_i$ to an axis $\mathcal{L}_i(A)$ and $\mathcal{B}_i$ to an axis $\mathcal{L}_i(B)$. Then the projection of X goes towards 0 along $\mathcal{L}_i(B)$ and starts from 0 along $\mathcal{L}_i(A)$, it has the direction of $s_a(y)$ at a point y of $\mathbb{R}^2$ near 0, where s_a is the planar reflexion that fixes $\mathcal{L}_i(A)$ and reverses $\mathcal{L}_i(B)$. See Figure 5.

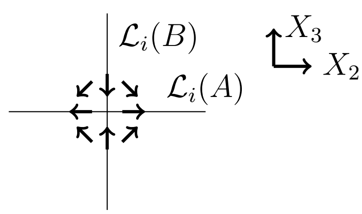

Figure 5: Projection of X

Then $X(y)$ is on the half great circle that contains $s_a(y)$, X_1 and $(-X_1)$. In Figure 6, γ_i is a vertical segment, all the other flow lines corresponding to crossings involving α_i go upward from a_i, and X is simply the upward vertical field.

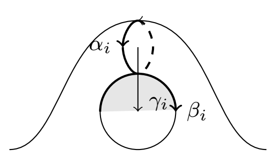

Figure 6: γ_i

3.2 On $p_1(X(w,\mathfrak{m}))$

Invariants p_1 of ∞-combings –or of *combings* that are homotopy classes of sections of the unit tangent bundle– of rational homology spheres M valued in $\mathbb{Q}$ have been introduced and studied in [Les13] as extensions of a relative first Pontrjagin class from parallelizations to combings.

For a combing that extends to a parallelization τ, the map p_1 coincides with the Hirzebruch defect (or Pontrjagin number) of the parallelization τ, studied in [Hir73, KM99, Les04a, Les12b]. For a parallelization $\tau\colon M\times\mathbb{R}^3\to TM$ of a 3-manifold M that bounds a connected oriented 4-dimensional manifold W with signature 0, $p_1(\tau)$ is defined as the evaluation at the fundamental class of $[W,\partial W]$ of the relative first Pontrjagin class of TW equipped with the trivialization of $TW_{|\partial W}$ that is the stabilization by the "outward normal exterior first" of τ. For ∞-combings that extend to parallelizations standard near ∞, p_1 is defined similarly by replacing W by a connected oriented signature 0 cobordism W_c with corners between $B(1)$ and the rational homology ball B_M. A neighborhood of the boundary

$$\partial W_c = -B(1)\cup_{\partial B(1)\sim 0\times B(1)} (-[0,1]\times\partial B(1))\cup_{\partial B_M\sim 1\times\partial B(1)} B_M,$$

of such a cobordism is naturally identified with an open subspace of one of the products $[0,1[\times B(1)$ or $]0,1]\times B_M$ near ∂W_c, so that the standard parallelization of $\mathbb{R}^3$ and τ induce a trivialization of $TW_{c|\partial W_c}$ by stabilizing by the "tangent vector to $[0,1]$ first". For more details, see [Les04a, Section 1.5].

Recall that any smooth compact oriented 3-manifold M can be equipped with a parallelization τ. When such a parallelization τ of M is given, two sections X and Y of $U\check{M}$ or UM induce a map $(X,Y)\colon \check{M}\to S^2\times S^2$. Such sections are said to be *transverse* if the graphs of the induced maps (X,Y) and $(X,-Y)$ are transverse to $\check{M}\times\mathrm{diag}(S^2\times S^2)$ in $\check{M}\times S^2\times S^2$. This is generic and independent of τ. For two transverse sections X and Y, let $L_{X=Y}$ be the preimage of the diagonal of S^2 under the map (X,Y). Thus $L_{X=Y}$ is an oriented link that is cooriented by the fiber of the normal bundle to the diagonal of $(S^2)^2$. It can be assumed to avoid $\infty\in M$, generically. In [Les13, Theorem 1.2], we proved that our extensions p_1 satisfy the following property that finishes defining them unambiguously.

Theorem 3.1. *When X and Y are two transverse ∞-combings (resp. combings) of a rational homology sphere M,*

$$p_1(Y)-p_1(X)=4lk(L_{X=Y},L_{X=-Y}).$$

In [Les13, Section 4.3], we also proved that for combings, p_1 coincides with the invariant θ_G defined by Gompf in [Gom98, Section 4].

The following properties of p_1 are easy to deduce from its definition.

Proposition 3.2.

- *A constant nonzero section N of $T\mathbb{R}^3$ is an ∞-combing of S^3 such that $p_1(N)=0$.*
- *Let M and M' be rational homology spheres equipped with ∞-combings X and X'. Assume that X' coincides with a constant section N of $B(1)$ on $\partial B_{M'}$ and that there is a*

standard ball $B(1)$ embedded in M where X coincides with N. Replacing this embedded ball $(B(1),N)$ by $(B_{M'},X')$ gives rise to an ∞-combing X'' of the obtained manifold whose p_1 is $p_1(X)+p_1(X')$.

- *Changing the orientation of M changes $p_1(X)$ to $-p_1(X)$.*

□

Let ξ be an oriented plane bundle over a compact oriented surface S and let σ be a nowhere vanishing section of ξ on ∂S. The *relative Euler number* $e(\xi,S,\sigma)$ of σ is the algebraic intersection of an extension of σ to S with the zero section of ξ. When S is connected, it is the obstruction to extending σ as a nowhere vanishing section of ξ. The following proposition is a direct corollary of consequences of Theorem 3.1 derived in [Les13].

Proposition 3.3. *Let $\mathfrak{m}$ and $\mathfrak{m}'$ be two matchings of $\mathcal{D}$. Let $L(\mathfrak{m}',\mathfrak{m})=L(\mathcal{D},\mathfrak{m}')-L(\mathcal{D},\mathfrak{m})$, and let $\Sigma(L(\mathfrak{m}',\mathfrak{m}))$ be a compact oriented surface bounded by $L(\mathfrak{m}',\mathfrak{m})$ in $M\setminus(S^3\setminus B(1))$. Consider the four following fields Y^{++}, Y^{+-}, $(Y^{-+}=-Y^{+-})$ and $(Y^{--}=-Y^{++})$ in a neighborhood of the $\gamma(c)$. Y^{++} and Y^{+-} are positive normals for $\mathcal{A}_i$ (that is oriented like $D(\alpha_i)$) on $\mathcal{A}_i\cap f_M^{-1}(]-\infty,3])$, and Y^{++} and Y^{-+} are positive normals for $\mathcal{B}_j$ on $\mathcal{B}_j\cap f_M^{-1}([3,+\infty[)$. These four fields are orthogonal to $X(w,\mathfrak{m})$ over $L(\mathfrak{m}',\mathfrak{m})$ and they define parallels $L(\mathfrak{m}',\mathfrak{m})_{\parallel Y^{\varepsilon,\eta}}$ of $L(\mathfrak{m}',\mathfrak{m})$ obtained by pushing in the $Y^{\varepsilon,\eta}$-direction. Then*

$$p_1(X(w,\mathfrak{m}'))-p_1(X(w,\mathfrak{m}))=-\sum_{(\varepsilon,\eta)\in\{+,-\}^2} lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}',\mathfrak{m})_{\parallel Y^{\varepsilon,\eta}})+E(w,\mathfrak{m}',\mathfrak{m})$$

where

$$E(w,\mathfrak{m}',\mathfrak{m})=-\sum_{(\varepsilon,\eta)\in\{+,-\}^2} e(X(w,\mathfrak{m})^{\perp},\Sigma(L(\mathfrak{m}',\mathfrak{m})),Y^{\varepsilon,\eta})$$

Proof: Set $L=L(\mathfrak{m}',\mathfrak{m})$. Construct a cable L_2 of L locally obtained by pushing one copy of L in each direction normal to the $\mathcal{B}_j$, except near the a_i where L_2 sits in $\mathcal{A}_i$. Define the field Z over L_2 such that at a point k of L_2, Z has the direction of the vector from the closest point to k on L towards k. Thus $X(w,\mathfrak{m}')=D(X(w,\mathfrak{m}),L,L_2,Z,-1)$ with the notation Proposition 4.21 in [Les13].

Then $((L_{\parallel Y^{+,+}},L_{\parallel Y^{-,-}}),(Y^{+,+},Y^{-,-}))$ is obtained from (L_2,Z) by some half-twists and

$$((L_{\parallel Y^{+,-}},L_{\parallel Y^{-,+}}),(Y^{+,-},Y^{-,+}))$$

is obtained from (L_2,Z) by the opposite half-twists. Then according to Proposition 4.21 in [Les13], with the notation of [Les13, Definition 4.16],

$$p_1(X(w,\mathfrak{m}'))=\frac{1}{2}\left(p_1(D(X(w,\mathfrak{m}),L,L_{\parallel Y^{+,+}},Y^{+,+},-1))+p_1(D(X(w,\mathfrak{m}),L,L_{\parallel Y^{+,-}},Y^{+,-},-1))\right).$$

Thus $p_1(X(w,\mathfrak{m}')) = \frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2} p_1(D(X(w,\mathfrak{m}),L,L_{\|Y^{\varepsilon,\eta}},Y^{\varepsilon,\eta},-1))$ and, according to [Les13, Proposition 4.18 and Lemma 4.14],

$$\begin{aligned} p_1(X(w,\mathfrak{m}')) - p_1(X(w,\mathfrak{m})) = \quad & -\sum_{(\varepsilon,\eta)\in\{+,-\}^2} lk(L(\mathfrak{m}',\mathfrak{m}), L(\mathfrak{m}',\mathfrak{m})_{\|Y^{\varepsilon,\eta}}) \\ & -\sum_{(\varepsilon,\eta)\in\{+,-\}^2} e(X(w,\mathfrak{m})^{\perp}, \Sigma(L(\mathfrak{m}',\mathfrak{m})), Y^{\varepsilon,\eta}). \end{aligned}$$

□

The following theorem will be proved in Subsection 4.5.

Theorem 3.4. *Let $L(w',w)$ be the union of the closures of the flow line through w' and the reversed flow line through w.*

$$p_1(X(w',\mathfrak{m})) - p_1(X(w,\mathfrak{m})) = 8lk(L(\mathcal{D},\mathfrak{m}), L(w',w))$$

4 On the variations of $p_1(X(w,\mathfrak{m}))$

4.1 More on the variation of p_1 when $\mathfrak{m}$ changes

Lemma 4.1. *Let $K = \sum_{c\in\mathcal{C}} k_c\gamma(c)$ be a cycle of M, and let $\Sigma(K)$ be a surface bounded by K in $\check{M}$. For $(\varepsilon,\eta)\in\{+,-\}^2$, let $Y^{\varepsilon,\eta}$ be the field defined in Proposition 3.3 along the $\gamma(c)$. Then*

$$\sum_{(\varepsilon,\eta)\in\{+,-\}^2} e(X(w,\mathfrak{m})^{\perp}, \Sigma(K), Y^{\varepsilon,\eta}) = -4\sum_{c\in\mathcal{C}} k_c d_e(c)$$

where d_e is defined before Lemma 2.1 with respect to our initial data that involve $(w,\mathfrak{m})$.

Proof: Set $X(\mathfrak{m}) = X(w,\mathfrak{m})$. Since M is a rational homology sphere, $e(X(\mathfrak{m})^{\perp},\Sigma(K),Y^{\varepsilon,\eta})$ does not depend on the surface $\Sigma(K)$. Choose the surface constructed in Lemma 2.5 with the points of $\mathfrak{m}$ as basepoints. After removing the neighborhood $N(\gamma(w))$ of the flow line through w, $f_M^{-1}(]-\infty,0])$ behaves as a product by the rectangle $R_{\mathcal{D}}$ of Figure 2 and has the product parallelization induced by the vertical vector field and the parallelization of $R_{\mathcal{D}}$. This parallelization extends to the one-handles of $H_{\mathcal{A}}$ as the standard parallelization of $\mathbb{R}^3$ in Figure 6 so that it naturally extends to $f_M^{-1}(]-\infty,3])$, it furthermore extends to the neighborhood of the favourite flow lines in Figure 6. The first vector of this parallelization is $X(\mathfrak{m})$ and its second vector is everywhere orthogonal to $D(\alpha_i)$. It can be chosen to be $Y^{\varepsilon,\eta}$. In a symmetric way, $X(\mathfrak{m})^{\perp}$ has a unit section that coincides with the second vector of the above parallelization on the neighborhoods of the favourite flow lines in Figure 6 and that is orthogonal to $D(\beta_i)$ on $f_M^{-1}([4,\infty[)\setminus N(\gamma(w))$. Thus $e(X(\mathfrak{m})^{\perp},\Sigma(K),Y^{\varepsilon,\eta})$ reads

$$\sum_{c\in\mathcal{C}} k_c e(X(\mathfrak{m})^{\perp}, |m_{j(c)},c|_{\beta}\times[3,4], Y^{\varepsilon,\eta}) - \sum_{(i,j,c)\in\underline{g}^2\times\mathcal{C}} \mathcal{J}_{ji}k_c\langle\alpha_i, |m_{j(c)},c|_{\beta}\rangle e(X(\mathfrak{m})^{\perp},\beta_j\times[3,4],Y^{\varepsilon,\eta})$$

where $d_e(|m_{j(c)},c|_\beta)=-\frac14\sum_{(\varepsilon,\eta)\in\{+,-\}^2}e(X(\mathfrak{m})^\perp,|m_{j(c)},c|_\beta\times[3,4],\tilde{Y}^{\varepsilon,\eta})$ and

$$d_e(\beta_s)=-\frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2}e(X(\mathfrak{m})^\perp,\beta_s\times[3,4],\tilde{Y}^{\varepsilon,\eta})$$

with respect to our partial extensions $\tilde{Y}^{\varepsilon,\eta}$ of $Y^{\varepsilon,\eta}$. (See [Les12a, Lemma 7.5] for more details.)
□

We get the following proposition as a direct corollary of Lemma 4.1.

Proposition 4.2. *Under the hypotheses of Proposition 3.3, if* $\mathfrak{m}'=\{d_j\}_{j\in\underline{g}}$, *then*

$$E(w,\mathfrak{m}',\mathfrak{m})=4\sum_{j=1}^{g}d_e(d_j)$$

where d_e *is defined with respect to our initial data that involve* $(w,\mathfrak{m})$.

□

Note that Lemma 2.2 independently implies that $\sum_{j=1}^{g}d_e(d_j)$ only depends on $(w,\mathfrak{m},\mathfrak{m}')$.

Lemma 4.1 also yields the following second corollary that is [Les12a, Proposition 7.2], that in turn yields Corollary 4.4.

Corollary 4.3. *Let* $\Sigma(L(\mathcal{D},\mathfrak{m}))$ *be a surface bounded by* $L(\mathcal{D},\mathfrak{m})$ *in* $\check{M}$.

$$e(\mathcal{D},w,\mathfrak{m})=\frac{1}{4}\sum_{(\varepsilon,\eta)\in\{+,-\}^2}e(X(w,\mathfrak{m})^\perp,\Sigma(L(\mathcal{D},\mathfrak{m})),Y^{\varepsilon,\eta})$$

□

Corollary 4.4. $e(\mathcal{D},w,\mathfrak{m})$ *is unchanged when the roles of the curves* α *and the curves* β *are permuted.*

Proof: Permuting the roles of the curves α and the curves β reverses the orientation of $L(\mathcal{D},\mathfrak{m})$ and changes $X(w,\mathfrak{m})$ to its opposite while the set $\{Y^{\varepsilon,\eta}\}_{(\varepsilon,\eta)\in\{+,-\}^2}$ is preserved. □

4.2 Associating a closed combing to a combing

The Heegaard surface $f_M^{-1}(0)$ of our Morse function f_M is obtained by gluing the complement D_R of a rectangle in a sphere S^2 to the boundary of the rectangle $R_{\mathcal{D}}$ of Figure 2. Let $D_R\times[-2,7]$ denote the intersection of $f_M^{-1}([-2,7])$ with the flow lines through D_R so that f_M is the projection to $[-2,7]$ on $D_R\times[-2,7]$ and the flow lines read $\{x\}\times[-2,7]$ there. Similarly, our Morse function f_M reads as the projection on the interval on

$$f_M^{-1}([-2,0]\cup[6,8])=\big(S^2\times[-2,0]\big)\cup\big(S^2\times[6,8]\big)$$

while $f_M^{-1}([-3,-2])$ and $f_M^{-1}([7,9])$ are balls centered at a minimum and a maximum mapped to -3 and 9, respectively.

The combing $X(w,\mathfrak{m})$ of Subsection 3.1 of B_M can be extended as a closed combing $X(M,w,\mathfrak{m})$ that is obtained from the tangent X_ϕ to the flow lines outside B_M by reversing it along the line $\overline{\{w\}\times]-3,9[}$ as follows:

Let us first describe $X(M,w,\mathfrak{m})$ on $D_R \times [-2,8]$. Let D be a small disk of D_R centered at w. Reverse the flow on $\{w\}\times[-2,8]$ so that it coincides with the tangent X_ϕ to the flow outside $D\times[-2,8]$, and so that on a ray of $D\times\{t\}$ directed by a vector Z from the center, it describes the half great circle $[-X_\phi, X_\phi]_Z$ from $(-X_\phi)$ to X_ϕ through Z, if $t\in[-2,7]$. Then on $S^2\times\{-2\}$, X is naturally homotopic to the restriction to the boundary of a constant field of B^3. See Figure 7 for a vertical section of the ball centered at the minimum where the constant vector field points downward. We extend it as such.

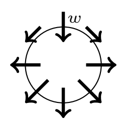

Figure 7: The vector field near a minimum in a planar section of $f_M^{-1}([-3,-2])$

Now, on $S^2\times\{7\}$, X looks like in Figure 8. It would naturally look as the restriction to the boundary of a constant field of B^3 if the half great circle $[-X_\phi, X_\phi]_Z$ from $(-X_\phi)$ to X_ϕ through Z went through $(-Z)$. Let $\rho_{X_\phi,\theta}$ denote the rotation with axis X_ϕ and with angle θ For $t\in[7,8]$, on a ray of $D\times\{t\}$ directed by a vector Z from the center, let X describe the half great circle $[-X_\phi, X_\phi]_{\rho_{X_\phi,(t-7)\pi}(Z)}$ from $(-X_\phi)$ to X_ϕ through $\rho_{X_\phi,(t-7)\pi}(Z)$. Now, we extend X as the constant field of B^3 that we see near the maximum, to obtain the closed combing $X(M,w,\mathfrak{m})$.

On $S^2\times\{7\}$ On $S^2\times\{8\}$

Figure 8: The vector field near a maximum

Lemma 4.5. $(p_1(X(M,w,\mathfrak{m})) - p_1(X(w,\mathfrak{m})))$ *is a constant independent of* M, w *and* $\mathfrak{m}$.

PROOF: Since the combing in the outside ball is unambiguously defined, and since it extends to a parallelization, this result follows from the definition of p_1 that includes the variation formula of Theorem 3.1. (See [Les13] for more details.) □

4.3 An abstract expression for the variation of p_1 when w varies

This section is devoted to the proof of the following proposition that describes the variation of the Pontrjagin class $p_1(X(w,\mathfrak{m}))$ when w varies.

Proposition 4.6. *Let w and w' be two exterior points of $\mathcal{D}$. Let $[w,w']_\alpha$ be a path on $\partial H_{\mathcal{A}}$ from w to w' disjoint from the α_i and let $[w',w]_\beta$ be a path on $\partial H_{\mathcal{A}}$ from w' to w disjoint from the β_j. Set $[w,w']_\beta = -[w',w]_\beta$. Assume that the tangent vectors of $[w,w']_\alpha$ and $[w,w']_\beta$ at w and w' coincide. Let*

$$L([w,w']_\alpha,[w',w]_\beta) = ([w,w']_\alpha \times \{2\}) \cup (\{w'\} \times [2,4]) \cup ([w',w]_\beta \times \{4\}) \cup (\{w\} \times [4,2])\,.$$

Let $\varepsilon = \pm 1$. Let Y be a vector field defined on $L([w,w']_\alpha,[w',w]_\beta)$ that is tangent to the Morse levels $\partial H_{\mathcal{A}} \times \{t\}$ and that is a ε-normal (positive if $\varepsilon = 1$ and negative otherwise) to $[w,w']_\alpha$ and a $(-\varepsilon)$ normal to $[w',w]_\beta$. Let $L([w,w']_\alpha,[w',w]_\beta)_{\parallel Y}$ be the induced parallel of $L([w,w']_\alpha,[w',w]_\beta)$. Let Σ be a surface bounded by $L([w,w']_\alpha,[w',w]_\beta)$. Then

$$\begin{aligned} p_1(X(w',\mathfrak{m}))) - p_1(X(w,\mathfrak{m}))) = \; & 4e(X(w,\mathfrak{m})^\perp,\Sigma,Y) \\ & -4lk(L([w,w']_\alpha,[w',w]_\beta),L([w,w']_\alpha,[w',w]_\beta)_{\parallel Y}). \end{aligned}$$

Proof: First note that $X(w,\mathfrak{m})$ directs $\{w'\} \times [2,4]$ and $\{w\} \times [4,2]$ so that the right-hand side of the equality above is independent of the field Y that satisfies the conditions of the statement. Let $L(w',w)$ be the knot of M that is the union of the closures of $\{w'\}\times]-3,9[$ and $\{w\}\times(-]-3,9[)$. Let $\tilde{X}(M,w',\mathfrak{m})$ be obtained from $X(M,w,\mathfrak{m})$ by reversing $X(M,w,\mathfrak{m})$, that is tangent to $L(w',w)$, along $L(w',w)$. In this situation, there is a standard way of reversing (namely the one that was used along $\{w\} \times [-2,7]$ in Subsection 4.2) by choosing a framing that determines both the parallel and the orthogonal field.

