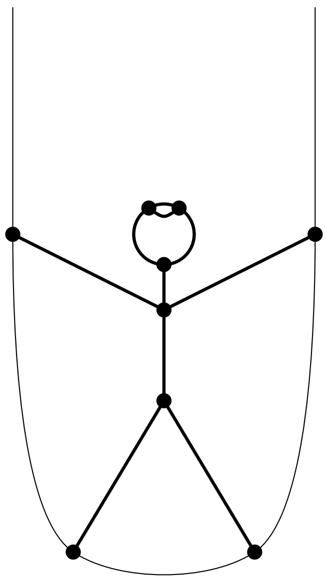
Invariants of links and 3–manifolds from graph configurations



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Contents

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À Lucien Guillou,
À nos 26 ans de bonheur ensemble,
À nos deux enfants, Gaëlle et Ronan.

Part I
Introduction

Chapter 1

Introductions

In this introductory chapter, we propose several introductions to this book

- a short abstract of this book for experts in Section 1.1,
- a slow informal introduction based on examples in Section 1.2, from which a wide audience can get the flavor of the studied topics and hopefully get interested,
- an independent more formal mathematical overview of the contents, to which the experts can go directly, in Section 1.3, and
- a section on the genesis of this book in Section 1.5.

I apologize for some repetitions due to this structure.

Unlike this first chapter, which has some parts written for experts, and which is sometimes imprecise, the rest of the book is precise, detailed and mostly self-contained. It relies only on the basic notions of algebraic topology and on the basic notions of de Rham cohomology that are surveyed in the appendices.

1.1 An abstract for experts

Very first conventions Unless otherwise mentioned, manifolds are smooth, but they may have boundary and corners. Let $\mathbb{K}$ be $\mathbb{Z}$ or $\mathbb{Q}$. In this book, a $\mathbb{K}$–*sphere* is a compact oriented 3-dimensional manifold with the same homology with coefficients in $\mathbb{K}$ as the standard unit sphere S^3 of $\mathbb{R}^4$. A $\mathbb{K}$–*ball* (resp. a $\mathbb{K}$–*cylinder*, a *genus* g $\mathbb{K}$–*handlebody*) is a compact oriented 3-dimensional manifold A with the same homology with coefficients in $\mathbb{K}$ as the standard unit ball B^3 of $\mathbb{R}^3$, (resp. the cylinder $D_1 \times [0,1]$, where D_1 is the unit disk of $\mathbb{C}$, the standard solid handlebody H_g of Figure 1.1), such that a neighborhood of the boundary of A (which is necessarily homeomorphic to the boundary of its model –ball, cylinder or handlebody–) is identified with a neighborhood of the boundary of its model by a smooth diffeomorphism.

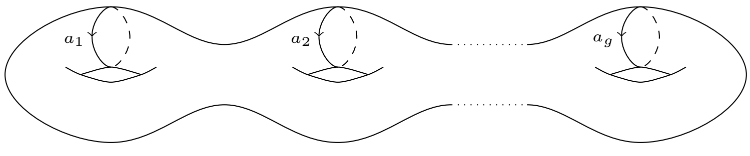


Figure 1.1: The genus g handlebody H_g

In "$\mathbb{Q}$–spheres", $\mathbb{Q}$ is a shortcut for *rational homology*, so $\mathbb{Q}$–spheres are also called *rational homology spheres* or *rational homology* 3-*spheres*, while $\mathbb{Z}$–spheres are also called *integer homology* 3-*spheres.*

Abstract In this book, following Witten [Wit89], Kontsevich [Kon94], Kuperberg and Thurston [KT99], we define an invariant $\mathcal{Z}$ of n-component links L in rational homology 3-spheres R, and we study its properties. The invariant $\mathcal{Z}$, which is often called the perturbative expansion of the Chern-Simons theory, is valued in a graded space $\mathcal{A}(\coprod_{j=1}^k S^1)$ generated by Jacobi diagrams Γ on $\coprod_{j=1}^k S^1$. These diagrams are a special kind of Feynman diagrams, they are unitrivalent. The invariant $\mathcal{Z}(L)$ is a combination $\mathcal{Z}(L) = \sum_\Gamma \mathcal{Z}_\Gamma(L)[\Gamma]$ for coefficients $\mathcal{Z}_\Gamma(L)$ that "count" embeddings of Γ in R that map the univalent vertices of Γ to L, in a sense that will be explained in the book, using integrals over configuration spaces, or, in a dual way, algebraic intersections in the same configuration spaces.

When $R = S^3$, the invariant $\mathcal{Z}$ is a universal Vassiliev link invariant studied by many authors including Guadagnini, Martellini and Mintchev [GMM90], Bar-Natan [BN95b], Bott and Taubes [BT94], Altschüler and Freidel [AF97], Dylan Thurston [Thu99] and Sylvain Poirier[Poi02]... This book contains a more flexible definition of this invariant.

We prove that the restriction of $\mathcal{Z}$ to $\mathbb{Q}$–spheres (equipped with empty links) is a universal finite type invariant with respect to *rational LP-surgeries,* which are replacements of rational homology handlebodies by other rational homology handlebodies, in a way that does not change the linking number of curves outside the replaced handlebodies. Together with recent results of Massuyeau [Mas14] and Moussard [Mou12], this implies that the restriction of $\mathcal{Z}$ to $\mathbb{Q}$–spheres contains the same information as the Le-Murakami-Ohtsuki

LMO invariant [LMO98] for these manifolds. This also implies that the degree one part of $\mathcal{Z}$ is the Casson-Walker invariant.

We extend $\mathcal{Z}$ to a functorial invariant of framed tangles in rational homology cylinders and we describe the behaviour of this functor under various operations including some cabling operations. We also compute iterated derivatives of this extended invariant with respect to discrete derivatives associated to the main theories of finite type invariants.

1.2 A slow informal introduction for beginners

In this introduction, we try to describe the contents of this book to a wide audience including graduate students, starting with examples, in order to give a flavor of the topics that are studied in this book, to introduce some of the ideas, some conventions and some methods, which will be used later.

This book is about invariants of links and 3-manifolds which count graph configurations. The first example of such an invariant goes back to Gauss in 1833 [Gau77]. It is the linking number of two knots. We discuss it in Section 1.2.1.

In Section 1.2.5, as a second example, we study the knot invariant w_2, which was first studied by Guadagnini, Martellini and Mintchev [GMM90] and Bar-Natan [BN95b] at the end of the eighties with the inspiration coming from Edward Witten's insight into the perturbative expansion of the Chern-Simons theory [Wit89], and next revisited and generalized by Kontsevich [Kon94], Bott and Taubes [BT94] [Bot96], Altschüler and Freidel [AF97], and others.

1.2.1 The linking number as a degree

Let S^1 denote the unit circle of the complex plane $\mathbb{C}$. $S^1 = \{z \mid z \in \mathbb{C}, |z| = 1\}$. Consider a C^∞ embedding

$$J \sqcup K \colon S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$$

of the disjoint union $S^1 \sqcup S^1$ of two circles in the ambient space $\mathbb{R}^3$, as that pictured in Figure 1.2. Such an embedding represents a 2*–component link.*

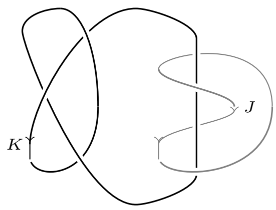
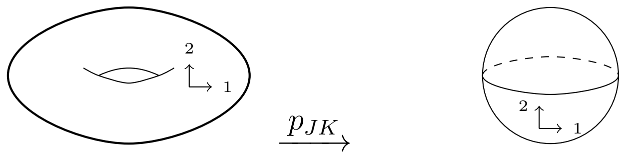


Figure 1.2: A 2–component link in $\mathbb{R}^3$

It induces the *Gauss map*



$$\begin{array}{llll} p_{JK}\colon & S^1 \times S^1 & \to & S^2 \\ & (w,z) & \mapsto & \frac{1}{\|K(z)-J(w)\|}(K(z)-J(w)) \end{array}$$

Definition 1.1. The *Gauss linking number* $lk_G(J,K)$ of the disjoint *knot embeddings* $J(S^1)$ and $K(S^1)$, which are simply denoted by J and K, is the degree of the Gauss map p_{JK}.

Let us give our favorite definition of the degree in this book, and, in order to do so, let us first agree on some conventions, which we will use in this book.

1.2.2 On orientations of manifolds and degrees of maps

We work with smooth manifolds with boundary, ridges and corners, which are locally diffeomorphic to open subspaces of $[0,1]^n$. These manifolds are described in more details in Sections 2.1.1 and 2.1.4. The cube $[0,1]^3$ is an example of such a manifold, its edges are *ridges*.

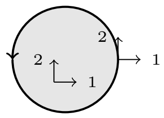
Conventions 1.2. Let M be such a manifold. The *interior* of M consists of the points with a neighborhood diffeomorphic to an open subspace of $\mathbb{R}^n$, and its *boundary* is the complement of its interior in M.

An *orientation* of a real vector space V of positive dimension is a basis of V up to a change of basis with positive determinant. When $V = \{0\}$, an orientation of V is an element of $\{-1,1\}$. An *orientation* of a smooth n–manifold is an orientation of its tangent space at each point of its interior, defined in a continuous way. (A local diffeomorphism h of $\mathbb{R}^n$ is orientation-preserving at x if and only if the Jacobian determinant of its derivative T_xh

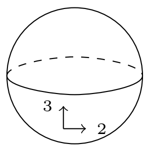
is positive. If the transition maps $\phi_j \circ \phi_i^{-1}$ of an *atlas* $(\phi_i)_{i\in I}$ of a manifold M (as in Section 2.1.1) are orientation-preserving (at every point) for $\{i,j\} \subseteq I$, then the manifold M is *oriented* by this atlas.) Unless otherwise mentioned, manifolds are smooth, oriented and compact, and considered up to orientation-preserving diffeomorphisms, in this book. Products are oriented by the order of the factors. More generally, unless otherwise mentioned, the order of appearance of coordinates or parameters orients manifolds. When M is an oriented manifold, $(-M)$ denotes the same manifold, equipped with the opposite orientation.

The boundary ∂M of an oriented manifold M is oriented by the *outward normal first* convention: If $x \in \partial M$ is not in a ridge, the outward normal to M at x followed by an oriented basis of $T_x\partial M$ induce the orientation of M.

For example, the standard orientation of the disk in the plane induces the standard orientation of the circle, counterclockwise, as the following picture shows.



As another example, the sphere S^2 is oriented as the boundary of the unit ball B^3 of $\mathbb{R}^3$, which has the standard orientation induced by (thumb, index finger (2), middle finger (3)) of the right hand.



Definitions 1.3. Let M and N be smooth manifolds, and let $p\colon M \to N$ be a smooth map from M to N. If the boundary of M is empty, then a point y is a *regular value* of p, if $y \in N$ and for any $x \in p^{-1}(y)$ the tangent map

$$T_xp\colon T_xM \to T_yN$$

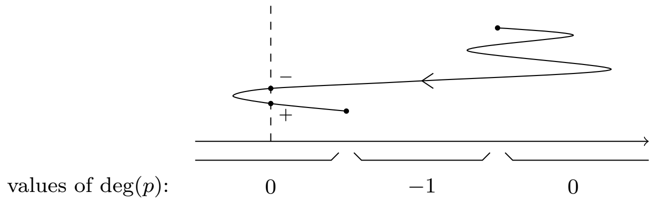
at x is surjective.[1] In general, M may have boundary, ridges and corners, and a point y is a *regular value* of p, if y is a regular value of the restrictions

[1] Our d–manifolds are covered by countably many open sets diffeomorphic to open subsets of $[0,1]^d$. According to the Morse-Sard theorem [Hir94, Chapter 3, Theorem 1.3, p. 69], the set of regular values of such a map p is dense. (It is even *residual*, i.e. it contains the intersection of a countable family of dense open sets.) If M is compact, it is furthermore open.

of p to the interior of M and to all the open, smooth faces (or strata) of M (of any codimension). In particular, if the dimensions of M and N coincide, then a regular value is not in the image of the boundary ∂M.

If M and N are oriented, if M is compact and if the dimension of M coincides with the dimension of N, then the *(differential) degree* of p at a regular value y of N is the (finite) sum running over the $x \in p^{-1}(y)$ of the signs of the determinants of $T_x p$ (in oriented charts). In this case, the differential degree of p extends as a continuous function $\deg(p)$ from $N \setminus p(\partial M)$ to $\mathbb{Z}$, as we prove in details in Lemma 2.3. In particular, if the boundary of M is empty and if N is connected, then $\deg(p)$ extends as a constant map from N to $\mathbb{Z}$, whose value is called the *degree* of p. See [Mil97, Chapter 5].

The following figure shows the values of $\deg(p)$ for the pictured vertical projection p from the interval $[0, 1]$ to $\mathbb{R}$.



Another easy example in higher dimensions is the case where p is an orientation-preserving embedding. In this case, $\deg(p)$ is 1 on the image of the interior of M and 0 outside the image of M.

1.2.3 Back to the linking number

The Gauss linking number $lk_G(J, K)$ can be computed from a link diagram as in Figure 1.2 as follows. It is the differential degree of p_{JK} at the vector Y that points towards us. The set $p_{JK}^{-1}(Y)$ is the set of pairs (w, z) of points for which the projections of $J(w)$ and $K(z)$ coincide, and $J(w)$ is under $K(z)$. They correspond to the *crossings* ${}^{J}\times{}^{K}$and ${}^{K}\times{}^{J}$ of the diagram.

In a diagram, a crossing is *positive* if we turn counterclockwise from the arrow at the end of the upper strand towards the arrow of the end of the lower strand like $\times$. Otherwise, it is *negative* like $\times$.

For the positive crossing ${}^{J}\times{}^{K}$, moving $J(w)$ along J following the orientation of J, moves $p_{JK}(w, z)$ towards the south-east direction, while moving $K(z)$ along K following the orientation of K, moves $p_{JK}(w, z)$ towards the north-east direction, so the local orientation induced by the image of p_{JK} around $Y \in S^2$ is $\langle^{Tpdz}_{Tpdw}$, which is $\langle^{2}_{1}$. Therefore, the contribution of a positive crossing to the degree is 1. It is easy to deduce that the contribution of a negative crossing is (-1).

We have proved the following formula

$$\deg_Y(p_{JK}) = \sharp\, {}^{J}\nwarrow\!\!\nearrow^{K} - \sharp^{K}\nwarrow\!\!\nearrow^{J},$$

where $\sharp$ stands for the cardinality. (In particular, $\sharp\, {}^{J}\nwarrow\!\!\nearrow^{K}$ is the number of occurences of ${}^{J}\nwarrow\!\!\nearrow^{K}$ in the diagram.) So

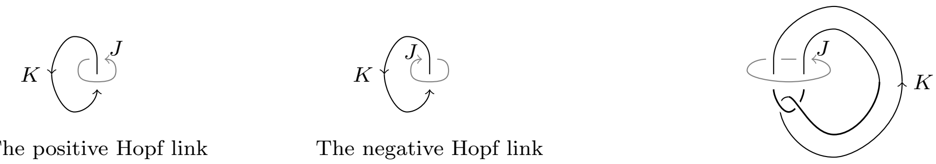
$$lk_G(J,K) = \sharp\, {}^{J}\nwarrow\!\!\nearrow^{K} - \sharp^{K}\nwarrow\!\!\nearrow^{J}.$$

Similarly, $\deg_{-Y}(p_{JK}) = \sharp^{K}\nwarrow\!\!\nearrow^{J} - \sharp\, {}^{J}\nwarrow\!\!\nearrow^{K}$. So

$$lk_G(J,K) = \sharp^{K}\nwarrow\!\!\nearrow^{J} - \sharp\, {}^{J}\nwarrow\!\!\nearrow^{K} = \frac{1}{2}\left(\sharp\, {}^{J}\nwarrow\!\!\nearrow^{K} + \sharp^{K}\nwarrow\!\!\nearrow^{J} - \sharp^{K}\nwarrow\!\!\nearrow^{J} - \sharp\, {}^{J}\nwarrow\!\!\nearrow^{K}\right)$$

and $lk_G(J,K) = lk_G(K,J)$.

In the example of Figure 1.2, $lk_G(J,K) = 2$. For the *positive Hopf link* of Figure 1.3, $lk_G(J,K) = 1$, for the *negative Hopf link,* $lk_G(J,K) = -1$, and, for the *Whitehead link*, $lk_G(J,K) = 0$.



The positive Hopf link The negative Hopf link The Whitehead link

Figure 1.3: The Hopf links and the Whitehead link

Since the differential degree of the Gauss map p_{JK} is constant on the set of regular values of p_{JK}, for any 2-form ω_S on S^2,

$$\int_{S^1\times S^1} p^*_{JK}(\omega_S) = \int_{S^2} \deg(p_{JK})\omega_S = lk_G(J,K)\int_{S^2}\omega_S.$$

Denote the standard area form of S^2 by $4\pi\omega_{S^2}$. So ω_{S^2} is the homogeneous volume form of S^2 such that $\int_{S^2}\omega_{S^2} = 1$. In 1833, Gauss defined the linking number of J and K, as an integral [Gau77]. In modern notation, his definition may be written as

$$lk_G(J,K) = \int_{S^1\times S^1} p^*_{JK}(\omega_{S^2}).$$

1.2.4 A first non-invariant count of graph configurations

The above Gauss linking number counts the *configurations* of the graph $v_J \longrightarrow v_K$, for which v_J is on the knot J (or more precisely on the image $J(S^1)$ of the knot embedding J), v_K is on the knot K, and the edge from v_J to v_K is a straight segment with an arbitrary generic fixed direction X in S^2. Here, *generic* means that X is a regular[2] point of p_{JK}, a *configuration* of $v_J \longrightarrow v_K$ is an injection $c\colon \{v_J, v_K\} \hookrightarrow (\mathbb{R}^3)^2$ such that $c(v_J) = J(z_J)$ for some $z_J \in S^1$ and $c(v_K) = J(z_K)$ for some $z_K \in S^1$, the corresponding *configuration space* is parametrized by $(z_J, z_K) \in S^1 \times S^1$ and it is diffeomorphic to $S^1 \times S^1$. The configurations such that the edge from v_J to v_K is a straight segment with direction X in S^2 are in one-to-one correspondence with $p_{JK}^{-1}(X)$, and the local degree of p_{JK} equips each of them with a sign. The *count* of configurations with direction X is the sum of these signs, which is nothing but the degree of p_{JK} at X, so that any choice of a generic X will give the same integral result, which will not be changed by a continuous deformation of our embedding among embeddings. The graph $v_J \longrightarrow v_K$ will also be denoted by $J \; \bullet\!\!\longrightarrow\!\!\bullet \; K$. In that diagram, the dashed circles show where the vertices should be.

Let $K\colon S^1 \hookrightarrow \mathbb{R}^3$ be a smooth embedding of the circle into $\mathbb{R}^3$. Such an embedding is called a *knot embedding*. An *isotopy* between two knot embeddings K and K_1 is smooth map $\psi\colon [0,1] \times S^1 \to \mathbb{R}^3$ such that the restriction $\psi(t,.)$ of ψ to $\{t\} \times S^1$ is a knot embedding for any $t \in [0,1]$, $\psi(0,.) = K$ and $\psi(1,.) = K_1$. When there exists such an isotopy, K and K_1 are said to be *isotopic* or in the same *isotopy class*. A *knot* is an isotopy class of knot embeddings.

Let us try to count the configurations

$$c\colon \{v_1, v_2\} \hookrightarrow (\mathbb{R}^3)^2$$

of the graph $v_1 \longrightarrow v_2$, for which $c(v_1)$ and $c(v_2)$ are two distinct points on the (image of) the knot embedding K, and the edge from v_1 to v_2 is a straight segment with an arbitrary direction X in S^2. The graph $v_1 \longrightarrow v_2$ is also denoted by $\overset{\frown}{\bullet\!\!\to\!\!\bullet}$ and the associated *configuration space* is

$$\check{C}(K; \overset{\frown}{\bullet\!\!\to\!\!\bullet}) = \{(K(z), K(z\exp(2i\pi t))) \mid z \in S^1, t \in]0,1[\},$$

it is naturally identified with the open annulus $S^1 \times]0,1[$. We have a *Gauss direction map*

$$\begin{array}{llll} G_K\colon & \check{C}(K; \overset{\frown}{\bullet\!\!\to\!\!\bullet}) & \to & S^2 \\ & c & \mapsto & \frac{1}{\|K(z\exp(2i\pi t)) - K(z)\|}(K(z\exp(2i\pi t)) - K(z)) \end{array}$$

[2]This is a generic condition thanks to the recalled Morse-Sard theorem.

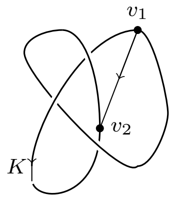


Figure 1.4: A configuration of a segment on K

and the degree of G_K at X makes sense as soon as X is a regular value of G_K whose preimage is finite.

The annulus $\check{C}(K;\leftrightarrow)$ can be compactified to the closed annulus $C(K;\leftrightarrow) = S^1 \times [0,1]$, to which G_K extends smoothly. The extended G_K, still denoted by G_K, maps $(z,0) \in S^1 \times \{0\}$ to the direction of the tangent vector to K at z and $(z,1) \in S^1 \times \{1\}$ to the opposite direction. The closed annulus $C(K;\leftrightarrow)$ is an example of a smooth manifold whose boundary is $S^1 \times \{0\}$ - $S^1 \times \{1\}$. The degree of G_K is a continuous map from $S^2 \setminus G_K(\partial C(K;\leftrightarrow))$ to $\mathbb{Z}$. Let us compute it for the following embeddings of the trivial knot.

Let O be an embedding of the circle to the horizontal plane. The image under G_O of the whole annulus is in the horizontal great circle of S^2. The set of regular values of G_O is the complement of this circle, and the degree of G_O is zero on all this set.

Let K_1 and K_{-1} be embeddings of S^1, which project to the horizontal plane as in Figure 1.5, which lie in the horizontal plane everywhere except when they cross over, and which lie in the union of two orthogonal planes.

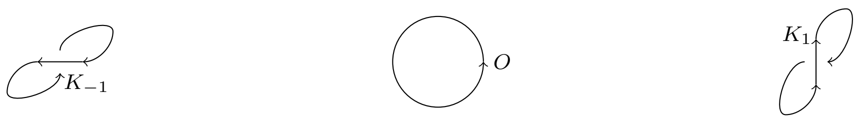
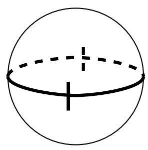


Figure 1.5: Diagrams of the trivial knot

The image of the boundary of $C(K_{\pm 1};\leftrightarrow) = S^1 \times [0,1]$ in S^2 lies in the union of the great circles of the two planes, or, more precisely, in the union of the horizontal plane and two vertical arcs, as in the following figure.



Therefore, when $K = K_\varepsilon$, for $\varepsilon = \pm 1$, the degree of G_{K_ε} (extends as a map, which) is constant on each side of our horizontal equator. Computing it at the North Pole $\vec{N}$ as in Subsection 1.2.3, we find that the degree of G_{K_ε} is ε on the Northern Hemisphere. We compute the degree of G_{K_ε} on the Southern Hemisphere similarly. It is also ε.

For any embedding $K\colon S^1 \hookrightarrow \mathbb{R}^3$, define

$$I_\theta(K) = \int_{\check{C}(K;\leftrightarrow)} G_K^*(\omega_{S^2}),$$

which is the integral of $\deg(G_K)\omega_{S^2}$ over S^2 and which can be seen as the *algebraic area* of $G_K(C(K;\leftrightarrow))$ The above degree evaluation allows us to compute the integrals $I_\theta(O) = 0$, $I_\theta(K_1) = 1$, and $I_\theta(K_{-1}) = -1$,

More generally, say that a knot embedding K that lies in the union of the horizontal plane and a finite union of vertical planes, so that the unit tangent vector to K is never vertical, is *almost horizontal.* The *writhe* of an almost horizontal knot embedding is the number of positive crossings minus the number of negative crossings of its orthogonal projection onto the horizontal plane. An almost horizontal embedding K has a natural parallel[3] $K_\parallel$ (up to isotopy) obtained from K by (slightly) pushing it down. For any almost horizontal knot embedding K, the degree of G_K may be extended to a constant function of S^2. More precisely, we have the following lemma:

Lemma 1.4. *For any almost horizontal knot embedding K, the degree of G_K is $lk(K, K_\parallel)$ at any regular value of G_K. So*

$$I_\theta(K) = \int_{\check{C}(K;\leftrightarrow)} G_K^*(\omega_{S^2}) = lk(K, K_\parallel)$$

is the writhe of K.

Proof: As in the previous examples, we see that an almost horizontal knot embedding has a constant I_θ-degree, which is its writhe. The parallel below $K_\parallel$ is isotopic in the complement of K to the parallel $K_{\parallel,\ell}$ on the left-hand side of K, and the formulas of Section 1.2.1 prove that $lk(K, K_{\parallel,\ell})$ is the writhe of K. $\square$

An (isotopy) knot invariant is a function of embeddings that takes the same value on isotopic knots. Unlike the Gauss linking number, the integral

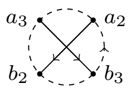
[3] A *parallel* of a knot embedding K in a 3–manifold M (such as $\mathbb{R}^3$) is a knot embedding $K_\parallel\colon S^1 \hookrightarrow M$ such that there exists an embedding f of $[0,1] \times S^1$ into M that restricts to $\{0\} \times S^1$ as K and to $\{1\} \times S^1$ as $K_\parallel$.

$I_\theta()$ is *not invariant under isotopy* since it takes distinct values on the isotopic knot embeddings K_{-1} and K_1.

It is easy to construct an embedding such that the degree of the direction map G_K extends as the constant map with value 0, in every isotopy class of embeddings of S^1 into $\mathbb{R}^3$, by adding kinks such as or to a horizontal projection. Since I_θ varies continuously under an isotopy of K, for any knot K of $\mathbb{R}^3$, I_θ maps the space of embeddings of S^1 into $\mathbb{R}^3$, isotopic to any knot K, onto $\mathbb{R}$. In particular, there are embeddings for which $I_\theta(K)$ is not an integer. For those embeddings, the degree of the direction map G_K cannot be extended to a constant map on S^2.

1.2.5 A first knot invariant which counts graph configurations

Let us now count configurations c of the following graph



for which c is an injection from the set $\{b_3, a_2, a_3, b_2\}$ of vertices of $\boxtimes$ to $\mathbb{R}^3$, the images $c(b_3)$, $c(a_2)$, $c(a_3)$ and $c(b_2)$ of the vertices b_3, a_2, a_3 and b_2 are on the knot K, we successively meet $c(b_3) = K(z)$, $c(a_2) = K(z\exp(2i\pi\alpha_2))$, $c(a_3) = K(z\exp(2i\pi\alpha_3))$ and $c(b_2) = K(z\exp(2i\pi\beta_2))$ along K, following the orientation of K, as on the dashed circle, which pictures this cyclic order of the four vertices.

The associated *configuration space* is

$$\check{C}(K;\boxtimes) = \left\{ \begin{array}{l} (K(z), K(z\exp(2i\pi\alpha_2)), K(z\exp(2i\pi\alpha_3)), K(z\exp(2i\pi\beta_2))) \\ \mid z \in S^1, (\alpha_2,\alpha_3,\beta_2) \in]0,1[^3, \alpha_2 < \alpha_3 < \beta_2 \end{array} \right\}.$$

For $i \in \underline{2} = \{1,2\}$, set $e_i = (a_i, b_i)$ and let $G_{e_i}(c) = \frac{c(b_i)-c(a_i)}{\|c(b_i)-c(a_i)\|}$ denote the direction of the image under c of the edge e_i, in S^2.

The open configuration space $\check{C}(K;\boxtimes)$ can be compactified naturally to

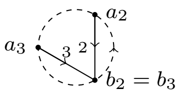
$$C(K;\boxtimes) = S^1 \times \{(\alpha_2,\alpha_3,\beta_2) \in [0,1]^3 \mid \alpha_2 \le \alpha_3 \le \beta_2\},$$

to which G_{e_2} and G_{e_3} extend smoothly as before, and

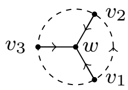
$$G_{\boxtimes} = (G_{e_2}, G_{e_3}) \colon C(K;\boxtimes) \to (S^2)^2$$

is a smooth map between two compact 4-manifolds.

The *codimension-one faces* $C(K;\boxtimes)$ are the four faces $(\alpha_2 = 0)$, $(\alpha_2 = \alpha_3)$, $(\alpha_3 = \beta_2)$, and $(\beta_2 = 1)$, in which c maps (at least) two consecutive vertices to the same point on K. When $G_{\boxtimes}$ is locally an embedding near such a face, the degree of $G_{\boxtimes}$ changes by ± 1 when we cross the image of this face[4], and it suffices to determine the images of the interiors of these codimension-one faces and the local degree at one regular point, to determine the degree of $G_{\boxtimes}$, as a map from $(S^2)^2 \setminus G_{\boxtimes}(\partial C(K;\boxtimes))$ to $\mathbb{Z}$. The following figure is associated to the face $(\beta_2 = 1)$ of $C(K;\boxtimes)$.



Let us now try to count configurations c of the following tripod $\curlyvee$



for which c is an injection from the set $\{w, v_1, v_2, v_3\}$ of vertices of the tripod $\curlyvee$ into $\mathbb{R}^3$, the image $c(w)$ of the vertex w is in $\mathbb{R}^3$, the images $c(v_1)$, $c(v_2)$ and $c(v_3)$ of the vertices v_1, v_2 and v_3 are on the knot K, and, we successively meet $c(v_1)$, $c(v_2)$ and $c(v_3)$ along K, following the orientation of K.

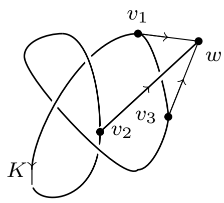


Figure 1.6: A configuration of the tripod on K

Such a *configuration* c maps w to $c(w) \in \mathbb{R}^3$, v_1 to $c(v_1) = K(z_1)$ for some $z_1 \in S^1$, v_2 to $c(v_2) = K(z_1 \exp(2i\pi t_2))$ and v_3 to $c(v_3) = K(z_1 \exp(2i\pi t_3))$. The set of these configurations is the *configuration space* $\check{C}(K;\curlyvee)$, which is an open 6–manifold parametrized by an open subspace of $\mathbb{R}^3 \times S^1 \times \{(t_2,t_3) \in]0,1[^2 \mid t_2 < t_3\}$. For $i \in \underline{3} = \{1,2,3\}$, set $e_i = (v_i, w)$ and let $G_{e_i}(c) = \frac{c(w)-c(v_i)}{\|c(w)-c(v_i)\|}$ denote the direction of the image under c of the edge e_i in S^2. These edge directions together provide a map

$$\begin{array}{llll} \check{G}_{\curlyvee}\colon & \check{C}(K;\curlyvee) & \to & (S^2)^3 \\ & c & \mapsto & (G_{e_1}(c), G_{e_2}(c), G_{e_3}(c)) \,. \end{array}$$

[4]See Lemma 2.3 for a precise statement.

from our open 6-manifold $\check{C}(K;\tripod)$ to the 6-manifold $(S^2)^3$.

For a regular point (X_1, X_2, X_3) of $\check{G}_{\tripod}$ whose preimage is finite, we can again count the configurations of the tripod such that the direction of the edge e_i is X_i, as the degree of $G_{\tripod}$ at (X_1, X_2, X_3).

In Chapter 8, using blow-up techniques, as Fulton, McPherson [FM94], Axelrod, Singer [AS92] and Kontsevich [Kon94] did, we construct a *compactification* $C(K;\tripod)$ of $\check{C}(K;\tripod)$ (and of many similar configuration spaces) such that $C(K;\tripod)$ is a smooth compact 6-manifold with boundary, ridges and corners whose interior is $\check{C}(K;\tripod)$, and such that $\check{G}_{\tripod}$ extends to a smooth map $G_{\tripod}$ over $C(K;\tripod)$. *Regular points* of $G_{\tripod}$, which are regular points of $\check{G}_{\tripod}$ in the complement of the image of the boundary $\partial C(K;\tripod) = C(K;\tripod) \setminus \check{C}(K;\tripod)$ under $G_{\tripod}$ form an open dense subset $\mathcal{O}$ of $(S^2)^3$, for which the differential degree of $\check{G}_{\tripod}$ makes sense. As mentioned above and proved in Lemma 2.3, this local integral degree may be extended to a continuous map on $(S^2)^3 \setminus G_{\tripod}(\partial C(K;\tripod))$. It becomes a constant map on the connected components of $(S^2)^3 \setminus G_{\tripod}(\partial C(K;\tripod))$. We compute this local degree explicitly, as an example, in Lemma 7.12, when K is the round circle O in a plane, and our computation proves that this degree cannot be extended to a constant map in this case.

Again, we can define the algebraic volume $\int_{(S^2)^3} \deg(G_{\tripod}) \wedge_{i=1}^3 p_i^*(\omega_{S^2})$ of the image of $C(K;\tripod)$ under $G_{\tripod}$, where $p_i^*(\omega_{S^2})$ denotes the pull-back under the projection p_i of the i^{th} factor of $(S^2)^3$ of ω_{S^2}, as

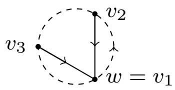
$$I(K;\tripod) = \int_{\check{C}(K;\tripod)} G_{\tripod}^* \left(\wedge_{i=1}^3 p_i^*(\omega_{S^2})\right).$$

but it has no reason to be a knot isotopy invariant and it is not.

In order to compute the map

$$\deg(G_{\tripod}) \colon (S^2)^3 \setminus G_{\tripod}(\partial C(K;\tripod)) \to \mathbb{Z},$$

we look at $G_{\tripod}(\partial C(K;\tripod))$. Let us describe the compactification $C(K;\tripod)$, near loci where $c(v_1)$ is far from $c(v_2)$ and $c(v_3)$ ($(t_2, t_3) \in]\alpha, 1-\alpha[^2$, for some $\alpha \in]0, 1/2[$), and $c(w) = c(v_1) + \eta x$ for some $\eta \in]0, \varepsilon[$ for a small $\varepsilon > 0$ and for some $x \in S^2$. Locally, the configuration space $\check{C}(K;\tripod)$ is diffeomorphic to $]0, \varepsilon[\times S^2 \times S^1 \times \{(t_2, t_3) \in]\alpha, 1-\alpha[^2 \mid t_2 < t_3\}$, and its compactification $C(K;\tripod)$ is diffeomorphic to $[0, \varepsilon[\times S^2 \times S^1 \times \{(t_2, t_3) \in]\alpha, 1-\alpha[^2 \mid t_2 \leq t_3\}$, (we close $]0, \varepsilon[$ at 0 – we also relax the inequality $t_2 < t_3$ but this is not important for us now –). In the compactification $C(K;\tripod)$, $c(w)$ is allowed to coincide with $c(v_1)$ (when $\eta = 0$), and the direction from $c(v_1)$ to $c(w)$ is still defined in this case (it is contained in the S^2 factor). The created local boundary is the 5-dimensional manifold $\{0\} \times (-S^2) \times S^1 \times \{(t_2, t_3) \in]\alpha, 1-\alpha[^2 \mid t_2 \leq t_3\}$, and its image is the product by S^2 of the image in $(S^2)^2$ of the configuration space associated to the following graph by the natural Gauss map "direction of the edges".

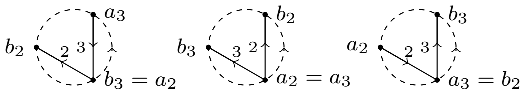


Again, the dashed circle represents the wanted cyclic order of $c(v_1)$, $c(v_2)$ and $c(v_3)$ along K. The image of the corresponding local boundary creates a "wall" in $(S^2)^3$ across which the local degree changes by ± 1. (See Lemma 2.3 for a more precise statement.)

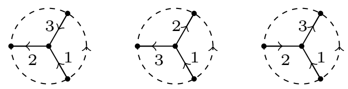
We recognize the picture associated to the face $(\beta_2 = 1)$ denoted by $F_{\beta_2=1}$ of $C(K; \boxtimes)$, and we observe that the image of the corresponding face under $G_{\circledast}$ coincides with the image of the face $S^2 \times F_{\beta_2=1}$ of $S^2 \times C(K; \boxtimes)$ under

$$G'_{\boxtimes} = 1_{S^2} \times G_{\boxtimes} \colon S^2 \times C(K; \boxtimes) \to (S^2)^3,$$

so the combination $\left(\deg(1_{S^2} \times G_{\boxtimes}) - \deg(G_{\circledast})\right)$ does not vary across the images of the corresponding faces (or at least not because of them). (There are some sign and orientation issues to check here, but they are carefully treated in a wider generality in Section 7.1 and in Lemma 9.14. Let the reader just trust the author that the signs are correct here.) We glued the images of $G'_{\boxtimes}$ and of $G_{\circledast}$ along $G'_{\boxtimes}(S^2 \times F_{\beta_2=1})$ to make the union of these images behave as the image of a manifold without boundary, locally. Unfortunately, the three other faces of $S^2 \times C(K; \boxtimes)$ created other walls in $(S^2)^3$ associated to the following figures:



In order to cancel those walls with the same type of faces of $C(K; \circledast)$ as before, we use Gauss maps associated with the following diagrams:

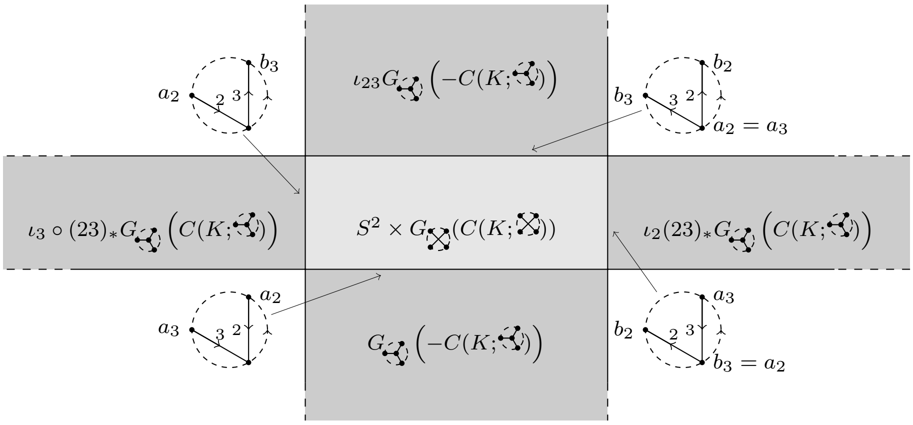


Let us describe these Gauss maps more precisely. For any subset I of $\underline{3}$, let ι_I denote the diffeomorphism of $(S^2)^3$ which maps (X_1, X_2, X_3) to $(\varepsilon_1 X_1, \varepsilon_2 X_2, \varepsilon_3 X_3)$, where $\varepsilon_i = -1$ when $i \in I$ and $\varepsilon_i = 1$ when $i \notin I$. For two elements i, j of $\underline{3}$ simply write $\iota_i = \iota_{\{i\}}$ and $\iota_{ij} = \iota_{\{i,j\}}$. For a permutation σ of $\underline{3}$, σ_* denotes the diffeomorphism of $(S^2)^3$ which maps (X_1, X_2, X_3)

to $(X_{\sigma^{-1}(1)}, X_{\sigma^{-1}(2)}, X_{\sigma^{-1}(3)})$. The Gauss maps associated to the above diagrams are $\iota_2 \circ (23)_* \circ G$, $\iota_{23} \circ G$ and $\iota_3 \circ (23)_* \circ G$, respectively, and the combination

$$\begin{array}{ll} \deg(1_{S^2} \times G) & -\deg(G) - \deg(\iota_{23} \circ G) \\ & +\deg(\iota_2 \circ (23)_* \circ G) + \deg(\iota_3 \circ (23)_* \circ G) \end{array}$$

does not vary across the boundary of the image of $1_{S^2} \times G$ as the following (very simplified) picture in $(S^2)^3$ (pictured as 2-dimensional) suggests.



Of course, doing so, we introduce other walls and we have not yet cancelled the walls due to the faces $c(v_2) = c(w)$ and $c(v_3) = c(w)$. However, the following proposition can be proved.

Proposition 1.5. *The map*

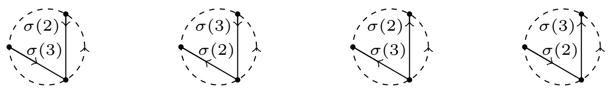
$$\begin{array}{l} \frac{1}{24}\sum_{\sigma\in\mathfrak{S}_3} deg\left(\sigma_*\left(1_{S^2} \times G\right)\right) \\ -\frac{1}{48}\sum_{I\subseteq \underline{3}}(-1)^{\sharp I}\left(deg(\iota_I \circ G) + deg(\iota_I \circ (23)_* \circ G)\right), \end{array}$$

which is well defined on an open dense subset of $(S^2)^3$, extends as a constant function of $(S^2)^3$ whose value $w_2(K)$ is in $\frac{1}{48}\mathbb{Z}$.

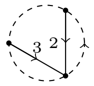
Sketch of proof: Let us just prove that the boundary

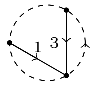
$$\sigma_*\left(S^2 \times G(\partial C(K;\ ))\right)$$

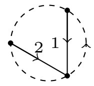
can be glued to the images of the faces $c(v_i) = c(w)$ of $C(K;\ )$ under the $\iota_I \circ G$ or the $\iota_I \circ (23)_* \circ G$, up to sign. For every permutation σ of $\underline{3}$, the boundary $\sigma_*\left(S^2 \times G(\partial C(K;\ ))\right)$ consists of the four faces



where the first one and the third one come with a coefficient $-\frac{1}{24}$, and the second one and the fourth one come with a coefficient $\frac{1}{24}$. The images of the (open) faces $(c(v_1) = c(w))$, $(c(v_2) = c(w))$ and $(c(v_3) = c(w))$ under G are

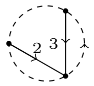


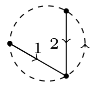


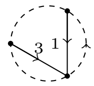
and


, respectively,

while the images of the (open) faces $(c(v_1) = c(w))$, $(c(v_2) = c(w))$ and $(c(v_3) = c(w))$ under $(23)_* \circ G$ are





and


, respectively.

In order to obtain the images under the compositions of these maps by some ι_I, we reverse the edge i when $i \in I$.

Furthermore, each of these faces appears twice (once for each orientation of the collapsed edge), with the same sign (since the antipodal map of S^2 reverses the orientation) with a coefficient $(-1)^{n(F)}\frac{1}{48}$, where $n(F)$ is the number of edges towards the bivalent vertex.

We leave the general discussion of signs to the reader and we do not discuss all the faces of $C(K;\,)$ since this proposition is a corollary of Theorem 7.32, which is proved with much more care, as it will be seen right after Theorem 7.32. Let us just mention that the image of the faces $c(v_1) = c(v_2)$ is contained in the codimension–two subspace of $(S^2)^3$, for which at least two S^2-coordinates are equal or opposite, so these faces do not create walls and may be forgotten. $\square$

Since the σ_* and the ι_I, when $\sharp I$ is even, preserve the volume of $(S^2)^3$, and since the ι_I multiply the volume by (-1), when $\sharp I$ is odd, the combination $w_2(K)$ in Proposition 1.5 may also be written as

$$w_2(K) = \frac{1}{4}\int_{C(K;\,)} G^*\left(\wedge_{i=1}^2 p_i^*(\omega_{S^2})\right) - \frac{1}{3}\int_{C(K;\,)} G^*\left(\wedge_{i=1}^3 p_i^*(\omega_{S^2})\right).$$

Since $w_2(.)$ is valued in $\frac{1}{48}\mathbb{Z}$, and since $w_2(K)$ varies continuously under an isotopy of K, $w_2(.)$ is an isotopy invariant. This invariant was first independently studied by Guadagnini, Martellini and Mintchev [GMM90] and Bar-Natan in 1990, under this integral form [BN95b]. In order to prove the isotopy invariance, one can alternatively use Stokes' theorem, together with compactifications, as those discussed above, of the one-parameter configuration space $\cup_{t\in[0,1]}C(K_t;\Gamma)$, to evaluate the variations of the integrals under a knot isotopy, as Bott and Taubes did in [BT94].

The above formulation of Proposition 1.5 presents the invariant $w_2(K)$ as a discrete count of configurations of and , as Dylan Thurston [Thu99] and Sylvain Poirier [Poi02] first did[5] independently. For a generic triple (a, b, c) of $(S^2)^3$, we count the configurations of , for which the edge directions are a pair of non-colinear vectors in $\{a, b, c, -a, -b, -c\}$, and the configurations of , for which the edge directions are a triple of pairwise non-colinear vectors in $\{a, b, c, -a, -b, -c\}$, with some coefficients and some signs determined by the corresponding local degrees above.

Since any knot of $\mathbb{R}^3$ is obtained from the trivial knot by (isotopies and) a finite number of crossing changes $\to$, in order to determine a knot invariant w, it suffices to know its value on the trivial knot and its variation $w(\;) - w(\;)$, which is denoted by $w(\;)$, under any such crossing change. (Here, and represent knot diagrams that coincide outside a disk that they intersect as in the figure, and represents a diagram that coincides with the former diagrams outside that disk.)

The variation $w(\;) = w(\;) - w(\;)$ can be thought of as a discrete derivative of the invariant, and the variation of this variation

$$\begin{aligned} w(\;\;) &= w(\;\;) - w(\;\;) \\ &= w(\;\;) - w(\;\;) - w(\;\;) + w(\;\;) \end{aligned}$$

under a disjoint crossing change is thought of as a discrete second derivative of this invariant. A knot invariant w is actually determined by its value on the trivial knot and the discrete second derivative. (If the second derivative is zero, the variation under a crossing change is independent of the knot and it is the same as $(w(\;) - w(\;) = 0)$.) In [BN95b] Bar-Natan computed this "discrete second derivative" for the invariant w_2 and he found:

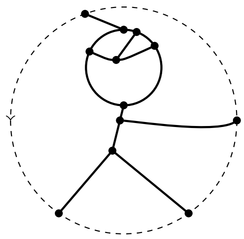
$$w_2(\;\;) = 0 \quad \text{and} \quad w_2(\;\;) = 1,$$

where the dashed lines indicate the connections inside K of the crossing strands. This allowed him to identify w_2 with $a_2 - \frac{1}{24}$, where $a_2(K) = \frac{1}{2}\Delta''(K)(1)$ is half the second derivative of the Alexander polynomial of K at one, since a_2 has the same discrete second derivative as w_2, $a_2(O) = 0$ and $w_2(O) = -\frac{1}{24}$ as Guadagnini, Martellini and Mintchev computed in [GMM90]. This is reproved in Examples 7.11 and Lemma 7.12.

[5]Polyak and Viro obtained a similar result [PV01, Theorem 3.A, Section 3.5] in the setting of long knots of $\mathbb{R}^3$, with fewer involved gluings, and with induced combinatorial formulae in terms of knot diagrams.

1.2.6 On other similar invariants of knots in $\mathbb{R}^3$

We can associate similar integrals over configuration space to every unitrivalent graph Γ whose univalent vertices are ordered cyclically, as in the following figure:



and exhibit other similar combinations that provide isotopy invariants of knots in $\mathbb{R}^3$, as several authors, including Maxim Kontsevich [Kon94], Daniel Altschüler and Laurent Freidel [AF97], Dylan Thurston [Thu99], did. These invariants are all finite type invariants with respect to the following definition. An invariant is of *degree less than* n if all its discrete derivatives of order n, which generalize the previously studied discrete second derivative, vanish. A knot invariant is a *Vassiliev invariant* or *a finite type invariant* if it is of degree less than some integer n. (A more precise definition is given in Section 6.1.)

Furthermore, as Altschüler and Freidel [AF97]- and Thurston [Thu99] independently - proved, every real-valued finite type knot invariant is a combination of integrals over configuration spaces associated to unitrivalent graphs.

This result is based on the *fundamental theorem of Vassiliev invariants* due to Bar-Natan and Kontsevich [Kon93, BN95a], which determines the space of finite type invariants as the dual of an algebra $\mathcal{A}$ generated by unitrivalent graphs, and on the construction of a *universal Vassiliev invariant* Z of knots in $\mathbb{R}^3$, valued in $\mathcal{A}$, by Altschüler, Freidel and Thurston. For any knot K of $\mathbb{R}^3$, $Z(K)$ is a combination of classes of unitrivalent graphs whose coefficients are integrals over corresponding configuration spaces. Any real-valued finite type invariant may be expressed as $\psi \circ Z$ for some linear form $\psi \colon \mathcal{A} \to \mathbb{R}$. It is still unknown whether finite type knot invariants distinguish all knots of $\mathbb{R}^3$, but many known polynomial invariants of knots of $\mathbb{R}^3$ such as the Alexander polynomial, the Jones polynomial, the HOMFLYPT polynomial also factor through Z.

1.2.7 On similar invariants of knots in other 3-manifolds

In this book, following ideas of Kontsevich [Kon94], Greg Kuperberg and Dylan Thurston [KT99], we generalize Z to links in more general 3-manifolds

$\check{R} = \check{R}(\mathcal{C})$ constructed from $\mathbb{R}^3$ by replacing the standard cylinder $D_1 \times [0,1]$ of $\mathbb{R}^3$ with a *rational homology cylinder* $\mathcal{C}$, which is a compact oriented 3-manifold with the same boundary and the same rational homology[6] as $D_1 \times [0,1]$ (or as the point). Such a more general 3-manifold $\check{R}$ is called a *rational homology* $\mathbb{R}^3$. It looks like $\mathbb{R}^3$ near ∞ and the linking number can be defined as follows for a two-component link embedding $J \sqcup K \colon S^1 \sqcup S^1 \to \check{R}$.

Two submanifolds A and B in a manifold M are *transverse* if at each intersection point $x \in A \cap B$, $T_xM = T_xA + T_xB$. If two transverse oriented submanifolds A and B in an oriented manifold M are of *complementary dimensions* (i.e. if the sum of their dimensions is the dimension of M), then the *sign of an intersection point* is $+1$ if $T_xM = T_xA \oplus T_xB$ as oriented vector spaces. Otherwise, the sign is -1. If A and B are compact, and if A and B are of complementary dimensions in M, then their *algebraic intersection* is the sum of the signs of the intersection points, it is denoted by $\langle A, B \rangle_M$.

When K bounds a compact oriented embedded surface Σ_K in $\check{R}$ transverse to J, the *linking number* of J and K is the algebraic intersection number of J and Σ_K in $\check{R}$. In general, there is an oriented surface Σ_{nK} immersed in K whose boundary is a positive multiple nK of K and $lk(J,K) = \frac{1}{n}\langle J, \Sigma_{nK}\rangle_{\check{R}}$.

For two-component links in $\mathbb{R}^3$, this definition coincides with Definition 1.1 of the Gauss linking number. See Proposition 2.9.

In the more general setting of rational homology $\mathbb{R}^3$, instead of counting unitrivalent graphs whose edge directions belong to a finite set of directions, we use the notion of propagator, precisely defined in Chapter 3. A propagator is a rational combination of oriented compact 4-manifolds in a suitable compactification $C_2(R)$ (defined in Section 3.2) of the configuration space

$$\check{C}_2(R) = \{(x,y) \in \check{R}^2 \mid x \neq y\}$$

that shares many properties with our *model propagator*

$$p_{S^2}^{-1}(X) = \overline{\{(x, x+tX) \mid x \in \mathbb{R}^3, t \in]0, +\infty[\}},$$

for $X \in S^2$.

With these model propagators $p_{S^2}^{-1}(X)$, the direction of a configured edge $(c(x), c(y))$ is X if and only if $(c(x), c(y))$ is in the propagator $p_{S^2}^{-1}(X)$. More general propagators produce similar codimension 2 constraints on configurations, and allow us to count unitrivalent graphs whose configured edges belong to a finite set of propagators, with signs, as above.

[6]This homological condition can be rephrased by saying that $\mathcal{C}$ is connected, and that for every knot embedding K, equipped with a compact tubular neighborhood $N(K)$ in $\mathcal{C}$ whose interior is $\mathring{N}(K)$, there is a compact oriented surface in $\mathcal{C} \setminus \mathring{N}(K)$ whose boundary is a curve in $\partial N(K)$ that does not bound a compact oriented surface in $N(K)$.

In general, the boundary of a propagator of $C_2(R)$ is in the boundary of $C_2(R)$, and, for any two-component link embedding $J \sqcup K \colon S^1 \sqcup S^1 \to \check{R}$ the algebraic intersection of $J \times K \subset \check{C}_2(R)$ with a propagator in $C_2(R)$ is the linking number of J and K. (Note that the linking number of two knots in $\mathbb{R}^3$ is indeed this algebraic intersection with a model propagator.)

Let us show examples of propagators in general rational homology $\mathbb{R}^3$.

1.2.8 Morse propagators

A rational homology $\mathbb{R}^3$ can be viewed as the union of two genus g handlebodies[7] as in Figure 1.7, where the two pieces are glued to each other by an a priori non-trivial diffeomorphism of ∂H_a.

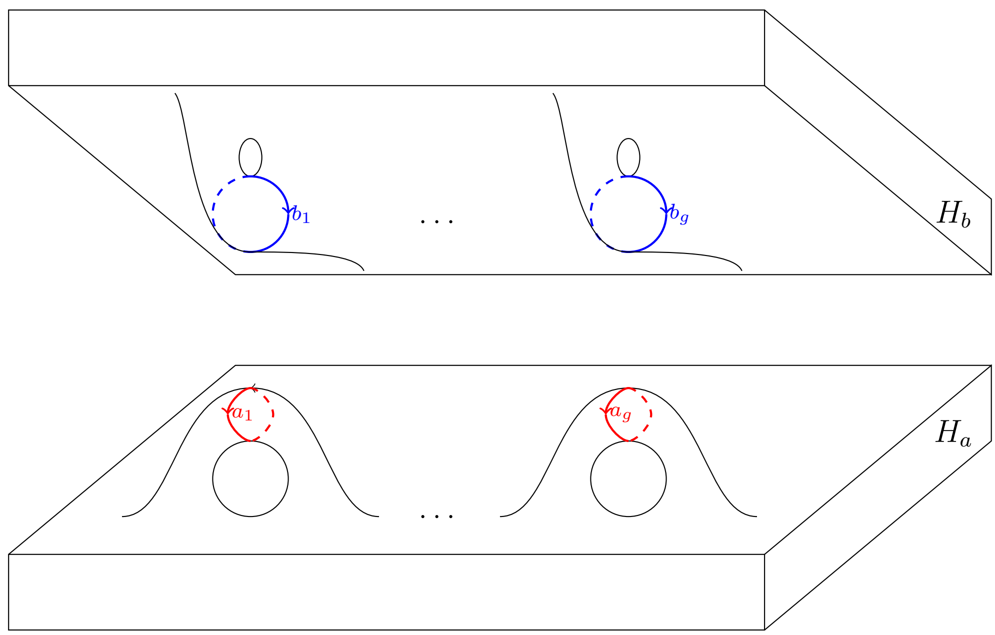


Figure 1.7: H_a and H_b

The handlebody H_a has g arbitrarily oriented meridian disks $D(a_i)$ centered at α_i, for i in $\underline{g} = \{1, 2, \dots, g\}$, and the handlebody H_b has g arbitrarily oriented meridian disks $D(b_j)$ centered at β_j, for $j \in \underline{g}$. The topology of $\check{R}$ is determined by the curves $a_i = \partial D(a_i)$ and $b_j = \partial D(b_j)$ in the surface ∂H_a. The data $(\partial H_a, (a_i)_{i \in \underline{g}}, (b_j)_{j \in \underline{g}})$ is called a *Heegaard diagram*. From such a data, we can construct

[7]Unlike the handlebodies in the rest of this book, which are as in Section 1.1, the handlebodies of this section are not compact.

- a *Morse function* $f\colon \check{R} \to \mathbb{R}$ whose *critical points* (the points at which the derivative of f vanishes) are the α_i, which have index one and are in $f^{-1}(1/3)$, and the β_j in $f^{-1}(2/3)$ whose index is 2, and such that $\partial H_a = f^{-1}(1/2)$

- a gradient vector field $\nabla\colon \check{R} \to T\check{R}$ associated to some metric $\mathfrak{g}$,

$$T_x f(y \in T_x\check{R}) = \langle \nabla(x), y\rangle_{\mathfrak{g}},$$

  and

- an associated *gradient flow*[8] $\phi\colon \mathbb{R} \times \check{R} \to \check{R}$ such that $\phi(0,.)$ is the Identity map and $\frac{\partial}{\partial t}\phi(t,x)_{(u,x)} = \nabla(\phi(u,x))$, whose *flow lines* are the $\phi(\mathbb{R} \times \{y\})$ for the y in the complement $\check{R} \setminus \mathcal{S}$ in $\check{R}$ of the set $\mathcal{S}$ of critical points of f.

For the standard height function f_0 and the standard metric of $\mathbb{R}^3$, the associated flow is
$$\left((\phi_t = \phi(t,.))\colon x \mapsto x + t\vec{N}\right),$$
where $\vec{N} = (0,0,1)$. Our rational homology $\mathbb{R}^3$ is assumed to coincide with $\mathbb{R}^3$ (equipped with its standard metric) outside $D_1 \times [0,1]$, and our Morse function (can be and) is assumed to coincide with f_0 outside $D_1 \times [0,1]$.

As a set, the complement $\check{R} \setminus \mathcal{S}$ is the disjoint union of the flow lines diffeomorphic to $\mathbb{R}$, which behave as follows. There are two flow lines $\mathcal{L}_+(\alpha_i)$ and $\mathcal{L}_-(\alpha_i)$ starting as vertical lines and ending at α_i that approach α_i at $+\infty$, as in Figure 1.8. (They start as vertical lines $\{.\}\times]-\infty,0[$.) The closure of their union is a line $\mathcal{L}(\alpha_i)$, which is called the *descending manifold* of α_i. It is oriented so that its algebraic intersection with $D(a_i)$ is 1. (This is not consistent with the orientation from bottom to top of one of the flow lines. If the flow line with the orientation of the positive normal to $D(a_i)$ is $\mathcal{L}_+(\alpha_i)$, then $\mathcal{L}(\alpha_i) = \overline{\mathcal{L}_+(\alpha_i)} \cup (-\mathcal{L}_-(\alpha_i))$.) There are two flow lines $\mathcal{L}_+(\beta_j)$ and $\mathcal{L}_-(\beta_j)$ that approach β_j at $-\infty$. The closure of their union is a line $\mathcal{L}(\beta_j)$, which is called the *ascending manifold* of β_j. It is oriented so that its algebraic intersection with $D(b_j)$ is 1.

The closure of the union of the flow lines that approach α_i at $-\infty$ is called the *ascending manifold* of α_i and is denoted by $\mathcal{A}_i$. Its intersection with H_a is $D(a_i)$ and it is oriented like $D(a_i)$. The closure of the union of the flow lines that approach β_j at $+\infty$ is called the *descending manifold* of β_j and is denoted by $\mathcal{B}_j$. Its intersection with H_b is $D(b_j)$ and it is oriented like $D(b_j)$. The ascending manifold $\mathcal{A}_i$ is an immersion of Figure 1.9, which

[8]The definition of this flow is justified in [Spi79, Chapter5], for example.

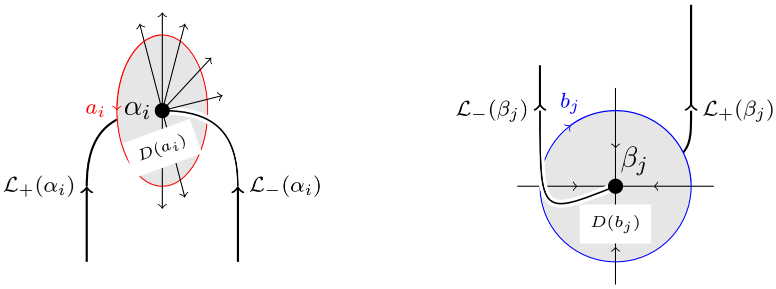


Figure 1.8: $\mathcal{L}_+(\alpha_i)$, $\mathcal{L}_-(\alpha_i)$, $\mathcal{L}_+(\beta_j)$, $\mathcal{L}_-(\beta_j)$

restricts to its interior as an embedding, where the flow lines $\gamma(c_k)$ are flow lines through crossings c_k of $a_i \cap b_j$, which approach α_i near $-\infty$ and β_j near $+\infty$. A figure for $\mathcal{B}_j$ is obtained by reversing the arrows, and by changing α_i to β_j.

Except for the flow lines of the descending manifolds of the α_i and the flow lines of the ascending manifolds of the β_j, each flow line intersects ∂H_a once, transversally, with a positive sign.

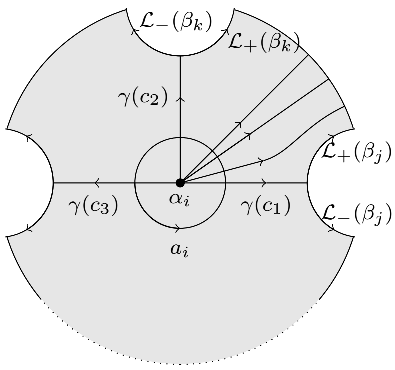


Figure 1.9: The interior of $\mathcal{A}_i$.

The flow lines that are not in the above descending or ascending manifolds begin as vertical half-lines $x \times]-\infty, 0[$ and end as vertical half-lines $y \times]1, \infty[$ for some x, y in $\mathbb{R}^2$. Except for the critical points, every point has a neighborhood diffeomorphic to a cube $]0,1[^3$, such that, with the induced identification, the flow maps (t,x) to $x + t\vec{N}$, for any (t,x) in $\mathbb{R} \times]0,1[^3$ such that $x + t\vec{N} \in]0,1[^3$.

Examples of propagators can be constructed from the gradient flow ($\phi_t = \phi(t,.)$) of a Morse function f without minima and maxima as above, as Greg

Kuperberg and the author did in [Les15a, Theorem 4.2]. Let P_ϕ denote the closure in $C_2(R)$ of $\{(x, \phi_t(x)) \mid x \in \check{R} \setminus \mathcal{S}, t \in]0, +\infty[\}$. Note that when $(\phi_t \colon x \mapsto x + t\vec{N})$ is the flow associated to the standard height function f_0 of $\mathbb{R}^3$, $P_\phi = p_{S^2}^{-1}(\vec{N})$ is one of our model propagators. Let

$$[\mathcal{J}_{ji}]_{(j,i)\in\{1,\dots,g\}^2} = [\langle a_i, b_j\rangle_{\partial H_a}]^{-1}$$

be the inverse matrix of the matrix of the algebraic intersection numbers $\langle a_i, b_j\rangle_{\partial H_a}$. (This matrix is invertible because of our homological conditions on $\check{R}$.)

Let $((\mathcal{B}_j \times \mathcal{A}_i) \cap C_2(R))$ denote the closure of $\big((\mathcal{B}_j \times \mathcal{A}_i) \cap (\check{R}^2 \setminus \text{diagonal})\big)$ in $C_2(R)$, then

$$P(f, \mathfrak{g}) = P_\phi + \sum_{(i,j)\in\{1,\dots,g\}^2} \mathcal{J}_{ji}\left((\mathcal{B}_j \times \mathcal{A}_i) \cap C_2(R)\right)$$

is an example of a propagator.

Pick four small generic perturbations P_1, P_2, P_3 and P_4 of such a propagator, (or of more general propagators as precisely defined in Section 3.3), the invariant w_2 can be extended to knots K in $\check{R}$ as follows. For $\Gamma =$, or , when the edges of Γ are oriented and numbered by the data of an injection j_E from the set $E(\Gamma)$ of the (plain) edges of Γ to $\underline{4}$, we can again count the configurations of Γ such that the configured oriented edge numbered by i (viewed as the ordered pair of its ends) is in P_i, with signs precisely defined. Denote the average over the choices of such edge-orientations and numberings by $I_a(K, \Gamma, (P_1, P_2, P_3, P_4))$, then

$$\begin{aligned} w_2(K) = \quad & I_a(K, \ , (P_1, P_2, P_3, P_4)) - I_a(K, \ , (P_1, P_2, P_3, P_4)) \\ & -2I_a(K, \ , (P_1, P_2, P_3, P_4)) \end{aligned}$$

does not depend on the chosen propagators, and it is again $\frac{1}{2}\Delta''(K)(1) - \frac{1}{24}$, if K is null-homologous, in this more general setting as we prove[9] in Theorem 18.41. When averaging, we divide by the number $2^{\sharp E(\Gamma)}$ of edge-orientations and by the number $\frac{4!}{(4-\sharp E(\Gamma))!}$ of numberings. Note that when we compute $w_2(K \subset \mathbb{R}^3)$ with model propagators, $I_a(K, \ , (P_1, P_2, P_3, P_4))$ vanishes because of the double edge, which gives contradictory constraints.

[9]David Leturcq generalized this result in [Let20], where he completely expresses the Alexander polynomial of null-homologous knots in $\mathbb{Q}$-spheres in terms of similar counts of configurations.

1.2.9 More about the contents of the book

The cited Altschüler-Freidel universal Vassiliev invariant of knots in $\mathbb{R}^3$ also extends to knots in rational homology $\mathbb{R}^3$. We describe that invariant Z in that generality in this book, with more general (and precisely defined) propagators. The above way of counting configurations in rational homology $\mathbb{R}^3$ was first proposed by Fukaya in [Fuk96] and further studied and made rigorous by other authors including Watanabe [Wat18a]. In this book, it can be viewed as a particular way of counting with the above propagators associated to Heegaard diagrams, which are only particular cases of the general notion of propagating chain described in Chapter 3. In this case, counts of configurations as above also yield invariants of rational homology 3-spheres, which are connected oriented closed[10] 3-manifolds, where knots have a nontrivial multiple which bounds an immersed oriented compact surface. They even yield a universal finite type invariant $\mathcal{Z}$ for these manifolds, with respect to the theory of finite type invariants developed by Delphine Moussard in [Mou12], for which the crossing changes are replaced with surgery operations –namely the *rational Lagrangian-preserving surgeries* described in Section 1.3.2 below– which replace a piece of a manifold with another such with respect to some constraints defined in Sections 1.1 and 1.3.2, thanks to surgery formulae proved in [Les04b]. The invariant $\mathcal{Z}$ also restricts to a universal finite type invariant for integer homology 3-spheres, which are connected oriented closed 3-manifolds where knots bound an embedded oriented compact surface, as in the standard 3-sphere S^3, with respect to the Ohtsuki theory of finite type invariants [Oht96] and other equivalent theories described in [GGP01], as first announced by Kuperberg and Thurston in [KT99].

Let us say a little more about the contents of this book, which is mostly self-contained. The used background material is described in the appendices. Unlike in this informal introduction, details and precise statements are given, and all the assertions are checked carefully. In order to define our general invariant Z, which is an infinite series of independent non-trivial invariants, we need to equip our rational homology $\mathbb{R}^3$ with parallelizations $\tau \colon \check{R} \times \mathbb{R}^3 \to T\check{R}$. The space of these parallelizations up to homotopy and the associated Pontrjagin numbers are described in Chapter 5.

This book has four parts. In its present first part, after stating conventions and known facts about 3-manifolds, we define propagating chains associated with parallelizations precisely, and we discuss the invariant Θ of parallelized rational homology 3–spheres. This invariant Θ is associated to

[10] A manifold is said to be *closed* if it is compact and connected, and if its boundary is empty.

the graph $\ominus$ and defined as the algebraic intersection of three transverse propagating chains. We explain how to get rid of the dependence of the parallelization of the rational homology 3-spheres with the help of relative Pontrjagin numbers to get an invariant of (unparallelized) rational homology 3-spheres in Section 4.3.

Our general invariant $\mathcal{Z}^f$ is an invariant of parallelized links[11] in rational homology 3-spheres, defined in the second part of the book. It takes its values in vector spaces generated by unitrivalent graphs as $\ominus$,  and . These spaces of diagrams have rich structures, which are useful to describe the properties of the invariants. They are described in Chapter 6. The general definition of $\mathcal{Z}^f$ for links is given in Chapter 7. It is first given in terms of integrals rather than in terms of discrete counts, because the results are easier to write and prove in the world of differential forms, where no genericity hypotheses are required. The proof that this definition is consistent as well as the first properties of $\mathcal{Z}^f$ are given in Chapters 9, 10 and 11, after the needed study of the compactifications of the involved configuration spaces in Chapter 8.

In order to compute and use an invariant of links or manifolds, it is interesting to cut links or manifolds into elementary pieces and understand how the invariant can be recovered from invariants of the small pieces. In the third part of this book, we achieve this task with elementary pieces that are *tangles* in rational homology cylinders as in Figure 1.10. These tangles, which are cobordisms between planar configurations are precisely defined in Section 13.1. They may be composed in many ways, horizontally and verti-

$L(\mathcal{L}) =$

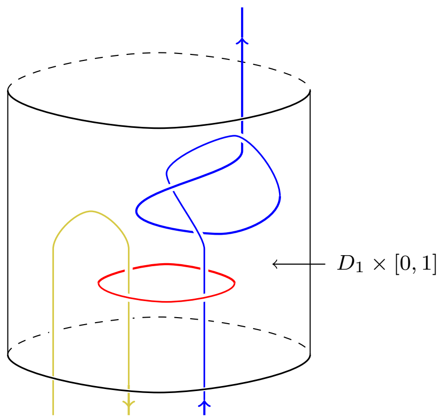


Figure 1.10: A tangle representative in $D_1 \times [0, 1]$

[11] *Framed links* or *parallelized links* are links equipped with a parallel (up to isotopy).

cally under some hypotheses, and by insertions in tubular neighborhoods of other tangle representatives. Such insertions are called *cablings*. We generalize $\mathcal{Z}^f$ to tangles and we describe the properties of our generalized $\mathcal{Z}^f$ under these compositions in the third part of this book.

As already mentioned, a fundamental property of $\mathcal{Z}^f$ is its universality among finite type invariants, when restricted to links in $\mathbb{R}^3$, with respect to the Vassiliev theory of finite type invariants based on crossing changes, and, when restricted to rational homology spheres, with respect to the Moussard theory based on rational Lagrangian-preserving surgeries. The proofs of universality involve computations of iterated discrete derivatives of $\mathcal{Z}^f$ in the same spirit as the Bar-Natan result recalled in the end of Section 1.2.5. These computations and some of their consequences are presented in the fourth part of the book for the general invariant $\mathcal{Z}^f$ of framed tangles in rational homology cylinders.

1.3 A quicker introduction

In this quicker introduction, which is independent of the first one, for experienced topologists, and which can also be read by beginners after the warm-up of the slower one, we describe the invariant $\mathcal{Z}$ of n-component links L in rational homology 3-spheres R, valued in a graded space generated by unitrivalent graphs, studied in this book, more precisely. We also specify some notions vaguely introduced in the slow introduction of Section 1.2 and we say more on the mathematical landscape around $\mathcal{Z}$.

1.3.1 On the construction of $\mathcal{Z}$

The invariant $\mathcal{Z}(L)$ is a sum $\mathcal{Z}(L) = \sum_\Gamma \mathcal{Z}_\Gamma(L)[\Gamma]$, running over unitrivalent graphs Γ, where the coefficient $\mathcal{Z}_\Gamma(L)$ "counts" embeddings of Γ in R that map the univalent vertices of Γ to L, in a sense that will be explained in the book, using integrals over configuration spaces, or, in a dual way, algebraic intersections in the same configuration spaces. Let us slightly specify that sense.

For technical reasons, we will remove a point ∞ from our rational homology 3-spheres R to transform them into open manifolds $\check{R}$. When R is the standard sphere S^3, $\check{R} = \mathbb{R}^3$.

Let $\Delta(\check{R}^2)$ denote the diagonal of $\check{R}^2$. Following Fulton, McPherson [FM94], Kontsevich [Kon94], Axelrod, Singer [AS94, Section 5] and others, we will introduce a suitable smooth compactification $C_2(R)$ (with boundary and ridges) of $\check{R}^2 \setminus \Delta(\check{R}^2)$ such that the map

$$\begin{array}{llll} p_{S^2}\colon & \left(\mathbb{R}^3\right)^2 \setminus \Delta\left(\left(\mathbb{R}^3\right)^2\right) & \to & S^2 \\ & (x,y) & \mapsto & \frac{1}{\|y-x\|}(y-x) \end{array}$$

extends to $C_2(S^3)$. We will introduce a notion of *propagating chain* and the dual notion of *propagating form* for R. A propagating chain is a 4-dimensional rational chain (i.e. a finite rational combination of oriented compact 4-manifolds with possible corners) of $C_2(R)$, while a propagating form is a closed 2–form on $C_2(R)$. Both of them have to satisfy some conditions on the boundary of $C_2(R)$, which make them share sufficiently many properties with our *model propagating chains* $p_{S^2}^{-1}(X)$, for $X \in S^2$, and our *model propagating forms* $p_{S^2}^*(\omega_S)$ for 2-forms ω_S on S^2 such that $\int_{S^2} \omega_S = 1$, which are defined when $R = S^3$. Propagating forms and propagating chains will both be called *propagators* when their nature is clear from the context.

