A propagator of $(C_2(M), \tau)$ is a 4-chain $P$ of $C_2(M)$ whose boundary is $q_\tau^{-1}(a)$ for some $a \in S^2$.

**Theorem (Kontsevich, math.GT/0411431)**

Let $M$ be a rational homology sphere. $B_\infty \subset M$. $\tau$ parallelization of $M = M \setminus \infty$ standard on $B_\infty \setminus \infty$. Let $P_a$, $P_b$ and $P_c$ be propagators of $(C_2(M), \tau)$ associated to pairwise distinct $a$, $b$ and $c$ of $S^2$. Then the algebraic intersection

$$\Theta(M, \tau) = \langle P_a, P_b, P_c \rangle_{C_2(M)}$$

of $P_a$, $P_b$ and $P_c$ in $C_2(M)$ is a topological invariant of $(M, \tau)$.

**Theorem (G. Kuperberg, D. Thurston 1999, math.GT/0411431)**

Let $M$ be a rational homology sphere

$$\Theta(M, \tau) = 6\lambda_{\text{Casson-Walker}}(M) + \frac{1}{4}p_1(\tau)$$

where $p_1(\tau)$ is a relative Pontryagin class.

$S^3 = \mathbb{R}^3 \cup \{\infty\}$. $B_0$ unit ball of $\mathbb{R}^3$. $B_\infty = \overline{S^3 \setminus B_0}$. $M$ closed 3-manifold, $H_*(M; \mathbb{Q}) = H_*(S^3; \mathbb{Q})$. Choose $i_\infty : B_\infty \to M$ and let $\infty = i_\infty(\infty)$. $\tilde{M} = M \setminus \{\infty\}$. $\tau : \tilde{M} \times \mathbb{R}^3 \to TM$ parallelization standard near $\infty$.

$C_2(M)$ is a compact 6-dimensional with corners

- the interior of $C_2(M)$ is $M^2 \setminus \text{diag}$
- $S(TM) \subset \partial C_2(M)$
- $\exists$ canonical $q_\tau : \partial C_2(M) \to S^2$ that restricts to the fiber of $S(TM)$ as the isomorphism induced by $\tau$
- $H_*(C_2(M); \mathbb{Q}) = H_*(S^2; \mathbb{Q})$
- $H_2(C_2(M); \mathbb{Q}) = \mathbb{Q}[\text{fiber of } S(TM)]$

A propagator of $(C_2(M), \tau)$ is a 4-chain $P$ of $C_2(M)$ whose boundary is $q_\tau^{-1}(a)$ for some $a \in S^2$.

Let $X$ be a section of $S(T\tilde{M})$ that is constant with value $a \in S^2$ on $B_\infty \setminus \infty$. A propagator of $(C_2(M), X)$ is a 4-chain of $C_2(M)$ whose boundary is $q_\tau^{-1}(a) \cup X(\tilde{M})$.

**Theorem (arXiv:1209.2785)**

Let $M$ be a rational homology sphere. Let $P_X$ and $P_{-X}$ be two transverse propagators associated to $X$ and $-X$.

$$[P_X \cap P_{-X}] = \Theta(M, X)[\text{fiber of } S(TM)] \text{ in } H_2(C_2(M); \mathbb{Q})$$

$\Theta(M, X)$ is a homotopical invariant of $(M, X)$ such that

$$\Theta(M, Y) - \Theta(M, X) = lk(L_{X=0}, L_{X=0}).$$
Theorem (Kontsevich, arXiv:1312.2566)

\[
Z_n(M, \tau) = \sum_{\Gamma} \frac{1}{2^{3n}(3n)!} l(\Gamma, (P_i)_{i=1,...,3n})[\Gamma] \in \mathbb{Q} \setminus \mathbb{Z}
\]
\[
\begin{array}{c}
\mathcal{V} > 0 \\
\mathcal{V} + \mathcal{V} + \mathcal{V} = 0
\end{array}
\]
does not depend on the choice of the \( P_i \).

Theorem (Kuperberg, Thurston 1999, math.GT/0411431, Massuyeau 2013, Moussard 2012)

\[
Z(M) = (Z_n(M, \tau))_{n \in \mathbb{N}} \exp(-p_1(\tau)\xi)
\]
is equivalent to \( \hat{Z}_{LMO} \) for rational homology spheres.

A few generalizations

- Variants of these theorems exist for framed links [arXiv:1312.2566] or tangles in rational homology spheres.

- Configurations of graphs in the exterior of a null-homologous knot \( K \) may also be counted with equivariant propagators Poincaré dual to the Blanchfield pairing in an equivariant configuration space [arXiv:1008.5026].