Proposition 4.6 is the direct consequence of Lemma 4.5 and of the following three lemmas.

Lemma 4.7. *There exists a constant C_0 independent of $(M,w,w',\mathfrak{m})$ such that*

$$\begin{aligned} p_1(\tilde{X}(M,w',\mathfrak{m})) - p_1(X(M,w,\mathfrak{m})) = \; & 4e(X(w,\mathfrak{m})^\perp,\Sigma,Y) + 4C_0 \\ & -4lk(L([w,w']_\alpha,[w',w]_\beta),L([w,w']_\alpha,[w',w]_\beta)_{\parallel Y}). \end{aligned}$$

Lemma 4.8. *There exists a constant C_1 independent of $(M,w,w',\mathfrak{m})$ such that*

$$p_1(\tilde{X}(M,w',\mathfrak{m})) - p_1(X(M,w',\mathfrak{m})) = 4C_1.$$

Lemma 4.9.

$$C_0 - C_1 = 0.$$

□

Proof of Lemma 4.7: Let $T([w,w']_\alpha)$ be the (closure of the) past of $[w,w']_\alpha \times \{2\}$ under the flow. This is a triangle and we can assume that it is smoothly embedded (near the minimum).

Similarly, let $T([w',w]_\beta)$ be the future of $[w',w]_\beta \times \{4\}$ under the flow, assume without loss that it intersects $S^2 \times \{7\}$ as a half-great circle, so that it intersects $f_M^{-1}([7,9])$ as a hemidisk denoted by $T_7([w',w]_\beta)$. Orient $T([w,w']_\alpha)$ and $T([w',w]_\beta)$ so that

$$\partial(\Sigma + T([w,w']_\alpha) + T([w',w]_\beta)) = L(w',w).$$

Then Y extends to $T([w,w']_\alpha)$ as the (ε)-normal on $T([w,w']_\alpha)$ that is in $X(M,w,\mathfrak{m})^\perp$. Similarly, Y extends to $T([w',w]_\beta)$ as the (ε)-normal on $T([w',w]_\beta)$, it is a unit vector field that is in $X(M,w,\mathfrak{m})^\perp$ outside the interior of $T_7([w',w]_\beta)$. Use Y to frame $L(w',w)$. Then, according to [Les13, Proposition 4.18 and Lemma 4.14] where $\eta = 1$,

$$\begin{aligned} & p_1(\tilde{X}(M,w',\mathfrak{m})) - p_1(X(M,w,\mathfrak{m})) \\ = {} & 4e(X(M,w,\mathfrak{m})^\perp, \Sigma + T_7([w',w]_\beta), Y) - 4lk(L(w',w), L(w',w)_{\|Y}) \end{aligned}$$

where

$$e(X(M,w,\mathfrak{m})^\perp, T_7([w',w]_\beta, Y) = C_0$$

for a constant C_0 independent of $(M,w,w',\mathfrak{m})$, and

$$lk(L(w',w), L(w',w)_{\|Y}) = lk(L([w,w']_\alpha, [w',w]_\beta), L([w,w']_\alpha, [w',w]_\beta)_{\|Y}).$$

□

Proof of Lemma 4.8: Recall that D is a small disk of $\partial H_{\mathcal{A}}$ centered at w. The vector fields $\tilde{X}(M,w',\mathfrak{m})$ and $X(M,w',\mathfrak{m})$ coincide outside $f_M^{-1}([-3,-2] \cup [7,9]) \cup D \times [-2,7]$. This is a ball where the definition of these fields is unambiguous and independent of $(M,w,w',\mathfrak{m})$. □

Proof of Lemma 4.9: According to the previous lemmas, for any $(M,w,w',\mathfrak{m})$,

$$\begin{aligned} p_1(X(M,w',\mathfrak{m})) - p_1(X(M,w,\mathfrak{m})) = {} & -4lk(L([w,w']_\alpha, [w',w]_\beta), L([w,w']_\alpha, [w',w]_\beta)_{\|Y}) \\ & +4e(X(w,\mathfrak{m})^\perp, \Sigma, Y) + 4(C_0 - C_1). \end{aligned}$$

When M is S^3 equipped with a Morse function with 2 extrema and no other critical points, and when w and w' are two points of S^2 related by a geodesic arc $[w,w']_\alpha = -[w',w]_\beta$, it is easy to check that the terms of the formula are zero, except for $(C_0 - C_1)$ that is therefore also 0. □

4.4 A combinatorial formula for the variation of p_1 when w varies

Now, we give an explicit formula for the right-hand side of Proposition 4.6.

Proposition 4.10. *Assume that w is on the upper side of the rectangle $R_{\mathcal{D}}$ of Figure 2. Assume that $[w,w']_\alpha$ and $[w,w']_\beta = -[w',w]_\beta$ point downward near w and w' and that $[w,w']_\beta$ is on the same side of $[w,w']_\alpha$ near w and w' as in Figure 9. Let $d_e^{(w)}([w,w']_\alpha)$ be the degree of the* *tangent map to* $[w,w']_\alpha$ *on the rectangle* $R_{\mathcal{D}}$ *of Figure 2. Let* $d_e^{(w)}([w,w']_\beta)$ *be the degree of the tangent map to* $[w,w']_\beta$ *on* $R_{\mathcal{D}}$*, where* $[w,w']_\beta$ *intersects the* α'_j *and the* α''_j *on their vertical portions opposite to the crossings of* $\mathfrak{m}$*, with horizontal tangencies. Then*

$$p_1(X(w',\mathfrak{m}))) - p_1(X(w,\mathfrak{m}))) = p'_1(\mathfrak{m};w,w')$$

where

$$\begin{array}{rl} p'_1(\mathfrak{m};w,w') = & 4d_e^{(w)}([w,w']_\alpha) - 4d_e^{(w)}([w,w']_\beta) \\ & +4\sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w,w']_\beta\rangle d_e^{(w)}(\beta_j) \\ & -4\langle]w,w'[_\alpha,]w,w'[_\beta\rangle \\ & +4\sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w,w']_\beta\rangle \langle [w,w']_\alpha, \beta_j\rangle \end{array}$$

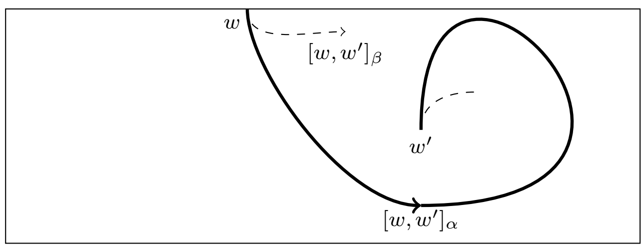

Figure 9: $[w,w']_\alpha$ and $[w,w']_\beta$

PROOF: Define the field Y of Proposition 4.6 along $\{w'\}\times[2,4]$ and $\{w\}\times[4,2]$, as the field pointing to the right in Figure 9 that is preserved by the flow along $\{w'\}\times[2,4]$ and $\{w\}\times[4,2]$, so that it is always normal to $[w,w']_\alpha\times[2,4]$ or $[w,w']_\beta\times[2,4]$ along $\{w'\}\times[2,4]$ and $\{w\}\times[4,2]$. Let $L=L([w,w']_\alpha,[w',w]_\beta)$ and let $L_\parallel = L_{\parallel Y}$. The proposition follows by applying Proposition 4.6, with the computations of Lemmas 4.11 and 4.13 below (replacing $[w,w']_\beta = -[w',w]_\beta$). □

Lemma 4.11.

$$lk\left(L,L_\parallel\right) = -\langle]w,w'[_\alpha,]w',w[_\beta\rangle + \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i,[w',w]_\beta\rangle\langle [w,w']_\alpha,\beta_j\rangle.$$

In order to prove Lemma 4.11, we shall use the following lemma.

Lemma 4.12. *There is a surface* $\Sigma([w,w']_\alpha,[w',w]_\beta)$ *in* $\partial H_{\mathcal{A}} \setminus \mathring{D}_R$ *such that*

$$\begin{array}{rl} \partial\Sigma([w,w']_\alpha,[w',w]_\beta) = & [w,w']_\alpha - \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle [w,w']_\alpha,\beta_j\rangle \alpha_i \\ & +[w',w]_\beta - \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i,[w',w]_\beta\rangle \beta_j. \end{array}$$

Let w'_E *be a point very close to* w' *on its right-hand side. Then*

$$\begin{array}{rl} \langle \Sigma([w,w']_\alpha,[w',w]_\beta), w'_E\rangle_{\partial H_{\mathcal{A}}} = & -\langle]w,w'[_\alpha,]w',w[_\beta\rangle \\ & +\sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i,[w',w]_\beta\rangle\langle [w,w']_\alpha,\beta_j\rangle. \end{array}$$

PROOF: Since the prescribed boundary $\partial\Sigma([w,w']_\alpha,[w',w]_\beta)$ is a cycle that does not intersect the α_i and the β_j, algebraically, the surface $\Sigma([w,w']_\alpha,[w',w]_\beta)$ exists. Let w_E be a point very close to w on its right-hand side. Along a path $]w_E,w'_E[_\alpha$ parallel to $]w,w'[_\alpha$, the intersection of a point with $\Sigma([w,w']_\alpha,[w',w]_\beta)$ starts with the value 0 and varies when the path meets $\partial\Sigma([w,w']_\alpha,[w',w]_\beta)$ so that

$$\begin{aligned}\langle\Sigma([w,w']_\alpha,[w',w]_\beta),w'_E\rangle_{\partial H_{\mathcal{A}}} &= -\langle]w_E,w'_E[_\alpha,\partial\Sigma([w,w']_\alpha,[w',w]_\beta)\rangle\\ &= -\langle]w_E,w'_E[_\alpha,]w',w[_\beta\rangle\\ &\quad+\sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\langle\alpha_i,[w',w]_\beta\rangle\langle]w_E,w'_E[_\alpha,\beta_j\rangle.\end{aligned}$$

□

PROOF OF LEMMA 4.11: L bounds

$$\begin{aligned}\Sigma_0 = {} & \Sigma([w,w']_\alpha,[w',w]_\beta)+([w,w']_\alpha\times[2,3])-([w',w]_\beta\times[3,4])\\ &+\sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\langle[w,w']_\alpha,\beta_j\rangle D(\alpha_i)+\sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\langle\alpha_i,[w',w]_\beta\rangle D(\beta_j).\end{aligned}$$

The link $L_{\|Y}=L([w,w']_\alpha,[w',w]_\beta)_{\|Y}$ does not meet the $D(\alpha_i)$ and the $D(\beta_j)$. Therefore its intersection with Σ_0 is the intersection of w'_E with $\Sigma([w,w']_\alpha,[w',w]_\beta)$ so that Lemma 4.12 yields the conclusion. □

Lemma 4.13. *Let Σ_1 be a surface bounded by L in M. Then*

$$\begin{aligned}e(X(w,\mathfrak{m})^\perp,\Sigma_1,Y) = {} & d_e^{(w)}([w,w']_\alpha)-d_e^{(w)}([w,w']_\beta)\\ &-\sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\langle\alpha_i,[w',w]_\beta\rangle d_e^{(w)}(\beta_j).\end{aligned}$$

PROOF: Let $\Sigma_2=\Sigma([w,w']_\alpha,[w',w]_\beta)\times\{2\}$ with the surface $\Sigma([w,w']_\alpha,[w',w]_\beta)\subset\partial H_{\mathcal{A}}$ of Lemma 4.12. The link L bounds

$$\begin{aligned}\Sigma_1 = {} & \Sigma_2-[w',w]_\beta\times[2,4]+\sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\langle[w,w']_\alpha,\beta_j\rangle D_{\leq 2}(\alpha_i)\\ &+\sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\langle\alpha_i,[w',w]_\beta\rangle D_{\geq 2}(\beta_j)\end{aligned}$$

where $D_{\leq 2}(\alpha_i)=D(\alpha_i)\cap f_M^{-1}([-3,2])$ and $D_{\geq 2}(\beta_j)=D(\beta_j)\cap f_M^{-1}([2,9])$. First extend Y to the product by $[2,4]$ of a short vertical segment $[w,w^{(S)}]$ from w to some point $w^{(S)}$ below w, such that $X(w,\mathfrak{m})$ directs $w\times[4,2]$ and $w^{(S)}\times[2,4]$, and $X(w,\mathfrak{m})$ is tangent to $[w,w^{(S)}]\times[2,4]$. Truncate the rectangle of Figure 9 so that $w^{(S)}$ is on its boundary and $w^{(S)}$ replaces w in the right-hand side of the equality of the statement without change. Now, $X(w,\mathfrak{m})$ is orthogonal to this rectangle, and the restriction to $\{w^{(S)},w'\}\times[2,4]$ of the field Y extends as the field Y_E that points East or to the right in Figures 2, 9 and 6 so that is normal to the $D(\alpha_i)$. This defines the standard extension associated with Figure 2 of Y_E on $f_M^{-1}([0,2])\setminus(D_R\times[0,2])$ that is extended to $[w',w]_\beta\times\{4\}$ so that it is normal to $[w',w]_\beta\times[2,4]$ along $[w',w]_\beta\times\{4\}$. The field Y_E can be extended independently to $D_{\geq 2}(\beta_j)$ and to $[w',w]_\beta\times[2,4]$ as a field normal to these surfaces. Then the Euler class of Y_E with respect to Σ_1 can be computed by comparing this extension to the standard one above on $[w,w']_\beta \times \{2\} + \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w',w]_\beta\rangle (\beta_j \times \{2\})$,

$$e(X(w,\mathfrak{m})^\perp, \Sigma_1, Y_E) = -d_e^{(w)}([w,w']_\beta) - \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w',w]_\beta\rangle d_e^{(w)}(\beta_j).$$

The fields Y_E and Y coincide on $L \setminus ([w,w']_\alpha \times \{2\})$ and

$$e(X(w,\mathfrak{m})^\perp, \Sigma_1, Y) = e(X(w,\mathfrak{m})^\perp, \Sigma_1, Y_E) + d_e^{(w)}([w,w']_\alpha).$$

□

4.5 Proof of Theorem 3.4

Thanks to Proposition 4.10, in order to prove Theorem 3.4, we are left with the proof that

$$p_1'(\mathfrak{m}; w, w') = 8lk(L(\mathcal{D}, \mathfrak{m}), L(w', w))$$

where $p_1'(\mathfrak{m}; w, w')$ is defined in the statement of Proposition 4.10 and $L(w', w)$ is the union of the closures of the flow line through w' and the reversed flow line through w. In order to prove this, fix an exterior point w_0 of $\mathcal{D}$, and define $p_1''(w)$, for any point exterior point w of $\mathcal{D}$, as

$$p_1''(w) = p_1'(\mathfrak{m}; w_0, w) - 8lk(L(\mathcal{D}, \mathfrak{m}), L(w, w_0))$$

Lemma 4.14. *p_1'' satisfies the following properties:*

- *$p_1''(w)$ only depends on the connected component of w in the complement of the α_i and the β_j in the closed surface $\partial H_{\mathcal{A}}$,*
- *$p_1''(w_0) = 0$,*
- *For any 4 points w, S, E, N located around a crossing $d \notin \mathfrak{m}$, as in Figure 10*

$$p_1''(N) + p_1''(S) = p_1''(w) + p_1''(E).$$

Proof: The first two properties come from the definition. Let us prove the third one.

Set

$$D = (p_1''(N) + p_1''(S) - (p_1''(w) + p_1''(E)))$$

Note that D is independent of w_0, thanks to Proposition 4.10, and that it reads $D = D_1 - 8D_2$ with

$$D_1 = p_1'(\mathfrak{m}; w, N) + p_1'(\mathfrak{m}; w, S) - p_1'(\mathfrak{m}; w, E) \text{ and } D_2 = lk(L(N,w) + L(S,E), L(\mathcal{D}, \mathfrak{m}))$$

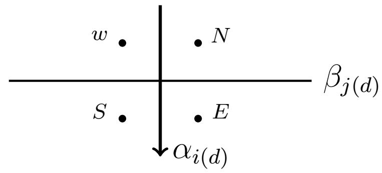

Figure 10: Near d

and

$$\begin{aligned} p_1'(\mathfrak{m}; w, w') = \quad & 4d_e^{(w)}([w,w']_\alpha) - 4d_e^{(w)}([w,w']_\beta) \\ & +4\sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w,w']_\beta\rangle d_e^{(w)}(\beta_j) \\ & -4\langle \Sigma([w,w']_\alpha, [w',w]_\beta), w'_E\rangle_{\partial H_{\mathcal{A}}} \end{aligned}$$

according to Proposition 4.10 and Lemma 4.12.

We are going to prove that

$$D_1 = 8D_2 = -8\sigma(d)\mathcal{J}_{j(d)i(d)}.$$

Let us first compute D_1. Its computation involves paths $[w,w']_\alpha$ and $[w,w']_\beta$ starting from w on the upper side of the rectangle $R_{\mathcal{D}}$ of Figure 2, before reaching a point $w' = N$, S or E. We assume that all these paths begin by following a first path $[w,\tilde{w}]$ that connects w to a point $\tilde{w}$ near d in the complement of the curves α_i and β_j in $R_{\mathcal{D}}$ and that this path $[w,\tilde{w}]$ has tangent vectors pointing downward at its ends. The degree of the path $[w,\tilde{w}]$ does not matter since it is counted twice with opposite sign in $(d_e^{(w)}([w,w']_\alpha) - d_e^{(w)}([w,w']_\beta))$. Thus we may change w to $\tilde{w}$ in $p_1'(\mathfrak{m}; w, w')$ or equivalently assume that w arises near d as in Figure 10 split along $\alpha_{i(d)}$ and embedded in Figure 2 as soon as we translate our initial conventions for tangencies near the boundaries. Now (keeping the first composition by $[w,\tilde{w}]$ in mind) we can draw our paths $[\tilde{w},S]_\alpha$ and $[\tilde{w},N]_\beta$ in Figure 11 below where $\tilde{w}$ is denoted by w. These paths together with the other drawn paths $[N,E]_\alpha$ and $[S,E]_\beta$ bound a "square" C around d. In Figure 11, there are also dashed paths $[w,N]_\alpha$ and $[w,S]_\beta$ that may be complicated outside the pictured neighborhood of our square but that meet this neighborhood as in the figure. We choose $[w,E]_\alpha$ (resp. $[w,E]_\beta$) to be the path composition of $[w,N]_\alpha$ and $[N,E]_\alpha$ (resp. $[w,S]_\beta$ and $[S,E]_\beta$).

With these choices, the contribution to D_1 of the parts

$$d_e^{(w)}([w,w']_\alpha) - d_e^{(w)}([w,w']_\beta) + \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [w,w']_\beta\rangle d_e^{(w)}(\beta_j)$$

cancel. When w' is E, N or S, let $\Sigma(w') = \Sigma([w,w']_\alpha, [w',w]_\beta)$, with the notation of Lemma 4.12. Then

$$\begin{aligned} D_1 &= 4\langle \Sigma(E), E_E\rangle - 4\langle \Sigma(N), N_E\rangle - 4\langle \Sigma(S), S_E\rangle \\ &= -4\langle \Sigma(N) + \Sigma(S) - \Sigma(E), E_E\rangle - 4\langle [N_E, E_E], \partial\Sigma(N)\rangle - 4\langle [S_E, E_E], \partial\Sigma(S)\rangle \end{aligned}$$

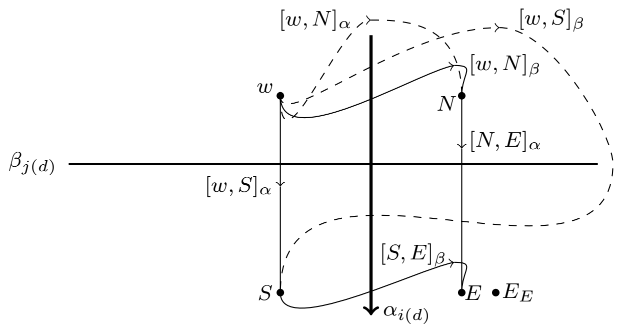

Figure 11: Near d

where $\partial(\Sigma(N)+\Sigma(S)-\Sigma(E)) = [w,S]_\alpha - [N,E]_\alpha + [N,w]_\beta - [E,S]_\beta$ so that $(\Sigma(N)+\Sigma(S)-\Sigma(E))$ is our square and $\langle \Sigma(N)+\Sigma(S)-\Sigma(E), E_E\rangle = 0$,

$$\begin{aligned}\langle [N_E,E_E]_\alpha, \partial\Sigma(N)\rangle &= \langle [N_E,E_E]_\alpha, [N,w]_\beta - \textstyle\sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle \alpha_i, [N,w]_\beta\rangle \beta_j\rangle \\ &= \sigma(d)\mathcal{J}_{j(d)i(d)}\end{aligned}$$

$$\begin{aligned}\langle [S_E,E_E]_\beta, \partial\Sigma(S)\rangle &= \langle -[w,S]_\alpha + \textstyle\sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle [w,S]_\alpha, \beta_j\rangle \alpha_i, [S_E,E_E]_\beta\rangle \\ &= \sigma(d)\mathcal{J}_{j(d)i(d)}\end{aligned}$$

Then $D_1 = -8\sigma(d)\mathcal{J}_{j(d)i(d)}$.