In particular, any propagating chain P and any propagating form ω satisfies the following property. For any two-component link $(J,K)\colon S^1 \sqcup S^1 \to \check{R}$,

$$\int_{J\times K \subset C_2(R)} \omega = \langle J \times K, P\rangle_{C_2(R)} = lk(J,K)$$

where $\langle .,. \rangle_{C_2(R)}$ stands for the algebraic intersection in $C_2(R)$ and lk is the linking number in R. The above equalities tell us in which way

propagators represent the linking form.

A *Jacobi diagram* Γ on $\coprod_{i=1}^n S^1$ is a unitrivalent graph Γ equipped with an isotopy class of injections from its set $U(\Gamma)$ of univalent vertices into the source $\coprod_{i=1}^n S^1$ of a link L. Let $V(\Gamma)$, $T(\Gamma)$ and $E(\Gamma)$ denote the set of

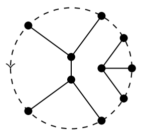


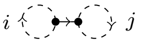
Figure 1.11: A (plain) Jacobi diagram on (the dashed) S^1

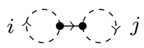
vertices, trivalent vertices, and edges of Γ, respectively. The *configuration space* $\check{C}(R,L;\Gamma)$ of injections from $V(\Gamma)$ to R that map the set $U(\Gamma)$ of univalent vertices of Γ to L, and induce the above isotopy class of injections, is an open submanifold of $\check{R}^{T(\Gamma)} \times L^{U(\Gamma)}$. When the edges of Γ are oriented, each edge e of Γ provides a natural restriction map $p(\Gamma,e)$ from $\check{C}(R,L;\Gamma)$

to $\check{R}^2 \setminus \Delta(\check{R}^2)$. When propagating forms $\omega(e)$ are associated to the edges, this allows one to define a real number

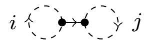
$$I(R, L, \Gamma, (\omega(e))_e) = \int_{\check{C}(R,L;\Gamma)} \bigwedge_{e \in E(\Gamma)} p(\Gamma, e)^*(\omega(e)).$$

Similarly, and dually, when propagating chains $P(e)$ in general position are associated to the edges, one can define a rational number $I(R, L, \Gamma, (P(e))_e)$ as the algebraic intersection of the codimension 2-chains $p(\Gamma, e)^{-1}(P(e))$ in $\check{C}(R, L; \Gamma)$.

For example, according to the given property of our propagators, when Γ is , which is an edge with a univalent vertex that must go to the component K_i of L and with its other univalent vertex that must go to another component K_j of L, the associated configuration space $\check{C}(R, L; \Gamma)$ is $K_i \times K_j$, and we have

$I(R, L,$  $, \omega) = \int_{K_i \times K_j \subset C_2(R)} \omega = lk(K_i, K_j)$

and

$I(R, L,$  $, P) = \langle K_i \times K_j, P \rangle_{C_2(R)} = lk(K_i, K_j)$

for any propagating form ω, and for any propagating chain P.

As another example, when Γ is the graph $\ominus$ with two trivalent vertices and three edges e_1, e_2 and e_3 from one vertex to the other one, and when R is a $\mathbb{Z}$-sphere, we will show how the propagators can be chosen so that

$$I(R, \emptyset, \ominus, (\omega(e))_{e \in \{e_1, e_2, e_3\}}) = I(R, \emptyset, \ominus, (P(e))_{e \in \{e_1, e_2, e_3\}}) = 6\lambda_{CW}(R),$$

where λ_{CW} is the Casson invariant normalized as in [AM90, GM92, Mar88]. The above propagating forms can be chosen so that $\omega(e_1) = \omega(e_2) = \omega(e_3)$. In particular, the Casson invariant, which may be written as

$$\lambda_{CW}(R) = \frac{1}{6} \int_{C_2(R)} \omega(e_1)^3,$$

may be viewed as a "cube of the linking number".

The real coefficient $\mathcal{Z}_\Gamma(L)$ in

$$\mathcal{Z}(L) = \sum_\Gamma \mathcal{Z}_\Gamma(L)[\Gamma]$$

is the product of $I(R, L, \Gamma, (\omega)_e)$ and a constant which depends only on the graph combinatorics, for a well-chosen propagating form ω. It can also be obtained by averaging some $I(R, L, \Gamma, (P(e))_e)$ over ways of equipping edges of Γ by propagating chains in a fixed set of generic propagating chains, and over ways of orienting the edges of Γ. The coefficients $\mathcal{Z}_\Gamma(L)$ depend on propagator choices, but relations among Jacobi diagrams in the target space $\mathcal{A}(\coprod_{i=1}^n S^1)$ of $\mathcal{Z}$ –which is generated by Jacobi diagrams– ensure that $\mathcal{Z}(L)$ is an isotopy invariant. The invariant $\mathcal{Z}$ may be thought of as a series of higher order linking invariants.

The definition of $\mathcal{Z}$ that is presented here is a generalization of the definition that was presented in details for $\mathbb{Q}$-spheres by the author in the unsubmitted preprint [Les04a], which was inspired by [KT99] and discussions with Dylan Thurston in Kyoto in 2001.

The present definition also includes links and tangles in $\mathbb{Q}$-spheres. Most of the additional arguments involved in the construction for links already appear in many places, but we repeat them to make the book as self-contained as possible. We present many variants of the definitions and make them as flexible as possible, because the flexibility has proved useful in many generalizations and applications of these constructions, such as equivariant constructions in [Les11, Les13], or the recent explicit computations of integrals over configuration spaces by David Leturcq [Let21a, Let20], which allowed him to get an expression of the Alexander polynomial of knots in $\mathbb{Q}$-spheres in terms of such integrals, in addition to the applications presented in this book.

In particular, with our flexible definition of a propagating chain, there is a natural propagating chain associated to a generic Morse function on a punctured $\mathbb{Q}$-sphere, and to a generic metric, as described in Section 1.2.8. The main part of such a propagator is the space of pairs of points on a gradient line such that the second point is after the first one. Such a Morse propagator was constructed by the author and Greg Kuperberg in [Les15a] for Morse functions without minima or maxima. Independent work of Tadayuki Watanabe [Wat18a] allows one to generalize these propagators to any Morse function. Thus, up to some corrections, $\mathcal{Z}$ counts embeddings of graphs whose edges embed in gradient lines of Morse functions as in a Fukaya article [Fuk96]. Similar constructions have been used by Watanabe in his recent construction of exotic homotopy classes of the group of diffeomorphisms of S^4 [Wat18b]. This book contains a complete framework to study these questions precisely.

We will also show how the construction of $\mathcal{Z}$ extends to tangles in rational homology cylinders, so that $\mathcal{Z}$ extends to a functor $\mathcal{Z}^f$ on a category of framed tangles with many important properties, which provide tools to reduce the computation of $\mathcal{Z}^f$ to its evaluation at elementary pieces. These properties of the functor $\mathcal{Z}^f$ are stated in Theorem 13.12, which is one of the main original theorems in this book.

1.3.2 More mathematical context

Finite type invariants The *finite type invariant* concept for knots was introduced in the 90's in order to classify knot invariants, with the work of Vassiliev, Goussarov and Bar-Natan, shortly after the birth of numerous quantum knot invariants, which are described by Turaev in [Tur10]. This very useful concept was extended to 3–manifold invariants by Ohtsuki [Oht96]. Theories of finite type invariants in dimension 3 are defined from a set $\mathcal{O}$ of operations on links or 3-manifolds. In the case of links in $\mathbb{R}^3$, $\mathcal{O}$ is the set $\mathcal{O}_V$ of crossing changes $\leftrightarrow$. The variation of an invariant λ under an operation of $\mathcal{O}$ may be thought of as a discrete derivative. When k independent operations $o_1, \dots, o_k$ on a pair (R, L) consisting of a link L in a $\mathbb{Q}$–sphere R are given, for a part I of $\{1, \dots, k\}$ with cardinality $\sharp I$, the pair $(R, L)((o_i)_{i \in I})$ is the pair obtained from (R, L) by applying the operations o_i for $i \in I$. Then the alternate sum

$$\sum_{I \subseteq \{1,\dots,k\}} (-1)^{\sharp I} \lambda((R, L)((o_i)_{i \in I}))$$

may be thought of as the k^{th} derivative of λ with respect to $\{o_1, \dots, o_k\}$ at (R, L). An *invariant of degree at most* k with respect to $\mathcal{O}$ is an invariant all degree $k+1$ derivatives of which vanish. A *finite type invariant* with respect to $\mathcal{O}$ is an invariant that is of degree at most k for some positive integer k. Finite type invariants of links in $\mathbb{R}^3$ with respect to the set of crossing changes are called *Vassiliev invariants*.

In this case of links in $\mathbb{R}^3$, Altschüler and Freidel [AF97] proved that the invariant $\mathcal{Z}$ described in this book is a *universal Vassiliev invariant*, meaning that all real-valued Vassiliev invariants of links in $\mathbb{R}^3$ factor through $\mathcal{Z}$. Since all the quantum invariants of [Tur10] can be viewed as sequences of finite type invariants, $\mathcal{Z}$ also contains all these invariants such as the Jones polynomial, its colored versions, the HOMFLY polynomial... Dylan Thurston proved similar universality results in [Thu99] independently, and he also proved that $\mathcal{Z}$ is rational. Further substantial work of Poirier in [Poi00] allowed the author to identify the invariant $\mathcal{Z}$ with the famous Kontsevich integral of links in $\mathbb{R}^3$ –described in [BN95a] and in [CDM12] by Chmutov, Duzhin, and Mostovoy– up to a change of variables described in [Les02] in terms of an "anomaly", which is sometimes called the Bott and Taubes anomaly.

The boundary ∂A of a genus g $\mathbb{Q}$-handlebody is the closed oriented genus g surface. The *Lagrangian* $\mathcal{L}_A$ of a compact 3-manifold A is the kernel of the map induced by the inclusion from $H_1(\partial A;\mathbb{Q})$ to $H_1(A;\mathbb{Q})$. (In Figure 1.1 of H_g, the Lagrangian of H_g is freely generated by the classes of the curves a_i.)

An *integral (resp. rational) Lagrangian-Preserving (or LP) surgery*

$$(A'/A)$$

is the replacement of an integral (resp. rational) homology handlebody A embedded in the interior of a 3-manifold M with another such A' whose boundary is identified with ∂A by an orientation-preserving diffeomorphism that sends $\mathcal{L}_{A'}$ to $\mathcal{L}_A$.

Theories of finite type invariants of integer (resp. rational) homology 3–spheres R can be defined from the set $\mathcal{O}_{\mathcal{L}}^{\mathbb{Z}}$ (resp. $\mathcal{O}_{\mathcal{L}}^{\mathbb{Q}}$) of integral (resp. rational) LP-surgeries. For $\mathbb{Z}$-spheres, results of Habiro [Hab00], Garoufalidis, Goussarov and Polyak [GGP01], and Auclair and the author [AL05] imply that the theory of real-valued finite type invariants with respect to $\mathcal{O}_{\mathcal{L}}^{\mathbb{Z}}$ is equivalent to the original theory defined by Ohtsuki in [Oht96] using surgeries on algebraically split links.

Greg Kuperberg and Dylan Thurston first showed that the restriction of $\mathcal{Z}$ to integer homology 3-spheres (equipped with empty links) is a *universal finite type invariant* of $\mathbb{Z}$–spheres with respect to $\mathcal{O}_{\mathcal{L}}^{\mathbb{Z}}$ in [KT99].

As in the case of links in $\mathbb{R}^3$, their proof of universality rests on a computation of the k^{th} derivatives of the degree k part $\mathcal{Z}_k$ of the invariant $\mathcal{Z} = (\mathcal{Z}_k)_{k\in\mathbb{Z}}$, which proves that $\mathcal{Z}_k$ is a degree k invariant whose k^{th} derivatives are universal in the following sense. All the k^{th} derivatives of degree k real-valued invariants factor through them.

The "Universality part" of this book will be devoted to a general computation of the k^{th} derivatives of the extension of $\mathcal{Z}_k$ to tangles with respect both to $\mathcal{O}_V$ and $\mathcal{O}_{\mathcal{L}}^{\mathbb{Q}}$ (which contains $\mathcal{O}_{\mathcal{L}}^{\mathbb{Z}}$). Their result stated in Theorem 17.32 and in Theorem 18.5 are crucial properties of $\mathcal{Z}$. Theorem 18.5 is one of the main original results of this book.

The splitting formulae, which compute the k^{th} derivatives of the degree k part $\mathcal{Z}_k$ with respect to $\mathcal{O}_{\mathcal{L}}^{\mathbb{Q}}$, were first proved by the author in [Les04b] (for the restriction of $\mathcal{Z}_k$ to $\mathbb{Q}$-spheres). They allowed Delphine Moussard to classify finite type invariants of $\mathbb{Q}$–spheres with respect to $\mathcal{O}_{\mathcal{L}}^{\mathbb{Q}}$ in [Mou12] and to prove that, when associated with the p-valuations of the cardinality $\sharp H_1$ of the torsion first homology group, $\mathcal{Z}$ is a *universal finite type invariant* of $\mathbb{Q}$–spheres with respect to $\mathcal{O}_{\mathcal{L}}^{\mathbb{Q}}$. Together with results of Gwénaël Massuyeau [Mas14] who proved that the LMO invariant of Le, Murakami and Ohtsuki [LMO98] satisfies the same formulae, the Moussard classification implies that $\mathcal{Z}$ and Z_{LMO} are equivalent in the sense that they distinguish the same $\mathbb{Q}$–spheres with identical $\sharp H_1$.

Thus, the invariant $\mathcal{Z}$ is as powerful as the famous LMO invariant for $\mathbb{Q}$-spheres, and as the famous Kontsevich integral for links. The LMO invariant and its generalizations for links in $\mathbb{Q}$-spheres [LMO98, BNGRT02a, BNGRT02b, BNGRT04] are defined from the Kontsevich integral of links in $\mathbb{R}^3$, in a combinatorial way. Any compact oriented 3-manifold can be presented by a framed link of $\mathbb{R}^3$, which is a link equipped with a favourite parallel, according to a theorem proved by Lickorish and Wallace, independently, and nicely reproved by Rourke in [Rou85]. The *Kirby moves* are specific modifications of framed links that do not change the presented manifold. According to a theorem of Kirby, two framed links present the same manifold if and only if they are related by a finite sequence of Kirby moves. The LMO invariant of a 3-manifold is defined from the Kontsevich integral of a framed link that presents such a manifold. The proof of its invariance relies on the cited Kirby theorem.

When restricted to braids, the Kontsevich integral has a natural geometric meaning. It measures how the strands turn around each other (see [CDM12] or [Les99, Section 1]) and it defines morphisms from braid groups to algebras of horizontal chord diagrams. Bar-Natan extended the Kontsevich integral to links [BN95a] and Le and Murakami extended the Kontsevich integral to a functor from framed tangles to a category of Jacobi diagrams [LM96]. Next, Le, Murakami and Ohtsuki defined the LMO invariant from the Le-Murakami-Kontsevich invariant of surgery presentations of the 3-manifolds using tricky algebraic manipulations of Jacobi diagrams with the help of Kirby calculus. Though some of the physical meaning of the LMO invariant can be recovered from its universality properties, a lot of it gets lost in the manipulations.

The presented construction of $\mathcal{Z}$ is much more physical, geometric and natural to the author, it does not rely on the Kirby theorem, and it provides information about graph embeddings in $\mathbb{Q}$-spheres.

1.4 Book organization

Chapter 2 completes our slow introduction to the linking number of Sections 1.2.1 and 1.2.3, and it contains more conventions and arguments that are used throughout this book. Chapter 3 introduces our definitions of propagators. These propagators are the basic ingredients of all our constructions. Most of the time, they are associated to a parallelization of the 3-manifold. The Theta invariant is the simplest 3-manifold invariant that can be derived from the techniques described in this book. We present it in details in Chapter 4. We first describe Θ as an invariant of a parallelized $\mathbb{Q}$-sphere R. (It is the intersection of three propagating chains associated to the given parallelization of R in $C_2(R)$, or, equivalently, the integral over $C_2(R)$ of the cube of a propagating chain associated to the given parallelization.) Next, we transform Θ to an invariant of $\mathbb{Q}$–spheres using relative Pontrjagin classes, also called Hirzebruch defects, as Kuperberg and Thurston did in [KT99]. Like Θ, $\mathcal{Z}$ will first appear as an invariant of parallelized links in parallelized $\mathbb{Q}$-spheres, constructed with associated propagators, before being corrected by a function of linking numbers associated to the link parallelizations, Pontrjagin numbers associated to the manifold parallelizations, and constants, which are called *anomalies*. Chapter 5 contains a detailed presentation of parallelizations of oriented 3–manifolds with boundaries and associated Pontrjagin numbers, and closes this introductory part.

After this detailed presentation of the degree one part of the graded invariant $\mathcal{Z}$ for links in $\mathbb{Q}$-spheres, which is determined by the linking numbers of the components and the Θ-invariant of the ambient manifold, we move to the second part of the book, which is devoted to the general presentation of $\mathcal{Z}$ for links in $\mathbb{Q}$-spheres. In this part, we first review various theories of finite type invariants for which various parts of $\mathcal{Z}$ will be universal finite type invariants. This allows us to introduce the spaces of Jacobi diagrams in which $\mathcal{Z}$ takes its values, in a natural way, in Chapter 6. The complete definitions of $\mathcal{Z}$ for links in $\mathbb{Q}$–spheres are given in Chapter 7 without proofs of consistency. The proofs that these definitions make sense and that they do not depend on the involved choices of propagating forms can be found in Chapters 9 and 10. They rely on the study of suitable compactifications of configuration spaces, which is presented in Chapter 8, and on some standard arguments of the subject, which already appear in many places starting with [Kon94], [BT94]... That second part of the book ends with discrete equivalent definitions of $\mathcal{Z}$ in terms of propagating chains and algebraic intersections rather than propagating forms and integrals, in Chapter 11. These definitions make clear that the invariant $\mathcal{Z}$ is rational. The other main properties of $\mathcal{Z}$ are precisely described in Sections 10.1, 10.6, 13.3 and Chapter 18. Some of them involve the definition of the extension of $\mathcal{Z}$ to tangles, which can be found in Theorem 12.7.

The third part of the book is devoted to this extension $\mathcal{Z}$ of $\mathcal{Z}$ to tangles, whose framed version, introduced in Definition 12.12, for framed tangles is denoted by $\mathcal{Z}^f$. The spirit of the definition is the same but its justification is more difficult since the involved compactified configuration spaces are not smooth compact manifolds with corners as before, and since they have additional types of faces. The definition and the properties of the extension are first presented without proofs in Chapters 12 and 13. They are justified in Chapters 14 and 17, respectively. In particular, Chapter 17 contains the proofs of many properties of the link invariant $\mathcal{Z}$. These proofs involve easy-to-discretize variants of the functor $\mathcal{Z}^f$, which are interesting on their own, and which are presented in Section 16.2. The third part of the book ends with the computation of the iterated derivatives of the generalized $\mathcal{Z}$ with respect to crossing changes, in Section 17.6. This computation proves that the restriction of $\mathcal{Z}$ to links in S^3 is a universal Vassiliev invariant

The fourth part focuses on the computation of the iterated derivatives of the generalized $\mathcal{Z}$ with respect to rational LP-surgeries. It begins with Chapter 18, which states the main results and their corollaries, and which reduces their proofs to the proofs of two key propositions, which are presented in Chapter 20. The proofs of these propositions involve the introduction of a more flexible definition of $\mathcal{Z}$ because the restriction of a parallelization of a $\mathbb{Q}$-sphere to the exterior of a $\mathbb{Q}$-handlebody does not necessarily extend to a $\mathbb{Q}$-handlebody that replaces the former one during a rational LP-surgery. Chapter 19 contains an extension of the notion of parallelization to a more flexible notion of *pseudo-parallelization*, and a corresponding more flexible definition of $\mathcal{Z}$. Pseudo-parallelizations also have associated propagators and Pontrjagin numbers and they easily extend to arbitrary $\mathbb{Q}$-handlebodies. More flexible variants of the definition of $\mathcal{Z}$ based on pseudo-parallelizations can be found in Chapter 21.

The book ends with Appendix A, which lists the basic results and techniques of algebraic and geometric topology that are used in the book, followed by Appendix B, which reviews the used properties of differential forms and de Rham cohomology.

Most chapters have their own detailed introduction and there are many cross-references to help the reader choose what she/he wants to read.

1.5 Book genesis

At first, this book aimed at presenting the results of two preprints [Les04a, Les04b] and lecture notes [Les15b], and it contains generalizations of the results of these preprints to wider settings. The preprints [Les04a, Les04b] have never been submitted for publication. The mathematical guidelines of the construction of the invariant $\mathcal{Z}$ presented in the abstract were inspired from Edward Witten's insight into the perturbative expansion of the Chern-Simons theory [Wit89]. They have been given by Maxim Kontsevich in [Kon94, Section 2]. Greg Kuperberg and Dylan Thurston developed these guidelines in [KT99]. They defined $\mathcal{Z}$ for $\mathbb{Q}$-spheres and they sketched a proof that the restriction of $\mathcal{Z}$ to $\mathbb{Z}$-spheres is a universal finite type invariant of $\mathbb{Z}$-spheres in the Ohtsuki-Goussarov-Habiro sense. This allowed them to identify the degree one part of $\mathcal{Z}$ with the Casson invariant, for $\mathbb{Z}$-spheres. I thank Dylan Thurston for explaining me his joint work with Greg Kuperberg in Kyoto in 2001.

In [Les04b], I proved splitting formulae for $\mathcal{Z}$. These formulae compute derivatives of $\mathcal{Z}$ with respect to rational LP-surgeries. They generalize similar Kuperberg-Thurston implicit formulae about Torelli surgeries. These formulae allowed me to identify the degree one part of $\mathcal{Z}$ with the Walker generalization of the Casson invariant for $\mathbb{Q}$-spheres, in [Les04b, Section 6]. They also allowed Delphine Moussard to classify finite type invariants with respect to these rational LP surgeries, and to prove that all such real-valued finite type invariants factor through some "augmentation" of $\mathcal{Z}$ by invariants derived from the order of the $H_1(.;\mathbb{Z})$, in [Mou12]. In [Mas14], Gwénaël Massuyeau proved that the LMO invariant of Thang Lê, Jun Murakami and Tomotada Ohtsuki [LMO98] satisfies the same splitting formulae as $\mathcal{Z}$. Thus, the Moussard classification implies that $\mathcal{Z}$ and Z_{LMO} are equivalent in the sense that they distinguish the same $\mathbb{Q}$–spheres with identical $\sharp H_1(.;\mathbb{Z})$. In order to write the proof of my splitting formulae, I needed to specify the definition of $\mathcal{Z}$ and I described the Kontsevich-Kuperberg-Thurston construction in details in [Les04a].

In [Les15b], mixing known constructions in the case of links in $\mathbb{R}^3$ with the construction of $\mathcal{Z}$ allowed me to define a natural extension of $\mathcal{Z}$ as an invariant of links in $\mathbb{Q}$-spheres, which also generalizes invariants of links in $\mathbb{R}^3$ defined by Guadagnini, Martellini and Mintchev [GMM90], Bar-Natan [BN95b] and by Bott and Taubes [BT94], which emerged after the Witten work [Wit89]. [12] It also allowed me to give more flexible definitions of $\mathcal{Z}$.

In addition to the revisited contents of the preprints [Les04a, Les04b] and of the notes [Les15b], this book contains an extension of $\mathcal{Z}$ as a functorial invariant of tangles in rational homology cylinders, and the proofs of many properties of this extension, which imply simpler properties for $\mathcal{Z}$ such as the multiplicativity of $\mathcal{Z}$ under connected sum, for example. This functorial extension, which generalizes the Poirier extension in [Poi00], and its properties are new and they appear only in this book (to my knowledge).

Most of the properties of $\mathcal{Z}$ are very intuitive and rather easy to accept after some hand-waving. Writing complete proofs is often more complicated than one would expect. I hope that I have succeeded in this task, which was much more difficult than I would have expected, in some cases.

[12] The relation between the perturbative expansion of the Chern-Simons theory of the Witten article and the configuration space integral viewpoint is explained by Polyak in [Pol05] and by Sawon in [Saw06].

1.6 Some open questions

1. A Vassiliev invariant is *odd* if it distinguishes some knot from the same knot with the opposite orientation. Are there odd Vassiliev invariants?

2. More generally, do Vassiliev invariants distinguish knots in S^3? In [Kup96], Greg Kuperberg proved that if they distinguish unoriented knots in S^3, then there exist odd Vassiliev invariants.

3. According to a theorem of Bar-Natan and Lawrence [BNL04], the LMO invariant fails to distinguish rational homology spheres with isomorphic H_1. So, according to a Moussard theorem [Mou12], rational finite type invariants fail to distinguish $\mathbb{Q}$-spheres. Do finite type invariants distinguish $\mathbb{Z}$-spheres?

4. Compute the anomalies α and β. For links in $\mathbb{R}^3$, the invariant $\mathcal{Z}$ is expressed as a function of the Kontsevich integral Z^K, described in [BN95a] and in [CDM12], and of the anomaly α, in [Les02]. The computation of α would finish clarifying the relationship between $\mathcal{Z}$ and Z^K, for links in $\mathbb{R}^3$.

5. Find surgery formulae for $\mathcal{Z}$. Do the surgery formulae that define Z_{LMO} from Z^K define $\mathcal{Z}$ from its restriction to links in $\mathbb{R}^3$?

6. Compare $\mathcal{Z}$ with the LMO invariant Z_{LMO} of Le, Murakami and Ohtsuki [LMO98].

7. Find relationships between $\mathcal{Z}$ or other finite type invariants and Heegaard Floer homologies. Recall the propagators associated with Heegaard diagrams of [Les15a] from Section 1.2.8.

8. Kricker defined a lift $\tilde{Z}^K$ of the Kontsevich integral Z^K (or the LMO invariant) for null-homologous knots in $\mathbb{Q}$-spheres [Kri00, GK04]. The Kricker lift is valued in a space $\tilde{A}$, which is a space of trivalent diagrams whose edges are decorated by rational functions whose denominators divide the Alexander polynomial. Compare the Kricker lift $\tilde{Z}^K$ with the equivariant configuration space invariant $\tilde{Z}^c$ of [Les11, Les13] valued in the same diagram space $\tilde{A}$.

9. Does one obtain $\mathcal{Z}$ from $\tilde{Z}^c$ in the same way as one obtains Z^K from $\tilde{Z}^K$?

Chapter 2

More on manifolds and on the linking number

The first section of this chapter specifies some basic notions of differential topology, quickly and sometimes vaguely introduced in Section 1.2.2. It also contains some additional notation and conventions. The second section completes our discussion of the linking number in Sections 1.2.1 and 1.2.3.

2.1 More background material on manifolds

2.1.1 Manifolds without boundary

This section presents a quick review of the notions of manifold and tangent bundle. The reader is referred to [Hir94, Chapter 1] for a clean and complete introduction.

A *topological n–dimensional manifold M without boundary* is a Hausdorff topological space that is a union of open subsets U_i labeled in a countable set I ($i \in I$), where every U_i is identified with an open subset V_i of $\mathbb{R}^n$ by a homeomorphism $\phi_i : U_i \to V_i$, called a *chart.* Such a collection $(\phi_i : U_i \to V_i)_{i\in I}$ of charts, for which $\cup_{i\in I} U_i = M$, is called an *atlas* of M. Manifolds are considered up to homeomorphism. So homeomorphic manifolds are considered identical.

For $r = 0, \dots, \infty$, the topological manifold M *has a C^r–structure* (induced by the atlas $(\phi_i)_{i\in I}$) or *is a C^r–manifold*, if, for each pair $\{i, j\} \subset I$, the *transition map* $\phi_j \circ \phi_i^{-1}$ defined on $\phi_i(U_i \cap U_j)$ is a C^r–diffeomorphism onto its image. The notion of C^s–maps, $s \leq r$, from such a manifold to another one can be induced naturally from the known case for which the manifolds are open subsets of some $\mathbb{R}^n$, thanks to the local identifications provided by the charts. Manifolds of class C^r are considered up to C^r–diffeomorphism. They are called *C^r–manifolds*. *Smooth manifolds* are C^∞–manifolds.

A *C^r embedding* from a C^r manifold A into a C^r manifold M is an injective C^r map j from A to M such that for any point a of A there exists a C^r diffeomorphism ϕ from an open neighborhood U of $j(a)$ in M to an open subset V of $\mathbb{R}^n$ and an open neighborhood U_A of a in A such that the restriction $j_{|U_A}$ of j to U_A is a C^r–diffeomorphism onto its image, which may be written as $j(U_A) = j(A) \cap U = \phi^{-1}\big(V \cap (\mathbb{R}^d \times \{(0,\dots,0)\})\big)$. A *submanifold* of a manifold M is the image of an embedding into M.

The *tangent space* T_xA to a C^r submanifold A of $\mathbb{R}^n$ at a point x of A, for $r \geq 1$ is the vector space of all tangent vectors to (a curve or 1-dimensional submanifold of) A at x. A well-known theorem [Hir94, Theorem 3.4, Chapter 1] asserts that any compact C^r–manifold, for $r \geq 1$ may be embedded in some $\mathbb{R}^d$, and thus viewed as a submanifold of $\mathbb{R}^d$. The *tangent bundle* TA to A is the union over the elements x of A of the T_xA. Its *bundle projection* $p\colon TA \to A$ maps an element v of T_xA to x. The tangent bundle to $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n \times \mathbb{R}^n$ canonically, and a C^r diffeomorphism between two open sets of $\mathbb{R}^n$, together with its derivatives induces a canonical C^{r-1}-diffeomorphism between their tangent bundles. The notion of tangent bundle of any C^r n-manifold, for $r \geq 1$, is naturally induced from the local identifications provided by the charts. A C^r map f from a C^r-manifold M to another one N has a well defined *tangent map*, which is a map $Tf\colon TM \to TN$, which restricts as a linear map $T_xf\colon T_xM \to T_{f(x)}N$.

2.1.2 More on low-dimensional manifolds

We now review classical results, which ensure that for $n = 0$, 1, 2 or 3, any topological n-manifold may be equipped with a unique smooth (i.e. C^∞) structure (up to diffeomorphism).

A topological manifold M as in the previous section *has a piecewise linear (or PL) structure* (induced by the atlas $(\phi_i)_{i\in I}$) or *is a PL–manifold*, if, for each pair $\{i,j\} \subset I$, the transition map $\phi_j \circ \phi_i^{-1}$ is a piecewise linear diffeomorphism onto its image. PL–manifolds are considered up to PL–diffeomorphism.

An n-dimensional simplex is the convex hull of $(n+1)$ points that are not contained in an affine subspace of dimension $(n-1)$ in some $\mathbb{R}^k$, with $k \geq n$. For example, a 1-dimensional simplex is a closed interval, a 2-dimensional simplex is a solid triangle and a 3-dimensional simplex is a solid tetrahedron. A topological space X has a *triangulation*, if it is a locally finite[1] union of

[1]A collection of sets in X is *locally finite* if each point of X has a neighborhood in X

k–simplices (closed in X), which are the simplices of the triangulation, such that (0) the simplices are embedded in X, (1) every face of a simplex of the triangulation is a simplex of the triangulation, (2) when two simplices of the triangulation are not disjoint, their intersection is a simplex of the triangulation.

PL manifolds always have such triangulations.

When $n \leq 3$, the above notion of PL-manifold coincides with the notions of smooth and topological manifold, according to the following theorem. This is no longer true when $n > 3$. See [Kui99].

Theorem 2.1. *When $n \leq 3$, the category of topological n–manifolds is naturally isomorphic to the category of PL n–manifolds and to the category of C^r n–manifolds, for $r = 1, \dots, \infty$.*

For example, according to this statement, which contains several theorems (see [Kui99]), any topological 3–manifold has a unique C^∞–structure. In the end of this subsection, $n = 3$.

The equivalence between the $C^i, i = 1, 2, \dots, \infty$-categories follows from work of Whitney in 1936 [Whi36]. In 1934, Cairns [Cai35] provided a map from the C^1–category to the PL category, which is the existence of a triangulation for C^1–manifolds, and he proved that this map is onto [Cai40, Theorem III] in 1940. Moise [Moi52] proved the equivalence between the topological category and the PL category in 1952. This diagram was completed by Munkres [Mun60, Theorem 6.3] and Whitehead [Whi61] in 1960 by their independent proofs of the injectivity of the natural map from the C^1–category to the topological category.

2.1.3 Connected sum

Let M_1 and M_2 be two smooth closed manifolds of dimension n. The *connected sum* $M_1 \sharp M_2$ of M_1 and M_2 is defined as follows. For $i \in \{1, 2\}$, let $\phi_i \colon 2\mathring{B}^n \hookrightarrow M_i$ be a smooth embedding of the open ball $2\mathring{B}^n$ of radius 2 of the Euclidean vector space $\mathbb{R}^n$ into M_i, such that ϕ_1 is orientation-preserving and ϕ_2 is orientation-reversing. The elements of $(2\mathring{B}^n \setminus \{0\})$ may be written as λx for a unique pair $(\lambda, x) \in]0, 2[\times S^{n-1}$, where S^{n-1} is the unit sphere of $\mathbb{R}^n$. Let $h \colon \phi_1(2\mathring{B}^n \setminus \{0\}) \to \phi_2(2\mathring{B}^n \setminus \{0\})$ be the diffeomorphism such that $h(\phi_1(\lambda x)) = \phi_2((2 - \lambda)x)$ for any $(\lambda, x) \in]0, 2[\times S^{n-1}$.

Then

$$M_1 \sharp M_2 = (M_1 \setminus \{\phi_1(0)\}) \cup_h (M_2 \setminus \{\phi_2(0)\})$$

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that intersects finitely many sets of the collection.

is the quotient space of $(M_1 \setminus \{\phi_1(0)\}) \sqcup (M_2 \setminus \{\phi_2(0)\})$, in which an element of $\phi_1(2\mathring{B}^n \setminus \{0\})$ is identified with its image under h. As a topological manifold,

$$M_1 \sharp M_2 = \left(M_1 \setminus \phi_1(\mathring{B}^n)\right) \cup_{\phi_1(S^{n-1}) \sim \phi_2(S^{n-1})} \left(M_2 \setminus \phi_2(\mathring{B}^n)\right).$$

2.1.4 Manifolds with boundary and ridges

A *topological n–dimensional manifold M with possible boundary* is a Hausdorff topological space that is a union of open subsets U_i labeled in a set I, $(i \in I)$, where every U_i is identified with an open subset V_i of $]-\infty, 0]^k \times \mathbb{R}^{n-k}$ by a chart $\phi_i : U_i \to V_i$. The *boundary* of $]-\infty, 0]^k \times \mathbb{R}^{n-k}$ consists of the points $(x_1, \dots, x_n)$ of $]-\infty, 0]^k \times \mathbb{R}^{n-k}$ such that there exists $i \leq k$ such that $x_i = 0$. The *boundary* of M consists of the points that are mapped to the boundary of $]-\infty, 0]^k \times \mathbb{R}^{n-k}$ by a chart.

A map from an open subset O of $]-\infty, 0]^k \times \mathbb{R}^{n-k}$ to an open subset of $]-\infty, 0]^{k'} \times \mathbb{R}^{n'-k'}$ is smooth (resp. C^r) at a point $x \in O$ if it extends as a smooth (resp. C^r) map from an open neighborhood of x in $\mathbb{R}^n$ to $\mathbb{R}^{n'}$.

The topological manifold M *is a smooth manifold with ridges (or with corners) (resp. with boundary)*, if, for each pair $\{i, j\} \subset I$, the map $\phi_j \circ \phi_i^{-1}$ defined on $\phi_i(U_i \cap U_j)$ is a smooth diffeomorphism onto its image (resp. and if furthermore $k \leq 1$, for any i). The *codimension j boundary* of such a manifold M, which is denoted by $\partial_j(M)$, consists of the points that are mapped to points $(x_1, \dots, x_n)$ of $]-\infty, 0]^k \times \mathbb{R}^{n-k}$ such that there are at least j indices $i \leq k$ such that $x_i = 0$. It is a closed subset of M, $\partial M = \partial_1(M)$.

The *codimension j faces* of such a smooth manifold M with corners are the connected components of $\partial_j(M) \setminus \partial_{j+1}(M)$. They are smooth manifolds of dimension $(n - j)$. The *interior* of M is $M \setminus \partial M$.

For $j \geq 2$, the codimension j faces are called *ridges* of M.

2.1.5 Algebraic intersections

We start this section with more orientation conventions and notation. The *normal bundle* to an oriented submanifold A in a manifold M, at a point x, is the quotient T_xM/T_xA of tangent bundles at x. It is denoted by N_xA. It is oriented so that (a lift of an oriented basis of) N_xA followed by (an oriented basis of) T_xA induce the orientation of T_xM. The orientation of $N_x(A)$ is a *coorientation* of A at x. The regular preimage of a submanifold under a map f is oriented so that f preserves the coorientations.

Two submanifolds A and B in a manifold M are *transverse* if at each $x \in A \cap B$, $T_xM = T_xA + T_xB$. As proved in [Hir94, Chapter 3 (Theorem

2.4 in particular)], transversality is a generic condition. The intersection of two transverse submanifolds A and B in a manifold M is a manifold, which is oriented so that the normal bundle to $A \cap B$ is $(N(A) \oplus N(B))$, fiberwise. In order to give a meaning to the sum $(N_x(A) \oplus N_x(B))$ at $x \in A \cap B$, pick a Riemannian metric on M, which canonically identifies $N_x(A)$ with $T_x(A)^\perp$, $N_x(B)$ with $T_x(B)^\perp$ and $N_x(A \cap B)$ with $T_x(A \cap B)^\perp = T_x(A)^\perp \oplus T_x(B)^\perp$. Since the space of Riemannian metrics on M is convex, and therefore connected, the induced orientation of $T_x(A \cap B)$ does not depend on the choice of Riemannian metric.

Let A, B, C be three pairwise transverse submanifolds in a manifold M such that $A \cap B$ is transverse to C. The oriented intersection $(A \cap B) \cap C$ is a well defined manifold. Our assumptions imply that at any $x \in A \cap B \cap C$, the sum $(T_xA)^\perp + (T_xB)^\perp + (T_xC)^\perp$ is a direct sum $(T_xA)^\perp \oplus (T_xB)^\perp \oplus (T_xC)^\perp$ for any Riemannian metric on M, so A is also transverse to $B \cap C$, and $(A \cap B) \cap C = A \cap (B \cap C)$. Thus, the intersection of transverse, oriented submanifolds is a well defined associative operation, where *transverse submanifolds* are manifolds such that the elementary pairwise intermediate possible intersections are well defined, as above. This intersection is also commutative when the codimensions of the submanifolds are even.

Recall from Subsection 1.2.7, that, for two transverse submanifolds A and B of complementary dimensions in a manifold M, the sign ± 1 of a point $x \in A \cap B$ is $+1$ if and only if $T_xM = T_xA \oplus T_xB$ as oriented vector spaces. That is equivalent to the condition that the orientation of the normal bundle to $x \in A \cap B$ coincides with the orientation of the ambient space M, that is that $T_xM = N_xA \oplus N_xB$ (as oriented vector spaces again, exercise). If A and B are compact and if A and B are of complementary dimensions in M, their algebraic intersection $\langle A, B\rangle_M$ is the sum of the signs of the intersection points.

Similarly, if $A_1, \dots, A_k$ are k transverse compact submanifolds of M whose codimension sum is the dimension of M, their *algebraic intersection* is defined to be $\langle A_1, \dots, A_k\rangle_M = \langle \cap_{i=1}^{k-1} A_i, A_k\rangle_M$. If M is a connected manifold, which contains a point x, the class of a 0-cycle in $H_0(M;\mathbb{Q}) = \mathbb{Q}[x] = \mathbb{Q}$ is a well defined number, and $\langle A_1, \dots, A_k\rangle_M$ can be defined as the homology class of the (oriented) intersection $\cap_{i=1}^{k} A_i$, equivalently. This algebraic intersection extends multilinearly to rational chains.

In a manifold M, a k-dimensional *chain (resp. rational chain)* is a finite combination with coefficients in $\mathbb{Z}$ (resp. in $\mathbb{Q}$) of (smooth, compact, oriented) k-dimensional submanifolds C of M with boundary and ridges, up to the identification of $(-1)C$ with $(-C)$.

Again, unless otherwise mentioned, manifolds are smooth, compact and oriented. The boundary ∂ of chains is the linear map that maps a submanifold to its oriented (by the usual outward normal first convention) boundary. This boundary is the sum of the closures of the codimension-one faces when there are ridges. The canonical orientation of a point is the sign $+1$, so $\partial[0,1] = \{1\} - \{0\}$.

The following lemma will be used in Subsection 2.2.1.

Lemma 2.2. *Let A and B be two transverse submanifolds of a d–dimensional manifold M, with respective dimensions α and β and with disjoint boundaries. Then*

$$\partial(A \cap B) = (-1)^{d-\beta} \partial A \cap B + A \cap \partial B.$$

PROOF: Note that $\partial(A \cap B) \subset \partial A \cup \partial B$. At a point $a \in \partial A$, T_aM is oriented by $(N_aA, o, T_a\partial A)$, where o is the outward normal to A. If $a \in \partial A \cap B$, then o is also an outward normal for $A \cap B$, and $\partial(A \cap B)$ is cooriented by (N_aA, N_aB, o), while $\partial A \cap B$ is cooriented by (N_aA, o, N_aB). At a point $b \in A \cap \partial B$, both $\partial(A \cap B)$ and $A \cap \partial B$ are cooriented by (N_bA, N_bB, o). $\square$

2.1.6 More on the degree

Here, we make the notion of *walls* in Section 1.2.5 more precise, by stating a lemma on the general behaviour of the degree. We prove it with the Morse-Sard theorem. The formula of the lemma could also be justified with Stokes's theorem.

Lemma 2.3. *Let $n \in \mathbb{N}$. Let $f \colon M \to N$ be a smooth map from a compact (oriented) n-manifold with possible boundary to a connected (oriented) n-manifold N. Then for any two distinct regular points a and b of f in the interior $Int(N)$ of N, there exists an embedding $\gamma \colon [0,1] \to Int(N)$ such that $\gamma(0) = a$, $\gamma(1) = b$ and, for any $x \in \partial M \cap f^{-1}(\gamma([0,1]))$, x is in an open face of ∂M (of codimension one in M), and $T_{f(x)}N = \eta(x) T_{f(x)}\gamma \oplus T_xf(T_x\partial M)$ as an oriented vector space, for some $\eta(x) = \pm 1$. For any such embedding γ,*

$$deg_a f - deg_b f = \langle \gamma, f(\partial M) \rangle_N = \sum_{x \in f^{-1}(\gamma) \cap \partial M} \eta(x).$$

PROOF: Let B^{n-1} be the unit ball of $\mathbb{R}^{n-1}$. Since N is connected, there exists an (orientation-preserving) embedding $\Psi \colon [-1,2] \times B^{n-1} \to \mathrm{Int}(N)$ such that $\Psi(0,0) = a$, $\Psi(1,0) = b$, and $\Psi(]-\varepsilon, \varepsilon[\times \varepsilon B^{n-1})$ and $\Psi(]1-\varepsilon, 1+\varepsilon[\times \varepsilon B^{n-1})$ consist of regular values of f, for some $\varepsilon \in]0, \frac{1}{100}[$.

If all the elements of $\Psi([0,1] \times \{0\})$ are regular values of f, then the preimage $f^{-1}(\Psi([0,1] \times \{0\}))$ is a disjoint union of intervals between a point of $f^{-1}(a)$ and $f^{-1}(b)$, along which the sign of $\det(T_xf)$ is constant, so $\deg_a f =$

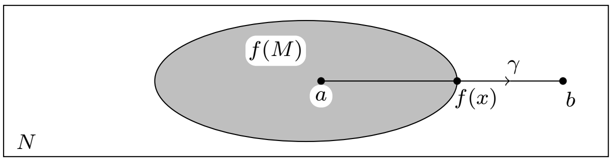


Figure 2.1: Lemma 2.3 for an embedding f from M to a rectangle N

$\deg_b f$. This proves that the degree is constant on any ball ot regular values. Let us go back to the general case and try to make a similar argument work. Set $N_\varepsilon = [0,1] \times \varepsilon \mathring{B}^{n-1}$, $M_\varepsilon = f^{-1}(\Psi(N_\varepsilon))$ and $f_\varepsilon = \Psi^{-1} \circ f_{|M_\varepsilon}$. For any $c \in \Psi(\mathring{N}_\varepsilon)$, $\deg_c f = \deg_{\Psi^{-1}(c)}(f_\varepsilon)$.

Let p_I and p_B respectively denote the natural projections of $[0,1] \times \varepsilon \mathring{B}^{n-1}$ to its factors $[0,1]$ and $\varepsilon \mathring{B}^{n-1}$. The Morse-Sard theorem guarantees the existence of a regular value y of $p_B \circ f_\varepsilon$ in $\varepsilon \mathring{B}^{n-1}$. Let $\gamma_y \colon [0,1] \to N_\varepsilon$ map t to (t,y). The preimage

$$f_\varepsilon^{-1}(\gamma_y) = f_\varepsilon^{-1}\left(\gamma_y([0,1])\right) = (-1)^{n-1}(p_B \circ f_\varepsilon)^{-1}(y)$$

is a submanifold of dimension 1 of M, cooriented by $(-1)^{n-1}(T_x f_\varepsilon)^{-1}(\mathbb{R}^{n-1} = T_y B^{n-1})$.

For $x \in (p_B \circ f_\varepsilon)^{-1}(y)$ such that f_ε is a local diffeomorphism near x, and,

$$T_{f_\varepsilon(x)} N_\varepsilon (= T_{f_\varepsilon(x)}\gamma_y \oplus \mathbb{R}^{n-1}) = \delta(x) T_x f_\varepsilon(T_x M)$$

as oriented vector spaces, with $\delta(x) = \pm 1$, $\delta(x) f_\varepsilon^{-1}(\gamma_y)$ is locally oriented by $p_I \circ f_\varepsilon$. The oriented boundary of $f_\varepsilon^{-1}(\gamma_y)$ is

$$\partial f_\varepsilon^{-1}(\gamma_y) = -f_\varepsilon^{-1}((0,y)) + f_\varepsilon^{-1}((1,y)) + \sum_{x \in f_\varepsilon^{-1}(\gamma_y) \cap \partial M} \eta(x) x.$$

Let us check the sign in front of $x \in f_\varepsilon^{-1}(\gamma_y) \cap \partial M$. Let $N_{x,\partial M}$ denote the outward normal to M at x, so $T_x M = \mathbb{R} N_{x,\partial M} \oplus T_x \partial M$,

$$T_{f_\varepsilon(x)} N_\varepsilon = \eta(x) T_{f_\varepsilon(x)}\gamma_y \oplus T_x f_\varepsilon(T_x \partial M) = \delta(x)\left(T_x f_\varepsilon(\mathbb{R} N_{x,\partial M}) \oplus T_x f_\varepsilon(T_x \partial M)\right),$$

the outward normal $N_{x,\partial M}$ to ∂M is oriented by $\eta(x)\delta(x) p_I \circ f_\varepsilon$, and the sign of the scalar product of $T_x\left(f_\varepsilon^{-1}(\gamma_y)\right)$ and $N_{x,\partial M}$ is $\eta(x)$. This makes clear that

$$\deg_{(0,y)} f_\varepsilon - \deg_{(1,y)} f_\varepsilon = \sum_{x \in f_\varepsilon^{-1}(\gamma_y) \cap \partial M} \eta(x) = \deg_y (p_B \circ f_\varepsilon)_{|M_\varepsilon \cap \partial M}$$

since $T_{f_\varepsilon(x)}N_\varepsilon = T_{f_\varepsilon(x)}\gamma_y \oplus \mathbb{R}^{n-1} = T_{f_\varepsilon(x)}\gamma_y \oplus \eta(x)T_x f_\varepsilon(T_x\partial M)$. This proves that the statement is true for $\Psi(a_y = \gamma_y(0))$, $\Psi(b_y = \gamma_y(1))$ and $\Psi \circ \gamma_y$. Since $\Psi(a_y)$ and a are in a common open ball B_a of regular points (in $\mathrm{Int}(N) \setminus f(\partial M)$), and since $\Psi(b_y)$ and b are in another common ball of regular points, we may modify $\Psi \circ \gamma_y$ to a smooth path γ from a to b with all the properties of the statement.

Let us finish by proving that for any smooth embedded path γ from a to b such that $T_{f(x)}N = \pm T_{f(x)}\gamma \oplus T_x f(T_x\partial M)$ for any $x \in \partial M \cap f^{-1}(\gamma)$, $\deg_a f - \deg_b f = \langle \gamma, f(\partial M)\rangle_N$. For any such path γ, we can construct a neighborhood embedding Ψ of γ as above, such that $\Psi(t,0) = \gamma(t)$ for all $t \in [0,1]$, and

$$p_B \circ \Psi^{-1} \circ f \colon \partial M \cap f^{-1}\left(\Psi\left(]-\varepsilon, 1+\varepsilon[\times \varepsilon \mathring{B}^{n-1}\right)\right) \to \varepsilon \mathring{B}^{n-1}$$

is a local diffeomorphism, whose degree $\langle \Psi([0,1] \times \{.\}), f(\partial M)\rangle_N$ is therefore constant on $\varepsilon \mathring{B}^{n-1}$. The previous study identifies this degree with $(\deg_a f - \deg_b f)$. $\square$

2.2 On the linking number, again

2.2.1 A general definition of the linking number

Lemma 2.4. *Let J and K be two rationally null-homologous disjoint cycles of respective dimensions j and k in a d-manifold M, where $d = j + k + 1$. There exists a rational $(j+1)$–chain Σ_J bounded by J transverse to K, and a rational $(k+1)$–chain Σ_K bounded by K transverse to J, and, for any two such rational chains Σ_J and Σ_K, $\langle J, \Sigma_K\rangle_M = (-1)^{j+1}\langle \Sigma_J, K\rangle_M$. In particular, $\langle J, \Sigma_K\rangle_M$ is a topological invariant of (J,K), which is denoted by $lk(J,K)$ and called the* linking number *of J and K.*

$$lk(J,K) = (-1)^{(j+1)(k+1)} lk(K,J).$$

Proof: Since K is rationally null-homologous, K bounds a rational $(k+1)$–chain Σ_K. Without loss of generality, Σ_K is assumed to be transverse to Σ_J, so $\Sigma_J \cap \Sigma_K$ is a rational 1–chain (which is a rational combination of circles and intervals). According to Lemma 2.2,

$$\partial(\Sigma_J \cap \Sigma_K) = (-1)^{d+k+1} J \cap \Sigma_K + \Sigma_J \cap K.$$

Furthermore, the sum of the coefficients of the points in the left-hand side must be zero, since this sum vanishes for the boundary of an interval. This proves that $\langle J, \Sigma_K\rangle_M = (-1)^{d+k}\langle \Sigma_J, K\rangle_M$, and therefore that this rational number is independent of the chosen Σ_J and Σ_K. Since $(-1)^{d+k}\langle \Sigma_J, K\rangle_M = (-1)^{j+1}(-1)^{k(j+1)}\langle K, \Sigma_J\rangle_M$, $lk(J,K) = (-1)^{(j+1)(k+1)}lk(K,J)$. $\square$

Remark 2.5. Our sign convention for the linking number differs from that in [ST80, Section 77, page 288], where the linking number of cycles J and K as in the lemma is defined as $\langle \Sigma_J, K\rangle_M$, instead. The reason for our sign convention is justified in Remark 2.10.

In particular, the *linking number* of two rationally null-homologous disjoint links J and K in a 3-manifold M is the algebraic intersection of a rational chain bounded by one of the links and the other one.

For $\mathbb{K} = \mathbb{Z}$ or $\mathbb{Q}$, in a $\mathbb{K}$-sphere or in a $\mathbb{K}$–ball (defined in Section 1.1), any knot is rationally null-homologous, so the linking number of two disjoint knots always makes sense.

A *meridian* of a knot K is the (oriented) boundary of a disk that intersects K once with a positive sign. See Figure 2.2. Since a chain Σ_J bounded by a knot J disjoint from K in a 3–manifold M provides a rational cobordism between J and a combination of meridians of K, $[J] = lk(J,K)[m_K]$ in $H_1(M \setminus K;\mathbb{Q})$, where m_K is a meridian of K.

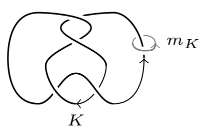


Figure 2.2: A meridian m_K of a knot K

Lemma 2.6. *When K is a knot in a $\mathbb{Q}$-sphere or a $\mathbb{Q}$-ball R, $H_1(R\setminus K;\mathbb{Q}) = \mathbb{Q}[m_K]$, so the equation $[J] = lk(J,K)[m_K]$ in $H_1(R \setminus K;\mathbb{Q})$ provides an alternative definition for the linking number.*

PROOF: Exercise. $\square$

The reader is also invited to check that the Gauss linking number lk_G of Section 1.2.1 coincides with the above linking number lk for two-component links of S^3, as an exercise. That is proved in the next subsection, see Proposition 2.9.

2.2.2 Generalizing the Gauss definition of the linking number and identifying the definitions

Let X and Y be two topological spaces. Recall that a *homotopy* from a continuous map f from X to Y to another such g is a continuous map $H\colon [0,1] \times X \to Y$ such that for any $x \in X$, $H(0,x) = f(x)$ and $H(1,x) = g(x)$. Two continuous maps f and g from X to Y are said to be *homotopic* if there exists a homotopy from f to g. A continuous map f from X to Y is a *homotopy equivalence* if there exists a continuous map g from Y to X such that $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y. The topological spaces X and Y are said to be *homotopy equivalent*, or *of the same homotopy type* if there exists a homotopy equivalence from X to Y.

Let $\Delta\left((\mathbb{R}^3)^2\right)$ denote the diagonal of $(\mathbb{R}^3)^2$.

Lemma 2.7. *The map*

$$\begin{array}{llll} p_{S^2}\colon & ((\mathbb{R}^3)^2 \setminus \Delta\left((\mathbb{R}^3)^2\right)) & \to & S^2 \\ & (x,y) & \mapsto & \frac{1}{\|y-x\|}(y-x) \end{array}$$

is a homotopy equivalence. In particular

$$H_i(p_{S^2})\colon H_i((\mathbb{R}^3)^2 \setminus \Delta\left((\mathbb{R}^3)^2\right); \mathbb{Z}) \to H_i(S^2;\mathbb{Z})$$

is an isomorphism for all integer i, $((\mathbb{R}^3)^2 \setminus \Delta\left((\mathbb{R}^3)^2\right))$ is a homology S^2, and $[S] = \left(H_2(p_{S^2})\right)^{-1}[S^2]$ is a canonical generator of

$$H_2((\mathbb{R}^3)^2 \setminus \Delta\left((\mathbb{R}^3)^2\right); \mathbb{Z}) = \mathbb{Z}[S].$$

PROOF: The configuration space $((\mathbb{R}^3)^2 \setminus \Delta\left((\mathbb{R}^3)^2\right))$ is homeomorphic to $\mathbb{R}^3 \times]0,\infty[\times S^2$ via the map

$$(x,y) \mapsto (x, \| y-x \|, p_{S^2}(x,y)).$$

□

As in Subsection 1.2.1, consider a two-component link $J \sqcup K : S^1 \sqcup S^1 \hookrightarrow \mathbb{R}^3$. This embedding induces an embedding

$$\begin{array}{llll} J \times K\colon & S^1 \times S^1 & \hookrightarrow & ((\mathbb{R}^3)^2 \setminus \Delta\left((\mathbb{R}^3)^2\right)) \\ & (w,z) & \mapsto & (J(w), K(z)) \end{array}$$

the map p_{JK} of Subsection 1.2.1 is the composition $p_{S^2} \circ (J \times K)$, and since $H_2(p_{JK})[S^1 \times S^1] = \deg(p_{JK})[S^2] = lk_G(J,K)[S^2]$ in $H_2(S^2;\mathbb{Z}) = \mathbb{Z}[S^2]$,

$$[(J \times K)(S^1 \times S^1)] = H_2(J \times K)[S^1 \times S^1] = lk_G(J,K)[S]$$

in $H_2((\mathbb{R}^3)^2 \setminus \Delta\left((\mathbb{R}^3)^2\right); \mathbb{Z}) = \mathbb{Z}[S]$. We will see that this definition of lk_G generalizes to links in rational homology spheres, and then we will prove that our generalized definition coincides with the general definition of linking numbers in this case.

For a manifold M, the normal bundle to the diagonal of M^2 in M^2 is identified with the tangent bundle to M, fiberwise, by the map

$$[(u,v)] \in \frac{(T_xM)^2}{\Delta\left((T_xM)^2\right)} \mapsto (v-u) \in T_xM.$$

A *parallelization* τ of an oriented 3-manifold M is a (smooth) bundle isomorphism $\tau\colon M \times \mathbb{R}^3 \longrightarrow TM$ that restricts to $x \times \mathbb{R}^3$ as an orientation-preserving linear isomorphism from $x \times \mathbb{R}^3$ to T_xM, for any $x \in M$. It has long been known that any oriented 3-manifold is parallelizable (i.e. admits a parallelization). It is proved in Subsection 5.2. Therefore, a tubular neighborhood of the diagonal $\Delta(M^2)$ in M^2 is diffeomorphic to $M \times \mathbb{R}^3$.

Lemma 2.8. *Let R be a rational homology sphere, let ∞ be a point of R. Let $\check{R} = R \setminus \{\infty\}$. Then $\check{R}^2 \setminus \Delta\left(\check{R}^2\right)$ has the same rational homology as S^2. Let B be a ball in $\check{R}$ and let x be a point inside B, then the class $[S]$ of $x \times \partial B$ is a canonical generator of $H_2(\check{R}^2 \setminus \Delta\left(\check{R}^2\right); \mathbb{Q}) = \mathbb{Q}[S]$.*

PROOF: In this proof, the homology coefficients are in $\mathbb{Q}$. The reader is referred to Section A.1. Since $\check{R}$ has the homology of a point, the Künneth Formula (Theorem A.10) implies that $\check{R}^2$ has the homology of a point. Now, by excision,

$$\begin{aligned} H_*(\check{R}^2, \check{R}^2 \setminus \Delta\left(\check{R}^2\right)) &\cong H_*(\check{R} \times \mathbb{R}^3, \check{R} \times (\mathbb{R}^3 \setminus 0)) \\ &\cong H_*(\mathbb{R}^3, S^2) \cong \begin{cases} \mathbb{Q} & \text{if } * = 3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Using the long exact sequence of the pair $(\check{R}^2, \check{R}^2 \setminus \Delta\left(\check{R}^2\right))$, we get that $H_*(\check{R}^2 \setminus \Delta\left(\check{R}^2\right)) \cong H_*(S^2)$. $\square$

Define the *Gauss linking number* of two disjoint links J and K in $\check{R}$, so that

$$[(J \times K)(S^1 \times S^1)] = lk_G(J,K)[S]$$

in $H_2(\check{R}^2 \setminus \Delta\left(\check{R}^2\right); \mathbb{Q})$. Note that the two definitions of lk_G coincide when $\check{R} = \mathbb{R}^3$.

Proposition 2.9. *For two disjoint links J and K in $\check{R}$,*

$$lk_G(J,K) = lk(J,K)$$

Proof: First recall that $lk(J,K)$ is the algebraic intersection $\langle J, \Sigma_K \rangle_R$ of J and a rational chain Σ_K bounded by K. Note that the definitions of $lk(J,K)$ and $lk_G(J,K)$ make sense when J and K are disjoint links. If J has several components J_i, for $i = 1, \dots, n$, then $lk_G(\coprod_{i=1}^n J_i, K) = \sum_{i=1}^n lk_G(J_i, K)$ and $lk(\coprod_{i=1}^n J_i, K) = \sum_{i=1}^n lk(J_i, K)$. There is no loss of generality in assuming that J is a knot for the proof, which we do.

The chain Σ_K provides a rational cobordism C in $\check{R} \setminus J$ between K and a combination of meridians of J, and a rational cobordism $C \times J$ in $\check{R}^2 \setminus \Delta\left(\check{R}^2\right)$, which allows us to see that $[J \times K] = lk(J,K)[J \times m_J]$ in $H_2(\check{R}^2 \setminus \Delta\left(\check{R}^2\right); \mathbb{Q})$. Similarly, Σ_J provides a rational cobordism between J and a meridian m_{m_J} of m_J, so $[J \times m_J] = [m_{m_J} \times m_J]$ in $H_2(\check{R}^2 \setminus \Delta\left(\check{R}^2\right); \mathbb{Q})$, and $lk_G(J,K) = lk(J,K) lk_G(m_{m_J}, m_J)$. Thus it remains to prove that $lk_G(m_{m_J}, m_J) = 1$ for a positive Hopf link (m_{m_J}, m_J), as in Figure 1.3, in a standard ball embedded in $\check{R}$. Now, there is no loss of generality in assuming that our link is a Hopf link in $\mathbb{R}^3$, so the equality follows from that for the positive Hopf link in $\mathbb{R}^3$. $\square$

Remark 2.10. Under the assumptions of Lemma 2.4, the reader can prove as an exercise that if M is connected and if B is a compact ball of M that contains a point x in its interior, then $J \times K$ is homologous to $lk(J,K)(x \times \partial B)$ in $M^2 \setminus \Delta\left(M^2\right)$. In particular, Proposition 2.9 generalizes to all pairs (J,K) as in Lemma 2.4 naturally. This justifies our sign convention in Lemma 2.4.

Chapter 3

Propagators

For a two-component link (J, K) in $\mathbb{R}^3$, the definition of the linking number $lk(J, K)$ can be rewritten as

$$lk(J,K) = \int_{J \times K} p_{S^2}^*(\omega) = \langle J \times K, p_{S^2}^{-1}(Y) \rangle_{(\mathbb{R}^3)^2 \setminus \Delta((\mathbb{R}^3)^2)}$$

for any 2-form ω of S^2 such that $\int_{S^2} \omega = 1$, and for any regular value Y of p_{JK}. Thus, $lk(J, K)$ is the integral of a 2-form $p_{S^2}^*(\omega)$ of $(\mathbb{R}^3)^2 \setminus \Delta\left((\mathbb{R}^3)^2\right)$ along the 2-cycle $[J \times K]$, or it is the intersection of the 2-cycle $[J \times K]$ with the 4-manifold $p_{S^2}^{-1}(Y)$. In order to adapt these definitions of the linking number to punctured rational homology 3–spheres $\check{R} = R \setminus \{\infty\}$, and to build other invariants of links and rational homology spheres R, we compactify $(\check{R})^2 \setminus \Delta\left((\check{R})^2\right)$ to a compact 6-manifold $C_2(R)$, in Section 3.2, using differential blow-ups described in Section 3.1. The above form $p_{S^2}^*(\omega)$ extends to $C_2(S^3 = \mathbb{R}^3 \cup \{\infty\})$ as a model *propagating form*, and the closure of $p_{S^2}^{-1}(Y)$ in $C_2(S^3)$ is a model *propagating chain*. We define general propagating forms and propagating chains in $C_2(R)$ in Section 3.3 by their behaviours on the created boundary of $C_2(R)$. The linking number in a rational homology sphere R is expressed in terms of these *propagators* as in the above equation, in Lemma 3.12. These propagators are the main ingredient in the definitions of the invariant $\mathcal{Z}$ studied in this book.

3.1 Blowing up in real differential topology

For a vector space T, $S(T)$ denotes the quotient $S(T) = \frac{T \setminus \{0\}}{\mathbb{R}^{+*}}$, where $\mathbb{R}^{+*}$ acts by scalar multiplication. Recall that the *unit normal bundle* of a submanifold C in a smooth manifold A is the fiber bundle whose fiber over $x \in C$ is $SN_x(C) = S(N_x(C))$.

In this book, *blowing up* a submanifold C in a smooth manifold A is a canonical process, which transforms A into a smooth manifold $B\ell(A, C)$ by replacing C with the total space of its unit normal bundle. Unlike blow-ups in algebraic geometry, this differential geometric blow-up, which amounts to remove an open tubular neighborhood (thought of as infinitely small) of C, topologically, creates boundaries. Let us define it formally.

A smooth *submanifold transverse to the ridges* of a smooth manifold A is a subset C of A such that for any point $x \in C$ there exists a smooth open embedding ϕ from $\mathbb{R}^c \times \mathbb{R}^e \times [0, 1[^d$ into A such that $\phi(0) = x$ and the image of ϕ intersects C exactly along $\phi(0 \times \mathbb{R}^e \times [0, 1[^d)$. Here c is the *codimension* of C, d and e are integers, which depend on x.

Definition 3.1. Let C be a smooth submanifold transverse to the ridges of a smooth manifold A. The *blow-up* $B\ell(A, C)$ is the unique smooth manifold $B\ell(A, C)$ (with possible ridges) equipped with a canonical smooth projection

$$p_b \colon B\ell(A, C) \to A$$

called the *blowdown map* such that

1. the restriction of p_b to $p_b^{-1}(A \setminus C)$ is a canonical diffeomorphism onto $A \setminus C$, which identifies $p_b^{-1}(A \setminus C)$ with $A \setminus C$ (we will simply regard $A \setminus C$ as a subset of $B\ell(A, C)$ via this identification),

2. there is a canonical identification of $p_b^{-1}(C)$ with the total space $SN(C)$ of the unit normal bundle to C in A,

3. the restriction of p_b to $p_b^{-1}(C) = SN(C)$ is the bundle projection from $SN(C)$ to C,

4. any smooth diffeomorphism ϕ from $\mathbb{R}^c \times \mathbb{R}^e \times [0, 1[^d$ onto an open subset $\phi(\mathbb{R}^c \times \mathbb{R}^e \times [0, 1[^d)$ of A whose image intersects C exactly along $\phi(0 \times \mathbb{R}^e \times [0, 1[^d)$, for natural integers c, e, d, provides a smooth embedding

$$\begin{array}{lcl} [0, \infty[\times S^{c-1} \times (\mathbb{R}^e \times [0, 1[^d) & \xrightarrow{\tilde{\phi}} & B\ell(A, C) \\ (\lambda \in]0, \infty[, v, x) & \mapsto & \phi(\lambda v, x) \\ (0, v, x) & \mapsto & T\phi(0, x)(v) \in SN(C) \end{array}$$

   with open image in $B\ell(A, C)$.

Proof that the definition is consistent: Use local diffeomorphisms of the form $\tilde{\phi}$ and charts on $A \setminus C$ to construct an atlas for $B\ell(A, C)$. These charts are obviously compatible over $A \setminus C$, and we need to check compatibility for charts $\tilde{\phi}$ and $\tilde{\psi}$ induced by embeddings ϕ and ψ as in the statement. For those, transition maps may be written as

$$(\lambda, u, x) \mapsto (\tilde{\lambda}, \tilde{u}, \tilde{x}),$$

where p_1 and p_2 denote the projections on the first and the second factor of $\mathbb{R}^c \times \left(\mathbb{R}^e \times [0,1[^d\right)$, respectively, and

$$\tilde{x} = p_2 \circ \psi^{-1} \circ \phi(\lambda u, x)$$
$$\tilde{\lambda} = \| \; p_1 \circ \psi^{-1} \circ \phi(\lambda u, x) \; \|$$
$$\tilde{u} = \begin{cases} \frac{p_1 \circ \psi^{-1} \circ \phi(\lambda u, x)}{\tilde{\lambda}} & \text{if } \lambda \neq 0 \\ \frac{T\left(p_1 \circ \psi^{-1} \circ \phi\right)(0,x)(u)}{\|T(p_1 \circ \psi^{-1} \circ \phi)(0,x)(u)\|} & \text{if } \lambda = 0 \end{cases}$$

In order to check that this is smooth, write

$$p_1 \circ \psi^{-1} \circ \phi(\lambda u, x) = \lambda \int_0^1 T\left(p_1 \circ \psi^{-1} \circ \phi\right)(t\lambda u, x)(u)dt,$$

where the integral does not vanish when λ is small enough.

More precisely, since the restriction to S^{c-1} of $T\left(p_1 \circ \psi^{-1} \circ \phi\right)(0, x)$ is an injection, for any $u_0 \in S^{c-1}$, there exists a neighborhood of $(0, u_0)$ in $\mathbb{R} \times S^{c-1}$ such that for any (λ, u) in this neighborhood, we have the following condition about the scalar product

$$\langle T\left(p_1 \circ \psi^{-1} \circ \phi\right)(\lambda u, x)(u), T\left(p_1 \circ \psi^{-1} \circ \phi\right)(0, x)(u)\rangle \; > 0.$$

Therefore, there exists $\varepsilon > 0$ such that for any $\lambda \in]-\varepsilon, \varepsilon[$, and for any $u \in S^{c-1}$,

$$\langle T\left(p_1 \circ \psi^{-1} \circ \phi\right)(\lambda u, x)(u), T\left(p_1 \circ \psi^{-1} \circ \phi\right)(0, x)(u)\rangle \; > 0.$$

Then

$$\tilde{\lambda} = \lambda \| \int_0^1 T\left(p_1 \circ \psi^{-1} \circ \phi\right)(t\lambda u, x)(u)dt \; \|$$

is a smooth function (defined even when $\lambda \leq 0$) and

$$\tilde{u} = \frac{\int_0^1 T\left(p_1 \circ \psi^{-1} \circ \phi\right)(t\lambda u, x)(u)dt}{\| \int_0^1 T\left(p_1 \circ \psi^{-1} \circ \phi\right)(t\lambda u, x)(u)dt \; \|}$$

is smooth, too. Thus our atlas is compatible and it defines $B\ell(A, C)$ together with its smooth structure. The projection p_b maps $\tilde{\phi}(\lambda, v, x)$ to $\phi(\lambda v, x)$ in a chart as above. Thus, it is obviously smooth and it has the wanted properties. □

Note the following immediate proposition.

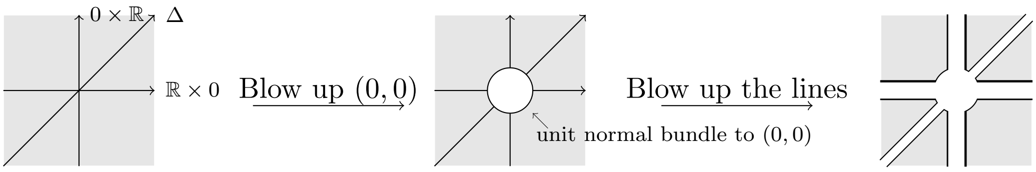


Figure 3.1: A composition of blow-ups

Proposition 3.2. *The blown-up manifold $B\ell(A, C)$ is homeomorphic to the complement in A of an open tubular neighborhood of C. In particular, $B\ell(A, C)$ is homotopy equivalent to $A \setminus C$. If C and A are compact, then $B\ell(A, C)$ is compact, it is a smooth compactification of $A \setminus C$.*

$\square$

In Figure 3.1, we see the result of blowing up $(0, 0)$ in $\mathbb{R}^2$, and the closures in $B\ell(\mathbb{R}^2, (0, 0))$ of $\{0\} \times \mathbb{R}^*$, $\mathbb{R}^* \times \{0\}$ and the diagonal of $(\mathbb{R}^*)^2$, in $B\ell(\mathbb{R}^2, (0, 0))$.

Proposition 3.3. *Let B and C be two smooth submanifolds transverse to the ridges of a C^∞ manifold A. Assume that C is a smooth submanifold of B transverse to the ridges of B.*

1. *The closure $\overline{B \setminus C}$ of $(B \setminus C)$ in $B\ell(A, C)$ is a submanifold of $B\ell(A, C)$, which intersects*
$$SN(C) \subseteq \partial B\ell(A, C)$$
as the unit normal bundle $SN_B(C)$ to C in B. It is canonically diffeomorphic to $B\ell(B, C)$.

2. *The blow-up $B\ell(B\ell(A, C), \overline{B \setminus C})$ of $B\ell(A, C)$ along $\overline{B \setminus C}$ has a canonical differential structure of a manifold with corners, and the preimage of $\overline{B \setminus C} \subset B\ell(A, C)$ in $B\ell(B\ell(A, C), \overline{B \setminus C})$ under the canonical projection*
$$B\ell(B\ell(A, C), \overline{B \setminus C}) \longrightarrow B\ell(A, C)$$
is the pull-back via the blowdown projection $(\overline{B \setminus C} \longrightarrow B)$ of the unit normal bundle to B in A.

