Counting graph configurations in 3-manifolds

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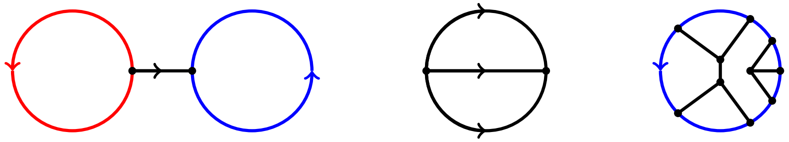

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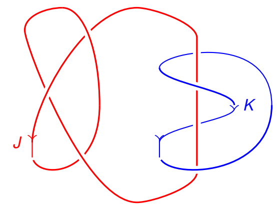
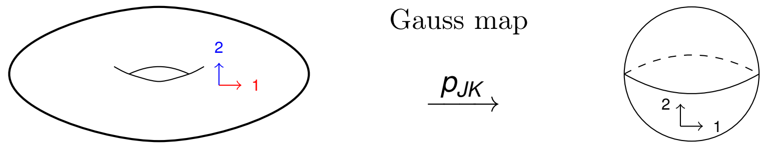
Let $J \coprod K$ be a 2–component link

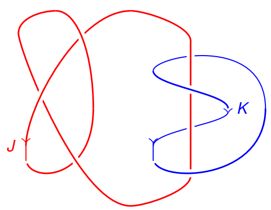


$J \coprod K \colon S^1 \coprod S^1 \hookrightarrow \mathbb{R}^3$

$$p_{JK} \colon \begin{array}{ccc} S^1 \times S^1 & \to & S^2 \\ (w,z) & \mapsto & \frac{1}{\|K(z)-J(w)\|}(K(z)-J(w)) \end{array}$$



The linking number $lk(J,K)$ of J and K is the degree of p_{JK}.



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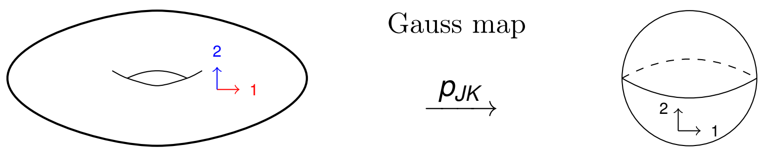


$lk(J,K) = \deg(p_{JK}) = \sharp\ {}^{J}$  ${}^{K} - \sharp\ {}^{K}$  ^{J}

$= \sharp\ {}^{K}$  ${}^{J} - \sharp\ {}^{J}$  ^{K}

$= \frac{1}{2}\Big(\sharp\ {}^{J}$  ${}^{K} - \sharp\ {}^{K}$  ${}^{J} + \sharp\ {}^{K}$  ${}^{J} - \sharp\ {}^{J}$  ${}^{K}\Big)$

$J \coprod K \colon S^1 \coprod S^1 \hookrightarrow \mathbb{R}^3$ induces $J \times K \colon S^1 \times S^1 \hookrightarrow (\mathbb{R}^3)^2 \setminus \mathrm{diag}$

$$q \colon \begin{array}{ccc} (\mathbb{R}^3)^2 \setminus \mathrm{diag} & \to & S^2 \\ (x,y) & \mapsto & \frac{1}{\|y-x\|}(y-x) \end{array}$$

$$p_{JK} = q \circ (J \times K) \colon \begin{array}{ccc} S^1 \times S^1 & \to & S^2 \\ (w,z) & \mapsto & \frac{1}{\|K(z)-J(w)\|}(K(z)-J(w)) \end{array}$$

$$\begin{aligned} lk(J,K) &= \langle J \times K, q^{-1}(a) \rangle_{(\mathbb{R}^3)^2 \setminus \mathrm{diag}} \\ &= \deg_{a \in S^2}(p_{JK} = q \circ (J \times K)) \end{aligned}$$

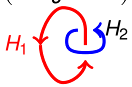
$J \times K = \mathrm{Im}(J \times K)$.

$lk(J,K)$ is the algebraic intersection number of $q^{-1}(a)$ and the torus $J \times K$ in the configuration space $(\mathbb{R}^3)^2 \setminus$ diag.