In order to compute D_2, construct a Seifert surface for $L(N,w)+L(S,E)$ made of

- two triangles parallel to the $D(\beta)$ with bottom boundaries $[w,N]_\beta$ and $[E,S]_\beta$,
- two triangles parallel to the $D(\alpha)$ with top edges $[S,w]_\alpha$ and $[N,E]_\alpha$,
- our square C bounded by $([N,w]_\beta \cup [w,S]_\alpha \cup [S,E]_\beta \cup [E,N]_\alpha)$ that is a meridian of $\gamma(d)$.

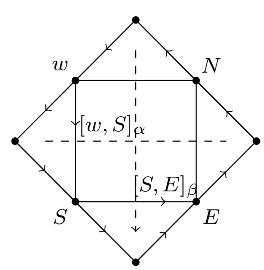

Figure 12: A Seifert surface of $L(N,w)+L(S,E)$

Therefore $D_2 = -\sigma(d)\mathcal{J}_{j(d)i(d)}$. □

Now, we conclude as follows. According to the above lemma, the variation of p_1'' across a curve α_i or β_j is constant so that the variation of p_1'' along a path γ reads

$$\sum_i v_i\langle\gamma,\alpha_i\rangle + \sum_j w_j\langle\gamma,\beta_j\rangle$$

Since this is zero for any loop γ, the v_i and the w_j vanish, and the function p_1'' is constant. Then it is identically zero and Theorem 3.4 is proved. □

5 Behaviour of $\tilde{\Theta}$ when w and $\mathfrak{m}$ vary

In this section, we compute the variations of $\tilde{\Theta}(w,\mathfrak{m})$ when w and $\mathfrak{m}$ change, and we find that these variations coincide with the variations of $\frac{1}{4}p_1(X(w,\mathfrak{m}))$ computed in the previous section. Thus we prove that $\left(\tilde{\Theta}(w,\mathfrak{m}) - \frac{1}{4}p_1(X(w,\mathfrak{m}))\right)$ is independent of $(w,\mathfrak{m})$.

5.1 Changing w

Let us first prove the following proposition that is similar to Proposition 2.6.

Proposition 5.1. *Let w and w' be two exterior points of $\mathcal{D}$. Let $L(w',w)$ be the union of the closures of the flow line through w' and the reversed flow line through w, let $[w,w']_\beta$ be a path from w to w' outside the β_j and let $[w,w']_\alpha$ be a path from w to w' outside the α_i. For any curve α_i (resp. β_j), choose a basepoint $p(\alpha_i)$ (resp. $p(\beta_j)$). For any 1−cycle $K = \sum_{c\in\mathcal{C}} k_c\gamma(c)$,*

$$\begin{aligned} lk(K, L(w',w)) \;=\; & \textstyle\sum_{c\in\mathcal{C}} k_c\langle[w,w']_\alpha, [p(\beta(c)), c|_\beta\rangle \\ & -\textstyle\sum_{(j,i)\in\underline{g}^2}\sum_{c\in\mathcal{C}} k_c\mathcal{J}_{ji}\langle\alpha_i, [p(\beta(c)), c|_\beta\rangle\langle[w,w']_\alpha,\beta_j\rangle \end{aligned}$$

where $\beta(c) = \beta_{j(c)}$.

Proof: As in Lemma 2.5, K bounds a chain

$$\begin{aligned} \Sigma(K) = \; & \Sigma_\Sigma(K) + \textstyle\sum_{c\in\mathcal{C}} k_c(T_\beta(c) + T_\alpha(c)) \\ & -\textstyle\sum_{(j,i)\in\underline{g}^2}\sum_{c\in\mathcal{C}} k_c\mathcal{J}_{ji}\left(\langle\alpha_i, |p(\beta(c)), c|_\beta\rangle D(\beta_j) - \langle|p(\alpha(c)), c|_\alpha, \beta_j\rangle D(\alpha_i)\right) \end{aligned}$$

where $\Sigma_\Sigma(K)$ is a chain of $\partial H_{\mathcal{A}}\setminus\{w\}$ with boundary

$$\begin{aligned} \partial\Sigma_\Sigma(K) = \; & \textstyle\sum_{c\in\mathcal{C}} k_c(|p(\alpha(c)), c|_\alpha - |p(\beta(c)), c|_\beta) \\ & +\textstyle\sum_{(j,i)\in\underline{g}^2}\sum_{c\in\mathcal{C}} k_c\mathcal{J}_{ji}\left(\langle\alpha_i, |p(\beta(c)), c|_\beta\rangle\beta_j - \langle|p(\alpha(c)), c|_\alpha, \beta_j\rangle\alpha_i\right). \end{aligned}$$

Now, $lk(L, L(w',w))$ is the intersection of w' and $\Sigma_\Sigma(K)$, that is $\langle -[w,w']_\alpha, \partial\Sigma_\Sigma(K)\rangle$. □

Lemma 5.2. *Let w and w' be two exterior points of $\mathcal{D}$. Let $L(w',w)$ be the union of the closures of the flow line through w' and the reversed flow line through w. Let $[w,w']_\alpha$ be a path of $\Sigma \setminus (\cup_{i=1}^{g}\alpha_i)$ from w to w'. Set*

$$\tilde{\Theta}' = \tilde{\Theta}(w',\mathfrak{m}) - \tilde{\Theta}(w,\mathfrak{m}) = e(\mathcal{D},w,\mathfrak{m}) - e(\mathcal{D},w',\mathfrak{m}).$$

Then

$$\tilde{\Theta}' = 2\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\left(\sum_{(r,s)\in \underline{g}^2} \mathcal{J}_{sr}\langle \alpha_r, |m_{j(c)},c|_\beta\rangle\langle [w,w']_\alpha,\beta_s\rangle - \langle [w,w']_\alpha, |m_{j(c)},c|_\beta\rangle\right).$$

Proof: Pick a vertical path $[w,w']_\alpha$ from a point w in the boundary of the rectangle of Figure 2 to the point w' that cuts horizontal parts of the β curves. When w is changed to w', the portions or arcs near the intersection points with $[w,w']_\alpha$ are transformed to arcs that turn around the whole picture of Figure 2. This operation adds 2 to the degree of an arc oriented from left to right. See Figure 13.

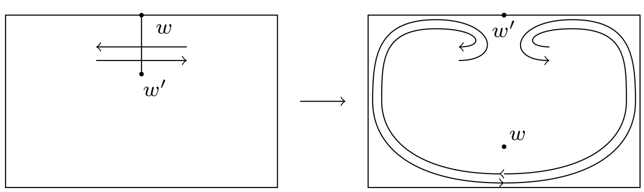

Figure 13: Changing w to w'

Therefore
$$d_e^{(w')}(\beta_s) - d_e^{(w)}(\beta_s) = 2\langle [w,w']_\alpha,\beta_s\rangle$$
and
$$d_e^{(w')}(|m_{j(c)},c|_\beta) - d_e^{(w)}(|m_{j(c)},c|_\beta) = 2\langle [w,w']_\alpha, |m_{j(c)},c|_\beta\rangle.$$

□

Corollary 5.3.

$$\tilde{\Theta}(w',\mathfrak{m}) - \tilde{\Theta}(w,\mathfrak{m}) = 2lk(L(\mathcal{D},\mathfrak{m}),L(w',w)) = \frac{1}{4}p_1(X(w',\mathfrak{m})) - \frac{1}{4}p_1(X(w,\mathfrak{m})).$$

This follows from Lemma 5.2, Proposition 5.1 and Theorem 3.4. □

5.2 Changing $\mathfrak{m}$

Let $\mathfrak{m}'=\{d_i \in \alpha_i \cap \beta_{\rho^{-1}(i)}\}$ be another matching for a permutation ρ. The matching $\mathfrak{m}'$ replaces our initial matching $\mathfrak{m}$ of positive crossings $m_i \in \alpha_i \cap \beta_i$.

Set $L(\mathfrak{m})=L(\mathcal{D},\mathfrak{m})$ and $L(\mathfrak{m}')=L(\mathcal{D},\mathfrak{m}')$.

Let $L(\mathfrak{m}',\mathfrak{m})=L(\mathfrak{m}')-L(\mathfrak{m})=\sum_{i=1}^{g}(\gamma(d_i)-\gamma_i)$.

This subsection is devoted to the proof the following proposition.

Proposition 5.4. *Under the assumptions above,*

$$\tilde{\Theta}(w,\mathfrak{m}')-\tilde{\Theta}(w,\mathfrak{m})=\frac{1}{4}p_1(X(w,\mathfrak{m}'))-\frac{1}{4}p_1(X(w,\mathfrak{m}))$$

This proposition is a direct corollary of Propositions 3.3, 4.2 and 5.5 so that we are left with the proof of Proposition 5.5 below.

Proposition 5.5. *Under the assumptions above,*

$$\begin{aligned}\tilde{\Theta}(w,\mathfrak{m}')-\tilde{\Theta}(w,\mathfrak{m}) &= lk(L(\mathfrak{m}'),L(\mathfrak{m}')_{\parallel})-lk(L(\mathfrak{m}),L(\mathfrak{m})_{\parallel})+e(\mathcal{D},w,\mathfrak{m})-e(\mathcal{D},w,\mathfrak{m}')\\ &= \sum_{i=1}^{g} d_e(d_i)-lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}',\mathfrak{m})_{\parallel}).\end{aligned}$$

Here, d_e is defined with respect to our initial data that involve w and $\mathfrak{m}$.

Proposition 5.5 is a direct consequence of Lemma 5.6 below and Lemma 5.7 that will be proved at the end of this subsection.

Lemma 5.6.

$$lk(L(\mathfrak{m}'),L(\mathfrak{m}')_{\parallel})-lk(L(\mathfrak{m}),L(\mathfrak{m})_{\parallel})=2lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}')_{\parallel})-lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}',\mathfrak{m})_{\parallel})$$

Proof: Use the symmetry of the linking number, and replace $L(\mathfrak{m})=L(\mathfrak{m}')-L(\mathfrak{m}',\mathfrak{m})$. □

Lemma 5.7.

$$e(\mathcal{D},w,\mathfrak{m}')-e(\mathcal{D},w,\mathfrak{m})=2lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}')_{\parallel})-\sum_{j=1}^{g}d_e(d_j)$$

where $d_e(d_{\rho(j)})=d_e(|m_j,d_{\rho(j)}|_\beta)-\sum_{s=1}^{g}\sum_{i=1}^{g}\mathcal{J}_{si}\langle\alpha_i,|m_j,d_{\rho(j)}|_\beta\rangle d_e(\beta_s)$.

Lemma 5.8.

$$\begin{aligned}lk(L(\mathfrak{m}',\mathfrak{m}),L(\mathfrak{m}')_{\parallel})= &\sum_{i=1}^{g}\sum_{c\in\mathcal{C}}\sigma(c)\mathcal{J}_{j(c)i(c)}\left(\sum_{(s,r)\in\underline{g}^2}\mathcal{J}_{sr}\langle|m_i,d_i|_\alpha,\beta_s\rangle\langle\alpha_r,|d_{\rho(j(c))},c|_\beta\rangle\right)\\ &-\sum_{i=1}^{g}\sum_{c\in\mathcal{C}}\sigma(c)\mathcal{J}_{j(c)i(c)}\left(\langle|m_i,d_i|_\alpha,|d_{\rho(j(c))},c|_\beta\rangle\right).\end{aligned}$$

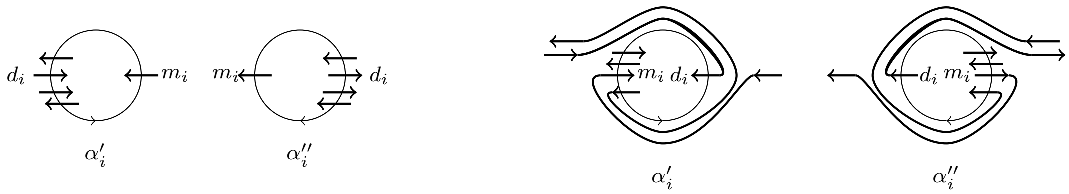

Figure 14: Making the crossings move around

PROOF: Use Proposition 2.7 with $p(\alpha_i) = m_i$, $p(\beta_j) = d_{\rho(j)}$, and $\tilde{\ell}$. □

PROOF OF LEMMA 5.7: Move the crossings of $[m_i, d_i]$, counterclockwise along α''_i and clockwise along α'_i as in Figure 14 so that m_i and d_i make half a loop and the crossings of $]m_i, d_i[$ make a (almost) full loop until they reach the standard position with respect to d_i.

As in the proof of Lemma 2.1, on both sides of each crossing c of $]m_i, d_i[$ the degree is incremented by $(-\sigma(c))$, and it is incremented by $(-\sigma(m_i)/2)$ on both sides of m_i and by $(-\sigma(d_i)/2)$ on both sides of d_i so that after this modification the degree $d'_e(\beta_j)$ of β_j reads

$$d'_e(\beta_j) = d_e(\beta_j) - 2\sum_{i=1}^{g} \langle |m_i, d_i|_\alpha, \beta_j \rangle.$$

Before this modification, the degree of the tangent to β_j from $d_{\rho(j(c))}$ to c was

$$d_e(|d_{\rho(j(c))}, c|_\beta) = \left\{ \begin{array}{ll} d_e(|d_{\rho(j(c))}, m_{j(c)}|_\beta)) + d_e(|m_{j(c)}, c|_\beta) & \text{if } c \in [m_{j(c)}, d_{\rho(j(c))}[_\beta \\ d_e(|d_{\rho(j(c))}, m_{j(c)}|_\beta)) + d_e(|m_{j(c)}, c|_\beta) - \ d_e(\beta_{j(c)}) & \text{if } c \in [d_{\rho(j(c))}, m_{j(c)}[_\beta \end{array} \right.$$

After the modification, it reads

$$d'_e(|d_{\rho(j(c))}, c|_\beta) = d_e(|d_{\rho(j(c))}, c|_\beta) - 2\sum_{i=1}^{g} \langle |m_i, d_i|_\alpha, |d_{\rho(j(c))}, c|_\beta \rangle.$$

Now

$$e(\mathcal{D}, w, \mathfrak{m}') = \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)} \sigma(c) d'_e(c)$$

where $d'_e(c) = d'_e(|d_{\rho(j(c))}, c|_\beta) - \sum_{(r,s) \in \underline{g}^2} \mathcal{J}_{sr} \langle \alpha_r, |d_{\rho(j(c))}, c|_\beta \rangle d'_e(\beta_s)$. Thus

$$e(\mathcal{D}, w, \mathfrak{m}') - e(\mathcal{D}, w, \mathfrak{m}) = e_1(w, \mathfrak{m}, \mathfrak{m}') - e_2(w, \mathfrak{m}, \mathfrak{m}')$$

where

$$e_1(w, \mathfrak{m}, \mathfrak{m}') = \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i(c)} \sigma(c) \left(d'_e(|d_{\rho(j(c))}, c|_\beta) - d_e(|m_{j(c)}, c|_\beta) \right)$$

and

$$e_2(w,\mathfrak{m},\mathfrak{m}') = \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c) \sum_{(r,s)\in\underline{g}^2} \mathcal{J}_{sr}\left(\langle\alpha_r, |d_{\rho(j(c))}, c|_\beta\rangle d'_e(\beta_s) - \langle\alpha_r, |m_{j(c)}, c|_\beta\rangle d_e(\beta_s)\right).$$

$$\begin{array}{rcl} e_1(w,\mathfrak{m},\mathfrak{m}') & = & \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c) d_e(|d_{\rho(j(c))}, m_{j(c)}|_\beta) - \sum_{c\in[d_{\rho(j(c))}, m_{j(c)}[_\beta} \mathcal{J}_{j(c)i(c)}\sigma(c) d_e(\beta_{j(c)}) \\ & & -2\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\sum_{i=1}^g \langle |m_i, d_i|_\alpha, |d_{\rho(j(c))}, c|_\beta\rangle \\ & = & \sum_{j=1}^g d_e(|d_{\rho(j)}, m_j|_\beta) - \sum_{j=1}^g\sum_{i=1}^g \mathcal{J}_{ji}\langle\alpha_i, [d_{\rho(j)}, m_j[_\beta\rangle d_e(\beta_j) \\ & & -2\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\sum_{i=1}^g \langle |m_i, d_i|_\alpha, |d_{\rho(j(c))}, c|_\beta\rangle. \end{array}$$

Since

$$\begin{array}{rcl} \langle\alpha_r, |d_{\rho(j(c))}, c|_\beta\rangle d'_e(\beta_s) - \langle\alpha_r, |m_{j(c)}, c|_\beta\rangle d_e(\beta_s) & = & -2\langle\alpha_r, |d_{\rho(j(c))}, c|_\beta\rangle \sum_{i=1}^g \langle |m_i, d_i|_\alpha, \beta_s\rangle \\ & & +\langle\alpha_r, |d_{\rho(j(c))}, m_{j(c)}|_\beta\rangle d_e(\beta_s) \\ & & -\chi_{[d_{\rho(j(c))}, m_{j(c)}[_\beta}(c)\langle\alpha_r, \beta_{j(c)}\rangle d_e(\beta_s), \end{array}$$

where $\chi_{[d_{\rho(j(c))}, m_{j(c)}[_\beta}(c) = \begin{cases} 1 & \text{if } c \in [d_{\rho(j(c))}, m_{j(c)}[_\beta \\ 0 & \text{otherwise,} \end{cases}$

$$\begin{array}{rcl} e_2(w,\mathfrak{m},\mathfrak{m}') & = & -2\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\sum_{(r,s,i)\in\underline{g}^3} \mathcal{J}_{sr}\langle |m_i, d_i|_\alpha, \beta_s\rangle\langle\alpha_r, |d_{\rho(j(c))}, c|_\beta\rangle \\ & & +\sum_{(r,s,j)\in\underline{g}^3} \mathcal{J}_{sr}\langle\alpha_r, |d_{\rho(j)}, m_j|_\beta\rangle d_e(\beta_s) \\ & & -\sum_{(r,s,j)\in\underline{g}^3} \mathcal{J}_{sr}\sum_{i=1}^g \mathcal{J}_{ji}\langle\alpha_i, [d_{\rho(j)}, m_j[_\beta\rangle\langle\alpha_r, \beta_j\rangle d_e(\beta_s) \\ & = & -2\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\sum_{(r,s,i)\in\underline{g}^3} \mathcal{J}_{sr}\langle |m_i, d_i|_\alpha, \beta_s\rangle\langle\alpha_r, |d_{\rho(j(c))}, c|_\beta\rangle \\ & & +\sum_{(r,s,j)\in\underline{g}^3} \mathcal{J}_{sr}\langle\alpha_r, |d_{\rho(j)}, m_j|_\beta\rangle d_e(\beta_s) \\ & & -\sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle\alpha_i, [d_{\rho(j)}, m_j[_\beta\rangle d_e(\beta_j). \end{array}$$

Therefore, according to Lemma 5.8, $e(\mathcal{D},w,\mathfrak{m}')-e(\mathcal{D},w,\mathfrak{m}) = 2lk(L(\mathfrak{m}',\mathfrak{m}), L(\mathfrak{m}')_\parallel)+V$ where

$$\begin{array}{rcl} V & = & \sum_{j=1}^g d_e(|d_{\rho(j)}, m_j|_\beta) - \sum_{(r,s,j)\in\underline{g}^3} \mathcal{J}_{sr}\langle\alpha_r, |d_{\rho(j)}, m_j|_\beta\rangle d_e(\beta_s) \\ & = & -\sum_{j=1}^g d_e(|m_j, d_{\rho(j)}|_\beta) + \sum_{(r,s,j)\in\underline{g}^3} \mathcal{J}_{sr}\langle\alpha_r, |m_j, d_{\rho(j)}|_\beta\rangle d_e(\beta_s) \\ & & +\sum_{j=1}^g d_e(\beta_j) - \sum_{(r,s,j)\in\underline{g}^3} \mathcal{J}_{sr}\langle\alpha_r, \beta_j\rangle d_e(\beta_s) \end{array}$$

Since the last line vanishes, we get the result. □

Corollary 5.3 and Proposition 5.4 allow us to define the function $\tilde{\lambda}$ of Heegaard diagrams

$$\tilde{\lambda}(\mathcal{D}) = \frac{\tilde{\Theta}(\mathcal{D},w,\mathfrak{m})}{6} - \frac{p_1(X(w,\mathfrak{m}))}{24}$$

that does not depend on the orientations and numberings of the curves α_i and β_j and that is also unchanged by permuting the roles of the α_i and β_j, thanks to Corollary 4.4.

6 Invariance of $\tilde{\lambda}$

In this section, we are first going to prove that $\tilde{\lambda}$ only depends on the Heegaard decomposition induced by $\mathcal{D}$ of M, and not on the curves α_i and β_j. Then it will be easily observed that $\tilde{\lambda}$ is additive under connected sum of Heegaard decompositions and that $\tilde{\lambda}$ maps the genus one Heegaard decomposition of S^3 to 0. Since according to the so-called Reidemeister-Singer theorem, two Heegaard decompositions of a 3-manifold become diffeomorphic after some connected sums with this Heegaard decomposition of S^3, we will conclude that $\tilde{\lambda}$ is an invariant of rational homology spheres, that is additive under connected sum.

6.1 Systems of meridians of a handlebody

A *handle slide* in a system $\{\alpha_i\}_{i\in\underline{g}}$ of meridians of a curve α_k across a curve α_j, with $j \neq k$, is defined as follows: Choose a path γ in $\partial H_{\mathcal{A}}$ from a point $\gamma(0) \in \alpha_k$ to a point $\gamma(1) \in \alpha_j$ such that $\gamma(]0,1[)$ does not meet $\cup_{i\in\underline{g}}\alpha_i$ and change α_k to the band sum α'_k of α_k and a parallel of α_j on the γ-side as in Figure 15.

