Moduli and Friends, Bucharest, September 2023

On topological invariants counting graph configurations

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I. The Theta invariant for 3-manifolds associated with 
(Mostly on blackboard, with included auxiliary slides).
II. The Watanabe invariant for $(D^4, \partial D^4)$-bundles associated with

For more about these invariants for 3-manifolds and links therein, see the book arXiv/2001.09929 (final version in preparation).

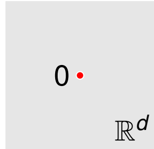
One more advertisement for a simple invariant of genus one knots arXiv/2304.04707.

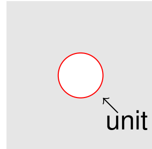
Blow-ups

Here, blowing up B properly embedded in A replaces B with its unit normal bundle UNB to produce

$$B\ell(A,B) = (A \setminus B) \cup UNB$$

homeomorphic to $A \setminus \mathring{N}(B)$.

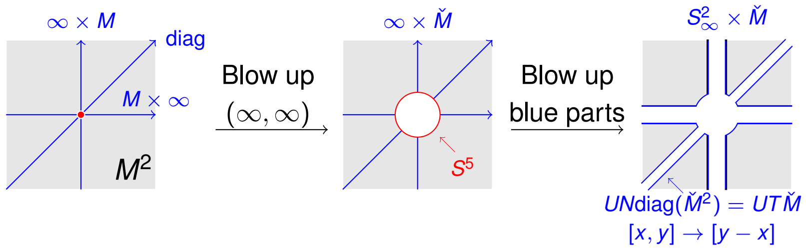


$B\ell(\mathbb{R}^d, 0) =$ 

$\mathbb{R}^d =]0,+\infty[\times S^{d-1} \cup \{0\}$ $\qquad B\ell(\mathbb{R}^d,0) = [0,+\infty[\times S^{d-1}$
$B\ell(\mathbb{R}^k \times \mathbb{R}^d, \mathbb{R}^k \times 0) = \mathbb{R}^k \times B\ell(\mathbb{R}^d, 0) \rightsquigarrow$ local models

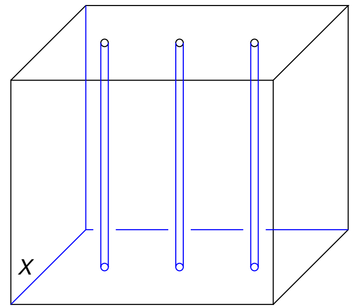
- $p_{B\ell(A,B)} : B\ell(A,B) \to A$ canonical,
- $B\ell(A,B)$ homotopy equivalent to $A \setminus B$,
- A compact $\Rightarrow B\ell(A,B)$ compact.

$C_2(M)$



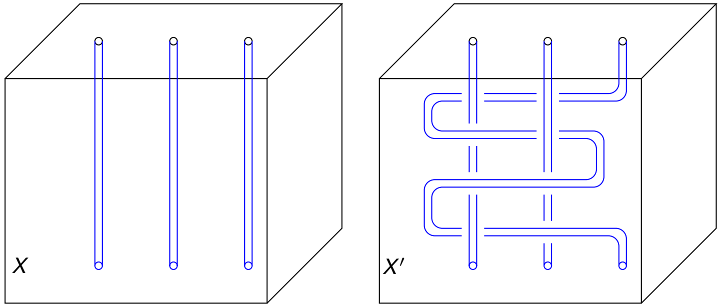
$$C_2(M) = B\ell\left(B\ell(M^2,(\infty,\infty)), \overline{\check{M}\times\infty} \coprod \overline{\infty\times\check{M}} \coprod \overline{\text{diag}(\check{M}^2)}\right)$$