Proof:

1. It is always possible to choose an embedding ϕ into A as in Definition 3.1 such that $\phi\left(\mathbb{R}^c \times \mathbb{R}^e \times [0, 1[^d\right)$ intersects C exactly along

$\phi(0 \times \mathbb{R}^e \times [0,1[^d)$ and it intersects B exactly along $\phi(0 \times \mathbb{R}^k \times [0,1[^d)$, $k > e$. (Choose an embedding suitable for B first and modify it so that it is suitable for C.) Look at the induced chart $\tilde{\phi}$ of $B\ell(A,C)$ near a point of $\partial B\ell(A,C)$.
The intersection of $(B \setminus C)$ with the image of $\tilde{\phi}$ is

$$\tilde{\phi}\left(]0,\infty[\times(0 \times S^{k-e-1} \subset S^{c-1}) \times \mathbb{R}^e \times [0,1[^d\right).$$

Thus, the closure of $(B \setminus C)$ intersects the image of $\tilde{\phi}$ as

$$\tilde{\phi}\left([0,\infty[\times(0 \times S^{k-e-1} \subset S^{c-1}) \times \mathbb{R}^e \times [0,1[^d\right).$$

2. Together with the above given charts of $\overline{B \setminus C}$, the smooth injective map

$$\begin{aligned} \mathbb{R}^{c-k+e} \times S^{k-e-1} &\longrightarrow S^{c-1} \\ (u,y) &\mapsto \frac{(u,y)}{\parallel (u,y) \parallel} \end{aligned}$$

identifies $\mathbb{R}^{c-k+e}$ with the fibers of the normal bundle to $\overline{B \setminus C}$ in $B\ell(A,C)$. The blow-up process will therefore replace $\overline{B \setminus C}$ by the quotient of the corresponding $(\mathbb{R}^{c-k+e} \setminus \{0\})$-bundle by $]0,\infty[$, which is of course the pull-back under the blowdown projection $(\overline{B \setminus C} \longrightarrow B)$ of the unit normal bundle to B in A.

$\square$

The fiber $SN_c(C)$ is oriented as the boundary of a unit ball of $N_c(C)$.

3.2 The configuration space $C_2(R)$

Regard S^3 as $\mathbb{R}^3 \cup \{\infty\}$ or as two copies of $\mathbb{R}^3$ identified along $\mathbb{R}^3 \setminus \{0\}$ by the (exceptionally orientation-reversing) diffeomorphism $x \mapsto x/\parallel x \parallel^2$.

Let $(-S^2_\infty)$ denote the unit normal bundle to ∞ in S^3, so $B\ell(S^3,\infty) = \mathbb{R}^3 \cup S^2_\infty$ and $\partial B\ell(S^3,\infty) = S^2_\infty$. There is a canonical orientation-preserving diffeomorphism $p_\infty \colon S^2_\infty \to S^2$, such that $x \in S^2_\infty$ is the limit, in $B\ell(S^3,\infty)$, of a sequence of points of $\mathbb{R}^3$ tending to ∞ along any line of $\mathbb{R}^3$ directed by $p_\infty(x) \in S^2$, in the direction of the line.

Fix a rational homology sphere R, a point ∞ of R, and $\check{R} = R \setminus \{\infty\}$. Identify a neighborhood of ∞ in R with the complement $\mathring{B}_{1,\infty}$ in S^3 of the closed ball $B(1)$ of radius 1 in $\mathbb{R}^3$. Let $\mathring{B}_{2,\infty}$ be the complement in S^3 of the closed ball $B(2)$ of radius 2 in $\mathbb{R}^3$. The ball $\mathring{B}_{2,\infty}$ is a smaller neighborhood of

∞ in R via the understood identification. Then $B_R = R \setminus \mathring{B}_{2,\infty}$ is a compact rational homology ball diffeomorphic to $B\ell(R,\infty)$.

Define the *configuration space* $C_2(R)$ to be the compact 6–manifold with boundary and ridges obtained from R^2 by blowing up (∞,∞) in R^2, and, the closures of $\{\infty\}\times\check{R}$, $\check{R}\times\{\infty\}$ and the diagonal of $\check{R}^2$ in $B\ell(R^2,(\infty,\infty))$, successively, as in Figure 3.1. Then $\partial C_2(R)$ contains the unit normal bundle $(\frac{T\check{R}^2}{\Delta\left(T\check{R}^2\right)}\setminus\{0\})/\mathbb{R}^{+*}$ to the diagonal of $\check{R}^2$. This bundle is identified with the unit tangent bundle $U\check{R}$ to $\check{R}$ by the map $([(x,y)]\mapsto[y-x])$.

Lemma 3.4. *Let $\check{C}_2(R) = \check{R}^2\setminus\Delta\left(\check{R}^2\right)$. The open manifold $C_2(R)\setminus\partial C_2(R)$ is $\check{C}_2(R)$ and the inclusion $\check{C}_2(R)\hookrightarrow C_2(R)$ is a homotopy equivalence. In particular, $C_2(R)$ is a compactification of $\check{C}_2(R)$ homotopy equivalent to $\check{C}_2(R)$, and it has the same rational homology as the sphere S^2. The manifold $C_2(R)$ is a smooth compact 6-dimensional manifold with boundary and ridges. There is a canonical smooth projection $p_{R^2}\colon C_2(R)\to R^2$.*

$$\partial C_2(R) = p_{R^2}^{-1}(\infty,\infty)\cup(S^2_\infty\times\check{R})\cup(-\check{R}\times S^2_\infty)\cup U\check{R}.$$

Proof: This lemma is a corollary of Propositions 3.2 and 3.3, and Lemma 2.8. We just give a few additional arguments to check that the three blow-ups in $B\ell(R^2,(\infty,\infty))$ can be performed simultaeously, and we take a closer look at the structure of $p_{R^2}^{-1}(\infty,\infty)$, below.

Let $B_{1,\infty}$ be the complement of the open ball of radius one of $\mathbb{R}^3$ in S^3. Blowing up (∞,∞) in $B^2_{1,\infty}$ transforms a neighborhood of (∞,∞) into the product $[0,1[\times S^5$. Explicitly, there is a map

$$\begin{array}{llll}\psi\colon & [0,1[\times S^5 & \to & B\ell(B^2_{1,\infty},(\infty,\infty))\\ & (\lambda\in]0,1[,(x\neq 0,y\neq 0)\in S^5\subset(\mathbb{R}^3)^2) & \mapsto & (\frac{1}{\lambda\|x\|^2}x,\frac{1}{\lambda\|y\|^2}y)\\ & (\lambda\in]0,1[,(0,y\neq 0)\in S^5\subset(\mathbb{R}^3)^2) & \mapsto & (\infty,\frac{1}{\lambda\|y\|^2}y)\\ & (\lambda\in]0,1[,(x\neq 0,0)\in S^5\subset(\mathbb{R}^3)^2) & \mapsto & (\frac{1}{\lambda\|x\|^2}x,\infty),\end{array}$$

which is a diffeomorphism onto its open image.

Here, the explicit image of $(\lambda\in]0,1[,(x\neq 0,y\neq 0)\in S^5\subset(\mathbb{R}^3)^2)$ is written in $\left(\mathring{B}_{1,\infty}\setminus\{\infty\}\right)^2\subset B\ell(\mathring{B}^2_{1,\infty},(\infty,\infty))$, where $\mathring{B}_{1,\infty}\setminus\{\infty\}\subset\mathbb{R}^3$. The image of ψ is a neighborhood of the preimage of (∞,∞) under the blowdown map $B\ell(R^2,(\infty,\infty))\xrightarrow{p_1}R^2$. This neighborhood respectively intersects $\infty\times\check{R}$, $\check{R}\times\infty$, and $\Delta\left(\check{R}^2\right)$ as $\psi(]0,1[\times 0\times S^2)$, $\psi(]0,1[\times S^2\times 0)$ and $\psi\left(]0,1[\times\left(S^5\cap\Delta\left((\mathbb{R}^3)^2\right)\right)\right)$. In particular, the closures of $\infty\times\check{R}$, $\check{R}\times\infty$, and $\Delta\left(\check{R}^2\right)$ in $B\ell(R^2,(\infty,\infty))$ intersect the boundary $\psi(0\times S^5)$ of $B\ell(R^2,(\infty,\infty))$ as three disjoint spheres in S^5, and they are isomorphic to $\infty\times B\ell(R,\infty)$,

$B\ell(R,\infty)\times\infty$ and $\Delta\left(B\ell(R,\infty)^2\right)$, respectively. Thus, the next steps will be three blow-ups along these three disjoint smooth manifolds.

These blow-ups will preserve the product structure $\psi([0,1[\times.)$. Thus, $C_2(R)$ is a smooth compact 6-dimensional manifold with boundary, with three *ridges* $S^2\times S^2$ in $p_{R^2}^{-1}(\infty,\infty)$. A neighborhood of these ridges in $C_2(R)$ is diffeomorphic to $[0,1[^2\times S^2\times S^2$. $\square$

Let ι_{S^2} denote the *antipodal map* of S^2, which sends x to $\iota_{S^2}(x)=-x$.

Lemma 3.5. *The map p_{S^2} of Lemma 2.7 extends smoothly to $C_2(S^3)$, and its extension p_{S^2} satisfies*

$$p_{S^2}=\begin{cases}\iota_{S^2}\circ p_\infty\circ p_1 & \text{on } S^2_\infty\times\mathbb{R}^3\\ p_\infty\circ p_2 & \text{on } \mathbb{R}^3\times S^2_\infty\\ p_2 & \text{on } U\mathbb{R}^3=\mathbb{R}^3\times S^2,\end{cases}$$

where p_1 and p_2 denote the projections on the first and second factor with respect to the above expressions.

Proof: Near the diagonal of $\mathbb{R}^3$, we have a chart of $C_2(S^3)$

$$\psi_d:\mathbb{R}^3\times[0,\infty[\times S^2\longrightarrow C_2(S^3),$$

which maps $(x\in\mathbb{R}^3,\lambda\in]0,\infty[,y\in S^2)$ to $(x,x+\lambda y)\in(\mathbb{R}^3)^2$. Here, p_{S^2} extends as the projection onto the S^2 factor.
Consider the orientation-reversing embedding ϕ_∞

$$\begin{array}{llll}\phi_\infty: & \mathbb{R}^3 & \longrightarrow & S^3\\ & \mu(x\in S^2) & \mapsto & \begin{cases}\infty & \text{if } \mu=0\\ \frac{1}{\mu}x & \text{otherwise.}\end{cases}\end{array}$$

Note that this chart induces the already given identification of the unit normal bundle S^2_∞ to $\{\infty\}$ in S^3 with S^2. When $\mu\neq 0$,

$$p_{S^2}(\phi_\infty(\mu x),y\in\mathbb{R}^3)=\frac{\mu y-x}{\parallel \mu y-x\parallel}.$$

Then p_{S^2} can be extended smoothly on $S^2_\infty\times\mathbb{R}^3$ (where $\mu=0$) by

$$p_{S^2}(x\in S^2_\infty,y\in\mathbb{R}^3)=-x.$$

Similarly, set $p_{S^2}(x\in\mathbb{R}^3,y\in S^2_\infty)=y$. Now, with the map ψ of the proof of Lemma 3.4, when x and y are not equal to zero and when they are distinct,

$$p_{S^2}\circ\psi((\lambda,(x,y)))=\frac{\frac{y}{\|y\|^2}-\frac{x}{\|x\|^2}}{\parallel\frac{y}{\|y\|^2}-\frac{x}{\|x\|^2}\parallel}=\frac{\parallel x\parallel^2 y-\parallel y\parallel^2 x}{\parallel\,\parallel x\parallel^2 y-\parallel y\parallel^2 x\parallel}$$

when $\lambda \neq 0$. This map extends to $\mathrm{B}\ell(R^2,(\infty,\infty))$ outside the boundaries of $\infty \times \mathrm{B}\ell(R,\infty)$, $\mathrm{B}\ell(R,\infty) \times \infty$ and $\Delta\left(\mathrm{B}\ell(R,\infty)^2\right)$, naturally, by keeping the same formula when $\lambda = 0$.

Let us check that the map p_{S^2} extends smoothly over the boundary of $\Delta\left(\mathrm{B}\ell(R,\infty)^2\right)$. There is a chart of $C_2(R)$ near the preimage of this boundary in $C_2(R)$

$$\psi_2 : [0,\infty[\times[0,\infty[\times S^2 \times S^2 \longrightarrow C_2(S^3),$$

which maps $(\lambda \in]0,\infty[, \mu \in]0,\infty[, x \in S^2, y \in S^2)$ to $(\phi_\infty(\lambda x), \phi_\infty(\lambda(x + \mu y)))$, where p_{S^2} may be written as

$$(\lambda,\mu,x,y) \mapsto \frac{y - 2\langle x,y\rangle x - \mu x}{\parallel y - 2\langle x,y\rangle x - \mu x \parallel},$$

and therefore extends smoothly along $\mu = 0$. We check that p_{S^2} extends smoothly over the boundaries of $(\infty \times \mathrm{B}\ell(R,\infty))$ and $(\mathrm{B}\ell(R,\infty) \times \infty)$ similarly. $\square$

Definition 3.6. Let τ_s denote the standard parallelization of $\mathbb{R}^3$. Say that a parallelization

$$\tau\colon \check{R} \times \mathbb{R}^3 \to T\check{R}$$

of $\check{R}$ that coincides with τ_s on $\mathring{B}_{2,\infty} \setminus \{\infty\}$ is *asymptotically standard.* According to Proposition 5.5, such a parallelization exists. Such a parallelization identifies $U\check{R}$ with $\check{R} \times S^2$.

Proposition 3.7. *For any asymptotically standard parallelization τ of $\check{R}$, there exists a smooth map $p_\tau\colon \partial C_2(R) \to S^2$ such that*

$$p_\tau = \begin{cases} p_{S^2} & \text{on } p_{R^2}^{-1}(\infty,\infty) \\ \iota_{S^2} \circ p_\infty \circ p_1 & \text{on } S^2_\infty \times \check{R} \\ p_\infty \circ p_2 & \text{on } \check{R} \times S^2_\infty \\ p_2 & \text{on } U\check{R} \stackrel{\tau}{=} \check{R} \times S^2, \end{cases}$$

where p_1 and p_2 denote the projections on the first and second factor with respect to the above expressions.

PROOF: This is a consequence of Lemma 3.5. $\square$

Definition 3.8. Define an *asymptotic rational homology* $\mathbb{R}^3$ to be a pair $(\check{R},\tau)$, in which $\check{R}$ is a 3-manifold that is decomposed as the union over $]1,2] \times S^2$ of a rational homology ball B_R and the complement $\mathring{B}_{1,\infty} \setminus \{\infty\}$ of the unit ball of $\mathbb{R}^3$, and τ is an asymptotically standard parallelization of $\check{R}$. Since such a pair $(\check{R},\tau)$ defines the rational homology sphere $R = \check{R} \cup \{\infty\}$ canonically, "Let $(\check{R},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$" is a shortcut for "Let R be a rational homology sphere, equipped with an embedding of $\mathring{B}_{1,\infty}$ into R and an asymptotically standard parallelization τ of the complement $\check{R}$ of the image of ∞ under this embedding, with respect to this embedding".

3.3 On propagators

A *volume-one form of* S^2 is a 2-form ω_S of S^2 such that $\int_{S^2} \omega_S = 1$. (See Appendix B for a short survey of differential forms and de Rham cohomology.)

Let $(\check{R},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Recall the map

$$p_\tau \colon \partial C_2(R) \to S^2$$

of Proposition 3.7.

Definition 3.9. A *propagating chain* of $(C_2(R),\tau)$ is a 4–chain P of $C_2(R)$ such that $\partial P = p_\tau^{-1}(a)$ for some $a \in S^2$. A *propagating form* of $(C_2(R),\tau)$ is a closed 2-form ω on $C_2(R)$ whose restriction to $\partial C_2(R)$ may be expressed as $p_\tau^*(\omega_S)$ for some volume-one form ω_S of S^2. Propagating chains and propagating forms will be simply called *propagators* when their nature is clear from the context.

Example 3.10. Recall the map $p_{S^2} \colon C_2(S^3) \to S^2$ of Lemma 3.5. For any $a \in S^2$, $p_{S^2}^{-1}(a)$ is a propagating chain of $(C_2(S^3),\tau_s)$, and for any 2-form ω_S of S^2 such that $\int_{S^2} \omega_S = 1$, $p_{S^2}^*(\omega_S)$ is a propagating form of $(C_2(S^3),\tau_s)$.

For our general $\mathbb{Q}$–sphere R, propagating chains of $(C_2(R),\tau)$ exist because the 3-cycle $p_\tau^{-1}(a)$ of $\partial C_2(R)$ bounds in $C_2(R)$ since $H_3(C_2(R);\mathbb{Q}) = 0$. Dually, propagating forms of $(C_2(R),\tau)$ exist because the restriction induces a surjective map $H^2(C_2(R);\mathbb{R}) \to H^2(\partial C_2(R);\mathbb{R})$ since

$$H^3(C_2(R), \partial C_2(R); \mathbb{R}) = 0.$$

When R is a $\mathbb{Z}$-sphere, there exist propagating chains of $(C_2(R),\tau)$ that are smooth 4-manifolds properly embedded in $C_2(R)$. See Theorem 11.9.

Definition 3.11. A *propagating form* of $C_2(R)$ is a closed 2-form ω_p on $C_2(R)$ whose restriction to $\partial C_2(R) \setminus UB_R$ is equal to $p_\tau^*(\omega)$ for some volume-one form ω_S of S^2 and some asymptotically standard parallelization τ. Similarly, a *propagating chain* of $C_2(R)$ is a 4–chain P of $C_2(R)$ such that $\partial P \subset \partial C_2(R)$ and $\partial P \cap (\partial C_2(R) \setminus UB_R)$ is equal to $p_\tau^{-1}(a)$ for some $a \in S^2$. (These definitions do not depend on τ.)

Explicit propagating chains associated with Heegaard splittings, which were constructed with Greg Kuperberg, are described in Section 1.2.8. They are integral chains multiplied by $\frac{1}{\sharp H_1(R;\mathbb{Z})}$, where $\sharp H_1(R;\mathbb{Z})$ is the cardinality of $H_1(R;\mathbb{Z})$.

Since $C_2(R)$ is homotopy equivalent to $(\check{R}^2 \setminus \Delta\left(\check{R}^2\right))$, Lemma 2.8 ensures that $H_2(C_2(R);\mathbb{Q}) = \mathbb{Q}[S]$, where the canonical generator $[S]$ is the homology class of a fiber of $U\check{R} \subset \partial C_2(R)$. For a two-component link (J, K) of $\check{R}$, the homology class $[J \times K]$ of $J \times K$ in $H_2(C_2(R);\mathbb{Q})$ is $lk(J,K)[S]$, according to Proposition 2.9.

Lemma 3.12. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let C be a two-cycle of $C_2(R)$. For any propagating chain P of $C_2(R)$ transverse to C and for any propagating form ω of $C_2(R)$,*

$$[C] = \int_C \omega[S] = \langle C, P\rangle_{C_2(R)}[S]$$

in $H_2(C_2(R);\mathbb{Q}) = \mathbb{Q}[S]$. In particular, for any two-component link (J, K) of $\check{R}$.

$$lk(J,K) = \int_{J\times K} \omega = \langle J \times K, P\rangle_{C_2(R)}.$$

PROOF: Fix a propagating chain P, the algebraic intersection $\langle C, P\rangle_{C_2(R)}$ depends only on the homology class $[C]$ of C in $C_2(R)$. Similarly, since ω is closed, $\int_C \omega$ depends only on $[C]$. (Indeed, if C and C' cobound a chain D transverse to P, $C \cap P$ and $C' \cap P$ cobound $\pm(D \cap P)$, and $\int_{\partial D = C' - C} \omega = \int_D d\omega$ according to Stokes's theorem.) Furthermore, the dependence on $[C]$ is linear. Therefore it suffices to check the lemma for a chain that represents the canonical generator $[S]$ of $H_2(C_2(R);\mathbb{Q})$. Any fiber of $U\check{R}$ is such a chain. $\square$

Definition 3.13. A propagating form ω of $C_2(R)$ is *homogeneous* if its restriction to $\partial C_2(R) \setminus UB_R$ is $p_\tau^*(\omega_{S^2})$ for the homogeneous volume form ω_{S^2} of S^2 of total volume 1.

Let ι be the involution of $C_2(R)$ that exchanges the two coordinates in $\check{R}^2 \setminus \Delta\left(\check{R}^2\right)$.

Lemma 3.14. *If ω_0 is a propagating form of $(C_2(R), \tau)$, then $(-\iota^*(\omega_0))$ and $\omega = \frac{1}{2}(\omega_0 - \iota^*(\omega_0))$ are propagating forms of $(C_2(R), \tau)$. Furthermore, $\iota^*(\omega) = -\omega$, and, if ω_0 is homogeneous, then $(-\iota^*(\omega_0))$ and $\omega = \frac{1}{2}(\omega_0 - \iota^*(\omega_0))$ are homogeneous.*

PROOF: There is a volume-one form ω_S of S^2 such that $\omega_{0|\partial C_2(R)} = p_\tau^*(\omega_S)$, so $(-\iota^*(\omega_0))_{|\partial C_2(R)} = p_\tau^*(-\iota_{S^2}^*(\omega_S))$, where ι_{S^2} is the antipodal map, which sends x to $\iota_{S^2}(x) = -x$, and $(-\iota_{S^2}^*(\omega_S))$ is a volume-one form of S^2. $\square$

Chapter 4

The Theta invariant

Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. We are now ready to define a topological invariant Θ for $(\check{R}, \tau)$ as the algebraic triple intersection of three propagating chains of $(C_2(R), \tau)$, in Section 4.1. In Section 4.3, we use relative Pontrjagin classes, introduced in Section 4.2, to turn Θ into a topological invariant of rational homology 3-spheres.

4.1 The Θ-invariant of (R, τ)

Recall from Section 2.1.5, that for three transverse compact submanifolds A, B, C in a manifold D such that the sum of the codimensions of A, B and C is the dimension of D, the algebraic intersection $\langle A, B, C\rangle_D$ is the sum over the intersection points a of $A \cap B \cap C$ of the associated signs, where the sign of a is positive if and only if the orientation of D is induced by the orientation of $N_aA \oplus N_aB \oplus N_aC$, where N_aA, N_aB and N_aC are identified with $(T_aA)^\perp$, $(T_aB)^\perp$ and $(T_aC)^\perp$, respectively, with the help of a Riemannian metric.

Theorem 4.1. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let P_a, P_b and P_c be three transverse propagators of $(C_2(R), \tau)$ with respective boundaries $p_\tau^{-1}(a)$, $p_\tau^{-1}(b)$ and $p_\tau^{-1}(c)$ for three distinct points a, b and c of S^2. Then*

$$\Theta(R, \tau) = \langle P_a, P_b, P_c\rangle_{C_2(R)}$$

does not depend on the chosen propagators P_a, P_b and P_c. It is a topological invariant of (R, τ). For any three propagating chains ω_a, ω_b and ω_c of $(C_2(R), \tau)$,

$$\Theta(R, \tau) = \int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c.$$

Proof: Since $H_4(C_2(R);\mathbb{Q}) = 0$, if the propagator P_a is replaced by a propagator P'_a with the same boundary, $(P'_a - P_a)$ bounds a 5-dimensional rational chain W transverse to $P_b \cap P_c$. The 1-dimensional chain $W \cap P_b \cap P_c$ does not meet $\partial C_2(R)$ since $P_b \cap P_c$ does not meet $\partial C_2(R)$. Therefore, up to a well-determined sign, the boundary of $W \cap P_b \cap P_c$ is $P'_a \cap P_b \cap P_c - P_a \cap P_b \cap P_c$, as in Figure 4.1. This proves that $\langle P_a, P_b, P_c \rangle_{C_2(R)}$ is independent of P_a when a is

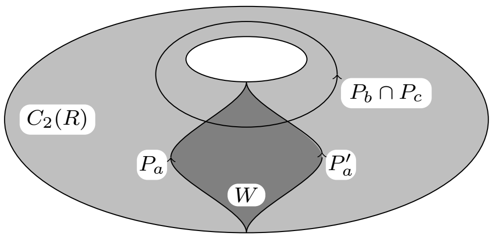


Figure 4.1: Proof of Theorem 4.1

fixed. Similarly, it is independent of P_b and P_c when b and c are fixed. Thus, $\langle P_a, P_b, P_c \rangle_{C_2(R)}$ is a rational function on the connected set of triples (a, b, c) of distinct point of S^2. It is easy to see that this function is continuous, and so, it is constant.

Let us prove that $\int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c$ is independent of the propagating forms ω_a, ω_b and ω_c, similarly.

Lemma 4.2. *Let ω_a and ω'_a be two propagating forms of $(C_2(R), \tau)$, which restrict to $\partial C_2(R)$ as $p_\tau^*(\omega_A)$ and $p_\tau^*(\omega'_A)$, respectively, for two volume-one forms ω_A and ω'_A of S^2. There exists a one-form η_A on S^2 such that $\omega'_A = \omega_A + d\eta_A$. For any such η_A, there exists a one-form η on $C_2(R)$ such that $\omega'_a - \omega_a = d\eta$, and the restriction of η to $\partial C_2(R)$ is $p_\tau^*(\eta_A)$.*

Proof of the lemma: Since ω_a and ω'_a are cohomologous, there exists a one-form η on $C_2(R)$ such that $\omega'_a = \omega_a + d\eta$. Similarly, since $\int_{S^2} \omega'_A = \int_{S^2} \omega_A$, there exists a one-form η_A on S^2 such that $\omega'_A = \omega_A + d\eta_A$. On $\partial C_2(R)$, $d(\eta - p_\tau^*(\eta_A)) = 0$. Thanks to the exact sequence with real coefficients

$$0 = H^1(C_2(R)) \longrightarrow H^1(\partial C_2(R)) \longrightarrow H^2(C_2(R), \partial C_2(R)) \cong H_4(C_2(R)) = 0,$$

we obtain $H^1(\partial C_2(R);\mathbb{R}) = 0$. Therefore, there exists a function f from $\partial C_2(R)$ to $\mathbb{R}$ such that

$$df = \eta - p_\tau^*(\eta_A)$$

on $\partial C_2(R)$. Extend f to a C^∞ map on $C_2(R)$ and replace η by $(\eta - df)$. $\square$

Then

$$\begin{aligned}\int_{C_2(R)} \omega'_a \wedge \omega_b \wedge \omega_c - \int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c &= \int_{C_2(R)} d(\eta \wedge \omega_b \wedge \omega_c)\\ &= \int_{\partial C_2(R)} \eta \wedge \omega_b \wedge \omega_c\\ &= \int_{\partial C_2(R)} p_\tau^*(\eta_A \wedge \omega_B \wedge \omega_C) = 0\end{aligned}$$

since any 5-form on S^2 vanishes. Thus, $\int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c$ is independent of the propagating forms ω_a, ω_b and ω_c. Now, we can choose the propagating forms ω_a, ω_b and ω_c supported in very small neighborhoods of P_a, P_b and P_c and Poincaré dual to P_a, P_b and P_c, respectively. So the intersection of the three supports is a very small neighborhood of $P_a \cap P_b \cap P_c$, from which it can easily be seen that $\int_{C_2(R)} \omega_a \wedge \omega_b \wedge \omega_c = \langle P_a, P_b, P_c\rangle_{C_2(R)}$. See Section 11.4, Section B.2 and Lemma B.4 in particular, for more details. $\square$

In particular, $\Theta(R, \tau)$ is equal to $\int_{C_2(R)} \omega^3$ for any propagating chain ω of $(C_2(R), \tau)$. Since such a propagating chain represents the linking number, $\Theta(R, \tau)$ can be thought of as the *cube of the linking number with respect to* τ.

When τ varies continuously, $\Theta(R, \tau)$ varies continuously in $\mathbb{Q}$. So $\Theta(R, \tau)$ is an invariant of the homotopy class of τ.

Remark 4.3. Define a *combing* of $\check{R}$ to be a section of $U\check{R}$ that coincides with $\tau_s(v)$ outside B_R, for some unit vector v of $\mathbb{R}^3$. For a combing X of R, define a propagating chain of $(C_2(R), X)$ as a propagating chain of $C_2(R)$ that intersects $U\check{R}$ along the image of X, and define $\tilde{\Theta}(R, X)$ as the algebraic intersection of a propagating chain of $(C_2(R), X)$, a propagating chain of $(C_2(R), -X)$ and any other propagating chain of $C_2(R)$. It is easily proved in [Les15a, Theorem 2.1] that $\tilde{\Theta}(R, X)$ depends only on R and on the homotopy class of X among combings. In particular, $\Theta(R, \tau) = \tilde{\Theta}(R, \tau(v))$ depends only on the homotopy class of the combing $\tau(v)$ of $U\check{R}$, for some unit vector v of $\mathbb{R}^3$. Further properties of the invariant $\tilde{\Theta}(R, .)$ of combings are studied in [Les15c]. An explicit formula for the invariant $\tilde{\Theta}(R, .)$ from a Heegaard diagram of R was found by the author in [Les15a]. See [Les15a, Theorem 3.8]. It was directly computed using the above definition of $\tilde{\Theta}(R, .)$ together with propagators associated with Morse functions, which were constructed by Greg Kuperberg and the author in [Les15a].

Example 4.4. Using (disjoint !) propagators $p_{S^2}^{-1}(a)$, $p_{S^2}^{-1}(b)$, $p_{S^2}^{-1}(c)$ associated to three distinct points a, b and c of $\mathbb{R}^3$, as in Example 3.10, it is clear that

$$\Theta(S^3, \tau_s) = \langle p_{S^2}^{-1}(a), p_{S^2}^{-1}(b), p_{S^2}^{-1}(c)\rangle_{C_2(S^3)} = 0.$$

4.2 Parallelizations of 3-manifolds and Pontrjagin classes

In this subsection, M denotes a smooth, compact oriented 3-manifold with possible boundary ∂M. Recall that a well-known theorem reproved in Section 5.2 ensures that such a 3-manifold is parallelizable.

Let $GL^+(\mathbb{R}^3)$ denote the group of orientation-preserving linear isomorphisms of $\mathbb{R}^3$. Let $[(M,\partial M),(GL^+(\mathbb{R}^3),1)]_m$ denote the set of (continuous) maps

$$g:(M,\partial M)\longrightarrow(GL^+(\mathbb{R}^3),1)$$

from M to $GL^+(\mathbb{R}^3)$ that send ∂M to the unit 1 of $GL^+(\mathbb{R}^3)$.
Let $[(M,\partial M),(GL^+(\mathbb{R}^3),1)]$ denote the group of homotopy classes of such maps, with the group structure induced by the multiplication of maps, using the multiplication in $GL^+(\mathbb{R}^3)$. For a map g in $[(M,\partial M),(GL^+(\mathbb{R}^3),1)]_m$, set

$$\begin{array}{rccc}\psi_{\mathbb{R}}(g): & M\times\mathbb{R}^3 & \longrightarrow & M\times\mathbb{R}^3\\ & (x,y) & \mapsto & (x,g(x)(y)).\end{array}$$

Let $\tau_M\colon M\times\mathbb{R}^3\to TM$ be a parallelization of M. Then any parallelization τ of M that coincides with τ_M on ∂M may be written as

$$\tau=\tau_M\circ\psi_{\mathbb{R}}(g)$$

for some $g\in[(M,\partial M),(GL^+(\mathbb{R}^3),1)]_m$.

Thus, fixing τ_M identifies the set of homotopy classes of parallelizations of M fixed on ∂M with the group $[(M,\partial M),(GL^+(\mathbb{R}^3),1)]$. Since $GL^+(\mathbb{R}^3)$ deformation retracts onto the group $SO(3)$ of orientation-preserving linear isometries of $\mathbb{R}^3$, the group $[(M,\partial M),(GL^+(\mathbb{R}^3),1)]$ is isomorphic to $[(M,\partial M),(SO(3),1)]$.

Definition 4.5. We regard S^3 as $B^3/\partial B^3$, where B^3 is the standard unit ball of $\mathbb{R}^3$ viewed as $([0,1]\times S^2)/(0\sim\{0\}\times S^2)$. Let $\chi_\pi\colon[0,1]\to[0,2\pi]$ be an increasing smooth bijection whose derivatives vanish at 0 and 1 such that $\chi_\pi(1-\theta)=2\pi-\chi_\pi(\theta)$ for any $\theta\in[0,1]$. Let $\rho\colon B^3\to SO(3)$ be the map that sends $(\theta\in[0,1],v\in S^2)$ to the rotation $\rho(\chi_\pi(\theta);v)$ with axis directed by v and with angle $\chi_\pi(\theta)$.

This map induces the double covering map $\tilde\rho\colon S^3\to SO(3)$, which orients $SO(3)$ and which allows one to deduce the first three homotopy groups of $SO(3)$ from those of S^3. The first three homotopy groups of $SO(3)$ are $\pi_1(SO(3))=\mathbb{Z}/2\mathbb{Z}$, $\pi_2(SO(3))=0$ and $\pi_3(SO(3))=\mathbb{Z}[\tilde\rho]$. For $v\in S^2$,

$\pi_1(SO(3))$ is generated by the class of the loop that maps $\exp(i\theta) \in S^1$ to the rotation $\rho(\theta; v)$. See Section A.2 and Theorem A.14 in particular.

Note that a map g from $(M, \partial M)$ to $(SO(3), 1)$ has a degree $\deg(g)$, which may be defined as the differential degree at a regular value (different from 1) of g. It can also be defined homologically, by $H_3(g)[M, \partial M] = \deg(g)[SO(3), 1]$.

The following theorem, for which no originality is claimed, is proved in Chapter 5 as a direct consequence of Definition 5.13, Lemmas 5.2, 5.7, 5.8 and Propositions 5.10, 5.22 and 5.25.

Theorem 4.6. *For any compact connected oriented* 3*-manifold* M*, the group*

$$[(M, \partial M), (SO(3), 1)]$$

is abelian, and the degree

$$deg\colon [(M, \partial M), (SO(3), 1)] \longrightarrow \mathbb{Z}$$

is a group homomorphism, which induces an isomorphism

$$deg\colon [(M, \partial M), (SO(3), 1)] \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}.$$

When $\partial M = \emptyset$*, (resp. when* $\partial M = S^2$*), there exists a canonical map* p_1 *from the set of homotopy classes of parallelizations of* M *(resp. that coincide with* τ_s *near* S^2*) to* $\mathbb{Z}$ *such that, for any map* g *in* $[(M, \partial M), (SO(3), 1)]_m$*, for any trivialization* τ *of* TM

$$p_1(\tau \circ \psi_{\mathbb{R}}(g)) - p_1(\tau) = 2deg(g), \quad \textit{and} \quad p_1((\tau_s)_{|B^3}) = 0.$$

Definition 5.13 of the map p_1 involves relative Pontrjagin classes. When $\partial M = \emptyset$, the map p_1 coincides with the map h that is studied by Hirzebruch in [Hir73, §3.1], and by Kirby and Melvin in [KM99] under the name of *Hirzebruch defect.*

Since $[(M, \partial M), (SO(3), 1)]$ is abelian, the set of parallelizations of M that are fixed on ∂M is an affine space with translation group $[(M, \partial M), (SO(3), 1)]$.

Recall the map $\rho\colon B^3 \to SO(3)$ from Definition 4.5. Let M be an oriented connected 3-manifold with possible boundary. For a ball B^3 embedded in M, let $\rho_M(B^3) \in [(M, \partial M), (SO(3), 1)]_m$ be a smooth map that coincides with ρ on B^3 and that maps the complement of B^3 to the unit of $SO(3)$. The homotopy class of $\rho_M(B^3)$ is well defined.

Lemma 4.7. $deg(\rho_M(B^3)) = 2$

PROOF: Exercise. □

4.3 Defining a $\mathbb{Q}$-sphere invariant from Θ

Recall that an asymptotic rational homology $\mathbb{R}^3$ is a pair $(\check{R}, \tau)$, in which $\check{R}$ is a 3-manifold decomposed as the union over $]1,2] \times S^2$ of a rational homology ball B_R and the complement $\mathring{B}_{1,\infty} \setminus \{\infty\}$ of the unit ball of $\mathbb{R}^3$, which is equipped with an asymptotically standard parallelization τ.

In this subsection, we prove the following proposition.

Proposition 4.8. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. For any map g in $[(B_R, B_R \cap \mathring{B}_{1,\infty}), (SO(3), 1)]_m$ trivially extended to $\check{R}$,*

$$\Theta(R, \tau \circ \psi_{\mathbb{R}}(g)) - \Theta(R, \tau) = \frac{1}{2} deg(g).$$

Theorem 4.6 allows us to derive the following corollary from Proposition 4.8.

Corollary 4.9. $\Theta(R) = \Theta(R, \tau) - \frac{1}{4} p_1(\tau)$ *is an invariant of* $\mathbb{Q}$*-spheres.*

$\square$

Since $p_1(\tau_s) = 0$, Example 4.4 shows that $\Theta(S^3) = 0$. We will prove that for any $\mathbb{Q}$-sphere R, $\Theta(-R) = -\Theta(R)$ in Proposition 5.15.

More properties of Θ will appear later in this book. We will first view this invariant as the degree one part of a much more general invariant $\mathcal{Z}$ or $\mathfrak{z}$ (introduced in Theorem 7.20 and in Corollary 10.9, respectively) in Corollary 10.11. The multiplicativity of $\mathcal{Z}$ under connected sum stated in Theorem 10.24 will imply that Θ is additive under connected sum. The invariant Θ will be identified with $6\lambda_{CW}$, where λ_{CW} is the Walker generalization of the Casson invariant to $\mathbb{Q}$-spheres, in Section 18.6. See Theorem 18.30. (The Casson-Walker invariant λ_{CW} is normalized as $\frac{1}{2}\lambda_W$ for rational homology spheres, where λ_W is the Walker normalisation in [Wal92].) The equality $\Theta = 6\lambda_{CW}$ will be obtained as a consequence of a *universality property* of $\mathcal{Z}$ with respect to a theory of finite type invariants. We will also present a direct proof of a surgery formula satisfied by Θ in Section 18.3.

Let us now prove Proposition 4.8.

Lemma 4.10. *The variation $\Theta(R, \tau \circ \psi_{\mathbb{R}}(g)) - \Theta(R, \tau)$ is independent of τ. Set $\Theta'(g) = \Theta(R, \tau \circ \psi_{\mathbb{R}}(g)) - \Theta(R, \tau)$. Then Θ' is a homomorphism from $[(B_R, B_R \cap \mathring{B}_{1,\infty}), (SO(3), 1)]$ to $\mathbb{Q}$.*

PROOF: For $d = a$, b or c, the propagator P_d of $(C_2(R), \tau)$ of Theorem 4.1 can be assumed to be a product $[-1,0] \times p^{-1}_{\tau|UB_R}(d)$ on a collar $[-1,0] \times UB_R$ of UB_R in $C_2(R)$. Since $H_3([-1,0] \times UB_R; \mathbb{Q}) = 0$,

$$\Big(\partial([-1,0] \times p^{-1}_{\tau|UB_R}(d)) \setminus (\{0\} \times p^{-1}_{\tau|UB_R}(d))\Big) \cup (\{0\} \times p^{-1}_{\tau \circ \psi_{\mathbb{R}}(g)|UB_R}(d))$$

bounds a chain G_d.

The chains G_a, G_b and G_c can be assumed to be transverse. Construct the propagator $P_d(g)$ of $(C_2(R), \tau\circ\psi_{\mathbb{R}}(g))$ from P_d by replacing $[-1,0]\times p^{-1}_{\tau|UB_R}(d)$ with G_d on $[-1,0]\times UB_R$. Then

$$\Theta(R, \tau\circ\psi_{\mathbb{R}}(g)) - \Theta(R,\tau) = \langle G_a, G_b, G_c\rangle_{[-1,0]\times UB_R}.$$

Using τ to identify UB_R with $B_R\times S^2$, ∂G_d may be written as

$$\partial G_d = \left((\partial([-1,0]\times B_R)\setminus(\{0\}\times B_R))\times\{d\}\right)\cup\{0\}\times\left(\cup_{m\in B_R}(m, g(m)(d))\right).$$

Therefore, $\Theta(R,\tau\circ\psi_{\mathbb{R}}(g)) - \Theta(R,\tau)$ is independent of τ. Then it is easy to see that Θ' is a homomorphism from $[(B_R, \partial B_R), (SO(3), 1)]$ to $\mathbb{Q}$. $\square$

According to Theorem 4.6 and to Lemma 4.7, it suffices to prove that $\Theta'(\rho_R(B^3)) = 1$ in order to prove Proposition 4.8. It is easy to see that $\Theta'(\rho_R(B^3)) = \Theta'(\rho)$. Thus, we are left with the proof of the following lemma.

Lemma 4.11. $\Theta'(\rho) = 1$.

We prove this lemma by computing the expression $\langle G_a, G_{-a}, G_c\rangle_{[-1,0]\times UB^3}$ from the above proof of Lemma 4.10 with explicit G_a and G_{-a} in $[-1,0]\times UB^3$. The chain G_a is constructed from a chain $G(a)$ of $B^3\times S^2$, described in Lemma 4.13.

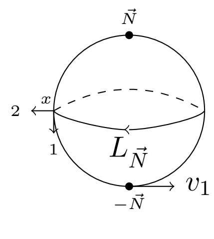
Again, we regard B^3 as $([0,1]\times S^2)/(0\sim\{0\}\times S^2)$. We first prove the following lemma:

Lemma 4.12. *Recall the map $\rho\colon B^3\to SO(3)$ from Definition 4.5. Let $a\in S^2$. The point $(-a)$ is regular for the map*

$$\begin{array}{cccc}\rho_a\colon & B^3 & \to & S^2\\ & m & \mapsto & \rho(m)(a)\end{array}$$

and its preimage (cooriented by S^2 via ρ_a) is the knot $L_a = -\{\frac{1}{2}\}\times S_a$, where S_a is the circle of S^2 of vectors orthogonal to a, which is oriented as the boundary of the hemisphere that contains a.

PROOF: We prove the lemma when a is the North Pole $\vec{N}$. It is easy to see that $\rho^{-1}_{\vec{N}}(-\vec{N}) = L_{\vec{N}}$, up to orientation.



Let $x \in L_{\vec{N}}$. When m moves along the great circle that contains $\vec{N}$ and x from x towards $(-\vec{N})$ in $\{\frac{1}{2}\} \times S^2$, $\rho(m)(\vec{N})$ moves from $(-\vec{N})$ in the same direction, which will be the direction of the tangent vector v_1 of S^2 at $(-\vec{N})$, counterclockwise in our picture, where x is on the left. Then in our picture, S^2 is oriented at $(-\vec{N})$ by v_1 and by the tangent vector v_2 at $(-\vec{N})$ towards us. In order to move $\rho(\theta; v)(\vec{N})$ in the v_2 direction, one increases θ, so $L_{\vec{N}}$ is cooriented and oriented as in the figure. □

Lemma 4.13. *Let $a \in S^2$. Recall the notation from Lemma 4.12 above. When $a \notin L_a$, let $[a, \rho(m)(a)]$ denote the unique geodesic arc of S^2 with length ($\ell \in [0, \pi[$) from a to $\rho(m)(a) = \rho_a(m)$. For $t \in [0,1]$, let $X_t(m) \in [a, \rho_a(m)]$ be such that the length of $[X_0(m) = a, X_t(m)]$ is $t\ell$. Let $G_h(a)$ be the closure of $\left(\cup_{t\in[0,1], m\in(B^3\setminus L_a)} (m, X_t(m))\right)$ in $B^3 \times S^2$. Let D_a be the disk $0 \cup]0, \frac{1}{2}] \times (-S_a)$ bounded by L_a in B^3. Set*

$$G(a) = G_h(a) + D_a \times S^2.$$

Then $G(a)$ is a chain of $B^3 \times S^2$ such that

$$\partial G(a) = -(B^3 \times a) + \cup_{m\in B^3}(m, \rho_a(m)).$$

Proof: The map X_t is well defined on $(B^3 \setminus L_a)$ and $X_1(m) = \rho_a(m)$. Let us show how the definition of X_t extends smoothly to the manifold $B\ell(B^3, L_a)$ obtained from B^3 by blowing up L_a. The map ρ_a maps the normal bundle to L_a to a disk of S^2 around $(-a)$, by an orientation-preserving diffeomorphism on every fiber (near the origin). In particular, ρ_a induces a map from the unit normal bundle to L_a to the unit normal bundle to $(-a)$ in S^2. This map preserves the orientation of the fibers. Then for an element y of the unit normal bundle to L_a in R, define $X_t(y)$ as before on the half great circle $[a, -a]_{\rho_a(-y)}$ from a to $(-a)$ that is tangent to $\rho_a(-y)$ at $(-a)$ (so $\rho_a(-y)$ is an outward normal to $[a, -a]_{\rho_a(-y)}$ at $(-a)$). This extends the definition of X_t, continuously. The whole sphere is covered with degree (-1) by the image of $([0,1] \times SN_x(L_a))$, where the fiber $SN_x(L_a)$ of the unit normal bundle to L_a is oriented as the boundary of a disk in the fiber of the normal bundle. Then

$$G_h(a) = \cup_{t\in[0,1], m\in B\ell(B^3, L_a)} \left(p_{B^3}(m), X_t(m)\right),$$

so

$$\partial G_h(a) = -(B^3 \times a) + \cup_{m\in B^3}(m, \rho_a(m)) + \cup_{t\in[0,1]} X_t(-\partial B\ell(S^3, L_a)),$$

where $(-\partial \mathrm{B}\ell(S^3, L_a))$ is oriented as $\partial N(L_a)$ so that the last summand may be written as $(-L_a \times S^2)$, because the sphere is covered with degree (-1) by the image of $([0,1] \times SN_x(L_a))$. □

PROOF OF LEMMA 4.11: We use the notation of the proof of Lemma 4.10 and we construct an explicit G_a in $[-1,0] \times UB^3 \stackrel{\tau_s}{=} [-1,0] \times B^3 \times S^2$. Let ι be the endomorphism of UB^3 over B^3 that maps a unit vector to the opposite one. Recall the chain $G(a)$ of Lemma 4.13 and set

$$\begin{array}{rl} G_a = & [-1,-2/3] \times B^3 \times a + \{-2/3\} \times G(a) \\ & +[-2/3,0] \times \cup_{m \in B^3}(m, \rho_a(m)) \\ \text{and} \quad G_{-a} & = [-1,-1/3] \times B^3 \times (-a) + \{-1/3\} \times \iota(G(a)) \\ & +[-1/3,0] \times \cup_{m \in B^3}(m, \rho(m)(-a)). \end{array}$$

Then

$$\begin{array}{rl} G_a \cap G_{-a} = & [-2/3,-1/3] \times L_a \times (-a) + \{-2/3\} \times D_a \times (-a) \\ & -\{-1/3\} \times \cup_{m \in D_a}(m, \rho_a(m)). \end{array}$$

Finally, according to the proof of Lemma 4.10, $\Theta'(\rho)$ is the algebraic intersection of $G_a \cap G_{-a}$ with $P_c(\rho)$ in $C_2(R)$. This intersection coincides with the algebraic intersection of $G_a \cap G_{-a}$ with any propagator of $C_2(R)$, according to Lemma 3.12. Therefore

$$\Theta'(\rho) = \langle P_c, G_a \cap G_{-a} \rangle_{[-1,0] \times B^3 \times S^2} = -\deg_c(\rho_a \colon D_a \to S^2)$$

for any regular value $c \neq -a$ of $\rho_{a|D_a}$. The image of

$$(-D_a) = 0 - \cup]0, \frac{1}{2}] \times L_a$$

under ρ_a covers the sphere with degree 1, so $\Theta'(\rho) = 1$. □

Chapter 5

Parallelizations of 3-manifolds and Pontrjagin classes

In this chapter, we study parallelizations of general oriented 3-manifolds with possible boundary, and we fix such a smooth oriented connected 3-manifold M with possible boundary. In particular, we prove Theorem 4.6 in Sections 5.1, 5.3 and 5.7. This theorem will be used in our general constructions of link invariants in 3-manifolds in the same way as it was used in the definition of Θ in Section 4.3. This chapter also describes further properties of Pontrjagin classes, which will be used in the fourth part of this book in some universality proofs. Section 5.9 describes the structure of the space of parallelizations of oriented 3-manifolds, more precisely. It is not used in other parts of the book.

5.1 $[(M, \partial M), (SO(3), 1)]$ is an abelian group.

Again, we regard S^3 as $B^3/\partial B^3$ and B^3 as $([0,1] \times S^2)/(0 \sim \{0\} \times S^2)$. Recall the map $\rho\colon B^3 \to SO(3)$ of Definition 4.5, which maps $(\theta \in [0,1], v \in S^2)$ to the rotation $\rho(\chi_\pi(\theta); v)$ with axis directed by v and with angle $\chi_\pi(\theta)$. Also recall that the group structure of $[(M, \partial M), (SO(3), 1)]$ is induced by the multiplication of maps, using the multiplication of $SO(3)$.

Any $g \in [(M, \partial M), (SO(3), 1)]_m$ induces a map

$$H_1(g; \mathbb{Z})\colon H_1(M, \partial M; \mathbb{Z}) \longrightarrow (H_1(SO(3), 1) = \mathbb{Z}/2\mathbb{Z}).$$

Since

$$\begin{aligned} H_1(M, \partial M; \mathbb{Z}/2\mathbb{Z}) &= H_1(M, \partial M; \mathbb{Z})/2H_1(M, \partial M; \mathbb{Z}) \\ &= H_1(M, \partial M; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}, \end{aligned}$$

$$
\begin{aligned}
\mathrm{Hom}(H_1(M,\partial M;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}) &= \mathrm{Hom}(H_1(M,\partial M;\mathbb{Z}/2\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})\\
&= H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z}),
\end{aligned}
$$

and the image of $H_1(g;\mathbb{Z})$ in $H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z})$ under the above isomorphisms is denoted by $H^1(g;\mathbb{Z}/2\mathbb{Z})$. (Equivalently, $H^1(g;\mathbb{Z}/2\mathbb{Z})$ denotes the image of the generator of $H^1(SO(3),1;\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}$ under $H^1(g;\mathbb{Z}/2\mathbb{Z})$ in $H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z})$.)

For $v\in S^2$, let γ_v denote the loop that maps $\exp(i\theta)\in S^1$ to the rotation $\rho(\theta;v)$ with axis directed by v and with angle θ. Let $\mathbb{R}P^2_S$ denote the projective plane embedded in $SO(3)$ consisting of the rotations of $SO(3)$ of angle π. Note that $\gamma_v(S^1)=\rho\left([0,1]\times\{v\}\right)$ and $\mathbb{R}P^2_S=\rho(\{1/2\}\times S^2)$ intersect once transversally. Thus, $H_1(SO(3);\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}[\gamma_v]$ and $H_2(SO(3);\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}[\mathbb{R}P^2_S]$.

Lemma 5.1. *The map*

$$
\begin{array}{llll}
H^1(.;\mathbb{Z}/2\mathbb{Z})\colon & [(M,\partial M),(SO(3),1)] & \to & H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z})\\
& [g] & \mapsto & H^1(g;\mathbb{Z}/2\mathbb{Z})
\end{array}
$$

is a group homomorphism.

PROOF: Let f and g be two elements of $[(M,\partial M),(SO(3),1)]_m$. In order to prove that $H^1(fg;\mathbb{Z}/2\mathbb{Z})=H^1(f;\mathbb{Z}/2\mathbb{Z})+H^1(g;\mathbb{Z}/2\mathbb{Z})$, it suffices to prove that, for any path $\gamma\colon[0,1]\to M$ such that $\gamma(\partial[0,1])\subset\partial M$,

$$H_1(fg;\mathbb{Z}/2\mathbb{Z})([\gamma])=H_1(f;\mathbb{Z}/2\mathbb{Z})([\gamma])+H_1(g;\mathbb{Z}/2\mathbb{Z})([\gamma]),$$

where $H_1(f;\mathbb{Z}/2\mathbb{Z})([\gamma])=\langle f\circ\gamma,\mathbb{R}P^2_S\rangle_{SO(3)}[\gamma_v]$ for the mod 2 algebraic intersection $\langle f\circ\gamma,\mathbb{R}P^2_S\rangle_{SO(3)}$. Both sides of this equation depend only on the homotopy classes of $f\circ\gamma$ and $g\circ\gamma$. Therefore, we may assume that the supports (closures of the preimages of $SO(3)\setminus\{1\}$) of $f\circ\gamma$ and $g\circ\gamma$ are disjoint and, in this case, the equality is easy to get. $\square$

Lemma 5.2. *Let M be an oriented connected 3-manifold with possible boundary. Recall that $\rho_M(B^3)\in[(M,\partial M),(SO(3),1)]_m$ is a map that coincides with ρ on a ball B^3 embedded in M and that maps the complement of B^3 to the unit of $SO(3)$.*

1. *Any homotopy class of a map g from $(M,\partial M)$ to $(SO(3),1)$, such that $H^1(g;\mathbb{Z}/2\mathbb{Z})$ is trivial, belongs to the subgroup $<[\rho_M(B^3)]>$ of $[(M,\partial M),(SO(3),1)]$ generated by $[\rho_M(B^3)]$.*
2. *For any $[g]\in[(M,\partial M),(SO(3),1)]$, $[g]^2\in<[\rho_M(B^3)]>$.*

3. The group $[(M, \partial M), (SO(3), 1)]$ *is abelian.*

PROOF: Let $g \in [(M, \partial M), (SO(3), 1)]_m$. Assume that $H^1(g; \mathbb{Z}/2\mathbb{Z})$ is trivial. Choose a cell decomposition of M relative to its boundary, with only one three-cell, no zero-cell if $\partial M \neq \emptyset$, one zero-cell if $\partial M = \emptyset$, one-cells, and two-cells. See [Hir94, Chapter 6, Section 3]. Then after a homotopy relative to ∂M, we may assume that g maps the one-skeleton of M to 1. Next, since $\pi_2(SO(3)) = 0$, we may assume that g maps the two-skeleton of M to 1, and therefore that g maps the exterior of some 3-ball to 1. Now g becomes a map from $B^3/\partial B^3 = S^3$ to $SO(3)$, and its homotopy class is $k[\tilde{\rho}]$ in $\pi_3(SO(3)) = \mathbb{Z}[\tilde{\rho}]$. Therefore g is homotopic to $\rho_M(B^3)^k$. This proves the first assertion.

Since $H^1(g^2; \mathbb{Z}/2\mathbb{Z}) = 2H^1(g; \mathbb{Z}/2\mathbb{Z})$ is trivial, the second assertion follows.

For the third assertion, first note that $[\rho_M(B^3)]$ belongs to the center of $[(M, \partial M), (SO(3), 1)]$ because it can be supported in a small ball disjoint from the support (preimage of $SO(3) \setminus \{1\}$) of a representative of any other element. Therefore, according to the second assertion, any square is in the center. In particular, if f and g are elements of $[(M, \partial M), (SO(3), 1)]$,

$$(gf)^2 = (fg)^2 = (f^{-1}f^2g^2f)(f^{-1}g^{-1}fg),$$

where the first factor is equal to $f^2g^2 = g^2f^2$. Exchanging f and g yields $f^{-1}g^{-1}fg = g^{-1}f^{-1}gf$. Then the commutator, which is a power of $[\rho_M(B^3)]$, thanks to Lemma 5.1 and to the first assertion, has a vanishing square, and thus a vanishing degree. Then it must be trivial. □

5.2 Any oriented 3-manifold is parallelizable.

In this subsection, we prove the following standard theorem. The spirit of our proof is the same as the Kirby proof in [Kir89, p.46]. But instead of assuming familiarity with the obstruction theory described by Steenrod in [Ste51, Part III], we use this proof as an introduction to this theory.

Theorem 5.3 (Stiefel)**.** *Any oriented* 3*-manifold is parallelizable.*

Lemma 5.4. *The restriction of the tangent bundle* TM *to an oriented* 3*-manifold* M*, to any closed surface* S*, orientable or not, immersed in* M*, is trivializable.*

PROOF: Let us first prove that this bundle is independent of the immersion. It is the direct sum of the tangent bundle to the surface and of its normal one-dimensional bundle. This normal bundle is trivial when S is orientable, and its unit bundle is the 2-fold orientation cover of the surface, otherwise. (The orientation cover of S is its 2-fold orientable cover that is trivial over annuli embedded in the surface). Then since any surface S can be immersed in $\mathbb{R}^3$, the restriction $TM_{|S}$ is the pull-back of the trivial bundle of $\mathbb{R}^3$ by such an immersion, and it is trivial. $\square$

Then using Stiefel-Whitney classes, the proof of Theorem 5.3 goes quickly as follows. Let M be an orientable smooth 3-manifold, equipped with a smooth triangulation. (A theorem of Whitehead proved in the Munkres book [Mun66] ensures the existence of such a triangulation.) By definition, the *first Stiefel-Whitney class* $w_1(TM) \in H^1(M;\mathbb{Z}/2\mathbb{Z} = \pi_0(GL(\mathbb{R}^3)))$ viewed as a map from $\pi_1(M)$ to $\mathbb{Z}/2\mathbb{Z}$ maps the class of a loop c embedded in M to 0 if $TM_{|c}$ is orientable and to 1 otherwise. It is the obstruction to the existence of a trivialization of TM over the one-skeleton of M. Since M is orientable, the first Stiefel-Whitney class $w_1(TM)$ vanishes and TM can be trivialized over the one-skeleton of M. The *second Stiefel-Whitney class* $w_2(TM) \in H^2(M;\mathbb{Z}/2\mathbb{Z} = \pi_1(GL^+(\mathbb{R}^3)))$ viewed as a map from $H_2(M;\mathbb{Z}/2\mathbb{Z})$ to $\mathbb{Z}/2\mathbb{Z}$ maps the class of a connected closed surface S to 0 if $TM_{|S}$ is trivializable and to 1 otherwise. The second Stiefel-Whitney class $w_2(TM)$ is the obstruction to the existence of a trivialization of TM over the two-skeleton of M, when $w_1(TM) = 0$. According to the above lemma, $w_2(TM) = 0$, and TM can be trivialized over the two-skeleton of M. Then since $\pi_2(GL^+(\mathbb{R}^3)) = 0$, any parallelization over the two-skeleton of M can be extended as a parallelization of M. $\square$

We detail the involved arguments without mentioning Stiefel-Whitney classes, (in fact, by almost defining $w_2(TM)$), below. The elementary proof below can be thought of as an introduction to the obstruction theory used above.

Elementary proof of Theorem 5.3: Let M be an oriented 3-manifold. Choose a triangulation of M. For any cell c of the triangulation, define an arbitrary trivialization $\tau_c \colon c \times \mathbb{R}^3 \to TM_{|c}$ such that τ_c induces the orientation of M. This defines a trivialization $\tau^{(0)} \colon M^{(0)} \times \mathbb{R}^3 \to TM_{|M^{(0)}}$ of M over the 0-skeleton $M^{(0)}$ of M, which is the set of 0-dimensional cells of the triangulation. Let $C_k(M)$ be the set of k–cells of the triangulation. Every cell is equipped with an arbitrary orientation. Let $e \in C_1(M)$ be an edge of the triangulation. On ∂e, $\tau^{(0)}$ may be written as $\tau^{(0)} = \tau_e \circ \psi_{\mathbb{R}}(g_e)$, for a map $g_e \colon \partial e \to GL^+(\mathbb{R}^3)$. Since $GL^+(\mathbb{R}^3)$ is connected, g_e extends to e, and $\tau^{(1)} = \tau_e \circ \psi_{\mathbb{R}}(g_e)$ extends $\tau^{(0)}$ to e. Doing so for all the edges extends $\tau^{(0)}$ to a trivialization $\tau^{(1)}$ of the one-skeleton $M^{(1)}$ of M, which is the union of the edges of the triangulation.

For an oriented triangle t of the triangulation, on ∂t, $\tau^{(1)}$ may be written as $\tau^{(1)} = \tau_t \circ \psi_{\mathbb{R}}(g_t)$, for a map $g_t \colon \partial t \to GL^+(\mathbb{R}^3)$. Let $E(t, \tau^{(1)})$ be the homotopy class of g_t in $(\pi_1(GL^+(\mathbb{R}^3)) = \pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z})$. Then $E(., \tau^{(1)}) \colon C_2(M) \to \mathbb{Z}/2\mathbb{Z}$ extends linearly to a cochain, which is independent of the τ_t. When $E(., \tau^{(1)}) = 0$, $\tau^{(1)}$ may be extended to a trivialization $\tau^{(2)}$ over the two-skeleton of M, as before. Since $\pi_2(GL^+(\mathbb{R}^3)) = 0$, $\tau^{(2)}$ can next be extended over the three-skeleton of M, that is over M.

Let us now study the obstruction cochain $E(., \tau^{(1)})$ whose vanishing guarantees the existence of a parallelization of M.

If the map g_e associated with e is changed to $d(e)g_e$ for some $d(e) \colon (e, \partial e) \to (GL^+(\mathbb{R}^3), 1)$, for every edge e, define the associated trivialization $\tau^{(1)\prime}$, and the cochain $D(\tau^{(1)}, \tau^{(1)\prime}) \colon (\mathbb{Z}/2\mathbb{Z})^{C_1(M)} \to \mathbb{Z}/2\mathbb{Z}$ that maps e to the homotopy class of $d(e)$. Then $(E(., \tau^{(1)\prime}) - E(., \tau^{(1)}))$ is the coboundary of $D(\tau^{(1)}, \tau^{(1)\prime})$. See Section A.1, before Theorem A.9.

Let us prove that $E(., \tau^{(1)})$ is a cocycle. Consider a 3-simplex T, then $\tau^{(0)}$ extends to T. Without loss of generality, assume that τ_T coincides with this extension, that for any face t of T, τ_t is the restriction of τ_T to t, and that the above $\tau^{(1)\prime}$ coincides with τ_T on the edges of ∂T. Then $E(., \tau^{(1)\prime})(\partial T) = 0$. Since a coboundary also maps ∂T to 0, $E(., \tau^{(1)})(\partial T) = 0$.

Now, it suffices to prove that the cohomology class of $E(., \tau^{(1)})$ (which is equal to $w_2(TM)$) vanishes, in order to prove that there is an extension $\tau^{(1)\prime}$ of $\tau^{(0)}$ on $M^{(1)}$ that extends on M.

Since $H^2(M; \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}(H_2(M; \mathbb{Z}/2\mathbb{Z}); \mathbb{Z}/2\mathbb{Z})$, it suffices to prove that $E(., \tau^{(1)})$ maps any 2–dimensional $\mathbb{Z}/2\mathbb{Z}$-cycle C to 0.

We represent the class of such a cycle C by a closed surface S, orientable or not, as follows. Let $N(M^{(0)})$ and $N(M^{(1)})$ be small regular neighborhoods of $M^{(0)}$ and $M^{(1)}$ in M, respectively, such that $N(M^{(1)}) \cap (M \setminus N(M^{(0)}))$ is a disjoint union, running over the edges e, of solid cylinders B_e identified with $]0,1[\times D^2$. The core $]0,1[\times\{0\}$ of $B_e =]0,1[\times D^2$ is a connected part of the interior of the edge e. ($N(M^{(1)})$ is thinner than $N(M^{(0)})$. See Figure 5.1.)

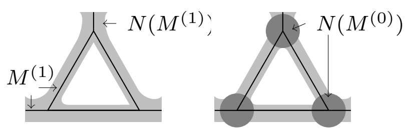


Figure 5.1: The neighborhoods $N(M^{(1)})$ and $N(M^{(0)})$

Construct S in the complement of $N(M^{(0)}) \cup N(M^{(1)})$ as the intersection of the support of C with this complement. Then the closure of S meets the part $[0,1] \times S^1$ of every $\overline{B_e}$ as an even number of parallel intervals from

$\{0\}\times S^1$ to $\{1\}\times S^1$. Complete S in $M\setminus N(M^{(0)})$ by connecting the intervals pairwise in $\overline{B_e}$ by disjoint bands. After this operation, the boundary of the closure of S is a disjoint union of circles in the boundary of $N(M^{(0)})$, where $N(M^{(0)})$ is a disjoint union of balls around the vertices. Glue disjoint disks of $N(M^{(0)})$ along these circles to finish the construction of S.

Extend $\tau^{(0)}$ to $N(M^{(0)})$, assume that $\tau^{(1)}$ coincides with this extension over $M^{(1)}\cap N(M^{(0)})$, and extend $\tau^{(1)}$ to $N(M^{(1)})$. The bundle $TM_{|S}$ is trivial, and we may choose a trivialization τ_S of TM over S that coincides with our extension of $\tau^{(0)}$ over $N(M^{(0)})$, over $S\cap N(M^{(0)})$. We have a cell decomposition of $(S, S\cap N(M^{(0)}))$ with only 1-cells and 2-cells, for which the 2-cells of S are in one-to-one canonical correspondence with the 2-cells of C, and one-cells correspond bijectively to bands connecting two-cells in the cylinders B_e. These one-cells are equipped with the trivialization of TM induced by $\tau^{(1)}$. Then we can define 2–dimensional cochains $E_S(., \tau^{(1)})$ and $E_S(.,\tau_S)$ as before, with respect to this cellular decomposition of S, where $(E_S(.,\tau^{(1)}) - E_S(.,\tau_S))$ is again a coboundary, and $E_S(.,\tau_S) = 0$, so $E_S(C,\tau^{(1)}) = 0$, and, since $E(C,\tau^{(1)}) = E_S(C,\tau^{(1)})$, $E(C,\tau^{(1)}) = 0$. $\square$

Theorem 5.3 has the following immediate corollary.

Proposition 5.5. *Any punctured oriented 3-manifold $\check{R}$ as in Definition 3.6 can be equipped with an asymptotically standard parallelization.*

Proof: The oriented manifold R admits a parallelization $\tau_0\colon R\times\mathbb{R}^3 \to TR$. Over $\mathring{B}_{1,\infty}\setminus\{\infty\}$, $\tau_s = \tau_0\circ\psi_{\mathbb{R}}(g)$ for a map $g\colon \mathring{B}_{1,\infty}\setminus\{\infty\} \to GL^+(\mathbb{R}^3)$. For $r\in[1,2]$, let $\mathring{B}_{r,\infty}$ (resp. $B_{r,\infty}$) be the complement in S^3 of the closed (resp. open) ball $B(r)$ of radius r in $\mathbb{R}^3$. Since $\pi_2(GL^+(\mathbb{R}^3)) = \{0\}$, the restriction of g to $B_{7/4,\infty}\setminus\mathring{B}_{2,\infty}$ extends to a map of $\mathring{B}_{1,\infty}\setminus\mathring{B}_{2,\infty}$ that maps $\mathring{B}_{1,\infty}\setminus\mathring{B}_{5/3,\infty}$ to 1. After smoothing, we get a smooth map $\tilde{g}\colon \mathring{B}_{1,\infty}\setminus\{\infty\} \to GL^+(\mathbb{R}^3)$ that coincides with g on $\mathring{B}_{2,\infty}$ and that maps $\mathring{B}_{1,\infty}\setminus\mathring{B}_{3/2,\infty}$ to 1. Extend $\tilde{g}$ to $\check{R}$ so that it maps $R\setminus\mathring{B}_{3/2,\infty}$ to 1, so $\tau_0\circ\psi_{\mathbb{R}}(\tilde{g})$ is an asymptotically standard parallelization as wanted. $\square$

5.3 The homomorphism induced by the degree

In this section, M is a compact connected oriented 3-manifold, with or without boundary. Let S be a closed surface, orientable or not, embedded in the interior of our manifold M, and let τ be a parallelization of our 3-manifold M. We define a twist $g(S,\tau)\in[(M,\partial M),(SO(3),1)]_m$ below.

The surface S has a tubular neighborhood $N(S)$, which is a $[-1,1]$–bundle over S. This bundle admits (orientation-preserving) bundle charts with domains $[-1,1]\times D$ for disks D of S, such that the changes of coordinates restrict to the fibers as $\pm$Identity. Then

$$g(S,\tau)\colon (M,\partial M) \longrightarrow (SO(3),1)$$

is the continuous map that maps $M \setminus N(S)$ to 1 such that $g(S,\tau)((t,s) \in [-1,1] \times D)$ is the rotation with angle $\pi(t+1)$ and with axis $p_2(\tau^{-1}(N_s) = (s, p_2(\tau^{-1}(N_s))))$, where $N_s = T_{(0,s)}([-1,1] \times s)$ is the tangent vector to the fiber $[-1,1] \times s$ at $(0,s)$. Since this rotation coincides with the rotation with opposite axis and with opposite angle $\pi(1-t)$, our map $g(S,\tau)$ is a well defined continuous map.

Clearly, the homotopy class of $g(S,\tau)$ depends only on the homotopy class of τ and on the isotopy class of S. When $M = B^3$, when τ is the standard parallelization of $\mathbb{R}^3$, and when $\frac{1}{2}S^2$ denotes the sphere $\frac{1}{2}\partial B^3$ inside B^3, the homotopy class of $g(\frac{1}{2}S^2,\tau)$ coincides with the homotopy class of ρ.

We will see later (Proposition 5.32) that the homotopy class of $g(S,\tau)$ depends only on the Euler characteristic $\chi(S)$ of S and on the class of S in $H_2(M;\mathbb{Z}/2\mathbb{Z})$. Thus, $g(S,\tau)$ will be simply denoted by $g(S)$. We will also see (Corollary 5.33) that any element of $[(M,\partial M),(SO(3),1)]$ can be represented by some $g(S)$.

Lemma 5.6. *The morphism $H^1(g(S,\tau);\mathbb{Z}/2\mathbb{Z})$ maps the generator of*

$$H^1(SO(3);\mathbb{Z}/2\mathbb{Z})$$

to the mod 2 intersection with S in

$$Hom(H_1(M,\partial M;\mathbb{Z}/2\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}) = H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z}).$$

The morphism $H^1(.;\mathbb{Z}/2\mathbb{Z})\colon [(M,\partial M),(SO(3),1)] \to H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z})$ is onto.

Proof: The first assertion is obvious, and the second one follows since $H^1(M,\partial M;\mathbb{Z}/2\mathbb{Z})$ is the Poincaré dual of $H_2(M;\mathbb{Z}/2\mathbb{Z})$ and since any element of $H_2(M;\mathbb{Z}/2\mathbb{Z})$ is the class of a closed surface. $\square$

Lemma 5.7. *The degree is a group homomorphism*

$$deg\colon [(M,\partial M),(SO(3),1)] \longrightarrow \mathbb{Z}$$

and $deg(\rho_M(B^3)^k) = 2k$.

Proof: It is easy to see that $\deg(fg) = \deg(f) + \deg(g)$ when f or g is a power of $[\rho_M(B^3)]$.

Let us prove that $\deg(f^2) = 2\deg(f)$ for any f. According to Lemma 5.6, there is an unoriented embedded surface S_f of the interior of C such that $H^1(f; \mathbb{Z}/2\mathbb{Z}) = H^1(g(S_f, \tau); \mathbb{Z}/2\mathbb{Z})$ for some trivialization τ of TM. Then, according to Lemmas 5.1 and 5.2, $fg(S_f, \tau)^{-1}$ is homotopic to some power of $\rho_M(B^3)$, and we are left with the proof that the degree of g^2 is $2\deg(g)$ for $g = g(S_f, \tau)$. This can be done easily, by noticing that g^2 is homotopic to $g(S_f^{(2)}, \tau)$, where $S_f^{(2)}$ is the boundary of the tubular neighborhood of S_f. In general, $\deg(fg) = \frac{1}{2}\deg((fg)^2) = \frac{1}{2}\deg(f^2g^2) = \frac{1}{2}\left(\deg(f^2) + \deg(g^2)\right)$, and the lemma is proved. $\square$

Lemmas 5.2 and 5.7 imply the following lemma.

Lemma 5.8. *The degree induces an isomorphism*

$$deg\colon [(M, \partial M), (SO(3), 1)] \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}.$$

Any group homomorphism $\psi\colon [(M, \partial M), (SO(3), 1)] \longrightarrow \mathbb{Q}$ may be expressed as

$$\frac{1}{2}\psi(\rho_M(B^3))\, deg.$$

$\square$

Proposition 5.9. *There is a canonical exact sequence*

$$0 \to H^3(M, \partial M; \pi_3(SO(3))) \xrightarrow{i} [(M, \partial M), (SO(3), 1)] \xrightarrow{P \circ H^1([.]; \mathbb{Z}/2\mathbb{Z})} H_2(M; \mathbb{Z}/2\mathbb{Z}) \to 0,$$

where $P\colon H^1(M, \partial M; \mathbb{Z}/2\mathbb{Z}) \to H_2(M; \mathbb{Z}/2\mathbb{Z})$ is a Poincaré duality isomorphism. Since M is connected (and oriented), $H^3(M, \partial M; \pi_3(SO(3)))$ is canonically isomorphic to $\mathbb{Z}$, $i(1) = [\rho_M(B^3)]$, and the morphism

$$\begin{array}{lcl} [(M, \partial M), (SO(3), 1)] \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & \mathbb{Q}[\rho_M(B^3)] \\ [g] \otimes 1 & \mapsto & \frac{deg(g)}{2}[\rho_M(B^3)] \end{array}$$

is an isomorphism.