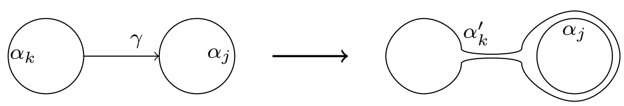

Figure 15: Handle slide in $\partial H_{\mathcal{A}}$

A *right-handed Dehn twist* about a simple closed curve $K(S^1)$ of a surface F is a homeomorphism of F that fixes the exterior of a collar $K(S^1) \times [-\pi,\pi]$ of K in F pointwise, and that maps $(K(\exp(i\theta)),t)$ to $(K(\exp(i(\theta+t+\pi))),t)$.

In order to prove that $\tilde{\lambda}$ only depends on the Heegaard decompositions and not on the chosen systems $\{\alpha_i\}_{i\in\underline{g}}$ and $\{\beta_j\}_{j\in\underline{g}}$ of meridians of $H_{\mathcal{A}}$ and $H_{\mathcal{B}}$ we shall use the following standard theorem.

Theorem 6.1. *Up to isotopy, renumbering of meridians, orientation reversals of meridians, two meridian systems of a handlebody are obtained from one another by a finite number of handle slides.*

PROOF: Let $\{\alpha_i\}_{i\in\underline{g}}$ and $\{\alpha'_i\}_{i\in\underline{g}}$ be two systems of meridians of $H_{\mathcal{A}}$. There exists an orientation-preserving diffeomorphism of $H_{\mathcal{A}}$ that maps the first system to the second one.

See $H_{\mathcal{A}}$ as the unit ball $B(1)$ of $\mathbb{R}^3$ with embedded handles $D(\alpha_i) \times [0,1]$ attached along $D(\alpha_i) \times \partial[0,1]$, so that there is a rotation ρ of angle $\frac{2\pi}{g}$ of $\mathbb{R}^3$ that maps $H_{\mathcal{A}}$ to itself and that permutes the handles, cyclically. See the meridians disks bounded by the α_i as disks $D(\alpha_i) = D(\alpha_i) \times \{\frac{1}{2}\}$ that cut the handles. Let H_i denote the handle of α_i. In [Suz77, Theorem 4.1], Suzuki proves that the group of isotopy classes of orientation-preserving diffeomorphisms of $H_{\mathcal{A}}$ is generated by 6 generators represented by the following diffeomorphisms

- the rotation ρ above of [Suz77, 3.1] that permutes the α_i, cyclically,

and the remaining 5-diffeomorphisms that fix all the handles H_i, for $i > 2$, pointwise,

- the knob interchange ρ_{12} of [Suz77, 3.4], that exchanges H_1 and H_2 and maps α_1 to α_2 and α_2 to α_1,
- the knob twist ω_1 of [Suz77, 3.2] that fixes H_2 pointwise and that maps α_1 to the curve with opposite orientation, (it is the final time of an ambient isotopy of $\mathbb{R}^3$ that performs a half-twist on a disk of $H_{\mathcal{A}}$ that contains the two feet ($D(\alpha_1) \times \{0\}$ and $D(\alpha_1) \times \{1\}$) of the handle H_1),
- the right-handed Dehn twist τ_1^{-1} of [Suz77, 3.3] along a curve parallel to α_1,
- the sliding ξ_{12} of [Suz77, 3.5 and 3.9], that is the final time of an ambient isotopy of $\mathbb{R}^3 \times \mathbb{R}$ that fixes the handles H_i, for $i > 2$, pointwise, and that lets one foot of H_1 slide along a circle parallel to α_2 once,
- the sliding θ_{12} of [Suz77, 3.5 and 3.8], that is the final time of an ambient isotopy of $\mathbb{R}^3$ that fixes the handles H_i, for $i > 2$, pointwise, and that lets one foot of H_1 slide along a circle a_2 that cuts α_2 once and that does not meet the interiors of the H_i, for $i \neq 2$.

All these generators are described more precisely in [Suz77, Section 3]. All of them except θ_{12} fix the set of curves α_i seen as unoriented curves, while θ_{12} fixes all the curves α_i, for $i \neq 2$ pointwise. When the foot of H_1 moves along the circle a_2, the curves that cross a_2 move with it, so that the meridian α_2 is changed as in Figure 16 that is a figure of a handle slide of α_2 across α_1. □

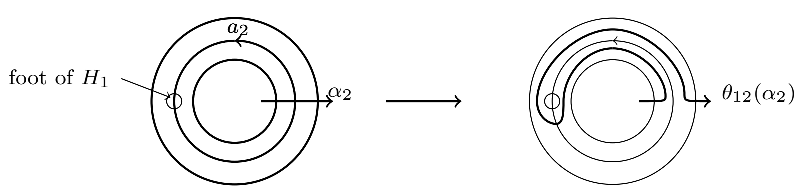

Figure 16: Action of θ_{12} on α_2

6.2 Isotopies of systems of meridians

When the α_i are fixed on $\partial H_{\mathcal{A}}$, and when the β_j vary by isotopy, the only generic encountered accidents are the births or deaths of bigons that modify the Heegaard diagram as in Figure 17 that represents the birth of a bigon between an arc of α_i and an arc of β_j.

Therefore, in order to prove that $\tilde{\lambda}$ is invariant when the β_j (or the α_i) are moved by an isotopy, it is enough to prove the following proposition:

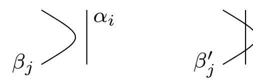

Figure 17: Birth of a bigon

Proposition 6.2. *For any Heegaard diagram $\mathcal{D}$ and $\mathcal{D}'$ such that $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a birth of a bigon as above.*

$$\tilde{\lambda}(\mathcal{D}') = \tilde{\lambda}(\mathcal{D}).$$

Since we know that changing the orientation of α_i does not modify $\tilde{\lambda}$, we assume that our born bigon is one of the two bigons shown in Figure 18, with two arcs going from a crossing e to a crossing f, without loss.

Figure 18: The considered two bigons

We fix a matching $\mathfrak{m}$ for $\mathcal{D} = ((\alpha_i), (\beta_j))$ and the same one for $\mathcal{D}'$, and an exterior point w of $\mathcal{D}'$ outside the bigon so that w is also an exterior point of $\mathcal{D}$.

Lemma 6.3.

$$p_1(X(\mathcal{D}, w, \mathfrak{m})) = p_1(X(\mathcal{D}', w, \mathfrak{m}))$$

Proof: The two fields $X(\mathcal{D}, w, \mathfrak{m})$ and $X(\mathcal{D}', w, \mathfrak{m})$ may be assumed to coincide outside a ball that contains the past and the future in $f_M^{-1}([-2, 7])$ of a disk of $H_{\mathcal{A}}$ around the bigon, with respect to a flow associated with $\mathcal{D}'$. Since both fields are positive normals to the level surfaces of f_M on this ball they are homotopic. □

Now, Proposition 6.2 is a direct consequence of Lemmas 6.4 and 6.5.

Lemma 6.4.

$$\ell_2(\mathcal{D}') = \ell_2(\mathcal{D}) + \mathcal{J}_{ji}/2.$$

$$s_\ell(\mathcal{D}', \mathfrak{m}) = s_\ell(\mathcal{D}, \mathfrak{m})$$

Proof: Let $\mathcal{C}$ be the set of crossings of $\mathcal{D}$. Note that $\sigma(f) = -\sigma(e)$. Then

$$\begin{aligned} G(\mathcal{D}') - G(\mathcal{D}) = \; & \sum_{c \in \mathcal{C}} \mathcal{J}_{j(c)i} \mathcal{J}_{ji(c)} \sigma(c)\sigma(f)\gamma(c) \times (\gamma(f) - \gamma(e))_{\parallel} \\ & + \sum_{d \in \mathcal{C}} \mathcal{J}_{ji(d)} \mathcal{J}_{j(d)i} \sigma(d)\sigma(f)(\gamma(f) - \gamma(e)) \times \gamma(d)_{\parallel} \\ & + \mathcal{J}_{ji}^2 (\gamma(f) - \gamma(e)) \times (\gamma(f) - \gamma(e))_{\parallel} \\ & - \mathcal{J}_{ji} \sigma(f)(\gamma(f) \times \gamma(f)_{\parallel} - \gamma(e) \times \gamma(e)_{\parallel}). \end{aligned}$$

Use Proposition 2.7 to compute $\ell^{(2)}(G(\mathcal{D}') - G(\mathcal{D}))$ with the basepoints of $\mathfrak{m}$, so that for any $c \in \mathcal{C}$,

$$\ell(c, f) - \ell(c, e) = \langle [p(\alpha(c)), c|_\alpha, |e, f|_\beta \rangle - \sum_{(k,\ell)\in \underline{g}^2} \mathcal{J}_{\ell k} \langle [p(\alpha(c)), c|_\alpha, \beta_\ell \rangle \langle \alpha_k, |e, f|_\beta \rangle = 0$$

since $\langle \alpha_k, |e, f|_\beta \rangle = 0$ for any k, and $\langle [p(\alpha(c)), c|_\alpha, |e, f|_\beta \rangle = 0$ for any $c \in \mathcal{C}$. Similarly, for any $d \in \mathcal{C}$, $\ell(e, d) = \ell(f, d)$ and

$$\ell(f - e, f - e) = \langle |e, f|_\alpha, |e, f|_\beta \rangle = 0.$$

Finally,

$$\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) = -\mathcal{J}_{ji}\sigma(f)(\ell(f, f) - \ell(e, e))$$

where

$$\ell(f, f) - \ell(e, e) = \langle [e, f|_\alpha, [e, f|_\beta \rangle - \langle [e, e|_\alpha, [e, e|_\beta \rangle = \sigma(e) + \frac{1}{4}\sigma(f) - \frac{1}{4}\sigma(e) = -\frac{1}{2}\sigma(f)$$

so that $\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) = \frac{1}{2}\mathcal{J}_{ji}$. Similarly, $s_\ell(\mathcal{D}', \mathfrak{m}) = s_\ell(\mathcal{D}, \mathfrak{m})$. □

Lemma 6.5.

$$e(\mathcal{D}', w, \mathfrak{m}) = e(\mathcal{D}, w, \mathfrak{m}) + \mathcal{J}_{ji}/2.$$

PROOF: Adding a bigon changes Figure 2 as in Figure 19.

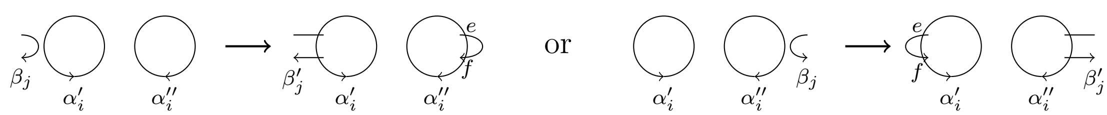

Figure 19: Adding a bigon

In particular, the $d_e(\beta_s)$ of Subsection 1.5 are unchanged, and so are the $d_e(c)$, for $c \in \mathcal{C}$. Then $e(\mathcal{D}', w, \mathfrak{m}) - e(\mathcal{D}, w, \mathfrak{m}) = \mathcal{J}_{ji}\sigma(f)d_e(|e, f|_\beta)$ that is $\frac{1}{2}\mathcal{J}_{ji}$ according to Figure 19. □

Remark 6.6. If the two arcs of the bigon did not begin at the same vertex, then $\mathcal{J}_{ji}$ would be replaced by $-\mathcal{J}_{ji}$ in the results of Lemmas 6.4 and 6.5.

6.3 Handle slides

This section is devoted to proving that $\tilde{\lambda}$ is invariant under handle slide. Since $\tilde{\lambda}$ depends neither on the orientations of the curves α_i and β_j, nor on their numberings, and since permuting the roles of the α_i and β_j does not change $\tilde{\lambda}$, it is sufficient to study a handle slide that transforms $\mathcal{D}$ to a diagram $\mathcal{D}'$ by changing β_1 to a band sum β'_1 of β_1 and the parallel β_2^+ of β_2 (on its

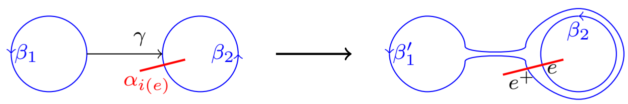

Figure 20: The considered handle slide

positive side) as in Figure 20. Up to the isotopies treated in the previous section, we may assume that the path γ from β_1 to β_2 does not meet the curves α_i, without loss, and we do. The first crossing on β_2^+ will be called e^+. It corresponds to a crossing $e \in \alpha_{i(e)} \cap \beta_2$ as in Figure 20.

Fix w outside a neighborhood of the path γ and β_2 so that it makes sense to say that w is the same for $\mathcal{D}$ and $\mathcal{D}'$. Fix a matching $\mathfrak{m}$ for $\mathcal{D}$. Assume $\mathfrak{m} = \{m_i\}_{i \in \underline{g}}$ and $m_i \in \alpha_i \cap \beta_i$ (by renumbering the α curves if necessary). The set $\mathcal{C}'$ of crossings of $\mathcal{D}'$ contains $\mathcal{C}$ so that $\mathfrak{m}$ is also a matching for $\mathcal{D}'$.

Under these assumptions, we are going to prove that $\tilde{\lambda}(\mathcal{D}') = \tilde{\lambda}(\mathcal{D})$ by proving the following lemmas.

Lemma 6.7.

$$p_1(X(\mathcal{D}, w, \mathfrak{m})) = p_1(X(\mathcal{D}', w, \mathfrak{m})).$$

Lemma 6.8.

$$\begin{aligned} \ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) &= \textstyle\sum_{c \in \beta_2, d \in [e,c|_\beta} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)} \\ &\overset{def}{=} \textstyle\sum_{c \in \beta_2, d \in [e,c[_\beta} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)} + \frac{1}{2}\sum_{c \in \beta_2} \mathcal{J}_{1i(c)}\mathcal{J}_{2i(c)}. \end{aligned}$$

Lemma 6.9.

$$s_\ell(\mathcal{D}', \mathfrak{m}) - s_\ell(\mathcal{D}, \mathfrak{m}) = \sum_{d \in \beta_2, c \in [e,d|_\beta} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)} - \sum_{c \in [e,m_2|_\beta} \sigma(c)\mathcal{J}_{1i(c)}.$$

Lemma 6.10.

$$e(\mathcal{D}', w, \mathfrak{m}) - e(\mathcal{D}, w, \mathfrak{m}) = \sum_{c \in]m_2,e[_\beta} \sigma(c)\mathcal{J}_{1i(c)}.$$

Since

$$\sum_{c \in]m_2,e[_\beta} \sigma(c)\mathcal{J}_{1i(c)} + \sum_{c \in [e,m_2|_\beta} \sigma(c)\mathcal{J}_{1i(c)} = \sum_{c \in \beta_2} \sigma(c)\mathcal{J}_{1i(c)} = \sum_{i=1}^{g} \mathcal{J}_{1i}\langle \alpha_i, \beta_2 \rangle = 0$$

and

$$\sum_{c \in \beta_2, d \in [e,c|_\beta} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)} + \sum_{d \in \beta_2, c \in [e,d|_\beta} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)} = \sum_{(c,d) \in \beta_2^2} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)} = 0,$$

these four lemmas imply that $\tilde{\lambda}(\mathcal{D}') = \tilde{\lambda}(\mathcal{D})$. $\square$

PROOF OF LEMMA 6.7: Let $X = X(\mathcal{D}, w, \mathfrak{m})$ and $X' = X(\mathcal{D}', w, \mathfrak{m})$. First note that X and X' coincide in $H_{\mathcal{A}}$. We describe a homotopy $(Y_t)_{t\in[0,1]}$ from $Y_0 = (-X)$ and $Y_1 = (-X')$ on $H_{\mathcal{B}}$.

See $(-X)$ in $H_{\mathcal{B}}$ as the upward vertical field in the first picture of Figure 22. This field is an outward normal to $H_{\mathcal{B}}$ except around w that is not shown in our figures, and around the crossings of $\mathfrak{m}$, more precisely on the gray disks D_i shown in Figure 21. Inside the disks D_i, $(-X)$ is an inward normal to $H_{\mathcal{B}}$. On the boundary of this disk, it is tangent to the surface. Our homotopy will fix $(-X)$ in the neighborhood of w where $(-X)$ is not an outward normal to $H_{\mathcal{B}}$, and the locus of $\partial H_{\mathcal{B}}$ where Y_t is a positive (resp. negative) normal to $H_{\mathcal{B}}$ will not depend on t. Thus this homotopy can be canonically modified (without changing the locus where Y_t is a positive (resp. negative) normal to $H_{\mathcal{B}}$) so that Y_t is fixed on $\partial H_{\mathcal{B}}$.

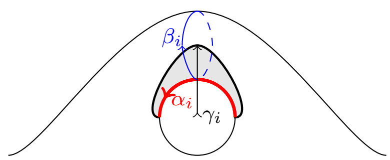

Figure 21: The front part of the disk D_i where the field points inward the surface

Observe that there is no loss in assuming that the path γ from β_1 to β_2 that parametrizes the handle slide is as in the first picture of Figure 22. The next pictures describe various positions of $H_{\mathcal{B}}$ under an ambient isotopy $(h_t)_{t\in[0,1]}$ of $\mathbb{R}^3$ that first moves the handle of β_2 upward (second picture), slides it over the handle of β_1 (fourth picture), moves the handle of β_1 upward (fifth picture) and replaces the slid foot of H_2 in its original position by letting it slide away from the handles (last picture). The isotopy $(h_t)_{t\in[0,1]}$ starts with h_0 that is the Identity and finishes with a homeomorphism h_1 of $\mathbb{R}^3$ that maps $H_{\mathcal{B}}$ to itself. Let $\vec{N}$ be the upward vector field of $\mathbb{R}^3$. Then $(h_t)_*^{-1}(\vec{N}_{|h_t(H_{\mathcal{B}})})$ defines a homotopy of nowhere zero vector fields from $Y_0 = (-X)$ and $Y_1 = (-X')$ on $H_{\mathcal{B}}$ that behaves as wanted on the boundary.

$\square$

Let us start with common preliminaries for the proofs of the remaining three lemmas.

Set $\mathcal{J}'_{2i} = \mathcal{J}_{2i} - \mathcal{J}_{1i}$. For any interval I of an α_i, $\langle I, \beta'_1\rangle = \langle I, \beta_1 + \beta_2^+\rangle$ and

$$\langle I, \mathcal{J}_{1i}\beta'_1 + \mathcal{J}'_{2i}\beta_2\rangle = \langle I, \mathcal{J}_{1i}\beta_1 + \mathcal{J}_{2i}\beta_2 + \mathcal{J}_{1i}(\beta_2^+ - \beta_2)\rangle.$$

Set $\mathcal{J}'_{ji} = \mathcal{J}_{ji}$ for any (j,i) such that $j \neq 2$. Every quantity associated with $\mathcal{D}'$ will have a prime superscript. Our definitions of the $\mathcal{J}'_{ji}$ ensure that

$$\langle \alpha_k, \sum_j \mathcal{J}'_{ji}\beta'_j\rangle = \langle \alpha_k, \sum_j \mathcal{J}_{ji}\beta_j\rangle = \delta_{ik},$$

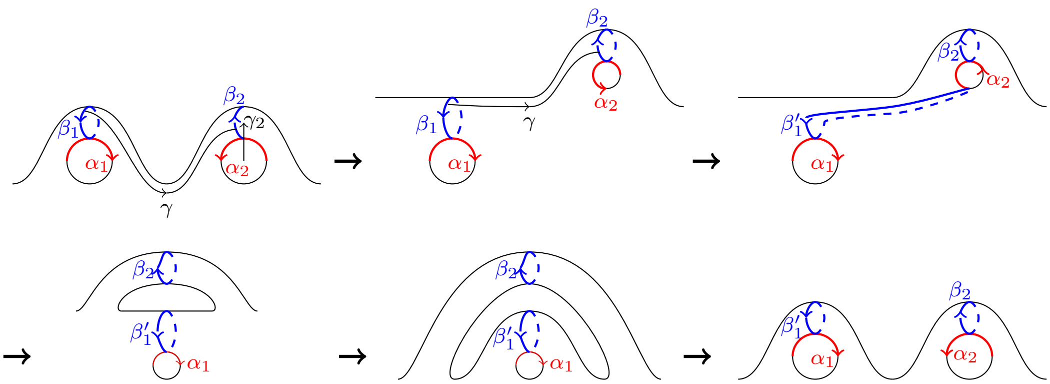

Figure 22: Handle slide

for any i and k, as required.

Let $\mathcal{C}_2$ be the set of crossings of $\mathcal{D}$ on β_2, and let $\mathcal{C}_2^+$ be the set of crossings of $\mathcal{D}'$ on β_2^+, $\mathcal{C}_2^+$ is in natural one-to-one correspondence with $\mathcal{C}_2$ and the crossing of $\mathcal{C}_2^+$ that corresponds to c will be denoted by c^+.

$$\mathcal{C}' = \mathcal{C} \cup \mathcal{C}_2^+.$$

Proof of Lemma 6.10: Without loss, assume that β_2 goes from right to left at the place of the band sum as in Figure 23. Then β_1 is above β_2 and it goes from left to right. Thus after the band sum, the degree of β'_1 is increased by $(-1/2)$ before and after β_2^+ and by $(1/2)$ before and after m_2.