Borromean surgery in dimension 3



X is the cube minus the three thin tubes

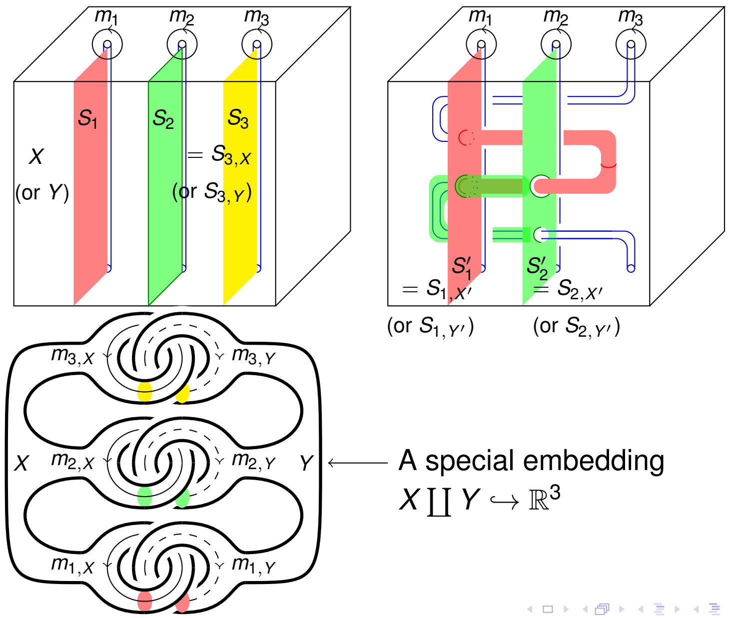
Borromean surgery in dimension 3



X and X' are the cubes minus the thin tubes. We have $\partial X = \partial X'$.
Assume $X \subset M$. Define $M_{\{X\}} = M \setminus \mathring{X} \cup_{\partial X = \partial X'} X' = M_X$.
Let Y be a copy of X.
Assume $X \sqcup Y \subset M$. Set $M_{\{X,Y\}} = (M_{\{X\}})_{\{Y\}} = M_{XY}$.
For $I \subset \{X, Y\}$, define M_I as above.
We compute $\Theta'' = \sum_{I \subset \{X,Y\}} (-1)^{|I|} \Theta(M_I)$

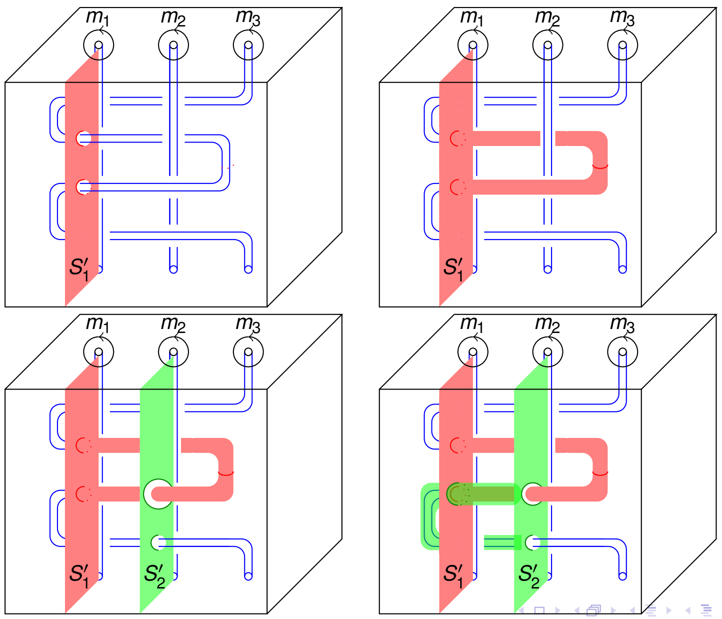
$$= \Theta(M_{XY}) - \Theta(M_Y) - \Theta(M_X) + \Theta(M).$$