Proof: The proposition is a consequence of Lemmas 5.2, 5.6 and 5.7. $\square$

5.4 On the groups $SU(n)$

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $n \in \mathbb{N}$. The stabilization maps induced by the inclusions

$$\begin{array}{rlcl} i: & GL(\mathbb{K}^n) & \longrightarrow & GL(\mathbb{K} \oplus \mathbb{K}^n) \\ & g & \mapsto & (i(g) : (x,y) \mapsto (x, g(y))) \end{array}$$

are denoted by i. Elements of $GL(\mathbb{K}^n)$ are represented by matrices whose columns contain the coordinates of the images of the basis elements, with respect to the standard basis of $\mathbb{K}^n$ View S^3 as the unit sphere of $\mathbb{C}^2$. So its elements are the pairs (z_1, z_2) of complex numbers such that $|z_1|^2 + |z_2|^2 = 1$. The group $SU(2)$ may be identified with S^3 by the homeomorphisms

$$\begin{array}{rlcl} m_r^{\mathbb{C}}\colon & S^3 & \to & SU(2) \\ & (z_1, z_2) & \mapsto & \begin{bmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{bmatrix} \end{array} \quad \text{and} \quad \begin{array}{rlcl} \overline{m}_r^{\mathbb{C}}\colon & S^3 & \to & SU(2) \\ & (z_1, z_2) & \mapsto & \begin{bmatrix} \overline{z}_1 & \overline{z}_2 \\ -z_2 & z_1 \end{bmatrix} \end{array},$$

so the first non-trivial homotopy group of $SU(2)$ is $\pi_3(SU(2)) = \mathbb{Z}[\overline{m}_r^{\mathbb{C}}]$, where $[\overline{m}_r^{\mathbb{C}}] = -[m_r^{\mathbb{C}}]$ and $\overline{m}_r^{\mathbb{C}}$ is a group homomorphism (it induces the group structure of S^3). The long exact sequence associated with the fibration

$$SU(n-1) \overset{i}{\hookrightarrow} SU(n) \to S^{2n-1},$$

described in Theorem A.14, proves that $i_*^n : \pi_j(SU(2)) \longrightarrow \pi_j(SU(n+2))$ is an isomorphism for $j \leq 4$ and $n \geq 0$, and in particular, that $\pi_j(SU(4)) = \{1\}$ for $j \leq 2$ and

$$\pi_3(SU(4)) = \mathbb{Z}[i_*^2(\overline{m}_r^{\mathbb{C}})],$$

where $i_*^2(\overline{m}_r^{\mathbb{C}})$ is the following map

$$\begin{array}{rccc} i_*^2(\overline{m}_r^{\mathbb{C}}): & (S^3 \subset \mathbb{C}^2) & \longrightarrow & SU(4) \\ & (z_1, z_2) & \mapsto & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{z}_1 & \overline{z}_2 \\ 0 & 0 & -z_2 & z_1 \end{bmatrix} \end{array}.$$

5.5 Definition of relative Pontrjagin numbers

Let M_0 and M_1 be two compact connected oriented 3-manifolds whose boundaries have collars that are identified by a diffeomorphism. Let $\tau_0\colon M_0 \times \mathbb{R}^3 \to TM_0$ and $\tau_1\colon M_1 \times \mathbb{R}^3 \to TM_1$ be two parallelizations (which respect the orientations) that agree on the collar neighborhoods of $\partial M_0 = \partial M_1$. Then the *relative Pontrjagin number* $p_1(\tau_0, \tau_1)$ is the Pontrjagin obstruction to extending the trivialization of $TW \otimes \mathbb{C}$ induced by τ_0 and τ_1 across the interior of a signature 0 cobordism W from M_0 to M_1. Details follow.

Let M be a compact connected oriented 3-manifold. A *special complex trivialization* of TM is a trivialization of $TM \otimes \mathbb{C}$ that is obtained from a trivialization $\tau_M \colon M \times \mathbb{R}^3 \to TM$ by composing $(\tau_M^{\mathbb{C}} = \tau_M \otimes_{\mathbb{R}} \mathbb{C}) \colon M \times \mathbb{C}^3 \to TM \otimes \mathbb{C}$ by

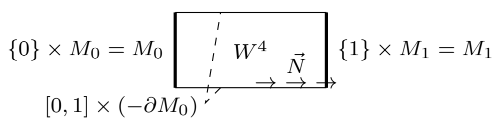
$$\begin{array}{llcl} \psi(G): & M \times \mathbb{C}^3 & \longrightarrow & M \times \mathbb{C}^3 \\ & (x,y) & \mapsto & (x, G(x)(y)) \end{array}$$

for a map $G \colon (M, \partial M) \to (SL(3,\mathbb{C}), 1)$, which is a map $G \colon M \to SL(3,\mathbb{C})$ that maps ∂M to 1. The definition and properties of relative Pontrjagin numbers $p_1(\tau_0, \tau_1)$, which are given with more details below, are valid for pairs (τ_0, τ_1) of special complex trivializations.

The *signature* of a 4-manifold is the signature of the intersection form on its $H_2(.;\mathbb{R})$ (number of positive entries minus number of negative entries in a diagonalised version of this form). It is well-known that any closed oriented three-manifold bounds a compact oriented 4-dimensional manifold. See [Rou85] for an elegant elementary proof. The signature of such a bounded 4-manifold may be changed arbitrarily, by connected sums with copies of $\mathbb{C}P^2$ or $-\mathbb{C}P^2$. A *cobordism from M_0 to M_1* is a compact oriented 4-dimensional manifold W with ridges such that

$$\partial W = -M_0 \cup_{\partial M_0 \sim 0 \times \partial M_0} (-[0,1] \times \partial M_0) \cup_{\partial M_1 \sim 1 \times \partial M_0} M_1,$$

which is identified with an open subspace of one of the products $[0,1[\times M_0$ or $]0,1] \times M_1$ near ∂W, as the following picture suggests.



Let $W = W^4$ be such a cobordism from M_0 to M_1, with signature 0. Consider the complex 4-bundle $TW \otimes \mathbb{C}$ over W. Let $\vec{N}$ be the tangent vector to $[0,1] \times \{\text{pt}\}$ over ∂W (under the above identifications), and let $\tau(\tau_0, \tau_1)$ denote the trivialization of $TW \otimes \mathbb{C}$ over ∂W that is obtained by stabilizing either τ_0 or τ_1 into $\vec{N} \oplus \tau_0$ or $\vec{N} \oplus \tau_1$. Then the obstruction to extending this trivialization to W is the relative first *Pontrjagin class*

$$p_1(W; \tau(\tau_0, \tau_1))[W, \partial W] \in H^4(W, \partial W; \mathbb{Z} = \pi_3(SU(4))) = \mathbb{Z}[W, \partial W]$$

of the trivialization.

Now, we specify our sign conventions for this Pontrjagin class. They are the same as in [MS74]. In particular, p_1 is the opposite of the second Chern class c_2 of the complexified tangent bundle. See [MS74, p. 174]. Let us describe those conventions. The determinant bundle of TW is trivial because W is oriented, and $\det(TW \otimes \mathbb{C})$ is also trivial. Our parallelization $\tau(\tau_0, \tau_1)$ over ∂W is special with respect to the trivialization of $\det(TW \otimes \mathbb{C})$. Equip M_0 and M_1 with Riemannian metrics that coincide near ∂M_0, and equip W with a Riemannian metric that coincides with the orthogonal product metric of one of the products $[0,1] \times M_0$ or $[0,1] \times M_1$ near ∂W. Equip $TW \otimes \mathbb{C}$ with the associated hermitian structure. Up to homotopy, assume that $\tau(\tau_0, \tau_1)$ is unitary with respect to the hermitian structure of $TW \otimes \mathbb{C}$ and the standard hermitian form of $\mathbb{C}^4$. Since $\pi_i(SU(4)) = \{0\}$ when $i < 3$, the trivialization $\tau(\tau_0, \tau_1)$ extends to a special unitary trivialization τ outside the interior of a 4-ball B^4 and defines

$$\tau \colon S^3 \times \mathbb{C}^4 \longrightarrow (TW \otimes \mathbb{C})_{|S^3}$$

over the boundary $S^3 = \partial B^4$ of this 4-ball B^4. Over this 4-ball B^4, the bundle $TW \otimes \mathbb{C}$ admits a special unitary trivialization

$$\tau_B \colon B^4 \times \mathbb{C}^4 \longrightarrow (TW \otimes \mathbb{C})_{|B^4}.$$

Then $\tau_B^{-1} \circ \tau(v \in S^3, w \in \mathbb{C}^4) = (v, \phi(v)(w))$, for a map $\phi : S^3 \longrightarrow SU(4)$ whose homotopy class may be written as

$$[\phi] = p_1(W; \tau(\tau_0, \tau_1))[i^2_*(\overline{m}^{\mathbb{C}}_r)] \in \pi_3(SU(4)),$$

where $i^2_*(\overline{m}^{\mathbb{C}}_r)$ was defined at the end of Section 5.4.

Define $p_1(\tau_0, \tau_1) = p_1(W; \tau(\tau_0, \tau_1))$.

Proposition 5.10. *Let M_0 and M_1 be two compact connected oriented* 3*-manifolds whose boundaries have collars that are identified by a diffeomorphism. Let $\tau_0 \colon M_0 \times \mathbb{C}^3 \to TM_0 \otimes \mathbb{C}$ and $\tau_1 \colon M_1 \times \mathbb{C}^3 \to TM_1 \otimes \mathbb{C}$ be two special complex trivializations (which respect the orientations) that agree on the collar neighborhoods of $\partial M_0 = \partial M_1$. The (first)* Pontrjagin number *$p_1(\tau_0, \tau_1)$ is well defined by the above conditions.*

Proof: According to the Nokivov additivity theorem, if a closed 4-manifold Y is decomposed as $Y = Y^+ \cup_X Y^-$, where Y^+ and Y^- are two 4-manifolds with boundary, embedded in Y, which intersect along a closed 3–manifold X (their common boundary, up to orientation), then

$$\mathrm{signature}(Y) = \mathrm{signature}(Y^+) + \mathrm{signature}(Y^-).$$

According to a Rohlin theorem (see [Roh52] or [GM86, p. 18]), when Y is a compact oriented 4–manifold without boundary, $p_1(Y) = 3\,\mathrm{signature}(Y)$.

We only need to prove that $p_1(\tau_0, \tau_1)$ is independent of the signature 0 cobordism W. Let W_E be a 4-manifold of signature 0 bounded by $(-\partial W)$. Then $W \cup_{\partial W} W_E$ is a 4-dimensional manifold without boundary whose signature is $(\mathrm{signature}(W_E) + \mathrm{signature}(W) = 0)$ by the Novikov additivity theorem. According to the Rohlin theorem, the first Pontrjagin class of $W \cup_{\partial W} W_E$ is also zero. On the other hand, this first Pontrjagin class is the sum of the relative first Pontrjagin classes of W and W_E with respect to $\tau(\tau_0, \tau_1)$. These two relative Pontrjagin classes are opposite and therefore the relative first Pontrjagin class of W with respect to $\tau(\tau_0, \tau_1)$ does not depend on W. $\square$

Similarly, it is easy to prove the following proposition.

Proposition 5.11. *Under the above assumptions except for the assumption on the signature of the cobordism W,*

$$p_1(\tau_0, \tau_1) = p_1(W; \tau(\tau_0, \tau_1)) - 3\,\mathrm{signature}(W).$$

$\square$

Remark 5.12. When $\partial M_1 = \emptyset$ and when $M_0 = \emptyset$, the map $p_1 (= p_1(\tau(\emptyset), .))$ coincides with the map h that is studied by Hirzebruch in [Hir73, §3.1], and by Kirby and Melvin in [KM99] under the name of *Hirzebruch defect.*

Definition 5.13. When $(\check{R}, \tau)$ is an asymptotic rational homology $\mathbb{R}^3$, set

$$p_1(\tau) = p_1\left((\tau_s)_{|B^3}, \tau_{|B_R}\right),$$

with the notation of Proposition 5.10.

Lemma 5.14. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$ as in Definition 3.8. The parallelization $\tau\colon \check{R} \times \mathbb{R}^3 \to T\check{R}$ induces the parallelization $\overline{\tau}\colon (-\check{R}) \times \mathbb{R}^3 \to T(-\check{R})$ such that $\overline{\tau}(x, v) = -\tau(x, v)$.*

Compose the orientation-preserving identification of a neighborhood of ∞ in R with $\mathring{B}_{1,\infty}$ by the (restriction of the) multiplication by (-1) in $\mathbb{R}^3 \cup \{\infty\}$ in order to get an orientation-preserving identification of a neighborhood of ∞ in $(-R)$ with $\mathring{B}_{1,\infty}$.

Then $(-\check{R} = \widecheck{(-R)}, \overline{\tau})$ is an asymptotic rational homology $\mathbb{R}^3$ and $p_1(\overline{\tau}) = -p_1(\tau)$.

Proof: Use a signature 0 cobordism W from $\{0\} \times B^3$ to $\{1\} \times B_R$ to compute $p_1(\tau)$. Extend the trivialization of $TW \otimes \mathbb{C}$ on ∂W, which may be expressed as $\vec{N} \oplus \tau_s$ on $\{0\} \times B^3 \cup (-[0,1] \times \partial B^3)$ and $\vec{N} \oplus \tau$ on $\{1\} \times B_R$, to a special trivialization on the complement of an open ball $\mathring{B}^4$ in W. Let $\overline{W}$ be the cobordism obtained from W by reversing the orientation of W. Equip $\overline{W \setminus \mathring{B}^4}$ with the trivialization obtained from the above trivialization by a composition by $\mathbf{1}_{\mathbb{R}} \times (-\mathbf{1})_{\mathbb{R}^3}$. Then the changes of trivializations $\phi\colon \partial B^4 \to SU(4)$ and $\overline{\phi}\colon \partial\overline{B^4} \to SU(4)$ are obtained from one another by the orientation-preserving conjugation by $\mathbf{1}_{\mathbb{R}} \times (-\mathbf{1})_{\mathbb{R}^3}$. Since $\partial\overline{B^4}$ and ∂B^4 have opposite orientations, we get the result. $\square$

Back to the invariant Θ defined in Corollary 4.9, we can now prove the following proposition.

Proposition 5.15. *For any $\mathbb{Q}$-sphere R, $\Theta(-R) = -\Theta(R)$.*

PROOF: If ω is a propagating form of $(C_2(R), \tau)$, then $\iota^*(\omega)$ is a propagating form of $(C_2(-R), \overline{\tau})$, so $\Theta(-R, \overline{\tau}) = \int_{C_2(-R)} \iota^*(\omega^3)$, where ι is the orientation-reversing diffeomorphism of $C_2(R)$ that exchanges the two coordinates in $\check{R}^2 \setminus \Delta\left(\check{R}^2\right)$, and $C_2(-R)$ is naturally identified with $C_2(R)$, in an orientation-preserving way. This proves that $\Theta(-R, \overline{\tau}) = -\Theta(R, \tau)$. Corollary 4.9 and Lemma 5.14 yield the conclusion. $\square$

5.6 On the groups $SO(3)$ and $SO(4)$

Let $\mathbb{H}$ denote the vector space $\mathbb{C} \oplus \mathbb{C}j$ and set $k = ij$. The *conjugate* of an element $(z_1 + z_2 j)$ of $\mathbb{H}$ is

$$\overline{z_1 + z_2 j} = \overline{z_1} - z_2 j.$$

Lemma 5.16. *The bilinear map that maps $(z_1 + z_2 j, z_1' + z_2' j)$ to $(z_1 z_1' - z_2 \overline{z_2'}) + (z_2 \overline{z_1'} + z_1 z_2')j$ maps (i,j) to k, (j,k) to i, (k,i) to j, (j,i) to $(-k)$, (k,j) to $(-i)$, (i,k) to $(-j)$, (i,i), (j,j) and (k,k) to (-1) and $(z_1 + z_2 j, \overline{z_1 + z_2 j})$ to $|z_1|^2 + |z_2|^2$. It defines an associative product on $\mathbb{H}$ such that $\mathbb{H}$ equipped with this product and with the addition is a field.*

PROOF: Exercise. $\square$

The noncommutative field $\mathbb{H}$, which contains $\mathbb{C}$ as a subfield, is the *field of quaternions*. It is equipped with the scalar product $\langle .,. \rangle$ that makes $(1, i, j, k)$ an orthonormal basis of $\mathbb{H}$. The associated norm, which maps $(z_1 + z_2 j)$ to $\sqrt{(z_1 + z_2 j)\overline{z_1 + z_2 j}}$, is multiplicative. The unit sphere of $\mathbb{H}$ is the sphere S^3, which is equipped with the group structure induced by the product of $\mathbb{H}$. The elements of $\mathbb{H}$ are the *quaternions*. The *real part* of a quaternion

$(z_1 + z_2 j)$ is the real part of z_1. The *pure quaternions* are the quaternions with zero real part.

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $n \in \mathbb{N}$, the $\mathbb{K}$ (euclidean or hermitian) oriented vector space with the direct orthonormal basis $(v_1, \ldots, v_n)$ is denoted by $\mathbb{K} < v_1, \ldots, v_n >$. The *cross product* or *vector product* of two elements v and w of $\mathbb{R}^3 = \mathbb{R} < i, j, k >$ is the element $v \times w$ of $\mathbb{R}^3$ such that for any $x \in \mathbb{R}^3$, $x \wedge v \wedge w = \langle x, v \times w \rangle i \wedge j \wedge k$ in $\bigwedge^3 \mathbb{R}^3 = \mathbb{R}$.

Lemma 5.17. *The product of two pure quaternions v and w is*

$$vw = -\langle v, w \rangle + v \times w.$$

Every element of S^3 may be expressed as $cos(\theta) + sin(\theta)v$ for a unique $\theta \in [0, \pi]$ and a pure quaternion v of norm 1, which is unique when $\theta \notin \{0, \pi\}$. For such an element, the restriction to $\mathbb{R} < i, j, k >$ of the conjugation

$$R(\theta, v) \colon w \mapsto (cos(\theta) + sin(\theta)v)w\overline{(cos(\theta) + sin(\theta)v)}$$

is the rotation with axis directed by v and with angle 2θ.

PROOF: It is easy to check the first assertion. The conjugation $R(\theta, v)$ preserves the scalar product of $\mathbb{H}$ and it fixes $\mathbb{R} \oplus \mathbb{R}v$, pointwise. Therefore, it restricts as an orthonormal transformation of $\mathbb{R} < i, j, k >$, which fixes v. Let w be a pure quaternion orthogonal to v.

$$R(\theta, v)(w) = (\cos(\theta) + \sin(\theta)v)w(\cos(\theta) - \sin(\theta)v)$$

is equal to

$$\begin{aligned} R(\theta, v)(w) &= \cos^2(\theta)w - \sin^2(\theta)vwv + \cos(\theta)\sin(\theta)(vw - wv) \\ &= \cos(2\theta)w + \sin(2\theta)v \times w. \end{aligned}$$

$\square$

Lemma 5.18. *The group morphism*

$$\begin{array}{llll} \tilde{\rho} \colon & S^3 & \to & SO(\mathbb{R} < i, j, k >) = SO(3) \\ & x & \mapsto & (w \mapsto x.w.\overline{x}) \end{array}$$

is surjective and its kernel is $\{-1, +1\}$. The morphism $\tilde{\rho}$ is a two-fold covering map, and this definition of $\tilde{\rho}$ coincides with the previous one (after Definition 4.5), up to homotopy.

Proof: According to Lemma 5.17, $\tilde{\rho}$ is surjective. Its kernel is the center of the group of unit quaternions, which is $\{-1, +1\}$. Thus $\tilde{\rho}$ is a two-fold covering map.

It is clear that this two-fold covering map coincides with the previous one, up to homotopy and orientation, since both classes generate $\pi_3(SO(3)) = \mathbb{Z}$. We take care of the orientation using the outward normal first convention to orient boundaries, as usual. Consider the diffeomorphism

$$\begin{array}{llll} \psi\colon &]0,\pi[\times S^2 & \to & S^3 \setminus \{-1,1\} \\ & (\theta, v) & \mapsto & \cos(\theta) + \sin(\theta)v. \end{array}$$

We study ψ at $(\pi/2, i)$. At $\psi(\pi/2, i)$, $\mathbb{H}$ is oriented as $\mathbb{R} \oplus \mathbb{R} < i, j, k >$, where $\mathbb{R} < i, j, k >$ is oriented by the outward normal to S^2, which coincides with the outward normal to S^3 in $\mathbb{R}^4$, followed by the orientation of S^2. In particular, since cos is an orientation-reversing diffeomorphism at $\pi/2$, ψ preserves the orientation near $(\pi/2, i)$, so ψ preserves the orientation everywhere, and the two maps $\tilde{\rho}$ are homotopic. $\square$

The following two group morphisms from S^3 to $SO(4)$ induced by the multiplication in $\mathbb{H}$

$$\begin{array}{llll} m_\ell\colon & S^3 & \to & (SO(\mathbb{H}) = SO(4)) \\ & x & \mapsto & (m_\ell(x)\colon v \mapsto x.v) \end{array}$$

$$\begin{array}{llll} \overline{m_r}\colon & S^3 & \to & SO(\mathbb{H}) \\ & y & \mapsto & (\overline{m_r}(y)\colon v \mapsto v.\overline{y}) \end{array}$$

together induce the surjective group morphism

$$\begin{array}{lll} S^3 \times S^3 & \to & SO(4) \\ (x, y) & \mapsto & (v \mapsto x.v.\overline{y}). \end{array}$$

The kernel of this group morphism is $\{(-1,-1), (1,1)\}$, so this morphism is a two-fold covering map. In particular, $\pi_3(SO(4)) = \mathbb{Z}[m_\ell] \oplus \mathbb{Z}[\overline{m_r}]$. Define

$$\begin{array}{llll} m_r\colon & S^3 & \to & (SO(\mathbb{H}) = SO(4)) \\ & y & \mapsto & (m_r(y)\colon v \mapsto v.y). \end{array}$$

Lemma 5.19. *In* $\pi_3(SO(4)) = \mathbb{Z}[m_\ell] \oplus \mathbb{Z}[\overline{m_r}]$,

$$i_*([\tilde{\rho}]) = [m_\ell] + [\overline{m}_r] = [m_\ell] - [m_r].$$

Proof: The π_3-product in $\pi_3(SO(4))$ coincides with the product induced by the group structure of $SO(4)$. $\square$

Lemma 5.20. *Recall that m_r denotes the map from the unit sphere S^3 of $\mathbb{H}$ to $SO(\mathbb{H})$ induced by the right-multiplication. Denote the inclusions $SO(n) \subset SU(n)$ by c. Then in $\pi_3(SU(4))$,*

$$c_*([m_r]) = 2[i_*^2(m_r^{\mathbb{C}})].$$

PROOF: Let $\mathbb{H}+I\mathbb{H}$ denote the complexification of $\mathbb{R}^4 = \mathbb{H} = \mathbb{R} < 1, i, j, k >$. Here, $\mathbb{C} = \mathbb{R} \oplus I\mathbb{R}$. When $x \in \mathbb{H}$ and $v \in S^3$, $c(m_r(v))(Ix) = Ix.v$, and $I^2 = -1$. Let $\varepsilon = \pm 1$, define

$$\mathbb{C}^2(\varepsilon) = \mathbb{C} < \frac{\sqrt{2}}{2}(1+\varepsilon Ii), \frac{\sqrt{2}}{2}(j+\varepsilon Ik) > .$$

Consider the quotient $\mathbb{C}^4/\mathbb{C}^2(\varepsilon)$. In this quotient, $Ii = -\varepsilon 1$, $Ik = -\varepsilon j$, and since $I^2 = -1$, $I1 = \varepsilon i$ and $Ij = \varepsilon k$. Therefore this quotient is isomorphic to $\mathbb{H}$ as a real vector space with its complex structure $I = \varepsilon i$. Then it is easy to see that $c(m_r(v))$ maps $\mathbb{C}^2(\varepsilon)$ to 0 in this quotient, for any $v \in S^3$. Thus $c(m_r(v))(\mathbb{C}^2(\varepsilon)) = \mathbb{C}^2(\varepsilon)$. Now, observe that $\mathbb{H}+I\mathbb{H}$ is the orthogonal sum of $\mathbb{C}^2(-1)$ and $\mathbb{C}^2(1)$. In particular, $\mathbb{C}^2(\varepsilon)$ is isomorphic to the quotient $\mathbb{C}^4/\mathbb{C}^2(-\varepsilon)$, which is isomorphic to $(\mathbb{H}; I = -\varepsilon i)$ and $c(m_r)$ acts on it by the right multiplication. Therefore, with respect to the orthonormal basis $\frac{\sqrt{2}}{2}(1-Ii, j-Ik, 1+Ii, j+Ik)$, $c(m_r(z_1+z_2j = x_1+y_1i+x_2j+y_2k))$ may be written as

$$c(m_r(x_1+y_1i+x_2j+y_2k)) = \begin{bmatrix} x_1+y_1I & -x_2+y_2I & 0 & 0 \\ x_2+y_2I & x_1-y_1I & 0 & 0 \\ 0 & 0 & x_1-y_1I & -x_2-y_2I \\ 0 & 0 & x_2-y_2I & x_1+y_1I \end{bmatrix}.$$

Therefore, the homotopy class of $c(m_r)$ is the sum of the homotopy classes of

$$(z_1+z_2j) \mapsto \begin{bmatrix} m_r^{\mathbb{C}}(z_1,z_2) & 0 \\ 0 & 1 \end{bmatrix} \text{ and } (z_1+z_2j) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & m_r^{\mathbb{C}} \circ \iota(z_1,z_2) \end{bmatrix},$$

where $\iota(z_1,z_2) = (\overline{z_1}, \overline{z_2})$. Since the first map is conjugate by a fixed element of $SU(4)$ to $i_*^2(m_r^{\mathbb{C}})$, it is homotopic to $i_*^2(m_r^{\mathbb{C}})$, and since ι induces the identity on $\pi_3(S^3)$, the second map is homotopic to $i_*^2(m_r^{\mathbb{C}})$, too. $\square$

The following lemma finishes to determine the maps

$$c_* : \pi_3(SO(4)) \longrightarrow \pi_3(SU(4))$$

and $c_*i_* : \pi_3(SO(3)) \longrightarrow \pi_3(SU(4))$.

Lemma 5.21.

$$c_*([\overline{m_r}]) = c_*([m_\ell]) = -2[i^2_*(m^{\mathbb{C}}_r)] = 2[i^2_*(\overline{m}^{\mathbb{C}}_r)].$$

$$c_*(i_*([\tilde{\rho}])) = 4[i^2_*(\overline{m}^{\mathbb{C}}_r)].$$

PROOF: According to Lemma 5.19, $i_*([\tilde{\rho}]) = [m_\ell]+[\overline{m}_r]$. Using the conjugacy of quaternions, $m_\ell(x)(v) = x.v = \overline{\overline{v}.\overline{x}} = \overline{\overline{m}_r(x)(\overline{v})}$. Therefore m_ℓ is conjugated to $\overline{m}_r$ via the conjugacy of quaternions, which lies in $(O(4) \subset U(4))$.

Since $U(4)$ is connected, the conjugacy by an element of $U(4)$ induces the identity on $\pi_3(SU(4))$. Thus,

$$c_*([m_\ell]) = c_*([\overline{m}_r]) = -c_*([m_r]) = -2[i^2_*(m^{\mathbb{C}}_r)] = 2[i^2_*(\overline{m}^{\mathbb{C}}_r)],$$

and $c_*(i_*([\tilde{\rho}])) = c_*([m_\ell]) + c_*([\overline{m}_r]) = 4[i^2_*(\overline{m}^{\mathbb{C}}_r)]$. □

5.7 Relating the Pontrjagin number to the degree

We finish proving Theorem 4.6 by proving the following proposition. See Lemmas 5.2, 5.7 and 5.8.

Proposition 5.22. *Let M_0 and M be two compact connected oriented 3-manifolds whose boundaries have collars that are identified by a diffeomorphism. Let $\tau_0\colon M_0 \times \mathbb{C}^3 \to TM_0 \otimes \mathbb{C}$ and $\tau\colon M \times \mathbb{C}^3 \to TM \otimes \mathbb{C}$ be two special complex trivializations (which respect the orientations) that coincide on the collar neighborhoods of $\partial M_0 = \partial M$. Let $[(M, \partial M), (SU(3), 1)]$ denote the group of homotopy classes of maps from M to $SU(3)$ that map ∂M to 1. For any*

$$g : (M, \partial M) \longrightarrow (SU(3), 1),$$

define

$$\begin{array}{rccc} \psi(g): & M \times \mathbb{C}^3 & \longrightarrow & M \times \mathbb{C}^3 \\ & (x, y) & \mapsto & (x, g(x)(y)) \end{array}$$

then $(p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau))$ is independent of τ_0 and τ. Set

$$p'_1(g) = p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau).$$

The map p'_1 induces an isomorphism from the group $[(M, \partial M), (SU(3), 1)]$ to $\mathbb{Z}$, and, if g is valued in $SO(3)$, then

$$p'_1(g) = 2\,deg(g).$$

Proof:

Lemma 5.23. *Under the hypotheses of Proposition 5.22,*

$$p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau) = p_1(\tau, \tau \circ \psi(g)) = -p_1(\tau \circ \psi(g), \tau)$$

is independent of τ_0 and τ.

Proof: Indeed, $(p_1(\tau_0, \tau \circ \psi(g)) - p_1(\tau_0, \tau))$ can be defined as the obstruction to extending the following trivialization of the complexified tangent bundle to $[0,1] \times M$ restricted to the boundary. The trivialization is $T[0,1] \oplus \tau$ on $(\{0\} \times M) \cup ([0,1] \times \partial M)$ and $T[0,1] \oplus \tau \circ \psi(g)$ on $\{1\} \times M$. Our obstruction is the obstruction to extending the map $\tilde{g}$ from $\partial([0,1] \times M)$ to $SU(4)$ that maps $(\{0\} \times M) \cup ([0,1] \times \partial M)$ to 1 and that coincides with $i \circ g$ on $\{1\} \times M$, viewed as a map from $\partial([0,1] \times M)$ to $SU(4)$, over $([0,1] \times M)$. This obstruction, which lies in $\pi_3(SU(4))$ since $\pi_i(SU(4)) = 0$, for $i < 3$, is independent of τ_0 and τ. $\square$

Lemma 5.23 guarantees that p_1' defines two group homomorphisms to $\mathbb{Z}$ from $[(M, \partial M), (SU(3), 1)]$ and from $[(M, \partial M), (SO(3), 1)]$. Since $\pi_i(SU(3))$ is trivial for $i < 3$ and since $\pi_3(SU(3)) = \mathbb{Z}$, the group of homotopy classes $[(M, \partial M), (SU(3), 1)]$ is generated by the class of a map that maps the complement of a 3-ball B to 1 and that factors through a map whose homotopy class generates $\pi_3(SU(3))$ on B. By definition of the Pontrjagin classes, p_1' sends such a generator to ± 1 and it induces an isomorphism from the group $[(M, \partial M), (SU(3), 1)]$ to $\mathbb{Z}$.

According to Lemma 5.8, the restriction of p_1' to $[(M, \partial M), (SO(3), 1)]$, is equal to $p_1'(\rho_M(B^3))\frac{\deg}{2}$, and it suffices to prove the following lemma.

Lemma 5.24.

$$p_1'(\rho_M(B^3)) = 4.$$

Let $g = \rho_M(B^3)$, we can extend $\tilde{g}$ (defined in the proof of Lemma 5.23) by the constant map with value 1 outside $[\varepsilon, 1] \times B^3 \cong B^4$, for some $\varepsilon \in]0,1[$, and, in $\pi_3(SU(4))$

$$[\tilde{g}_{|\partial B^4}] = p_1(\tau, \tau \circ \psi(g))[i_*^2(\overline{m}_r^{\mathbb{C}})].$$

Since $\tilde{g}_{|\partial B^4}$ is homotopic to $c \circ i \circ \tilde{\rho}$, Lemma 5.21 allows us to conclude. $\square$

$\square$

5.8 Properties of Pontrjagin numbers

Proposition 5.25. *Let M_0 and M_1 be two compact connected oriented 3-manifolds whose boundaries have collars that are identified by a diffeomorphism. Let $\tau_0 \colon M_0 \times \mathbb{C}^3 \to TM_0 \otimes \mathbb{C}$ and $\tau_1 \colon M_1 \times \mathbb{C}^3 \to TM_1 \otimes \mathbb{C}$ be two special complex trivializations (which respect the orientations) that agree on the collar neighborhoods of $\partial M_0 = \partial M_1$.*

The first Pontrjagin number $p_1(\tau_0, \tau_1)$ satisfies the following properties.

1. *Let M_2 be a compact 3-manifold whose boundary has a collar neighborhood identified with a collar neighborhood of ∂M_0. Let τ_2 be a special complex trivialization of TM_2 that agrees with τ_0 near ∂M_2. If two of the Lagrangians of M_0, M_1 and M_2 coincide in $H_1(\partial M_0; \mathbb{Q})$, then*

   $$p_1(\tau_0, \tau_2) = p_1(\tau_0, \tau_1) + p_1(\tau_1, \tau_2).$$

   In particular, since $p_1(\tau_0, \tau_0) = 0$, $p_1(\tau_1, \tau_0) = -p_1(\tau_0, \tau_1)$.

2. *Let D be a connected compact 3-manifold that contains M_0 in its interior, and let τ_D be a special complex trivialization of TD that restricts as the special complex trivialization τ_0 on TM_0, let D_1 be obtained from D_0 by replacing M_0 with M_1, and let τ_{D_1} be the trivialization of TD_1 that agrees with τ_1 on TM_1 and with τ_D on $T(D \setminus M_0)$. If the Lagrangians of M_0 and M_1 coincide, then*

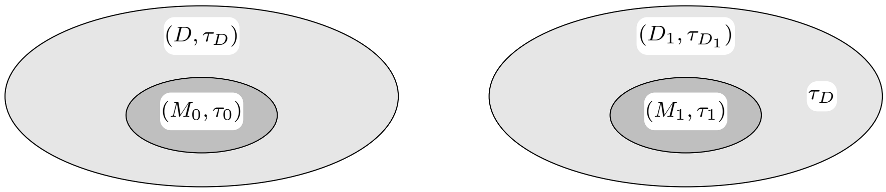
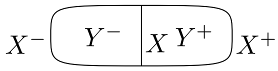


Figure 5.2: The manifolds D and D_1 of Proposition 5.25

   $$p_1(\tau_D, \tau_{D_1}) = p_1(\tau_0, \tau_1).$$

The proof uses a weak form of the Wall Non-Additivity theorem. We quote the weak form we need.

Theorem 5.26 ([Wal69])**.** *Let Y be a compact oriented 4–manifold (with possible boundary), and let X be a 3–manifold properly embedded in Y that separates Y and that induces the splitting $Y = Y^+ \cup_X Y^-$, for two 4-manifolds Y^+ and Y^- in Y, whose intersection is X, which is oriented as part of the boundary of Y^-, as in the following figure of Y:*



Set

$$X^+ = \overline{\partial Y^+ \setminus (-X)} \quad \textit{and} \quad X^- = -\overline{\partial Y^- \setminus X}.$$

Let $\mathcal{L}$, $\mathcal{L}^-$ and $\mathcal{L}^+$ denote the Lagrangians of X, X^- and X^+, respectively. They are Lagrangian subspaces of $H_1(\partial X, \mathbb{Q})$. Then

$$\big(\textit{signature}(Y) - \textit{signature}(Y^+) - \textit{signature}(Y^-)\big)$$

is the signature of an explicit quadratic form on

$$\frac{\mathcal{L} \cap (\mathcal{L}^- + \mathcal{L}^+)}{(\mathcal{L} \cap \mathcal{L}^-) + (\mathcal{L} \cap \mathcal{L}^+)}.$$

Furthermore, this space is isomorphic to $\frac{\mathcal{L}^+ \cap \left(\mathcal{L} + \mathcal{L}^-\right)}{(\mathcal{L}^+ \cap \mathcal{L}) + (\mathcal{L}^+ \cap \mathcal{L}^-)}$ and $\frac{\mathcal{L}^- \cap \left(\mathcal{L} + \mathcal{L}^+\right)}{(\mathcal{L}^- \cap \mathcal{L}) + (\mathcal{L}^- \cap \mathcal{L}^+)}$.

We use this theorem in cases when the above space is trivial. That is why we do not make the involved quadratic form explicit.

Proof of Proposition 5.25: Let us prove the first property. Let $Y^- = W$ be a signature 0 cobordism from $X^- = M_0$ to $X = M_1$, and let Y^+ be a signature 0 cobordism from M_1 to $X^+ = M_2$. Then it is enough to prove that the signature of $Y = Y^+ \cup_X Y^-$ is zero. With the notation of Theorem 5.26, under our assumptions, the space $\frac{\mathcal{L} \cap \left(\mathcal{L}^- + \mathcal{L}^+\right)}{(\mathcal{L} \cap \mathcal{L}^-) + (\mathcal{L} \cap \mathcal{L}^+)}$ is trivial, therefore, according to the Wall theorem, the signature of Y is zero. The first property follows.

We now prove that under the assumptions of the second property,

$$p_1(\tau_D, \tau_{D_1}) = p_1(\tau_0, \tau_1).$$

Let $Y^+ = \left([0,1] \times (D \setminus \mathring{M}_0)\right)$, let $Y^- = W$ be a signature 0 cobordism from M_0 to M_1, and let $X = -[0,1] \times \partial M_0$. Note that the signature of Y^+ is zero. In order to prove the wanted equality, it is enough to prove that the signature of $Y = Y^+ \cup_X Y^-$ is zero. Here, $H_1(\partial X; \mathbb{Q}) = H_1(\partial M_0) \oplus H_1(\partial M_0)$. Let $j \colon H_1(\partial M_0) \to H_1(D \setminus \mathring{M}_0)$ and let $j_{\partial D} \colon H_1(\partial D) \to H_1(D \setminus \mathring{M}_0)$ be the maps induced by inclusions. With the notation of Theorem 5.26,

$$\begin{array}{lll}
\partial X & = -\left(\partial[0,1]\right) \times \partial M_0, \\
X^- & = -\{1\} \times M_1 \cup \left(\{0\} \times M_0\right), \\
X^+ & = -[0,1] \times \partial D \cup \Big((\partial[0,1]) \times (D \setminus \mathring{M}_0) \Big), \\
\mathcal{L} & = \{(x,-x) \mid x \in H_1(\partial M_0)\} \\
\mathcal{L}^- & = \{(x,y) \mid x \in \mathcal{L}_{M_0}, y \in \mathcal{L}_{M_1}\} \\
\mathcal{L}^+ & = \{(x,y) \mid (j(x),j(y)) = (j_{\partial D}(z \in H_1(\partial D)), -j_{\partial D}(z))\} \\
 & = \{(y,-y) \mid j(y) \in \mathrm{Im}(j_{\partial D})\} \oplus \{(x,0) \mid j(x) = 0\} \\
\mathcal{L} \cap \mathcal{L}^- & = \{(x,-x) \mid x \in (\mathcal{L}_{M_0} \cap \mathcal{L}_{M_1} = \mathcal{L}_{M_0})\} \\
\mathcal{L} \cap \mathcal{L}^+ & = \{(x,-x) \mid j(x) \in \mathrm{Im}(j_{\partial D})\}
\end{array}$$

Let us prove that $\mathcal{L} \cap (\mathcal{L}^- + \mathcal{L}^+) = (\mathcal{L} \cap \mathcal{L}^-) + (\mathcal{L} \cap \mathcal{L}^+)$. For a subspace K of $H_1(\partial M_0;\mathbb{Q})$, set $j_{MV}(K) = \{(x,-x) \mid x \in K\} \subset H_1(\partial X;\mathbb{Q})$. Then $\mathcal{L} = j_{MV}(H_1(\partial M_0))$ and $\mathcal{L} \cap \mathcal{L}^+ = j_{MV}\left(j^{-1}\left(\mathrm{Im}(j_{\partial D})\right)\right)$.

$$\mathcal{L} \cap (\mathcal{L}^- + \mathcal{L}^+) = \mathcal{L} \cap \mathcal{L}^+ + j_{MV}\left(\mathcal{L}_{M_1} \cap (\mathcal{L}_{M_0} + \mathrm{Ker}(j))\right).$$

Since $\mathcal{L}_{M_0} = \mathcal{L}_{M_1}$, $(\mathcal{L}_{M_1} \cap (\mathcal{L}_{M_0} + \mathrm{Ker}(j))) = \mathcal{L}_{M_0}$, so $\mathcal{L} \cap (\mathcal{L}^- + \mathcal{L}^+) = (\mathcal{L} \cap \mathcal{L}^+) + j_{MV}(\mathcal{L}_{M_0})$. Then the second property is proved, thanks to Wall's theorem, which guarantees the additivity of the signature in this case. $\square$

The parallelizations of S^3 As a Lie group, S^3 has two natural homotopy classes of parallelizations τ_ℓ and τ_r, which we describe below. Identify the tangent space T_1S^3 to S^3 at 1 with $\mathbb{R}^3$ (arbitrarily, with respect to the orientation). For $g \in S^3$, the multiplication induces two diffeomorphisms $m_\ell(g)$ and $m_r(g)$ of S^3, $m_\ell(g)(h) = gh$ and $m_r(g)(h) = hg$. Let $T(m_\ell(g))$ and $T(m_r(g))$ denote their respective tangent maps at 1. Then $\tau_\ell(h \in S^3, v \in \mathbb{R}^3 = T_1S^3) = (h, T(m_\ell(h))(v))$ and $\tau_r(h \in S^3, v \in \mathbb{R}^3 = T_1S^3) = (h, T(m_r(h))(v))$.

Proposition 5.27. *$p_1(\tau_\ell) = 2$ and $p_1(\tau_r) = -2$.*

Proof: Regard S^3 as the unit sphere of $\mathbb{H}$. So $T_1S^3 = \mathbb{R} < i,j,k >$. The unit ball $B(\mathbb{H})$ of $\mathbb{H}$ has the standard parallelization of a real vector space equipped with a basis, and the trivialization $\tau(\tau_\ell)$ induced by τ_ℓ on $\partial B(\mathbb{H})$ is such that $\tau_\ell(h \in S^3, v \in \mathbb{H}) = (h, hv) \in (S^3 \times \mathbb{H} = T\mathbb{H}_{|S^3})$, so $c_*([m_\ell]) = p_1(\tau_\ell)[i^2_*(\overline{m}^{\mathbb{C}}_r)]$ in $\pi_3(SU(4))$ by Definition 5.13 of p_1. According to Lemma 5.21, $p_1(\tau_\ell) = 2$. Similarly, $p_1(\tau_r) = -2$. $\square$

On the image of p_1 For $n \geq 3$, a *spin structure* of a smooth n–manifold is a homotopy class of parallelizations over a 2-skeleton of M (or, equivalently, over the complement of a point, if $n = 3$ and if M is connected).

The class of the covering map $\tilde{\rho}$ described after Definition 4.5 is the standard generator of $\pi_3(SO(3)) = \mathbb{Z}[\tilde{\rho}]$. Recall the map $\rho_M(B^3)$ of Lemma 4.7. Set $[\tilde{\rho}][\tau] = [\tau\psi_{\mathbb{R}}(\rho_M(B^3))]$. The set of homotopy classes of parallelizations that induce a given spin structure form an affine space with translation group $\pi_3(SO(3))$. According to Theorem 4.6 and Lemma 4.7, $p_1([\tilde{\rho}][\tau]) = p_1(\tau) + 4$.

Definition 5.28. The *Rohlin invariant* $\mu(M, \sigma)$ of a smooth closed 3-manifold M, equipped with a spin structure σ, is the mod 16 signature of a compact 4-manifold W, bounded by M, equipped with a spin structure that restricts to M as a stabilization of σ.

The *first Betti number* of M is the dimension of $H_1(M; \mathbb{Q})$. It is denoted by $\beta_1(M)$. Kirby and Melvin proved the following theorem [KM99, Theorem 2.6].

Theorem 5.29. *For any closed oriented* 3*-manifold M, for any parallelization τ of M,*

$$(p_1(\tau) - \mathit{dimension}(H_1(M; \mathbb{Z}/2\mathbb{Z})) - \beta_1(M)) \in 2\mathbb{Z}.$$

Let M be a closed 3*-manifold equipped with a given spin structure σ. Then p_1 is a bijection from the set of homotopy classes of parallelizations of M that induce σ to*

$$2\left(\mathit{dimension}(H_1(M; \mathbb{Z}/2\mathbb{Z})) + 1\right) + \mu(M, \sigma) + 4\mathbb{Z}$$

When M is a $\mathbb{Z}$–sphere, p_1 is a bijection from the set of homotopy classes of parallelizations of M to $(2 + 4\mathbb{Z})$.

Thanks to Proposition 5.25(2), Theorem 5.29 implies Proposition 5.30 below.

Proposition 5.30. *Let M_0 be the unit ball of $\mathbb{R}^3$ and let τ_s be the standard parallelization of $\mathbb{R}^3$,*

- *for any given $\mathbb{Z}$-ball M, $p_1(.) = p_1((\tau_s)_{|B^3}, .)$ defines a bijection from the set of homotopy classes of parallelizations of M that are standard near $\partial M = S^2$ to $4\mathbb{Z}$.*
- *For any $\mathbb{Q}$-ball M, for any trivialization τ of M that is standard near $\partial M = S^2$,*

$$(p_1(\tau) - \mathit{dimension}(H_1(M; \mathbb{Z}/2\mathbb{Z}))) \in 2\mathbb{Z}.$$

5.9 More on $[(M, \partial M), (SO(3), 1)]$

This section is a complement to the study of $[(M, \partial M), (SO(3), 1)]$ started in Sections 5.1 and 5.3. It is not used later in this book. We show how to describe all the elements of $[(M, \partial M), (SO(3), 1)]$ as twists across surfaces, and we describe the structure of $[(M, \partial M), (SO(3), 1)]$ precisely, by proving the following theorem.

Theorem 5.31. *Let M be a compact oriented 3-manifold.*

If all the closed surfaces embedded in M have an even Euler characteristic, then $[(M, \partial M), (SO(3), 1)]$ is canonically isomorphic to $H^3(M, \partial M; \mathbb{Z}) \oplus H_2(M; \mathbb{Z}/2\mathbb{Z})$, and the degree maps $[(M, \partial M), (SO(3), 1)]$ onto $2\mathbb{Z}$.

If $H_1(M; \mathbb{Z})$ has no 2-torsion, then all the closed surfaces embedded in M have an even Euler characteristic.

If M is connected, and if there exists a closed surface S of M with odd Euler characteristic, then the degree maps $[(M, \partial M), (SO(3), 1)]$ onto $\mathbb{Z}$, and $[(M, \partial M), (SO(3), 1)]$ is isomorphic to $\mathbb{Z} \oplus Ker(e\partial_B)$, where

$$e\partial_B \colon H_2(M; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$$

maps the class of a surface to its Euler characteristic mod 2 and the kernel of $e\partial_B$ has a canonical image in $[(M, \partial M), (SO(3), 1)]$.

Representing the elements of $[(M, \partial M), (SO(3), 1)]$ by surfaces

Proposition 5.32. *Let S be a non-necessarily orientable surface embedded in a 3-manifold M equipped with a parallelization τ. Recall the map $g(S, \tau)$ from the beginning of Section 5.3. If M is connected, then $g(S, \tau)^2$ is homotopic to $\rho_M(B^3)^{\chi(S)}$. In particular, the homotopy class of $g(S, \tau)$ depends only on $\chi(S)$ and on the class of S in $H_2(M; \mathbb{Z}/2\mathbb{Z})$.*

PROOF: Assume that S is connected and oriented. Perform a homotopy of τ, so that τ^{-1} maps the positive normal $N^+(S) = T_{(u,s)}([-1, 1] \times s)$ to $u \times S$ to a fixed vector v of S^2, on $[-1, 1] \times (S \setminus D)$, for a disk D of S. Then, there is a homotopy from $[0, 1] \times [-1, 1] \times (S \setminus D)$ to $SO(3)$,

- which factors through the projection onto $[0, 1] \times [-1, 1]$,
- which maps $(1, u, s)$ to the rotation $g(S, \tau)^2(u, s)$ with axis v and angle $2\pi(u + 1)$, for any $(u, s) \in [-1, 1] \times (S \setminus D)$,
- which maps $(\partial([0, 1] \times [-1, 1]) \setminus (\{1\} \times [-1, 1])) \times (S \setminus D)$ to 1.

This homotopy extends to a homotopy $h\colon [0,1] \times [-1,1] \times S \to SO(3)$ from $h_0 = \rho_{[-1,1]\times S}(B^3)^k$ for some $k \in \mathbb{Z}$, to $h_1 = g(S,\tau)^2_{|[-1,1]\times S}$, such that $h([0,1] \times \{-1,1\} \times D) = 1$. Thus $g(S,\tau)^2$ is homotopic to $\rho_M(B^3)^k$, where k depends only on the homotopy class of the restriction of τ to D (because that class determines the homotopy class of the restriction of τ to $[-1,1] \times D$ relatively to $[-1,1] \times \partial D$). Since $\pi_2(SO(3))$ is trivial, that homotopy class depends only on the homotopy class of the restriction of τ to ∂D, which depends only on the homotopy class of the Gauss map from $(D, \partial D)$ to (S^2, v), which maps $s \in D$ to $p_2(\tau^{-1}(N^+(s)))$, because the map from $\pi_2(S^2)$ to $\pi_1(SO(2) = S^1)$ in the long exact sequence associated with the fibration

$$SO(2) \stackrel{i}{\hookrightarrow} SO(3) \to S^2,$$

described in Theorem A.14, is injective. Hence k depends only on the degree d of this Gauss map. Cutting D into smaller disks shows that k depends linearly on the degree d. Note that d is the degree of the Gauss map from S to S^2 before the homotopy of τ. Since for a standard sphere S^2, $d = 1$, and $g(S^2,\tau) = \rho_M(B^3)$, $k = 2d$. It remains to see that the degree of the Gauss map is $\frac{\chi(S)}{2}$. This is easily observed for a standard embedding of S into $\mathbb{R}^3$ equipped with its standard trivialization. Up to homotopy, the trivializations of $TM_{|S}$ are obtained from that one by compositions by rotations with fixed axis v supported in neighborhoods of curves outside the preimage of v. So the degree (at v) is independent of the trivialization. Thus, the proposition is proved when S is orientable and connected. When S is not orientable, according to Lemma 5.2, $g(S,\tau)^2$ is homotopic to $\rho_M(B^3)^k$, for some k. Furthermore, $g(S,\tau)^2$ is homotopic to $g(S^{(2)},\tau)$, where $S^{(2)}$ is the boundary of the tubular neighborhood of S, which is orientable, and whose Euler characteristic is $2\chi(S)$. Then $g(S,\tau)^4$ is homotopic to $\rho_M(B^3)^{2k}$ and to $\rho_M(B^3)^{2\chi(S)}$. Since the arguments are local, they extend to the disconnected case and prove that $g(S,\tau)^2$ is homotopic to $\rho_M(B^3)^{\chi(S)}$ for any S. Then Proposition 5.9 and Lemma 5.6 allow us to conclude that the homotopy class of $g(S,\tau)$ depends only on $\chi(S)$ and on the class of S in $H_2(M;\mathbb{Z}/2\mathbb{Z})$. □

Hence, $g(S,\tau)$ will be denoted by $g(S)$. Lemma 5.6 and Propositions 5.9 and 5.32 easily imply the following corollary.

Corollary 5.33. *All elements of $[(M,\partial M),(SO(3),1)]$ can be represented by $g(S)$ for some embedded disjoint union of closed surfaces S of M.*

□

Structure of $[(M, \partial M), (SO(3), 1)]$ Tensoring a free chain complex $C_*(M; \mathbb{Z})$ whose homology is $H_*(M; \mathbb{Z})$ by the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

yields the associated long exact homology sequence

$$\cdots \to H_*(M; \mathbb{Z}) \xrightarrow{\times 2} H_*(M; \mathbb{Z}) \to H_*(M; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\partial_B} H_{*-1}(M; \mathbb{Z}) \to \dots ,$$

where ∂_B is the *Bockstein morphism*.

Definition 5.34. The *self-linking number* of a torsion element x of $H_1(M; \mathbb{Z})$ is the linking number $lk(c, c')$ of a curve c that represents x and a parallel c' of c, mod $\mathbb{Z}$.

Proposition 5.35. *There is a canonical group homomorphism*

$$e\partial_B \colon H_2(M; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

that admits the following two equivalent definitions:

1. *For any embedded surface S, $e\partial_B$ maps the class of S to the Euler characteristic of S mod 2.*

2. *The map $e\partial_B$ is the composition of the Bockstein morphism*

$$\partial_B \colon H_2(M; \mathbb{Z}/2\mathbb{Z}) \longrightarrow 2\text{-}torsion\ of\ H_1(M; \mathbb{Z})$$

and the map

$$e \colon 2\text{-}torsion\ of\ H_1(M; \mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

that maps the class of a curve x to 1 if the self-linking number of x is $\frac{1}{2}$ mod $\mathbb{Z}$, and to 0 otherwise.

PROOF: The map $e\partial_B$ is well defined by the second definition and it is a group homomorphism. Let S be a connected closed surface. If S is orientable, then the long exact sequence proves that $\partial_B([S]) = 0$ and the Euler characteristic of S is even. Otherwise, there is a curve x (Poincaré dual to $w_1(S)$) such that $S \setminus x$ is orientable, and the boundary of the closure of the source of $S \setminus x$ maps to $(\pm 2x)$, so $\partial_B([S]) = [x]$, by definition. The characteristic curve x may be assumed to be connected. Then the tubular neighborhood of x in S is either a Moebius band or an annulus. In the first case, $e([x]) = 1$ and $\chi(S)$ is odd; otherwise, $e([x]) = 0$ and $\chi(S)$ is even. $\square$

Proposition 5.36. *Let M be an oriented connected 3-manifold. Then*

$$[(M, \partial M), (SO(3), 1)] \cong \mathbb{Z} \oplus Ker(e\partial_B),$$

and the degree maps $[(M, \partial M), (SO(3), 1)]$ onto $2\mathbb{Z}$ when $e\partial_B = 0$ and onto $\mathbb{Z}$ otherwise.

Proof of Proposition 5.36 and Theorem 5.31:
The class in $H_2(M; \mathbb{Z}/2\mathbb{Z})$ of a surface with even Euler characteristic can be represented by a surface S with null Euler characteristic by disjoint union of trivial bounding surfaces. According to Proposition 5.32, for such an S, the class of $g(S)$ is a 2-torsion element of $[(M, \partial M), (SO(3), 1)]$ called $\sigma([S])$. This defines a canonical partial section

$$\sigma\colon\ (\mathrm{Ker}(e\partial_B) \subset H_2(M; \mathbb{Z}/2\mathbb{Z})) \to \mathrm{Ker}(deg\colon [(M, \partial M), (SO(3), 1)] \to \mathbb{Z})$$

of the sequence of Proposition 5.9. Therefore, if $e\partial_B = 0$,

$$[(M, \partial M), (SO(3), 1)] = \mathbb{Z}[\rho_M(B^3)] \oplus \sigma\left(\mathrm{Ker}(e\partial_B) = H_2(M; \mathbb{Z}/2\mathbb{Z})\right).$$

If $e\partial_B \neq 0$, there exists a closed surface S_1 with $\chi(S_1) = 1$ in M. Since the degree is a group homomorphism, Proposition 5.32 implies that $\deg(g(S_1)) = 1$ for such an S_1. Thus

$$[(M, \partial M), (SO(3), 1)] = \mathbb{Z}[g(S_1)] \oplus \sigma(\mathrm{Ker}(e\partial_B)).$$

$\square$

Part II
The general invariants

Chapter 6

Introduction to finite type invariants and Jacobi diagrams

In this chapter, we introduce the target space of the invariant $\mathcal{Z}$ studied in this book. It is a space generated by unitrivalent graphs, called Jacobi diagrams. This space has been studied by Bar-Natan in his fundational article [BN95a], where most of the results of this chapter come from.

6.1 Definition of finite type invariants

Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$.

A $\mathbb{K}$–valued *invariant* of oriented 3-manifolds is a function from the set of 3-manifolds, considered up to orientation-preserving diffeomorphism, to $\mathbb{K}$. Let $\sqcup_{i=1}^{n} S_i^1$ denote a disjoint union of n circles, where each S_i^1 is a copy of S^1. Here, an *n–component link* in a 3-manifold R is an equivalence class of smooth embeddings $L\colon \sqcup_{i=1}^{n} S_i^1 \hookrightarrow R$ under the equivalence relation[1] that identifies two embeddings L and L' if and only if there is an orientation-preserving diffeomorphism h of R such that $h(L) = L'$. A *knot* is a one-component link. A *link invariant* (resp. a *knot invariant*) is a function of links (resp. knots). For example, Θ is an invariant of $\mathbb{Q}$-spheres and the linking number is a rational invariant of two-component links in rational homology spheres.

In order to study a function, it is common to study its derivative, and the derivatives of its derivative. The derivative of a function is defined from its variations. For a function f from $\mathbb{Z}^d = \oplus_{i=1}^{d}\mathbb{Z}e_i$ to $\mathbb{K}$, one can define its

[1]This relation is equivalent to the usual equivalence relation defined by isotopies when R is $\mathbb{R}^3$ or S^3. In general 3-manifolds, two equivalent links are not necessarily isotopic, but the link invariants described in this book are invariant under the above equivalence relation.

first order derivatives $\frac{\partial f}{\partial e_i}\colon \mathbb{Z}^d \to \mathbb{K}$ by

$$\frac{\partial f}{\partial e_i}(z) = f(z+e_i) - f(z)$$

and check that all the first order derivatives of f vanish if and only if f is constant. Inductively define an n-order derivative to be a first order derivative of an $(n-1)$-order derivative for a positive integer n. Then it can be checked that all the $(n+1)$-order derivatives of a function vanish if and only if f is a polynomial in the coordinates, of degree not greater than n. In order to study topological invariants, we can similarly study their variations under *simple operations.*

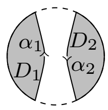
Below, X denotes one of the following sets

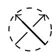
- $\mathbb{Z}^d$,
- the set $\mathcal{K}$ of knots in $\mathbb{R}^3$, the set $\mathcal{K}_k$ of k-component links in $\mathbb{R}^3$,
- the set $\mathcal{M}$ of $\mathbb{Z}$-spheres, the set $\mathcal{M}_Q$ of $\mathbb{Q}$-spheres (up to orientation-preserving diffeomorphism).

and $\mathcal{O}(X)$ denotes a set of *simple operations* acting on some elements of X.

For $X = \mathbb{Z}^d$, $\mathcal{O}(X)$ consists of the operations $(z \to z \pm e_i)$

For knots or links in $\mathbb{R}^3$, the *simple operations* are *crossing changes*. A *crossing change ball* of a link L is a ball B of the ambient space, where $L \cap B$ is a disjoint union of two arcs α_1 and α_2 properly embedded in B, and there exist two disjoint topological disks D_1 and D_2 embedded in B, such that, for $i \in \{1,2\}$, $\alpha_i \subset \partial D_i$ and $(\partial D_i \setminus \alpha_i) \subset \partial B$ as in the following picture.



After an isotopy, a projection of (B, α_1, α_2) looks like  or , a *crossing change* is a change that does not change L outside B and that modifies it inside B by a local move ( $\to$ ) or ( $\to$ ). For the move ( $\to$ ), the crossing change is *positive*, it is *negative* for the move ( $\to$ ).

For integer (resp. rational) homology 3-spheres, the simple operations are integral (resp. rational) *LP-surgeries*, which are defined in Subsection 1.3.2, and $\mathcal{O}(\mathcal{M})$ (resp. $\mathcal{O}(\mathcal{M}_Q)$) is denoted by $\mathcal{O}^{\mathbb{Z}}_{\mathcal{L}}$ (resp. $\mathcal{O}^{\mathbb{Q}}_{\mathcal{L}}$).

Say that crossing changes are *disjoint* if they sit inside disjoint 3-balls. Say that LP-surgeries (A'/A) and (B'/B) in a manifold R are *disjoint* if A

and B are disjoint in R. Two operations on $\mathbb{Z}^d$ are always *disjoint* (even if they look identical). In particular, disjoint operations *commute* (their result does not depend on which one is performed first). Let $\underline{n} = \{1, 2, \ldots, n\}$. Consider the vector space $\mathcal{F}_0(X) = \mathcal{F}_0(X; \mathbb{K})$ freely generated by X over $\mathbb{K}$. For an element x of X and n pairwise disjoint operations $o_1, \ldots, o_n$ acting on x, define

$$[x; o_1, \ldots, o_n] = \sum_{I \subseteq \underline{n}} (-1)^{\sharp I} x((o_i)_{i \in I}) \in \mathcal{F}_0(X),$$

where $\underline{n} = \{1, 2, \ldots, n\}$, and $x((o_i)_{i \in I})$ denotes the element of X obtained by performing the operations o_i on x for $i \in I$. Then define $\mathcal{F}_n(X) = \mathcal{F}_n(X; \mathbb{K})$ as the $\mathbb{K}$-subspace of $\mathcal{F}_0(X)$ generated by the $[x; o_1, \ldots, o_n]$, for all $x \in X$ equipped with n pairwise disjoint simple operations $o_1, \ldots, o_n$ acting on x. Since

$$[x; o_1, \ldots, o_n, o_{n+1}] = [x; o_1, \ldots, o_n] - [x(o_{n+1}); o_1, \ldots, o_n],$$

$\mathcal{F}_{n+1}(X) \subseteq \mathcal{F}_n(X)$, for all $n \in \mathbb{N}$.

Definition 6.1. A $\mathbb{K}$–valued function f on X, extends uniquely as a $\mathbb{K}$–linear map on

$$\mathcal{F}_0(X)^* = \mathrm{Hom}(\mathcal{F}_0(X); \mathbb{K}),$$

which is still denoted by f. For an integer $n \in \mathbb{N}$, the invariant (or function) f is of *degree* $\leq n$ if and only if $f(\mathcal{F}_{n+1}(X)) = 0$. The *degree* of such an invariant is the smallest integer $n \in \mathbb{N}$ such that $f(\mathcal{F}_{n+1}(X)) = 0$. An invariant is of *finite type* if it is of degree n for some $n \in \mathbb{N}$. This definition depends on the chosen set of operations $\mathcal{O}(X)$. We fixed our choices for our sets X, but other choices could lead to different notions. See [GGP01].

Let $\mathcal{I}_n(X) = (\mathcal{F}_0(X)/\mathcal{F}_{n+1}(X))^*$ be the space of invariants of degree at most n. Of course, for all $n \in \mathbb{N}$, $\mathcal{I}_n(X) \subseteq \mathcal{I}_{n+1}(X)$.

Example 6.2. $\mathcal{I}_n(\mathbb{Z}^d)$ is the space of polynomials of degree at most n on $\mathbb{Z}^d$. (Exercise).

Lemma 6.3. *If $f \in \mathcal{I}_m(X)$ and $g \in \mathcal{I}_n(X)$, then $fg \in \mathcal{I}_{m+n}(X)$.*

Proof: Let $[x; (o_i)_{i \in \underline{m+n+1}}] \in \mathcal{F}_{m+n+1}(X)$. The lemma is a direct consequence of the equality

$$fg([x; (o_i)_{i \in \underline{m+n+1}}]) = \sum_{J \subseteq \underline{m+n+1}} f([x; (o_j)_{j \in J}])g([x((o_j)_{j \in J}); (o_i)_{i \in \underline{m+n+1} \setminus J}]),$$

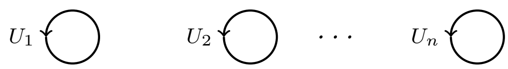
which is proved as follows. The right-hand side is equal to:

$$\sum_{J\subseteq \underline{m+n+1}} (-1)^{|J|} \left(\sum_{K|K\subseteq J} (-1)^{|K|} f(x((o_i)_{i\in K})) \right) \left(\sum_{L|J\subseteq L} (-1)^{|L|} g(x((o_i)_{i\in L})) \right)$$

$$= \sum_{(K,L)|K\subseteq L\subseteq \underline{m+n+1}} (-1)^{|K|+|L|} f(x((o_i)_{i\in K}))g(x((o_i)_{i\in L})) \left(\sum_{J|K\subseteq J\subseteq L} (-1)^{|J|} \right),$$

where $\sum_{J|K\subseteq J\subseteq L}(-1)^{|J|} = \begin{cases} 0 & if\ K \subsetneq L \\ (-1)^{|K|} & if\ K = L. \end{cases}$ □

Lemma 6.4. *Any n-component link in $\mathbb{R}^3$ can be transformed to the trivial n-component link below by a finite number of disjoint crossing changes.*



Proof: Let L be an n-component link in $\mathbb{R}^3$. Since $\mathbb{R}^3$ is simply connected, there is a homotopy that carries L to the trivial link. Such a homotopy $h\colon [0,1] \times \sqcup_{i=1}^n S_i^1 \to \mathbb{R}^3$ can be chosen to be smooth and such that $h(t,.)$ is an embedding, except for finitely many times t_i, $0 < t_1 < \cdots < t_i < t_{i+1} < \cdots < 1$, at which $h(t_i,.)$ is an immersion with one double point and no other multiple points, and the link $h(t,.)$ changes exactly by a crossing change when t crosses a t_i. (For an alternative elementary proof of this fact, see [Les05, Subsection 7.1] before Definition 7.5, for example). □

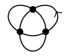
In particular, a degree 0 invariant of n-component links of $\mathbb{R}^3$ must be constant, since it is not allowed to vary under a crossing change.

Exercise 6.5. 1. Check that $\mathcal{I}_1(\mathcal{K}) = \mathbb{K}c_0$, where c_0 is the constant map that maps any knot to 1.
2. Check that the linking number is a degree 1 invariant of 2–component links of $\mathbb{R}^3$.
3. Check that $\mathcal{I}_1(\mathcal{K}_2) = \mathbb{K}c_0 \oplus \mathbb{K}lk$, where c_0 is the constant map that maps any two-component link to 1.

6.2 Introduction to chord diagrams

Let f be a knot invariant of degree at most n.

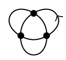
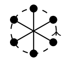
We want to evaluate $f([K; o_1, \ldots, o_n])$, where the o_i are disjoint negative crossing changes $\to$ to be performed on a knot K. Such a

$[K; o_1, \dots, o_n]$ is usually represented as a *singular knot with* n *double points*, which is an immersion of a circle with n transverse double points like  , where each double point  can be *desingularized* in two ways the *positive* one  and the *negative* one  , and K is obtained from the singular knot by desingularizing all the crossings in the positive way, which is  in our example. Note that the sign of the desingularization is defined from the orientation of the ambient space. Thus, singular knots represent elements of $\mathcal{F}_0(\mathcal{K})$ and three singular knots that coincide outside a ball, inside which they look as in the following *skein relation*

 $=$  $-$ 

satisfy this relation in $\mathcal{F}_0(\mathcal{K})$.

Define the *chord diagram* $\Gamma_C([K; o_1, \dots, o_n])$ associated to $[K; o_1, \dots, o_n]$ as follows. Draw the preimage of the associated singular knot with n double points as an oriented dashed circle equipped with the $2n$ preimages of the double points and join the pairs of preimages of a double point by a plain segment called a *chord*.

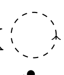
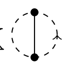
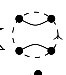
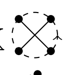
$\Gamma_C($  $) =$ 

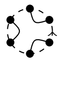
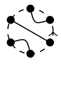
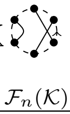
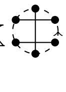
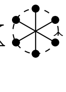
Formally, a *chord diagram* with n chords is a cyclic order of the $2n$ ends of the n chords, up to a permutation of the chords and up to exchanging the two ends of a chord.

Lemma 6.6. *When f is a knot invariant of degree at most n, $f([K; o_1, \dots, o_n])$ depends only on $\Gamma_C([K; o_1, \dots, o_n])$.*

Proof: Since f is of degree n, $f([K; o_1, \dots, o_n])$ is invariant under a crossing change outside the balls of the o_i, that is outside the double points of the associated singular knot. Therefore, $f([K; o_1, \dots, o_n])$ depends only on the cyclic order of the $2n$ arcs involved in the o_i on K. (A more detailed proof can be found in [Les05, Subsection 7.3].) □

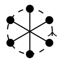
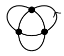
Let $\mathcal{D}_n$ be the $\mathbb{K}$-vector space freely generated by the n chord diagrams on S^1.

$\mathcal{D}_0 = \mathbb{K}$ , $\mathcal{D}_1 = \mathbb{K}$ , $\mathcal{D}_2 = \mathbb{K}$  $\oplus \mathbb{K}$ ,

$\mathcal{D}_3 = \mathbb{K}$  $\oplus \mathbb{K}$  $\oplus \mathbb{K}$  $\oplus \mathbb{K}$  $\oplus \mathbb{K}$ .

Lemma 6.7. *The map ϕ_n from $\mathcal{D}_n$ to $\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}$ that maps an n–chord diagram Γ to some $[K; o_1, \dots, o_n]$ whose diagram is Γ is well defined and surjective.*

PROOF: Use the arguments of the proof of Lemma 6.6. □

As an example, $\phi_3($$) = [$  $]$. Lemma 6.7 implies that

$$\phi_n^*\colon \left(\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}\right)^* \to \mathcal{D}_n^*$$

is injective. The kernel of the restriction below

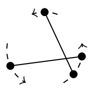
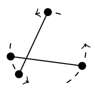
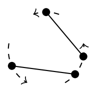
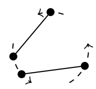
$$\mathcal{I}_n(\mathcal{K}) = \left(\frac{\mathcal{F}_0(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}\right)^* \to \left(\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}\right)^*$$

is $\mathcal{I}_{n-1}(\mathcal{K})$. Thus, $\frac{\mathcal{I}_n(\mathcal{K})}{\mathcal{I}_{n-1}(\mathcal{K})}$ injects into $\mathcal{D}_n^*$ and $\mathcal{I}_n(\mathcal{K})$ is finite dimensional for all n. In particular, $\frac{\mathcal{F}_0(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}$ is finite-dimensional and the above restriction is surjective. Therefore,

$$\frac{\mathcal{I}_n(\mathcal{K})}{\mathcal{I}_{n-1}(\mathcal{K})} = \mathrm{Hom}(\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})};\mathbb{K}).$$

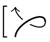
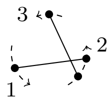
An *isolated chord* in a chord diagram is a chord between two points of S^1 that are consecutive on the circle.

Lemma 6.8. *Let D be a diagram on S^1 that contains an isolated chord. Then $\phi_n(D) = 0$. Let D^1, D^2, D^3, D^4 be four n-chord diagrams that are identical outside three portions of circles, inside which they look like:*

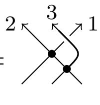
$D^1 =$  , $D^2 =$  , $D^3 =$  *and* $D^4 =$  .

then

$$\phi_n(-D^1 + D^2 + D^3 - D^4) = 0.$$

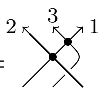
PROOF: For the first assertion, observe that $\phi_n($$) = [$$] - [$$]$. Let us prove the second one. We may represent $D^1 =$ 

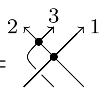
by a singular knot K^1 with n double points, which intersects a ball as

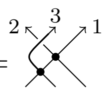
$K^1 =$ 

.

Let K^2, K^3, K^4 be the singular knots with n double points that coincide with K^1 outside this ball, and that intersect this ball as shown in the picture:

$K^2 =$ 

, $K^3 =$ 

, $K^4 =$ 

.

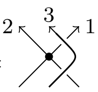
Then $D(K^2) = D^2$, $D(K^3) = D^3$ and $D(K^4) = D^4$. Therefore, $\phi_n(-D^1 + D^2 + D^3 - D^4) = -[K^1] + [K^2] + [K^3] - [K^4]$.

Thus, it is enough to prove that we have

$$-[K^1] + [K^2] + [K^3] - [K^4] = 0$$

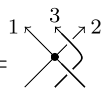
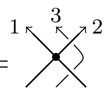
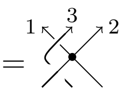
in $\mathcal{F}_n(\mathcal{K})$. Let us prove this.

Let K_0 be the singular knot with $(n-1)$ double points that intersects our ball as

$K_0 =$ 

and that coincides with K^1 outside this ball.

The strands 1 and 2 involved in the pictured double point are in the horizontal plane and they orient it. The strand 3 is vertical and intersects the horizontal plane in a positive way between the tails of 1 and 2. Now, make 3 turn around the double point counterclockwise, so that it successively becomes the knots with $(n-1)$ double points:

$K_1 =$  , $K_2 =$  and $K_3 =$  .