$J \coprod K \colon S^1 \coprod S^1 \hookrightarrow \mathbb{R}^3$ induces $J \times K \colon S^1 \times S^1 \hookrightarrow (\mathbb{R}^3)^2 \setminus \mathrm{diag}$,
$lk(J,K) = \langle J \times K, q^{-1}(a)\rangle_{(\mathbb{R}^3)^2\setminus\mathrm{diag}}$

$$\begin{array}{llll} q\colon & (\mathbb{R}^3)^2 \setminus \mathrm{diag} & \to & S^2 \\ & (x,y) & \mapsto & \frac{1}{\|y-x\|}(y-x) \end{array}$$

- q homotopy equivalence
- $H_2\left((\mathbb{R}^3)^2 \setminus \mathrm{diag}\right) = \mathbb{Q}[0 \times \partial B_0]$ where B_0 unit ball of $\mathbb{R}^3$
- $[J \times K] = lk(J,K)[0 \times \partial B_0]$ in $H_2\left((\mathbb{R}^3)^2 \setminus \mathrm{diag}\right)$
- If Σ_J is a Seifert surface of J ($\partial \Sigma_J = J$), then



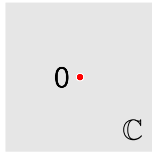
$[J \times K] = \langle \Sigma_J, K\rangle_{\mathbb{R}^3}[H_1 \times H_2]$
- $lk(J,K) = \langle \Sigma_J, K\rangle_{\mathbb{R}^3}$

$$\begin{array}{llll} (\mathbb{R}^3)^2 \setminus \mathrm{diag} & \xrightarrow{\text{homeo}} & \mathbb{R}^3 \times \mathbb{R}^{+*} \times S^2 \\ (x,y) & \mapsto & \left(x, \| y-x \|, q(x,y) = \frac{1}{\|y-x\|}(y-x)\right) \end{array}$$

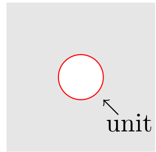
Here, blowing up a submanifold B properly embedded in a manifold A replaces B by its unit normal bundle $S(NB)$ to produce

$$B\ell(A,B) = (A \setminus B) \cup S(NB).$$

(At $x \in B$, $S(NB)_x = (N_xB \setminus \{0\})/\mathbb{R}^{+*}$ with N_xB fiber of normal bundle.)



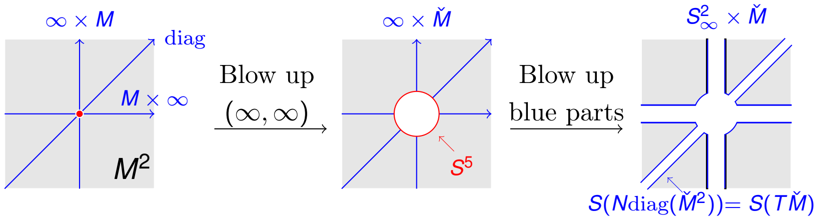
$B\ell(\mathbb{C},0) =$



$\mathbb{C} =]0,+\infty[\times S^1 \cup \{0\}$ $\qquad$ $B\ell(\mathbb{C},0) = [0,+\infty[\times S^1$

- $B\ell(A,B)$ homeomorphic to $A \setminus \mathring{N}(B)$
- $B\ell(A,B)$ homotopy equivalent to $A \setminus B$.
- A compact $\Rightarrow B\ell(A,B)$ compact.
- When $\partial A = \emptyset$, $\partial B\ell(A,B) = -S(NB)$. Here, manifolds are oriented.
- $p_{B\ell(A,B)} \colon B\ell(A,B) \to A$ canonical

$S^3 = \mathbb{R}^3 \cup \{\infty\}$. B_0 unit ball of $\mathbb{R}^3$. $B_\infty = \overline{S^3 \setminus B_0}$.
M closed (oriented) 3–manifold.
Choose $i_\infty \colon B_\infty \hookrightarrow M$ and let $\infty = i_\infty(\infty)$. $\check{M} = M \setminus \{\infty\}$.
$C_1(M) = B\ell(M,\infty)$, $C_1(S^3) \cong B^3$. $\partial C_1(M) = S^2_\infty \cong S^2$.
$C_2(M) = B\ell\left(B\ell(M^2,(\infty,\infty)), \overline{\check{M} \times \infty} \coprod \overline{\infty \times \check{M}} \coprod \overline{\mathrm{diag}(\check{M}^2)}\right)$

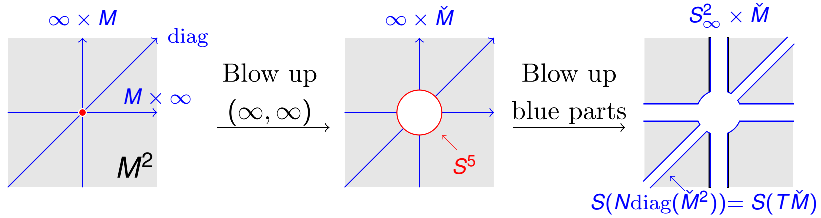


Lemma (math.GT/0411088)

$$\begin{array}{llll} q\colon & (\mathbb{R}^3)^2 \setminus \mathrm{diag} & \to & S^2 \\ & (x,y) & \mapsto & \frac{1}{\|y-x\|}(y-x) \end{array}$$

extends to $C_2(S^3)$.