Surfaces dual to the meridians

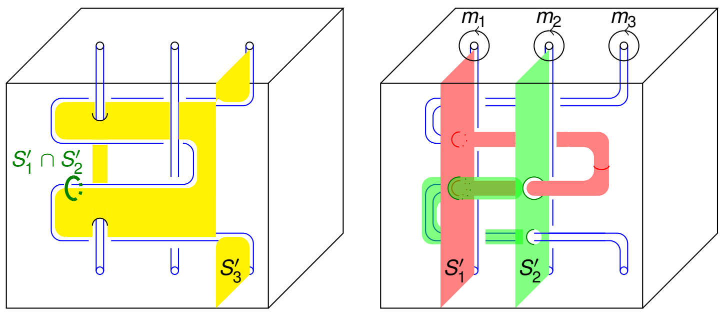


A special embedding
$X \coprod Y \hookrightarrow \mathbb{R}^3$

The surfaces S'_1 and S'_2



The surface S'_3



$$\langle S'_1, S'_2, S'_3 \rangle = \pm 1$$

Watanabe's disproof of the $4d$-Smale conjecture

$I=[0,1]$.
In 2018, Watanabe constructed a non-trivial $(D^4,\partial D^4)$-bundle over $(I^2,\partial I^2)$

$$\begin{array}{ccc}(\mathbb{R}^4,\mathbb{R}^4\setminus \mathring{D}^4) & \hookrightarrow & E\\ & & \downarrow\\ & & I^2\end{array}$$

(i.e. a $(D^4,\partial D^4)$-bundle over I^2 equipped with a trivialization along ∂I^2 that does not extend to I^2)
$(\Rightarrow \pi_1(\mathrm{Diff}(D^4,\partial D^4))\neq\{1\}\Rightarrow \pi_1(\mathrm{Diff}(S^4))\neq \pi_1(SO(5)))$
He detected such a counter-example to the $4d$-Smale conjecture with an invariant, which he called a

Kontsevich characteristic class.

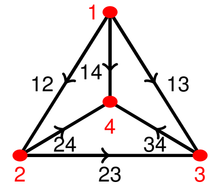
A Kontsevich-Watanabe invariant

$$\begin{array}{ccc}(\mathbb{R}^4,\mathbb{R}^4\setminus \mathring{D}^4)\hookrightarrow E & C_2(S^4)\hookrightarrow C_2(E) & C_2(E)\xrightarrow{\iota} C_2(E)\\ \downarrow & \downarrow & (x,y)\mapsto(y,x)\\ I^2 & I^2 & I^2\end{array}$$

trivialized over ∂I^2
vertically framed

associated $C_2(S^4)$-bundle
$\rightsquigarrow G\colon \partial C_2(E)\to S^3$

A propagator of E is a codimension 3 cycle P of $(C_2(E),\partial C_2(E))$ such that $\partial P=G^{-1}(a)$ for some $a\in S^3$



$$\begin{array}{ccc}C_4(S^4)\hookrightarrow C_4(E) & \xrightarrow{p_{ij}} & C_2(E)\\ (x_1,x_2,x_3,x_4) & \mapsto & (x_i,x_j)\end{array}$$

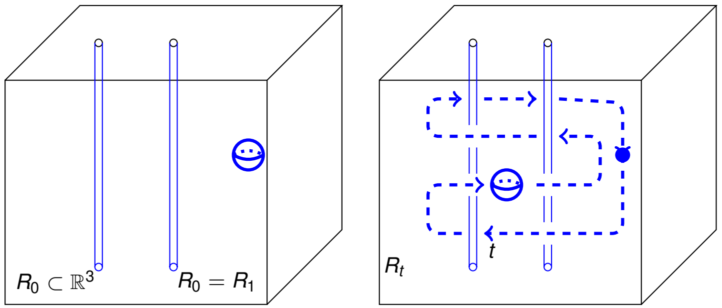
$(P_a,P_b,\dots,P_f)$ generic family of 6 propagators of E

$$W(E)=\sum \frac{1}{2^6 6!}\left\langle p_{12}^{-1}\left(P_{g(12)}+\iota(P_{g(12)})\right),\dots,p_{34}^{-1}\left(P_{g(34)}+\iota(P_{g(34)})\right)\right\rangle_{C_4(E)}$$

