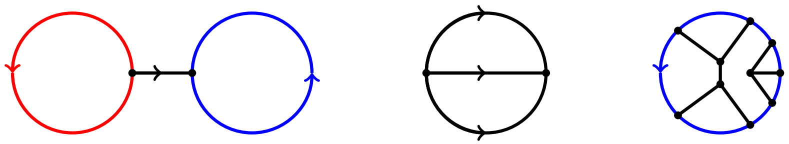
Counting graph configurations in 3-manifolds (II)

Christine Lescop

CNRS, Institut Fourier, Grenoble



International Conference “Quantum Topology”

Lake Bannoye Magnitogorsk

July 15th, 2014

$S^3 = \mathbb{R}^3 \cup \{\infty\}$. B_0 unit ball of $\mathbb{R}^3$. $B_\infty = \overline{S^3 \setminus B_0}$.
M closed 3–manifold. $H_*(M;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$
Choose $i_\infty \colon B_\infty \hookrightarrow M$ and let $\infty = i_\infty(\infty)$. $\check{M} = M \setminus \{\infty\}$.
$\tau \colon \check{M} \times \mathbb{R}^3 \to T\check{M}$ parallelization standard near ∞.

$C_2(M)$ is a compact 6-dimensional with corners

- the interior of $C_2(M)$ is $\check{M}^2 \setminus$ diag
- $S(TM) \subset \partial C_2(M)$
- $\exists$ canonical $q_\tau \colon \partial C_2(M) \to S^2$ that restricts to the fiber of $S(TM)$ as the isomorphism induced by τ
- $H_*(C_2(M);\mathbb{Q}) = H_*(S^2;\mathbb{Q})$
- $H_2(C_2(M);\mathbb{Q}) = \mathbb{Q}[\text{fiber of } S(TM)]$

A propagator of $(C_2(M),\tau)$ is a 4-chain $\mathcal{P}$ of $C_2(M)$ whose boundary is $q_\tau^{-1}(a)$ for some $a \in S^2$.

A propagator of $(C_2(M),\tau)$ is a 4-chain $\mathcal{P}$ of $C_2(M)$ whose boundary is $q_\tau^{-1}(a)$ for some $a \in S^2$.

Theorem (Kontsevich, math.GT/0411431)

Let M be a rational homology sphere.
$B_\infty \subset M$, τ parallelization of $\check{M} = M \setminus \infty$ standard on $B_\infty \setminus \infty$.
Let $\mathcal{P}_a$, $\mathcal{P}_b$ and $\mathcal{P}_c$ be propagators of $(C_2(M),\tau)$ associated to pairwise distinct a, b and c of S^2. Then the algebraic intersection

$$\Theta(M,\tau) = \langle \mathcal{P}_a, \mathcal{P}_b, \mathcal{P}_c \rangle_{C_2(M)}$$

of $\mathcal{P}_a$, $\mathcal{P}_b$ and $\mathcal{P}_c$ in $C_2(M)$ is a topological invariant of (M,τ).

Theorem (G. Kuperberg, D. Thurston 1999, math.GT/0411431)

Let M be a rational homology sphere

$$\Theta(M,\tau) = 6\lambda_{\text{Casson-Walker}}(M) + \frac{1}{4}p_1(\tau)$$

where $p_1(\tau)$ is a relative Pontrjagin class.

Let X be a section of $S(T\check{M})$ that is constant with value $a \in S^2$ on $B_\infty \setminus \infty$. A propagator of $(C_2(M),X)$ is a 4-chain of $C_2(M)$ whose boundary is $q^{-1}_{\tau|\partial C_2(M)\setminus S(T\check{M})}(a) \cup X(\check{M})$.

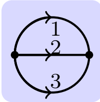
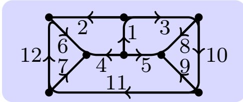
Theorem (arXiv:1209.2785)

Let M be a rational homology sphere.
Let $\mathcal{P}_X$ and $\mathcal{P}_{-X}$ be two transverse propagators associated to X and $-X$.

$$[\mathcal{P}_X \cap \mathcal{P}_{-X}] = \Theta(M,X)[\text{fiber of } S(TM)] \text{ in } H_2(C_2(M);\mathbb{Q})$$

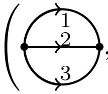
$\Theta(M,X)$ is a homotopical invariant of (M,X) such that

$$\Theta(M,Y) - \Theta(M,X) = lk(L_{X=Y}, L_{X=-Y}).$$

$n \in \mathbb{N}$. Fix $3n$ transverse propagators $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{3n}$ of $(C_2(M), \tau)$.
Let Γ be a trivalent graph with $3n$ oriented edges numbered from 1 to $3n$ without like  or  .