On its way, it goes successively through our four knots K^1, K^2, K^3 and K^4 with n double points, which appear inside matching parentheses, in the following obvious identity in $\mathcal{F}_{n-1}(\mathcal{K})$:

$$([K_1] - [K_0]) + ([K_2] - [K_1]) + ([K_3] - [K_2]) + ([K_0] - [K_3]) = 0.$$

Now, $[K^i] = \pm([K_i] - [K_{i-1}])$, where the sign $\pm$ is $+$ when the vertical strand goes through an arrow from K_{i-1} to K_i, and minus when it goes through a tail. Therefore the above equality can be written as

$$-[K^1] + [K^2] + [K^3] - [K^4] = 0$$

and finishes the proof of the lemma. $\square$

Let $\overline{\mathcal{A}}_n$ denote the quotient of $\mathcal{D}_n$ by the *four–term relation,* which is the quotient of $\mathcal{D}_n$ by the vector space generated by the $(-D^1 + D^2 + D^3 - D^4)$ for all the 4-tuples (D^1, D^2, D^3, D^4) as in Lemma 6.8. Call $(1T)$ the relation that identifies a diagram with an isolated chord with 0, so $\overline{\mathcal{A}}_n/(1T)$ is the quotient of $\overline{\mathcal{A}}_n$ by the vector space generated by diagrams with an isolated chord.

According to Lemma 6.8 above, the map ϕ_n induces a map

$$\overline{\phi}_n \colon \overline{\mathcal{A}}_n/(1T) \longrightarrow \frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}.$$

The fundamental theorem of *Vassiliev invariants* (which are finite type knot invariants) can now be stated.

Theorem 6.9 (Bar-Natan, Kontsevich)**.** *There exists a family of linear maps* $\left(\overline{\check{\mathcal{Z}}}_n \colon \mathcal{F}_0(\mathcal{K}) \to \overline{\mathcal{A}}_n/(1T)\right)_{n\in\mathbb{N}}$ *such that*

- $\overline{\check{\mathcal{Z}}}_n(\mathcal{F}_{n+1}(\mathcal{K})) = 0$*,*
- *Let* $\overline{\overline{\mathcal{Z}}}_n$ *be the map induced by* $\overline{\check{\mathcal{Z}}}_n$ *from* $\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})}$ *to* $\overline{\mathcal{A}}_n/(1T)$*. Then* $\overline{\overline{\mathcal{Z}}}_n \circ \overline{\phi}_n$ *is the identity map of* $\overline{\mathcal{A}}_n/(1T)$*.*

In particular $\frac{\mathcal{F}_n(\mathcal{K})}{\mathcal{F}_{n+1}(\mathcal{K})} \cong \overline{\mathcal{A}}_n/(1T)$ *are identified by the inverse isomorphisms* $\overline{\overline{\mathcal{Z}}}_n$ *and* $\overline{\phi}_n$*, and* $\frac{\mathcal{I}_n(\mathcal{K})}{\mathcal{I}_{n-1}(\mathcal{K})} \cong (\overline{\mathcal{A}}_n/(1T))^*$*.*

This theorem has been proved by Kontsevich and Bar-Natan using the *Kontsevich integral* $Z^K = (Z^K_n)_{n\in\mathbb{N}}$ [BN95a], for $\mathbb{K} = \mathbb{R}$. It is also true when $\mathbb{K} = \mathbb{Q}$. It is reproved in Section 17.6 using the invariant $\mathcal{Z}$ studied in this book. An invariant $\overline{\check{\mathcal{Z}}}$ as in the above statement has the following universality property.

Corollary 6.10. *For any real-valued degree n invariant f of knots in $\mathbb{R}^3$, there exist linear forms $\psi_i \colon \overline{\mathcal{A}}_i/(1T) \to \mathbb{R}$, for $i = 0, \dots, n$ such that:*

$$f = \sum_{i=0}^{n} \psi_i \circ \overline{\check{\mathcal{Z}}}_i.$$

Proof: Let $\psi_n = f_{|\mathcal{F}_n} \circ \overline{\phi}_n$, then $f - \psi_n \circ \overline{\check{\mathcal{Z}}}_n$ is an invariant of degree at most $n-1$. Conclude by induction. $\square$

By projection (or up to (1T)), such an invariant $\overline{\check{\mathcal{Z}}}$ therefore defines a universal Vassiliev knot invariant, with respect to the following definition.

An invariant $Y \colon \mathcal{F}_0(\mathcal{K}) \to \prod_{n\in\mathbb{N}} \overline{\mathcal{A}}_n/(1T)$ such that

- $Y_n(\mathcal{F}_{n+1}(\mathcal{K})) = 0$, and
- Y_n induces a left inverse to $\overline{\phi}_n$ from $\frac{\mathcal{F}_n(\mathcal{K};\mathbb{R})}{\mathcal{F}_{n+1}(\mathcal{K};\mathbb{R})}$ to $\overline{\mathcal{A}}_n/(1T)$

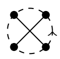
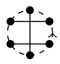
is called a *universal Vassiliev knot invariant*.

The terminology is justified because such an invariant contains all the real-valued Vassiliev knot invariants as in Corollary 6.10.

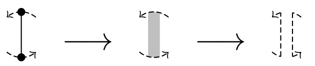
As proved by Altschüler and Freidel [AF97], the restriction of the invariant $\mathcal{Z} = (\mathcal{Z}_n)_{n\in\mathbb{N}}$ to knots of $\mathbb{R}^3$ also satisfies the properties of Theorem 6.9, so it is also a *universal Vassiliev knot invariant.* We will give alternative proofs of generalizations of this result in this book.

Similar characterizations of the spaces of finite type invariants of links in $\mathbb{R}^3$, integer homology 3-spheres and rational homology 3-spheres will be presented in Section 17.6 and Chapter 18, respectively. For integer homology 3-spheres and rational homology 3-spheres, the most difficult parts of the proofs will be viewed as consequences of the splitting formulae satisfied by $\mathcal{Z}$, which are stated in Theorem 18.5.

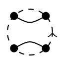
Let us end this subsection with an example of a nontrivial linear form on $\overline{\mathcal{A}}_n/(1T)$.

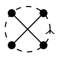
Example 6.11. Let us first define a function $\breve{w}_C$ of chord diagrams. Let Γ be a chord diagram. Immerse Γ in the unit disk D_1 of the plane so that the chords of Γ are embedded and attached to the left-hand side of the boundary S^1 of D_1, as in our former pictures like  or .

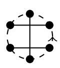
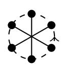
Embed an oriented band $[0,1]^2$ in the plane around each chord, so that $(\partial[0,1]) \times [0,1]$ is a neighborhood of the ends of the chord in the dashed circle S^1, and perform the *surgery* on the dashed circle S^1 that replaces $(\partial[0,1]) \times [0,1]$ with $[0,1] \times \partial[0,1]$ as in the following figure.

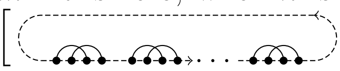


If the resulting naturally oriented one-manifold is connected, then $\breve{w}_C(\Gamma) = 1$. Otherwise, $\breve{w}_C(\Gamma) = 0$. For example,

$\breve{w}_C$ () $= 1$, $\breve{w}_C$ () $= 0$, $\breve{w}_C$ () $= 0$.

Since  is connected, $\breve{w}_C$ () $= 1$

$\breve{w}_C$ () $= \breve{w}_C$ () $= 0$.

More generally, the reader can check that one of our surgeries changes the mod 2 congruence class of the number of connected components of the surgered manifold, so $\breve{w}_C(\Gamma) = 0$ for any chord diagram Γ with an odd number of chords. Extend $\check{w}_C$ linearly over $\mathcal{D}_n$. For 4-tuples (D^1, D^2, D^3, D^4) as in Lemma 6.8, the reader can check that the extended $\check{w}_C$ maps $(-D^1 + D^2 + D^3 - D^4)$ to zero. It also maps diagrams with isolated chords to zero. Therefore, $\check{w}_C$ induces a linear map w_C on $\overline{\mathcal{A}}_n$ (and on $\overline{\mathcal{A}}_n/(1T)$) for any n. For any chord diagram Γ, $w_C([\Gamma]) = \check{w}_C(\Gamma)$. This linear map was first studied by Bar-Natan and Garoufalidis in [BNG96]. They called it the *Conway weight system.* It is zero, when n is odd, and it is not zero when n is even, since $w_C\Big(\Big[$  $\Big]\Big) \neq 0$.

6.3 More spaces of diagrams

Definition 6.12. A *uni-trivalent graph* Γ is a 6-tuple

$$(H(\Gamma), E(\Gamma), U(\Gamma), T(\Gamma), p_E, p_V),$$

where

- $H(\Gamma)$, $E(\Gamma)$, $U(\Gamma)$ and $T(\Gamma)$ are finite sets, which are called the set of half-edges of Γ, the set of edges of Γ, the set of univalent vertices of Γ and the set of trivalent vertices of Γ, respectively,

- $p_E \colon H(\Gamma) \to E(\Gamma)$ is a two-to-one map (every element of $E(\Gamma)$ has two preimages under p_E), and,

- $p_V \colon H(\Gamma) \to U(\Gamma) \sqcup T(\Gamma)$ is a map such that every element of $U(\Gamma)$ has one preimage under p_V and every element of $T(\Gamma)$ has three preimages under p_V,

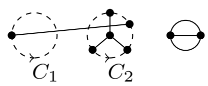
up to isomorphism. In other words, Γ is a set $H(\Gamma)$ equipped with two partitions, a partition into pairs (induced by p_E), and a partition into singletons and triples (induced by p_V), up to the bijections that preserve the partitions. These bijections are the *automorphisms* of Γ.

Definition 6.13. Let C be a one-manifold, oriented or not. A *Jacobi diagram* Γ *with support* C, also called *Jacobi diagram on* C, is a finite uni-trivalent graph Γ equipped with an isotopy class $[i_\Gamma]$ of injections i_Γ from the set $U(\Gamma)$ of univalent vertices of Γ into the interior of C. For such a Γ, a Γ-*compatible injection* is an injection in the class $[i_\Gamma]$. An *orientation* of a trivalent vertex of Γ is a cyclic order on the set of the three half-edges that meet at this vertex. An *orientation* of a univalent vertex u of Γ is an orientation of the connected component $C(u)$ of $i_\Gamma(u)$ in C, for a choice of Γ-compatible i_Γ, associated to u. This orientation is also called (and thought[2] of as) a *local orientation of* C *at* u. When C is oriented, the orientation of C orients the univalent vertices of Γ naturally.

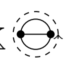
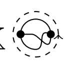
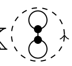
A *vertex-orientation* of a Jacobi diagram Γ is an *orientation* of every vertex of Γ. A Jacobi diagram is *oriented* if it is equipped with a vertex-orientation[3].

Unless otherwise mentioned, the supports of Jacobi diagrams are oriented and the induced orientations of univalent vertices are used without being mentioned. But the above notion of local orientations will prove useful to state some properties of the invariant Z studied in this book, such as the behaviour under cablings in Theorem 13.12.

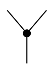
Such an oriented Jacobi diagram Γ is represented by a planar immersion of $\Gamma \cup C = \Gamma \cup_{U(\Gamma)} C$, where the univalent vertices of $U(\Gamma)$ are located at their images under a Γ-compatible injection i_Γ, the (oriented) one-manifold C is represented by dashed lines, whereas the edges of the diagram Γ are represented by plain segments. The vertices are represented by big points. The orientation of a trivalent vertex is represented by the counterclockwise order of the three half-edges that meet at the vertex. Here is an example of a picture of a Jacobi diagram Γ on the disjoint union $M = S^1 \sqcup S^1$ of two (oriented) circles:



The *degree* of such a diagram is half the number of all its vertices. Note that a chord diagram of $\mathcal{D}_n$ is a degree n Jacobi diagram on S^1 without trivalent vertices. For an (oriented) one-manifold C, $\mathcal{D}_n(C)$ denotes the $\mathbb{K}$-vector space freely generated by the degree n oriented Jacobi diagrams on C. For the (oriented) circle S^1,

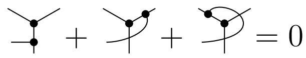
$\mathcal{D}_1(S^1) = \mathbb{K}$  $\oplus\, \mathbb{K}$  $\oplus\, \mathbb{K}$  $\oplus\, \mathbb{K}$  $\oplus\, \mathbb{K}$ 

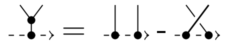
For an (oriented) one-manifold C, $\mathcal{A}_n(C)$ denotes the quotient of $\mathcal{D}_n(C)$ by the following relations AS, Jacobi and STU:

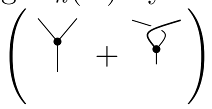
AS (or antisymmetry):  $+$  $= 0$

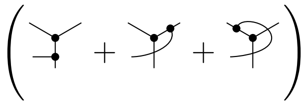
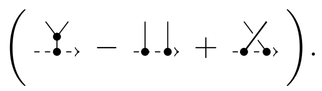
[2]A *local orientation* of C is simply an orientation of $C(u)$, but since different vertices are allowed to induce different orientations, we think of these orientations as being *local*, i.e. defined in a neighborhood of $i_\Gamma(u)$ for a choice of Γ-compatible i_Γ.

[3]When C is oriented, it suffices to specify the orientations of the trivalent vertices, since the univalent vertices are oriented by C.

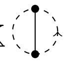
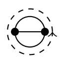
Jacobi: 

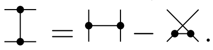
STU: 

Each of these relations relate oriented Jacobi diagrams which are identical outside the pictures. The quotient $\mathcal{A}_n(C)$ is the largest quotient of $\mathcal{D}_n(C)$ in which these relations hold. It is obtained by quotienting $\mathcal{D}_n(C)$ by the vector space generated by elements of $\mathcal{D}_n(C)$ of the form ,

 or .

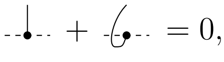
Example 6.14.

$\mathcal{A}_1(S^1) = \mathbb{K}$  $\oplus\, \mathbb{K}$ .

Remark 6.15. When $\partial C = \emptyset$, Lie algebras provide nontrivial linear maps, called *weight systems* from $\mathcal{A}_n(C)$ to $\mathbb{K}$, see [BN95a], [CDM12, Chapter 6] or [Les05, Section 6]. In the weight system constructions, the Jacobi relation for the Lie bracket ensures that the maps defined for oriented Jacobi diagrams factor through the Jacobi relation. In [Vog11], Pierre Vogel proved that the maps associated with Lie (super)algebras are sufficient to detect nontrivial elements of $\mathcal{A}_n(C)$ until degree 15, and he exhibited a non-trivial element of $\mathcal{A}_{16}(\emptyset)$ that cannot be detected by such maps. The Jacobi relation was originally called IHX , by Bar-Natan in [BN95a] because, up to AS, it can be written as . Note that the four entries in this IHX relation play the same role, up to AS.

Definition 6.16. The orientation of a univalent vertex u of a Jacobi diagram on a non-oriented one-manifold C corresponds to the counterclockwise cyclic order of the three half-edges that meet at u in a planar immersion of $\Gamma \cup_{U(\Gamma)} C$, where the half-edge of u in Γ is attached to the left-hand side of C, with respect to the local orientation of C at u, as in the following pictures.

For a non-oriented one-manifold C, $\mathcal{D}_n(C)$ is the $\mathbb{K}$-vector space generated by the degree n oriented Jacobi diagrams on C –where there are further orientation choices for univalent vertices–, and $\mathcal{A}_n(C)$ is the quotient of $\mathcal{D}_n(C)$ by the previous relations AS, Jacobi and STU together with the additional antisymmetry relation



where the (unoriented) STU relation may be written as:

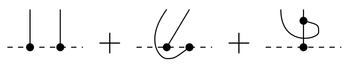
STU: 

Lemma 6.17. *Let C be an oriented one-manifold, and let C^u denote the non-oriented manifold obtained from C by forgetting the orientation. Let Γ^u be an oriented Jacobi diagram on C^u. A univalent vertex u of Γ^u is C-oriented if it induces the orientation of C. Otherwise, it is $(-C)$-oriented. Let $\Gamma(\Gamma^u, C)$ be the oriented Jacobi diagram on C obtained from Γ^u by reversing the local orientation of the $(-C)$-oriented univalent vertices, and let $k(\Gamma^u, C)$ be the number of $(-C)$-oriented univalent vertices of Γ^u. The linear map from $\mathcal{D}_n(C^u)$ into $\mathcal{D}_n(C)$ that maps any oriented Jacobi diagram Γ^u to $(-1)^{k(\Gamma^u,C)}\Gamma(\Gamma^u, C)$, and the linear canonical injection from $\mathcal{D}_n(C)$ into $\mathcal{D}_n(C^u)$ induce canonical isomorphisms between $\mathcal{A}_n(C)$ and $\mathcal{A}_n(C^u)$, which are inverse to each other, for any integer $n \in \mathbb{N}$.*

Proof: Exercise. $\square$

The above lemma justifies the use of the same notation $\mathcal{A}_n(.)$ for oriented and unoriented supports. We draw Jacobi diagrams on oriented supports, by attaching the half-edges of univalent vertices to the left-hand side of the support, in order to avoid confusion, and in order to get rid of the orientation of the support more easily.

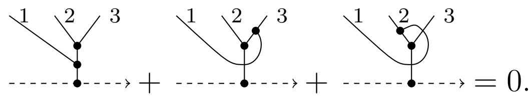
Remark 6.18. Note that the unoriented *STU* relation above can be drawn like the Jacobi relation up to AS

 $= 0.$

Definition 6.19. When $C \neq \emptyset$, let $\check{\mathcal{A}}_n(C) = \check{\mathcal{A}}_n(C; \mathbb{K})$ denote the quotient of $\mathcal{A}_n(C) = \mathcal{A}_n(C; \mathbb{K})$ by the vector space generated by the diagrams that have at least one connected component without univalent vertices. Then $\check{\mathcal{A}}_n(C)$ is generated by the degree n oriented Jacobi diagrams whose (plain) connected components contain at least one univalent vertex.

Lemma 6.20. *$\check{\mathcal{A}}_n(S^1)$ is the quotient of the vector space generated by the degree n oriented Jacobi diagrams whose connected components contain at least one univalent vertex, by the relations AS and STU. In other words, the Jacobi relation is a consequence of the relations AS and STU in this vector space.*

Proof: We want to prove that the Jacobi relation is true in the quotient by *AS* and *STU* of the space of unitrivalent diagrams on S^1 with at least one univalent vertex in each connected component. Consider three diagrams that are represented by three immersions which coincide outside a disk D, inside which they are as in the pictures involved in the Jacobi relation. Use STU as much as possible to remove all trivalent vertices that can be removed without changing the two vertices in D, on the three diagrams simultaneously. This transforms the Jacobi relation to be proved to a sum of similar relations, where one of the four entries of the disk is directly connected to S^1. Thus, since the four entries play the same role in the Jacobi relation, we may assume that the Jacobi relation to be proved is



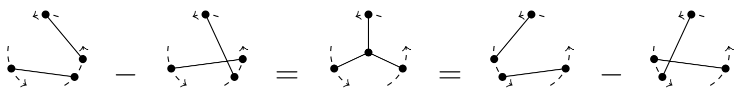
Using STU twice and AS transforms the summands of the left-hand side to diagrams that can be represented by three straight lines from the entries 1, 2, 3 to three fixed points of the horizontal line numbered from left to right. When the entry $i \in \{1, 2, 3\}$ is connected to the point $\sigma(i)$ of the horizontal dashed line, where σ is a permutation of $\{1, 2, 3\}$, the corresponding diagram is denoted by $(\sigma(1)\sigma(2)\sigma(3))$. Thus, the expansion of the left-hand side of the above equation is

$$\begin{array}{r} ((123) - (132) - (231) + (321)) \\ -((213) - (231) - (132) + (312)) \\ -((123) - (213) - (312) + (321)) \end{array},$$

which vanishes and the lemma is proved. □

Proposition 6.21. *The natural map from $\mathcal{D}_n$ to $\check{\mathcal{A}}_n(S^1)$ induces an isomorphism from the space $\overline{\mathcal{A}}_n$ of chord diagrams to $\check{\mathcal{A}}_n(S^1)$.*

First part of the proof: The natural map from $\mathcal{D}_n$ to $\check{\mathcal{A}}_n(S^1)$ factors though $4T$ since, according to STU,



in $\check{\mathcal{A}}_n(S^1)$. Since STU allows us to write any oriented Jacobi diagram whose connected components contain at least a univalent vertex as a combination of chord diagrams, inductively, the induced map from $\overline{\mathcal{A}}_n$ to $\check{\mathcal{A}}_n(S^1)$ is surjective. Injectivity will be proved in Section 6.4 by constructing an inverse map.

6.4 Multiplying diagrams

Set $\mathcal{A}(C) = \prod_{n\in\mathbb{N}} \mathcal{A}_n(C)$, $\check{\mathcal{A}}(C) = \prod_{n\in\mathbb{N}} \check{\mathcal{A}}_n(C)$ and $\overline{\mathcal{A}} = \prod_{n\in\mathbb{N}} \overline{\mathcal{A}}_n$.

Assume that a one-manifold C is decomposed as a union of two one-manifolds $C = C_1 \cup C_2$ whose interiors in C do not intersect. Define the *product associated with this decomposition*:

$$\mathcal{A}(C_1) \times \mathcal{A}(C_2) \longrightarrow \mathcal{A}(C)$$

to be the continuous bilinear map that maps $([\Gamma_1], [\Gamma_2])$ to $[\Gamma_1 \sqcup \Gamma_2]$, if Γ_1 is a diagram with support C_1 and if Γ_2 is a diagram with support C_2, where $\Gamma_1 \sqcup \Gamma_2$ denotes their disjoint union[4] on $C_1 \cup C_2$. In particular, the disjoint union of diagrams turns $\mathcal{A}(\emptyset)$ into a commutative algebra graded by the degree, and it turns $\mathcal{A}(C)$ to an $\mathcal{A}(\emptyset)$-module, for any 1-dimensional manifold C.

An orientation-preserving diffeomorphism from a manifold C to another one C' induces a natural isomorphism from $\check{\mathcal{A}}_n(C)$ to $\check{\mathcal{A}}_n(C')$, for all n. Let $I = [0, 1]$ be the compact oriented interval. If $I = C$, and if we identify I with $C_1 = [0, 1/2]$ and with $C_2 = [1/2, 1]$ with respect to the orientation, then the above process turns $\check{\mathcal{A}}(I)$ into an algebra, the elements of which with non-zero degree zero part admit an inverse.

Proposition 6.22. *The algebra $\check{\mathcal{A}}([0, 1])$ is commutative. The projection from $[0, 1]$ to $S^1 = [0, 1]/(0 \sim 1)$ induces an isomorphism from $\check{\mathcal{A}}_n([0, 1])$ to $\check{\mathcal{A}}_n(S^1)$ for all n, so $\check{\mathcal{A}}(S^1)$ inherits a commutative algebra structure from this isomorphism. The choice of an oriented connected component C_j of C equips $\check{\mathcal{A}}(C)$ with an $\check{\mathcal{A}}([0, 1])$-*module structure $\sharp_j$ *, induced by the orientation-preserving inclusion from $[0, 1]$ to a small part of C_j outside the vertices, and the insertion of diagrams with support $[0, 1]$ there.*

In order to prove this proposition, we present a useful trick in diagram spaces.

Lemma 6.23. *Let C be a non-oriented one-manifold. Let Γ_1 be an oriented Jacobi diagram (resp. a chord diagram) with support C as in Definitions 6.13 and 6.16. Assume that $\Gamma_1 \cup C$ is immersed in the plane so that $\Gamma_1 \cup C$ meets an open annulus A embedded in the plane exactly along $n + 1$ embedded arcs α_1, α_2, $\dots$, α_n and β, and one vertex v, as in the examples below, so that*

1. *the α_i are disjoint, they may be dashed or plain (they are dashed in the case of chord diagrams), they go from a boundary component of A to the other one,*

[4]Formally, a $\Gamma_1 \sqcup \Gamma_2$-compatible injection restricts to $U(\Gamma_i)$ as a Γ_i -compatible injection, for $i \in \{1, 2\}$.

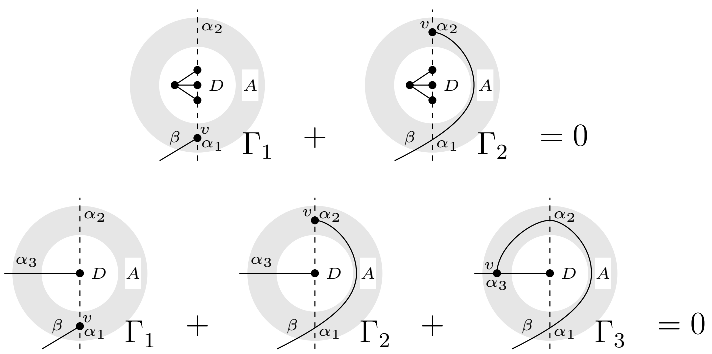
2. *β is a plain arc that goes from the boundary of A to $v \in \alpha_1$,*

3. *the bounded component D of the complement of A does not contain a boundary point of C,*

4. *the vertex-orientations are induced by the planar immersion by the local counterclockwise orders in the neighborhoods of vertices (as usual).*

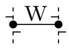
Let Γ_i be the diagram obtained from Γ_1 by attaching the endpoint v of β to α_i instead of α_1 on the same side, where the side of an arc is its side when going from the outside boundary component of A to the inside one ∂D, as in the examples below. Then $\sum_{i=1}^n \Gamma_i = 0$ in $\mathcal{A}(C)$ (resp. in the space $\overline{\mathcal{A}}$ of Section 6.2).

Examples 6.24.



Proof: The second example shows that the STU relation is equivalent to the relation of the statement, when the bounded component D of $\mathbb{R}^2 \setminus A$ intersects Γ_1 in the neighborhood of a univalent vertex on C. Similarly, the Jacobi relation is easily seen as given by that relation when D intersects Γ_1 in the neighborhood of a trivalent vertex. Also note that AS corresponds to the case when D intersects Γ_1 along a dashed or plain arc. Let us now give the Bar-Natan [BN95a, Lemma 3.1] proof. See also [Vog11, Lemma 3.3]. Assume, without loss of generality, that v is always attached on the left-hand side of the α's.

In the case of trivalent diagrams, add the trivial (by Jacobi and STU) contribution of the sum of the diagrams obtained from Γ_1 by attaching v to each of the three (dashed or plain) half-edges of each vertex w of $\Gamma_1 \cup C$ in D on the right-hand side when the half-edges are oriented towards w (i.e. by attaching v to the hooks in w), to the sum. Now, group the terms of the obtained sum by edges of $\Gamma_1 \cup C$, where v is attached, and observe that the sum is zero, edge by edge, by AS.

For chord diagrams, similarly add the trivial (by 4T) contribution of the sum of the diagrams obtained from Γ_1 by attaching v to each of the four (dashed) half-edges adjacent to each chord W of $\Gamma_1 \cup C$ in D, on the right-hand side when the half-edges are oriented towards W (i.e. by attaching v to the hooks in 

), to the sum. Again, group the terms of the obtained sum by dashed edges of $\Gamma_1 \cup C$, where v is attached, and observe that the sum is zero, edge by edge, by AS. $\square$

End of proof of Proposition 6.21:

As promised, we construct a map f from $\check{\mathcal{A}}_n(S^1)$ to the space $\overline{\mathcal{A}}_n$ of chord diagrams up to $(4T)$ and (AS), and we prove that it is an inverse of the natural surjective map g from $\overline{\mathcal{A}}_n$ to $\check{\mathcal{A}}_n(S^1)$. Let $\mathcal{D}_{n,k}$ denote the vector space generated by the oriented unitrivalent degree n diagrams on S^1 that have at most k trivalent vertices, and at least one per connected component.

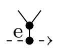
We will define linear maps λ_k from $\mathcal{D}_{n,k}$ to $\overline{\mathcal{A}}_n$ by induction on k so that

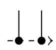
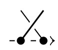
1. λ_0 maps a chord diagram to its class in $\overline{\mathcal{A}}_n$,
2. the restriction of λ_k to $\mathcal{D}_{n,k-1}$ is λ_{k-1}, and,
3. λ_k maps all the relations AS and STU that involve only elements of $\mathcal{D}_{n,k}$ to zero.

It is clear that when we have succeeded in such a task, the linear map from the space of oriented unitrivalent degree n diagrams on S^1 that have at least one univalent vertex per connected component that maps a diagram d with k trivalent vertices to $\lambda_k(d)$ will factor through STU and AS, and that the induced map $\overline{\lambda}$ will provide the wanted inverse map and allow us to conclude the proof. Now, let us define our maps λ_k with the announced properties.

Let $k \geq 1$, assume that λ_{k-1} is defined on $\mathcal{D}_{n,k-1}$ and that λ_{k-1} maps all the relations AS and STU that involve only elements of $\mathcal{D}_{n,k-1}$ to zero. We want to extend λ_{k-1} on $\mathcal{D}_{n,k}$ to a linear map λ_k that maps all the relations AS and STU that involve only elements of $\mathcal{D}_{n,k}$ to zero.

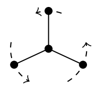
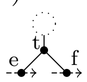
Let d be a diagram with k trivalent vertices, and let e be an edge of d that contains one univalent vertex and one trivalent vertex. Set

$$\lambda\left((d,e) = \right.$$


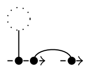
$$\left.\right) = \lambda_{k-1}\left(\right.$$

$$-$$

$$\left.\right).$$

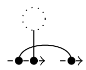
It suffices to prove that $\lambda(d,e)$ is independent of our chosen edge e to conclude the proof by defining the linear map λ_k, which will obviously satisfy the wanted properties, by

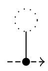
$$\lambda_k(d) = \lambda(d,e).$$

Assume that there are two different edges e and f of d that connect a trivalent vertex to a univalent vertex. We prove that $\lambda(d, e) = \lambda(d, f)$. If e and f are disjoint, then the fact that λ_{k-1} satisfies STU allows us to express both $\lambda(d, e)$ and $\lambda(d, f)$ as the same combination of four diagrams with $(k - 2)$ vertices, and we are done. Thus, we assume that e and f are two different edges that share a trivalent vertex t. If there exists another trivalent vertex that is connected to S^1 by an edge g, then $\lambda(d, e) = \lambda(d, g) = \lambda(d, f)$ and we are done. Thus, we furthermore assume that t is the unique trivalent vertex that is connected to S^1 by an edge. So, either t is the unique trivalent vertex, and its component is necessarily like  and the fact that $\lambda(d, e) = \lambda(d, f)$ is a consequence of (4T), or the component of t is of the form  where the dotted circle represents a dashed diagram with only one pictured entry. Thus,

$$\lambda(d, e) = \lambda_{k-1}\left(\quad - \quad \right).$$





Now, this is zero because the expansion of  as a sum of chord diagrams commutes with any vertex in $\overline{\mathcal{A}}_n$, according to Lemma 6.23. Similarly, $\lambda(d, f) = 0$. Thus, $\lambda(d, e) = \lambda(d, f)$ in this last case and we are done. $\square$

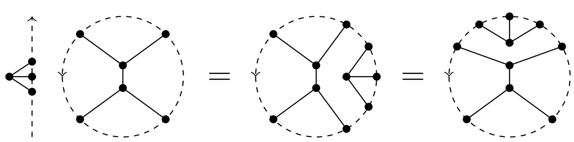
Lemma 6.25. *For any one-manifold C, the class of a Jacobi diagram with one univalent vertex vanishes in $\mathcal{A}_n(C)$.*

Proof: Exercise. (Use Lemma 6.23.) $\square$

Proof of Proposition 6.22: With each choice of a connected component C_j of C, we associate an $\check{\mathcal{A}}(I)$*-module structure* $\sharp_j$ *on* $\mathcal{A}(C)$, which is given by the continuous bilinear map:

$$\check{\mathcal{A}}(I) \times \mathcal{A}(C) \longrightarrow \mathcal{A}(C)$$

such that: If Γ' is a diagram with support C and if Γ is a diagram with support I, then $([\Gamma], [\Gamma'])$ is mapped to the class of the diagram obtained by inserting Γ along C_j outside the vertices of Γ, according to the given orientation. For example,



As shown in the first example that illustrates Lemma 6.23, the independence of the choice of the insertion locus is a consequence of Lemma 6.23, where Γ_1 is the disjoint union $\Gamma \sqcup \Gamma'$ and Γ_1 intersects D along $\Gamma \cup I$. This also proves that $\check{\mathcal{A}}(I)$ is a commutative algebra. Now, it suffices to prove that the morphism from $\check{\mathcal{A}}(I)$ to $\check{\mathcal{A}}(S^1)$ induced by the identification of the two endpoints of I is an isomorphism. This is proved in the more general proposition below. $\square$

Proposition 6.26. *Let $n \in \mathbb{N}$. Let $\mathcal{L}$ be a disjoint union of circles. The projection from $[0,1]$ to $S^1 = [0,1]/(0 \sim 1)$ induces an isomorphism from $\check{\mathcal{A}}_n([0,1] \sqcup \mathcal{L})$ to $\check{\mathcal{A}}_n(S^1 \sqcup \mathcal{L})$.*

PROOF: The morphism from $\check{\mathcal{A}}([0,1] \sqcup \mathcal{L})$ to $\check{\mathcal{A}}(S^1 \sqcup \mathcal{L})$ induced by the identification of the two endpoints of $[0,1]$ amounts to mod out $\check{\mathcal{A}}([0,1] \sqcup \mathcal{L})$ by the relation that identifies two diagrams that are obtained from one another by moving the nearest univalent vertex to an endpoint of $[0,1]$ near the other endpoint. Applying Lemma 6.23 (with β coming from the inside boundary of the annulus) shows that this relation is a consequence of the relations in $\check{\mathcal{A}}([0,1] \sqcup \mathcal{L})$. So that morphism is an isomorphism from $\check{\mathcal{A}}([0,1] \sqcup \mathcal{L})$ to $\check{\mathcal{A}}(S^1 \sqcup \mathcal{L})$. $\square$

Lemma 6.27. *If $\pi : C' \longrightarrow C$ is a smooth map between two unoriented one-manifolds C and C' such that $\pi(\partial C') \subset \partial C$, then one can unambiguously define the linear degree–preserving map $\pi^* : \check{\mathcal{A}}(C) \longrightarrow \check{\mathcal{A}}(C')$, which depends only on the homotopy class of π, such that: If Γ is (the class of) an oriented Jacobi diagram on C (as in Definitions 6.13 and 6.16) with univalent vertices that avoid the critical values of π, then $\pi^*(\Gamma)$ is the sum of all diagrams on C' obtained from Γ by lifting each univalent vertex to one of its preimages under π. (These diagrams have the same vertices and edges as Γ, and the local orientations at univalent vertices are induced naturally by the local orientations of the corresponding univalent vertices of Γ.)*

PROOF: It suffices to see that this operation is compatible with STU, and that nothing bad happens when a univalent vertex of C moves across a critical value of π. $\square$

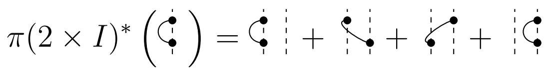
Notation 6.28. Let C be a one-manifold, and let C_0 be a connected component of C. Let

$$C(r \times C_0) = (C \setminus C_0) \sqcup \left(\sqcup_{i=1}^r C_0^{(i)}\right)$$

be the manifold obtained from C by *duplicating* C_0 (r-1) times, that is by replacing C_0 with r copies of C_0 and let $\pi(r \times C_0) : C(r \times C_0) \longrightarrow C$ be the associated map, which is the identity on $(C \setminus C_0)$, and the trivial r-fold covering from $\sqcup_{i=1}^{r} C_0^{(i)}$ to C_0. The associated map is the *duplication map:*

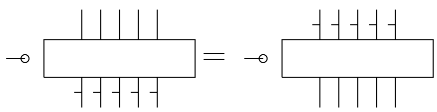
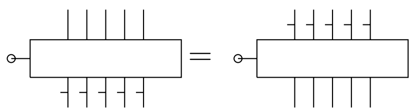
$$\pi(r \times C_0)^* : \mathcal{A}(C) \longrightarrow \mathcal{A}(C(r \times C_0)).$$

Example 6.29.



Note the following lemma.

Lemma 6.30. *When $\mathcal{L}$ is a disjoint union of r intervals, an r-duplicated vertex commutes with an element of $\mathcal{A}(\mathcal{L})$. This sentence is explained by the pictures below. In the first picture, there is a Jacobi diagram on $\mathcal{L}$ inside the rectangle and the picture represents the sum of the diagrams obtained by attaching the free end of an edge (that with the empty circle) of some other part of a Jacobi diagram to each of the hooks attached to the vertical strands. The second picture is similar except that the edge with the free end is a part of a Jacobi diagram, which is inside the box apart from this half-edge.*

 *and* 

PROOF: This is a direct consequence of Lemma 6.23 as the pictures show. □

6.5 Coproduct on $\mathcal{A}(C)$

Below, all tensor products are over the ground field $\mathbb{K}$. The canonical identification of $V \otimes \mathbb{K}$ with V for a finite dimensional vector space over $\mathbb{K}$ will always be implicit. In this section, C denotes a one-manifold.

For $n \in \mathbb{N}$, set

$$(\mathcal{A}(C) \otimes \mathcal{A}(C))_n = \oplus_{i=0}^{n} \mathcal{A}_i(C) \otimes \mathcal{A}_{n-i}(C)$$

and

$$\mathcal{A}(C)\hat{\otimes}\mathcal{A}(C) = \prod_{n \in \mathbb{N}} (\mathcal{A}(C) \otimes \mathcal{A}(C))_n.$$

The topological product $\mathcal{A}(C)$ is equipped with the following collection of linear maps
$$\Delta_n \colon \mathcal{A}_n(C) \to (\mathcal{A}(C) \otimes \mathcal{A}(C))_n.$$
The image of the class of a Jacobi diagram $\Gamma = \sqcup_{i\in I}\Gamma_i$ with $\sharp I$ non-empty (plain) connected components Γ_i, numbered arbitrarily in a set I, is

$$\Delta_n([\Gamma]) = \sum_{J\subseteq I}[\sqcup_{i\in J}\Gamma_i] \otimes [\sqcup_{i\in(I\setminus J)}\Gamma_i].$$

It is easy to check that Δ_n is well defined. The family $\Delta = (\Delta_n)_{n\in\mathbb{N}}$ defines a degree–preserving map from $\mathcal{A}(C)$ to $\mathcal{A}(C)\hat{\otimes}\mathcal{A}(C)$.

There is a well defined continuous linear map $\varepsilon \colon \mathcal{A}(C) \to \mathbb{K}$ that maps $\mathcal{A}_i(C)$ to 0 for any $i > 0$ and that maps the class of the empty diagram to 1. The ground field $\mathbb{K}$ is considered as a degree 0 vector space. So the map ε is a degree–preserving homomorphism.

In the following statement, degrees are omitted, but the following identities, which express the fact that Δ is a graded *coproduct* with associated *counit* ε, are collections of identities between collections of degree–preserving linear maps between finite dimensional vector spaces. For example, the *coassociativity* identity

$$(\Delta \otimes \text{Identity}) \circ \Delta = (\text{Identity} \otimes \Delta) \circ \Delta$$

means that for any $n \in \mathbb{N}$

$$(\Delta \otimes \text{Identity})_n \circ \Delta_n = (\text{Identity} \otimes \Delta)_n \circ \Delta_n,$$

where both maps are valued in

$$(\mathcal{A}(C) \otimes \mathcal{A}(C) \otimes \mathcal{A}(C))_n = \oplus_{i,j,k|(i,j,k)\in\mathbb{N}^3, i+j+k=n}\mathcal{A}_i(C) \otimes \mathcal{A}_j(C) \otimes \mathcal{A}_k(C).$$

Lemma 6.31. $(\varepsilon \otimes \textit{Identity}) \circ \Delta = (\textit{Identity} \otimes \varepsilon) \circ \Delta = \textit{Identity}$

$$(\Delta \otimes \textit{Identity}) \circ \Delta = (\textit{Identity} \otimes \Delta) \circ \Delta$$

PROOF: Exercise. $\square$

Let
$$\begin{array}{llll} \tau_n \colon & (\mathcal{A}(C) \otimes \mathcal{A}(C))_n & \to & (\mathcal{A}(C) \otimes \mathcal{A}(C))_n \\ & x \otimes y & \mapsto & y \otimes x. \end{array}$$
Then we also immediately have the identity

$$\tau \circ \Delta = \Delta,$$

which expresses the *cocommutativity* of Δ.

6.6 Hopf algebra structures

Definition 6.32. A *connected, finite type, commutative and cocommutative Hopf algebra* over a field $\mathbb{K}$ is the topological product $\mathcal{H} = \prod_{n\in\mathbb{N}} \mathcal{H}_n$ of finite dimensional vector spaces $\mathcal{H}_n$ over $\mathbb{K}$ equipped with families $(m, \Delta, \upsilon, \varepsilon)$ of degree–preserving linear maps

- a *multiplication* $m = (m_n\colon (\mathcal{H}\otimes\mathcal{H})_n \to \mathcal{H}_n)_{n\in\mathbb{N}}$, where $(\mathcal{H}\otimes\mathcal{H})_n = \oplus_{i=0}^n \mathcal{H}_i \otimes \mathcal{H}_{n-i}$
- a *coproduct* $\Delta = (\Delta_n\colon \mathcal{H}_n \to (\mathcal{H}\otimes\mathcal{H})_n)_{n\in\mathbb{N}}$
- a *unit* $\upsilon\colon \mathbb{K} \to \mathcal{H}$, which maps $\mathbb{K}$ to $\mathcal{H}_0$ and which is an isomorphism from $\mathbb{K}$ to $\mathcal{H}_0$ (connectedness)
- a *counit* $\varepsilon\colon \mathcal{H} \to \mathbb{K}$, where $\mathbb{K}$ is again assumed to be of degree 0

that satisfy

- the following identities, which express that (m, υ) is an associative and commutative product with unit $\upsilon(1)$:

  $$\begin{aligned}&m \circ (m \otimes \text{Identity}) = m \circ (\text{Identity} \otimes m)\\&m \circ (\upsilon \otimes \text{Identity}) = m \circ (\text{Identity} \otimes \upsilon) = \text{Identity}\\&m \circ \tau = m,\end{aligned}$$

  where $\tau_n\colon (\mathcal{H}\otimes\mathcal{H})_n \to (\mathcal{H}\otimes\mathcal{H})_n$ maps $(x \otimes y)$ to $(y \otimes x)$.

- the following identities, which express that (Δ, ε) is a coassociative and cocommutative product with counit ε:

  $$\begin{aligned}&(\Delta \otimes \text{Identity}) \circ \Delta = (\text{Identity} \otimes \Delta) \circ \Delta\\&(\varepsilon \otimes \text{Identity}) \circ \Delta = (\text{Identity} \otimes \varepsilon) \circ \Delta = \text{Identity}\\&\tau \circ \Delta = \Delta.\end{aligned}$$

- the following *compatibility identity,* which expresses the fact that Δ is an algebra morphism and that m is a coalgebra morphism

  $$\Delta \circ m = (m \otimes m) \circ (\text{Identity} \otimes \tau \otimes \text{Identity}) \circ (\Delta \otimes \Delta),$$

  where the product on $(\mathcal{H}\hat{\otimes}\mathcal{H}) = \prod_{n\in\mathbb{N}} (\mathcal{H}\otimes\mathcal{H})_n$ is defined from m so that it maps $(a \otimes b) \otimes (a' \otimes b')$ to $m(a \otimes a') \otimes m(b \otimes b')$.

Lemma 6.33. *In a connected, finite type, commutative and cocommutative Hopf algebra,* $\varepsilon \circ \upsilon =$ *Identity and* $\Delta(\upsilon(1)) = \upsilon(1) \otimes \upsilon(1)$*. The element* $\upsilon(1)$ *with be denoted by* 1.

PROOF: Since $(\varepsilon \otimes \text{Identity}) \circ \Delta = \text{Identity}$, Δ is injective and $\varepsilon(\upsilon(1)) = k \neq 0$. Furthermore, since Δ is degree–preserving, $\Delta(\upsilon(1)) = k'\upsilon(1) \otimes \upsilon(1)$, where $kk' = 1$. Then applying the compatibility identity to $\upsilon(1) \otimes x$ yields $\Delta(x) = k'\Delta(x)$, so $k' = 1 = k$. $\square$

In a connected, finite type, commutative and cocommutative Hopf algebra, a *primitive* element is an element such that $\Delta(x) = 1 \otimes x + x \otimes 1$ and a *group-like* element is an element such that $\Delta(x) = x \otimes x$ and $\varepsilon(x) \neq 0$.

The proof of the following lemma is straightforward and left to the reader.

Lemma 6.34. *Equipped with the product of Section 6.4, with the coproduct of Section 6.5, and with the counit that maps the class of the empty diagram to 1, $\check{\mathcal{A}}(S^1)$, $\mathcal{A}(S^1)$ and $\mathcal{A}(\emptyset)$ are connected, finite type, commutative and cocommutative Hopf algebras. The unit $\upsilon(1)$ of these algebras is the class of the empty diagram. Furthermore, connected Jacobi diagrams are primitive elements of these algebras.*

$\square$

Note the elementary lemma.

Lemma 6.35. *If y is a primitive element of a connected, finite type, commutative and cocommutative Hopf algebra, then $\exp(y)$ is group-like.*

PROOF: It suffices to prove that $\Delta(y^n) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} y^k \otimes y^{n-k}$. When $n = 0$, this is Lemma 6.33. According to the compatibility identity $\Delta(y^n) = \Delta(y^{n-1})(y \otimes \upsilon(1) + \upsilon(1) \otimes y)$. $\square$

We can now state a version of the Milnor-Moore theorem.

Theorem 6.36. *Let $(\mathcal{H}; m, \Delta, \upsilon, \varepsilon)$ be a connected, finite type, commutative and cocommutative Hopf algebra over a field $\mathbb{K}$. Let $\mathcal{P}_n$ denote the set of primitive elements of $\mathcal{H}_n$. It is a finite dimensional vector space. Pick a basis b_n of each $\mathcal{P}_n$, for each n. For all $n \in \mathbb{N}$, $\mathcal{H}_n$ is freely generated by the degree n monomials[5] in the elements of $b_{\leq n} = \cup_{k \in \mathbb{N} | k \leq n} b_k$, as a vector space.*

PROOF: Since $\mathcal{P}_0 = \{0\}$, and $\mathcal{H}_0$ is freely generated by the empty monomial, the theorem holds for $n = 0$. Let $n \geq 1$, let d_n be the set of degree n monomials in the elements of $b_{\leq n-1}$. We want to prove that $\mathcal{H}_n$ is freely generated by $d_n \sqcup b_n$, by induction on n.

[5] By monomials, we mean monomials with coefficient one. So degree n monomials in the elements of $b_{\leq n}$ are of the form $\prod_{i \in I} p_i^{r(i)}$, for elements p_i of $b_{\leq n}$ of degree $d(i)$, and positive integers $r(i)$ such that $\sum_{i \in I} d(i) r(i) = n$.

For $x \in \mathcal{H}_n$, set $\Delta'(x) = \Delta(x) - x \otimes \upsilon(1) - \upsilon(1) \otimes x$. According to Lemma 6.33, since $(\varepsilon \otimes \text{Identity}) \circ \Delta = (\text{Identity} \otimes \varepsilon) \circ \Delta = \text{Identity}$,

$$\Delta'(x) \in \oplus_{i=1}^{n-1} \mathcal{A}_i(C) \otimes \mathcal{A}_{n-i}(C)$$

and $\mathcal{P}_n$ is the kernel of Δ'.

By induction, $\mathcal{A}_i(C)$ (resp. $\mathcal{A}_{n-i}(C)$) has a basis consisting of degree i (resp. $(n-i)$) monomials in the elements of $b_{\leq n-1}$. Thus $\mathcal{A}_i(C) \otimes \mathcal{A}_{n-i}(C)$ has a basis consisting of tensor products of these monomials. Multiplying two such monomials yields an element

$$\prod_{i \in I} p_i^{r(i)}$$

of d_n, where the p_i are distinct elements of $b_{\leq n-1}$, and the $r(i)$ are positive integers.

$$\Delta(\prod_{i \in I} p_i^{r(i)}) =$$
$$\sum_{k\colon I \to \mathbb{N} | 0 \leq k(i) \leq r(i), \forall i} \left(\prod_{i \in I} \frac{r(i)!}{k(i)!(r(i)-k(i))!} \right) \left(\prod_{i \in I} p_i^{k(i)} \right) \otimes \left(\prod_{i \in I} p_i^{r(i)-k(i)} \right).$$

This formula proves that Δ' injects the vector space freely generated by the degree n monomials in the elements of $\cup_{k \in \mathbb{N} | k < n} b_k$ into $\oplus_{i=1}^{n-1} \mathcal{A}_i(C) \otimes \mathcal{A}_{n-i}(C)$. Thus, the degree n monomials in the elements of $\cup_{k \in \mathbb{N} | k \leq n} b_k$ form a free system of $\mathcal{H}_n$, and it suffices to prove that they generate $\mathcal{H}_n$. In order to do so, it is enough to check that for every $x \in \mathcal{H}_n$, for every element $d \in d_n$, there exists a constant $a(x,d)$ such that

$$\Delta'(x) = \sum_{d \in d_n} a(x,d) \Delta'(d).$$

Indeed, in that case, $\left(x - \sum_{d \in d_n} a(x,d)d\right)$ is primitive. Fix $x \in \mathcal{H}_n$.

Let $d = \prod_{i \in I} p_i^{r_d(i)} \in \mathcal{H}_n$, where $r_d(i) > 0$, $\forall i \in I$. Let $E(d)$ be the set of maps $k\colon I \to \mathbb{N}$ such that $0 < \sum_{i \in I} k(i)$, $k(i) \leq r_d(i)$, and $d(k) \stackrel{\text{def}}{=} \prod_{i \in I} p_i^{k(i)} \in \mathcal{H}_j$, for some j such that $0 < j < n$. Then $\Delta'(d) = \sum_{k \in E(d)} c_{k,r_d} d(k) \otimes d(r_d - k)$, where

$$c_{k,r_d} = \prod_{i \in I} \frac{r_d(i)!}{k(i)!(r_d(i)-k(i))!} \neq 0,$$

while

$$\Delta'(x) = \sum_{d \in d_n} \left(\sum_{k \in E(d)} c_k(x,d) d(k) \otimes d(r_d - k) \right).$$

Now, it suffices to prove that for any $d \in d_n$, there exists $a(x,d)$ such that for any $k \in E(d)$, $c_k(x,d) = a(x,d)c_{k,r_d}$. Fix $d \in d_n$ and set $r = r_d$.

Thanks to the coassociativity of Δ, $(\Delta \otimes \text{Identity}) \circ \Delta(x) = (\text{Identity} \otimes \Delta) \circ \Delta(x)$. Therefore, if $h(i) \leq k(i)$ for all $i \in I$, and $\sum_{i \in I} h(i) > 0$, then the coefficient

$$c_{h,k}c_k(x,d) = c_h(x,d)c_{k-h,r-h}$$

of $d(h) \otimes d(k-h) \otimes d(r-k)$ in $(\Delta \otimes \text{Identity}) \circ \Delta(x)$ determines both the coefficient $c_h(x,d)$ of $d(h) \otimes d(r-h)$ and the coefficient $c_k(x,d)$ in $\Delta'(x)$, and we have

$$c_h(x,d) = \frac{c_{h,k}}{c_{k-h,r-h}} c_k(x,d).$$

The coassociativity applied to d implies similarly that

$$c_h(d,d) = \frac{c_{h,k}}{c_{k-h,r-h}} c_k(d,d),$$

where $c_h(d,d) = c_{h,r}$. So

$$c_h(x,d) = \frac{c_{h,r}}{c_{k,r}} c_k(x,d).$$

Choose $j \in I$ and let $\delta_j \in E(d)$ be such that $\delta_j(j) = 1$ and $\delta_j(i) = 0$ for all $i \in I \setminus \{j\}$. Set $a(x,d) = \frac{c_{\delta_j}(x,d)}{c_{\delta_j,r}}$. Then for all $k \in E(d)$ such that $k(j) \neq 0$, $a(x,d) = \frac{c_k(x,d)}{c_{k,r}}$. If $\sum_{i \in I} r(i) > 2$, then for any $i \in I \setminus \{j\}$, the map $\delta_i + \delta_j$ is in $E(d)$, $a(x,d) = \frac{c_{\delta_i}(x,d)}{c_{\delta_i,r}}$ for any $i \in I$ and therefore $a(x,d) = \frac{c_k(x,d)}{c_{k,r}}$ for any $k \in E(d)$. The only untreated case is $I = \{i,j\}$ with $r(i) = r(j) = 1$. In this case, the cocommutativity of Δ leads to the result. □

Corollary 6.37. *Under the assumptions of Theorem 6.36, there is a well defined unique linear projection p^c from $\mathcal{H}$ to $\mathcal{P} = \prod_{n \in \mathbb{N}} \mathcal{P}_n$ that maps the products of two elements of positive degree to* 0*, and that maps $\mathcal{H}_0$ to* 0*.*

□

Theorem 6.38. *Let $\mathcal{H}$ be a connected, finite type, commutative and cocommutative Hopf algebra. Let $\mathcal{P}$ be the space of its primitive elements and let $p^c \colon \mathcal{H} \to \mathcal{P}$ be the projection of Corollary 6.37. Any group-like element x of $\mathcal{H}$ is the exponential of a unique primitive element of $\mathcal{P}$. This element is $p^c(x)$.*

PROOF: First note that $p^c(\exp(y)) = y$ for any primitive element y of $\mathcal{H}$. Therefore, if $\exp(y) = \exp(y')$ for two primitive elements y and y' of $\mathcal{H}$, then $y = y'$.

Set $y_n = p^c(x)_n$ and let us prove $x = \exp(y)$. Since $\varepsilon(x) \neq 0$, $x_0 = k\upsilon(1)$ with $k \neq 0$, and $k = 1$ since $\Delta(x) = x \otimes x$. Then the equality $x = \exp(y)$ is true in degree 0. Assume that it is true until degree $(n-1)$. Then $\Delta_n(x_n) = \sum_{i=0}^{n} x_i \otimes x_{n-i}$, while $\Delta_n(\exp(y)_n) = \sum_{i=0}^{n} \exp(y)_i \otimes \exp(y)_{n-i}$, according to Lemma 6.35. Thus, $\Delta_n(x_n - \exp(y)_n) = 1 \otimes (x_n - \exp(y)_n) + (x_n - \exp(y)_n) \otimes 1$. Therefore $(x_n - \exp(y)_n)$ is primitive and $(x_n - \exp(y)_n) = p^c(x_n - \exp(y)_n) = 0$. $\square$

Chapter 7

First definitions of Z

In this chapter, we introduce the invariant $\mathcal{Z}$ of links in $\mathbb{Q}$-spheres, which is the main object of this book. We illustrate the required definitions by many examples, which may be skipped by a reader who only wants the definition.

7.1 Configuration spaces of Jacobi diagrams in 3-manifolds

Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$, as in Definition 3.8. Let $\mathcal{L}$ be a disjoint union of k circles S^1_i, $i \in \underline{k}$ and let

$$L : \mathcal{L} \longrightarrow \check{R}$$

denote a C^∞ embedding from $\mathcal{L}$ to $\check{R}$. The embedding L is a *link embedding*. Let Γ be a Jacobi diagram with support $\mathcal{L}$ as in Definition 6.13. Let $U = U(\Gamma)$ denote the set of univalent vertices of Γ, and let $T = T(\Gamma)$ denote the set of trivalent vertices of Γ. A *configuration* of Γ (with respect to L) is an embedding

$$c : U \cup T \hookrightarrow \check{R}$$

whose restriction $c_{|U}$ to U may be written as $L \circ j$ for some Γ-compatible injection

$$j : U \hookrightarrow \mathcal{L}.$$

Denote the set of these configurations by $\check{C}(R, L; \Gamma)$,

$$\check{C}(R, L; \Gamma) = \left\{ c : U \cup T \hookrightarrow \check{R} \,; \exists j \in [i_\Gamma], c_{|U} = L \circ j \right\}.$$

In $\check{C}(R, L; \Gamma)$, the univalent vertices move along $L(\mathcal{L})$, while the trivalent vertices move in the ambient space $\check{R}$, and $\check{C}(R, L; \Gamma)$ is an open submanifold of $\mathcal{L}^U \times \check{R}^T$, naturally.

An *orientation* of a set of cardinality at least 2 is a total order of its elements up to an even permutation. When L is oriented, such an orientation of $V(\Gamma)$ orients $\check{C}(R,L;\Gamma)$ naturally, since it orders the oriented odd-dimensional factors of $\mathcal{L}^U \times \check{R}^T$. Below, we associate an orientation of $\check{C}(R,L;\Gamma)$ to a vertex-orientation of Γ and an orientation of the set $H(\Gamma)$ of half-edges of Γ.

Cut each edge of Γ into two half-edges. When an edge is oriented, define its *first* half-edge and its *second* one, so that following the orientation of the edge, the first half-edge is met first. When the edges of Γ are oriented, the orientations of the edges of Γ induce the following orientation of the set $H(\Gamma)$ of half-edges of Γ: Order $E(\Gamma)$ arbitrarily, and order the half-edges as (First half-edge of the first edge, second half-edge of the first edge, …, second half-edge of the last edge). The induced orientation of $H(\Gamma)$ is called the *edge-orientation* of $H(\Gamma)$. Note that it does not depend on the order of $E(\Gamma)$.

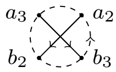
Lemma 7.1. *When Γ is equipped with a vertex-orientation, orientations of the manifold $\check{C}(R,L;\Gamma)$ are in canonical one-to-one correspondence with orientations of the set $H(\Gamma)$.*

Proof: Since $\check{C}(R,L;\Gamma)$ is an open submanifold of $\mathcal{L}^U \times \check{R}^T$, naturally, it inherits $\mathbb{R}^{\sharp U+3\sharp T}$-valued charts from $\mathbb{R}$-valued orientation-preserving charts of $\mathcal{L}$, with respect to the (possibly local as in Definition 6.13) orientation(s) of $\mathcal{L}$, and $\mathbb{R}^3$-valued orientation-preserving charts of $\check{R}$. In order to define the orientation of $\mathbb{R}^{\sharp U+3\sharp T}$, it suffices to identify its factors and order them (up to even permutation). Each of the factors may be labeled by an element of $H(\Gamma)$: the $\mathbb{R}$-valued local coordinate of an element of $\mathcal{L}$ corresponding to the image under j of an element u of U sits in the factor labeled by the half-edge that contains u; the 3 cyclically ordered (by the orientation of $\check{R}$) $\mathbb{R}$-valued local coordinates of the image under a configuration c of an element t of T belong to the factors labeled by the three half-edges that contain t, which are ordered cyclically by the vertex-orientation of Γ, so that the cyclic orders match. □

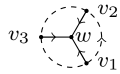
We will use Lemma 7.1 to orient $\check{C}(R,L;\Gamma)$ as summarized in the following immediate corollary.

Corollary 7.2. *If Γ is equipped with a vertex-orientation $o(\Gamma)$ and if the edges of Γ are oriented, then the induced edge-orientation of $H(\Gamma)$ orients $\check{C}(R,L;\Gamma)$, via the canonical correspondence described in the proof of Lemma 7.1.*

Examples 7.3. The orientations of the configuration spaces $\check{C}(K;\boxtimes)$ and $\check{C}(K;\circledast)$ (induced by the order of the given coordinates) in Section 1.2.5 are respectively induced by the edge-orientations, and by the orientation of the vertex w, in the following figures.



and



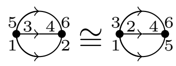
Recall that $\check{C}(K;\circledast)$, which may be described as

$$\left\{c\colon \{w,v_1,v_2,v_3\}\hookrightarrow \mathbb{R}^3 \mid \begin{array}{l} c(v_i)=K(z_i\in S^1), z_2=\exp(2i\pi t_2)z_1, \\ z_3=\exp(2i\pi t_3)z_1, 0<t_2<t_3<1 \end{array}\right\},$$

was regarded as an open submanifold of $\mathbb{R}^3\times(S^1)^3=\{(X_1,X_2,X_3,z_1,z_2,z_3)\}$, where $(X_1,X_2,X_3)=c(w)$.

Let us check that the above coordinates orient $\check{C}(K;\circledast)$ as the orientation of edges does. For $i\in\underline{3}$, let $e(i)$ denote the edge from v_i to w. This orders the three factors of $(S^2)^{E(\Gamma)}$. Distribute the coordinates X_1, X_2, X_3 so that X_i is on the second half-edge of $e(i)$. Then the vertex-orientation and the edge-orientation of Γ induce the orientation of $H(\circledast)$ represented by $(z_1,X_1,z_2,X_2,z_3,X_3)$, which is the same as the orientation represented by $(X_1,X_2,X_3,z_1,z_2,z_3)$. The case of $\check{C}(K;\boxtimes)$ is left to the reader.

Example 7.4. Equip the diagram $\ominus$ with its vertex-orientation induced by the picture. Orient its three edges so that they start from the same vertex. Then the orientation of $\check{C}(R,L;\ominus)$ induced by this edge-orientation of $\ominus$ matches the orientation of $(\check{R}\times\check{R})\setminus\Delta$ induced by the order of the two factors, where the first factor corresponds to the position of the vertex from which the three edges start, as shown in the following picture.



Remark 7.5. For a Jacobi diagram Γ equipped with a vertex-orientation $o(\Gamma)$, an orientation of $V(\Gamma)$ induces the following orientation of $H(\Gamma)$. Fix a total order of $V(\Gamma)$ that induces its given orientation. Then the corresponding orientation of $H(\Gamma)$ is induced by a total order which starts with the half-edges adjacent to the first vertex, ordered with respect to $o(\Gamma)$ if the vertex is trivalent, and continues with the half-edges adjacent to the second vertex, to the third one, ... This orientation is called the *vertex-orientation* of $H(\Gamma)$ associated to $o(\Gamma)$ and to the orientation of $V(\Gamma)$. In particular, an orientation of $H(\Gamma)$ (such as the edge-orientation of $H(\Gamma)$ when the edges of Γ are oriented) and a vertex-orientation $o(\Gamma)$ together induce an orientation of $V(\Gamma)$, namely the orientation of $V(\Gamma)$ such that the induced vertex-orientation of $H(\Gamma)$ matches the given orientation of $H(\Gamma)$.

The dimension of $\check{C}(R,L;\Gamma)$ is

$$\sharp U(\Gamma)+3\sharp T(\Gamma)=2\sharp E(\Gamma),$$

where $E=E(\Gamma)$ denotes the set of edges of Γ. Since the degree of Γ is $n=n(\Gamma)=\frac{1}{2}(\sharp U(\Gamma)+\sharp T(\Gamma))$,

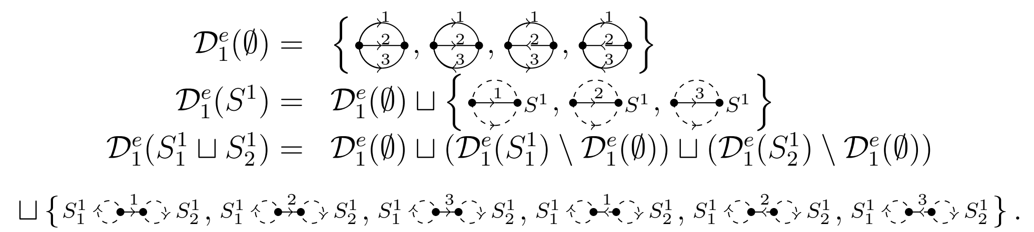
$$\sharp E(\Gamma)=3n-\sharp U(\Gamma).$$

7.2 Configuration space integrals

Definition 7.6. Let A be a finite set. An *A-numbered Jacobi diagram* is a Jacobi diagram Γ whose edges are oriented, equipped with an injection $j_E\colon E(\Gamma)\hookrightarrow A$. Such an injection numbers the edges when $A\subset\mathbb{N}$. Let $\mathcal{D}^e_n(\mathcal{L})$ denote the set of $\underline{3n}$-numbered degree n Jacobi diagrams with support $\mathcal{L}$ without *looped edges* like .

Note that the injection j_E is a bijection for any diagram of $\mathcal{D}^e_n(\mathcal{L})$ without univalent vertices.

Examples 7.7.



$$\mathcal{D}^e_1(S^1_1\sqcup S^1_2)=\mathcal{D}^e_1(\emptyset)\sqcup(\mathcal{D}^e_1(S^1_1)\setminus\mathcal{D}^e_1(\emptyset))\sqcup(\mathcal{D}^e_1(S^1_2)\setminus\mathcal{D}^e_1(\emptyset))$$

Let Γ be a numbered degree n Jacobi diagram with support $\mathcal{L}$. An edge e oriented from a vertex v_1 to a vertex v_2 of Γ induces the following canonical map

$$\begin{array}{llll} p_e\colon & \check{C}(R,L;\Gamma) & \to & C_2(R)\\ & c & \mapsto & (c(v_1),c(v_2)). \end{array}$$

Let $o(\Gamma)$ be a vertex-orientation of Γ. For any $i\in\underline{3n}$, let $\omega(i)$ be a propagating form of $(C_2(R),\tau)$. Define

$$I\left(R,L,\Gamma,o(\Gamma),(\omega(i))_{i\in\underline{3n}}\right)=\int_{(\check{C}(R,L;\Gamma),o(\Gamma))}\bigwedge_{e\in E(\Gamma)}p_e^*(\omega(j_E(e))),$$

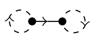
where $(\check{C}(R,L;\Gamma),o(\Gamma))$ denotes the manifold $\check{C}(R,L;\Gamma)$, equipped with the orientation induced by $o(\Gamma)$ and by the edge-orientation of Γ, as in Corollary 7.2. The convergence of this integral is a consequence of the following proposition, which will be proved in Chapter 8. (See the end of Section 8.2.)

Proposition 7.8. *There exists a smooth compactification of $\check{C}(R, L; \Gamma)$, which will be denoted by $C(R, L; \Gamma)$, to which the maps p_e extend smoothly.*

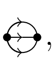
According to this proposition, $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$ extends smoothly to $C(R, L; \Gamma)$, and

$$\int_{(\check{C}(R,L;\Gamma),o(\Gamma))} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e))) = \int_{(C(R,L;\Gamma),o(\Gamma))} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e))).$$

Examples 7.9. For any three propagating forms $\omega(1)$, $\omega(2)$ and $\omega(3)$ of $(C_2(R), \tau)$,

$$I(R, K_i \sqcup K_j \colon S^1_i \sqcup S^1_j \hookrightarrow \check{R}, {}_{S^1_i}$$  $${}_{S^1_j}, (\omega(i))_{i \in \underline{3}}) = lk(K_i, K_j)$$

and

$$I(R, \emptyset,$$  $$, (\omega(i))_{i \in \underline{3}}) = \Theta(R, \tau)$$

for any numbering of the (plain) diagrams (exercise).

Examples 7.10. For any trivalent numbered degree n Jacobi diagram

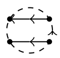
$$I(\Gamma) = I\left(S^3, \emptyset, \Gamma, o(\Gamma), (p^*_{S^2}(\omega_{S^2}))_{i \in \underline{3n}}\right) = 0.$$

Indeed, $I(\Gamma)$ is equal to

$$\int_{(\check{C}(S^3,\emptyset;\Gamma),o(\Gamma))} \left(\prod_{e \in E(\Gamma)} p_{S^2} \circ p_e \right)^* \left(\bigwedge_{e \in E(\Gamma)} \omega_{S^2} \right),$$

where

- $\bigwedge_{e \in E(\Gamma)} \omega_{S^2}$ is a product volume form of $(S^2)^{E(\Gamma)}$ with total volume one.
- $\check{C}(S^3, \emptyset; \Gamma)$ is the space $\check{C}_{\underline{3n}}(\mathbb{R}^3)$ of injections of $\underline{3n}$ into $\mathbb{R}^3$,
- the degree of $\wedge_{e \in E(\Gamma)} \omega_{S^2}$ is equal to the dimension of $\check{C}(S^3, \emptyset; \Gamma)$, and
- The map $\left(\prod_{e \in E(\Gamma)} p_{S^2} \circ p_e\right)$ is never a local diffeomorphism since it is invariant under the action of global translations on $\check{C}(S^3, \emptyset; \Gamma)$.

Examples 7.11. Let us now compute $I\left(S^3, O, \Gamma, o(\Gamma), (p^*_{S^2}(\omega_{S^2}))_{i \in \underline{3n}}\right)$, where O denotes the representative of the unknot of S^3, that is the image of the embedding of the unit circle S^1 of $\mathbb{C}$ regarded as $\mathbb{C} \times \{0\}$, into $\mathbb{R}^3$ regarded as $\mathbb{C} \times \mathbb{R}$, for the following graphs $\Gamma_1 =$ , $\Gamma_2 =$ , $\Gamma_3 =$ ,

$\Gamma_4 =$ . Since all edges are equipped with the same standard propagating form $p_{S^2}^*(\omega_{S^2})$, we do not number the edges. For $i \in \underline{4}$, set $I(\Gamma_i) = I\left(S^3, O, \Gamma_i, o(\Gamma_i), (p_{S^2}^*(\omega_{S^2}))\right)$. We are about to prove that $I(\Gamma_1) = I(\Gamma_2) = I(\Gamma_3) = 0$ and that $I(\Gamma_4) = \frac{1}{8}$.

For $i \in \underline{4}$, set $\Gamma = \Gamma_i$, $I(\Gamma)$ is again equal to

$$\int_{(\check{C}(S^3,O;\Gamma),o(\Gamma))} \left(\prod_{e\in E(\Gamma)} p_{S^2} \circ p_e\right)^* \left(\bigwedge_{e\in E(\Gamma)} \omega_{S^2}\right).$$

When $i \in \underline{2}$, the image of $\prod_{e\in E(\Gamma)} p_{S^2} \circ p_e$ lies in the subset of $(S^2)^2$ consisting of the pair of horizontal vectors. Since the interior of this subset is empty, the form $\left(\prod_{e\in E(\Gamma)} p_{S^2} \circ p_e\right)^* \left(\omega_{S^2}^{E(\Gamma)}\right)$ vanishes identically, and $I(\Gamma_i) = 0$. When $i = 3$, the two edges that have the same endpoints must have the same direction, so the image of $\prod_{e\in E(\Gamma)} p_{S^2} \circ p_e$ lies in the subset of $(S^2)^{E(\Gamma)}$, where two S^2-coordinates are identical (namely those in the S^2-factors corresponding to the above pair of edges), and $I(\Gamma_3) = 0$ as before.