$g\colon\{12,13,14,23,24,34\}\hookrightarrow\{a,b,\dots,f\}$

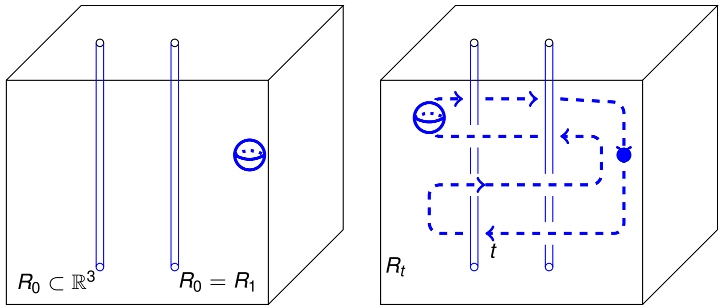
... if the vertical framing of E over ∂E extends to E.

Watanabe's first (blue) surgery



R_0 is the cube minus the two thin tubes
and a tubular neighborhood of the sphere.
R_t is the cube minus the two thin tubes
and a tubular neighborhood of the travelling sphere at time $t\in I$.
$R=R_0\times I$ $\qquad R'=\cup_{t\in I}R_t\times\{t\}$
This model R is in $\mathbb{R}^4$.

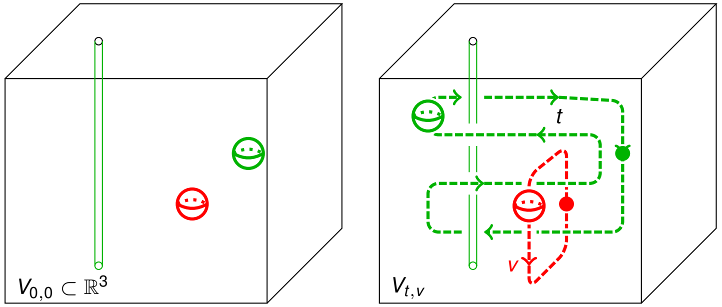
Watanabe's first (blue) surgery



R_t is the cube minus the two thin tubes
and a tubular neighborhood of the travelling sphere at time $t\in I$
$R=R_0\times I$ $\qquad R'=\cup_{t\in I}R_t\times\{t\}$
When R is embedded in some M^4,

$$M^4_R=M^4(R'/R)=(M^4\setminus\mathring{R})\cup_{\partial R}R'.$$

Watanabe's parametrized (green) surgery

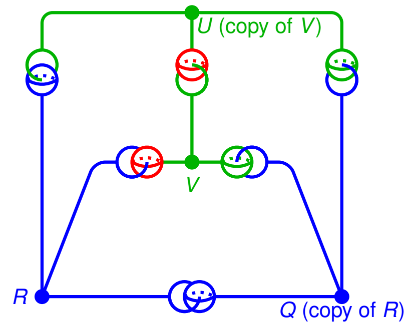


$V_{t,v}$ is the cube minus the thin tube
and tubular neighborhoods of the two travelling spheres
at $(t,v) \in I^2$.

$V_v = V_{0,v} \times I$ is identified with $V_0 = V_{0,0} \times I$

$V'_v = \cup_{t \in I} V_{t,v} \times \{t\}$.

The Watanabe counter-example



$$E_{QRUV} = \cup_{(u,v)\in I^2} \mathbb{R}^4 \left(Q'/Q, R'/R, U'_u/U_u, V'_v/V_v\right) \times \{u,v\}$$

$$W(E_{QRUV}) = \sum_{I \subset \{Q,R,U,V\}} (-1)^{|I|} W(E_I) \neq 0$$

https://if-summer2024.sciencesconf.org/