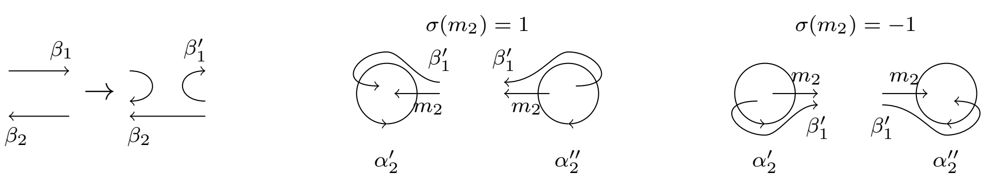

Figure 23: Variation of d_e

Therefore $d'_e(\beta'_1) = d_e(\beta_1) + d_e(\beta_2)$, and, for any i,

$$\sum_{j=1}^{g} \mathcal{J}'_{ji} d'_e(\beta'_j) = \sum_{j=1}^{g} \mathcal{J}_{ji} d_e(\beta_j).$$

Then for any c that is not in $[e^+, m_1[_{\beta'_1}$, $d'_e(c) = d_e(c)$. Since $d_e(\beta_2) - \sum_{(r,s)\in \underline{g}^2} \mathcal{J}_{sr}\langle \alpha_r, \beta_2\rangle d_e(\beta_s) = 0$, for any $c \in [e^+, m_1[_{\beta'_1} \setminus \beta_2^+$, $d'_e(c) = d_e(c)$, too, so that

$$e(\mathcal{D}', w, \mathfrak{m}) - e(\mathcal{D}, w, \mathfrak{m}) = \sum_{c\in\beta_2} \mathcal{J}_{1i(c)} \sigma(c)(d'_e(c^+) - d_e(c)).$$

For $c \in \beta_2$,

$$d'_e(c^+) - d'_e(e^+) = \begin{cases} d_e(c) - d_e(e) & \text{if } c \in]e, m_2[\\ d_e(c) - d_e(e) + 1 & \text{if } c \in]m_2, e[\\ d_e(c) - d_e(e) + \frac{1}{2} & \text{if } c = m_2. \end{cases}$$

□

For the remaining two lemmas, for any 2–cycle $G = \sum_{(c,d)\in(\mathcal{C}')^2} g_{cd}(\gamma(c) \times \gamma(d)_{\parallel})$ of M^2, we compute $\ell^{(2)}(G)$ with Proposition 2.7 with

$$\ell(c, d) = \langle [p(\alpha(c)), c|_\alpha, [p(\beta(d)), d|_\beta\rangle - \sum_{(i,j)\in \underline{g}^2} \mathcal{J}'_{ji}\langle [p(\alpha(c)), c|_\alpha, \beta'_j\rangle\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle$$

where $p(\beta_2) = e$ and $p(\beta'_1)$ is the first crossing of β'_1 after β_2^+ on β_1, the $p(\alpha_i)$ are not on β_2^+, and, if $p(\alpha_i) \in \beta_2$, then $\sigma(p(\alpha_i)) = 1$ (up to changing the orientation of α_i). This map $\ell^{(2)}$ may be used for any 2–cycle $G = \sum_{(c,d)\in\mathcal{C}^2} g_{cd}(\gamma(c) \times \gamma(d)_{\parallel})$ of M^2, as well, and we use it.

Lemma 6.11. *Recall $\mathcal{C}' = \mathcal{C} \cup \mathcal{C}_2^+$. Let $(c, d) \in \mathcal{C}^2$. If $c \in \beta_2$, then*

$$\ell(c^+, d) - \ell(c, d) + \frac{1}{2}\sum_{i=1}^{g} \mathcal{J}_{2i}\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle = \begin{cases} 0 & \text{if } d \notin \beta_2 \\ 0 & \text{if } d \in \beta_2 \text{ and } c \notin [e, d]_\beta \\ \frac{1}{2} & \text{if } d \in \beta_2 \text{ and } c \in [e, d[_\beta \\ \frac{1}{4} & \text{if } c = d. \end{cases}$$

If $d \in \beta_2$, then

$$\ell(c, d^+) - \ell(c, d) = \begin{cases} -\frac{1}{2} & \text{if } c \in [e, d[_\beta \\ -\frac{1}{4} & \text{if } c = d \\ 0 & \text{if } c \notin [e, d]_\beta. \end{cases}$$

If $(c, d) \in \mathcal{C}_2^2$, then

$$\ell(c^+, d^+) - \ell(c^+, d) = \ell(c, d^+) - \ell(c, d).$$

PROOF: Let $(c, d) \in \mathcal{C}^2$. Assume $c \in \beta_2$. If $\sigma(c) = 1$, then

$$\begin{aligned} \ell(c^+, d) - \ell(c, d) &= \langle |c, c^+|_\alpha, [p(\beta(d)), d|_\beta\rangle - \textstyle\sum_{(i,j)\in\underline{g}^2} \mathcal{J}'_{ji}\langle |c, c^+|_\alpha, \beta'_j\rangle\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle \\ &= \langle |c, c^+|_\alpha, [p(\beta(d)), d|_\beta\rangle - \tfrac{1}{2}\textstyle\sum_{i=1}^{g}(\mathcal{J}'_{2i} + \mathcal{J}'_{1i})\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle. \end{aligned}$$

If $\sigma(c) = -1$,

$$\ell(c^+, d) - \ell(c, d) = -\langle |c^+, c|_\alpha, [p(\beta(d)), d|_\beta\rangle - \frac{1}{2}\sum_{i=1}^{g} \mathcal{J}_{2i}\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle.$$

Let $d \in \beta_2$. For any interval I of an α_i, $\langle I, [p(\beta'_1), d^+|_\beta\rangle = \langle I, \beta_1 + [e^+, d^+|_\beta\rangle$.
If $c \notin \beta_2$,

$$\begin{aligned} \ell(c, d^+) - \ell(c, d) &= \langle [p(\alpha(c)), c|_\alpha, \beta_1\rangle - \textstyle\sum_{(i,j)\in\underline{g}^2} \mathcal{J}'_{ji}\langle [p(\alpha(c)), c|_\alpha, \beta'_j\rangle\langle \alpha_i, \beta'_1 - \beta'_2\rangle \\ &= \langle [p(\alpha(c)), c|_\alpha, \beta_1\rangle - \textstyle\sum_{j\in\underline{g}}(\delta_{j1} - \delta_{j2})\langle [p(\alpha(c)), c|_\alpha, \beta'_j\rangle \\ &= \langle [p(\alpha(c)), c|_\alpha, \beta_1\rangle - \langle [p(\alpha(c)), c|_\alpha, \beta'_1 - \beta'_2\rangle \\ &= 0. \end{aligned}$$

When $c \in \beta_2$, we similarly get

$$\ell(c,d^+) - \ell(c,d) = \langle [p(\alpha(c)), c|_\alpha, [e^+,d^+|_\beta - [e,d|_\beta \rangle = \begin{cases} -\frac{1}{2} & \text{if } c \in [e,d[_\beta \\ -\frac{1}{4} & \text{if } c = d \\ 0 & \text{if } c \notin [e,d]_\beta. \end{cases}$$

Since $\langle [p(\alpha(c)), c|_\alpha, [e^+,d^+|_\beta - [e,d|_\beta \rangle = \langle [p(\alpha(c)), c^+|_\alpha, [e^+,d^+|_\beta - [e,d|_\beta \rangle$, $\ell(c^+,d^+) - \ell(c^+,d) = \ell(c,d^+) - \ell(c,d)$. ☐

Proof of Lemma 6.9: Set $L = L(\mathcal{D},\mathfrak{m}) = \sum_{i=1}^g \gamma_i - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\gamma(c)$ and $L' = L(\mathcal{D}',\mathfrak{m})$. Then

$$L' - L = \sum_{c\in\mathcal{C}_2} \mathcal{J}_{1i(c)}\sigma(c)(\gamma(c) - \gamma(c^+))$$

is a cycle and

$$lk(L', L'_{\|}) - lk(L, L_{\|}) = \ell((L'-L)\times(L'-L)) + 2\ell((L'-L)\times L)$$

thanks to the symmetry of the linking number in Proposition 2.6.

The last assertion of Lemma 6.11 guarantees that

$$\ell((L'-L)\times(L'-L)) = 0.$$

Now, $\ell((L'-L)\times L) = \ell_1 + \ell_2$ with

$$\ell_1 = \sum_{c\in\mathcal{C}_2, i\in\underline{g}} \mathcal{J}_{1i(c)}\sigma(c)(\ell(c,m_i) - \ell(c^+,m_i))$$

where $\mathfrak{m} = \{m_i\}_{i\in\underline{g}}$ and $m_i \in \alpha_i \cap \beta_i$ and

$$\ell_2 = \sum_{c\in\beta_2, d\in\mathcal{C}} \mathcal{J}_{1i(c)}\sigma(c)\mathcal{J}_{j(d)i(d)}\sigma(d)(\ell(c^+,d) - \ell(c,d)).$$

Since the part $\left(\frac{1}{2}\sum_{i=1}^g \mathcal{J}_{2i}\langle \alpha_i, [p(\beta(d)), d|_\beta\rangle\right)$ that occurs in the expressions of $(\ell(c^+,d) - \ell(c,d))$ in Lemma 6.11 is independent of c, the factor $\sum_{c\in\beta_2}\mathcal{J}_{1i(c)}\sigma(c)$ that vanishes makes it disappear so that

$$\ell((L'-L)\times L) = \tilde{\ell}_1 + \tilde{\ell}_2$$

where

$$\tilde{\ell}_1 = -\frac{1}{2}\left(\sum_{c\in[e,m_2[_\beta} \sigma(c)\mathcal{J}_{1i(c)} + \frac{1}{2}\sigma(m_2)\mathcal{J}_{12}\right) = -\frac{1}{2}\sum_{c\in[e,m_2|_\beta} \sigma(c)\mathcal{J}_{1i(c)}$$

and

$$\tilde{\ell}_2 = \frac{1}{2}\sum_{d\in\beta_2, c\in[e,d|_\beta} \sigma(c)\sigma(d)\mathcal{J}_{1i(c)}\mathcal{J}_{2i(d)}.$$

□

Proof of Lemma 6.8: Recall

$$\ell_2(\mathcal{D}) = \sum_{(c,d)\in\mathcal{C}^2} \mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)\ell(c,d) - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\ell(c,c).$$

Define the projection $q\colon \mathcal{C}' \to \mathcal{C}$ such that $q(c) = c$ if $c \in \mathcal{C}$ and $q(c^+) = c$ if $c \in \beta_2$. Since a crossing c of β_2 gives rise to two crossings c and c^+ of $\mathcal{C}'$ whose coefficients $\mathcal{J}'_{2r}$ and $\mathcal{J}'_{1r}$ add up to $\mathcal{J}_{2r}$,

$$\ell_2(\mathcal{D}) = \sum_{(c,d)\in(\mathcal{C}')^2} \mathcal{J}'_{j(c)i(d)}\mathcal{J}'_{j(d)i(c)}\sigma(c)\sigma(d)\ell(q(c),q(d)) - \sum_{c\in\mathcal{C}'} \mathcal{J}'_{j(c)i(c)}\sigma(c)\ell(q(c),q(c))$$

so that

$$\begin{aligned}\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) = \ & \textstyle\sum_{(c,d)\in(\mathcal{C}')^2} \mathcal{J}'_{j(c)i(d)}\mathcal{J}'_{j(d)i(c)}\sigma(c)\sigma(d)(\ell(c,d) - \ell(q(c),q(d))) \\ & - \textstyle\sum_{c\in\mathcal{C}_2} \mathcal{J}'_{1i(c)}\sigma(c)(\ell(c^+,c^+) - \ell(c,c)).\end{aligned}$$

Write $\ell(c,d) - \ell(q(c),q(d)) = \ell(c,d) - \ell(c,q(d)) + \ell(c,q(d)) - \ell(q(c),q(d))$.

$$\ell(c,d^+) - \ell(c,d) = \ell(q(c),d^+) - \ell(q(c),d) = \begin{cases} -\frac{1}{2} & \text{if } q(c) \in [e,d[_\beta \\ -\frac{1}{4} & \text{if } q(c) = d \\ 0 & \text{if } q(c) \notin [e,d]_\beta \end{cases}$$

for $d \in \mathcal{C}_2$ so that

$$\begin{aligned}\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) = \ & -\tfrac{1}{2}\textstyle\sum_{d\in\mathcal{C}_2}\sum_{c\in[e,d|_\beta}(\mathcal{J}'_{2i(d)} + \mathcal{J}'_{1i(d)})\mathcal{J}'_{1i(c)}\sigma(c)\sigma(d) + A \\ & +\tfrac{1}{4}\textstyle\sum_{c\in\mathcal{C}_2}\mathcal{J}'_{1i(c)}\sigma(c) - \sum_{c\in\mathcal{C}_2}\mathcal{J}'_{1i(c)}\sigma(c)(\ell(c^+,c) - \ell(c,c))\end{aligned}$$

where

$$\begin{aligned}A &= \textstyle\sum_{(c,d)\in(\mathcal{C}')^2} \mathcal{J}'_{j(c)i(d)}\mathcal{J}'_{j(d)i(c)}\sigma(c)\sigma(d)(\ell(c,q(d)) - \ell(q(c),q(d))) \\ &= \textstyle\sum_{(c,d)\in\mathcal{C}'\times\mathcal{C}} \mathcal{J}'_{j(c)i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)(\ell(c,d) - \ell(q(c),d)) \\ &= \textstyle\sum_{(c,d)\in\mathcal{C}_2\times\mathcal{C}} \mathcal{J}'_{1i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)(\ell(c^+,d) - \ell(c,d)) \\ &= -\tfrac{1}{2}\textstyle\sum_{(c,d)\in\mathcal{C}_2\times\mathcal{C}} \mathcal{J}'_{1i(d)}\mathcal{J}_{j(d)i(c)}\sigma(c)\sigma(d)(\sum_{i=1}^{g}\mathcal{J}_{2i}\langle\alpha_i,[p(\beta(d)),d|_\beta\rangle) \\ &\quad +\tfrac{1}{2}\textstyle\sum_{d\in\mathcal{C}_2,c\in[e,d|_\beta} \mathcal{J}'_{1i(d)}\mathcal{J}_{2i(c)}\sigma(c)\sigma(d) \\ &= -\tfrac{1}{2}\textstyle\sum_{i=1}^{g}\sum_{d\in\mathcal{C}_2} \mathcal{J}'_{1i(d)}\sigma(d)\mathcal{J}_{2i}\langle\alpha_i,[p(\beta(d)),d|_\beta\rangle \\ &\quad +\tfrac{1}{2}\textstyle\sum_{d\in\mathcal{C}_2,c\in[e,d|_\beta} \mathcal{J}'_{1i(d)}\mathcal{J}_{2i(c)}\sigma(c)\sigma(d) \\ &= 0.\end{aligned}$$

$$\begin{aligned}\textstyle\sum_{c\in\mathcal{C}_2}\mathcal{J}_{1i(c)}\sigma(c)(\ell(c^+,c) - \ell(c,c)) &= -\tfrac{1}{2}\textstyle\sum_{c\in\mathcal{C}_2}\mathcal{J}_{1i(c)}\sigma(c)(\sum_{i=1}^{g}\mathcal{J}_{2i}\langle\alpha_i,[p(\beta(c)),c|_\beta\rangle) \\ &\quad +\tfrac{1}{4}\textstyle\sum_{c\in\mathcal{C}_2}\mathcal{J}_{1i(c)}\sigma(c) \\ &= -\tfrac{1}{2}\textstyle\sum_{(c,d)\in\mathcal{C}_2^2; d\in[e,c|_\beta}\mathcal{J}_{2i(d)}\mathcal{J}_{1i(c)}\sigma(c)\sigma(d).\end{aligned}$$

$$\begin{aligned}\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) &= -\frac{1}{2}\sum_{d\in\mathcal{C}_2}\sum_{c\in[e,d|_\beta}\mathcal{J}_{2i(d)}\mathcal{J}_{1i(c)}\sigma(c)\sigma(d)\\ &\quad +\frac{1}{2}\sum_{(c,d)\in\mathcal{C}_2^2;d\in[e,c|_\beta}\mathcal{J}_{2i(d)}\mathcal{J}_{1i(c)}\sigma(c)\sigma(d).\end{aligned}$$

For $r,s \in \underline{g}$, set

$$V_{r,s} = \sum_{c\in\mathcal{C}_2}\sum_{d\in[e,c|_\beta}\mathcal{J}_{ri(d)}\mathcal{J}_{si(c)}\sigma(c)\sigma(d).$$

Note that $V_{r,s} + V_{s,r} = \delta_{r2}\delta_{s2}$ (recall the argument after the statement of Lemma 6.10). Thus $\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) = \frac{1}{2}(V_{2,1} - V_{1,2}) = V_{2,1}$. □

6.4 Connected sums and stabilizations

The previous subsections guarantee that $\tilde{\lambda}$ is an invariant of Heegaard decompositions.

Lemma 6.12. *Let*

$$S^3 = T_A \cup_{\partial T_A \sim -\partial T_B} T_B$$

be the genus one decomposition of S^3 as a union of two solid tori T_A and T_B glued along their boundaries so that the meridian α_1 of T_A meets the meridian β_1 of T_B once.

$$\tilde{\lambda}(T_A \cup_{\partial T_A \sim -\partial T_A} T_B) = 0.$$

PROOF: Orient α_1 and β_1 so that $\langle \alpha_1, \beta_1 \rangle_{\partial T_A} = 1$. Then $\mathcal{J}_{11} = 1$. Let $\mathfrak{m} = \{\alpha_1 \cap \beta_1\}$ be the unique matching. Let w be a point of the connected $\partial T_A \setminus (\alpha_1 \cup \beta_1)$. Then the intersection of T_A with B^3 can be embedded in B^3 as in Figure 6, and T_B is its complement in B^3. In particular, $X(w,\mathfrak{m})$ is the vertical field of $\mathbb{R}^3$ and $p_1(X(w,\mathfrak{m})) = 0$. The rectangular picture of the Heegaard diagram is simply Figure 24 so that $e(\mathcal{D}, w, \mathfrak{m}) = 0$. Since $G(\mathcal{D}) = \emptyset$ and

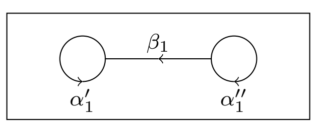

Figure 24: Genus one Heegaard diagram of S^3

$L(\mathcal{D},\mathfrak{m}) = \emptyset$, $\ell_2(\mathcal{D}) = 0$ and $s_\ell(\mathcal{D},\mathfrak{m}) = 0$. □

The *connected sum* $M\sharp M'$ of two connected closed manifolds M and M' of dimension d is obtained by removing the interior of an open ball from M and from M' and by gluing the obtained manifolds along their spherical boundaries

$$M\sharp M' = (M \setminus \mathring{B}^d) \cup_{S^{d-1}} (M' \setminus \mathring{B}'^d).$$

When the manifolds are 3–manifolds equipped with Heegaard decompositions $M = H_{\mathcal{A}} \cup_{\partial H_{\mathcal{A}}} H_{\mathcal{B}}$ and $M' = H'_{\mathcal{A}} \cup_{\partial H'_{\mathcal{A}}} H'_{\mathcal{B}}$, the *connected sum* of the Heegaard decompositions is the Heegaard decomposition

$$M \sharp M' = H_{\mathcal{A}} \sharp_{\partial} H'_{\mathcal{A}} \cup_{\partial H_{\mathcal{A}} \sharp \partial H'_{\mathcal{A}}} H_{\mathcal{B}} \sharp_{\partial} H'_{\mathcal{B}}$$

where the open ball B (resp. B') removed from M (resp. from M') intersects the Heegaard surface $\partial H_{\mathcal{A}}$ (resp. $\partial H'_{\mathcal{A}}$) as a properly embedded two dimensional disk that separates B into two half-balls $\mathring{H}_{\mathcal{A}} \cap B$ and $\mathring{H}_{\mathcal{B}} \cap B$ (resp. $\mathring{H}_{\mathcal{A}}{}' \cap B'$ and $\mathring{H}_{\mathcal{B}}{}' \cap B'$), the connected sum along the boundaries

$$H_{\mathcal{A}} \sharp_{\partial} H'_{\mathcal{A}} = \left(H_{\mathcal{A}} \setminus (H_{\mathcal{A}} \cap \mathring{B})\right) \cup_{H_{\mathcal{A}} \cap \partial B \sim (-H'_{\mathcal{A}} \cap \partial B')} \left(H'_{\mathcal{A}} \setminus (H'_{\mathcal{A}} \cap \mathring{B}')\right)$$

is homeomorphic to the manifold obtained by identifying $H_{\mathcal{A}}$ and $H'_{\mathcal{A}}$ along a two-dimensional disk of the boundary, and $H_{\mathcal{B}} \sharp_{\partial} H'_{\mathcal{B}}$ is defined similarly.