$S^3 = \mathbb{R}^3 \cup \{\infty\}$. B_0 unit ball of $\mathbb{R}^3$. $B_\infty = \overline{S^3 \setminus B_0}$.
M closed (oriented) 3–manifold.
Choose $i_\infty \colon B_\infty \hookrightarrow M$ and let $\infty = i_\infty(\infty)$. $\check{M} = M \setminus \{\infty\}$.
$\tau \colon \check{M} \times \mathbb{R}^3 \to T\check{M}$ parallelization standard near ∞.
$C_2(M) = B\ell\left(B\ell(M^2,(\infty,\infty)), \overline{\check{M} \times \infty} \coprod \overline{\infty \times \check{M}} \coprod \overline{\mathrm{diag}(\check{M}^2)}\right)$



$$\begin{array}{llll} S(N\mathrm{diag}(M^2)) = \frac{TM^2}{T\mathrm{diag}(M^2)} = \frac{(TM)^2}{\mathrm{diag}(TM)^2} & \xrightarrow{\cong} & S(TM) \\ [(x,y)] & \mapsto & \frac{1}{\|y-x\|}(y-x) \end{array}$$

$\exists q_\tau \colon \partial C_2(M) \to S^2$ canonical, similar to $q_{|\partial C_2(S^3)}$.

M closed (oriented) 3–manifold. $H_*(M;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$.
Choose $i_\infty\colon B_\infty \hookrightarrow M$ and let $\infty = i_\infty(\infty)$. $\check{M} = M \setminus \{\infty\}$.
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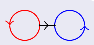
$C_2(M) = B\ell\left(B\ell(M^2,(\infty,\infty)), \overline{\check{M}\times\infty} \coprod \overline{\infty\times\check{M}} \coprod \overline{\mathrm{diag}(\check{M}^2)}\right)$

$C_2(M)$ is a compact 6-dimensional with corners
- the interior of $C_2(M)$ is $\check{M}^2 \setminus \mathrm{diag}$
- $\exists$ canonical $q_\tau\colon \partial C_2(M) \to S^2$
- $H_*(C_2(M);\mathbb{Q}) = H_*(S^2;\mathbb{Q})$
- $H_2(C_2(M);\mathbb{Q}) = \mathbb{Q}[\text{fiber of } S(TM)]$

A propagator of $(C_2(M),\tau)$ is a 4-chain $\mathcal{P}$ of $C_2(M)$ whose boundary is $q_\tau^{-1}(a)$ for some $a \in S^2$.
Example: $q^{-1}(a)$ propagator of $(C_2(S^3),\tau_{\text{standard}})$.

Propagators exist and are Poincaré dual to [fiber of $S(TM)$] in $C_2(M)$.

For $J \coprod K\colon S^1 \coprod S^1 \hookrightarrow \check{M}$, $lk(J,K) = \langle J\times K, \mathcal{P}\rangle_{C_2(M)}$



A propagator of $(C_2(M),\tau)$ is a 4-chain $\mathcal{P}$ of $C_2(M)$ whose boundary is $q_\tau^{-1}(a)$ for some $a \in S^2$.

Theorem (Witten, Kontsevich, math.GT/0411431)

Let M be a rational homology sphere
(i.e. a closed 3–manifold such that $H_*(M;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$).
$B_\infty \subset M$, τ parallelization of $\check{M} = M\setminus\infty$ standard on $B_\infty\setminus\infty$.
Let $\mathcal{P}_a$, $\mathcal{P}_b$ and $\mathcal{P}_c$ be propagators of $(C_2(M),\tau)$ associated to pairwise distinct a, b and c of S^2. Then the algebraic intersection

$$\Theta(M,\tau) = \langle \mathcal{P}_a, \mathcal{P}_b, \mathcal{P}_c\rangle_{C_2(M)}$$

of $\mathcal{P}_a$, $\mathcal{P}_b$ and $\mathcal{P}_c$ in $C_2(M)$ is a topological invariant of (M,τ).

Theorem (G. Kuperberg, D. Thurston 1999)

Let M be an integer homology sphere.

$$\Theta(M,\tau) = 6\lambda_{\text{Casson}}(M) + \tfrac{1}{4}p_1(\tau)$$

where $p_1(\tau)$ is a relative Pontrjagin class.

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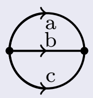

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Theorem (G. Kuperberg, D. Thurston 1999, math.GT/0411431)

Let M be a rational homology sphere.

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