$\check{C}_{V(\Gamma)}(M)$ set of injections from the set $V(\Gamma)$ of vertices to $\check{M}$.
$p_i \colon \check{C}_{V(\Gamma)}(M) \to C_2(M)$ restriction to the ends of Edge $\sharp i$.

$$I(\Gamma, (\mathcal{P}_i)_{i=1,\ldots,3n}) = \langle p_1^{-1}(\mathcal{P}_1), \ldots, p_{3n}^{-1}(\mathcal{P}_{3n})\rangle_{\check{C}_{V(\Gamma)}(M)}.$$

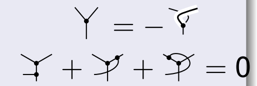
Example: $I\Big($  $, (\mathcal{P}_i)_{i=1,2,3}\Big) = \langle \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\rangle_{C_2(M)} = \Theta(M, \tau)$.

Γ trivalent graph with $3n$ oriented numbered edges, no .
$\check{C}_{V(\Gamma)}(M)$ set of injections from the set $V(\Gamma)$ of vertices to $\check{M}$.
$p_i \colon \check{C}_{V(\Gamma)}(M) \to C_2(M)$ restriction to the ends of Edge $\sharp i$.

$$I(\Gamma, (\mathcal{P}_i)_{i=1,\ldots,3n}) = \langle p_1^{-1}(\mathcal{P}_1), \ldots, p_{3n}^{-1}(\mathcal{P}_{3n})\rangle_{\check{C}_{V(\Gamma)}(M)}.$$

Theorem (Witten, Kontsevich, arXiv:1312.2566)

$$Z_n(M,\tau) = \sum_\Gamma \frac{1}{2^{3n}(3n)!} I(\Gamma, (\mathcal{P}_i)_{i=1,\ldots,3n})[\Gamma] \in \mathbb{Q}<\Gamma>$$



does not depend on the choice of the $\mathcal{P}_i$.

Theorem (Kuperberg, Thurston 1999)

For a constant ξ called anomaly,

$$Z(M) = ((Z_n(M,\tau))_{n\in\mathbb{N}})\exp(-p_1(\tau)\xi)$$

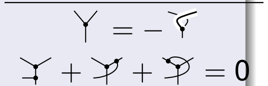
is an invariant of M equivalent to the Le-Murakami-Ohtsuki invariant Z_{LMO} for integer homology spheres.

Γ trivalent graph with $3n$ oriented numbered edges, no .
$\check{C}_{V(\Gamma)}(M)$ set of injections from the set $V(\Gamma)$ of vertices to $\check{M}$.
$p_i \colon \check{C}_{V(\Gamma)}(M) \to C_2(M)$ restriction to the ends of Edge $\sharp i$.

$$I(\Gamma, (\mathcal{P}_i)_{i=1,\ldots,3n}) = \langle p_1^{-1}(\mathcal{P}_1), \ldots, p_{3n}^{-1}(\mathcal{P}_{3n})\rangle_{\check{C}_{V(\Gamma)}(M)}.$$

Theorem (Kontsevich, arXiv:1312.2566)

$$Z_n(M,\tau) = \sum_\Gamma \frac{1}{2^{3n}(3n)!} I(\Gamma, (\mathcal{P}_i)_{i=1,\ldots,3n})[\Gamma] \in \mathbb{Q}<\Gamma>$$



does not depend on the choice of the $\mathcal{P}_i$.

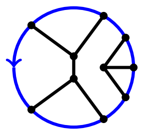
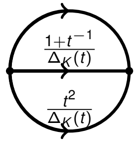
Theorem (Kuperberg, Thurston 1999, math.GT/0411431, Massuyeau 2013, Moussard 2012)

$$Z(M) = ((Z_n(M,\tau))_{n\in\mathbb{N}})\exp(-p_1(\tau)\xi)$$

is equivalent to Z_{LMO} for rational homology spheres.

A few generalizations

- Variants of these theorems exist for framed links [arXiv:1312.2566] or tangles in rational homology spheres.

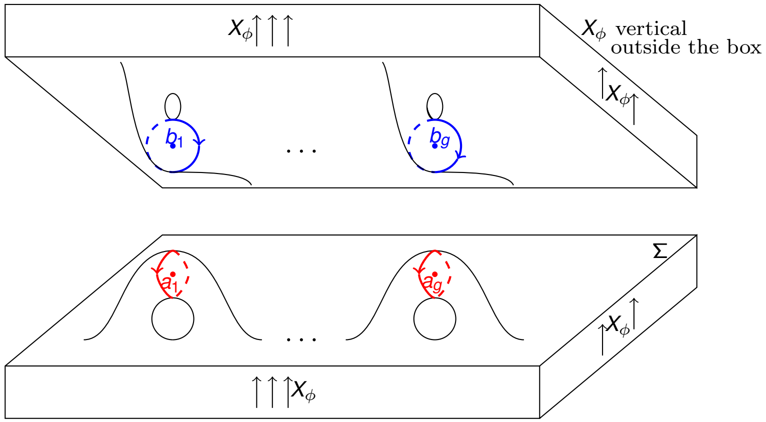
 

- Configurations of graphs in the exterior of a null-homologous knot K may also be counted with equivariant propagators Poincaré dual to the Blanchfield pairing in an equivariant configuration space [arXiv:1008.5026].
  This provides a knot invariant that shares Garoufalidis-Rozansky universality properties with the Kricker lift of the Kontsevich integral [arXiv:1306.1705] and that determines Z(branched covers along the knot).