Proposition 6.13. *Under the hypotheses above, if M and M' are rational homology spheres, then*

$$\tilde{\lambda}(H_{\mathcal{A}} \sharp_{\partial} H'_{\mathcal{A}} \cup_{\partial H_{\mathcal{A}} \sharp \partial H'_{\mathcal{A}}} H_{\mathcal{B}} \sharp_{\partial} H'_{\mathcal{B}}) = \tilde{\lambda}(H_{\mathcal{A}} \cup_{\partial H_{\mathcal{A}}} H_{\mathcal{B}}) + \tilde{\lambda}(H'_{\mathcal{A}} \cup_{\partial H'_{\mathcal{A}}} H'_{\mathcal{B}})$$

Proof: When performing such a connected sum on manifolds equipped with Heegaard diagrams $\mathcal{D} = (\partial H_{\mathcal{A}}, (\alpha_i)_{i \in \underline{g}}, (\beta_j)_{j \in \underline{g}})$ and $\mathcal{D}' = (\partial H'_{\mathcal{A}}, (\alpha'_i)_{i \in \underline{g'}}, (\beta'_j)_{j \in \underline{g'}})$ and with exterior points w and w' of $\mathcal{D}$ and $\mathcal{D}'$, we assume that the balls D and D' meet the Heegaard surfaces inside the connected component of w or w' outside the diagram curves, without loss, and we choose a basepoint w'' in the corresponding region of $\partial H_{\mathcal{A}} \sharp \partial H'_{\mathcal{A}}$. Then we obtain the obvious Heegaard diagram

$$\mathcal{D}'' = (\partial H_{\mathcal{A}} \sharp \partial H'_{\mathcal{A}}, (\alpha''_i)_{i \in \underline{g''}}, (\beta''_j)_{j \in \underline{g''}})$$

where $g'' = g + g'$, $\alpha''_i = \alpha_i$ and $\beta''_i = \beta_i$ when $i \leq g$ and $\alpha''_i = \alpha'_{i-g}$ and $\beta''_i = \beta'_{i-g}$ when $i > g$, with the associated intersection matrix and its inverse that are diagonal with respect to the two blocks corresponding to the former matrices associated with $\mathcal{D}$ and $\mathcal{D}'$.

When $\mathcal{D}$ and $\mathcal{D}'$ are furthermore equipped with matchings $\mathfrak{m}$ and $\mathfrak{m}'$, $\mathfrak{m}'' = \mathfrak{m} \cup \mathfrak{m}'$ is a matching for $\mathcal{D}''$ and a rectangular figure for $(\mathcal{D}'', w'', \mathfrak{m}'')$ similar to Figure 2 is obtained from the corresponding figures for $\mathcal{D}$ and $\mathcal{D}'$ by juxtapositions of the two rectangles of $\mathcal{D}$ and $\mathcal{D}'$. In particular,

$$e(\mathcal{D}'', w'', \mathfrak{m}'') = e(\mathcal{D}', w', \mathfrak{m}') + e(\mathcal{D}, w, \mathfrak{m}).$$

Furthermore, we can see $B_{M''}$ as the juxtaposition of two half-balls glued along a vertical disk equipped with the vertical field (over the intersection of the two rectangles above) such that the two half-balls are obtained from B_M and $B_{M'}$ by removing standard vertical half-balls equipped with the vertical field, so that the vector field $X(w'', \mathfrak{m}'')$ coincides with $X(w, \mathfrak{m})$ on the remaining part of B_M and with $X(w', \mathfrak{m}')$ on the remaining part of $B_{M'}$. This makes clear that

$$p_1(X(w'', \mathfrak{m}'')) = p_1(X(w, \mathfrak{m})) + p_1(X(w', \mathfrak{m}')).$$

Now it is easy to observe that $G(\mathcal{D}'') = G(\mathcal{D}) + G(\mathcal{D}')$, that

$$\ell_2(\mathcal{D}'') = \ell_2(\mathcal{D}) + \ell_2(\mathcal{D}'),$$

that $L(\mathcal{D}'', \mathfrak{m}'') = L(\mathcal{D}, \mathfrak{m}) + L(\mathcal{D}', \mathfrak{m}')$ and that

$$s_\ell(\mathcal{D}'', \mathfrak{m}'') = s_\ell(\mathcal{D}, \mathfrak{m}) + s_\ell(\mathcal{D}', \mathfrak{m}').$$

□

A connected sum of a Heegaard decomposition with the genus one decomposition of S^3 is called a *stabilization*. A well-known Reidemeister-Singer theorem proved by Siebenmann in [Sie80], asserts that any two Heegaard decompositions of the same 3-manifold become isomorphic after some stabilizations. This Reidemeister-Singer theorem can also be proved using Cerf theory [Cer70] as in [OS04, Proposition 2.2].

Together with Proposition 6.13 and Lemma 6.12, it implies that $\tilde{\lambda}$ does not depend on the Heegaard decomposition and allows us to prove the following theorem.

Theorem 6.14. *There exists a unique invariant $\tilde{\lambda}$ of $\mathbb{Q}$-spheres such that for any Heegaard diagram $\mathcal{D}$ of a $\mathbb{Q}$-sphere M, equipped with a matching $\mathfrak{m}$ and with an exterior point w,*

$$24\tilde{\lambda}(M) = 4\ell_2(\mathcal{D}) + 4s_\ell(\mathcal{D}, \mathfrak{m}) - 4e(\mathcal{D}, w, \mathfrak{m}) - p_1(X(w, \mathfrak{m})).$$

Furthermore, $\tilde{\lambda}$ satisfies the following properties.

- *For any two rational homology spheres M_1 and M_2,*

$$\tilde{\lambda}(M_1 \sharp M_2) = \tilde{\lambda}(M_1) + \tilde{\lambda}(M_2).$$

- *For any rational homology sphere M, if $(-M)$ denotes the manifold M equipped with the opposite orientation, then*

$$\tilde{\lambda}(-M) = -\tilde{\lambda}(M).$$

Proof: The invariance of $\tilde{\lambda}$ is already proved. Proposition 6.13 now implies that $\tilde{\lambda}$ is additive under connected sum. Reversing the orientation of M reverses the orientation of the surface that contains a diagram $\mathcal{D}$ of M. This changes the signs of the intersection points and reverses the sign of $\mathcal{J}$. Thus $L(\mathcal{D}, \mathfrak{m})$, $G(\mathcal{D})$ and $X(w, \mathfrak{m})$ are unchanged, while ℓ is changed to its opposite. Changing the orientation of the ambient manifold reverses the sign of p_1. A rectangular diagram of $(-M)$ as in Figure 2 is obtained from the diagram of M by a orthogonal symmetry that fixes a vertical line so that the d_e are changed to their opposites. Thus all the terms of the formula are multiplied by (-1) when the orientation of M is reversed.

□

7 The Casson surgery formula for $\tilde{\lambda}$

7.1 The statement and its consequences

In this section, we prove that $\tilde{\lambda}$ coincides with the Casson invariant for integral homology spheres by proving that it satisfies the same surgery formula. More precisely, we prove the following theorem.

Theorem 7.1. *Let K be a null-homologous knot in a $\mathbb{Q}$-sphere M.*

Let Σ be an oriented connected surface of genus $g(\Sigma)$ in M bounded by K such that the closure of the complement of a collar

$$H_{\mathcal{A}} = \Sigma \times [-1,1]$$

of $\Sigma = \Sigma \times \{0\}$ in M is homeomorphic to a handlebody $H_{\mathcal{B}}$. This gives rise to the Heegaard decomposition

$$M = H_{\mathcal{A}} \cup_{\Psi_M} H_{\mathcal{B}}$$

where Ψ_M is an orientation reversing diffeomorphism from $\partial H_{\mathcal{B}}$ to $\partial H_{\mathcal{A}}$. Let $M(K)$ be the manifold obtained from M by surgery of coefficient 1 along K that can be defined by its Heegaard decomposition

$$M(K) = H_{\mathcal{A}} \cup_{\Psi_M \circ \mathfrak{t}_K} H_{\mathcal{B}}$$

where $\mathfrak{t}_K$ is the right-handed Dehn twist of $(-\partial H_{\mathcal{B}})$ about K. Let $g = 2g(\Sigma)$. Let $(z_i)_{i \in \underline{g}}$ be closed curves of $\Sigma = \Sigma \times \{0\}$ that form a geometric symplectic basis of $H_1(\Sigma)$ as in Figure 25, and let $z_i^+ = z_i \times \{1\}$. For any $i \in \underline{g(\Sigma)}$, $\langle z_{2i-1}, z_{2i} \rangle = 1$. Then

$$\tilde{\lambda}(M(K)) - \tilde{\lambda}(M) = \sum_{(i,r) \in \underline{g(\Sigma)}^2} \left(lk(z_{2i}^+, z_{2r}) lk(z_{2i-1}^+, z_{2r-1}) - lk(z_{2i}^+, z_{2r-1}) lk(z_{2i-1}^+, z_{2r}) \right).$$

We will prove the theorem exactly as it is stated. A *Seifert surface* Σ of K as in the statement is said to be *unknotted*. It is well-known that any Seifert surface can be transformed to an unknotted one by adding some tubes (to remove unwanted 2-handles from its exterior). (See [Mar88, Lemme 5.1], [AM90, p.84] or [GM92, Lemme 4.1] in the original surveys [Mar88, AM90, GM92] of the Casson invariant, for example.) Thus any null-homologous knot bounds an unknotted surface as in the statement. The manifold $M(K; \frac{p}{q})$ obtained from M by *Dehn surgery* with coefficient p/q along K, for two coprime integers p and q, is usually defined as

$$M(K; \frac{p}{q}) = \left(M \setminus \mathring{N}(K) \right) \cup_{\partial N(K) \sim \partial D^2 \times S^1} \left(D^2 \times S^1 \right)$$

where $N(K)$ is a tubular neighborhood of K, and the gluing homeomorphism from $\partial D^2 \times S^1$ to $\partial N(K)$ identifies the meridian $\partial D^2 \times \{x\}$ of $D^2 \times S^1$ with a curve homologous to $pm(K) + q\ell(K)$ where $m(K)$ is the meridian of K such that $lk(m(K), K) = 1$ and $\ell(K)$ is the curve parallel

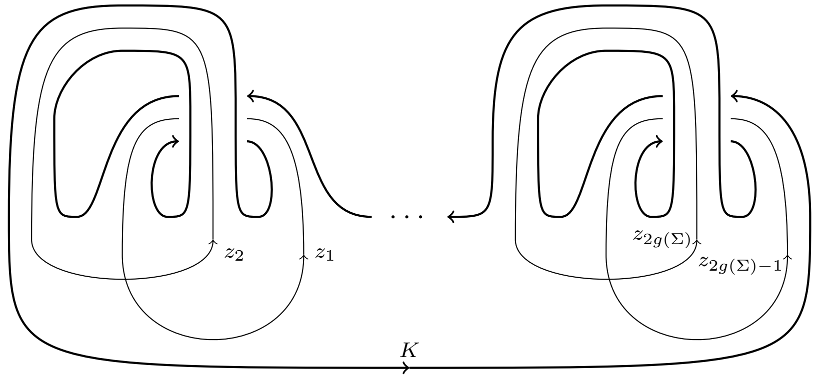

Figure 25: Curves on the surface Σ

to K such that $lk(\ell(K), K) = 0$. In our case, for $n \in \mathbb{Z} \setminus \{0\}$, the manifold $M(K; \frac{1}{n})$ obtained from M by surgery of coefficient $\frac{1}{n}$ along K can also be defined by its Heegaard decomposition

$$M(K; \frac{1}{n}) = H_{\mathcal{A}} \cup_{\Psi_M \circ t_K^n} H_{\mathcal{B}},$$

and it is easy to observe that the variation $(\tilde{\lambda}(M(K; \frac{1}{n})) - \tilde{\lambda}(M))$ can be deduced from the general knowledge of $(\tilde{\lambda}(M(K)) - \tilde{\lambda}(M))$. In our case, Theorem 7.1 implies that

$$\tilde{\lambda}(M(K; \frac{1}{n})) - \tilde{\lambda}(M) = n(\tilde{\lambda}(M(K)) - \tilde{\lambda}(M)).$$

In our proof, we will obtain the variation $\tilde{\lambda}(M(K)) - \tilde{\lambda}(M)$ as it is stated, directly, so that our proof shows that

$$\lambda' = \sum_{(i,r) \in \underline{g(\Sigma)}^2} \left(lk(z_{2i}^+, z_{2r}) lk(z_{2i-1}^+, z_{2r-1}) - lk(z_{2i}^+, z_{2r-1}) lk(z_{2i-1}^+, z_{2r}) \right)$$

is a knot invariant. In Lemma 7.18, we will identify λ' with $\frac{1}{2}\Delta_K''(1)$ where Δ_K denotes the Alexander polynomial of K so that the surgery formula of Theorem 7.1 coincides with the Casson surgery formula of [Mar88, Thm. 1.1 (v)], [AM90, p. (xii)] or [GM92, Thm. 1.5]. Since any integral homology sphere can be obtained from S^3 by a finite sequence of surgeries with coefficients ± 1 (see [Mar88, Section 4] or [GM92, Lemme 2.1], for example), it follows that $\tilde{\lambda}$ coincides with the Casson invariant for integral homology spheres.

Our proof will also yield the following theorem. Recall that the *Euler class* of a nowhere zero vector field of a 3-manifold M is the Euler class of its orthogonal plane bundle in M.

Theorem 7.2. *Let F be a genus $g(F)$ oriented compact surface with connected boundary embedded in an oriented compact 3-manifold M whose boundary ∂M is either empty or identified* *with $\partial B(1)$. Let $[-2,2]\times F$ be a neighborhood of $\mathring{F}=\{0\}\times\mathring{F}$ in M, and let X be a nowhere zero vector field of M whose Euler class is a torsion element of $H^2(M;\mathbb{Z})$, that is tangent to $[-2,2]\times\{x\}$ at any point (u,x) of $[-2,2]\times F$, and that is constant on $\partial B(1)$ when $\partial M=\partial B(1)$. Let K be a parallel of ∂F inside F, and let $([-2,2]\times F)(K)$ be obtained from $[-2,2]\times F$ by $+1$–Dehn surgery along K. Let $\mathfrak{t}_K$ denote the right-handed Dehn twist about K. Then*

$$([-2,2]\times F)(K)=[-2,0]\times F\cup_{\{0\}\times F\overset{\mathfrak{t}_K}{\leftarrow}\{0\}\times F^+}[0,2]\times F^+$$

where F^+ is a copy of F and $(0,x)\in\{0\}\times F^+$ is identified with $(0,\mathfrak{t}_K(x))\in\{0\}\times F$. Define the diffeomorphism

$$\begin{array}{llll}\psi_F\colon & ([-2,2]\times F)(K) & \to & [-2,2]\times F & \\ & (t,x) & \mapsto & (t,x) & \textit{if } (t,x)\in[-2,0]\times F\\ & (t,x) & \mapsto & (t,\mathfrak{t}_K(x)) & \textit{if } (t,x)\in[0,2]\times F^+\end{array}$$

and let Y be a nowhere zero vector field of $M(K)$ that coincides with X outside $]-1,1[\times\mathring{F}$ and that is normal to $\psi_F^{-1}(\{t\}\times F)$ on $\psi_F^{-1}(\{t\}\times F)$ for any $t\in[-2,2]$. Then

$$p_1(Y)-p_1(X)=(4g(F)-1)g(F).$$

7.2 A preliminary lemma on Pontrjagin numbers

Lemma 7.3. *Under the assumptions of Theorem 7.2, the variation $(p_1(Y)-p_1(X))$ does not depend on M, K and F, it only depends on $g(F)$. It will be denoted by $p_1(g(F))$.*

PROOF: Let $\tau_F\colon F\times\mathbb{R}^2\to TF$ be a parallelization of F such that the parallelization $X\oplus\tau_F$ of $[-2,2]\times F$ extends to a trivialization τ of M –that is standard on ∂M if $\partial M=S^2$–. (Since M is parallelizable and since $\pi_1(SO(3))$ is generated by a loop of rotations with arbitrary fixed axis, there exists a parallelization of M that has this prescribed form on $[-2,2]\times F$.) Observe that the degree of the tangent map to K is $(1-2g)$ with respect to τ_F. (This degree does not depend on τ_F and can be computed in Figure 25.) Let $K\times[-1,1]$ be a tubular neighborhood of K in F such that $K\times\{-1\}=\partial F$. Then $[-1,1]\times K\times[-1,1]$ is a neighborhood $N_\square(K)$ of K in M that has a standard parallelization $\tau_\nu=(X,TK,\nu)$ where TK stands for the unit tangent vector to K and ν is tangent to $\{(h,x)\}\times[-1,1]$. We assume that

$$\tau_\nu^{-1}\tau((t,k=\exp(2i\pi\theta),u),v\in\mathbb{R}^3)=((t,k,u),\rho_{(2g-1)\theta}(v))$$

where $\rho_{(2g-1)\theta}$ is the rotation whose axis is directed by the first basis vector e_1 of $\mathbb{R}^3$ with angle $(2g-1)\theta$.

Let $\hat{K}$ be the image of K (that is fixed by $\mathfrak{t}_K$) in $M(K)$. The neighborhood $N_\square(\hat{K})=\psi_F^{-1}(N_\square(K))$ of $\hat{K}$ in $([-2,2]\times F)(K)$ is also equipped with a standard parallelization $\hat{\tau}_\nu=(Y,T\hat{K},\hat{\nu})=\psi_{F*}^{-1}\circ\tau_\nu$.

Define the parallelization τ' of $M(K)$ that coincides with τ outside $]-1,1[\times\mathring{F}$ and that is the following stabilization of the positive normal Y to F on $[-1,1]\times F$. Let $\check{F}=F\setminus(K\times[-1,1[)$. On $[-1,1]\times\check{F}$,

$$\tau'(t,x,v\in\mathbb{R}^3)=\tau(t,x,\rho_{(1-2g)\pi(h+1)}(v)).$$

This parallelization extends to $N_\square(\hat{K})$ as a stabilization of Y because it extends to a square bounded by the following square meridian μ_K of $\hat{K}$

$$\mu_K=\{-1\}\times\{k\}\times[-1,1]+([-1,1]\times(k,1))-\{1\}\times\mathfrak{t}_K^{-1}(\{k\}\times[-1,1])-([-1,1]\times(k,-1))$$

written with respect to coordinates of $\partial N_\square(K)$.

Write a (round) tubular neighborhood $N(K)$ in $N_\square(K)$ as $S^1\times D^2=\partial D^2\times D^2$ so that μ_K induces the same parallelization of K as the longitude $(\{x\}\times D^2)$. Let

$$W_F=\big(([0,1]\times[-1,1]\times F)\cup_{\{1\}\times N(K)\sim\partial D^2\times D^2}D^2\times D^2\big)\sharp(-\mathbb{C}P^2)$$

be a cobordism from $[-1,1]\times F$ to $([-1,1]\times F)(K)$ obtained from $[0,1]\times[-1,1]\times F$ by gluing a 2-handle $D^2\times D^2$ along $N(K)$ using the identification of $N(K)$ with $\partial D^2\times D^2$ above, by smoothing in a standard way, and by next performing a connected sum with a copy of $(-\mathbb{C}P^2)$ in the interior of the 2-handle. We compute $(p_1(\tau')-p_1(\tau))$ by using the cobordism W_F completed to a signature 0 cobordism by the product $[0,1]\times(M\setminus\mathrm{Int}([-1,1]\times F))$ where $T[0,1]\oplus\tau$ extends both τ and τ'. Since $\pi_1(SU(2))$ is trivial, the induced complex parallelization over $\partial([0,1]\times[-1,1])\times\check{F}$ extends as a stabilization of $T[0,1]\oplus X$ whose restriction to $[0,1]\times[-1,1]\times\partial\check{F}$ only depends on the genus of F. Thus $(p_1(\tau')-p_1(\tau))$ is the obstruction to extending this extension to $\big([0,1]\times N_\square(K)\cup_{\{1\}\times N(K)\sim\partial D^2\times D^2}D^2\times D^2\big)\sharp(-\mathbb{C}P^2)$ and it only depends on $g(F)$. Call it $p_1(g(F))$.

Now compose τ and τ' by a small rotation whose axis is the second basis vector e_2 of $\mathbb{R}^3$ around $[-2,2]\times F$, so that $X\neq\pm\tau(e_1)$ on $[-1,1]\times F$, and X and $\tau(e_1)$ are transverse. Then $L_{X=\tau(e_1)}=L_{Y=\tau'(e_1)}$, $L_{X=-\tau(e_1)}=L_{Y=-\tau'(e_1)}$. Furthermore, since $L_{X=\tau(e_1)}$ does not meet $[-1,1]\times F$, and since it is rationally null-homologous –because the Euler class of X is a torsion element of $H^2(M;\mathbb{Z})$ (see [Les13, Theorem 1.1] for details)–, $L_{X=\tau(e_1)}$ bounds a Seifert surface disjoint from $N_\square(K)$ and $L_{Y=\tau'(e_1)}$ bounds the same Seifert surface in $M(K)\setminus N_\square(\hat{K})$ so that

$$lk(L_{X=\tau(e_1)},L_{X=-\tau(e_1)})=lk(L_{Y=\tau'(e_1)},L_{Y=-\tau'(e_1)})$$

and

$$p_1(X)-p_1(\tau)=p_1(Y)-p_1(\tau')$$

according to Theorem 3.1, if $H_1(M;\mathbb{Q})=0$, and according to [Les13, Theorem 1.2], more generally. $\square$

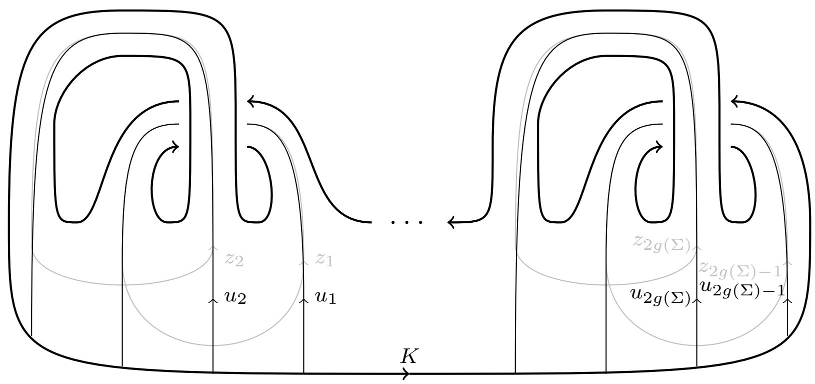

Figure 26: The curves u_i on the surface Σ

7.3 Introduction to the proof of the surgery formula

Let us now begin our proof of Theorem 7.1 by fixing the Heegaard diagrams that we are going to use.