Lemma 7.12. *Let* $\Gamma =$ . *Then*

$$I\left(S^3, O, \Gamma, o(\Gamma), (p_{S^2}^*(\omega_{S^2}))\right) = \frac{1}{8}.$$

Proof: Let $(S^1)^3_+$ be the subset of $(S^1)^3$ consisting of triples (z_1, z_2, z_3) of pairwise distinct elements of S^1 such that the orientation of S^1 induces the cyclic order (z_1, z_2, z_3) among them. Then

$$\check{C} = \check{C}(S^3, O; \Gamma) = \{(X, z_1, z_2, z_3) \mid (z_1, z_2, z_3) \in (S^1)^3_+, X \in \mathbb{R}^3 \setminus \{z_1, z_2, z_3\}\}.$$

Furthermore, these coordinates orient $\check{C}$ as the orientation of edges does, as explained in Example 7.3. Let $\check{C}^+ = \{(X, z_1, z_2, z_3) \in \check{C} \mid X_3 > 0\}$. The orthogonal symmetry σ_h with respect to the horizontal plane acts on $\check{C}$ by an orientation reversing diffeomorphism, which changes X_3 to $(-X_3)$ and leaves the other coordinates unchanged. It also acts on S^2 by an orientation-reversing diffeomorphism, which preserves the volume up to sign. Therefore,

$$I(\Gamma) = 2\int_{\check{C}^+} \left(\prod_{e\in E(\Gamma)} p_{S^2} \circ p_e\right)^* \left(\bigwedge_{e\in E(\Gamma)} \omega_{S^2}\right).$$

Let S^2_+ denote the set of elements of S^2 with positive *height* (i.e. third coordinate), and let $(S^2)^3_+$ be the set of elements of $\left(S^2_+\right)^3$ which form a direct basis. Note that the volume of $(S^2)^3_+$ is $\frac{1}{16}$ (with respect to $\bigwedge_{e\in E(\Gamma)}\omega_{S^2}$). Indeed the volume of $\left(S^2_+\right)^3$ is $\frac{1}{8}$. This is also the volume of the subset of $\left(S^2_+\right)^3$ consisting of triples of non-coplanar vectors. The involution that exchanges the last two vectors in the latter set sends the direct bases to the indirect ones, and it is volume preserving. Therefore, in order to prove that $I(\Gamma_4)=\frac{1}{8}$, it suffices to prove that

$$\begin{array}{llll}\Psi\colon & \check{C}^+ & \to & (S^2)^3\\ & c & \mapsto & \left(\prod_{e\in E(\Gamma)} p_{S^2}\circ p_e\right)(c)\end{array}$$

is an orientation-preserving diffeomorphism onto $(S^2)^3_+$. Let $c=(X,z_1,z_2,z_3)$ be a point of $\check{C}^+$. Let us first check that $\Psi(c)\in(S^2)^3_+$. For $j\in\underline{3}$, the vector $\overrightarrow{z_jX}$ may be written as λ_jV_j for some $\lambda_j\in]0,+\infty[$ and for some $V_j\in S^2_+$. Since $(V_1,\overrightarrow{z_1z_2}=\lambda_1V_1-\lambda_2V_2,\overrightarrow{z_1z_3}=\lambda_1V_1-\lambda_3V_3)$ is a direct basis of $\mathbb{R}^3$, so is (V_1,V_2,V_3). Let us now compute the sign of the Jacobian of Ψ at c. Let $T_c\Psi$ denote the tangent map to Ψ at c. For $j\in\underline{3}$, let Z_j denote the unit tangent vector to S^1 at z_j, and let $p_j\colon(\mathbb{R}^3)^3\to\mathbb{R}^3/\mathbb{R}V_j$ be the projection onto the j^{th} factor composed by the projection onto the tangent space $\mathbb{R}^3/\mathbb{R}V_j$ to S^2 at V_j. Then $p_j(T_c\Psi(Z_j))=-Z_j$ in the tangent space to S^2 at V_j, which is generated by (the projections onto $\mathbb{R}^3/\mathbb{R}V_j$ of) $-Z_j$ and any vector W_j such that $\det(V_j,-Z_j,W_j)=1$. Reorder the oriented basis $(-Z_1,W_1,-Z_2,W_2,-Z_3,W_3)$ of $T(S^2)^3_+$ at (V_1,V_2,V_3) by a positive permutation of the coordinates as $(W_1,W_2,W_3,-Z_1,-Z_2,-Z_3)$. Writing the matrix of $T_c\Psi$ with respect to this basis $(W_1,W_2,W_3,-Z_1,-Z_2,-Z_3)$ of the target space, and the basis (V_1,V_2,V_3) of $T_X\mathbb{R}^3$ followed by (Z_1,Z_2,Z_3) for the source $T_c\check{C}^+$ produces a matrix whose last three columns contain 1 on the diagonals as only nonzero entries. In the quotient $\frac{T(S^2)^3_+}{\mathbb{R}(-Z_1,0,0)\oplus\mathbb{R}(0,-Z_2,0)\oplus\mathbb{R}(0,0,-Z_3)}$, $T_c\Psi(V_1)$ may be expressed as

$$T_c\Psi(V_1)=\det(V_2,-Z_2,V_1)W_2+\det(V_3,-Z_3,V_1)W_3,$$

where $\det(V_2,-Z_2,V_1)=\det(Z_2,V_2,V_1)$. Let us prove that $\det(Z_2,V_2,V_1)>0$. When $z_2=-1$, projecting c on $\left(\mathbb{R}=\frac{\mathbb{C}}{i\mathbb{R}}\right)\times\mathbb{R}$ produces a picture as in Figure 7.1 that makes this result clear. The general result follows easily. Finally, the Jacobian of Ψ at (X,z_1,z_2,z_3) is the determinant of

$$\begin{bmatrix} 0 & \det(Z_1,V_1,V_2) & \det(Z_1,V_1,V_3)\\ \det(Z_2,V_2,V_1) & 0 & \det(Z_2,V_2,V_3)\\ \det(Z_3,V_3,V_1) & \det(Z_3,V_3,V_2) & 0\end{bmatrix},$$

which is

$$\begin{array}{ll} & \det(Z_1,V_1,V_2)\det(Z_2,V_2,V_3)\det(Z_3,V_3,V_1)\\ + & \det(Z_1,V_1,V_3)\det(Z_2,V_2,V_1)\det(Z_3,V_3,V_2),\end{array}$$

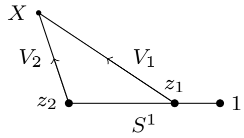


Figure 7.1: Partial projection of (X, z_1, z_2, z_3) on $\frac{\mathbb{C}}{i\mathbb{R}} \times \mathbb{R}$ when $z_2 = -1$

which is positive since all the involved terms are. Let us finally check that every element (V_1, V_2, V_3) of $(S^2)^3_+$ has a unique element in its preimage. Construct the three lines of $\mathbb{R}^3$ directed by V_1, V_2 and V_3 through the origin of $\mathbb{R}^3$. The line directed by V_i intersects the horizontal plane at height (-1) at a point w_i. There is a unique circle in this horizontal plane that contains w_1, w_2 and w_3. Let $\frac{1}{\lambda}$ be the radius of this circle, and let w_0 be its center. Then the unique element of $\Psi^{-1}(V_1, V_2, V_3)$ is $(-\lambda w_0, \lambda w_1 - \lambda w_0, \lambda w_2 - \lambda w_0, \lambda w_3 - \lambda w_0)$. □

7.3 Configuration space integrals associated to a chord

Let us now study the case of $I(\ S^1_j, (\omega(i))_{i \in \underline{3}})$, which depends on the chosen propagating forms, and on the diagram numbering.

A *dilation* is a homothety with positive ratio. Let U^+K_j denote the fiber space over K_j consisting of the tangent vectors to the knot K_j of $\check{R}$ that orient K_j, up to dilation. The fiber of U^+K_j consists of one point, so the total space of this *unit positive tangent bundle to K_j* is K_j. Let U^-K_j denote the fiber space over K_j consisting of the opposite tangent vectors to K_j, up to dilation.

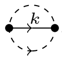
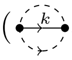
For a knot K_j in $\check{R}$,

$$\check{C}(K_j;\ S^1_j) = \{(K_j(z), K_j(z\exp(i\theta))) \mid (z, \theta) \in S^1 \times]0, 2\pi[\}.$$

Let $C_j = C(K_j;\ S^1_j)$ be the closure of $\check{C}(K_j;\ S^1_j)$ in $C_2(R)$. This closure is diffeomorphic to $S^1 \times [0, 2\pi]$, where $S^1 \times \{0\}$ is identified with U^+K_j, $S^1 \times \{2\pi\}$ is identified with U^-K_j, and $\partial C(K_j;\ S^1_j) = U^+K_j - U^-K_j$.

Lemma 7.13. *For any $i \in \underline{3}$, let $\omega(i)$ and $\omega'(i)$ be propagating forms of $(C_2(R), \tau)$, which restrict to $\partial C_2(R)$ as $p_\tau^*(\omega(i)_{S^2})$ and $p_\tau^*(\omega'(i)_{S^2})$, respectively. Then, there exists a one-form $\eta(i)_{S^2}$ on S^2 such that $\omega'(i)_{S^2} =$*

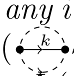
$\omega(i)_{S^2} + d\eta(i)_{S^2}$, *and*

$I($$S^1_j, (\omega'(i))_{i\in\underline{3}}) - I($$S^1_j, (\omega(i))_{i\in\underline{3}}) = \int_{U^+K_j} p_\tau^*(\eta(k)_{S^2})$
$-\int_{U^-K_j} p_\tau^*(\eta(k)_{S^2}).$

PROOF: For any i, according to Lemma 4.2, $\eta(i)_{S^2}$ exists, and there exists a one-form $\eta(i)$ on $C_2(R)$ such that $\omega'(i) = \omega(i) + d\eta(i)$ and the restriction of $\eta(i)$ to $\partial C_2(R)$ is $p_\tau^*(\eta(i)_{S^2})$. Apply Stokes's theorem to $\int_{C_j}(\omega'(k) - \omega(k)) = \int_{C_j} d\eta(k)$. □

Exercise 7.14. Find a knot K_j of $\mathbb{R}^3$ and a form $\eta(k)$ of $C_2(\mathbb{R}^3)$ such that the right-hand side of Lemma 7.13 does not vanish. (Use Lemma 4.2, hints can be found in Section 7.5.)

Recall that a propagating form ω of $(C_2(R), \tau)$ is *homogeneous* if its restriction to $\partial C_2(R)$ is $p_\tau^*(\omega_{S^2})$ for the homogeneous volume form ω_{S^2} of S^2 of total volume 1.

Lemma 7.15. *For any $i \in \underline{3}$, let $\omega(i)$ be a homogeneous propagating form of $(C_2(R), \tau)$. Then $I($$S^1_j, (\omega(i))_{i\in\underline{3}})$ does not depend on the choices of the $\omega(i)$, it is denoted by $I_\theta(K_j, \tau)$.*

PROOF: Apply Lemma 4.2 with $\eta_A = 0$, so $\eta(k)_{S^2} = 0$ in Lemma 7.13. □

7.4 First definition of Z

From now on, the coefficients of our spaces of Jacobi diagrams are in $\mathbb{R}$. ($\mathbb{K} = \mathbb{R}$.) Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $L\colon \mathcal{L} \hookrightarrow \check{R}$ be a link embedding.

Let $[\Gamma, o(\Gamma)]$ denote the class in $\mathcal{A}_n(\mathcal{L})$ of a numbered Jacobi diagram Γ, in the space $\mathcal{D}^e_n(\mathcal{L})$ of Definition 7.6, equipped with a vertex-orientation $o(\Gamma)$, then

$$I(R, L, \Gamma, o(\Gamma), (\omega(i))_{i\in\underline{3n}})[\Gamma, o(\Gamma)] \in \mathcal{A}_n(\mathcal{L})$$

is independent of the vertex-orientation $o(\Gamma)$ of Γ, it is simply denoted by

$$I(R, L, \Gamma, (\omega(i))_{i\in\underline{3n}})[\Gamma].$$

Notation 7.16. For $\Gamma \in \mathcal{D}^e_n(\mathcal{L})$, set

$$\zeta_\Gamma = \frac{(3n - \sharp E(\Gamma))!}{(3n)!2^{\sharp E(\Gamma)}}.$$

Recall the definitions of propagating forms from Section 3.3. For any $i \in \underline{3n}$, let $\omega(i)$ be a propagating form of $C_2(R)$. For $n \in \mathbb{N}$, set

$$Z_n(\check{R}, L, (\omega(i))_{i\in\underline{3n}}) = \sum_{\Gamma\in\mathcal{D}^e_n(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i\in\underline{3n}})[\Gamma] \in \mathcal{A}_n(\mathcal{L}).$$

This $Z_n(\check{R}, L, (\omega(i))_{i\in\underline{3n}})$ is the hero of this book. Let us describe some of its variants.

Let $\mathcal{A}^c_n(\emptyset)$ denote the subspace of $\mathcal{A}_n(\emptyset)$ generated by connected trivalent Jacobi diagrams. Set $\mathcal{A}^c(\emptyset) = \prod_{n\in\mathbb{N}} \mathcal{A}^c_n(\emptyset)$ and let $p^c \colon \mathcal{A}(\emptyset) \to \mathcal{A}^c(\emptyset)$ be the projection that maps the empty diagram and diagrams with several connected components to 0. Let $\mathcal{D}^c_n$ denote the subset of $\mathcal{D}^e_n(\emptyset)$ that contains the connected diagrams of $\mathcal{D}^e_n(\emptyset)$. For $n \in \mathbb{N}$, set

$$z_n(\check{R}, (\omega(i))_{i\in\underline{3n}}) = p^c(Z_n(\check{R}, \emptyset, (\omega(i)))).$$

$$z_n(\check{R}, (\omega(i))) = \sum_{\Gamma\in\mathcal{D}^c_n} \zeta_\Gamma I(R, \emptyset, \Gamma, (\omega(i))_{i\in\underline{3n}})[\Gamma] \in \mathcal{A}^c_n(\emptyset).$$

When all the forms $\omega(i)$ are equal to $\omega(1)$, $Z_n(\check{R}, L, \omega(1))$ and $z_n(\check{R}, \omega(1))$ denote $Z_n(\check{R}, L, (\omega(i))_{i\in\underline{3n}})$ and $z_n(\check{R}, (\omega(i))_{i\in\underline{3n}})$, respectively.

We also use the projection $\check{p} \colon \mathcal{A}(\mathcal{L}) \to \check{\mathcal{A}}(\mathcal{L})$ that maps the diagrams with connected components without univalent vertices to zero and that maps the other diagrams to themselves. Set $\check{Z}_n = \check{p}\circ Z_n$. For example, $\check{Z}_n(\check{R}, L, \omega(1)) = \check{p}\left(Z_n(\check{R}, L, \omega(1))\right)$. We also remove the subscript n to denote the collection (or the sum) of the Z_n for $n \in \mathbb{N}$. For example,

$$\check{Z}\left(\check{R}, L, \omega(1)\right) = \left(\check{Z}_n(\check{R}, L, \omega(1))\right)_{n\in\mathbb{N}} = \sum_{n\in\mathbb{N}} \check{Z}_n\left(\check{R}, L, \omega(1)\right) \in \check{\mathcal{A}}(\mathcal{L}).$$

As a first example, let us prove the following proposition.

Proposition 7.17. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$, then for any triple $(\omega(i))_{i\in\underline{3}}$ of propagating forms of $(C_2(R), \tau)$,*

$$Z_1(\check{R}, \emptyset, (\omega(i))_{i\in\underline{3}}) = z_1(\check{R}, (\omega(i))) = \frac{\Theta(R,\tau)}{12}[\ominus]$$

in $\mathcal{A}_1(\emptyset) = \mathcal{A}_1(\emptyset; \mathbb{R}) = \mathbb{R}[\ominus]$.

PROOF: The diagram $\ominus$ is the only trivalent diagram with 2 vertices without looped edges, and it is easy to check that $\mathcal{A}_1(\emptyset) = \mathbb{R}[\ominus]$. Each of the three edges go from one vertex to the other one. There are 4 elements in $\mathcal{D}^e_1(\emptyset)$, depending on whether the orientations of Edge 2, and Edge 3 coincide with the orientation of Edge 1. See Example 7.7. When the three edges start from the same vertex, call the corresponding element θ_{++}, order the vertices so that the vertex from which the edges start is first. Recall from Example 7.4 that the vertex-orientation $o(\theta_{++})$ of $\ominus$ induced by the picture $\ominus$ and the edge-orientation of $H(\theta_{++})$ orient $C_2(R)$ as $\check{R}^2 \setminus \Delta(\check{R}^2)$, as in Corollary 7.2. Then, according to Theorem 4.1, for any 3 propagating forms $(\omega(i))_{i\in\underline{3n}}$ of $(C_2(R),\tau)$,

$$I(\theta_{++}, o(\theta_{++}), (\omega(i))_{i\in\underline{3n}}) = \Theta(R,\tau).$$

Reversing the edge-orientation of Edge 2 transforms θ_{++} to θ_{-+}, $\omega(2)$ to $\iota^*(\omega(2))$, and it changes the edge-orientation of $H(\theta_{\pm+})$. Since $(-\iota^*(\omega(2)))$ is a propagating form according to Lemma 3.14,

$$I(\theta_{++}, (\omega(i))_{i\in\underline{3n}})[\theta_{++}] = I(\theta_{-+}, (\omega(i))_{i\in\underline{3n}})[\theta_{-+}].$$

Similarly, the four graphs of $\mathcal{D}_1^e(\emptyset)$ will contribute in the same way to

$$Z_1(\check{R}, \emptyset, (\omega(i))_{i\in\underline{3}}) = \frac{4}{3!2^3}\Theta(R,\tau)\,[\ominus]\,.$$

□

Examples 7.18. According to the computations in Example 7.10,

$$Z_n(\mathbb{R}^3, \emptyset, p_{S^2}^*(\omega_{S^2})) = z_n(\mathbb{R}^3, p_{S^2}^*(\omega_{S^2})) = 0$$

for $n > 0$, and $Z_0(\mathbb{R}^3, \emptyset, p_{S^2}^*(\omega_{S^2})) = [\emptyset]$, while $z_0(\mathbb{R}^3, p_{S^2}^*(\omega_{S^2})) = 0$.

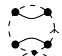
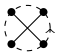
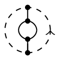
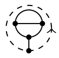
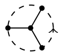
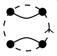
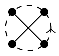
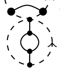
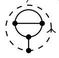
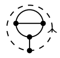
For the embedding O of the trivial knot in $\mathbb{R}^3$ of Example 7.11,

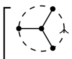
$$Z_0(\mathbb{R}^3, O, p_{S^2}^*(\omega_{S^2})) = 1 = [\emptyset].$$

Since $I_\theta(O,\tau_s) = 0$, $Z_1(\mathbb{R}^3, O, p_{S^2}^*(\omega_{S^2})) = 0$.

Let us now see that

$Z_2\left(\mathbb{R}^3, O, (p_{S^2}^*(\omega_{S^2}))_{i\in\underline{6}}\right) = \frac{1}{24}\Big[$  $\Big]$.

First observe that reversing the orientation of an edge does not change $I(S^3, O, \Gamma, (p_{S^2}^*(\omega_{S^2}))_{i\in\underline{6}})[\Gamma]$, for a degree 2 numbered Jacobi diagram Γ, since it changes both the orientation of $\check{C}(S^3, 0; \Gamma)$ and the sign of the form to be integrated, because $\iota_{S^2}^*(\omega_{S^2}) = -\omega_{S^2}$, where ι_{S^2} is the antipodal map of S^2. Thus $I(S^3, O, \Gamma, (p_{S^2}^*(\omega_{S^2}))_{i\in\underline{6}})[\Gamma]$ depends only on the underlying Jacobi diagram. The degree 2 Jacobi diagrams all components of which have univalent vertices and without looped edges are , , ,  and . As proved in Example 7.11, , ,  do not contribute to $Z_2\left(\mathbb{R}^3, O, (p^*_{S^2}(\omega_{S^2}))_{i\in\underline{6}}\right)$. Since $[$$] = 0$ by Lemma 6.25,  does not contribute either. Lemma 7.12 ensures that

$I\Big(S^3, O,$ $, (p^*_{S^2}(\omega_{S^2}))\Big)\Big[$$\Big] = \frac{1}{8}\Big[$$\Big].$

When $\Gamma =$ , $\zeta_\Gamma = \frac{3!}{6!2^3}$ and there are $\frac{1}{3}\frac{6!2^3}{3!}$ numbered graphs of $\mathcal{D}^e_2(S^1)$ that are isomorphic to Γ as a Jacobi diagram.

See also Proposition 7.26 and Example 7.28. Alternative computations of similar quantities have been performed by Guadagnini, Martellini and Mintchev in [GMM90].

The following theorem is proved in Chapter 9. See Section 9.1, and Corollary 9.4 and Lemma 9.1, in particular.

Theorem 7.19. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $n \in \mathbb{N}$. For any $i \in \underline{6n}$, let $\omega(i)$ be a propagating form of $C_2(R)$. Then $z_{2n}(\check{R}, (\omega(i))_{i\in\underline{6n}})$ is independent of the chosen $\omega(i)$, it depends only on the diffeomorphism class of R. It is denoted by $z_{2n}(R)$.*

For odd n, $z_n(\check{R}, (\omega(i))_{i\in\underline{3n}})$ depends on the chosen $\omega(i)$. Theorem 7.20 explains how to deal with this dependence when the $\omega(i)$ are homogeneous propagating forms of $(C_2(R), \tau)$. Alternative compensations of this dependence have been studied by Tadayuki Watanabe in [Wat18a] and by Tatsuro Shimizu in [Shi16].

We are going to prove the following theorem in the next chapters. The proof will be concluded in the end of Section 10.5.

Theorem 7.20. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $\mathcal{L} = \sqcup_{j=1}^k S^1_j$. Let $L\colon \mathcal{L} \hookrightarrow \check{R}$ be an embedding. Let $n \in \mathbb{N}$. For any $i \in \underline{3n}$, let $\omega(i)$ be a homogeneous propagating form of $(C_2(R), \tau)$.*

Then $Z_n(\check{R}, L, (\omega(i)))$ is independent of the chosen $\omega(i)$, it depends only on the diffeomorphism class of (R, L), on $p_1(\tau)$ and on the $I_\theta(K_j, \tau)$, for the components K_j of L. It is denoted by $Z_n(\check{R}, L, \tau)$. More precisely, set

$$Z(\check{R}, L, \tau) = (Z_n(\check{R}, L, \tau))_{n\in\mathbb{N}} \in \mathcal{A}(\sqcup_{j=1}^k S^1_j).$$

There exist two constants $\alpha \in \check{\mathcal{A}}(S^1; \mathbb{R})$ and $\beta \in \mathcal{A}(\emptyset; \mathbb{R})$, which are called anomalies, *such that*

- *for any $n \in \mathbb{N}$, $\alpha_{2n} = 0$ and $\beta_{2n} = 0$, and*

- $\exp(-\frac{1}{4}p_1(\tau)\beta)\prod_{j=1}^k(\exp(-I_\theta(K_j,\tau)\alpha)\sharp_j)\,Z(\check{R},L,\tau)=\mathcal{Z}(R,L)$ *depends only on the diffeomorphism class of* (R,L).

Here $\exp(-I_\theta(K_j,\tau)\alpha)$ *acts on* $Z(\check{R},L,\tau)$*, on the copy* S^1_j *of* S^1 *as indicated by the subscript* j.

Furthermore, if $\check{R}=\mathbb{R}^3$, then the projection $\check{\mathcal{Z}}(S^3,\ L)$ of $Z(S^3,L)$ on $\check{\mathcal{A}}(\sqcup_{j=1}^k S^1_j)$ is a *universal finite type invariant of links in* $\mathbb{R}^3$, i.e. the projection of $\check{\mathcal{Z}}_n$ onto $\overline{\mathcal{A}}_n/(1T)$ satisfies the properties stated for $\overline{\check{\mathcal{Z}}}_n$ in Theorem 6.9. This result, which is due to Altschüler and Freidel [AF97], is proved in Section 17.6. See Theorem 17.32. Let $\check{\mathcal{Z}}(R,L)$ denote the projection of $\mathcal{Z}(R,L)$ on $\check{\mathcal{A}}(\sqcup_{j=1}^k S^1_j)$, and set

$$\mathcal{Z}(R)=\mathcal{Z}(R,\emptyset).$$

Then Theorem 7.20 implies that

$$\mathcal{Z}(R,L)=\mathcal{Z}(R)\check{\mathcal{Z}}(R,L).$$

(See Remark 7.27.)

The invariant $\mathcal{Z}$ of rational homology 3–spheres is the Kontsevich configuration space invariant studied by Kuperberg and Thurston in [KT99] and described in [Les04a] as Z_{KKT}.

We will see in Chapter 11 that the anomalies α and β are rational, that is that $\alpha\in\check{\mathcal{A}}(S^1;\mathbb{Q})$ (in Proposition 11.1) and $\beta\in\mathcal{A}(\emptyset;\mathbb{Q})$ (in Theorem 11.8).

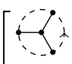
Examples 7.21. According to Example 7.18, $Z(\mathbb{R}^3,\emptyset,\tau_s)=1=[\emptyset]$. Since $p_1(\tau_s)=0$, $\mathcal{Z}(S^3)=1$, too.

For the embedding O of the trivial knot in $\mathbb{R}^3$ of Example 7.11,

$$I_\theta(O,\tau_s)=0.$$

So, according to Example 7.18, $\mathcal{Z}_0(S^3,O)=1$, $\mathcal{Z}_1(S^3,O)=0$ and

$$\mathcal{Z}_2(S^3,O)=\frac{1}{24}\left[\quad\right].$$



Definition 7.22. Recall that ι is the continuous involution of $C_2(R)$ that maps (x,y) to (y,x) on $\check{R}^2\setminus$ diagonal. An *antisymmetric* propagating form is a propagating form such that $\iota_*(\omega)=-\omega$.

Example 7.23. The propagating form $p^*_{S^2}(\omega_{S^2})$ of $(C_2(S^3),\tau)$ is antisymmetric.

Lemma 3.14 ensures the existence of antisymmetric homogeneous propagating forms ω of $(C_2(R),\tau)$.

Definition 7.24. Let Γ be a Jacobi diagram on an oriented one-manifold $\mathcal{L}$ as in Definition 6.13. An *automorphism* of Γ is a permutation of the set $H(\Gamma)$ of half-edges of Γ that maps a pair of half-edges of an edge to another such and a triple of half-edges that contain a vertex to another such, and such that, for the induced bijection b of the set $U(\Gamma)$ of univalent vertices equipped with the injection $j\colon U(\Gamma) \hookrightarrow \mathcal{L}$ into the support $\mathcal{L}$ of Γ, $j \circ b$ is isotopic to j. (In other words, the automorphisms preserve the components of the univalent vertices, and they also preserve their linear order on intervals and their cyclic order on circles.) Let $\mathrm{Aut}(\Gamma)$ denote the set of automorphisms of Γ.

Examples 7.25. There are 6 automorphisms of $\ominus$ that fix each vertex. They correspond to the permutations of the edges. The cardinality $\sharp\mathrm{Aut}(\ominus)$ of $\mathrm{Aut}(\ominus)$ is 12, $\sharp\mathrm{Aut}($  $) = 1$, and $\sharp\mathrm{Aut}($$) = 3$.

Recall from Notation 7.16 that $\zeta_\Gamma = \frac{(3n-\sharp E(\Gamma))!}{(3n)!2^{\sharp E(\Gamma)}}$. Let $\mathcal{D}^u_n(\mathcal{L})$ denote the set of unnumbered, unoriented degree n Jacobi diagrams on $\mathcal{L}$ without looped edges.

Proposition 7.26. *Under the assumptions of Theorem 7.20, let ω be an antisymmetric homogeneous propagating form of $C_2(R)$. Then*

$$Z_n(\check{R}, L, \omega) = \sum_{\Gamma \in \mathcal{D}^u_n(\mathcal{L})} \frac{1}{\sharp Aut(\Gamma)} I(R, L, \Gamma, (\omega)_{i \in \underline{3n}})[\Gamma].$$

Proof: Set $\omega(i) = \omega$ for any i. For a *numbered* graph Γ (i.e. a graph equipped with the structure described in Definition 7.6), there are $\frac{1}{\zeta_\Gamma}$ ways of *renumbering* it (i.e. changing this structure), and $\sharp\mathrm{Aut}(\Gamma)$ of them will produce the same numbered graph. Therefore

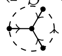
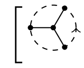
$$\sum_{\Gamma \in \mathcal{D}^e_n(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i \in \underline{3n}})[\Gamma] = \sum_{\Gamma \in \mathcal{D}^u_n(\mathcal{L})} \frac{1}{\sharp\mathrm{Aut}(\Gamma)} I(R, L, \Gamma, (\omega)_{i \in \underline{3n}})[\Gamma].$$

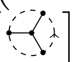
$\square$

Remark 7.27. Let ω be an antisymmetric homogeneous propagating form of $(C_2(R), \tau)$. The homogeneous definition of $Z_n(\check{R}, L, \tau) = Z_n(\check{R}, L, \omega)$ above makes clear that $Z_n(\check{R}, L, \tau)$ is a measure of graph configurations, where a graph configuration is an embedding of the set of vertices of a uni-trivalent graph into $\check{R}$ that maps univalent vertices to $\mathcal{L}(L)$. The embedded vertices are connected by a set of abstract plain edges, which represent the measuring form. The factor $\frac{1}{\sharp\mathrm{Aut}(\Gamma)}$ ensures that every such configuration of an unnumbered, unoriented graph is measured once.

Example 7.28. For the embedding O of the trivial knot in $\mathbb{R}^3$ of Examples 7.11, according to the computations performed in the series of examples 7.18, and with the notation of Theorem 7.20 and Notation 7.16, as in Example 7.21,

$$\mathcal{Z}_2\left(S^3, O\right) = Z_2\left(\mathbb{R}^3, O, (p^*_{S^2}(\omega_{S^2}))_{i \in \underline{3n}}\right)$$

$= \frac{1}{3} I \Big(S^3, O,$  $, o($  $), (p^*_{S^2}(\omega_{S^2}))\Big) \Big[$  $\Big]$

$= \frac{1}{24} \Big[$  $\Big].$

We end this section by stating Theorems 7.30 and 7.32 about the numerical invariants obtained from $\check{\mathcal{Z}}_n$ by applying the Conway weight system w_C of Example 6.11. For $n = 2$, we get the invariant w_2 discussed in Section 1.2.5.

Since w_C is multiplicative, and since w_C sends elements of odd degree to zero, w_C sends $\exp(-I_\theta(K_j, \tau)\alpha)$ to the unit of $\check{\mathcal{A}}(S^1)$. So, with the notation of Theorem 7.20, for any $n \in \mathbb{N}$, $w_C\left(\check{\mathcal{Z}}_n(R, L)\right) = w_C\left(\check{Z}_n(\check{R}, L, \tau)\right)$, and we can forget the anomaly for $w_C \circ \check{\mathcal{Z}}$. Theorem 7.30 tells us that we can furthermore omit the homogeneity assumptions on the forms for $w_C \circ \check{\mathcal{Z}}$ and reduce our averaging process. We will average only over some degree n graphs whose edges are numbered in $\underline{3n-2}$, or even in $\underline{3}$ when $n = 2$, in some cases.

(Since $\check{p} \colon \mathcal{A}(\mathcal{L}) \to \check{\mathcal{A}}(\mathcal{L})$ maps all graphs with less than 2 univalent vertices to zero (thanks to Lemma 6.25), the graphs that contribute to $\check{Z}_n(\check{R}, L, \omega(1))$ have at most $3n - 2$ edges. So it is natural to average only over these graphs.)

Notation 7.29. For a finite set A, let $\mathcal{D}^e_{n,A}(\mathcal{L})$ denote the set of A-numbered degree n Jacobi diagrams with support $\mathcal{L}$ without looped edges, as in Definition 7.6. (These diagrams have at most $\sharp A$ edges.) The coefficient ζ_Γ associated to a diagram $\Gamma \in \mathcal{D}^e_{n,A}(\mathcal{L})$ is

$$\zeta_\Gamma = \frac{(\sharp A - \sharp E(\Gamma))!}{(\sharp A)! 2^{\sharp E(\Gamma)}}.$$

For any $i \in A$, let $\omega(i)$ be a propagating form of $C_2(R)$ and set

$$Z_{n,A}(\check{R}, L, (\omega(i))_{i \in A}) = \sum_{\Gamma \in \mathcal{D}^e_{n,A}(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i \in A})[\Gamma] \in \mathcal{A}_n(\mathcal{L}).$$

For $m \in \mathbb{N}$, set $\mathcal{D}^e_{n,m}(\mathcal{L}) = \mathcal{D}^e_{n,\underline{m}}(\mathcal{L})$ and

$$Z_{n,m}(\check{R}, L, (\omega(i))_{i \in \underline{m}}) = Z_{n,\underline{m}}(\check{R}, L, (\omega(i))_{i \in \underline{m}}).$$

Note that $\mathcal{D}^e_n = \mathcal{D}^e_{n,3n}$ and that $Z_{n,3n} = Z_n$. With the projection $\check{p} \colon \mathcal{A}_n(S^1) \to \check{\mathcal{A}}_n(S^1)$ of Notation 7.16, set

$$\check{Z}_{n,m}\left(\check{R}, K, (\omega(i))_{i \in \underline{m}}\right) = \check{p}\left(Z_{n,m}\left(\check{R}, K, (\omega(i))_{i \in \underline{m}}\right)\right).$$

Theorem 7.30. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $K\colon S^1 \hookrightarrow \check{R}$ be an embedding. Let $n \in \mathbb{N}$. For any $i \in \underline{3n-2}$, let $\omega(i)$ be a propagating form of $C_2(R)$.*

Let p^c be the projection given by Corollary 6.37 from $\check{\mathcal{A}}(S^1)$ to the space $\check{\mathcal{A}}^c(S^1)$ of its primitive elements. Recall the linear form $w_C\colon (\check{\mathcal{A}}_n(S^1) = \overline{\mathcal{A}}_n) \to \mathbb{R}$ induced by the Conway weight system of Example 6.11.

Then $w_C\left(\check{Z}_{n,3n-2}\left(\check{R}, K, (\omega(i))_{i\in\underline{3n-2}}\right)\right)$ and

$$w_C \circ p^c\left(\check{Z}_{n,3n-2}\left(\check{R}, K, (\omega(i))_{i\in\underline{3n-2}}\right)\right)$$

are independent of the chosen $\omega(i)$, they depend only on the diffeomorphism class of $(\check{R}, K)$, and they are respectively denoted by $w_C\check{\mathcal{Z}}_n(R,K)$ and

$$w_C p^c \check{\mathcal{Z}}_n(R,K).$$

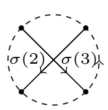
At the end of Subsection 9.1, the proof of Theorem 7.30 will be reduced to the proof of Proposition 9.7, which is proved in Subsection 9.3.

Remark 7.31. Assuming both Theorems 7.30 and 7.20, there is no notation conflict between them. Indeed, with the notation of Theorem 7.20, for any $n \in \mathbb{N}$, $w_C\left(\check{\mathcal{Z}}_n(R,L)\right) = w_C\left(\check{Z}_n(\check{R},L,\tau)\right) = w_C\left(\check{Z}_n(\check{R},L,\omega)\right) = \sum_{\Gamma\in\mathcal{D}^u_n(\mathcal{L})} \frac{1}{\sharp\mathrm{Aut}(\Gamma)} I(R,L,\Gamma,(\omega)_{i\in\underline{3n}}) w_C([\Gamma])$ for any homogeneous propagating form ω of $(C_2(R),\tau)$, thanks to Proposition 7.26. When $\omega(i) = \omega$ for all i, that is also the expression of $w_C\left(\check{Z}_{n,3n-2}\left(\check{R}, K, (\omega(i))_{i\in\underline{3n-2}}\right)\right)$.

The following theorem, which implies Proposition 1.5, is proved at the end of Section 9.3.

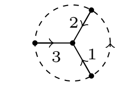
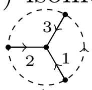
Theorem 7.32. *Let $K\colon S^1 \hookrightarrow \mathbb{R}^3$ be a knot embedding. For $i \in \underline{3}$, let $\omega_{S^2}(i)$ be a volume-one form of S^2, then $w_C\left(\check{Z}_{2,3}(\mathbb{R}^3, K, (p^*_{S^2}(\omega_{S^2}(i)))_{i\in\underline{3}})\right)$ is independent of the chosen $\omega_{S^2}(i)$. It is an isotopy invariant of K, which coincides with $w_C\check{\mathcal{Z}}_2(S^3,K)$ and $w_C p^c\check{\mathcal{Z}}_2(S^3,K)$.*

Proof of Proposition 1.5 assuming Theorem 7.32: The only degree 2 Jacobi diagrams on S^1 without looped edges, with at most three edges, and with no trivalent component, are , and . We have $w_C([\ ]) = 1$, $w_C([\ ]) = 0$, $w_C([\ ]) = w_C([\ ]) - w_C([\ ]) = -1$. There are six elements of $\mathcal{D}^e_{2,3}(S^1)$ isomorphic to , one for each permutation σ of $\underline{3}$. They may be drawn as

$$\Gamma(\sigma) = $$ 

and $\zeta_{\Gamma(\sigma)} = \frac{1}{3!\times 4}$. For such a graph $\Gamma(\sigma)$,

$$I(S^3, K, \Gamma(\sigma), (p^*_{S^2}(\omega_{S^2}(i)))_{i\in\underline{3}}) = \int_{(S^2)^3} \deg\left(\sigma_*\left(1_{S^2} \times G\right)\right) \wedge_{i=1}^3 p_i^*(\omega_{S^2}(i)).$$

There are 16 elements of $\mathcal{D}^e_{2,3}(S^1)$ isomorphic to , they are obtained from the two diagrams  and  by reversing the directions of some edges. For those diagrams $\zeta_\Gamma = \frac{1}{3!\times 8}$. Theorem 7.32 implies that for any embedding $K\colon S^1 \hookrightarrow \mathbb{R}^3$,

$$w_C\check{\mathcal{Z}}_2(S^3, K) = \int_{(S^2)^3} w_2(K) \wedge_{i=1}^3 p_i^*(\omega_{S^2}(i))$$

for the locally constant degree map $w_2(K)$ of Proposition 1.5 defined on an open dense subset of $(S^2)^3$. Any point (X_1, X_2, X_3) of this open dense subset of regular points of $(S^2)^3$, has an open connected neighborhood $\prod_{i=1}^3 W_i$ of regular points, and, there are volume-one 2-forms $\omega_{S^2}(i)$ supported on W_i. For such forms, $w_C\check{\mathcal{Z}}_2(S^3, K) = w_2(K)(X_1, X_2, X_3)$. So Theorem 7.32 implies that $w_2(K)$ is constant. $\square$

7.5 Straight links

A one-cycle c of S^2 is *algebraically trivial* if, for any two points x and y outside its support, the algebraic intersection of an arc from x to y transverse to c with c is zero, or, equivalently, if the integral of any one form of S^2 along c is zero.

Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $K\colon S^1 \hookrightarrow \check{R}$ be a knot embedding. Set $C_K = C(K; \; _{S^1})$, $C_K \subset C_2(R)$.

Lemma 7.33. *If $p_\tau(\partial C_K)$ is algebraically trivial, then for any propagating chain F of $(C_2(R), \tau)$ transverse to C_K and for any propagating form ω_p of $(C_2(R), \tau)$,*

$$\int_{C_K} \omega_p = \langle C_K, F\rangle_{C_2(R)} = I_\theta(K, \tau),$$

with respect to the definition of $I_\theta(K,\tau)$ in Lemma 7.15. In particular, $I_\theta(K,\tau) \in \mathbb{Q}$, and $I_\theta(K,\tau) \in \mathbb{Z}$ when R is an integral homology sphere.

Proof: According to Lemma 7.13, changing the propagating form ω_p to ω'_p adds some $\int_{\partial C_K} p_\tau^*(\eta_S) = \int_{p_\tau(\partial C_K)} \eta_S$ for some one-form on S^2. Then by definition, $\int_{C_K} \omega_p$ is independent of the propagating form ω_p of $(C_2(R), \tau)$.

For a propagating chain F of $(C_2(R), \tau)$ transverse to C_K, one can choose a propagating form dual to F and supported near F such that $\int_{C_K} \omega_p = \langle C_K, F \rangle_{C_2(R)}$. (See the end of Section B.2 and Lemma B.4 in particular, for details.) The rationality of $I_\theta(K, \tau)$ follows from the rationality of F, and since F can be chosen to be an integral chain when R is a $\mathbb{Z}$-sphere, $I_\theta(K, \tau) \in \mathbb{Z}$ in this case. $\square$

Remark 7.34. One could have proved that $\langle C_K, F \rangle_{C_2(R)}$ is independent of the chosen propagating chain F of $(C_2(R), \tau)$ transverse to C_K directly, as follows. For any such two propagating chains F and F' with respective boundaries $p_\tau^{-1}(a)$ and $p_\tau^{-1}(a')$, there exists a rational chain W of $C_2(R)$ transverse to C_K and to $\partial C_2(R)$, whose boundary is $F' - F - p_\tau^{-1}([a, a'])$, where $[a, a']$ is a path from a to a' in S^2 transverse to $p_\tau(\partial C_K)$. Then

$$\langle C_K, F' - F - p_\tau^{-1}([a, a']) \rangle_{C_2(R)} = 0.$$

So

$$\begin{aligned}\langle C_K, F' \rangle_{C_2(R)} - \langle C_K, F \rangle_{C_2(R)} &= \langle C_K, p_\tau^{-1}([a, a']) \rangle_{C_2(R)} \\ &= \langle \partial C_K, p_\tau^{-1}([a, a']) \rangle_{\partial C_2(R)} \\ &= \langle p_\tau(\partial C_K), [a, a'] \rangle_{S^2} = 0.\end{aligned}$$

Lemma 7.35. *Assume that $p_\tau(\partial C_K)$ is algebraically trivial. Let $Y \in S^2 \setminus p_\tau(\partial C_K)$. Let $Z \colon K \to S^2$ map $k \in K$ to the vector $Z(k)$ of S^2 orthogonal to $p_\tau(T_k K)$ in the half great circle of S^2 that contains $p_\tau(T_k K)$, Y and $p_\tau(-T_k K)$. Define the parallel $K_{\|,\tau,Y}$ by pushing a point k of K in the direction $\tau(Z(k))$. Then*

$$I_\theta(K, \tau) = lk(K, K_{\|,\tau,Y}).$$

PROOF: Thanks to Lemma 3.12, $lk(K, K_{\|,\tau,Y})$ is the evaluation of any propagator of $C_2(R)$ on $K \times K_{\|,\tau,Y}$. Let ω_p be a propagating form of $C_2(R)$ (as in Definition 3.11), which may be expressed as $p_\tau^*(\omega_{-Y})$ on $UN(K)$, for a 2–form ω_{-Y} of S^2 supported in a geometric disk in $S^2 \setminus p_\tau(\partial C_K)$ centered at $(-Y)$. Observe that the intersection of such a disk with the half great circle of S^2 that contains $p_\tau(T_k K)$, Y and $p_\tau(-T_k K)$ is empty for any $k \in K$.

Compute $lk(K, K_{\|,\tau,Y})$ as the limit of $I(K \quad K_{\|,\tau,Y}, \omega_p)$, when $K_{\|,\tau,Y}$ tends to K. The configuration space $K \times K_{\|,\tau,Y}$ is a torus of $C_2(R)$. When $K_{\|,\tau,Y}$ tends to K, this torus tends to the union of the annulus $C(K; \quad K)$ and an annulus $K \times_{\tau,Y} J$ contained in $UR_{|K}$, which fibers over K and whose fiber over $k \in K$ contains all the limit directions from k to a close point on $K_{\|,\tau,Y}$. This fiber is the half great circle of $UR_{|k}$ that p_τ maps to the half great circle of S^2 that contains $p_\tau(T_k K)$, Y and $p_\tau(-T_k K)$. Thus $K \times K_{\|,\tau,Y}$ is homologous to the torus

$$T = C_K \cup (K \times_{\tau,Y} J).$$

The integral of ω_p on $K \times_{\tau,Y} J$ is the integral of ω_{-Y} along $p_\tau(K \times_{\tau,Y} J)$, which is zero since $p_\tau(K \times_{\tau,Y} J)$ does not meet the support of ω_{-Y}. Therefore, $lk(K, K_{\|,\tau,Y}) = \int_{C_K} \omega_p = I_\theta(K,\tau)$, thanks to Lemma 7.33. □

An isotopy class of parallels of a knot is called a *framing* of a knot.

Corollary 7.36. *A knot embedding K such that $p_\tau(\partial C_K)$ is algebraically trivial, with respect to a parallelization τ, has a canonical framing induced by τ, which is the framing induced by a parallel $K_{\|,\tau,Y}$ for an arbitrary $Y \in S^2 \setminus p_\tau(\partial C_K)$. For such a knot embedding K, for any propagating chain F of $(C_2(R),\tau)$ transverse to C_K and for any propagating form ω_p of $(C_2(R),\tau)$,*

$$\int_{C_K} \omega_p = \langle C_K, F\rangle_{C_2(R)} = I_\theta(K,\tau) = lk(K, K_{\|,\tau,Y}).$$

PROOF: Since the linking number $lk(K, K_{\|,\tau,Y})$ determines the framing, the corollary is a direct consequence of Lemmas 7.33 and 7.35. □

A knot embedding $K \colon S^1 \hookrightarrow \check{R}$ is *straight* with respect to τ if the curve $p_\tau(U^+K)$ of S^2 is algebraically trivial (recall the notation from Proposition 3.7 and Section 7.3). A link embedding is *straight* with respect to τ if all its components are. Straight knot embeddings and almost horizontal knot embeddings in $\mathbb{R}^3$ are examples of knot embeddings K such that $p_\tau(\partial C_K)$ is algebraically trivial. Therefore Lemma 1.4 is a particular case of the above corollary. As a second corollary, we get the following lemma.

Lemma 7.37. *For any knot embedding K in $\check{R}$, there exists an asymptotically standard parallelization $\tilde{\tau}$ homotopic to τ such that the first vector $\tilde{\tau}(.;(1,0,0))$ of $\tilde{\tau}$ is tangent to K and orients K. In this case, let $K_{\tilde{\tau}}$ be the parallel of K obtained by pushing K in the direction of the second vector $\tilde{\tau}(.;(0,1,0))$ of $\tilde{\tau}$. Then*

$$I_\theta(K,\tilde{\tau}) = lk(K, K_{\tilde{\tau}}).$$

PROOF: In order to obtain $\tilde{\tau}$, it suffices to perform a homotopy of τ around the image of K so that the first vector of $\tilde{\tau}$ becomes tangent to K along K. Thus K is straight with respect to $(R,\tilde{\tau})$. Apply Corollary 7.36. □

Lemma 7.38. *Let $K_0 \colon \{0\} \times S^1 \to \check{R}$ be a straight embedding with respect to τ. Let $K \colon [0,1] \times S^1 \to \check{R}$ be an embedding such that its restriction $K_1 \colon \{1\} \times S^1 \to \check{R}$ is straight with respect to an asymptotically standard parallelization τ_1 homotopic to τ, then*

$$I_\theta(K_1,\tau_1) - I_\theta(K_0,\tau) \in 2\mathbb{Z}.$$

For any rational number x congruent to $I_\theta(K_0,\tau)$ mod 2, there exists a straight embedding K_1 isotopic to K_0 such that $I_\theta(K_1,\tau) = x$.

Proof: Let $H\colon t \mapsto \tau_t$ be a smooth homotopy from $\tau = \tau_0$ to τ_1. Let $p_H\colon [0,1]\times\partial C_2(R) \to S^2$ be the smooth map that restricts to $\{t\}\times\partial C_2(R)$ as p_{τ_t}. There is a closed 2-form ω on $[0,1]\times C_2(R)$ that restricts to $[0,1]\times\partial C_2(R)$ as $p_H^*(\omega_{S^2})$. (Such a form may be obtained by modifying $p^*_{C_2(R)}p^*_\tau(\omega_{S^2})$ in a collar neighborhood of $U\check{R}$ using the homotopy H.) Then the integral of ω over

$$\partial \cup_{t\in[0,1]} C(K; K_t) = C(K; K_1) - C(K; K_0) - \cup_{t\in[0,1]}\partial C(K; K_t)$$

vanishes. So

$$I_\theta(K_1,\tau_1) - I_\theta(K_0,\tau) = \int_{\cup_{t\in[0,1]}\partial C(K; K_t)} \omega.$$

This is the area in S^2 of the integral cycle $\cup_{t\in[0,1]}p_{\tau_t}(\partial C(K; K_t))$. This cycle is the union of the two integral cycles

$$\cup_{t\in[0,1]}p_{\tau_t}(U^+(K_t)) \quad \text{and} \quad \cup_{t\in[0,1]}\, p_{\tau_t}(-U^-(K_t))$$

which have the same integral area. Thus $I_\theta(K_1,\tau_1) - I_\theta(K_0,\tau) \in 2\mathbb{Z}$.

Adding two small almost horizontal kinks in a standard ball as in the end of Section 1.2.4 which turn in opposite direction and contribute with the same crossing sign like (and) or (and) preserves straightness and adds ± 2 to I_θ. □

7.6 Second definition of Z

Let us state another version of Theorem 7.20 using straight links instead of homogeneous propagating forms. Recall $\zeta_\Gamma = \frac{(3n-\sharp E(\Gamma))!}{(3n)!2^{\sharp E(\Gamma)}}$.

Theorem 7.39. *Let $(\check{R},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $L\colon \mathcal{L} \hookrightarrow \check{R}$ be a straight embedding with respect to τ. For any $i \in \underline{3n}$, let $\omega(i)$ be a propagating form of $(C_2(R),\tau)$.*

Then the element $Z_n(\check{R},L,(\omega(i))_{i\in\underline{3n}})$ of $\mathcal{A}_n(\mathcal{L})$ defined in Notation 7.16 is independent of the chosen propagating forms $\omega(i)$ of $(C_2(R),\tau)$. It is denoted by $Z^s_n(\check{R},L,\tau)$.

In particular, with the notation of Theorem 7.20,

$$Z^s_n(\check{R},L,\tau) = Z_n(\check{R},L,\tau).$$

The proof of this theorem will be given in Section 10.4.

A more general definition without the straightness assumption could be extracted from Theorem 16.7.

Since straight links L with respect to τ are framed links $(L, L_{\|,\tau})$ according to Corollary 7.36 and Lemma 7.35, we can keep the information from the link framing and define the invariant of straight links

$$\mathcal{Z}^f(\check{R}, L, L_{\|,\tau}) = \exp(-\frac{1}{4}p_1(\tau)\beta)Z(\check{R}, L, \tau),$$

which depends only on $(\check{R}, L)$ and on the $lk(K_j, K_{j\|,\tau})$ for the components K_j of L, according to Theorem 7.20 and Corollary 7.36.

Definition 7.40. Define the framed link invariant $\mathcal{Z}^f$ such that

$$\mathcal{Z}^f(\check{R}, \sqcup_{j=1}^k K_j, \sqcup_{j=1}^k K_{j\|}) = \prod_{j=1}^k \left(\exp(lk(K_j, K_{j\|})\alpha)\sharp_j\right) \mathcal{Z}(R, \sqcup_{j=1}^k K_j),$$

for a link $\sqcup_{j=1}^k K_j$ equipped with a parallel $K_{j\|}$ for each component K_j, from the invariant $\mathcal{Z}$ of Theorem 7.20.

Thanks to Theorem 7.20, Corollary 7.36 and Theorem 7.39, both definitions coincide for straight framed links $(L, L_{\|,\tau})$.

Again, we can reduce our averaging process when projecting to $\check{\mathcal{A}}(\mathcal{L})$ and get the following theorem, which will also be proved in Section 10.4.

Theorem 7.41. *Under the assumptions of Theorem 7.39,*

$$\check{p}\left(Z_n^s(\check{R}, L, \tau)\right) = \sum_{\Gamma \in \mathcal{D}^e_{n,3n-2}(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i \in \underline{3n-2}})\check{p}([\Gamma]).$$

Chapter 8

Compactifications of configuration spaces

Compactifications of our open configuration spaces $\check{C}(R,L;\Gamma)$ are useful to study the behaviour of our integrals and their dependence on the choice of propagating forms. Indeed, the convergence of the integrals involved in the definitions of Z are proved by finding a smooth compactification (with boundary and ridges), to which the integrated forms extend smoothly. The variation of an integral under the addition of an exact form $d\eta$ is the integral of η on the codimension-one faces of the boundary, which need to be identified precisely. Therefore, the proofs of Theorems 7.39 and 7.20 (and their variants with reduced averages) require a deeper knowledge of configuration spaces. We give all the useful statements in Sections 8.2 to 8.4. All of them are proved in Sections 8.5 to 8.8.

Before giving all the required general statements, we introduce the main ideas involved in the compactifications, which are due to Fulton and McPherson in [FM94], Axelrod and Singer [AS94, Section 5], and also investigated by Kontsevich [Kon94], Bott and Taubes [BT94], by presenting the main features of the compactifications in some examples.

8.1 An informal introduction

Our first example of a compactification $C(R,L;\Gamma)$ of a configuration space $\check{C}(R,L;\Gamma)$ is the closed annulus $C(K;\ominus) = C(S^3,K;\ominus)$ of Section 1.2.4. Our (more interesting) second example is the compactification $C_2(R) = C(R,\emptyset;\ominus)$ studied in Section 3.2. Note that $C(K;\ominus)$ is the closure of $\check{C}(K;\ominus)$ in $C_2(R)$.

As in the example of $C_2(R) = C(R,\emptyset;\ominus)$, for a trivalent Jacobi diagram

Γ, our general compactification $C(R, \emptyset; \Gamma)$ of the space $\check{C}(R, \emptyset; \Gamma) = \check{C}_{V(\Gamma)}(R)$ of injective maps from $V(\Gamma)$ to $\check{R}$ depends only on the finite set $V = V(\Gamma)$ of vertices of Γ, it is denoted by $C_V(R)$. As in the example of $C(S^3, K; \bullet\!\!-\!\!\bullet)$, for a Jacobi diagram Γ on the source of a link L of $\check{R}$, the compactification $C(R, L; \Gamma)$ of $\check{C}(R, L; \Gamma)$ is defined in Proposition 8.6 as the closure of $\check{C}(R, L; \Gamma)$ in $C_{V(\Gamma)}(R)$. That is why we first study $C_V(R)$ by generalizing the construction of $C_2(R)$ performed in Section 3.2. In this general case, we start with R^V and blow up all the diagonals and all the loci which involve ∞, in the sense of Section 3.1, following the process that is precisely described in Theorem 8.4.

In this informal introduction, we forget about ∞ and first discuss how the diagonals are successively blown up in the manifold $(\mathbb{R}^3)^V$ of maps from V to $\mathbb{R}^3$, in order to get to get the preimage $C_V[\mathbb{R}^3]$ of $(\mathbb{R}^3)^V$ in $C_{V(\Gamma)}(S^3)$, under the composition of the blowdown maps valued in $(S^3)^V$.

8.1.1 On the configuration space $C_V[\mathbb{R}^3]$ of four points in $\mathbb{R}^3$

The diagonal $\Delta_V((\mathbb{R}^3)^V)$ is the set of constant maps from V to $\mathbb{R}^3$. The fiber of the normal bundle to the vector space $\Delta_V((\mathbb{R}^3)^V)$ in $(\mathbb{R}^3)^V$ is the vector space $(\mathbb{R}^3)^V/\Delta_V((\mathbb{R}^3)^V)$. It consists of maps c from V to $\mathbb{R}^3$ up to global translation[1]. The fiber of the unit normal bundle to the diagonal $\Delta_V((\mathbb{R}^3)^V)$ is the space $\overline{\mathcal{S}}_V(\mathbb{R}^3)$ of non-constant maps from $V = \{v_1, v_2, \dots, v_{\sharp V}\}$ to $\mathbb{R}^3$ up to (global) translation, and up to dilation[2]. The space $\overline{\mathcal{S}}_V(\mathbb{R}^3)$, which is studied in Section 8.3, can be identified with the space of maps $w \colon V \to \mathbb{R}^3$ which map v_1 to 0 and such that $\sum_{i=2}^{\sharp V} \| w(v_i) \|^2 = 1$. It is diffeomorphic to a sphere of dimension $3\sharp V - 4$ and we have a diffeomorphism

$$\psi \colon \mathbb{R}^3 \times [0, \infty[\times \overline{\mathcal{S}}_V(\mathbb{R}^3) \to B\ell((\mathbb{R}^3)^V, \Delta_V((\mathbb{R}^3)^V)),$$

which maps (u, μ, w) to the map $c \colon V \to \mathbb{R}^3$ such that $c(v_i) = u + \mu w(v_i)$ for any $i \in \underline{\sharp V}$. This map c is furthermore equipped with the data of the map w when c is constant, or, equivalently, when $\mu = 0$. In particular, this first blow-up equips each constant map c_0 in the manifold $B\ell_1 = B\ell((\mathbb{R}^3)^V, \Delta_V((\mathbb{R}^3)^V))$ with the additional data of a non-constant map $w \colon V \to \mathbb{R}^3$ up to translation and dilation. Let c_0 be the constant map which maps V to u, and let (c_0, w) denote $\psi(u, 0, w)$, then (c_0, w) is the limit in $B\ell_1$ of $\psi(u, t, w)$, when $t > 0$ tends to 0. Therefore, the map w can be thought of as an infinitesimal

[1] A map c is identified with the map $(v \mapsto c(v) + W)$ for any $W \in \mathbb{R}^3$.

[2] In $\overline{\mathcal{S}}_V(\mathbb{R}^3)$, a map c is furthermore identified with the map $(v \mapsto \lambda c(v))$ for any $\lambda \in]0, +\infty[$.

configuration, and the first blow up provides a magnifying glass, which allows us to see this infinitesimal configuration w of the vertices which are mapped to the same point u in $\mathbb{R}^3$.

When $\sharp V = 2$, we are done and $B\ell_1$ is the preimage $C_V[\mathbb{R}^3]$ of $(\mathbb{R}^3)^2$ in $C_2(S^3)$. In general, we blow up the other *diagonals* $\Delta_A((\mathbb{R}^3)^V)$ for all subsets A of V, where $\Delta_A((\mathbb{R}^3)^V)$ is the subspace of $(\mathbb{R}^3)^V$ consisting of maps c that map A to a single element and such that $c(V\setminus A) \subset \mathbb{R}^3\setminus c(A)$, as in Section 8.2.

Let us describe the process when $V = V(\Gamma) = \{v_1, v_2, v_3, v_4\}$. From now on, we restrict to this case in this subsection, and we blow up the closures of the diagonals $\Delta_A((\mathbb{R}^3)^V)$ in $B\ell_1$ for the subsets A of V of cardinality 3. Note that these closures are disjoint, since the involved non-constant maps w are constant on A in the closure of $\Delta_A((\mathbb{R}^3)^V)$. So these blow-ups can be performed in an arbitrary order. The fiber of the unit normal bundle of $\Delta_A((\mathbb{R}^3)^V)$ in $B\ell_1$ is the space $\overline{\mathcal{S}}_A(\mathbb{R}^3)$ of non-constant maps from A to $\mathbb{R}^3$ up to (global) translation and dilation. Let $B = A_{123} = \{v_1, v_2, v_3\}$. View $\overline{\mathcal{S}}_{123}(\mathbb{R}^3) = \overline{\mathcal{S}}_B(\mathbb{R}^3)$ as the space of maps $w_{123}\colon V \to \mathbb{R}^3$ that map v_1 and v_4 to 0 and such that $\parallel w_{123}(v_2) \parallel^2 + \parallel w_{123}(v_3) \parallel^2 = 1$. We have a smooth embedding

$$\psi_2\colon \mathbb{R}^3 \times [0,\infty[^2 \times S^2 \times \overline{\mathcal{S}}_B(\mathbb{R}^3) \to B\ell(B\ell_1, \overline{\Delta_B((\mathbb{R}^3)^V)})$$

that maps $(u, \mu, \mu_{123}, W_4, w_{123})$ to the map $c\colon V \to \mathbb{R}^3$ such that $c(v_4) = u + \frac{\mu}{\sqrt{1+\mu_{123}^2}} W_4$ and $c(v_i) = u + \frac{\mu}{\sqrt{1+\mu_{123}^2}} \mu_{123} w_{123}(v_i)$ for $i \in \underline{3}$. The image of this embedding contains a neighborhood of the blown-up $\overline{\Delta_B((\mathbb{R}^3)^V)}$, which may be written as $\psi_2\left(\mathbb{R}^3 \times [0,\infty[\times \{0\} \times S^2 \times \overline{\mathcal{S}}_B(\mathbb{R}^3)\right)$, where $\mu_{123} = 0$.

Here, this blow-up equips a map c of $\Delta_B((\mathbb{R}^3)^V)$ with the additional data of the (infinitesimal, non-constant) configuration $w_{123|B}\colon B \to \mathbb{R}^3$ up to translation and dilation, and it equips a constant map c_0 with value u in the closure of $\Delta_B((\mathbb{R}^3)^V)$ in $B\ell_1$ with such a configuration $w_{123|B}$ in addition to the former w, which maps B to 0 and v_4 to some $W_4 \in S^2$. In this case, the obtained configuration $\psi_2(u, 0, 0, W_4, w_{123})$ is denoted by $(c, w, w_{123|B})$, and we have three observation scales, a first scale, where all the $c(v_i)$ coincide, a second (infinitely smaller) scale w in $B\ell_1$, where the $w(v_i)$ coincide for $i \in \underline{3}$, but $w(v_4) - w(v_1)$ is not zero, and its direction is known as a vector $W_4 \in S^2$, and a third scale (infinitely smaller than the second one) in $B\ell(B\ell_1, \overline{\Delta_B((\mathbb{R}^3)^V)})$, where the configuration $w_{123|B}$ of $\{v_1, v_2, v_3\}$ is visible up to global translation and dilation, as in the first three pictures of Figure 8.1.

Let $B\ell_2$ be the manifold obtained by blowing up the four closures of the diagonals $\Delta_A((\mathbb{R}^3)^V)$ in $B\ell_1$ for the subsets A of V of cardinality 3. We have

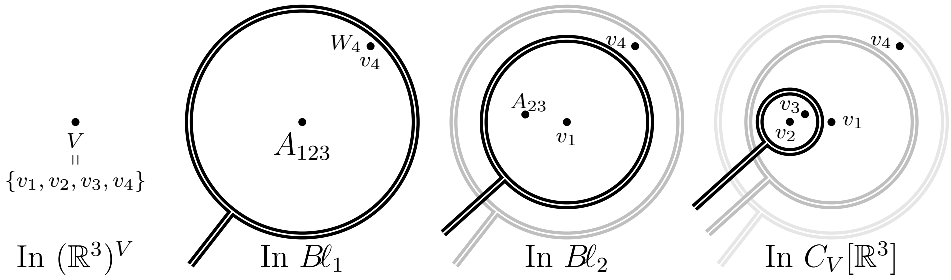


Figure 8.1: The magnifying glasses provided by the successive blow-ups from $(\mathbb{R}^3)^V$ to $C_V[\mathbb{R}^3]$, for a configuration (c, w, w_{123}, w_{23})

local charts similar to ψ_2 for $B\ell_2$ in the neighborhoods of the disjoint blown up loci.

Finally, we blow up (the preimages under the composition of the previous blowdown maps of) the closures of the diagonals $\Delta_A((\mathbb{R}^3)^V)$ in $B\ell_2$ for the subsets A of V of cardinality 2, in an arbitrary order, to get the manifold $C_V[\mathbb{R}^3]$ of Section 8.6. Here the diagonals $\Delta_{\{v_1,v_2\}}((\mathbb{R}^3)^V)$ and $\Delta_{\{v_3,v_4\}}((\mathbb{R}^3)^V)$ are no longer disjoint. Nevertheless the blow-up operations associated to $A_{12} = \{v_1, v_2\}$ and $A_{34} = \{v_3, v_4\}$, which act on different coordinates, commute. In the neighborhood of the intersection of the corresponding blown-up loci in $C_V[\mathbb{R}^3]$, we have an embedding

$$\psi_3 \colon \mathbb{R}^3 \times [0, \infty[\times[0, \frac{1}{3}[^2 \times (S^2)^3 \to C_V[\mathbb{R}^3],$$

which maps

$$(u, \mu', \mu_{12}, \mu_{34}, W_3, W_{12}, W_{34})$$

to the map $c\colon V \to \mathbb{R}^3$ such that $c(v_1) = u$, $c(v_2) = u + \mu'\mu_{12}W_{12}$, $c(v_3) = u + \mu' W_3$, $c(v_4) = u + \mu'(W_3 + \mu_{34}W_4)$. Here, $W_3 \in S^2 = \overline{\mathcal{S}}_{\{A_{12},A_{34}\}}(\mathbb{R}^3)$, $W_{12} \in S^2 = \overline{\mathcal{S}}_{A_{12}}(\mathbb{R}^3)$, $W_{34} \in S^2 = \overline{\mathcal{S}}_{A_{34}}(\mathbb{R}^3)$. The configuration c is equipped with the map w up to translation and dilation, which maps v_1, v_2, v_3 and v_4 to 0, $\mu_{12}W_{12}$, W_3, and $W_3 + \mu_{34}W_4$, respectively, when c is constant. The configuration c is equipped with W_{12} and/or W_{34}, when the restriction of w (or c) to A_{12} and/or A_{34} is constant as in Figure 8.2, which shows the three magnifying glasses that have popped up for a configuration $(c, w, W_{12}, W_{34}) = \psi_3(u, 0, 0, 0, W_3, W_{12}, W_{34})$.

For a constant map c, whose associated infinitesimal w is constant on A_{123}, and whose next associated w_{123} is constant on $A_{23} = \{v_2, v_3\}$, the third

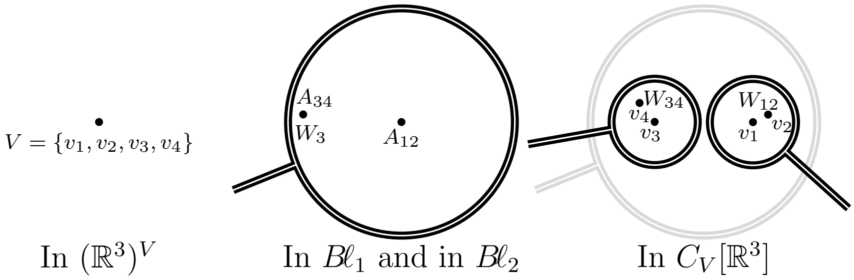


Figure 8.2: The magnifying glasses provided by the successive blow-ups from $(\mathbb{R}^3)^V$ to $C_V[\mathbb{R}^3]$, for a configuration (c, w, W_{12}, W_{34})

blow-up family provides a fourth smaller observation scale, where we see a non-constant map $w_{23}\colon A_{23} \to \mathbb{R}^3$ up to global translation and dilation, which shows the direction from $w_{23}(v_2)$ to $w_{23}(v_3)$ as in Figure 8.1.

8.1.2 More configuration spaces and their stratifications

In general, for a finite set V, and for an integer $d \in \mathbb{N} \setminus \{0\}$ we transform $(\mathbb{R}^d)^V$ to a manifold $C_V[\mathbb{R}^d]$ by successively blowing up the closures of (the preimages under the composition of the previous blowdown maps of) the diagonals $\Delta_A((\mathbb{R}^d)^V)$ associated to the subsets A of V of cardinality k, for $k = \sharp V, \sharp V - 1, \dots, 2$ in this decreasing order, in Section 8.6. It will follow from Theorem 8.30 and Proposition 8.31 that there are natural smooth (quotients of) restriction maps from $C_V[\mathbb{R}^d]$ to the space $\overline{\mathcal{S}}_A(\mathbb{R}^d)$ of non-constant maps from A to $\mathbb{R}^d$ up to translation and dilation, for every subset A of V of cardinality at least 2. These restriction maps are smooth extensions of the natural (quotients of) restriction maps from the space $\check{C}_V[\mathbb{R}^d]$ of injective maps from V to $\mathbb{R}^d$ to the compact space $\overline{\mathcal{S}}_A(\mathbb{R}^d)$, and $C_V[\mathbb{R}^d]$ could be defined simply to be the closure of the image of $\check{C}_V[\mathbb{R}^d]$ in the product $(\mathbb{R}^d)^V \times \prod_{A \subseteq V; \sharp A \geq 2} \overline{\mathcal{S}}_A(\mathbb{R}^d)$, or in smaller spaces, as Sinha did in [Sin04], but the differential structures of our configuration spaces are essential for our purposes. That is why we study them in details in this chapter.

Definition 8.1. The *partition associated to a map f* from a finite set V to some set X is the following set $K(V; f)$ of subsets of V.

$$K(V; f) = \{f^{-1}(x); x \in f(V)\}.$$

In this book, a *partition* of a finite set V is a set of disjoint non-empty subsets of V whose union is V. The elements of a partition $K(V)$ are called the *kids* of V (with respect to the partition). (We do not call them children because the initial of children is already used in the notation of configuration spaces.) The *daughters* of V with respect to such a partition are its kids with cardinality at least 2, its *sons* are the singletons of $K(V)$. The set of daughters of V is denoted by $D(V, K(V))$, or by $D(V)$ when $K(V)$ is understood, $D(V, K(V; f))$ is simply denoted by $D(V; f)$.

Definition 8.2. A *parenthesization* $\mathcal{P}$ of a set V is a set $\mathcal{P} = \{A_i; i \in I\}$ of subsets of V, each of cardinality greater than one, such that, for any two distinct elements i, j of I one of the following holds $A_i \subset A_j$, $A_j \subset A_i$ or $A_i \cap A_j = \emptyset$.

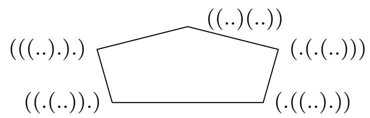
Every element x of the space $C_V[\mathbb{R}^d]$ defines the following parenthesization $\mathcal{P}(x)$ of V. The maximal elements (with respect to the inclusion) of $\mathcal{P}(x)$ are the daughters of V with respect to $(p_b(x) \in (\mathbb{R}^d)^V)$. For any element A of $\mathcal{P}(x)$, the maximal strict subsets of A in $\mathcal{P}(x)$ are the daughters of A with respect to the restriction of x to A. In our examples of Figures 8.1 and 8.2, the parenthetizations are $\{V, A_{123}, A_{23}\}$ and $\{V, A_{12}, A_{34}\}$, respectively. They are in one-to-one correspondences with the magnifying glasses provided by the iterated blow-ups, or, equivalently, with the blow-ups that affected x. For a parenthesization $\mathcal{P}$ of V, define the *stratum* $C_{V,\mathcal{P}}[\mathbb{R}^d] = \{x \in C_V[\mathbb{R}^d] \mid \mathcal{P}(x) = \mathcal{P}\}$. As it can be seen in the above examples, and as it is stated in a larger generality in Proposition 8.32, such a stratum is a smooth manifold of codimension $\sharp\mathcal{P}$ in $C_V[\mathbb{R}^d]$. These strata define a *stratification* of $C_V[\mathbb{R}^d]$, which is, as a set, the disjoint union of the strata $C_{V,\mathcal{P}}[\mathbb{R}^d]$. The open codimension-one faces of $C_V[\mathbb{R}^d]$ are the strata of codimension one, they are in one-to-one correspondence with the subsets A of V of cardinality greater than 1. Such an open face consists of the pairs $(c \in \Delta_A((\mathbb{R}^d)^V), w_A \in \overline{\mathcal{S}}_A(\mathbb{R}^d))$ such that the restriction of c to $V \setminus A$ and the map w_A are injective.

The space $\overline{\mathcal{S}}_V(\mathbb{R}^d)$ of non-constant maps from V to $\mathbb{R}^d$ up to translation and dilation is the preimage of a constant map under the blowdown map from $B\ell((\mathbb{R}^d)^V, \Delta_V((\mathbb{R}^d)^V))$ to $(\mathbb{R}^d)^V$. The diagonals $\Delta_A(\overline{\mathcal{S}}_V(\mathbb{R}^d))$ of this space, for strict subsets A of V of cardinality greater than 1, are successively blown-up as above, in Theorem 8.11, to transform $\overline{\mathcal{S}}_V(\mathbb{R}^d)$ to the preimage $\mathcal{S}_V(\mathbb{R}^d)$ of a constant map under the composed blowdown map from $C_V[\mathbb{R}^d]$ to $(\mathbb{R}^d)^V$. The space $\mathcal{S}_V(\mathbb{R}^d)$ is presented in details in Section 8.3. It is a compactification of the space $\check{\mathcal{S}}_V(\mathbb{R}^d)$ of injective maps from V to $\mathbb{R}^d$ up to translation and dilation.

The spaces $\check{\mathcal{S}}_{\underline{2}}(\mathbb{R}^d)$, $\overline{\mathcal{S}}_{\underline{2}}(\mathbb{R}^d)$, $\mathcal{S}_{\underline{2}}(\mathbb{R}^d)$ are identical, they consist of the classes of the maps $w_X \colon \underline{2} \to \mathbb{R}^d$ such that $w_X(1) = 0$ and $w_X(2) = X$ for $X \in S^{d-1}$. So they are diffeomorphic to the unit sphere S^{d-1} of $\mathbb{R}^d$. In particular, the space $\overline{\mathcal{S}}_{\underline{2}}(\mathbb{R})$ has two elements, which are the classes of w^+ and w^-, where w^+ and w^- map 1 to 0, $w^+(2) = 1$ and $w^-(2) = -1$.

Let us discuss the case $d = 1$ in more details.

Example 8.3. In general, for an integer $k \geq 2$, $\check{\mathcal{S}}_{\underline{k}}(\mathbb{R})$ and its compactification $\mathcal{S}_{\underline{k}}(\mathbb{R})$ have $k!$ components, which correspond to the orders of the $c(i)$ in $\mathbb{R}$, for configurations c in $\check{\mathcal{S}}_{\underline{k}}(\mathbb{R})$. Denote the connected component of $\check{\mathcal{S}}_{\underline{k}}(\mathbb{R})$ in which the configurations c satisfy $c(1) < c(2) < \cdots < c(k)$, by $\check{\mathcal{S}}_{<,\underline{k}}(\mathbb{R})$, and its respective closures in $\mathcal{S}_{\underline{k}}(\mathbb{R})$ and in $\overline{\mathcal{S}}_{\underline{k}}(\mathbb{R})$ by $\mathcal{S}_{<,\underline{k}}(\mathbb{R})$ and $\overline{\mathcal{S}}_{<,\underline{k}}(\mathbb{R})$. Then $\check{\mathcal{S}}_{<,\underline{3}}(\mathbb{R}) = \{(0,t,1) \mid t \in]0,1[\}$, and $\mathcal{S}_{<,\underline{3}}(\mathbb{R}) = \overline{\mathcal{S}}_{<,\underline{3}}(\mathbb{R})$ is its natural compactification $[0,1]$, where $t \in]0,1[$ represents the injective configuration $(0,t,1)$, 0 represents the limit configuration $((..).) = \lim_{t\to 0}(0,t,1)$ and 1 represents the limit configuration $(.(..)) = \lim_{t\to 0}(0,1-t,1)$. The configuration space $\mathcal{S}_{<,\underline{4}}(\mathbb{R})$ is diffeomorphic to the following well-known pentagon.