Propagator associated to a Morse function

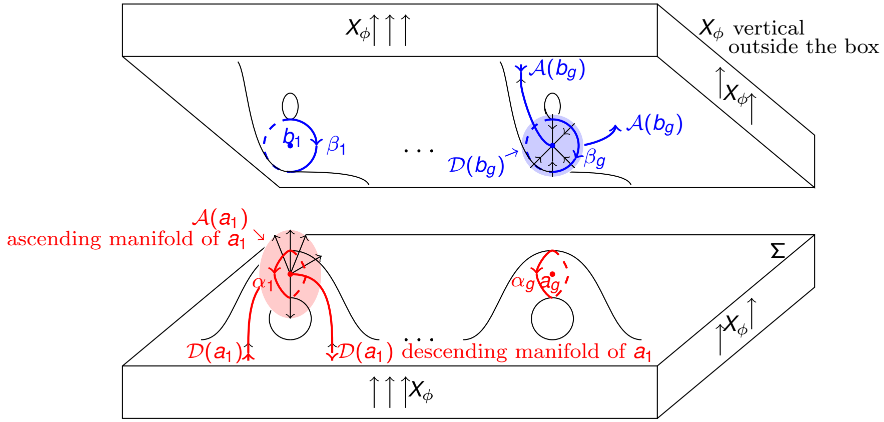
ϕ gradient flow associated to a Morse function of $\check{M}$ with
g index one critical points a_i "under"
g index two critical points b_j, and no other critical points.

$$P_\phi = \overline{\{(x, \phi_t(x)); x \in \check{M}, t \in]0, +\infty[\}} \subset C_2(M).$$



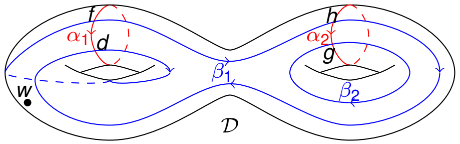
Propagator associated to a Morse function

$$P_\phi = \overline{\{(x, \phi_t(x)); x \in \check{M}, t \in]0, +\infty[\}} \subset C_2(M).$$
$$\partial P_\phi \cap \mathrm{Int}(C_2(M)) = -\sum_{i=1}^{g} \mathcal{D}(a_i) \times \mathcal{A}(a_i) - \sum_{j=1}^{g} \mathcal{D}(b_j) \times \mathcal{A}(b_j)$$



$$P_\phi = \overline{\{(x, \phi_t(x)); x \in \check{M}, t \in]0, +\infty[\}} \subset C_2(M).$$
$$\partial P_\phi \cap \mathrm{Int}(C_2(M)) = -\sum_{i=1}^{g} \mathcal{D}(a_i) \times \mathcal{A}(a_i) - \sum_{j=1}^{g} \mathcal{D}(b_j) \times \mathcal{A}(b_j)$$

In $\Sigma \cup$ (disk around w),



Heegaard diagram

$\mathcal{C} = \{crossings\}$.

$c \in \mathcal{C}$. If $c = {}^{\beta}\!\nearrow\!\!\!\nwarrow^{\alpha}$, then $\sigma(c) = 1$. If $c = {}^{\alpha}\!\nwarrow\!\!\!\nearrow^{\beta}$, $\sigma(c) = -1$.

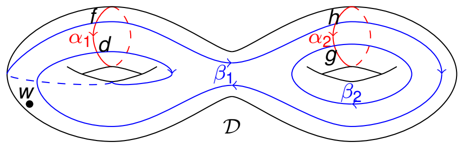
$\langle \alpha_i, \beta_j \rangle_\Sigma = \sum_{c \in \alpha_i \cap \beta_j} \sigma(c)$, $[(\mathcal{J}_{ji})_{i,j}] = [(\langle \alpha_i, \beta_j \rangle_\Sigma)_{i,j}]^{-1}$

Theorem (with G. Kuperberg, arXiv:1209.3219)

$\left(\mathcal{P} = P_\phi + \sum_{i,j} \mathcal{J}_{ji} \mathcal{D}(b_j) \times \mathcal{A}(a_i)\right)$ is a propagator of $C_2(M)$.
(i.e. $\partial \mathcal{P} \subset \partial C_2(M)$ and $\mathcal{P}$ Poincaré dual to [fiber of $S(TM)$])

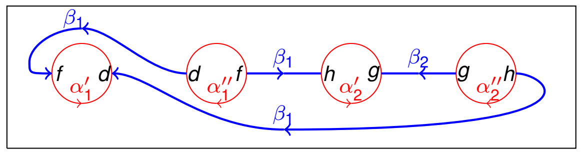
Related work by Fukaya 1996, Watanabe 2012, Shimizu 2013.