Let u_i be non-intersecting curves of Σ as in Figure 26 with boundaries in $\partial\Sigma$ such that u_i is homologous to z_i in $H_1(\Sigma, \partial\Sigma)$. Then the $u_i \times [-1,1]$ form a system of (topological) meridian disks for the handlebody $H_{\mathcal{A}}$. Set $\alpha_i = -\partial(u_i \times [-1,1])$. Fix a system of meridians $(\beta_j)_{j\in\underline{g}}$ that meet the α curves transversally and that meet $K \times [-1,1]$ as a product by $[-1,1]$. Set $\Sigma^+ = \Sigma \times \{1\}$ and $\Sigma^- = \Sigma \times \{-1\}$. Assume that the Heegaard diagram $\mathcal{D} = ((\alpha_i)_{i\in\underline{g}}, (\beta_j)_{j\in\underline{g}})$ has a matching $\mathfrak{m} = \{m_i\}_{i\in\underline{g}}$ where $m_i \in \alpha_i \cap \beta_i$ and $m_i \in \Sigma^-$ (up to isotopies of the curves β). The invariant $\tilde{\lambda}(M)$ will be computed with the diagram $\mathcal{D}$, and the invariant $\tilde{\lambda}(M(K))$ will be computed with the diagram

$$\mathcal{D}' = ((\alpha_i)_{i\in\underline{g}}, (\beta'_j = \mathfrak{t}_K(\beta_j))_{j\in\underline{g}}).$$

We fix a common exterior point w for $\mathcal{D}$ and $\mathcal{D}'$ in Σ^-.

Lemma 7.4. *The variation* $(p_1(X(\mathcal{D}', w, \mathfrak{m})) - p_1(X(\mathcal{D}, w, \mathfrak{m})))$ *is equal to the number* $p_1(g(\Sigma))$ *defined in Lemma 7.3.*

PROOF: Apply Lemma 7.3 to

$$F = \Sigma^+ \cup_{K\times\{1\}} (K \times [-1,1]) \subset \partial H_{\mathcal{A}},$$

$X = X(\mathcal{D}, w, \mathfrak{m})$ and $Y = X(\mathcal{D}', w, \mathfrak{m})$. □

Let u_i also denote $u_i \times \{1\} = \alpha_i \cap \Sigma^+$.

Assume that along K, from some basepoint of K, we first meet all the intersection points of K with the β_j and next the intersection points of K with the α_i, that correspond to the endpoints of the u_i, as in Figure 27.

Recall $\lambda' = \sum_{(i,r)\in\underline{g(\Sigma)}^2} \left(lk(z^+_{2i}, z_{2r})lk(z^+_{2i-1}, z_{2r-1}) - lk(z^+_{2i}, z_{2r-1})lk(z^+_{2i-1}, z_{2r})\right)$.

We are going to prove the following lemmas.

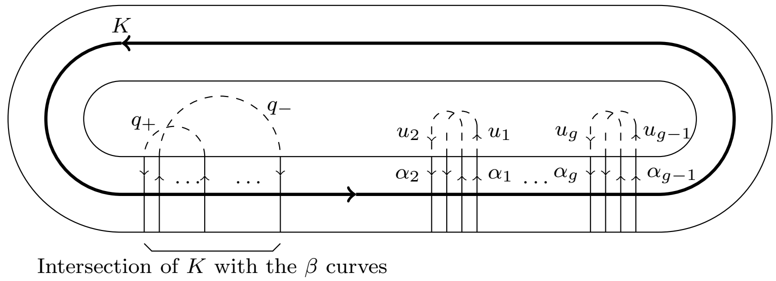

Figure 27: The intersections of K with the curves of $\mathcal{D}$

Lemma 7.5.

$$\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) = 8\lambda'.$$

Lemma 7.6.

$$s_\ell(\mathcal{D}', \mathfrak{m}) - s_\ell(\mathcal{D}, \mathfrak{m}) = -g(\Sigma)^2 - 2\lambda'.$$

Lemma 7.7.

$$e(\mathcal{D}', w, \mathfrak{m}) - e(\mathcal{D}, w, \mathfrak{m}) = (1-g)g(\Sigma).$$

It follows from these lemmas that

$$24\tilde{\lambda}(M(K)) - 24\tilde{\lambda}(M) = 24\lambda' + 4g(\Sigma)(g(\Sigma)-1) - p_1(g(\Sigma))$$

Applying this formula to a trivial knot U seen as the boundary of a genus $g(\Sigma)$ surface Σ_U for which $\lambda' = 0$ shows that

$$p_1(g(\Sigma)) = 4g(\Sigma)(g(\Sigma)-1)$$

since $M(U)$ is diffeomorphic to M.

Thus Lemmas 7.5, 7.6 and 7.7 imply Theorems 7.1 and 7.2, and we are left with their proofs that occupy most of the end of this section.

7.4 Preliminaries for the proofs of the remaining three lemmas

Set $\overline{2r} = 2r-1$ and $\overline{2r-1} = 2r$.

Lemma 7.8. *For any* $(i, r) \in \underline{g}^2$,

$$\sum_{j=1}^{g} \mathcal{J}_{ji}\langle u_r, \beta_j\rangle = \langle z_i, z_{\bar{i}}\rangle lk(z_r^+, z_{\bar{i}}).$$

PROOF: Think of $H_{\mathcal{A}}$ as a thickening of a wedge of the z_i. Let $m(z_i)$ denote a meridian of z_i on $\partial H_{\mathcal{A}}$. Then $z_r^+ = \sum_{k=1}^{g} lk(z_r^+, z_k)m(z_k)$ in $H_1(H_{\mathcal{B}};\mathbb{Q})$. Since $m(z_k)$ is homologous to $\langle z_{\overline{k}}, z_k\rangle(z_{\overline{k}}^+ - z_{\overline{k}}^-)$ in $\partial H_{\mathcal{A}}$,

$$\begin{aligned}\langle m(z_k), \beta_j\rangle &= \langle z_{\overline{k}}, z_k\rangle\langle u_{\overline{k}} - u_{\overline{k}}^-, \beta_j\rangle\\ &= \langle z_{\overline{k}}, z_k\rangle\langle \alpha_{\overline{k}}, \beta_j\rangle\end{aligned}$$

$\langle u_r, \beta_j\rangle = \langle z_r^+, \beta_j\rangle = \sum_{k=1}^{g} lk(z_r^+, z_k)\langle z_{\overline{k}}, z_k\rangle\langle \alpha_{\overline{k}}, \beta_j\rangle$, and $\sum_{j=1}^{g} \mathcal{J}_{ji}\langle u_r, \beta_j\rangle = \langle z_i, z_{\overline{i}}\rangle lk(z_r^+, z_{\overline{i}})$. □

Lemma 7.9.

$$\sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji}\langle u_i, \beta_j\rangle = g(\Sigma).$$

PROOF: $\sum_{i=1}^{g}\langle z_i, z_{\overline{i}}\rangle lk(z_i^+, z_{\overline{i}}) = \sum_{r=1}^{g(\Sigma)}(lk(z_{2r-1}^+, z_{2r}) - lk(z_{2r}^+, z_{2r-1}))$. □

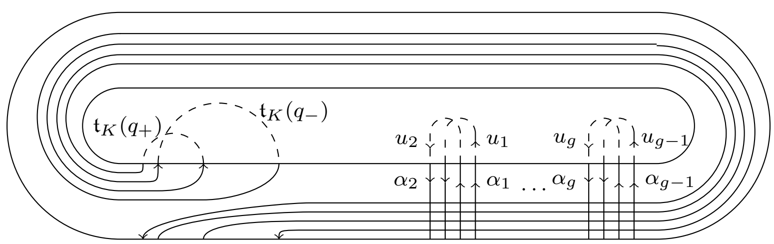

Figure 28: The diagram $\mathcal{D}'$ in a neighborhood of K on $\partial H_{\mathcal{A}}$

For $j \in \underline{g}$, let Q_j denote the set of connected components of $\beta_j' \cap (\Sigma^+ \cup (\partial\Sigma \times [-1,1])$. Let $Q = \cup_{j=1}^{g} Q_j$. For an arc q of Q_j, set $j(q) = j$. The intersection of an arc q of Q with $\Sigma^+ \times \{1\}$ will be denoted by q^+. Let $\mathcal{C}$ and $\mathcal{C}'$ denote the set of crossings of $\mathcal{D}$ and $\mathcal{D}'$, respectively.

For each $(q,i) \in Q \times \underline{g}$, there is a set $\mathcal{C}(q,i) = \alpha_i \cap q$ of 4 crossings. Then

$$\mathcal{C}' = \mathcal{C} \coprod \coprod_{(q,i)\in Q\times \underline{g}} \mathcal{C}(q,i).$$

Denote $\mathcal{C}(q,i) = \{d_1(q,i), d_2(q,i), d_3(q,i), d_4(q,i)\}$ where following α_i from m_i, $d_1(q,i)$, $d_2(q,i)$, $d_3(q,i)$ and $d_4(q,i)$ are met in this order. Set $\sigma(q,i) = \sigma(d_2(q,i))$. Then

$$\sigma(q,i) = \sigma(d_2(q,i)) = \sigma(d_4(q,i)) = -\sigma(d_1(q,i)) = -\sigma(d_3(q,i)).$$

Let $t(q,i)$ denote the (tail) arc of q before q^+ with its ends in $\mathcal{C}(q,i)$ and let $h(q,i)$ denote the (head) arc of q after q^+ with its ends in $\mathcal{C}(q,i)$.

If $\sigma(q,i) = -1$, then q goes from left to right as q_- in Figures 27 and 28, following q we meet $\mathcal{C}(q,i)$ in the order d_3, d_2, d_1, d_4, $t(q,i) = |d_3, d_2|_\beta$ and $h(q,i) = |d_1, d_4|_\beta$.

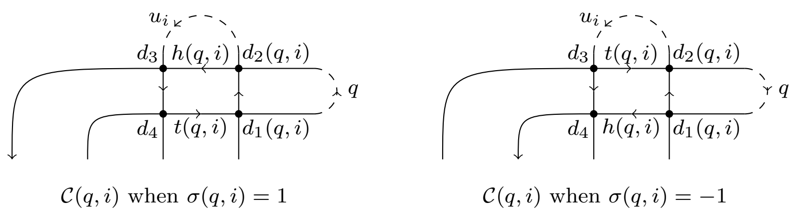

Figure 29: The new crossings of $\mathcal{D}'$

If $\sigma(q,i)=1$, then q goes from right to left as q_+ in Figures 27 and 28, following q we meet $\mathcal{C}(q,i)$ in the reversed order d_4, d_1, d_2, d_3, $t(q,i)=|d_4,d_1|_\beta$ and $h(q,i)=|d_2,d_3|_\beta$. Thus $t(q,i)$ begins at $d_{b(q,i)}(q,i)$ where $b(q,i)=3$ if $\sigma(q,i)=-1$ and $b(q,i)=4$ if $\sigma(q,i)=1$.

Note that for any $(i,j)\in \underline{g}^2$, $\langle\alpha_i,\beta_j\rangle=\langle\alpha_i,\beta'_j\rangle$ so that the coefficients $\mathcal{J}_{ji}$ are the same for $\mathcal{D}$ and $\mathcal{D}'$.

The set of crossings of $\mathcal{D}$ on Σ^+ (resp. on Σ^-) will be denoted by $\mathcal{C}^+$ (resp. by $\mathcal{C}^-$).

Proof of Lemma 7.7: On the rectangle $R_{\mathcal{D}}$ of Figure 2 for $(\mathcal{D},w,\mathfrak{m})$, let p'_i (resp. p''_i) denote the other end of the diameter of α'_i (resp. α''_i) that contains the crossing m_i of $\mathfrak{m}$. Draw the knot K on a picture of the Heegaard diagram as in Figure 2 so that K meets the curves α' and α'' as the β_j do, away from the points of $\mathfrak{m}$, with horizontal tangent vectors near the p'_i and the p''_i. Let $N(\mathfrak{m})$ denote an open tubular neighborhood of $\mathfrak{m}$ in $\partial H_{\mathcal{A}}$ made of $2g(\Sigma)$ open disks. See $\partial H_{\mathcal{A}}\setminus \mathring{N}(\mathfrak{m})$ as obtained from the rectangle $R_{\mathcal{D}}$ with holes bounded by the α'_i and the α''_i, by gluing horizontal thin rectangles D_i along their two vertical small sides that are neighborhoods of p'_i or p''_i in α'_i or α''_i. The standard parallelization of this picture equips $\partial H_{\mathcal{A}}\setminus \mathring{N}(\mathfrak{m})$ with a parallelization so that the degree $d_e(K)$ of the tangent to K is $1-2g(\Sigma)$ in this figure. A similar picture for $(\mathcal{D}',w,\mathfrak{m})$ is obtained by performing the Dehn twist about K on the β-curves in this figure. Since these curves do not intersect K algebraically, the $d_e(\beta_j)$ are unchanged by this operation. Similarly, for any crossing c of $\mathcal{C}^-$, $d_e(|m_{j(c)},c|_\beta)$ is unchanged and so is $d_e(c)$. For any crossing c of $\mathcal{C}^+$, we have $d'_e(c)=d_e(c)+1-2g(\Sigma)$ since $\langle K,|m_{j(c)},c|_\beta\rangle=1$. Now, let $(q,i)\in Q\times\underline{g}$. The contribution of $\mathcal{C}(q,i)$ to $(e(\mathcal{D}',w,\mathfrak{m})-e(\mathcal{D},w,\mathfrak{m}))$ is

$$\pm\mathcal{J}_{j(q)i}\left(d_e(h(q,i))+d_e(t(q,i))-\sum_{(r,s)\in\underline{g}^2}\mathcal{J}_{sr}\langle\alpha_r,h(q,i)+t(q,i)\rangle d_e(\beta_s)\right)$$

that is zero. Finally,

$$\begin{aligned}e(\mathcal{D}',w,\mathfrak{m})-e(\mathcal{D},w,\mathfrak{m}) &= (1-g)\textstyle\sum_{c\in\mathcal{C}^+}\mathcal{J}_{j(c)i(c)}\sigma(c)\\ &= (1-g)\textstyle\sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\langle u_i,\beta_j\rangle\\ &= (1-g)g(\Sigma)\end{aligned}$$

according to Lemma 7.9. $\square$

7.5 Study of ℓ

Let ℓ and ℓ' be the maps of Proposition 2.7 associated with $\mathcal{D}$ and $\mathcal{D}'$, respectively, with respect to the basepoints m_i of Σ^-.

$$\ell(c,d) = \langle [m_{i(c)}, c|_\alpha, [m_{j(d)}, d|_\beta \rangle - \sum_{(i,j)\in \underline{g}^2} \mathcal{J}_{ji} \langle [m_{i(c)}, c|_\alpha, \beta_j \rangle \langle \alpha_i, [m_{j(d)}, d|_\beta \rangle$$

Lemma 7.10. *Let* $(c,d) \in \mathcal{C}^2$. *If* $(c,d) \in (\mathcal{C}^+)^2$, *then*

$$\ell'(c,d) = \ell(c,d) - 1$$

Otherwise,

$$\ell'(c,d) = \ell(c,d)$$

PROOF: Recall that the m_i are in Σ^-. Note that $\mathfrak{t}_K([m_{j(d)}, d|_\beta)$ is obtained from $[m_{j(d)}, d|_\beta$ by adding some multiple of K located in $K \times [-1,1]$, algebraically, so that

$$\langle \alpha_i, \mathfrak{t}_K([m_{j(d)}, d|_\beta)\rangle = \langle \alpha_i, [m_{j(d)}, d|_\beta \rangle$$

for any $i \in \underline{g}$. Since $\mathfrak{t}_K(\beta_j)$ differs from β_j by an algebraically null sum of copies of K in $K \times [-1,1]$,

$$\langle [m_{i(c)}, c|_\alpha, \beta'_j \rangle = \langle [m_{i(c)}, c|_\alpha, \beta_j \rangle$$

for any $j \in \underline{g}$. Thus in any case,

$$\ell'(c,d) - \ell(c,d) = \langle [m_{i(c)}, c|_\alpha, \mathfrak{t}_K([m_{j(d)}, d|_\beta) - [m_{j(d)}, d|_\beta \rangle.$$

If $d \in \Sigma^-$, $\mathfrak{t}_K([m_{j(d)}, d|_\beta)$ differs from $[m_{j(d)}, d|_\beta$ by an algebraically null sum of copies of K in $K \times [-1,1]$ so that $\ell'(c,d) = \ell(c,d)$. If $d \in \Sigma^+$, $\ell'(c,d) - \ell(c,d) = \langle [m_{i(c)}, c|_\alpha, K \rangle$. If $c \in \Sigma^-$, the arc $[m_{i(c)}, c|_\alpha$ meets $K \times [-1,1]$ as the empty set or as two parallel arcs with opposite direction and $\ell'(c,d) = \ell(c,d)$. If $c \in \Sigma^+$, then the arc $[m_{i(c)}, c|_\alpha$ meets $K \times [-1,1]$ as an arc that crosses K once with a negative sign. □

Lemma 7.11. *Let* $c \in \mathcal{C}$ *and let* $(q,i) \in Q \times \underline{g}$.

$$\sum_{d \in \mathcal{C}(q,i)} \sigma(d) \ell'(c,d) = \sum_{d \in \mathcal{C}(q,i)} \sigma(d) \ell'(d,c) = 0.$$

PROOF: For any interval I of a β'-curve,

$$\sum_{d \in \mathcal{C}(q,i)} \sigma(d) \langle [m_i, d|_\alpha, I \rangle = \sigma(q,i) \langle |d_1(q,i), d_2(q,i)|_\alpha + |d_3(q,i), d_4(q,i)|_\alpha, I \rangle$$

that is zero if I has no end points in $K \times [-1,1]$. This shows that $\sum_{d \in \mathcal{C}(q,i)} \sigma(d)\ell'(d,c) = 0$. For any interval I of an α-curve,

$$\sum_{d \in \mathcal{C}(q,i)} \sigma(d)\langle I, [m_{j(q)}, d|_\beta, \rangle = -\langle I, t(q,i) + h(q,i)\rangle.$$

Again, this is zero if I has no end points in $K \times [-1,1]$. $\square$

Lemma 7.12. *Let (q,i) and (q',r) belong to $Q \times \underline{g}$. If $q \neq q'$ and $i \neq r$, then*

$$\sum_{(c,d) \in \mathcal{C}(q,i) \times \mathcal{C}(q',r)} \sigma(c)\sigma(d)\ell'(c,d) = -lk_{K \times \{1\}}(\partial q^+, \partial q'^+) lk_{K \times \{1\}}(\partial u_i, \partial u_r).$$

If $q = q'$ or $i = r$, then $\sum_{(c,d) \in \mathcal{C}(q,i) \times \mathcal{C}(q',r)} \sigma(c)\sigma(d)\ell'(c,d) = 0$.

Proof: Set $A = \sum_{(c,d) \in \mathcal{C}(q,i) \times \mathcal{C}(q',r)} \sigma(c)\sigma(d)\ell'(c,d)$. As in the proof of Lemma 7.11,

$$\begin{aligned} A &= -\textstyle\sum_{c \in \mathcal{C}(q,i)} \sigma(c)\langle [m_i, c|_\alpha, t(q',r) + h(q',r)\rangle \\ &= -\sigma(q,i)\langle |d_1(q,i), d_2(q,i)|_\alpha + |d_3(q,i), d_4(q,i)|_\alpha, t(q',r) + h(q',r)\rangle \end{aligned}$$

This is zero unless $q \neq q'$, $i \neq r$ and $lk(\partial q, \partial q') lk(\partial u_i, \partial u_r) \neq 0$. When the sign of q' changes, so does the result. Furthermore, the result is symmetric when (q,i) and (q',r) are exchanged, thanks to the symmetry of the linking number (see Proposition 2.6).

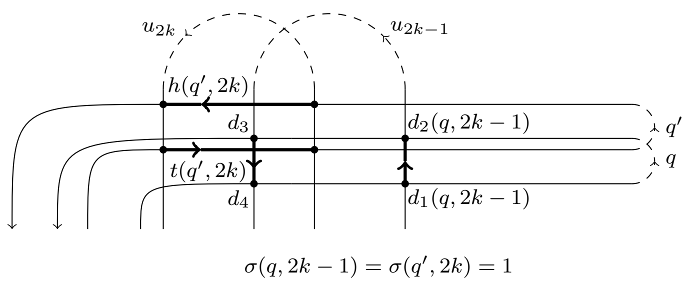

Figure 30: Computation of $lk(\partial q, \partial q') lk(\partial u_{2k-1}, \partial u_{2k})$

Therefore, it suffices to prove the lemma when $\sigma(q,i) = \sigma(q',r) = 1$ and $(i,r) = (2k-1, 2k)$. When we have the order $h(q',r)h(q,i)t(q',r)t(q,i)$ on K that coincides with $h(u_r)h(u_i)t(u_r)t(u_i)$, we get $A = -1$ as in Figure 30. For the order $h(q,i)h(q',r)t(q,i)t(q',r)$, we get $A = 1$. $\square$

Lemma 7.13. *When $i \neq r$, $lk_{K \times \{1\}}(\partial u_i, \partial u_r) = -\langle z_i, z_r\rangle$.*
When $q \neq q'$,

$$lk_{K \times \{1\}}(\partial q^+, \partial q'^+) = -\sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle \langle u_k, q\rangle \langle u_{\overline{k}}, q'\rangle$$

and, for any $q \in Q$, $\sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle \langle u_k, q\rangle \langle u_{\overline{k}}, q'\rangle = 0$.