The edge from $((.(..)).)$ to $(((..).).)$ is the preimage of the diagonal $\Delta_{\underline{3}}(\overline{\mathcal{S}}_{<,V}(\mathbb{R}))$ under the composed blowdown map from $\mathcal{S}_{<,\underline{4}}(\mathbb{R})$ to $\overline{\mathcal{S}}_{<,\underline{4}}(\mathbb{R})$, it is naturally diffeomorphic to $\mathcal{S}_{<,\underline{3}}(\mathbb{R})$. The edge from $(((..).).)$ to $((..)(..))$ is the closure of the preimage of the diagonal $\Delta_{\underline{2}}(\overline{\mathcal{S}}_{<,V}(\mathbb{R}))$, its interior is naturally diffeomorphic to $\check{\mathcal{S}}_{<,\{2,3,4\}}(\mathbb{R})$.

In general, for $k \geq 3$, the configuration space $\mathcal{S}_{<,\underline{k}}(\mathbb{R})$ is a *Stasheff polyhedron* [Sta63] of dimension $(k-2)$ whose corners are labeled by *non-associative words* in the letter $\bullet$, as in the above example. For any integer $k \geq 2$, a non-associative word w with k letters represents a limit configuration $w = \lim_{t\to 0} w(t)$, where $w(t) = (w_1(t) = 0, w_2(t), \dots, w_{k-1}(t), w_k(t) = 1)$ is an injective configuration for $t \in]0, \frac{1}{2}[$, and, if w is the product uv of a non-associative word u of length $j \geq 1$ and a non-associative word v of length $(k-j) \geq 1$, then $w_i(t) = tu_i(t)$ when $1 < i \leq j$ and $w_i(t) = 1 - t + tv_{i-j}(t)$ when $j < i < k$. For example, $(((..).).)(t) = (0, t^2, t, 1)$. In a limit configuration associated to such a non-associative word, points inside matching parentheses are thought of as infinitely closer to each other than they are to points outside these matching parentheses. The parenthesization associated as above to a non-associative word is the set of strict[3] subsets inside matching parentheses.

[3]In Theorem 8.26, we will rather associate the set of all subsets inside matching parentheses.

8.2 General presentation of $C_V(R)$ and $C_{V(\Gamma)}(R)$

Let V denote a finite set. We use this notation for a generic finite set since our sets will end up being sets of vertices of Jacobi diagrams. The space of maps from V to X is denoted by X^V as usual. For a subset A of V of cardinality at least 2, recall that the subspace of X^V consisting of maps c that map A to a single element and such that $c(V \setminus A) \subset X \setminus c(A)$ is a diagonal denoted by $\Delta_A(X^V)$. In particular, if $\sharp V \geq 2$, the *small diagonal* consisting of constant maps is denoted by $\Delta_V(X^V)$.

Also recall that $\check{C}_V(R)$ is the space of injective maps from V to $\check{R}$.

Theorem 8.4. *Define a compactification $C_V(R)$ of $\check{C}_V(R)$ as follows. For a non-empty $A \subseteq V$, let Ξ_A be the set of maps from V to R that map A to ∞ and $V \setminus A$ to $R \setminus \{\infty\}$. Start with R^V. Blow up Ξ_V (which is reduced to the point $m = \infty^V$ such that $m^{-1}(\infty) = V$).*
Then for $k = \sharp V, \sharp V - 1, \dots, 3, 2$, in this decreasing order, successively blow up the closures of (the preimages under the composition of the previous blowdown maps of) the $\Delta_A(\check{R}^V)$ such that $\sharp A = k$ (choosing an arbitrary order among them) and, next, the closures of (the preimages under the composition of the previous blowdown maps of) the Ξ_J such that $\sharp J = k - 1$ (again choosing an arbitrary order among them). The successive manifolds that are blown-up in the above process are smooth, at a given step, and transverse to the ridges. The obtained manifold $C_V(R)$ is a smooth compact $(3\sharp V)$-manifold with ridges independent of the possible order choices in the process. The interior of $C_V(R)$ is $\check{C}_V(R)$, and the composition of the blowdown maps gives rise to a canonical smooth blowdown projection $p_b \colon C_V(R) \to R^V$.

This theorem is proved in Section 8.7 (see Theorem 8.33). Its proof involves the results of Sections 8.5 and 8.6.

Let $C_n(R) = C_{\underline{n}}(R)$.

With the above definition $C_1(R) = B\ell(R, \infty)$, and as announced, $C_2(R)$ is the compactification studied in Section 3.2. In particular, Theorem 8.4 is true when $\sharp V \leq 2$.

Theorem 8.5. *1. Under the assumptions of Theorem 8.4, for $A \subset V$, the restriction map*

$$p_A \colon \check{C}_V(R) \to \check{C}_A(R)$$

extends to a smooth restriction map still denoted by p_A from $C_V(R)$ to

$C_A(R)$ such that the following square commutes:

$$\begin{tikzcd} C_V(R) \arrow[r, "p_A"] \arrow[d, "p_b"'] & C_A(R) \arrow[d, "p_b"] \\ R^V \arrow[r, "p_A"] & R^A \end{tikzcd}$$

2. *For an open subset U of R, let $C_V(U)$ denote $p_b^{-1}(U^V)$. If $V = \sqcup_{i \in I} A_i$, and if $(U_i)_{i\in I}$ is a family of disjoint open sets of R, then the product*

$$p_b^{-1}(\prod_{i\in I} U_i^{A_i}) \xrightarrow{\prod_{i\in I} p_{A_i}} \prod_{i\in I} C_{A_i}(U_i)$$

of the above restriction maps is a diffeomorphism.

The first part of this theorem is a direct consequence of Proposition 8.44, with the notation of Theorem 8.33. Its second part comes from the locality of the blow-up operations. The spaces $C_V(\check{R})$, which involve only blow-ups along the diagonals have been studied by Axelrod and Singer [AS94, Section 5], and with more details by Sinha [Sin04]. Their properties that are useful in this book are proved in Sections 8.5 and 8.6. Similar compactifications of configuration spaces in an algebraic geometry setting have been studied by Fulton and McPherson in [FM94].

Recall

$$\check{C}(R, L; \Gamma) = \left\{c : U \cup T \hookrightarrow \check{R} \,; \exists j \in [i_\Gamma], c_{|U} = L \circ j\right\}.$$

Proposition 8.6. *The closure of $\check{C}(R, L; \Gamma)$ in $C_{V(\Gamma)}(R)$ is a smooth compact submanifold of $C_{V(\Gamma)}(R)$ transverse to the ridges, which is denoted by $C(R, L; \Gamma)$.*

Proposition 8.6 is proved in Section 8.8. Theorems 8.4 and 8.5 and Proposition 8.6 imply Proposition 7.8.

8.3 Configuration spaces associated to unit normal bundles to diagonals

For a vector space T, recall from Section 3.1 that $S(T)$ denotes the quotient $S(T) = \frac{T\setminus\{0\}}{\mathbb{R}^{+*}}$ of $T \setminus \{0\}$ by the dilations. If T is equipped with a Euclidean norm, then $\mathbb{S}(T)$ denotes the unit sphere of T. In this case, $S(T)$ and $\mathbb{S}(T)$ are diffeomorphic.

Definition 8.7. Let V denote a finite set of cardinality at least 2. Let T be a vector space of dimension δ. We use the notation T for a generic vector space since T will end up being a tangent space. Let

$$\overline{\mathcal{S}}_V(T) = S(T^V/\Delta_V(T^V)) = \frac{\left(T^V/\Delta_V(T^V)\right) \setminus \{0\}}{\mathbb{R}^{+*}}$$

be the space of non-constant maps from V to T up to translation and dilation.

Lemma 8.8. *Let A be a subset of V. The fiber of the unit normal bundle to $\Delta_A(R^V)$ in R^V over a configuration m is $\overline{\mathcal{S}}_A(T_{m(A)}R)$.*

PROOF: Exercise. $\square$

As expected for a unit normal bundle fiber, we have the following lemma.

Lemma 8.9. *The topological space $\overline{\mathcal{S}}_V(T)$ has a canonical smooth structure. It is diffeomorphic to a sphere of dimension $((\sharp V - 1)\delta - 1)$.*

PROOF: Choosing a basepoint $b(V)$ for V and a basis for T identifies $\overline{\mathcal{S}}_V(T)$ with the set $\mathbb{S}\left(T^{V\setminus\{b(V)\}}\right)$ of maps $c\colon V \to T$ such that

- $c(b(V)) = 0$, and,
- $\sum_{v\in V} \parallel c(v) \parallel^2 = 1$

with respect to the norm for which our basis is orthonormal. It is easy to see that changes of basepoints, or bases of T give rise to diffeomorphisms of spheres. $\square$

Convention 8.10. In this chapter, we do not take care of orientations. Later, we will associate the following orientation of $\overline{\mathcal{S}}_V(T)$ to an order of V and an orientation of T. Assume that $V = \{v_1, \dots, v_n\}$ and that T is oriented. The order on V orients T^V and $T^{V\setminus\{v_1\}}$. Then the map from the boundary of the unit ball of $T^{V\setminus\{v_1\}}$ to $\overline{\mathcal{S}}_V(T)$ that maps an element $(x_2, \dots, x_n)$ of $\mathbb{S}\left(T^{V\setminus\{v_1\}}\right)$ to the class of $(0, x_2, \dots, x_n)$ is an orientation-preserving diffeomorphism.

When $V = \underline{2}$, fixing $b(\underline{2}) = 1$ identifies $\overline{\mathcal{S}}_V(T)$ with the unit sphere $\mathbb{S}(T)$ of T, if T is equipped with a Euclidean norm.

For a strict subset A of V of cardinality at least 2, define the diagonal $\Delta_A(\overline{\mathcal{S}}_V(T))$ as the subset of $\overline{\mathcal{S}}_V(T)$ consisting of classes of maps c from V to T that are constant on A and such that $c(V \setminus A) \cap c(A) = \emptyset$. Let $\check{\mathcal{S}}_V(T)$ denote the subspace of $\overline{\mathcal{S}}_V(T)$ consisting of *injective* maps from V to T up to translation and dilation. The following theorem defines a bigger compactification of $\check{\mathcal{S}}_V(T)$, which is also used in our study of the variations Z, and especially in the definition of the anomalies. The following two theorems are proved in Section 8.5 (see Theorem 8.26 and Proposition 8.29).

Theorem 8.11. *Start with* $\overline{\mathcal{S}}_V(T)$. *For* $k = \sharp V - 1, \dots, 3, 2$, *in this decreasing order, successively blow up the closures of (the preimages under the composition of the previous blowdown maps of) the* $\Delta_A(\overline{\mathcal{S}}_V(T))$ *such that* $\sharp A = k$. *The successive manifolds which are blown-up in the above process are smooth, and transverse to the ridges. The resulting iterated blown-up manifold does not depend on the order choices.*

Thus this process gives rise to a canonical compact smooth manifold $\mathcal{S}_V(T)$ *with ridges whose interior is* $\check{\mathcal{S}}_V(T)$.

Note that if $\sharp V = 2$, then $\mathcal{S}_V(T) = \check{\mathcal{S}}_V(T) = \overline{\mathcal{S}}_V(T)$. The manifold $\mathcal{S}_V(T)$ satisfies the following properties.

Theorem 8.12. *With the notation of Theorem 8.11, for any subset* A *of* V, *the restriction from* $\check{\mathcal{S}}_V(T)$ *to* $\check{\mathcal{S}}_A(T)$ *extends to a smooth map from* $\mathcal{S}_V(T)$ *to* $\mathcal{S}_A(T)$.

The codimension-one faces of $\mathcal{S}_V(T)$ will be the loci for which only one blow-up along some $\Delta_A(\overline{\mathcal{S}}_V(T))$ is involved, for a strict subset A of V such that $\sharp A \geq 2$. The blowdown map maps the interior $f(A)(T)$ of such a face into $\Delta_A(\overline{\mathcal{S}}_V(T))$, and thus to $\check{\mathcal{S}}_{\{a\}\cup(V\setminus A)}(T)$ for an arbitrary element a of A. As it will be seen in Lemma 8.21, the fiber of the unit normal bundle of $\Delta_A(\overline{\mathcal{S}}_V(T))$ is $\overline{\mathcal{S}}_A(T)$. Thus, the following proposition will be clear in the end of Section 8.5, where a one-line proof is given.

Proposition 8.13. *The codimension-one faces of* $\mathcal{S}_V(T)$ *are in one-to-one correspondence with the strict subsets* A *of* V *with cardinality at least* 2. *The (open) face* $f(A)(T)$ *corresponding to such an* A *is*

$$f(A)(T) = \check{\mathcal{S}}_A(T) \times \check{\mathcal{S}}_{\{a\}\cup(V\setminus A)}(T)$$

for an element a *of* A. *For a subset* e *of cardinality* 2 *of* V, *the restriction to* $f(A)(T)$ *of the extended restriction*

$$p_e \colon \mathcal{S}_V(T) \to \mathcal{S}_e(T)$$

may be described as follows,

- *if* $e \subseteq A$, *then* p_e *is the composition of the natural projections*

$$f(A)(T) \longrightarrow \check{\mathcal{S}}_A(T) \longrightarrow S_e(T),$$

- *if* $e \subseteq (V \setminus A) \cup \{a\}$, *then* p_e *is the composition of the natural projections*

$$f(A)(T) \longrightarrow \check{\mathcal{S}}_{\{a\}\cup(V\setminus A)}(T) \longrightarrow S_e(T),$$

- *if* $e \cap A = \{a'\}$, *let* $\tilde{e}$ *be obtained from* e *by replacing* a' *with* a, *then* $p_e = p_{\tilde{e}}$.

The space $\mathcal{S}_V(T_x\check{R})$ described in the above theorem is related to $C_V(R)$ by the following proposition, which is a corollary of Theorem 8.30.

Proposition 8.14. *For* $x \in \check{R}$, *for* $p_b \colon C_V(R) \to R^V$,

$$\mathcal{S}_V(T_x\check{R}) = p_b^{-1}(x^V).$$

8.4 The codimension-one faces of $C(R, L; \Gamma)$

Recall that our terminology in Section 2.1.4 makes codimension-one faces open in the boundary of a manifold with ridges. Our codimension-one faces may be called *open codimension-one faces* in other references. We describe the codimension-one faces of $C(R, L; \Gamma)$ below with an outline of justification. Details are provided in Sections 8.5 to 8.8. Again, the codimension-one faces are the loci for which only one blow-up is involved.

In this book, the sign $\subset$ stands for "is a strict subset of" or "$\subseteq$ and $\neq$".

Faces that involve blow-ups along diagonals Δ_A Let A be a subset of the set $V(\Gamma)$ of vertices of a Jacobi diagram Γ on the source $\mathcal{L}$ of a link L, $\sharp A \geq 2$. Let us describe the (open) face $F(A, L, \Gamma)$, which comes from the blow-up along $\Delta_A(\check{R}^{V(\Gamma)})$. Such a face contains limit configurations that map A to a point of $\check{R}$. Elements of such a face are described by their images m under the blowdown map

$$p_b \colon C_{V(\Gamma)}(R) \to R^{V(\Gamma)},$$

which maps $F(A, L, \Gamma)$ to $\Delta_A(\check{R}^{V(\Gamma)})$, together with elements of the fiber

$$\overline{\mathcal{S}}_A(T_{m(A)}\check{R}).$$

Let $a \in A$. Let $\check{\Delta}_A(\check{R}^{V(\Gamma)})$ denote the set of maps of $\Delta_A(\check{R}^{V(\Gamma)})$ whose restriction to $\{a\} \cup (V(\Gamma) \setminus A)$ is injective, and set

$$B(A, L, \Gamma) = \check{\Delta}_A(\check{R}^{V(\Gamma)}) \cap p_b(C(R, L; \Gamma)).$$

The face $F(A, L, \Gamma)$ fibers over the subspace $B(A, L, \Gamma)$. When A contains no univalent vertex, the fiber over a point m is $\check{\mathcal{S}}_A(T_{m(A)})$.

Let i_Γ be a Γ-compatible injection. Let $\mathcal{L}_1$ be an oriented[4] connected component of $\mathcal{L}$ and let $U_1 = i_\Gamma^{-1}(\mathcal{L}_1)$. The restriction of i_Γ to U_1 into

[4]All what is actually used below, is a local orientation.

$\mathcal{L}_1$ induces a permutation σ of U_1, such that, travelling along $\mathcal{L}_1$, we meet $i_\Gamma(v)$, $i_\Gamma(\sigma(v))$, ..., $i_\Gamma(\sigma^{\sharp U_1}(v) = v)$, successively, for any element v of U_1. A *set of consecutive elements* of U_1, with respect to i_Γ, is a subset A_U of U_1 that may be written as $\{v, \sigma(v), \ldots, \sigma^k(v)\}$ for some element $v \in U_1$ and for $k \leq \sharp U_1 - 1$. If $A_U \neq U_1$, the first element v in such an A_U is unique and σ induces the following unique linear order

$$v < \sigma(v) < \cdots < \sigma^k(v)$$

on such a set A_U of consecutive elements U_1, which is said to be *compatible* with the isotopy class $[i_\Gamma]$ of i_Γ. If $A_U = U_1$, every choice of an element v in A_U induces a linear order

$$v < \sigma(v) < \cdots < \sigma^k(v),$$

which is said to be *compatible* with $[i_\Gamma]$.

Let $A_U = A \cap U(\Gamma)$. When A contains univalent vertices, if $F(A, L, \Gamma)$ is non-empty, then A_U must be a set of consecutive vertices on a component $\mathcal{L}_1$ of $\mathcal{L}$ with respect to the given class $[i_\Gamma]$ of injections of $U(\Gamma)$ into $\mathcal{L}_1$. The fiber over a point m is the subset $\check{\mathcal{S}}_A(T_{m(A)}\check{R}, L, \Gamma)$ of $\check{\mathcal{S}}_A(T_{m(A)}\check{R})$ consisting of injections that map A_U on a line directed by $T_{m(A)}L$, so that the order induced by the line on A_U coincides with

- the linear order induced by $[i_\Gamma]$, if $A \cap U(\Gamma)$ is not the whole $i_\Gamma^{-1}(\mathcal{L}_1)$,
- one of the $\sharp i_\Gamma^{-1}(\mathcal{L}_1)$ linear orders compatible with the cyclic order induced by $[i_\Gamma]$, if $A_U = i_\Gamma^{-1}(\mathcal{L}_1)$.

In this latter case, neither the fiber, nor $F(A, L, \Gamma)$ is connected. Their connected components are in one-to-one correspondence with the compatible orders.

According to Theorem 8.5, for any pair e of $V(\Gamma)$, there exists a smooth restriction map from $C_{V(\Gamma)}$ to $C_e(R)$. An order on e identifies $C_e(R)$ with $C_2(R)$. We describe the natural restriction p_e to the (open) face $F(A, L, \Gamma)$ below for a pair e of $V(\Gamma)$.

- If $\sharp(e \cap A) \leq 1$, then p_e is the composition of the natural projections
$$F(A, L, \Gamma) \longrightarrow \check{\Delta}_A(\check{R}^{V(\Gamma)}) \longrightarrow \check{C}_e(R).$$
- If $e \subseteq A$, then p_e maps an element of $\check{\mathcal{S}}_A(T_{m(A)}\check{R}, L, \Gamma)$ to its projection in $\check{\mathcal{S}}_e(T_{m(A)}\check{R}) \subset C_e(R)$.

Faces $F(V(\Gamma), L, \Gamma)$ The faces of the previous paragraph for which L is a knot embedding, and a connected graph Γ with at least one univalent vertex collapses, will play a particular role. Such a face $F(V(\Gamma), L, \Gamma)$ has one connected component for each linear order of $U(\Gamma)$ that is compatible with the cyclic order of $U(\Gamma)$. A Jacobi diagram $\check{\Gamma}$ on $\mathbb{R}$ yields a diagram $c\ell(\check{\Gamma})$ on S^1, which is viewed as $\mathbb{R} \cup \{\infty\}$, by adding ∞ to $\mathbb{R}$. A linear order of $U(\Gamma)$ that is compatible with the cyclic order of $U(\Gamma)$ can be represented by lifting Γ as a Jacobi diagram $\check{\Gamma}$ on $\mathbb{R}$, whose univalent vertices are ordered by their inclusion into $\mathbb{R}$, such that $c\ell(\check{\Gamma}) = \Gamma$. The corresponding connected component of $F(V(\Gamma), L, \Gamma)$ is denoted by $F(V(\Gamma), L, \check{\Gamma})$

Such a connected component fibers over the source S^1 of link. The fiber over $z \in S^1$ will be denoted by $\check{\mathcal{S}}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$, where $\vec{t}_{L(z)}$ denotes an oriented tangent vector to L at $L(z)$.

The space $\check{\mathcal{S}}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$ is the space of injections of the vertices of $\check{\Gamma}$ into the vector space $T_{L(z)}\check{R}$ that map the univalent vertices of $\check{\Gamma}$ to the oriented line $\mathbb{R}\vec{t}_{L(z)}$ directed by $\vec{t}_{L(z)}$ with respect to the linear order of $U(\check{\Gamma})$, up to dilation and translation with respect to a vector in $\mathbb{R}\vec{t}_{L(z)}$. It is naturally a subspace of $\check{\mathcal{S}}_{V(\Gamma)}(T_{L(z)}\check{R})$, since it is equivalent to mod out by all translations, or to only consider configurations that map a univalent vertex to $\mathbb{R}\vec{t}_{L(z)}$, and to mod out by translations along $\mathbb{R}\vec{t}_{L(z)}$.

Lemma 8.15. *The closure $\overline{F}(V(\Gamma), L, \check{\Gamma})$ of $F(V(\Gamma), L, \check{\Gamma})$ in $C(R, L; \Gamma)$ is a manifold with ridges. The closure $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$ of $\check{\mathcal{S}}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$ in $C(R, L; \Gamma)$ is canonically diffeomorphic to its closure in $\mathcal{S}_{V(\Gamma)}(T_{L(z)}\check{R})$. It is a smooth manifold with ridges.*

Proof: The first assertion comes from Proposition 8.6 and from the fact that the closed faces of manifolds with ridges are manifolds with ridges (or from the proof of Lemma 8.16 at the end of Section 8.5). The space $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$ is the closure of $\check{\mathcal{S}}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$ in the fiber over $L(z)^{V(\Gamma)}$ of $C_{V(\Gamma)}(\check{R})$, which is $\mathcal{S}_{V(\Gamma)}(T_{L(z)}\check{R})$ according to Proposition 8.14.

Now, $\overline{F}(V(\Gamma), L, \check{\Gamma})$ fibers over S^1, and the fiber over S^1 is also a manifold with ridges. $\square$

Let A be a strict subset of $V(\check{\Gamma})$ with cardinality at least 2 whose univalent vertices are consecutive on $\mathbb{R}$. Let $a \in A$. Let $\check{\Delta}_A(\mathcal{S}_{V(\Gamma)}(T_{L(z)}\check{R}))$ denote the set of (classes of) maps of $\mathcal{S}_{V(\Gamma)}(T_{L(z)}\check{R})$ whose restriction to A is constant and whose restriction to $\{a\} \cup (V(\Gamma) \setminus A)$ is injective. Set

$$B(A, \vec{t}_{L(z)}; \check{\Gamma}) = \mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma}) \cap \check{\Delta}_A(\mathcal{S}_{V(\Gamma)}(T_{L(z)}\check{R})).$$

Define the (open) face $f(A, \vec{t}_{L(z)}; \check{\Gamma})$ of $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$ to be the space that fibers over the subspace $B(A, \vec{t}_{L(z)}; \check{\Gamma})$, whose fiber is the space of injections of A into $T_{L(z)}\check{R}$ that map the univalent vertices of A to the oriented line $\mathbb{R}\vec{t}_{L(z)}$ with respect to the linear order of $U(\check{\Gamma}) \cap A$, up to dilation and translation by a vector in $\mathbb{R}\vec{t}_{L(z)}$.

The following lemma will be proved at the end of Section 8.5.

Lemma 8.16. *The codimension-one faces of $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$ are the faces $f(A, \vec{t}_{L(z)}; \check{\Gamma})$ for the strict subsets A of $V(\check{\Gamma})$ with cardinality at least 2 whose univalent vertices are consecutive on $\mathbb{R}$. The faces $f(A, \vec{t}_{L(z)}; \check{\Gamma})$ are the intersections of $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$ with the codimension-one faces $f(A)(T_{L(z)}\check{R})$ of $\mathcal{S}_{V(\check{\Gamma})}(T_{L(z)}\check{R})$ listed in Proposition 8.13. In particular, the restriction maps p_e from $f(A)(T_{L(z)}\check{R})$ to $\mathcal{S}_e(T_{L(z)}\check{R})$ of Proposition 8.13 restrict as restriction maps still denoted by p_e from $f(A, \vec{t}_{L(z)}; \check{\Gamma})$ to $\mathcal{S}_e(T_{L(z)}\check{R})$.*

Faces that involve ∞. Let A be a non-empty subset of the set $V(\Gamma)$ of vertices of a Jacobi diagram Γ. Let $a \in A$. Let us describe the (open) face $F_\infty(A, L, \Gamma)$ that comes from the blow-up along Ξ_A. It contains limit configurations, which map A to ∞. If it is non-empty, then A contains no univalent vertices.

Let $\check{\Xi}_A$ denote the set of maps of Ξ_A that restrict to injective maps on $\{a\} \cup (V(\Gamma) \setminus A)$, and set

$$B_\infty(A, L, \Gamma) = \check{\Xi}_A \cap p_b(C(R, L; \Gamma)).$$

The face $F_\infty(A, L, \Gamma)$ fibers over the subspace $B_\infty(A, L, \Gamma)$.

Notation 8.17. Let $\check{\mathcal{S}}(T_\infty R, A)$ denote the set of injective maps from A to $(T_\infty R \setminus 0)$ up to dilation.

Note that $\check{\mathcal{S}}(T_\infty R, A)$ is an open submanifold of the unit normal bundle of Ξ_A, which is nothing but $S((T_\infty R)^A)$. Then

$$F_\infty(A, L, \Gamma) = B_\infty(A, L, \Gamma) \times \check{\mathcal{S}}(T_\infty R, A).$$

For a pair e of $V(\Gamma)$, the natural restriction to $F_\infty(A, L, \Gamma)$ of

$$p_e \colon C_{V(\Gamma)} \to C_e(R)$$

behaves in the following way.

- If $e \subseteq V(\Gamma) \setminus A$, then p_e is the composition of the natural maps

$$F_\infty(A, L, \Gamma) \to B_\infty(A, L, \Gamma) \to \check{C}_{(V(\Gamma)\setminus A)}(R) \to C_e(R).$$

- If $e \subseteq A$, then p_e is the composition of the natural maps

$$F_\infty(A, L, \Gamma) \longrightarrow \check{\mathcal{S}}(T_\infty R, A) \longrightarrow \check{\mathcal{S}}(T_\infty R, e) \hookrightarrow C_e(R).$$

- If $e \cap A = \{a'\}$, then p_e is the composition of the natural maps

$$F_\infty(A, L, \Gamma) \longrightarrow \check{C}_{(e\setminus\{a'\})}(R) \times \check{\mathcal{S}}(T_\infty R, \{a'\}) \longrightarrow$$
$$\longrightarrow \check{R}^{(e\setminus\{a'\})} \times S(T_\infty R^{\{a'\}}) \hookrightarrow C_e(R).$$

Summary We have just outlined a proof of the following proposition:

Proposition 8.18. *The disjoint union of*

- *the $F_\infty(A, L, \Gamma)$ for non-empty subsets A of $T(\Gamma)$,*
- *the $F(A, L, \Gamma)$ for subsets A of $V(\Gamma)$ with cardinality at least 2 such that $A \cap U(\Gamma)$ is a (possibly empty) set of consecutive vertices on a connected component of $\mathcal{L}$,*

described above, embeds canonically in $C(R, L; \Gamma)$, and the image of its canonical embedding is the open codimension-one boundary

$$\partial_1 C(R, L; \Gamma) \setminus \partial_2 C(R, L; \Gamma)$$

of $C(R, L; \Gamma)$. Furthermore, for any ordered pair e of $V(\Gamma)$, the restriction of the map

$$p_e \colon C(R, L; \Gamma) \to C_2(R),$$

given by Theorem 8.4, to this codimension-one boundary, is as described above.

Proposition 8.18 is proved in Section 8.8. It will be used to prove that Z^s and Z are independent of the used propagating forms of Theorems 7.20 and 7.39, in Chapters 9 and 10.

All the results stated so far in this chapter are sufficient to understand the proofs of Theorems 7.20 and 7.39. They are proved in details in the rest of this chapter, which can be skipped by a reader who is already convinced by the outline above.

8.5 Detailed study of $\mathcal{S}_V(T)$

In this section, we study the configuration space $\mathcal{S}_V(T)$ presented in Section 8.3. We first prove Theorem 8.11. Let us first describe the transformations operated by the first blow-ups, locally.

Fix T, equip T and its powers with Euclidean norms. Let $\tilde{w}_0 \in \overline{\mathcal{S}}_V(T)$. Identify $\overline{\mathcal{S}}_V(T)$ with the unit sphere $\mathbb{S}\left(T^{V\setminus\{b(V)\}}\right)$ of $T^{V\setminus\{b(V)\}}$ for an element $b(V)$ of V. Then $\tilde{w}_0$ is viewed as a map from V to T such that $\tilde{w}_0(b(V)) = 0$ and $\sum_{v\in V} \parallel \tilde{w}_0(v) \parallel^2 = 1$. Recall Definition 8.1 for a partition associated to a map and the associated notation.

Definition 8.19. A *based partition* of a finite set V equipped with a basepoint $b(V) \in V$ is a partition $K(V)$ of V into non-empty subsets A equipped with basepoints $b(A)$ such that:

- $b(A) \in A$,
- if $b(V) \in A$, then $b(A) = b(V)$.

Fix $\tilde{w}_0$, and let $K(V) = K(V; \tilde{w}_0)$ be the associated fixed partition. Fix associated basepoints so that $K(V)$ becomes a based partition.

In general for a based subset A of V equipped with a based partition $(K(A), b)$, define the set $O(A, K(A), b, T)$ of maps $w : V \longrightarrow T$ such that

- $\sum_{B \in K(A)} \parallel w(b(B)) \parallel^2 = 1$
- $w(b(A)) = 0$, $w(V \setminus A) = \{0\}$, and
- two elements of A that belong to different kids of A are mapped to different points of T by w.

There is a straightforward identification of $O(V, K(V), b, T)$ with an open subset of $\overline{\mathcal{S}}_V(T)$, which contains $\tilde{w}_0$. Let $w_0 \in O(V, K(V), b, T)$ denote the element that corresponds to $\tilde{w}_0$ under this identification. Set

$$W_V = O(V, K(V), b, T) \cap \cap_{A \in D(V)} \Delta_A \left(\overline{\mathcal{S}}_V(T)\right).$$

Note that $K(V)$ is a set that is naturally based by the element $b(K(V))$ of $K(V)$ that contains $b(V)$. Then W_V is an open subset of $\mathbb{S}(T^{K(V) \setminus \{b(K(V))\}})$. It is the image $\check{\mathbb{S}}(T^{K(V) \setminus \{b(K(V))\}})$ of $\check{\mathcal{S}}_{K(V)}(T)$ under the canonical identification of $\overline{\mathcal{S}}_{K(V)}(T)$ with $\mathbb{S}(T^{K(V) \setminus \{b(K(V))\}})$.

For $A \in K(V)$, view the elements of $T^{A \setminus \{b(A)\}}$ as the maps from V to T that map $(V \setminus A) \cup \{b(A)\}$ to 0, and denote $T^{A \setminus \{b(A)\}}_{<\varepsilon}$ the ball of its elements of norm smaller than ε. Note the easy lemma.

Lemma 8.20. *Let $K(V)$ be a based partition of V. Let $w_0 \in W_V$, then there exists an open neighborhood $N(w_0)$ of w_0 in W_V, and an $\varepsilon \in]0, \infty[$ such that the map*

$$\begin{array}{lcl} N(w_0) \times \prod_{A \in D(V)} T^{A \setminus \{b(A)\}}_{<\varepsilon} & \to & \overline{\mathcal{S}}_V(T) \\ (w, (\mu_A \tilde{w}_A)_{A \in D(V)}) & \mapsto & w + \sum_{A \in D(V)} \mu_A \tilde{w}_A \end{array}$$

is an open embedding, whose image does not meet diagonals that do not correspond to (non-necessarily strict) subsets B of daughters of V.

□

In particular, the first blow-ups that will affect this neighborhood of $\tilde{w}_0$ in $\overline{\mathcal{S}}_V(T)$ are blow-ups along diagonals corresponding to daughters of V. For any daughter A of V, the above identification identifies the normal bundle to $\Delta_A\left(\overline{\mathcal{S}}_V(T)\right)$ with $T^{A\setminus\{b(A)\}}$, and the corresponding blow-up replaces the factor $T^{A\setminus\{b(A)\}}_{<\varepsilon}$ with $[0,\varepsilon[\times\mathbb{S}(T^{A\setminus\{b(A)\}})$. Thus it is clear that the blow-ups corresponding to different daughters of V commute. Note that our argument also proves the following lemma.

Lemma 8.21. *Let A be a subset of V. The fiber of the unit normal bundle to $\Delta_A(\overline{\mathcal{S}}_V(T))$ is $\overline{\mathcal{S}}_A(T)$.*

□

When performing the blow-ups along the diagonals corresponding to the daughters of V, we replace $\mu_A\tilde{w}_A \in T^{A\setminus\{b(A)\}}$, for $\mu_A \in [0,\varepsilon[$ and $\tilde{w}_A \in \mathbb{S}(T^{A\setminus\{b(A)\}})$ with $(\mu_A,\tilde{w}_A)$. Thus we replace 0 with the set of normal vectors $\tilde{w}_A$ that pop up during the blow-up.

Lemma 8.22. *In particular, with the notation of Lemma 8.20, we get a chart of*

$$\mathcal{B}\ell\left(O(V,K(V),b,T),(\Delta_A(\overline{\mathcal{S}}_V(T))\cap O(V,K(V),b,T))_{A\in D(V)}\right),$$

which maps

$$\left(w,(\mu_A,\tilde{w}_A)_{A\in D(V)}\right)\in N(w_0)\times\prod_{A\in D(V)}([0,\varepsilon[\times\mathbb{S}(T^{A\setminus\{b(A)\}}))$$

to the element

$$\left(w+\sum_{A\in D(V)}\mu_A\tilde{w}_A,(\tilde{w}_A)_{A\in D(V)|\mu_A=0}\right)$$

of

$$\mathcal{B}\ell\left(O(V,K(V),b,T),\left(\Delta_A(\overline{\mathcal{S}}_V(T))\,(\cap O(V,K(V),b,T)\right)_{A\in D(V)}\right),$$

where the $\tilde{w}_A$ are the normal vectors viewed in $\overline{\mathcal{S}}_A(T)$ that popped up during the blow-ups.

We can construct an atlas of $\mathcal{B}\ell\left(O(V,K(V),b,T),(\Delta_A(\overline{\mathcal{S}}_V(T)))_{A\in D(V)}\right)$ with charts of this form.

In order to conclude and get charts of manifolds with ridges, we blow up the

$$\overline{\mathcal{S}}_A(T) \cong \mathbb{S}(T^{A\setminus\{b(A)\}})$$

for the daughters A of V, and we iterate. Such an iteration produces a parenthesization of V as in Definition 8.2.

Definition 8.23. A Δ-*parenthesization* of V is a parenthesization $\mathcal{P}$ of V such that $V \in \mathcal{P}$. The *daughters* of an element A of a parenthesization $\mathcal{P}$ (with respect to $\mathcal{P}$) are the maximal elements of $\mathcal{P}$ strictly included in A. The *mother* of an element A in $\mathcal{P}$ is the smallest element of $\mathcal{P}$ that contains A strictly. A Δ-parenthesization $\mathcal{P}$ is organized as a tree, in which the vertices are the elements of $\mathcal{P}$ and the edges are in one-to-one correspondence with the pairs (daughter,mother) of $\mathcal{P}^2$. We orient its edges from the daughter to her mother, as in Figure 8.3. The *sons* of an element A of $\mathcal{P}$ are the singletons consisting of elements of A that do not belong to a daughter of $\mathcal{P}$. Any element A of $\mathcal{P}$ is equipped with the set $K(A,\mathcal{P})$ ($= K(A)$ when $\mathcal{P}$ is fixed) of the *kids* of A, which are its daughters and its sons.

Example 8.24. The trees associated to the parenthetizations $\{V, A_{123}, A_{23}\}$ and $\{V, A_{12}, A_{34}\}$, which correspond to Figures 8.1 and 8.2 in Section 8.1, are pictured in Figure 8.3. With respect to the parenthetization $\{V, A_{123}, A_{23}\}$, the daughter A_{123} of V has two kids, its son $\{1\}$ and its daughter A_{23}.

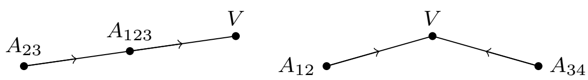


Figure 8.3: Trees associated to Δ-parenthesizations

Let $\mathcal{T}(V)$ (resp. $\mathcal{T}_\Delta(V)$) denote the set of parenthesizations (resp. Δ-parenthesizations) of V. Fix $\mathcal{P} \in \mathcal{T}_\Delta(V)$. For any $A \in \mathcal{P}$ choose a basepoint $b(A) = b(A;\mathcal{P})$, such that if A and B are in $\mathcal{P}$, if $B \subset A$, and if $b(A) \in B$, then $b(B) = b(A)$. When $A \in \mathcal{P}$, $D(A)$ denotes the set of daughters of A. The basepoint $b(B)$ of a son B of $A \in \mathcal{P}$ is its unique point. A Δ-parenthesization equipped with basepoints as above is called a *based* Δ-*parenthesization*.

Recall the canonical identification of the set $O(A, K(A), b, T)$ of Definition 8.19 with an open subset of $\overline{\mathcal{S}}_A(T)$. Set

$$W_A = O(A, K(A), b, T) \cap \cap_{B \in D(A)} \Delta_B(\overline{\mathcal{S}}_A(T)).$$

Note that W_A may be identified canonically with an open subset of the sphere $\overline{\mathcal{S}}_{K(A)}(T)$.

For $((\mu_A)_{A\in\mathcal{P}\setminus\{V\}},(w_A)_{A\in\mathcal{P}}) \in (\mathbb{R}^+)^{\mathcal{P}\setminus\{V\}} \times \prod_{A\in\mathcal{P}} W_A$, and for $B \in \mathcal{P}$, define $v_B((\mu_A)_{A\in\mathcal{P}\setminus\{V\}},(w_A)_{A\in\mathcal{P}})$ as the following map from B to T.

$$\begin{aligned} v_B((\mu_A),(w_A)) &= \sum_{C\in\mathcal{P}|C\subseteq B}\left(\prod_{D\in\mathcal{P}|C\subseteq D\subset B}\mu_D\right) w_C \\ &= w_B + \sum_{C\in D(B)}\mu_C\left(w_C + \sum_{D\in D(C)}\mu_D\left(w_D+\dots\right)\right). \end{aligned}$$

The construction of v_B is illustrated in Figure 8.4. It may be seen on the subgraph of the tree that corresponds to the subsets of B, as the sum over the vertices C of this tree of the maps w_C, associated to its vertices, multiplied by the products of the coefficients μ_D, associated to the edges of the path from C to B.

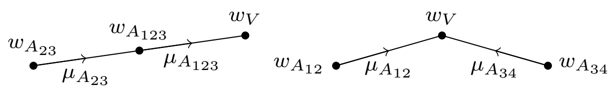


Figure 8.4: About the construction of v_B

Note that $v_B((\mu_A),(w_A)) \in O(B,K(B),b,T)$, when the μ_A are small enough. Also note the following easy lemma.

Lemma 8.25. *For any $(w_A^0)_{A\in\mathcal{P}} \in \prod_{A\in\mathcal{P}} W_A$, there exists a neighborhood $N((w_A^0))$ of $0 \times (w_A^0)$ in $(\mathbb{R}^+)^{\mathcal{P}\setminus\{V\}} \times \prod_{A\in\mathcal{P}} W_A$ such that,*
for any $((\mu_A)_{A\in\mathcal{P}\setminus\{V\}},(w_A)_{A\in\mathcal{P}}) \in N((w_A^0))$,

- *if $v_V((\mu_A),(w_A))$ is constant on B for a subset B of V, then B is (non-necessarily strictly) included in a daughter of V,*
- *if $\mu_A \neq 0$ for any $A \in \mathcal{P}$, then $v_V((\mu_A),(w_A))$ is an injective map from V to T.*

□

When the construction of Theorem 8.11 makes sense, a point of our configuration space $\mathcal{S}_V(T)$ is denoted as a tuple

$$\left(v_V((\mu_A),(w_A)),(v_B((\mu_A),(w_A)))_{B|\mu_B=0}\right),$$

which contains its blowdown projection $v_V((\mu_A),(w_A))$ in $\overline{\mathcal{S}}_V(T)$ followed by the normal vectors $v_B((\mu_A),(w_A))$ that have popped up during the blow-ups. Lemma 8.21 ensures that the normal vectors are non-constant maps from some B to T up to translation and dilation, which are elements of $\overline{\mathcal{S}}_B(T)$.

Theorem 8.26. *Theorem 8.11 is correct and defines $\mathcal{S}_V(T)$. As a set, $\mathcal{S}_V(T)$ is the disjoint union over the Δ-parenthesizations $\mathcal{P}$ of V of the $\mathcal{S}_{V,\mathcal{P}}(T)$, where $\mathcal{S}_{V,\mathcal{P}}(T)$ is canonically diffeomorphic to*

$$\prod_{A\in\mathcal{P}} \check{\mathcal{S}}_{K(A)}(T)$$

and contains the elements that have been transformed by the blow-ups along the closures of the $(\Delta_A(T))_{A\in\mathcal{P}\setminus\{V\}}$, and that have not been transformed by other blow-ups. The composed blowdown map maps $((w_A)_{A\in\mathcal{P}})$ to the map that maps an element of a kid B of V to $w_V(B)$, and the other w_A are (similarly identified with) the unit normal vectors that have appeared during the blow-ups.

Any based Δ-parenthesization $\mathcal{P}$ of V and any $(w_A^0)_{A\in\mathcal{P}} \in \prod_{A\in\mathcal{P}} W_A$ provide a smooth open embedding $\psi(\mathcal{P},(w_A^0)_{A\in\mathcal{P}})$ from a neighborhood $N((w_A^0))$ as in Lemma 8.25 to $\mathcal{S}_V(T)$:

$$\begin{array}{lcl} N((w_A^0)) & \hookrightarrow & \mathcal{S}_V(T) \\ ((\mu_A)_{A\in\mathcal{P}\setminus\{V\}},(w_A)_{A\in\mathcal{P}}) & \mapsto & \left(\begin{array}{l} v_V((\mu_A),(w_A)), \\ (v_B((\mu_A),(w_A)))_{\{B\in\mathcal{P}\setminus\{V\}\mid\mu_B=0\}} \end{array}\right), \end{array}$$

which restricts to $N((w_A^0)) \cap \left((\mathbb{R}^{+*})^{\mathcal{P}\setminus\{V\}} \times \prod_{A\in\mathcal{P}} W_A\right)$ as a diffeomorphism onto an open subset of $\check{\mathcal{S}}_V(T)$. Furthermore, the open images of the embeddings $\psi(\mathcal{P},(w_A^0)_{A\in\mathcal{P}})$ for different $(\mathcal{P},(w_A^0)_{A\in\mathcal{P}})$ cover $\mathcal{S}_V(T)$, and the codimension of $\mathcal{S}_{V,\mathcal{P}}(T)$ in $\mathcal{S}_V(T)$ equals the cardinality of $\mathcal{P}\setminus\{V\}$.

PROOF: Note that the images of the embeddings that correspond to $\mathcal{P} = \{V\}$ cover $\check{\mathcal{S}}_V(T)$ trivially. The theorem is obviously true when $\sharp V = 2$.

Proceed by induction on $\sharp V$, by multiplying the charts of Lemma 8.22 by the charts provided by the theorem for the daughters of V. Let A be a daughter of V. By induction, $\mathcal{S}_A(T)$ is covered by charts $\psi(\mathcal{P}_A,(w_B^0)_{B\in\mathcal{P}_A})$ from $N\left((w_B^0)_{B\in\mathcal{P}_A}\right)$ to some open subset U_A of $\mathcal{S}_A(T)$ associated with parenthetisations $\mathcal{P}_A$ of A, as in the theorem.

Then elements of U_A will be "multiplied" by real numbers in some interval $[0,\eta[$. This multiplication makes sense with the chosen normalizations, which are different in the theorem and in the statement of Lemma 8.22, where $\mathcal{S}_A(T)$ was identified with $\mathbb{S}(T^{A\setminus\{b(A)\}})$. An element w of $O(A,K(A),b,T)$ must be multiplied by $\frac{1}{g(w)}$, where

$$g(w) = \sqrt{\sum_{a\in A} \parallel w(a) \parallel^2},$$

to become an element of $\mathbb{S}(T^{A\setminus\{b(A)\}})$. Note that g is a smooth function and that

$$1 \leq g(v_A((\mu_B),(w_B))) \leq 2\sqrt{\sharp A}$$

as soon as $N\left((w_B^0)_{B\in\mathcal{P}_A}\right)$ is a subset of $[0,\frac{1}{2}]^{\mathcal{P}_A\setminus\{A\}} \times \prod_{B\in\mathcal{P}_A} W_B$; that is assumed without loss of generality. Then the charts of the statement are obtained from the charts of Lemma 8.22 by the smooth replacement of

$$(ug(w), \frac{1}{g(w)}w) \in [0,\varepsilon[\times\mathbb{S}(T^{A\setminus\{b(A)\}})$$

with

$$(u,w) \in [0,\frac{\varepsilon}{2\sqrt{\sharp A}}[\times N\left((w_B^0)_{B\in\mathcal{P}_A}\right),$$

for $(u,w) \in [0,\frac{\varepsilon}{2\sqrt{\sharp A}}[\times N\left((w_B^0)_{B\in\mathcal{P}_A}\right)$, for charts as above, for each $A \in D(V)$. Such charts cover the images of

$$N(w_0) \times \prod_{A\in D(V)} \left([0,\frac{\varepsilon}{2\sqrt{\sharp A}}[\times\mathbb{S}(T^{A\setminus\{b(A)\}})\right)$$

in the charts of Lemma 8.22. Thus all together, our charts cover $\mathcal{S}_V(T)$. The other assertions are easy to check from the proof. $\square$

The above proof also proves the following two propositions, with the notation ∂_r introduced in the beginning of Section 2.1.4.

Proposition 8.27. *Let $\mathcal{T}_{r,\Delta}(V)$ be the subset of $\mathcal{T}_\Delta(V)$ consisting of the Δ-parenthesizations of V of cardinality r. Then*

$$\partial_{r-1}(\mathcal{S}_V(T)) \setminus \partial_r(\mathcal{S}_V(T)) = \sqcup_{\mathcal{P}\in\mathcal{T}_{r,\Delta}(V)}\mathcal{S}_{V,\mathcal{P}}(T).$$

For two Δ-parenthesizations $\mathcal{P}$ and $\mathcal{P}'$, if $\mathcal{P} \subset \mathcal{P}'$, then $\mathcal{S}_{V,\mathcal{P}'}(T) \subset \overline{\mathcal{S}_{V,\mathcal{P}}(T)}$.

Proof: The first assertion can be deduced from the charts of Theorem 8.26. Let $c_0 = (w_A)_{A\in\mathcal{P}'} \in \mathcal{S}_{V,\mathcal{P}'}(T)$ and let $\Psi = \psi(\mathcal{P}',(w_A)_{A\in\mathcal{P}'})\colon N((w_A)) \to \mathcal{S}_V(T)$ be a smooth open embedding as in Theorem 8.26. Let $\varepsilon \in]0,\infty[$ be such that $[0,\varepsilon]^{\mathcal{P}'\setminus\{V\}} \times \{(w_A)_{A\in\mathcal{P}'}\} \subset N((w_A)_{A\in\mathcal{P}'})$. For $t \in [0,1]$, set

$$\mu_A(t) = \begin{cases} \varepsilon t & \text{if } A \in \mathcal{P}' \setminus (\{V\}\cup\mathcal{P}) \\ 0 & \text{if } A \in \mathcal{P} \end{cases}$$

and let $c(t) = \Psi\left((\mu_A(t))_{A\in\mathcal{P}'\setminus\{V\}},(w_A)_{A\in\mathcal{P}'}\right)$. Then $c(t) \in \mathcal{S}_{V,\mathcal{P}}(T)$ for any $t \in]0,1]$ and $\lim_{t\to 0} c(t) = c_0$. $\square$

Proposition 8.28. *Any injective linear map ϕ from a vector space T to another such T' induces a canonical embedding $\phi_*\colon \mathcal{S}_V(T) \to \mathcal{S}_V(T')$. This embedding maps an element $((w_A)_{A\in\mathcal{P}})$ of $\mathcal{S}_{V,\mathcal{P}}(T)$ to the element $((\phi \circ w_A)_{A\in\mathcal{P}})$ of $\mathcal{S}_{V,\mathcal{P}}(T')$. If ψ is another injective linear map from a vector space T' to a third vector space T'', then $(\psi\circ\phi)_* = \psi_* \circ \phi_*$.*

□

Finally, let us check the following proposition, which implies Theorem 8.12.

Proposition 8.29. *Let A be a finite subset of cardinality at least 2 of a finite set V. For $\mathcal{P} \in \mathcal{T}_\Delta(V)$, define*

$$\mathcal{P}_A = \{B \cap A \mid B \in \mathcal{P}, \sharp(B\cap A) \geq 2\}$$

in $\mathcal{T}_\Delta(A)$, and for $C \in \mathcal{P}_A$, let $\hat{C}$ be the smallest element of $\mathcal{P}$ that contains C or is equal to C. For

$$w = (w_B \in \check{\mathcal{S}}_{K(B)}(T))_{B\in\mathcal{P}} \in \mathcal{S}_{V,\mathcal{P}}(T),$$

and for $C \in \mathcal{P}_A$, define w'_C as the natural restriction of $w_{\hat{C}}$ to $K(C,\mathcal{P}_A)$. Then set

$$p_A(w) = ((w'_C)_{C\in\mathcal{P}_A}) \in \mathcal{S}_{A,\mathcal{P}_A}(T).$$

This consistently defines a smooth map

$$p_A\colon \mathcal{S}_V(T) \to \mathcal{S}_A(T).$$

The map p_A is the unique continuous extension from $\mathcal{S}_V(T)$ to $\mathcal{S}_A(T)$ of the restriction map from $\check{\mathcal{S}}_V(T)$ to $\check{\mathcal{S}}_A(T)$.

PROOF: It is easy to see that this restriction map is well defined. In order to prove that it is smooth, use the charts of Theorem 8.26. Fix a Δ-parenthesization $\mathcal{P}$ of V, and the induced Δ-parenthesization $\mathcal{P}_A$ of A. Fix basepoints b_A for $\mathcal{P}_A$ according to the rule stated in Definition 8.23, and fix basepoints for $\mathcal{P}$ so that when $B \in \mathcal{P}$ and when $B \cap A \in \mathcal{P}_A$, then $b(B) = b_A(A \cap B)$, and if $B \cap A \neq \emptyset$, then $b(B) \in B \cap A$. According to Theorem 8.26, it suffices to prove smoothness in charts involving the based Δ-parenthesizations $\mathcal{P}$ and $\mathcal{P}_A$. So it suffices to prove that the projections on the factors that contain the restrictions maps w'_C are smooth, and that the projections on the factors that contain the dilation factors μ_C are smooth, for $C \in \mathcal{P}_A \setminus \{A\}$.

Again, with our charts and with our conditions on the basepoints, for any $C \in \mathcal{P}_A$,

$$w'_C = \frac{1}{g(C, w_{\hat{C}})} w_{\hat{C}|K(C,\mathcal{P}_A)},$$

where

$$g(C, w_{\hat{C}}) = \sqrt{\sum_{D \in K(C,\mathcal{P}_A)} \parallel w_{\hat{C}}(D) \parallel^2}$$

is not zero, since $w_{\hat{C}}$ is non constant on $K(C, \mathcal{P}_A)$, and we immediately see that the projection on the factor of w'_C is smooth.

Let $C \in \mathcal{P}_A \setminus \{A\}$, and let $m(C)$ denote the *mother* of C in $\mathcal{P}_A$. Let E be the set of basepoints of the kids of $m(C)$ distinct from C with respect to $\mathcal{P}_A$. Consider the elements $B_i \in \mathcal{P}$, for $i = 1, 2, \ldots, k(B)$, such that $C = B_i \cap A$ for any $i \leq k(C)$, where $(\hat{C} = B_1) \subset B_2 \cdots \subset B_{k(C)}$.

Then the restrictions of the configurations $w'_{m(C)} + \mu_C w'_C$ and $w_{\widehat{m(C)}} + \left(\prod_{i=1}^{k(C)} \mu_{B_i}\right) w_{\hat{C}}$ to $C \cup E$ coincide up to dilation. So $g(m(C), w_{\widehat{m(C)}})\mu_C w'_C$ coincides with $\left(\prod_{i=1}^{k(C)} \mu_{B_i}\right) g(C, w_{\hat{C}}) w'_C$ on C, and

$$\mu_C = \left(\prod_{i=1}^{k(C)} \mu_{B_i}\right) \frac{g(C, w_{\hat{C}})}{g(m(C), w_{\widehat{m(C)}})}.$$

Thus μ_C is smooth (it is defined even when the μ_{B_i} are negative). $\square$

Proposition 8.13 follows from Propositions 8.27 and 8.29. $\square$

Proof of Lemma 8.16: The structure of a smooth manifold with ridges of $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$ in Lemma 8.15 can be deduced from the charts of Theorem 8.26, alternatively. These charts also show that $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$ is a submanifold transverse to the ridges of $\mathcal{S}_{V(\check{\Gamma})}(T_{L(z)}\check{R})$. So its codimension-one faces are the intersections of $\mathcal{S}(T_{L(z)}\check{R}, \vec{t}_{L(z)}; \check{\Gamma})$ with the codimension-one faces of $\mathcal{S}_{V(\check{\Gamma})}(T_{L(z)}\check{R})$. Then Lemma 8.16 follows from Proposition 8.13. $\square$

8.6 Blowing up diagonals

In the rest of this chapter, M is a smooth manifold without boundary of dimension $\delta > 0$. It is not necessarily oriented. The set of injective maps from V to M is denoted by $\check{C}_V[M]$ with brackets instead of parentheses (so $\check{C}_V[\check{R}] = \check{C}_V(\check{R})$, but $\check{C}_V[R] \neq \check{C}_V(R)$).

Theorem 8.30. *Let M be a manifold of dimension δ. Let V be a finite set.*

Start with M^V. For $k = \sharp V, \ldots, 3, 2$, in this decreasing order, successively blow up the closures of (the preimages under the composition of the previous blowdown maps of) the $\Delta_A(M^V)$ such that $\sharp A = k$. The successive manifolds that are blown-up in the above process are smooth, and transverse to the *ridges. The resulting iterated blown-up manifold does not depend on the order choices. Thus this process gives rise to a canonical smooth manifold* $C_V[M]$ *with ridges equipped with its composed blowdown projection*

$$p_b \colon C_V[M] \to M^V.$$

- *Let* $f \in M^V$ *be a map from* V *to* M*. Then* $p_b^{-1}(f)$ *is canonically diffeomorphic to* $\prod_{A \in D(V;f)} \mathcal{S}_A(T_{f(A)}M)$*. An element* $x \in p_b^{-1}(f)$ *of* $C_V[M]$ *will be denoted as* $(f \in M^V, (w_A \in \mathcal{S}_A(T_{f(A)}M))_{A \in D(V;f)})$*, with the notation of Definition 8.19.*

- *For any open subset* U *of* M*,* $C_V[U] = p_b^{-1}(U^V)$*.*

- $\check{C}_V[M]$ *is dense in* $C_V[M]$*.*

- *If* M *is compact, then* $C_V[M]$ *is compact, too.*

- *Any choice of a basepoint* $b(V)$ *of* V *and of an open embedding*

  $$\phi \colon \mathbb{R}^\delta \to M$$

  induces the diffeomorphism

  $$\psi(\phi, b) \colon \mathbb{R}^\delta \times \mathbb{R}^+ \times \mathcal{S}_V(\mathbb{R}^\delta) \to C_V[\phi(\mathbb{R}^\delta)]$$

  described below.
  Smoothly identify $\check{\mathcal{S}}_V(\mathbb{R}^\delta)$ *with an open subset of* $\mathbb{S}((\mathbb{R}^\delta)^{V \setminus \{b(V)\}})$*. Let* $u \in \mathbb{R}^\delta$*,* $\mu \in \mathbb{R}^+$ *and* $n \in \mathcal{S}_V(\mathbb{R}^\delta)$*. Then*

  $$p_b(\psi(\phi, b)(u, \mu, n)) = \phi \circ (u + \mu p_b(n)),$$

  where $p_b(n)$ *is viewed as a map from* V *to* $\mathbb{R}^\delta$*, such that* $p_b(n)(b(V)) = 0$ *and* $\sum_{v \in V} \parallel p_b(n)(v) \parallel^2 = 1$*, and* $(u + \mu p_b(n))$ *denotes the map from* V *to* $\mathbb{R}^\delta$*, obtained from* $p_b(n)$ *by composition by the homothety with ratio* μ*, followed by the translation of vector* u*. For* $u \in \mathbb{R}^\delta$ *and* $n \in \check{\mathcal{S}}_V(\mathbb{R}^\delta)$*,* $p_b(\psi(\phi, b)(u, 0, n))$ *is a constant map with value* $\phi(u)$ *and*

  $$\psi(\phi, b)(u, 0, n) = (\phi(u)^V, (T_u\phi)_*(n)).$$

  The restriction of $\psi(\phi, b)$ *to* $\mathbb{R}^\delta \times \mathbb{R}^{+*} \times \check{\mathcal{S}}_V(\mathbb{R}^\delta)$ *is a diffeomorphism onto* $\check{C}_V[\phi(\mathbb{R}^\delta)]$*.*

- *An embedding* ϕ *of a manifold* M_1 *into another such* M_2 *induces the following canonical embedding* ϕ_* *from* $C_V[M_1]$ *to* $C_V[M_2]$*.*

  $$\phi_*(f \in M_1^V, (w_A \in \mathcal{S}_A(T_{f(A)}M_1))_{A \in D(V;f)}) = (\phi \circ f, (T_{f(A)}\phi)_*(w_A)).$$

  If ψ *is an embedding from* M_2 *to another manifold* M_3*, then* $(\psi \circ \phi)_* = \psi_* \circ \phi_*$*.*

Proof: Start with $M = \mathbb{R}^\delta$ equipped with its usual Euclidean norm, and fix $b(V)$. Any map f from V to $\mathbb{R}^\delta$ may be written as $f(b(V)) + y$, for a unique element $y \in (\mathbb{R}^\delta)^{V\setminus\{b(V)\}}$. Then blowing up $(\mathbb{R}^\delta)^V$ along the diagonal $\Delta_V((\mathbb{R}^\delta)^V)$, described by the equation $y = 0$, replaces $(\mathbb{R}^\delta)^{V\setminus\{b(V)\}}$ by $\mathbb{R}^+ \times \mathbb{S}((\mathbb{R}^\delta)^{V\setminus\{b(V)\}})$ and provides a diffeomorphism from $\mathbb{R}^\delta \times \mathbb{R}^+ \times \mathbb{S}((\mathbb{R}^\delta)^{V\setminus\{b(V)\}})$ to $B\ell((\mathbb{R}^\delta)^V, \Delta_V((\mathbb{R}^\delta)^V))$.

The diagonals corresponding to strict subsets of V are products by $\mathbb{R}^\delta \times \mathbb{R}^+$ of the diagonals corresponding to the same subsets for the manifold $\mathbb{S}((\mathbb{R}^\delta)^{V\setminus\{b(V)\}}) \cong \overline{\mathcal{S}}_V(\mathbb{R}^\delta)$, which was studied in the previous subsection.

Thus $C_V[\mathbb{R}^\delta]$ is well defined, and our diffeomorphism lifts to a diffeomorphism from $\mathbb{R}^\delta \times \mathbb{R}^+ \times \mathcal{S}_V(\mathbb{R}^\delta)$ to $C_V[\mathbb{R}^\delta]$. So the composition of this diffeomorphism with the product of the charts obtained in Theorem 8.26 by the identity map yields an atlas of $C_V[\mathbb{R}^\delta]$.

For a diffeomorphism $\phi\colon \mathbb{R}^\delta \to U$, where U is an open subspace of a manifold, the diffeomorphism

$$\phi^V\colon (\mathbb{R}^\delta)^V \to U^V$$

preserves diagonals. So $C_V[U]$ is well defined for any open subset U diffeomorphic to $\mathbb{R}^\delta$, in a manifold M. Furthermore, ϕ^V lifts as a natural diffeomorphism $\phi_*\colon C_V[\mathbb{R}^\delta] \to C_V[U]$.

Note that the elements of $C_V[\mathbb{R}^\delta]$ have the prescribed form. Since the normal bundle to a diagonal $\Delta_A(\check{C}_V[U])$ is $\left(\frac{TU^A}{\Delta_A(TU)} \setminus \{0\}\right)/\mathbb{R}^{+*}$, the diffeomorphism ϕ_* from $C_V[\mathbb{R}^\delta]$ to $C_V[U]$ maps

$$x = \big(f \in (\mathbb{R}^\delta)^V, (w_A \in \mathcal{S}_A(\mathbb{R}^\delta))_{A\in D(V;f)}\big)$$

to

$$\phi_*(x) = (\phi \circ f, (T_{f(A)}\phi)_*(w_A)).$$

Then the elements of $C_V[U]$ have the prescribed form, too.

In order to prove that $C_V[M]$ is well defined for a manifold M, it suffices to see that it is well defined over an open neighborhood of any point c of M^V. Such a map c defines the partition $K(V;c)$. There exist pairwise disjoint open neighborhoods U_A diffeomorphic to $\mathbb{R}^\delta$ of the $c(A)$, for $A \in K(V) = K(V;c)$. It is easy to see that $C_V[M]$ is well defined over $\prod_{A\in K(V)} U_A^A$ and that it is canonically isomorphic to $\prod_{A\in K(V)} C_A[U_A]$, there. Thus $C_V[M]$ is well defined and its elements have the prescribed form.

If ϕ is a diffeomorphism from a manifold M_1 to another such M_2, then the diffeomorphism

$$\phi^V\colon M_1^V \to M_2^V$$

preserves diagonals. So it lifts as a natural diffeomorphism $\phi_*\colon C_V[M_1] \to C_V[M_2]$, which behaves as stated.

Then the study of the map induced by an embedding from a manifold M_1 into another such M_2 can be easily reduced to the case of a linear embedding from $\mathbb{R}^k$ to $\mathbb{R}^\delta$. For those, use the identification of $C_V[\mathbb{R}^\delta]$ with $\mathbb{R}^\delta \times \mathbb{R}^+ \times \mathcal{S}_V(\mathbb{R}^\delta)$.

The other statements are easy to check. $\square$

Proposition 8.31. *Let B be a subset of V, then the restriction from $\check{C}_V[M]$ to $\check{C}_B[M]$ extends uniquely to a smooth map p_B from $C_V[M]$ to $C_B[M]$. Let $f \in M^V$. The elements A of $D(B; f_{|B})$ are of the form $B \cap \hat{A}$ for a unique $\hat{A}$ in $D(V; f)$. Then*

$$p_B(f, (w_C \in \mathcal{S}_C(T_{f(C)}M))_{C \in D(V;f)}) = (f_{|B}, (w_{\hat{A}|A})_{A \in D(B; f_{|B})}).$$

Proof: It is obvious when $\sharp B = 1$. Assume $\sharp B \geq 2$. In order to check that this restriction map is smooth, it is enough to prove that it is smooth when $M = \mathbb{R}^\delta$. Assume $b(V) \in B$. Use the diffeomorphism of Theorem 8.30 to write an element of $C_V[\mathbb{R}^\delta]$ as $(u \in \mathbb{R}^\delta, \mu \in \mathbb{R}^+, n \in \mathcal{S}_V(\mathbb{R}^\delta))$ Then the restriction maps (u, μ, n) to $(u, \| p_b(n)_{|B} \| \mu, n_{|B})$, and it is smooth according to Proposition 8.29. $\square$

An element $x = (f \in M^V, (w_A \in \mathcal{S}_A(T_{f(A)}M))_{A \in D(V;f)})$ of $C_V[M]$ induces the parenthesization $\mathcal{P}(x)$ of V which is the union over the elements A of $D(V; f)$ of the $\mathcal{P}_A(x)$ such that $w_A \in \mathcal{S}_{V,\mathcal{P}_A(x)}(T_{f(A)}M)$. (See Theorem 8.26.) Let $\mathcal{P}$ be a parenthesization of V. Set $C_{V,\mathcal{P}}[M] = \{x \in C_{V,\mathcal{P}}[M] \mid \mathcal{P}(x) = \mathcal{P}\}$.

The following proposition is easy to observe. (See Proposition 8.27.)

Proposition 8.32. *Let $\mathcal{T}_r(V)$ be the subset of $\mathcal{T}(V)$ consisting of the parenthesizations of V of cardinality r. Then*

$$\partial_r(C_{V,\mathcal{P}}[M]) \setminus \partial_{r+1}(C_{V,\mathcal{P}}[M]) = \sqcup_{\mathcal{P} \in \mathcal{T}_r(V)} C_{V,\mathcal{P}}[M]$$

For two parenthesizations $\mathcal{P}$ and $\mathcal{P}'$, if $\mathcal{P} \subset \mathcal{P}'$, then $C_{V,\mathcal{P}'}[M] \subset \overline{C_{V,\mathcal{P}}[M]}$.

$\square$

8.7 Blowing up ∞

We state and prove the following generalization of Theorem 8.4.

Theorem 8.33. *Let V be a finite set. Let M be a manifold without boundary of dimension δ, and let $\infty \in M$. Set $\check{M} = M\setminus\{\infty\}$. For a non-empty $A \subseteq V$, let Ξ_A be the set of maps from V to M that map A to ∞, and $V \setminus A$ to $\check{M}$.*

Start with M^V. Blow up Ξ_V, which is reduced to the point $m = \infty^V$ such that $m^{-1}(\infty) = V$.
Then for $k = \sharp V, \sharp V-1, \dots, 3, 2$, in this decreasing order, successively blow up the closures of (the preimages under the composition of the previous blowdown maps of) the $\Delta_A(\check{M}^V)$ such that $\sharp A = k$ (choosing an arbitrary order among them) and, next, the closures of (the preimages under the composition of the previous blowdown maps of) the Ξ_J such that $\sharp J = k-1$ (again choosing an arbitrary order among them). The successive manifolds that are blown-up in the above process are smooth, at a given step, and transverse to the ridges. The obtained manifold $C_V[M,\infty]$ is a smooth $(\delta\sharp V)$-manifold, with ridges, independent of the possible order choices in the process. It is compact if M is compact. The interior of $C_V[M,\infty]$ is the space $\check{C}_V[M,\infty]$ of injective maps from V to $M \setminus \infty$, and the composition of the blowdown maps gives rise to a canonical smooth blowdown projection $p_b \colon C_V[M,\infty] \to M^V$.

When $(M,\infty) = (R,\infty)$, $\check{C}_V[\check{R}] = \check{C}_V(\check{R}) = \check{C}_V[M,\infty]$, and $C_V(R) = C_V[M,\infty]$.

Assume that $M = \mathbb{R}^\delta$, which is equipped with its standard Euclidean norm, and that $\infty = 0 \in \mathbb{R}^\delta$. Set $T = \mathbb{R}^\delta$. Then the first blow-up transforms T^V to

$$B\ell(T^V, \Xi_V) = \mathbb{R}^+ \times \mathbb{S}(T^V),$$

where $\mathbb{S}(T^V)$ is the set of maps $f_{\mathbb{S}}$ from V to T, such that $\sum_{v\in V} \parallel f_{\mathbb{S}}(v) \parallel^2 = 1$, and a pair $(u, f_{\mathbb{S}})$, where $u \in \mathbb{R}^{+*}$ and $f_{\mathbb{S}} \in \mathbb{S}(T^V)$, is nothing but the element $uf_{\mathbb{S}}$ of T^V. Note that the closures of the diagonals and the Ξ_A closures are products by $\mathbb{R}^+$ in $B\ell(T^V, \Xi_V)$. The next blow-ups are products by this factor $\mathbb{R}^+$ of blow-ups of $\mathbb{S}(T^V)$. We study how the latter blow-ups affect a neighborhood of $f_{\mathbb{S}}$ in $\mathbb{S}(T^V)$, or rather how these blow-ups affect a neighborhood of the class f_0 of $f_{\mathbb{S}}$ in $S(T^V) \cong \mathbb{S}(T^V)$ since we will use other normalizations.

Definition 8.34. A *special partition* of V is a partition at most one element of which is called special, where the special element cannot be the set V. Any $f_{\mathbb{S}} \in \mathbb{S}(T^V)$ defines the special partition $K^s(V) = K^s(V; f_{\mathbb{S}})$, which is the partition $K(V; f_{\mathbb{S}})$, where the possible preimage of 0 under $f_{\mathbb{S}}$ is called special.

A *based special partition* is a special partition that is based so that $b(V)$ is in the special element if there is a special element. Let A be a subset of V equipped with a based special partition $K^s(A)$. If $K^s(A)$ has a special element, let A^s denote this special element. Otherwise, set $A^s = \emptyset$. The set of non-special elements of $K^s(A)$ is denoted by $K^s_d(A)$, and the set of non-special elements of $K^s(A)$ with cardinality at least 2 is denoted by $D^s_d(A)$.

Define the open subset $O^s(A, K^s(A), b, T)$ of $S(T^A)$ to be the set of maps $f\colon V \longrightarrow T$ such that

- $\sum_{B \in K^s(A)} \parallel f(b(B)) \parallel^2 = 1$
- $f(V \setminus A) = \{0\}$,
- two elements of A that belong to different kids of A are mapped to different points of T by f, and
- $0 \notin f(A \setminus A^s)$.

For an element $f_{\mathbb{S}}$ of $\mathbb{S}(T^V)$, $O^s(V, K^s(V; f_{\mathbb{S}}), b, T)$ is an open neighborhood of the class f_0 of $f_{\mathbb{S}}$ in $S(T^V)$. The only diagonal closures that intersect $\mathbb{R}^+ \times O^s(V, K^s(V) = K^s(V; f_{\mathbb{S}}), b, T)$ correspond to non-strict subsets of the elements of $K^s(V)$, and if the closure of Ξ_B intersects $\mathbb{R}^+ \times O^s(V, K^s(V), b, T)$, then $B \subseteq V^s$.

Fix a based special partition $K^s(V)$. Set

$$W^s(K^s(V)) = O^s(V, K^s(V), b, T) \cap \Xi_{V^s} \cap \cap_{A \in D^s_d(V)} \Delta_A(T^V),$$

where $\Xi_\emptyset = S(T^V)$. Note that W^s is an open subset of $S(T^{K^s_d(V)})$. For $A \subset V$, view the elements of T^A as the maps from V to T that map $(V \setminus A)$ to 0, and let $T^A_{<\varepsilon}$ denote the open ball of radius ε of T^A. Note the easy lemma.

Lemma 8.35. *Let $K^s(V)$ be a based special partition of V. Let f_0 be an element of $W^s(K^s(V))$. Assume that there is no special element in $K^s(V)$. Then there exists an open neighborhood $N(f_0)$ of f_0 in $W^s(K^s(V))$, and an $\varepsilon \in]0, \infty[$ such that the map*

$$\begin{array}{lcl} N(f_0) \times \prod_{A \in D^s_d(V)} T^{A \setminus \{b(A)\}}_{<\varepsilon} & \to & O^s(V, K^s(V), b, T) \\ (f, (\mu_A \tilde{w}_A)_{A \in D^s_d(V)}) & \mapsto & f + \sum_{A \in D^s_d(V)} \mu_A \tilde{w}_A \end{array}$$

is an open embedding, whose image does not meet diagonals that do not correspond to (non-necessarily strict) subsets B of daughters of V, and does not meet any Ξ_B either.