A combinatorial formula for Θ



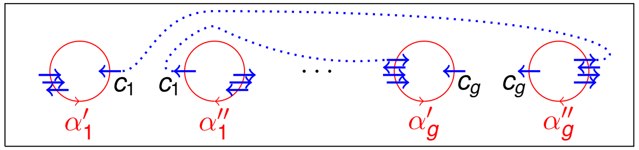
Heegaard diagram

Choose $c_i \in \alpha_i \cap \beta_i$ for any $i = 1, \ldots, g$ (up to renumbering the β_i). $\mathfrak{m} = \{c_1, \ldots, c_g\}$.



Example: $\mathfrak{m} = \{d, g\}$.

Choose $c_i \in \alpha_i \cap \beta_i$ for any $i = 1, \ldots, g$. $\mathfrak{m} = \{c_1, \ldots, c_g\}$.



For $c \in \mathcal{C}$, $c \in \alpha_{i(c)} \cap \beta_{j(c)}$.
$\tilde{\delta}(c) = \delta(|c_{j(c)}, c|_\beta) - \sum_{(r,s)\in\{1,\ldots,g\}^2} \mathcal{J}_{sr}\langle \alpha_r, |c_{j(c)}, c|_\beta\rangle \delta(\beta_s)$,
where δ is the degree of the tangent map in $\frac{1}{2}\mathbb{Z}$.

$$e(\mathcal{D}, w, \mathfrak{m}) = \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\tilde{\delta}(c).$$

$$\ell(c,d) = \langle |c_{i(c)}, c|_\alpha, |d_{j(d)}, d|_\beta\rangle - \sum_{(i,j)\in\{1,\ldots,g\}^2} \mathcal{J}_{ji}\langle |c_{i(c)}, c|_\alpha, \beta_j\rangle\langle \alpha_i, |d_{j(d)}, d|_\beta\rangle$$

$$\ell(\mathcal{D}, \mathfrak{m}) = \sum_{(c,d)\in\mathcal{C}^2} \left(\mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)} + \mathcal{J}_{j(c)i(c)}\mathcal{J}_{j(d)i(d)}\right)\sigma(c)\sigma(d)\ell(c,d) - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\ell(c,c)$$

A combinatorial formula for Θ

Choose $c_i \in \alpha_i \cap \beta_i$ for any $i = 1, \ldots, g$. $\mathfrak{m} = \{c_1, \ldots, c_g\}$.

Theorem (arXiv:1209.3219, arXiv:1402.2261)

$$\Theta(M, X_{\phi,w,\mathfrak{m}}) = \ell(\mathcal{D}, \mathfrak{m}) - e(\mathcal{D}, w, \mathfrak{m})$$

For $c \in \mathcal{C}$, $c \in \alpha_{i(c)} \cap \beta_{j(c)}$.
$\tilde{\delta}(c) = \delta(|c_{j(c)}, c|_\beta) - \sum_{(r,s)\in\{1,\ldots,g\}^2} \mathcal{J}_{sr}\langle \alpha_r, |c_{j(c)}, c|_\beta\rangle \delta(\beta_s)$,
where δ is the degree of the tangent map in $\frac{1}{2}\mathbb{Z}$.

$$e(\mathcal{D}, w, \mathfrak{m}) = \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\tilde{\delta}(c).$$

$$\ell(c,d) = \langle |c_{i(c)}, c|_\alpha, |d_{j(d)}, d|_\beta\rangle - \sum_{(i,j)\in\{1,\ldots,g\}^2} \mathcal{J}_{ji}\langle |c_{i(c)}, c|_\alpha, \beta_j\rangle\langle \alpha_i, |d_{j(d)}, d|_\beta\rangle$$

$$\ell(\mathcal{D}, \mathfrak{m}) = \sum_{(c,d)\in\mathcal{C}^2} \left(\mathcal{J}_{j(c)i(d)}\mathcal{J}_{j(d)i(c)} + \mathcal{J}_{j(c)i(c)}\mathcal{J}_{j(d)i(d)}\right)\sigma(c)\sigma(d)\ell(c,d) - \sum_{c\in\mathcal{C}} \mathcal{J}_{j(c)i(c)}\sigma(c)\ell(c,c)$$