PROOF: Let $\gamma(q')$ be a curve on $K \times \{1\}$ that does not meet the α-curves, such that $\partial\gamma(q') = \partial q'^{+}$. Then $lk_{K\times\{1\}}(\partial q^+, \partial q'^+) = \langle \partial q^+, \gamma(q')\rangle_K$. Since q and q' do not intersect, this also reads $lk_{K\times\{1\}}(\partial q^+, \partial q'^+) = -\langle q, q'^+ - \gamma(q')\rangle_{\partial H_{\mathcal{A}}}$ where $(q'^+ - \gamma(q'))$ is a closed curve of Σ^+ whose homology class reads

$$(q'^+ - \gamma(q')) = \sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle \langle q'^+ - \gamma(q'), z_{\overline{k}}\rangle_{\Sigma^+} z_k = \sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle \langle q', u_{\overline{k}}\rangle_{\partial H_{\mathcal{A}}} z_k.$$

□

Lemma 7.14.

$$\sum_{(q,q')\in Q_j\times Q_s} \sum_{(c,d)\in\mathcal{C}(q,i)\times\mathcal{C}(q',r)} \sigma(c)\sigma(d)\ell'(c,d) = -\langle z_i, z_r\rangle \sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle \langle u_k, \beta_j\rangle \langle u_{\overline{k}}, \beta_s\rangle.$$

PROOF: According to Lemmas 7.12 and 7.13,

$$\sum_{(c,d)\in\mathcal{C}(q,i)\times\mathcal{C}(q',r)} \sigma(c)\sigma(d)\ell'(c,d) = -\langle z_i, z_r\rangle \sum_{k=1}^{g} \langle z_k, z_{\overline{k}}\rangle \langle u_k, q\rangle \langle u_{\overline{k}}, q'\rangle.$$

□

Lemma 7.15.

$$\begin{aligned}\textstyle\sum_{c\in\mathcal{C}'\setminus\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\ell'(c,c) \quad = \quad & \textstyle\sum_{(i,j,r,s)\in\underline{g}^4} \mathcal{J}_{ji}\mathcal{J}_{sr}\langle u_i, \beta_s\rangle\langle u_r, \beta_j\rangle - g(\Sigma) \\ & \textstyle -\sum_{(i,j,k,s)\in\underline{g}^4} \mathcal{J}_{ji}\mathcal{J}_{s\overline{i}}\langle z_i, z_{\overline{i}}\rangle\langle z_k, z_{\overline{k}}\rangle\langle u_k, \beta_j\rangle\langle u_{\overline{k}}, \beta_s\rangle\end{aligned}$$

PROOF: Let us fix $(q,i) \in Q \times \underline{g}$ and compute $\sum_{c\in\mathcal{C}(q,i)} \sigma(c)\ell'(c,c)$.

Since the arc $[m_i, d_1(q,i)[_\alpha$ does not intersect the arcs $[d_{b(q,i)}(q,i), c]_\beta$,

$$\begin{aligned}\textstyle\sum_{c\in\mathcal{C}(q,i)} \sigma(c)\langle [m_i, c|_\alpha, [m_{j(q)}, c|_\beta\rangle \quad = \quad & \textstyle\sum_{c\in\mathcal{C}(q,i)} \sigma(c)\langle [d_1(q,i), c|_\alpha, [d_{b(q,i)}(q,i), c|_\beta\rangle \\ & \textstyle +\sum_{c\in\mathcal{C}(q,i)} \sigma(c)\langle [d_1(q,i), c|_\alpha, [m_{j(q)}, d_{b(q,i)}(q,i)[_\beta\rangle\end{aligned}$$

where $\sum_{c\in\mathcal{C}(q,i)} \sigma(c)\langle [d_1(q,i), c|_\alpha, [m_{j(q)}, d_{b(q,i)}(q,i)[_\beta\rangle$ equals

$$\sigma(q,i)\langle |d_1(q,i), d_2(q,i)|_\alpha + |d_3(q,i), d_4(q,i)|_\alpha, [m_{j(q)}, d_{b(q,i)}(q,i)[_\beta\rangle = 0$$

since the arc $[m_{j(q)}, d_{b(q,i)}(q,i)[_\beta$ intersects the arcs $[d_1(q,i), c]_\alpha$ as whole arcs q' of $Q_{j(c)}$.

Thus

$$\sum_{c\in\mathcal{C}(q,i)} \sigma(c)\langle [m_i, c|_\alpha, [m_{j(q)}, c|_\beta\rangle = 1 + \sum_{c\in\mathcal{C}(q,i)} \sigma(c)\langle [d_1(q,i), c[_\alpha, [d_{b(q,i)}(q,i), c[_\beta\rangle$$

Note that neither $d_1(q,i)$ nor $d_{b(q,i)}$ contributes to the new sum.

$$\langle [d_1(q,i), d_2(q,i)[_\alpha, [d_{b(q,i)}(q,i), d_2(q,i)[_\beta\rangle = \begin{cases} \sigma(d_1(q,i)) = -1 & \text{if } \sigma(q,i)=1 \\ 0 & \text{if } \sigma(q,i)=-1. \end{cases}$$

If $\sigma(q,i)=1$, we are left with the computation of

$$\langle [d_1(q,i), d_3(q,i)[_\alpha, [d_{b(q,i)}(q,i), d_3(q,i)[_\beta\rangle = \langle u_i, q\rangle.$$

If $\sigma(q,i)=-1$, we are left with the computation of

$$\langle [d_1(q,i), d_4(q,i)[_\alpha, [d_{b(q,i)}(q,i), d_4(q,i)[_\beta\rangle = \langle u_i, q\rangle + \sigma(d_3(q,i)).$$

In any case,

$$\sum_{c\in\mathcal{C}(q,i)} \sigma(c)\langle [m_i, c|_\alpha, [m_{j(q)}, c|_\beta\rangle = -\langle u_i, q\rangle.$$

Let us fix $(r,s)\in \underline{g}^2$ and compute $A = \sum_{c\in\mathcal{C}(q,i)} \sigma(c)\langle [m_i, c|_\alpha, \beta'_s\rangle\langle \alpha_r, [m_{j(q)}, c|_\beta\rangle$. Observe

$$\langle [m_i, d_4(q,i)|_\alpha, \beta'_s\rangle = \langle [m_i, d_1(q,i)|_\alpha, \beta'_s\rangle + \langle u_i, \beta_s\rangle$$

and

$$\langle [m_i, d_3(q,i)|_\alpha, \beta'_s\rangle = \langle [m_i, d_2(q,i)|_\alpha, \beta'_s\rangle + \langle u_i, \beta_s\rangle.$$

Let $B = \sigma(q,i)\langle u_i,\beta_s\rangle \left(\langle \alpha_r, [m_{j(q)}, d_4(q,i)|_\beta\rangle - \langle \alpha_r, [m_{j(q)}, d_3(q,i)|_\beta\rangle\right) = -\langle u_i,\beta_s\rangle\langle \alpha_r, q^+\rangle$.

$$\begin{aligned} A - B &= \sigma(q,i)\langle [m_i, d_1(q,i)|_\alpha, \beta'_s\rangle \left(\langle \alpha_r, [m_{j(q)}, d_4(q,i)|_\beta\rangle - \langle \alpha_r, [m_{j(q)}, d_1(q,i)|_\beta\rangle\right) \\ &\quad +\sigma(q,i)\langle [m_i, d_2(q,i)|_\alpha, \beta'_s\rangle \left(\langle \alpha_r, [m_{j(q)}, d_2(q,i)|_\beta\rangle - \langle \alpha_r, [m_{j(q)}, d_3(q,i)|_\beta\rangle\right) \\ &= \sigma(q,i)\left(\langle [m_i, d_1(q,i)|_\alpha, \beta'_s\rangle - \langle [m_i, d_2(q,i)|_\alpha, \beta'_s\rangle\right)\langle \alpha_r, h(q,i)\rangle \\ &= -\sigma(q,i)\langle |d_1(q,i), d_2(q,i)|_\alpha, \beta'_s\rangle\langle \alpha_r, h(q,i)\rangle \end{aligned}$$

where $\langle \alpha_r, h(q,i)\rangle = lk_{K\times\{1\}}(\partial u_r, \partial u_i)$ when $r\neq i$, so that $\langle \alpha_r, h(q,i)\rangle = \langle z_i, z_r\rangle$ in any case. Summarizing, we get

$$\begin{aligned} \textstyle\sum_{c\in\mathcal{C}(q,i)} \sigma(c)\ell'(c,c) = \; & -\langle u_i, q\rangle \\ & +\textstyle\sum_{(r,s)\in\underline{g}^2} \mathcal{J}_{sr}\left(\langle u_i,\beta_s\rangle\langle u_r, q\rangle + \sigma(q,i)\langle |d_1(q,i), d_2(q,i)|_\alpha, \beta_s\rangle\langle z_i, z_r\rangle\right). \end{aligned}$$

where

$$\begin{aligned} \sigma(q,i)\langle |d_1(q,i), d_2(q,i)|_\alpha, \beta_s\rangle &= -\textstyle\sum_{q'\in Q_s; q'\neq q} lk_{K\times\{1\}}(\partial q', \partial q) \\ &= \textstyle\sum_{q'\in Q_s; q'\neq q} lk_{K\times\{1\}}(\partial q, \partial q') \\ &= -\textstyle\sum_{k=1}^{g}\langle z_k, z_{\overline{k}}\rangle\langle u_k, q\rangle\langle u_{\overline{k}}, \beta_s\rangle \end{aligned}$$

according to Lemma 7.13.

Now, let us fix $j \in \underline{g}$ and compute

$$\begin{aligned}
\sum_{q\in Q_j}\sum_{c\in\mathcal{C}(q,i)}\sigma(c)\ell'(c,c) &= -\langle u_i,\beta_j\rangle + \sum_{(r,s)\in\underline{g}^2}\mathcal{J}_{sr}\langle u_i,\beta_s\rangle\langle u_r,\beta_j\rangle \\
&\quad -\sum_{(r,s)\in\underline{g}^2}\mathcal{J}_{sr}\langle z_i,z_r\rangle\sum_{k=1}^{g}\langle z_k,z_{\overline{k}}\rangle\langle u_k,\beta_j\rangle\langle u_{\overline{k}},\beta_s\rangle \\
&= -\langle u_i,\beta_j\rangle + \sum_{(r,s)\in\underline{g}^2}\mathcal{J}_{sr}\langle u_i,\beta_s\rangle\langle u_r,\beta_j\rangle \\
&\quad -\sum_{(k,s)\in\underline{g}^2}\mathcal{J}_{s\bar{i}}\langle z_i,z_{\bar{i}}\rangle\langle z_k,z_{\overline{k}}\rangle\langle u_k,\beta_j\rangle\langle u_{\overline{k}},\beta_s\rangle
\end{aligned}$$

$$\begin{aligned}
\sum_{c\in\mathcal{C}'\setminus\mathcal{C}}\mathcal{J}_{j(c)i(c)}\sigma(c)\ell'(c,c) &= \sum_{(i,j)\in\underline{g}^2}\mathcal{J}_{ji}\left(\sum_{(r,s)\in\underline{g}^2}\mathcal{J}_{sr}\langle u_i,\beta_s\rangle\langle u_r,\beta_j\rangle - \langle u_i,\beta_j\rangle\right) \\
&\quad -\sum_{(i,j,k,s)\in\underline{g}^4}\mathcal{J}_{ji}\mathcal{J}_{s\bar{i}}\langle z_i,z_{\bar{i}}\rangle\langle z_k,z_{\overline{k}}\rangle\langle u_k,\beta_j\rangle\langle u_{\overline{k}},\beta_s\rangle.
\end{aligned}$$

Conclude with Lemma 7.9. □

7.6 Proofs of the remaining two lemmas

Lemma 7.16.

$$\begin{aligned}
2\lambda' &= \sum_{(i,r)\in\underline{g}^2} lk(z_r^+,z_i)lk(z_{\overline{r}}^+,z_{\bar{i}})\langle z_i,z_{\bar{i}}\rangle\langle z_r,z_{\overline{r}}\rangle \\
&= \sum_{(i,j,k,s)\in\underline{g}^4}\mathcal{J}_{ji}\mathcal{J}_{s\bar{i}}\langle z_i,z_{\bar{i}}\rangle\langle z_k,z_{\overline{k}}\rangle\langle u_k,\beta_j\rangle\langle u_{\overline{k}},\beta_s\rangle
\end{aligned}$$

Proof: Let C be the expression of the second line. Computing C with Lemma 7.8 yields

$$C = \sum_{(i,k)\in\underline{g}^2} lk(z_k^+,z_{\bar{i}})\langle z_i,z_{\bar{i}}\rangle^2 lk(z_{\overline{k}}^+,z_i)\langle z_{\bar{i}},z_i\rangle\langle z_k,z_{\overline{k}}\rangle.$$

□

Proof of Lemma 7.6: Use Lemmas 7.10, 7.11, 7.14, 7.9 and 7.16 above to compute

$$\begin{aligned}
s_\ell(\mathcal{D}',\mathfrak{m}) - s_\ell(\mathcal{D},\mathfrak{m}) &= -\sum_{(c,d)\in(\mathcal{C}^+)^2}\mathcal{J}_{j(c)i(c)}\sigma(c)\mathcal{J}_{j(d)i(d)}\sigma(d) \\
&\quad -\sum_{(i,j,k,s)\in\underline{g}^4}\mathcal{J}_{ji}\mathcal{J}_{s\bar{i}}\langle z_i,z_{\bar{i}}\rangle\langle z_k,z_{\overline{k}}\rangle\langle u_k,\beta_j\rangle\langle u_{\overline{k}},\beta_s\rangle \\
&= -\sum_{(i,j,r,s)\in\underline{g}^4}\mathcal{J}_{ji}\langle u_i,\beta_j\rangle\mathcal{J}_{sr}\langle u_r,\beta_s\rangle - 2\lambda' \\
&= -g(\Sigma)^2 - 2\lambda'.
\end{aligned}$$

□

Lemma 7.17. *Set*

$$\lambda'_+(\Sigma) = \sum_{(i,r)\in\underline{g}^2} lk(z_r^+,z_i)lk(z_{\bar{i}}^+,z_{\overline{r}})\langle z_i,z_{\bar{i}}\rangle\langle z_r,z_{\overline{r}}\rangle.$$

Then

$$\lambda'_+(\Sigma) = -\sum_{(i,j,r,s)\in\underline{g}^4}\mathcal{J}_{ji}\mathcal{J}_{sr}\langle u_i,\beta_s\rangle\langle u_r,\beta_j\rangle = 2\lambda' - g(\Sigma).$$

PROOF: Using Lemma 7.8, we get

$$\begin{aligned}
\sum_{(i,j,r,s)\in\underline{g}^4}(-\mathcal{J}_{ji}\mathcal{J}_{sr}\langle u_i,\beta_s\rangle\langle u_r,\beta_j\rangle) &= -\sum_{(i,r)\in\underline{g}^2} lk(z_r^+,z_{\overline{i}})\langle z_i,z_{\overline{i}}\rangle lk(z_i^+,z_{\overline{r}})\langle z_r,z_{\overline{r}}\rangle \\
&= \sum_{(i,r)\in\underline{g}^2} lk(z_r^+,z_i)lk(z_{\overline{i}}^+,z_{\overline{r}})\langle z_i,z_{\overline{i}}\rangle\langle z_r,z_{\overline{r}}\rangle \\
&= \sum_{(i,r)\in\underline{g}^2} lk(z_i^+,z_r)lk(z_{\overline{i}}^+,z_{\overline{r}})\langle z_i,z_{\overline{i}}\rangle\langle z_r,z_{\overline{r}}\rangle \\
&\quad -\sum_{(i,r)\in\underline{g}^2}\langle z_i,z_r\rangle lk(z_{\overline{i}}^+,z_{\overline{r}})\langle z_i,z_{\overline{i}}\rangle\langle z_r,z_{\overline{r}}\rangle \\
&= 2\lambda' + \sum_{i\in\underline{g}} lk(z_{\overline{i}}^+,z_i)\langle z_i,z_{\overline{i}}\rangle = 2\lambda' - g(\Sigma).
\end{aligned}$$

□

PROOF OF LEMMA 7.5: According to Lemmas 7.15, 7.16 and 7.17,

$$\sum_{c\in\mathcal{C}'\setminus\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\ell'(c,c) = -\lambda'_+(\Sigma) - g(\Sigma) - 2\lambda' = -4\lambda'.$$

Therefore

$$\sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\ell(c,c) - \sum_{c\in\mathcal{C}'} \mathcal{J}_{j(c)i(c)}\sigma(c)\ell'(c,c) = 4\lambda' + \sum_{(i,j)\in\underline{g}^2} \mathcal{J}_{ji}\langle u_i,\beta_j\rangle = 4\lambda' + g(\Sigma)$$

according to Lemmas 7.10 and 7.9. Using Lemmas 7.10, 7.11 and 7.14 again, we get

$$\begin{aligned}
\ell_2(\mathcal{D}') - \ell_2(\mathcal{D}) &= -\sum_{(i,j,k,s)\in\underline{g}^4} \mathcal{J}_{j\overline{i}}\mathcal{J}_{si}\langle z_i,z_{\overline{i}}\rangle\langle z_k,z_{\overline{k}}\rangle\langle u_k,\beta_j\rangle\langle u_{\overline{k}},\beta_s\rangle \\
&\quad -\sum_{(i,j,k,s)\in\underline{g}^4} \mathcal{J}_{jr}\mathcal{J}_{si}\langle u_i,\beta_j\rangle\langle u_r,\beta_s\rangle \\
&\quad +4\lambda' + g(\Sigma) \\
&= 2\lambda' + \lambda'_+(\Sigma) + 4\lambda' + g(\Sigma) = 8\lambda'
\end{aligned}$$

thanks to Lemma 7.17. □

Finally, we identify λ' to $\frac{1}{2}\Delta''_K(1)$ where

$$\Delta_K(t) = t^{-g(\Sigma)}\det\left(\left[tlk(z_r^+,z_s) - lk(z_s^+,z_r)\right]_{(r,s)\in\underline{g}^2}\right)$$

denotes the Alexander polynomial of K.

Lemma 7.18.

$$\frac{1}{2}\Delta''_K(1) = \lambda'.$$

PROOF: Note $tlk(z_r^+,z_s) - lk(z_s^+,z_r) = (t-1)lk(z_r^+,z_s) + \langle z_r,z_s\rangle$.

$$\Delta_K(t) = t^{-g(\Sigma)} + t^{-g(\Sigma)}(t-1)\sum_{i\in\underline{g}} lk(z_i^+,z_{\overline{i}})\langle z_i,z_{\overline{i}}\rangle + t^{-g(\Sigma)}(t-1)^2 A + O(t-1)^3$$

where $\sum_{i\in\underline{g}} lk(z_i^+, z_{\overline{i}})\langle z_i, z_{\overline{i}}\rangle = g(\Sigma)$ (see Lemma 7.9) and

$$\begin{aligned} A &= \textstyle\sum_{\{i,r\}\subset \underline{g}}\langle z_i, z_{\overline{i}}\rangle\langle z_r, z_{\overline{r}}\rangle \left(lk(z_i^+, z_{\overline{i}})lk(z_r^+, z_{\overline{r}}) - lk(z_i^+, z_{\overline{r}})lk(z_r^+, z_{\overline{i}})\right)\\ &= \tfrac{1}{2}\textstyle\sum_{(i,r)\in\underline{g}^2}\langle z_i, z_{\overline{i}}\rangle\langle z_r, z_{\overline{r}}\rangle \left(lk(z_i^+, z_{\overline{i}})lk(z_r^+, z_{\overline{r}}) - lk(z_i^+, z_{\overline{r}})lk(z_r^+, z_{\overline{i}})\right)\\ &= \tfrac{g(\Sigma)^2}{2} + \tfrac{1}{2}\textstyle\sum_{(i,r)\in\underline{g}^2} lk(z_r^+, z_i)lk(z_{\overline{i}}^+, z_{\overline{r}})\langle z_i, z_{\overline{i}}\rangle\langle z_r, z_{\overline{r}}\rangle\\ &= \tfrac{1}{2}\left(g(\Sigma)^2 + \lambda'_+(\Sigma)\right)\end{aligned}$$

thanks to Lemma 7.17.
$\Delta'_K(t) = -g(\Sigma)t^{-g(\Sigma)-1} + g(\Sigma)(t^{-g(\Sigma)} - g(\Sigma)t^{-g(\Sigma)-1}(t-1)) + 2t^{-g(\Sigma)}(t-1)A + O(t-1)^2.$

$$\Delta''_K(1) = g(\Sigma)(g(\Sigma) + 1 - 2g(\Sigma)) + 2A = g(\Sigma) + \lambda'_+(\Sigma) = 2\lambda'$$

according to Lemma 7.17. □

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