This provides a knot invariant that shares Garoufalidis-Rozansky universality properties with the Krickert lift of the Kontsevich integral [arXiv:1306.1705] and that determines \( \mathcal{Z} \) (branched covers along the knot).
Propagator associated to a Morse function

\(\phi\) gradient flow associated to a Morse function of \(M\) with

\(g\) index one critical points \(a_i\) “under”

\(g\) index two critical points \(b_i\), and no other critical points.

\[ P_\phi = \{(x, \phi(t(x))); x \in M, t \in [0,+\infty] \} \subset C_2(M). \]

\(x_i\) \(x_o\) vertical outside the box

\(\Sigma\)

\[ \partial (\Sigma) = (\alpha, \beta) \]

\[ \partial \phi \cap \text{Int}(C_2(M)) = - \sum_{j=1}^g D(a_j) \times A(a_j) - \sum_{j=1}^g D(b_j) \times A(b_j) \]

In \(\Sigma \cup (\text{disk around } w)\),

Heegaard diagram

\[ C = \{\text{crossings}\}. \]

Choose \(c_i \in \alpha_i \cap \beta_i\) for any \(i = 1, \ldots, g\) (up to renumbering the \(\beta_i\)) \(m = \{c_1, \ldots, c_g\}\).

Example: \(m = \{d, g\}\).

Counting graphs in 3–manifolds

A combinatorial formula for \(\Theta\)

\[ \Theta = P_\phi + \sum_{i,j} J_{ij} D(b_j) \times A(a_j) \]

A combinatorial formula for $\Theta$

Choose $c_i \in \alpha_i \cap \beta_i$ for any $i = 1, \ldots, g$. $m = \{c_1, \ldots, c_g\}$.

For $c \in C$, $c \in \alpha_i \cap \beta_j(c)$.

\[ \delta(c) = \delta(\{c_i, c_j\}) - \sum_{(r,s) \in \{1, \ldots, g\}^2} \mathcal{J}_r \langle \alpha, c_i, c_j \rangle \delta(\beta)_s, \]

where $\delta$ is the degree of the tangent map in $\frac{1}{2}Z$.

\[ e(D, w, m) = \sum_{c \in C} \mathcal{J}_r \langle c_i, c_j, d_r \rangle \delta(c). \]

\[ \ell(c, d) = \langle c_i, c_j, d_r \rangle - \sum_{(r,s) \in \{1, \ldots, g\}^2} \mathcal{J}_s \langle c_i, c_j, d_r \rangle \delta(\beta)_s. \]

\[ \ell(D, m) = \sum_{c \in C} \langle \mathcal{J}_r \langle c_i, d_r \rangle \mathcal{J}_s \delta(c) + \mathcal{J}_s \delta(c), \sigma(c) \rangle, \]

\[ - \sum_{c \in C} \mathcal{J}_r \langle c_i, \delta(c) \rangle \langle c_r, d_r \rangle = \sum_{c \in C} \mathcal{J}_r \langle c_i, \delta(c) \rangle \langle c_r, d_r \rangle. \]

\[ \Theta(M, X_{\phi, w, m}) = \ell(D, m) - e(D, w, m), \]

For $c \in C$, $c \in \alpha_i \cap \beta_j(c)$.

\[ \delta(c) = \delta(\{c_i, c_j\}) - \sum_{(r,s) \in \{1, \ldots, g\}^2} \mathcal{J}_r \langle \alpha, c_i, c_j \rangle \delta(\beta)_s. \]

where $\delta$ is the degree of the tangent map in $\frac{1}{2}Z$.

\[ e(D, w, m) = \sum_{c \in C} \mathcal{J}_r \langle c_i, c_j, d_r \rangle \delta(c). \]

\[ \ell(c, d) = \langle c_i, c_j, d_r \rangle - \sum_{(r,s) \in \{1, \ldots, g\}^2} \mathcal{J}_s \langle c_i, c_j, d_r \rangle \delta(\beta)_s. \]

\[ \ell(D, m) = \sum_{c \in C} \langle \mathcal{J}_r \langle c_i, d_r \rangle \mathcal{J}_s \delta(c) + \mathcal{J}_s \delta(c), \sigma(c) \rangle, \]

\[ - \sum_{c \in C} \mathcal{J}_r \langle c_i, \delta(c) \rangle \langle c_r, d_r \rangle = \sum_{c \in C} \mathcal{J}_r \langle c_i, \delta(c) \rangle \langle c_r, d_r \rangle. \]