□

Then the only blow-ups that will affect this neighborhood of f_0 in $S(T^V)$ are blow-ups along diagonals corresponding to daughters of V and to their subsets, which will replace the factors $T^{A \setminus \{b(A)\}}_{<\varepsilon}$ with $[0, \varepsilon[\times \mathcal{S}_A(T)$, as before.

Lemma 8.36. *Let $K^s(V)$ be a based special partition of V. Let f_0 be an element of $W^s(K^s(V))$. Assume that there is a special element V^s in $K^s(V)$. Then there exists an open neighborhood $N(f_0)$ of f_0 in $W^s(K^s(V))$, and an $\varepsilon \in]0,\infty[$ such that the map*

$$\begin{array}{lcl} N(f_0) \times \prod_{A\in D^s_d(V)} T^{A\setminus\{b(A)\}}_{<\varepsilon} \times T^{V^s}_{<\varepsilon} & \to & O^s(V,K^s(V),b,T) \\ (f,(\mu_A\tilde{w}_A)_{A\in D^s_d(V)},u_s\tilde{f}_s) & \mapsto & f+\sum_{A\in D^s_d(V)}\mu_A\tilde{w}_A + u_s\tilde{f}_s \end{array}$$

is an open embedding, whose image does not meet diagonals that do not correspond to (non-necessarily strict) subsets B of daughters of V and does not meet any Ξ_B such that B is not a (non-necessarily strict) subset of V^s either.

□

Proof of Theorem 8.33: The only blow-ups that will affect the neighborhood of f_0 in $S(T^V)$ that corresponds to the neighborhood $N(f_0)$ of Lemma 8.36 are blow-ups along diagonals corresponding to non-special daughters of V and to their subsets, which will replace the factors $T^{A\setminus\{b(A)\}}_{<\varepsilon}$ by $[0,\varepsilon[\times\mathcal{S}_A(T)$, as before, and blow-ups, which will act on the factor $T^{V^s}_{<\varepsilon}$. These latter blow-ups are the studied blow-ups, where V is replaced by the subset V^s of smaller cardinality. The theorem follows by induction on $\sharp V$ when M is $\mathbb{R}^\delta$, and $\infty = 0$.

Let ϕ be a diffeomorphism from $\mathbb{R}^\delta$ onto an open subset U of M such that $\phi(0)=\infty$, then $\phi^V\colon (\mathbb{R}^\delta)^V \to U^V$ preserves both diagonals and submanifolds Ξ_B. So the theorem follows, when $M=U$, and again it easily follows that the process is well defined over products $\prod_{A\in K(V)} U_A^A$ for a partition $K(V)$ of V, and disjoint open subsets U_A of M, such that, if one of them contains ∞, then it is contained in U. □

Since the construction of Theorem 8.33 makes sense, a point of our configuration space $C_V[M,\infty]$ is denoted as a tuple, which contains its blowdown projection in M^V followed by the normal vectors that have popped up during the blow-ups. Here, the normal bundle to Ξ_B is the set $S((T_\infty M)^B)$ of nonzero maps from B to $T_\infty M$ up to dilation.

In order to describe an element x of

$$p_b^{-1}(\infty^V) \subset C_V[M,\infty],$$

explicitly, we record the blow-ups that affected x by recording the set $\mathcal{P}_s(x)$ of subsets B of V such that the element was transformed by the blow-up along (the closure of the preimage of) Ξ_B and the set $\mathcal{P}_d(x)$ of subsets B of V such that the element was transformed by the blow-up along (the closure of the preimage of) $\Delta_B(M^V)$.

Definition 8.37. An ∞-*parenthesization* $(\mathcal{P}_s, \mathcal{P}_d)$ of a set V is a set $\mathcal{P} = \{A_i \mid i \in I\}$ of subsets of V, each of cardinality at least one, such that,

- for any two distinct elements i, j of I one of the following holds $A_i \subset A_j$, $A_j \subset A_i$ or $A_i \cap A_j = \emptyset$,
- $V \in \mathcal{P}$,
- $\mathcal{P}$ is equipped with a non-empty subset
$$\mathcal{P}_s = \{V = V(1), V(2), \ldots, V(\sigma)\},$$
where $V(i+1) \subset V(i)$, of *special elements* such that if $A \in \mathcal{P}_s$, $B \in \mathcal{P}$ and $A \subset B$, then $B \in \mathcal{P}_s$,
- $\mathcal{P}$ is equipped with a subset $\mathcal{P}_d$ of *degenerate elements*, which is either $\mathcal{P} \setminus \mathcal{P}_s$ or $(\mathcal{P} \setminus \mathcal{P}_s) \cup \{V(\sigma)\}$,
- the cardinality of all elements of $\mathcal{P} \setminus \{V(\sigma)\}$ is greater than one, and if $\sharp V(\sigma) = 1$, then $V(\sigma) \notin \mathcal{P}_d$.

Note that $\mathcal{P} = \mathcal{P}_s \cup \mathcal{P}_d$ in any case. Again, the daughters of an element of A in $\mathcal{P}_d$ are the maximal elements of $\mathcal{P}_d$ strictly included in A (i.e. the elements B of $\mathcal{P}_d$, strictly included in A, such that there is no other element of $\mathcal{P}_d$ between A and B). Their set is again denoted by $D(A)$. The *sons* of an element A of $\mathcal{P}_d$ are the singletons consisting of elements of A that do not belong to a daughter of $\mathcal{P}_d$, the *kids* of such an A are its daughters and its sons, and $K(A)$ denotes the set of kids of A.

For any $i \in \underline{\sigma}$, the set $D^s_d(V(i))$ of non-special daughters of $V(i)$ is the set of maximal elements of $\mathcal{P}_d$ (non-necessarily strictly) included in $V(i) \setminus V(i)^s$, where $V(i)^s = V(i+1)$ if $i < \sigma$ and $V(\sigma)^s = \emptyset$. (If $V(\sigma) \in \mathcal{P}_d$, then $D^s_d(V(\sigma)) = \{V(\sigma)\}$.) The set $K^s_d(V(i))$ of non-special kids of $V(i)$ is the union of $D^s_d(V(i))$ and the set of the non-special sons of $V(i)$, which are the singletons consisting of elements of $V(i) \setminus V(i)^s$ that do not belong to a daughter of $V(i)$. If $i \neq \sigma$, $(V(i+1) = V(i)^s) \in K^s(V(i))$ is the special kid of $V(i)$, and $K^s(V(i)) = K^s_d(V(i)) \cup \{V(i+1)\}$. $K^s_d(V(\sigma)) = K^s(V(\sigma))$.

Example 8.38. For an ∞-parenthesization $(\mathcal{P}_s, \mathcal{P}_d)$, the elements of $\mathcal{P}_s \sqcup \mathcal{P}_d$ may be drawn as the vertices of a tree, as in Figure 8.5, where the elements of $\mathcal{P}_s$ are on the horizontal top line. The element $V(\sigma)$ corresponds to two vertices of the tree when $V(\sigma) \in \mathcal{P}_d$, one on the top line represents $V(\sigma)$ as an element of $\mathcal{P}_s$, and the other one represents $V(\sigma)$ as an element of $\mathcal{P}_d$, as in the right part of Figure 8.5.

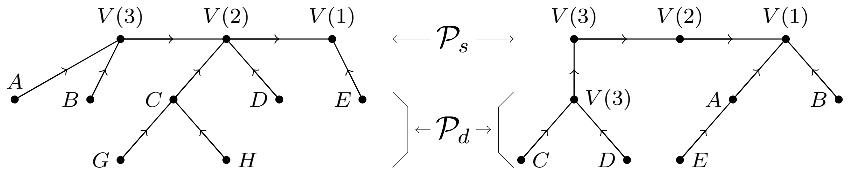


Figure 8.5: Trees associated to ∞-parenthesizations

An ∞-parenthesization $(\mathcal{P}_s,\mathcal{P}_d)$ of V such that $\mathcal{P}_s=\{V(1),\dots,V(\sigma)\}$ is a *based ∞-parenthesization* of V if any $A\in\mathcal{P}=\mathcal{P}_s\cup\mathcal{P}_d$ is equipped with a basepoint $b(A)=b(A;\mathcal{P})$, such that

- $b(V(i))=b(V(\sigma))$ for any $i=1,\dots,\sigma$, and,
- if A and B are in $\mathcal{P}$, if $B\subset A$, and if $b(A)\in B$, then $b(B)=b(A)$.

Let $(\mathcal{P}_s,\mathcal{P}_d)$ be such a based ∞-parenthesization. For $i\in\underline{\sigma}$, set

$$W_i^s=O^s(V(i),K^s(V(i)),b,\mathbb{R}^\delta)\cap\Xi_{V(i)^s}\cap\cap_{B\in D_d^s(V(i))}\Delta_B(\mathbb{R}^\delta).$$

The set W_i^s is canonically identified with an open submanifold of the quotient $S((\mathbb{R}^\delta)^{K_d^s(V(i))})$.

For any $A\in\mathcal{P}_d$, recall the definition of $O(A,K(A),b,\mathbb{R}^\delta)$ from the beginning of Section 8.5, and set

$$W_A=O(A,K(A),b,\mathbb{R}^\delta)\cap\cap_{B\in D(A)}\Delta_B(\overline{\mathcal{S}}_A(\mathbb{R}^\delta))$$

as in Section 8.5. For

$$((u_i)_{i\in\underline{\sigma}},(\mu_A)_{A\in\mathcal{P}_d},(f_i)_{i\in\underline{\sigma}},(w_A)_{A\in\mathcal{P}_d})\in(\mathbb{R}^+)^\sigma\times(\mathbb{R}^+)^{\mathcal{P}_d}\times\prod_{i\in\underline{\sigma}}W_i^s\times\prod_{A\in\mathcal{P}_d}W_A,$$

define $c((u_i),(\mu_A),(f_i),(w_A))$ as the map from V to $\mathbb{R}^\delta$ such that
$c((u_i),(\mu_A),(f_i),(w_A))=$

$$\sum_{V(k)\in\mathcal{P}_s}\left(\prod_{i|V(k)\subseteq V(i)}u_i\right)f_k+\sum_{C\in\mathcal{P}_d}\left(\prod_{i|C\subseteq V(i)}u_i\right)\left(\prod_{D\in\mathcal{P}_d|C\subseteq D}\mu_D\right)w_C.$$

For $V(j)\in\mathcal{P}_s$, define $v_j((u_i),(\mu_A),(f_i),(w_A))$ as the map from $V(j)$ to $\mathbb{R}^\delta$ such that
$v_j((u_i),(\mu_A),(f_i),(w_A))=$

$$\begin{aligned}&\textstyle\sum_{V(k)\in\mathcal{P}_s|V(k)\subseteq V(j)}\left(\prod_{i|V(k)\subseteq V(i)\subset V(j)}u_i\right)f_k\\+\ &\textstyle\sum_{C\in\mathcal{P}_d|C\subseteq V(j)}\left(\prod_{i|C\subseteq V(i)\subset V(j)}u_i\right)\left(\prod_{D\in\mathcal{P}_d|C\subseteq D\subseteq V(j)}\mu_D\right)w_C.\end{aligned}$$

For $B \in \mathcal{P}_d$, define $v_B((u_i),(\mu_A),(f_i),(w_A))$ as the map from B to $\mathbb{R}^\delta$ such that

$$v_B((u_i),(\mu_A),(f_i),(w_A)) = \sum_{C \in \mathcal{P}_d | C \subseteq B} \left(\prod_{D \in \mathcal{P}_d | C \subseteq D \subset B} \mu_D \right) w_C.$$

Again, we can picture the construction of the maps v_j and v_B on the tree of Figure 8.5, as in Figure 8.6. For example, v_j is the sum, over the vertices C that are connected to the vertex $V(j)$ of the top line by an oriented path from C to $V(j)$, of the associated map w_C or f_k, multiplied by the products of the coefficients μ_D or u_i associated to the edges of the path from C to $V(j)$.

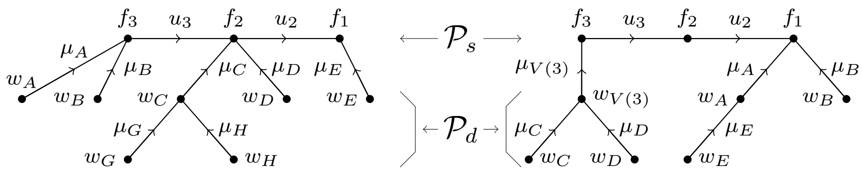


Figure 8.6: Constructions of the maps v_j and v_B

Note the following easy lemma.

Lemma 8.39. *For any* $((f_i^0)_{i \in \underline{\sigma}}, (w_A^0)_{A \in \mathcal{P}_d}) \in \prod_{i \in \underline{\sigma}} W_i^s \times \prod_{A \in \mathcal{P}_d} W_A$, *there exists a neighborhood* $N((f_i^0),(w_A^0))$ *of* $0 \times ((f_i^0),(w_A^0))$ *in* $(\mathbb{R}^+)^\sigma \times (\mathbb{R}^+)^{\mathcal{P}_d} \times \prod_{i \in \underline{\sigma}} W_i^s \times \prod_{A \in \mathcal{P}_d} W_A$ *such that:*
for any $((u_i)_{i \in \underline{\sigma}}, (\mu_A)_{A \in \mathcal{P}_d}, (f_i)_{i \in \underline{\sigma}}, (w_A)_{A \in \mathcal{P}_d}) \in N((f_i^0),(w_A^0))$ *such that* $\mu_A \neq 0$ *for any* $A \in \mathcal{P}_d$ *and* $u_i \neq 0$ *for any* $i \in \underline{\sigma}$, $c((u_i),(\mu_A),(f_i),(w_A))$ *is an injective map from* V *to* $\mathbb{R}^\delta \setminus \{0\}$.

Let $\phi \colon \mathbb{R}^\delta \to M$ be an open embedding which maps 0 to ∞. Again, when the construction of Theorem 8.4 makes sense, a point $x_t \in C_{V_t}[M,\infty]$ is denoted as a tuple, which contains its blowdown projection in M^{V_t} followed by the normal vectors that have popped up during the blow-ups. Let $V = p_b(x_t)^{-1}(\infty)$ and let $x = x_{t|V}$ denote the restriction of x_t to V. The normal vectors that have popped up during the blow-ups of x are labeled by elements of the ∞-parenthesization $(\mathcal{P}_s(x), \mathcal{P}_d(x))$ of V. For an element $V(k)$ of $\mathcal{P}_s(x)$, the unit normal bundle to $\Xi_{V(k)}$ is the set $S((T_\infty M)^{V(k)})$ of nonzero maps $T_0\phi \circ f_k$ from $V(k)$ to $T_\infty M$, up to dilation. Note that $T_0\phi$ identifies $O^s(V(k), K^s(V(k)), b, \mathbb{R}^\delta)$ with an open subset of $S((T_\infty M)^{V(k)})$.

Notation 8.40. For a given ∞-parenthesization $(\mathcal{P}_s, \mathcal{P}_d)$, and for $A \in \mathcal{P}_d$, define $k(A)$ as the maximal integer among the integers k such that $A \subseteq V(k)$.

For a set A of $\mathcal{P}_d(x)$, the image $x(b(A))$ of A under x in $\mathcal{B}\ell(M,\infty)$ is the dilation class $[T_0\phi\left(f_{k(A)}(A)\right)]$ of $T_0\phi\left(f_{k(A)}(A)\right)$ in

$$\left(S(T_\infty M) = U_\infty M = p_b^{-1}(\infty)\right) \subset \mathcal{B}\ell(M,\infty).$$

The unit normal bundle to $\overline{\Delta_A((M\setminus\infty)^V)}$ in the space obtained by the previous blowups in the neighborhood of x is canonically isomorphic to the unit normal bundle to $\Delta_A(\mathcal{B}\ell(M,\infty))$ in $\mathcal{B}\ell(M,\infty)^A$, whose fiber over x is

$$\overline{\mathcal{S}}_A(T_{x(b(A))}\mathcal{B}\ell(M,\infty)).$$

Its elements may be written as $T_{\phi_*^{-1}(x(b(A)))}\phi_* \circ w_A$ with respect to the map $\phi_* \colon \mathcal{B}\ell(\mathbb{R}^\delta, 0) \to \mathcal{B}\ell(M,\infty)$ induced by ϕ.

Proposition 8.41. *For any based ∞-parenthesization $(\mathcal{P}_s, \mathcal{P}_d)$ of V, and for any $((f_i^0)_{i\in\underline{\sigma}}, (w_A^0)_{A\in\mathcal{P}_d}) \in \prod_{i\in\underline{\sigma}} W_i^s \times \prod_{A\in\mathcal{P}_d} W_A$, there is a smooth open embedding $\psi(\mathcal{P}_s,\mathcal{P}_d,(f_i^0)_{i\in\underline{\sigma}},(w_A^0)_{A\in\mathcal{P}_d},\phi)$*

$$\begin{array}{lll} N\left((f_i^0),(w_A^0)\right) & \to & C_V[M,\infty] \\ ((u_i),(\mu_A),(f_i),(w_A)) & \mapsto & \left(\begin{array}{l} \phi\circ(c((u_i),(\mu_A),(f_i),(w_A))), \\ (T_0\phi\circ v_i)_{i|u_i=0}, (T_{c(b(A))}\phi_*\circ v_A)_{A|\mu_A=0}) \end{array}\right), \end{array}$$

where $c(b(A))$ denotes the image of $b(A)$ under c in $\mathcal{B}\ell(\mathbb{R}^\delta,0)$, as above. The embedding $\psi(\mathcal{P}_s,\mathcal{P}_d,(f_i^0)_{i\in\underline{\sigma}},(w_A^0)_{A\in\mathcal{P}_d},\phi)$ restricts to

$$N\left((f_i^0),(w_A^0)\right) \cap \left((\mathbb{R}^{+*})^\sigma \times (\mathbb{R}^{+*})^{\mathcal{P}_d} \times \prod_{i\in\underline{\sigma}} W_i^s \times \prod_{A\in\mathcal{P}_d} W_A\right)$$

as a diffeomorphism onto an open subset of $\check{C}_V[M,\infty]$. Furthermore, the open images of the embeddings $\psi(\mathcal{P}_s,\mathcal{P}_d,(f_i^0)_{i\in\underline{\sigma}},(w_A^0)_{A\in\mathcal{P}_d},\phi)$ for different $(\mathcal{P}_s,\mathcal{P}_d,(f_i^0)_{i\in\underline{\sigma}},(w_A^0)_{A\in\mathcal{P}_d})$ cover $p_b^{-1}(U(\infty^V))$ in $C_V[M,\infty]$, for an open neighborhood $U(\infty^V)$ of ∞^V in M^V.

Proof: Follow the charts in the proof of Theorem 8.33. $\square$

Recall that $\check{\mathcal{S}}(T_\infty M, A)$ denotes the set of injective maps from A to $(T_\infty M\setminus 0)$ up to dilation, from Notation 8.17. The proof of Theorem 8.33 also proves the following proposition.

Proposition 8.42. *For an ∞-parenthesization $(\mathcal{P}_s, \mathcal{P}_d)$ of V, set*

$$C_{V,\mathcal{P}_s,\mathcal{P}_d}[M,\infty] = \{x \in p_b^{-1}\left(\infty^V\right) \subseteq C_V[M,\infty] \mid \mathcal{P}_s(x) = \mathcal{P}_s \text{ and } \mathcal{P}_d(x) = \mathcal{P}_d\}.$$

The space $C_{V,\mathcal{P}_s,\mathcal{P}_d}[M,\infty]$ fibers over $\prod_{V(k)\in\mathcal{P}_s} \check{\mathcal{S}}(T_\infty M, K_d^s(V(k)))$, and its fiber is $\prod_{A\in\mathcal{P}_d} \check{S}_{K(A)}(T_{f_{M,k(A)}(A)} B\ell(M,\infty))$, where $f_{M,k(A)}(A)$ denotes the image[5] *of A in the above factor $\check{\mathcal{S}}(T_\infty M, K_d^s(V(k(A))))$. (Recall Notation 8.40).*

The stratum $C_{V,\mathcal{P}_s,\mathcal{P}_d}[M,\infty]$ is an open part of

$$\partial_{\sharp\mathcal{P}_s+\sharp\mathcal{P}_d}(C_V[M,\infty]) \setminus \partial_{\sharp\mathcal{P}_s+\sharp\mathcal{P}_d+1}(C_V[M,\infty]).$$

Corollary 8.43. *An element of $C_V[M,\infty]$ is a map c from V to M, equipped with*

- *an element $w_{c^{-1}(y)} \in \mathcal{S}_{c^{-1}(y)}(T_yM)$ for each $y \in M \setminus \{\infty\}$ that has several preimages under c,*
- *an ∞-parenthesization $(\mathcal{P}_s, \mathcal{P}_d)$ of $c^{-1}(\infty)$, if $c^{-1}(\infty) \neq \emptyset$, and in this case,*
  - *an element $f_{M,i}$ of $\check{\mathcal{S}}(T_\infty M, K_d^s(V(i)))$ for any element $V(i)$ of $\mathcal{P}_s$,*
  - *an element $w_{M,A}$ of $\check{S}_{K(A)}(T_{f_{M,k(A)}(A)} B\ell(M,\infty))$ for each $A \in \mathcal{P}_d$, with the notation of Proposition 8.42.*

Finally, let us prove Theorem 8.5 by checking the following proposition. For a finite set V, let $\mathcal{P}_{\geq 1}(V)$ (resp. $\mathcal{P}_{\geq 2}(V)$) denote the set of (non-strict) subsets of V of cardinality at least 1 (resp. at least 2).

Proposition 8.44. *Let A be a finite subset of a finite set V. The map $p_A \colon C_V[M,\infty] \to C_A[M,\infty]$ that maps an element of $C_V[M,\infty]$ as in Corollary 8.43 to the element of $C_A[M,\infty]$ consisting of*

- *the restriction $c_{|A}$,*
- *for each $y \in M \setminus \{\infty\}$ that has several preimages under $c_{|A}$, the restriction to $c_{|A}^{-1}(y)$ of the element $w_{c^{-1}(y)} \in \mathcal{S}_{c^{-1}(y)}(T_yM)$,*
- *if $\infty \in c(A)$, the ∞-parenthesization $(\mathcal{P}_{A,s}, \mathcal{P}_{A,d})$ of $c_{|A}^{-1}(\infty)$ such that*

$$\mathcal{P}_{A,s} = \{B \cap A \mid B \in \mathcal{P}_s, \sharp(B \cap A) \geq 1\}$$

$$\mathcal{P}_{A,d} = \{B \cap A \mid B \in \mathcal{P}_d, \sharp(B \cap A) \geq 2\}$$

[5] This image is an element of $S(T_\infty M)$, which is $p_b^{-1}(\infty)$ in $B\ell(M,\infty)$.

- *for each $B \in \mathcal{P}_{A,d}$, the "restriction" of $w_{M,\hat{B}}$ to $K(B)$, where $\hat{B}$ is the smallest element of $\mathcal{P}_d$ that contains B or is equal to B,*

- *for each $B \in \mathcal{P}_{A,s}$, the "restriction" of $f_{M,i(B)}$ to $K_d^s(B)$, where $V(i(B))$ is the smallest element of $\mathcal{P}_s$ that contains B or is equal to B,*

is well defined and smooth. It is the unique continuous map such that the following square commutes.

$$\begin{tikzcd} C_V[M,\infty] \arrow[r, "p_A"] \arrow[d, "p_b"'] & C_A[M,\infty] \arrow[d, "p_b"] \\ M^V \arrow[r, "p_A"] & M^A \end{tikzcd}$$

Proof: It is easy to see that this restriction map is well defined, its restriction to $C_V[M \setminus \{\infty\}]$ obviously coincides with the map defined on $C_V[M \setminus \{\infty\}]$ in Proposition 8.31, and it suffices to prove that it is smooth over a neighborhood of ∞^V in M^∞, with the charts of Proposition 8.41. The proof is similar to the proof of Proposition 8.29 and is left to the reader. $\square$

8.8 Finishing the proofs of the statements of Sections 8.2 and 8.4

Proof of Proposition 8.6: In order to study the closure of $\check{C}(R,L;\Gamma)$ in $C_{V(\Gamma)}(R)$, we study its intersection with some $p_b^{-1}(\prod_{i\in I} U_i^{V_i})$ for disjoint small compact U_i, again, where at most one U_i contains ∞ and this U_i does not meet the link. So the corresponding V_i does not contain univalent vertices and the structure of the corresponding factor is given by Theorem 8.4.

Thus, it is enough to study $p_A\left(\overline{\check{C}(R,L;\Gamma)}\right) \cap C_A(\phi(\mathbb{R}^3))$ when

- ϕ is an embedding from $\mathbb{R}^3$ to $\check{R}$ that maps the vertical line $\mathbb{R}\vec{v}$ through the origin oriented from bottom to top onto $\phi(\mathbb{R}^3) \cap L$, so that ϕ identifies $(\mathbb{R}^3, \mathbb{R}\vec{v})$ with $(\phi(\mathbb{R}^3), \phi(\mathbb{R}^3) \cap L)$,

- the univalent vertices of A form a non-empty subset $A_U = U(\Gamma) \cap A$ of consecutive vertices on a given component L_i of L.

Let $\mathcal{O}(A_U)$ denote the set of the linear orders $<$ on A_U compatible with $[i_\Gamma]$. (This is a singleton unless A_U contains all the univalent vertices of L_i.) Via the natural maps induced by ϕ, $\check{C}_A(\phi(\mathbb{R}^3))$ is identified with the set of injections from A to $\mathbb{R}^3$ and $p_A\left(\check{C}(R,L;\Gamma)\right) \cap \check{C}_A(\phi(\mathbb{R}^3))$ is identified with the disjoint union over $\mathcal{O}(A_U)$ of the subsets $\check{C}_A(\mathbb{R}^3, A_U, <\in \mathcal{O}(A_U))$ of injections that map A_U to $\mathbb{R}\vec{v}$ so that the order induced by $\mathbb{R}\vec{v}$ coincides with $<$. Fix $< \in \mathcal{O}(A_U)$, and write $A_U = \{v_1, \dots, v_k\}$ so that $v_1 < v_2 \cdots < v_k$.

Fix $b(A) = v_1$, and study the closure of $\check{C}_A(\mathbb{R}^3, A_U, <)$ in $\mathbb{R}^3 \times \mathbb{R}^+ \times \mathcal{S}_A(\mathbb{R}^3)$, using the diffeomorphism $\psi(\phi, v_1)$ of Theorem 8.30. This closure is the closure of $\mathbb{R}\vec{v} \times \mathbb{R}^+ \times \check{\mathcal{S}}_A(\mathbb{R}^3, A_U, <)$, where $\check{\mathcal{S}}_A(\mathbb{R}^3, A_U, <)$ is the quotient of $\check{C}_A(\mathbb{R}^3, A_U, <)$ by translations and dilations. Then the charts of Theorem 8.26 (used with basepoints that are as much as possible in A_U) make clear that the closure of $\check{\mathcal{S}}_A(\mathbb{R}^3, A_U, <)$ in $\mathcal{S}_A(\mathbb{R}^3)$ consists of the limit configurations c such that $c(v_j) - c(v_i)$ is non-negatively colinear with $\vec{v}$ at any scale (i.e. in any infinitesimal configuration w that has popped up during blow-ups) for any i and j in $\{1, \dots, k\}$ such that $i < j$, and that this closure is a smooth submanifold of $\mathcal{S}_A(\mathbb{R}^3)$ transverse to the ridges. $\square$

This proof also proves the following lemma.

Lemma 8.45. *The codimension-one faces of $C(R, L; \Gamma)$ are the intersections of $C(R, L; \Gamma)$ with the codimension-one faces of $C_{V(\Gamma)}(R)$.*

$\square$

Proposition 8.18 then follows, with the help of Propositions 8.44, 8.42 and 8.32. $\square$

8.9 Alternative descriptions of configuration spaces

Apart from Lemma 8.46, this section will not be used in this book. It mentions other presentations of the configuration spaces studied in Sections 8.5 to 8.7, without proofs, which are left to the reader as exercises.

Let V be a finite set of cardinality at least 2. Recall that $\mathcal{P}_{\geq 2} = \mathcal{P}_{\geq 2}(V)$ is the set of its (non-strict) subsets of cardinality at least 2. The smooth blow-down projection from $\mathcal{S}_A(T)$ to $\overline{\mathcal{S}}_A(T)$ for an $A \in \mathcal{P}_{\geq 2}(V)$ may be composed by the smooth restriction map from $\mathcal{S}_V(T)$ to $\mathcal{S}_A(T)$ to produce a smooth map π_A from $\mathcal{S}_V(T)$ to $\overline{\mathcal{S}}_A(T)$.

Lemma 8.46. *The product over the subsets A of V with cardinality at least 2 of the π_A is a smooth embedding*

$$\prod_{A \in \mathcal{P}_{\geq 2}} \pi_A \colon \mathcal{S}_V(T) \hookrightarrow \prod_{A \in \mathcal{P}_{\geq 2}} \overline{\mathcal{S}}_A(T)$$

and the image of $\mathcal{S}_V(T)$ is the closure of the image of the restriction of $\prod_{A \in \mathcal{P}_{\geq 2}} \pi_A$ to $\check{\mathcal{S}}_V(T)$.

PROOF: The injectivity of $\prod_{A\in\mathcal{P}_{\geq 2}} \pi_A$ can be seen from the description of $\mathcal{S}_V(T)$ as a set, which is given in Theorem 8.26. □

Proposition 8.47. *The image* $\left(\prod_{A\in\mathcal{P}_{\geq 2}} \pi_A\right)(\mathcal{S}_V(T))$ *is the subset of*

$$\prod_{A\in\mathcal{P}_{\geq 2}} \overline{\mathcal{S}}_A(T)$$

consisting of the elements $((c_A)_{A\in\mathcal{P}_{\geq 2}})$ *such that for any two elements* A *and* B *of* $\mathcal{P}_{\geq 2}$ *such that* $B \subset A$*, the restriction of* c_A *to* B *coincides with* c_B *if it is not constant.*

PROOF: Exercise. □

Thus, $\mathcal{S}_V(T)$ can be defined as its image described in the above proposition. Similar definitions involving only cardinality 2 and 3 subsets of V can be found in the Sinha article [Sin04].

For a smooth manifold M without boundary, we have similar smooth maps π_A from $C_V[M]$ to $\mathrm{B}\ell(M^A, \Delta_A(M^A))$, which also define a smooth map

$$\prod_{A\in\mathcal{P}_{\geq 2}} \pi_A \colon C_V[M] \hookrightarrow \prod_{A\in\mathcal{P}_{\geq 2}} \mathrm{B}\ell(M^A, \Delta_A(M^A))$$

The elements of $\mathrm{B}\ell(M^A, \Delta_A(M^A))$ are maps c from A to M that are equipped with an element $w \in \overline{\mathcal{S}}_A(T_{c(A)}M)$ when they are constant.

Proposition 8.48. *The map*

$$\prod_{A\in\mathcal{P}_{\geq 2}} \pi_A \colon C_V[M] \hookrightarrow \prod_{A\in\mathcal{P}_{\geq 2}} B\ell(M^A, \Delta_A(M^A))$$

is an embedding. Its image is the subset of $\prod_{A\in\mathcal{P}_{\geq 2}} B\ell(M^A, \Delta_A(M^A))$ *consisting of the elements* $((c_A)_{A\in\mathcal{P}_{\geq 2}})$ *such that*
for any two elements A *and* B *of* $\mathcal{P}_{\geq 2}$ *such that* $B \subset A$*,*

- *the restriction to* B *of the map* $p_b(c_A)\colon A \to M$ *coincides with* $p_b(c_B)$*, and,*
- *if* $p_b(c_A)$ *is constant, then the restriction to* B *of* $w_A(c_A) \in \overline{\mathcal{S}}_A(T_{c_A(A)}M)$ *is* $w_B(c_B)$ *if this restriction is not constant.*

PROOF: Exercise. $\square$

Again, $C_V[M]$ can be defined as its image described in the above proposition, and similar definitions involving only cardinality 2 and 3 subsets of V can be found in [Sin04].

Below, we give similar statements for $C_V[M,\infty]$, where $\infty \in M$. For a finite set V such that $\sharp V \geq 2$, define

$$C(M,\infty,V) = B\ell(B\ell(M^V,\infty^V),\overline{\Delta_V(\check{M}^V)}),$$

which is the manifold obtained from M^V by the first two blow-ups. For a singleton V, set $C(M,\infty,V) = B\ell(M,\infty)$. In particular, there is a canonical smooth composition of blow-down maps from $C_A(M)$ to $C(M,\infty,A)$, for any subset A of the finite set V, and by composition with the restriction map from $C_V(M)$ to $C_A(M)$, there is a canonical smooth projection

$$\pi_A \colon C_V(M) \to C(M,\infty,A).$$

Description of $C(M,\infty,V)$ as a set Recall that $B\ell(M^V,\infty^V)$ is the manifold obtained from M^V by blowing up M^V at $\infty^V = (\infty,\infty,\dots,\infty)$. As a set, $B\ell(M^V,\infty^V)$ is the union of $M^V \setminus \infty^V$ with the unit tangent bundle $S\left((T_\infty M)^V\right)$ of M^V at ∞^V. Let $\overline{\Delta_V(\check{M}^V)}$ denote the closure in $B\ell(M^V,\infty^V)$ of the small diagonal of $\check{M}^V$ consisting of the constant maps. The boundary of $B\ell(M^V,\infty^V)$ is $S((T_\infty M)^V)$ and the boundary of $\overline{\Delta_V(\check{M}^V)}$ is the small diagonal of $(T_\infty M \setminus 0)^V$ up to dilation. This allows us to view all the elements of $\overline{\Delta_V(\check{M}^V)}$ as constant maps from V to $B\ell(M,\infty)$, and provides a canonical diffeomorphism $p_1 : \overline{\Delta_V(\check{M}^V)} \longrightarrow B\ell(M,\infty)$ as in Proposition 3.3.

Now, $C(M,\infty,V)$ is obtained from $B\ell(M^V,\infty^V)$ by blowing up $\overline{\Delta_V(\check{M}^V)}$. Thus as a set, $C(M,\infty,V)$ is the union of

- the set of non-constant maps from V to M,
- the space $\{\infty^V\} \times \frac{(T_\infty M)^V \setminus \Delta((T_\infty M)^V)}{]0,\infty[}$, where $]0,\infty[= \mathbb{R}^{+*}$ acts by multiplication and,
- the bundle over $\overline{\Delta_V(\check{M}^V)} = B\ell(M,\infty)$ whose fiber at a constant map with value $x \in B\ell(M,\infty)$ is $\overline{\mathcal{S}}_V(T_x B\ell(M,\infty))$, according to Lemma 8.8.

An element c of $C(M,\infty,V)$ will be denoted by a tuple formed by its (composed) blowdown projection $p_b(c)$ to M^V followed by

- nothing, if $p_b(c) \notin \Delta_V(M^V)$

- an element w_V of $\overline{\mathcal{S}}_V(T_xM)$ if $p_b(c) = x^V \in \Delta_V((M \setminus \{\infty\})^V)$
- an element f_M of $S(T_\infty M^V)$ if $p_b(c) = \infty^V$ followed by
  - nothing, if f_M is non constant,
  - an element $w_{M,V}$ of $\overline{\mathcal{S}}_V(T_{[f_M(V)]}B\ell(M,\infty))$, otherwise.

Then this tuple contains the unit normal vectors that popped up during the blow-ups that affected c.

Differentiable structure of $C(M,\infty,V)$ Let us study the differentiable structure of $C(M,\infty,V)$ near the preimage of ∞^V, more precisely. Use an open embedding $\phi\colon \mathbb{R}^\delta \hookrightarrow M$ that maps 0 to ∞. Define the following finite open covering $\{\tilde{O}^s_2(V,b(V))\}_{b(V)\in V}$ of $S((\mathbb{R}^\delta)^V)$. For $b(V) \in V$, $\tilde{O}^s_2(V,b(V))$ is the set of maps f from V to $\mathbb{R}^\delta$ such that $f(b(V)) \neq 0$ up to dilation. It is first identified with the set $O^s_2(V,b(V)) = S^{\delta-1} \times (\mathbb{R}^\delta)^{V\setminus\{b(V)\}}$ of maps $f\colon V \to \mathbb{R}^\delta$ such that $\parallel f(b(V)) \parallel = 1$. The first blow-up at ∞^V yields a smooth open embedding $\phi_{2,b(V)}$ from $\mathbb{R}^+ \times O^s_2(V,b(V))$ onto an open subset of $B\ell(M^V,\infty^V)$, which maps $(u_1 \neq 0, f)$ to $\phi\circ(u_1 f)$. The union of the images of the embeddings $\phi_{2,b(V)}$ over the elements $b(V)$ of V cover $p_b^{-1}\left((\phi(\mathbb{R}^3))^V\right)$ in $B\ell(M^V,\infty^V)$. Define the diffeomorphism

$$\begin{array}{llll}\psi_{b(V)}\colon & O^s_2(V,b(V)) & \to & S^{\delta-1}\times(\mathbb{R}^\delta)^{V\setminus\{b(V)\}}\\ & f & \mapsto & \left(f(b(V)),(f(x)-f(b(V)))_{x\in V\setminus\{b(V)\}}\right)\end{array}$$

The closure of $\Delta_V(\check{M}^V)$ meets the image of $\phi_{2,b(V)}\circ\left(\mathbf{1}_{\mathbb{R}^+}\times\psi^{-1}_{b(V)}\right)$ as the image of $\mathbb{R}^+\times S^{\delta-1}\times 0$. So the blow-up along $\overline{\Delta_V(\check{M}^V)}$ respects the factorization by $\mathbb{R}^+\times S^{\delta-1}$ and is diffeomorphic to $\mathbb{R}^+\times S^{\delta-1}\times B\ell((\mathbb{R}^\delta)^{V\setminus\{b(V)\}},0)$ in a neighborhood of the blown-up diagonal closure.

Proposition 8.49. *Let V be a finite set with at least two elements and let $b(V)\in V$.*

Identify $\overline{\mathcal{S}}_V(\mathbb{R}^\delta)$ with the sphere $\mathbb{S}((\mathbb{R}^\delta)^{V\setminus\{b(V)\}})$ of maps from V to $\mathbb{R}^\delta$ that map $b(V)$ to 0 and that have norm 1 for the usual Euclidean norm on $(\mathbb{R}^\delta)^V$. There is a smooth diffeomorphism from $(\mathbb{R}^+)^2\times S^{\delta-1}\times\mathbb{S}((\mathbb{R}^\delta)^{V\setminus\{b(V)\}})$ onto an open subset of $C(M,\infty,V)$, which maps $(u_1,\mu_V,\vec{f},w_V)$ to

$$\begin{cases}\phi\circ(u_1(\vec{f}+\mu_V w_V)) & \text{if } u_1\mu_V\neq 0,\\ \left(\phi\circ(u_1(\vec{f}+\mu_V w_V)),T_0\phi\circ(\vec{f}+\mu_V w_V)\right) & \text{if } u_1=0 \text{ and } \mu_V\neq 0,\\ \left(\phi\circ(u_1(\vec{f}+\mu_V w_V)),T_{u_1\vec{f}}\phi\circ w_V\right) & \text{if } u_1\neq 0 \text{ and } \mu_V=0,\\ \left(\phi\circ(u_1(\vec{f}+\mu_V w_V)),T_0\phi\circ(\vec{f}^V),T_{[\vec{f}]}\phi_*\circ w_V\right) & \text{if } u_1=\mu_V=0,\end{cases}$$

where $\phi\circ(u_1(\vec{f}+\mu_V w_V))$ *maps an element* v *of* V *to* $\phi(u_1(\vec{f}+\mu_V w_V(v)))$*, and* $T_{[\vec{f}]}\phi_*$ *maps an element of* $\mathbb{R}^\delta$ *to an element of* $T_{\phi_*\left([\vec{f}]\in U_0\mathbb{R}^\delta\right)\in U_\infty M}B\ell(M,\infty)$*. Furthermore, the open images of these* $(\sharp V)$ *diffeomorphisms cover*

$$p_b^{-1}\left(\phi(\mathbb{R}^\delta)^V\right).$$

PROOF: The proof is straightforward. □

The image of $C_V[M,\infty]$

Proposition 8.50. *The map* $\prod_{A\in\mathcal{P}_{\geq 1}}\pi_A\colon C_V[M,\infty]\hookrightarrow\prod_{A\in\mathcal{P}_{\geq 1}}C(M,\infty,A)$ *is an embedding. Its image is the subset of* $\prod_{A\in\mathcal{P}_{\geq 1}}C(M,\infty,A)$ *consisting of the elements* $((c_A)_{A\in\mathcal{P}_{\geq 1}})$ *such that*
for any two elements A *and* B *of* $\mathcal{P}_{\geq 1}$ *such that* $B\subset A$,

- *the restriction to* B *of the map* $p_b(c_A)\colon A\to M$ *coincides with* $p_b(c_B)$*, and,*
- *if* $p_b(c_A)$ *is constant such that* $p_b(c_A)(A)\neq\{\infty\}$*, then the restriction to* B *of* $w_A\in\overline{\mathcal{S}}_A(T_{c_A(A)}M)$ *is* w_B *if this restriction is not constant,*
- *if* $p_b(c_A)(A)=\{\infty\}$*, then the restriction to* B *of* $f_{M,A}\in S((T_\infty M)^A)$ *is* $f_{M,B}\in S((T_\infty M)^B)$ *if this restriction is not identically zero, and, if furthermore,* $f_{M,A}$ *is constant, then the restriction to* B *of* $w_{M,A}\in\overline{\mathcal{S}}_A(T_{[f_{M,A}(A)]}B\ell(M,\infty))$ *is* $w_{M,B}$ *if this restriction is not constant.*

PROOF: Exercise □

Again, $C_V[M,\infty]$ could have been defined as the above image, which is the closure of the image of $\check{C}_V[M,\infty]$, but its differentiable structure had to be studied anyway.

More information about the homotopy groups and the homology of the configuration spaces $\check{C}_V[\mathbb{R}^d]$ and $\check{C}_V[S^d]$ can be found in the book [FH01] by Fadell and Husseini.

Chapter 9

Dependence on the propagating forms

In this chapter, we show how our combinations of integrals over configuration spaces depend on the chosen propagating forms.

9.1 Introduction

In this section, we give a first general description of the variation of Z when propagating forms change, in Proposition 9.2, whose typical proof will occupy the next sections. Then we show how Proposition 9.2 and the preliminary lemma 9.1 apply to prove Theorem 7.19, and two other lemmas 9.5 and 9.6, about independence of chosen propagating forms as in Definition 3.11.

Again, any closed 2-form on $\partial C_2(R)$ extends to $C_2(R)$ because the restriction induces a surjective map $H^2(C_2(R);\mathbb{R}) \to H^2(\partial C_2(R);\mathbb{R})$ since $H^3(C_2(R),\partial C_2(R);\mathbb{R}) = 0$.

Lemma 9.1. *Let $(\check{R},\tau_0)$ be an asymptotic rational homology $\mathbb{R}^3$, as in Definition 3.8. Let $\tau\colon [0,1]\times\check{R}\times\mathbb{R}^3 \to T\check{R}$ be a smooth map whose restriction to $\{t\}\times\check{R}\times\mathbb{R}^3$ is an asymptotically standard parallelisation τ_t of $\check{R}$ for any $t\in[0,1]$. Define $p_\tau\colon [0,1]\times\partial C_2(R)\to[0,1]\times S^2$ by $p_\tau(t,x) = (t, p_{\tau_t}(x))$.*

Let ω_0 and ω_1 be two propagating forms of $C_2(R)$ that restrict to $\partial C_2(R)\setminus UB_R$ as $p^*_{\tau_0}(\omega_{0,S^2})$ and as $p^*_{\tau_1}(\omega_{1,S^2})$, respectively, for two forms ω_{0,S^2} and ω_{1,S^2} of S^2 such that $\int_{S^2}\omega_{0,S^2} = \int_{S^2}\omega_{1,S^2} = 1$.

Then there exist

- *a closed 2-form $\tilde{\omega}_{S^2}$ on $[0,1]\times S^2$ whose restriction to $\{t\}\times S^2$ is ω_{t,S^2} for $t\in\{0,1\}$,*

- *for any such $\tilde{\omega}_{S^2}$, a closed 2-form ω^∂ on $[0,1]\times\partial C_2(R)$ whose restriction to $\{t\} \times \partial C_2(R)$ is $\omega_{t|\partial C_2(R)}$ for $t \in \{0,1\}$, and whose restriction to $[0,1] \times (\partial C_2(R) \setminus U(B_R))$ is $p_\tau^*(\tilde{\omega}_{S^2})$.*
- *and, for any such compatible $\tilde{\omega}_{S^2}$ and ω^∂, a closed 2-form ω on $[0,1] \times C_2(R)$ whose restriction to $\{t\} \times C_2(R)$ is ω_t for $t \in \{0,1\}$, and whose restriction to $[0,1] \times \partial C_2(R)$ is ω^∂.*

If ω_0 and ω_1 are propagating forms of $(C_2(R), \tau_0)$ and $(C_2(R), \tau_1)$, we may choose $\omega^\partial = p_\tau^*(\tilde{\omega}_{S^2})$ on $[0,1] \times \partial C_2(R)$.

Proof: As in Lemma 4.2, there exists a one form η_{S^2} on S^2 such that $d\eta_{S^2} = \omega_{1,S^2} - \omega_{0,S^2}$. Define the closed 2-form $\tilde{\omega}_{S^2}$ on $[0,1] \times S^2$ by

$$\tilde{\omega}_{S^2} = p_{S^2}^*(\omega_{0,S^2}) + d(tp_{S^2}^*(\eta_{S^2})),$$

where t is the coordinate on $[0,1]$.

Now, the form ω^∂ is defined on the boundary of $[0,1] \times U(B_R)$, and it extends as a closed 2-form ω^∂ as wanted there because the restriction induces a surjective map $H^2([0,1] \times U(B_R); \mathbb{R}) \to H^2\left(\partial\left([0,1] \times U(B_R)\right); \mathbb{R}\right)$ since $H^3\left([0,1] \times U(B_R), \partial\left([0,1] \times U(B_R)\right); \mathbb{R}\right) = 0$.

Finally, the wanted form ω is defined on the boundary of $[0,1] \times C_2(R)$ and it extends similarly as a closed 2-form to $[0,1] \times C_2(R)$. $\square$

When A is a subset of the set of vertices $V(\Gamma)$ of a numbered Jacobi diagram Γ with support a one-manifold $\mathcal{L}$, $E(\Gamma_A)$ denotes the set of edges of Γ between two elements of A (edges of Γ are plain), and Γ_A is the subgraph of Γ consisting of the vertices of A and the edges of $E(\Gamma_A)$ together with the natural restriction to $U(\Gamma) \cap A$ of the isotopy class of injections from $U(\Gamma)$ to $\mathcal{L}$ associated to Γ.

The following proposition, whose proof occupies most of this chapter, is crucial in the study of variations of Z.

Proposition 9.2. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $L = \sqcup_{j=1}^k K_j$ be an embedding of $\mathcal{L} = \sqcup_{j=1}^k S_j^1$ into $\check{R}$. Let $\tau\colon [0,1] \times \check{R} \times \mathbb{R}^3 \to T\check{R}$ be a smooth map[1] whose restriction to $\{t\} \times \check{R} \times \mathbb{R}^3$ is an asymptotically standard parallelisation τ_t of $\check{R}$ for any $t \in [0,1]$. Define $p_\tau\colon [0,1]\times\partial C_2(R) \to [0,1] \times S^2$ by $p_\tau(t,x) = (t, p_{\tau_t}(x))$.*

Let n and m be positive integers. For $i \in \underline{m}$, let $\tilde{\omega}(i)$ be a closed 2-form on $[0,1] \times C_2(R)$ whose restriction to $\{t\} \times C_2(R)$ is denoted by $\tilde{\omega}(i,t)$, for any $t \in [0,1]$.

[1]This homotopy τ is not useful for this statement, but this notation will be used later.

Assume that $\tilde{\omega}(i)$ *restricts to* $[0,1] \times (\partial C_2(R) \setminus UB_R)$ *as* $p_\tau^*(\tilde{\omega}_{S^2}(i))$*, for some closed two-form* $\tilde{\omega}_{S^2}(i)$ *on* $[0,1] \times S^2$ *such that* $\int_{\{t\}\times S^2} \tilde{\omega}_{S^2}(i) = 1$ *for* $t \in [0,1]$*. Recall the notation from Section 7.4 and set*

$$Z_{n,m}(t) = \sum_{\Gamma \in \mathcal{D}^e_{n,m}(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\tilde{\omega}(i,t))_{i \in \underline{m}})[\Gamma] \in \mathcal{A}_n(\sqcup_{j=1}^k S^1_j)$$

and $Z_n(t) = Z_{n,3n}(t)$*. Let* $\mathcal{D}^{e,F}_{n,m}(\mathcal{L})$ *denote the set of pairs* (Γ, A) *such that*

- $\Gamma \in \mathcal{D}^e_{n,m}(\mathcal{L})$*,*
- $A \subseteq V(\Gamma)$*,* $\sharp A \geq 2$*,*
- Γ_A *is a connected component of* Γ*,*
- $\sharp A \equiv 2$ *mod* 4 *if* $A \cap U(\Gamma) = \emptyset$*, and*
- $A \cap U(\Gamma)$ *is a set of consecutive vertices on one component* $\mathcal{L}_A$ *of* $\mathcal{L}$ *if* $A \cap U(\Gamma) \neq \emptyset$*.*

For $(\Gamma, A) \in \mathcal{D}^{e,F}_{n,m}(\mathcal{L})$*, set*

$$I(\Gamma, A) = \int_{[0,1]\times F(A,L,\Gamma)} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e)))[\Gamma],$$

where $p_e \colon [0,1] \times C(R, L; \Gamma) \to [0,1] \times C_2(R)$ *is the product by the identity map* $\mathbf{1}_{[0,1]}$ *of* $[0,1]$ *of the previous* p_e*, and the face* $F(A, L, \Gamma)$ *of* $C(R, L; \Gamma)$ *is described in Section 8.4. Set* $\mathcal{D}^{e,F}_n(\mathcal{L}) = \mathcal{D}^{e,F}_{n,3n}(\mathcal{L})$*. Then*

$$Z_n(1) - Z_n(0) = \sum_{(\Gamma,A) \in \mathcal{D}^{e,F}_n(\mathcal{L})} \zeta_\Gamma I(\Gamma, A)$$

and

$$Z_{n,3n-2}(1) - Z_{n,3n-2}(0) = \sum_{(\Gamma,A) \in \mathcal{D}^{e,F}_{n,3n-2}(\mathcal{L})} \zeta_\Gamma I(\Gamma, A).$$

This statement simplifies as in Corollary 9.4 when $L = \emptyset$ using the projection $p^c \colon \mathcal{A}(\emptyset) \to \mathcal{A}^c(\emptyset)$, which maps diagrams with several connected components to 0. Recall that $\mathcal{D}^c_n$ is the subset of $\mathcal{D}^e_n(\emptyset)$ whose elements are the numbered diagrams of $\mathcal{D}^e_n(\emptyset)$ with one connected component.

For an oriented connected diagram Γ, the face $F(V(\Gamma), \emptyset, \Gamma)$ fibers over $\check{R}$, and the fiber over $x \in \check{R}$ is the space $\check{\mathcal{S}}_{V(\Gamma)}(T_x\check{R})$ of injections from $V(\Gamma)$ to $T_x\check{R}$, up to translation and dilation. See Section 8.3. This face is denoted by $\check{\mathcal{S}}_{V(\Gamma)}(T\check{R})$.

Lemma 9.3. *Let $\Gamma \in \mathcal{D}_n^c$ be equipped with a vertex-orientation, which induces an orientation of $C(R, \emptyset; \Gamma)$, as in Corollary 7.2. These orientations induce the orientation of $V(\Gamma)$ that is described in Remark 7.5. The orientation of $F(V(\Gamma), \emptyset, \Gamma)$ as part of the boundary of $C(R, \emptyset; \Gamma)$ can be described alternatively as follows. The face $F(V(\Gamma), \emptyset, \Gamma)$ is oriented as the local product $\check{R} \times$ fiber, where the fiber is oriented as in Convention 8.10, using the above orientation of $V(\Gamma)$.*

PROOF: The dilation factor for the quotient $\check{\mathcal{S}}_{V(\Gamma)}(T_x\check{R})$ plays the role of an inward normal for $C(R, \emptyset; \Gamma)$. The orientation of $C(R, \emptyset; \Gamma)$ near the boundary is given by the orientation of $\check{R}$, followed by this inward normal, followed by the fiber orientation. □

For any pair e of $V(\Gamma)$, we have a natural restriction map

$$p_e \colon \check{\mathcal{S}}_{V(\Gamma)}(T\check{R}) \to \check{\mathcal{S}}_e(T\check{R}) \cong U\check{R},$$

which provides natural restriction maps

$$p_e \colon [0,1] \times \check{\mathcal{S}}_{V(\Gamma)}(T\check{R}) \to [0,1] \times \check{\mathcal{S}}_e(T\check{R})$$

by multiplication by $\mathbf{1}_{[0,1]}$.

Proposition 9.2 has the following corollary.

Corollary 9.4. *Using Notation 7.16, under the assumptions of Proposition 9.2, when $L = \emptyset$, set $z_n(t) = p^c(Z_n(t))$. So*

$$z_n(t) = z_n\left(R, (\tilde{\omega}(i,t))_{i \in \underline{3n}}\right).$$

Define

$$z_n\left([0,1] \times UB_R; (\tilde{\omega}(i))_{i \in \underline{3n}}\right) = \sum_{\Gamma \in \mathcal{D}_n^c} I(\Gamma, V(\Gamma)),$$

where

$$I(\Gamma, V(\Gamma)) = \zeta_\Gamma \int_{[0,1] \times \check{\mathcal{S}}_{V(\Gamma)}(TB_R)} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e)))[\Gamma].$$

Then $z_n(1) = z_n(0)$ if n is even, and

$$z_n(1) - z_n(0) = z_n\left([0,1] \times UB_R; (\tilde{\omega}(i))_{i \in \underline{3n}}\right)$$

for any odd integer n.

Proof: $\int_{[0,1]\times \check{\mathcal{S}}_{V(\Gamma)}(T(\check{R}\setminus B_R))} \bigwedge_{e\in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e))) = 0$ because the integrated form factors through $[0,1]\times \check{\mathcal{S}}_{V(\Gamma)}(\mathbb{R}^3)$ via a map induced by p_τ, which is fixed and independent of τ, there. In particular, $z_n\left([0,1]\times UB_R; (\tilde{\omega}(i))_{i\in \underline{3n}}\right)$ depends only on $\left(\tilde{\omega}(i)_{|[0,1]\times UB_R}\right)_{i\in\underline{3n}}$. □

Proof of Theorem 7.19 assuming Proposition 9.2: Changing propagating forms $\omega(i)_0$ of $C_2(R)$ to other ones $\omega(i)_1$ provides forms $\tilde{\omega}(i)$ on $[0,1]\times C_2(R)$ as in Lemma 9.1. Then Corollary 9.4 guarantees that $z_{2n}(\check{R},\emptyset,(\omega(i)))$ does not depend on the used propagating forms (which are not normalized on $U(B_R)$, and hence do not depend on parallelizations). □

Lemma 9.5. *Let $(\check{R},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $L\colon \mathcal{L}\hookrightarrow \check{R}$ be a link embedding. For any $i\in\underline{3n}$, let $\omega(i)$ be a homogeneous propagating form of $(C_2(R),\tau)$. Then, as stated in the unproved theorem 7.20, $Z_n(\check{R},L,(\omega(i)))$ is independent of the chosen $\omega(i)$. Denote it by $Z_n(\check{R},L,\tau)$.*

Proof of Lemma 9.5 assuming Proposition 9.2: In order to prove this lemma, it suffices to prove that if some homogeneous $\omega(i)=\tilde{\omega}(i,0)$ is changed to another homogeneous propagating form $\tilde{\omega}(i,1)$, then $Z_n(\check{R},L,(\omega(i)))$ is unchanged. According to Lemma 4.2, under these assumptions, there exists a one-form η on $C_2(R)$ such that

- $\tilde{\omega}(i,1)=\tilde{\omega}(i,0)+d\eta$, and,
- $\eta_{|\partial C_2(R)}=0$.

Let $p_{C_2}\colon [0,1]\times C_2(R)\to C_2(R)$ denote the projection on the second factor. Define closed 2-forms $\tilde{\omega}(j)$ on $[0,1]\times C_2(R)$ by

- $\tilde{\omega}(j)=p_{C_2}^*(\omega(j))$, if $j\neq i$, and
- $\tilde{\omega}(i)=p_{C_2}^*(\tilde{\omega}(i,0))+d(tp_{C_2}^*(\eta))$.

Then the variation of $Z_n(\check{R},L,(\omega(i)))$ is $Z_n(1)-Z_n(0)$, with the notation of Proposition 9.2, where all the forms involved in some $I(\Gamma,A)$, except $p_{e(i)}^*(\tilde{\omega}(i))$, for the possible edge $e(i)$ such that $j_E(e(i))=i$, factor through $p_{C_2}^*$. Thus if $i\notin \mathrm{Im}(j_E)$, then all the forms factor through $p_{C_2}^*$, and $I(\Gamma,A)$ vanishes. Locally, $F(A,L,\Gamma)$ is diffeomorphic to the product of $F(A,L,\Gamma_A)$ by $C(R,L;\Gamma\setminus\Gamma_A)$.

If $e(i)$ is not an edge of Γ_A, then the form $\bigwedge_{e\in E(\Gamma_A)} p_e^*(\tilde{\omega}(j_E(e)))$ factors through $F(A,L,\Gamma_A)$ whose dimension is $2\sharp E(\Gamma_A)-1$, and the form vanishes. If $e(i)$ is an edge of Γ_A, then the part $p_{e(i)}^*(d(tp_{C_2}^*(\eta)))$ vanishes since η vanishes on $\partial C_2(R)$. Thus $\bigwedge_{e\in E(\Gamma_A)} p_e^*(\omega(j_E(e)))$ still factors through $F(A, L, \Gamma_A)$. Thus when (R, L, τ) is fixed, $Z_n(\check{R}, L, (\omega(i)))$ is independent of the chosen homogeneous $\omega(i)$. $\square$

Lemma 9.6. *Let $(\check{R}, \tau_0)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $n \in \mathbb{N}$. For any $i \in \underline{3n}$, let $\omega(i)$ be a propagating form of $(C_2(R), \tau_0)$. Then $Z_n(\check{R}, \emptyset, (\omega(i))) = Z_n(\check{R}, \emptyset, \tau_0)$ with the notation of Lemma 9.5. Furthermore, $Z_n(\check{R}, \emptyset, \tau_0)$ depends only on the homotopy class of τ_0.*

PROOF ASSUMING PROPOSITION 9.2: Let $\tau\colon [0,1] \times \check{R} \times \mathbb{R}^3 \to T\check{R}$ be a smooth map whose restriction to $\{t\} \times \check{R} \times \mathbb{R}^3$ is an asymptotically standard parallelisation τ_t of $\check{R}$ for any $t \in [0, 1]$. Define $p_\tau\colon [0,1]\times\partial C_2(R) \to [0,1]\times S^2$ by $p_\tau(t, x) = (t, p_{\tau_t}(x))$. For any $i \in \underline{3n}$, let $\omega(i)_0$ be a (non-necessarily homogeneous) propagating form of $(C_2(R), \tau_0)$ and let $\omega(i)_1$ be a propagating form of $(C_2(R), \tau_1)$. It suffices to prove that

$$Z_n(\check{R}, \emptyset, (\omega(i)_1)) - Z_n(\check{R}, \emptyset, (\omega(i)_0)) = 0.$$

Use forms $\omega(i)$ on $[0, 1] \times C_2(R)$ provided by Lemma 9.1, which restrict to $[0, 1] \times \partial C_2(R)$ as $p_\tau^*(\tilde{\omega}_{S^2})$, to express this variation as in Proposition 9.2. Here, a face $F(A, \emptyset, \Gamma)$ is an open dense subset of the product of $F(A, \emptyset, \Gamma_A)$ by $\check{C}(R, \emptyset; \Gamma\setminus\Gamma_A)$ and τ identifies $[0,1]\times F(A, \emptyset, \Gamma_A)$ with $[0,1]\times\check{R}\times\check{\mathcal{S}}_A(\mathbb{R}^3)$. The form $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$ pulls back through $[0,1]\times\check{\mathcal{S}}_A(\mathbb{R}^3)\times\check{C}(R, \emptyset; \Gamma\setminus \Gamma_A)$, and it vanishes. $\square$

The following variant of Proposition 9.2 implies Theorem 7.30. It is proved in Section 9.3.

Proposition 9.7. *Under the assumptions of Proposition 9.2, the following statement is also true. For $\Gamma \in \mathcal{D}^e_{n,m}(\mathcal{L})$, and for a connected component Γ_A of Γ (with no univalent vertices or) whose univalent vertices are consecutive on one component $\mathcal{L}_A$ of $\mathcal{L}$, let $\Gamma^{rev}(A)$ denote the graph obtained from Γ by reversing the order*[2] *on the univalent vertices of Γ_A induced by i_Γ. Then*

$$Z_n(1) - Z_n(0) = \sum_{(\Gamma,A)\in\mathcal{D}^{e,F}_n(\mathcal{L})} \zeta_\Gamma I'(\Gamma, A)$$

and

$$Z_{n,3n-2}(1) - Z_{n,3n-2}(0) = \sum_{(\Gamma,A)\in\mathcal{D}^{e,F}_{n,3n-2}(\mathcal{L})} \zeta_\Gamma I'(\Gamma, A),$$

[2]When $\mathcal{L}_A$ is oriented, this order is a linear order if $U(\Gamma)$ has vertices of $V(\Gamma)\setminus A$ on $\mathcal{L}_A$; it is cyclic, otherwise. When $\mathcal{L}_A$ is not oriented, the order is not defined, but reversing the order is well defined in any case. When $\mathcal{L}_A$ is oriented, $[\Gamma] = [\Gamma_A\sharp_{\mathcal{L}_A}\Gamma_{V(\Gamma)\setminus A}]$ and $[\Gamma^{\mathrm{rev}}(A)] = [\Gamma_A^{\mathrm{rev}}(A)\sharp_{\mathcal{L}_A}\Gamma_{V\setminus A}]$.

where

$$I'(\Gamma,A)=\frac{1}{2}\int_{[0,1]\times F(A,L,\Gamma)}\bigwedge_{e\in E(\Gamma)}p_e^*(\tilde{\omega}(j_E(e)))\frac{1}{2}\left([\Gamma]-(-1)^{\sharp E(\Gamma_A)}[\Gamma^{rev}(A)]\right).$$

Proof of Theorem 7.30 assuming Proposition 9.7: There is a map rev from $\mathcal{A}(S^1)$ to itself that sends the class of a diagram Γ to the class of the diagram obtained from Γ by reversing the order of the univalent vertices on S^1 and by multiplying the class by $(-1)^{\sharp T(\Gamma)}$. The composition $w_C \circ \mathrm{rev}$ equals w_C. Furthermore, w_C sends odd degree diagrams to zero, and w_C is multiplicative with respect to the multiplication of $\mathcal{A}(S^1)$. So for any $(\Gamma,A)\in\mathcal{D}^{e,F}_{n,m}(S^1)$ as in the above statement, such that Γ has no component without univalent vertices, $\left(w_C([\Gamma])-(-1)^{\sharp E(\Gamma_A)}w_C([\Gamma^{\mathrm{rev}}(A)])\right)$ is equal to

$$w_C([\Gamma_{V(\Gamma)\setminus A}])\left(w_C([\Gamma_A])-(-1)^{\sharp E(\Gamma_A)+\sharp T(\Gamma_A)}w_C(\mathrm{rev}([\Gamma_A]))\right)=0,$$

since $\sharp E(\Gamma_A)+\sharp T(\Gamma_A)$ is even when the degree of Γ_A is even. Therefore, $w_C(I'(\Gamma,A))$ is always zero, and $w_C\left(\check{Z}_{n,3n-2}(\check{R},K,(\omega(i))_{i\in\underline{3n-2}}\right)$ is independent of the chosen $\omega(i)$. The variation of $p^c\left(\check{Z}_{n,3n-2}(\check{R},K,(\omega(i))_{i\in\underline{3n-2}}\right)$ is

$$\sum_{(\Gamma,A)\in\mathcal{D}^{e,F}_{n,3n-2}(S^1)}\zeta_\Gamma p^c\check{p}\left(I'(\Gamma,A)\right),$$

where w_C sends $p^c\check{p}\left(I'(\Gamma,A)\right)$ to zero, for the same reasons as above. $\square$

9.2 Sketch of proof of Proposition 9.2

According to Stokes's theorem, for any $\Gamma\in\mathcal{D}^e_{n,m}(\mathcal{L})$, where $m=3n$ or $m=3n-2$,

$$\begin{aligned}I(R,L,\Gamma,(\tilde{\omega}(i,1))_{i\in\underline{m}}) = {} & I(R,L,\Gamma,(\tilde{\omega}(i,0))_{i\in\underline{m}})\\ & +\textstyle\sum_F\int_{[0,1]\times F}\bigwedge_{e\in E(\Gamma)}p_e^*(\tilde{\omega}(j_E(e))),\end{aligned}$$

where the sum runs over the codimension-one faces F of $C(R,L;\Gamma)$, which are described in Proposition 8.18. Let $\tilde{p}_e\colon C(R,L;\Gamma)\to C_e(R)$ be the natural restriction and set

$$p(\Gamma)=\mathbf{1}_{[0,1]}\times\prod_{e\in E(\Gamma)}\tilde{p}_e\colon [0,1]\times C(R,L;\Gamma)\to[0,1]\times\prod_{e\in E(\Gamma)}C_e(R).$$

Let $p_{j_E(e_0)}\colon [0,1]\times\prod_{e\in E(\Gamma)} C_e(R) \to [0,1]\times C_2(R)$ be the composition of the natural projection onto $[0,1]\times C_{e_0}(R)$ and the natural identification of $[0,1]\times C_{e_0}(R)$ with $[0,1]\times C_2(R)$, and define the form

$$\Omega_{E(\Gamma)} = \bigwedge_{e\in E(\Gamma)} p^*_{j_E(e)}(\tilde{\omega}(j_E(e)))$$

on $[0,1]\times\prod_{e\in E(\Gamma)} C_e(R)$. Set

$$I(\Gamma,A) = \int_{[0,1]\times F(A,L,\Gamma)} p(\Gamma)^*(\Omega_{E(\Gamma)})[\Gamma]$$

for any subset A of $V(\Gamma)$ of cardinality at least 2 (where $F(A,L,\Gamma)$ is empty (and hence $I(\Gamma,A)=0$) if $A\cap U(\Gamma)$ is not a set of consecutive vertices on one component of $\mathcal{L}$). Set

$$I(\Gamma,A,\infty) = \int_{[0,1]\times F_\infty(A,L,\Gamma)} p(\Gamma)^*(\Omega_{E(\Gamma)})[\Gamma]$$

for any subset A of $V(\Gamma)$ of cardinality at least 1 (where $F_\infty(A,L,\Gamma)$ is empty (and hence $I(\Gamma,A)=0$) if $A\cap U(\Gamma)$ is not empty).

$$Z_{n,m}(1) - Z_{n,m}(0) =$$
$$\sum_{\Gamma\in\mathcal{D}^e_{n,m}(\mathcal{L})}\left(\sum_{A\in\mathcal{P}_{\geq 2}(V(\Gamma))}\zeta_\Gamma I(\Gamma,A) + \sum_{A\in\mathcal{P}_{\geq 1}(V(\Gamma))}\zeta_\Gamma I(\Gamma,A,\infty)\right)$$

In order to prove Proposition 9.2, it suffices to prove that the codimension-one faces F of the $C(R,L;\Gamma)$ that do not appear in the statement of Proposition 9.2 do not contribute.

This is the consequence of Lemmas 9.8 to 9.14, together with the analysis before Lemma 9.13.

Lemma 9.8. *For any $\Gamma\in\mathcal{D}^e_{n,m}(\mathcal{L})$, For any non-empty subset A of $V(\Gamma)$, $I(\Gamma,A,\infty)=0$.*

Proof: Recall from Section 8.4 that

$$F_\infty(A,L,\Gamma) = B_\infty(A,L,\Gamma)\times\check{\mathcal{S}}(T_\infty R,A).$$

Let E_C be the set of the edges of Γ that contain an element of $V(\Gamma)\setminus A$ and an element of A. Let p_2 denote the projection of $F_\infty(A,L,\Gamma)$ onto $\check{\mathcal{S}}(T_\infty R,A)$.

For $e \in E_A \cup E_C$, $P_e : (S^2)^{E_A \cup E_C} \longrightarrow S^2$ is the projection onto the factor indexed by e. We prove that there exists a smooth map

$$g : \check{\mathcal{S}}(T_\infty R, A) \longrightarrow (S^2)^{E_A \cup E_C}$$

such that

$$\bigwedge_{e \in E_A \cup E_C} p_e^*(\tilde{\omega}(j_E(e))) = (\mathbf{1}_{[0,1]} \times (g \circ p_2))^* \left(\bigwedge_{e \in E_A \cup E_C} (\mathbf{1}_{[0,1]} \times P_e)^*(\tilde{\omega}_{S^2}(j_E(e))) \right).$$

If $e \in E_A \cup E_C$, then $p_e(F_\infty(A, L, \Gamma)) \subset \partial C_2(R) \setminus U(\check{R})$, we have

$$(\mathbf{1}_{[0,1]} \times p_e)^*(\tilde{\omega}(j_E(e))) = (\mathbf{1}_{[0,1]} \times (p_\tau \circ p_e))^*(\tilde{\omega}_{S^2}(j_E(e))),$$

and $p_\tau \circ p_e$ factors through $\check{\mathcal{S}}(T_\infty R, A)$ (and therefore may be expressed as $((P_e \circ g) \circ p_2))$. Indeed, if $e \in E_C$, $p_\tau \circ p_e$ depends only on the projection on $S(T_\infty R)$ of the vertex at ∞ (of A), while, if $e \in E_A$, $p_\tau \circ p_e$ factors through $\check{\mathcal{S}}(T_\infty R, e)$.

Therefore if the degree of the form $\left(\bigwedge_{e \in E_A \cup E_C} p_e^*(\tilde{\omega}_{S^2}(j_E(e)))\right)$ is bigger than the dimension $3\sharp A$ of $[0,1] \times \check{\mathcal{S}}(T_\infty R, A)$, this form vanishes on $F_\infty(A, L, \Gamma)$. The degree of the form is $(2\sharp E_A + 2\sharp E_C)$,

$$3\sharp A = 2\sharp E_A + \sharp E_C.$$

Therefore, the integral vanishes unless E_C is empty. In this case, all the $p_\tau \circ p_e$, for $e \in E_A$ factor through the conjugates under the inversion $(x \mapsto x/\| x \|^2)$ of the translations which make sense, and the form $\left(\bigwedge_{e \in E_A} p_e^*(\tilde{\omega}_{S^2}(j_E(e)))\right)$ factors through the product by $[0,1]$ of the quotient of $\check{\mathcal{S}}(T_\infty R, A)$ by these translation conjugates. So it vanishes, too. $\square$

If there exists a smooth map from $[0,1] \times F(A, L, \Gamma)$ to a manifold of strictly smaller dimension that factorizes the restriction of

$$p(\Gamma) = \left(\mathbf{1}_{[0,1]} \times \prod_{e \in E(\Gamma)} \tilde{p}_e \right) : [0,1] \times C(R, L; \Gamma) \to [0,1] \times \prod_{e \in E(\Gamma)} C_e(R)$$

to $[0,1] \times F(A, L, \Gamma)$, then $I(\Gamma, A) = 0$. We use this principle to get rid of some faces.

Lemma 9.9. *Let $\Gamma \in \mathcal{D}^e_{n,m}(\mathcal{L})$. For any subset A of $V(\Gamma)$ such that the graph Γ_A defined before Proposition 9.2 is not connected, and Γ_A is not a pair of univalent vertices, $I(\Gamma, A) = 0$.*

Proof: In the fiber $\check{\mathcal{S}}_A(T_{m(A)}\check{R}, L, \Gamma)$ of $F(A, L, \Gamma)$ we may translate one connected component of Γ_A whose set of vertices is C, independently, without changing the restriction of $p(\Gamma)$ to $F(A, L, \Gamma)$, by translation, in the direction of $T_{m(A)}L$ when C contains univalent vertices. Unless C and $A \setminus C$ are reduced to a univalent vertex, the quotient of $\check{\mathcal{S}}_A(T_{m(A)}\check{R}, L, \Gamma)$ by these translations has a smaller dimension than $\check{\mathcal{S}}_A(T_{m(A)}\check{R}, L, \Gamma)$, and $p(\Gamma)$ factors through the corresponding quotient of $[0, 1] \times F(A, L, \Gamma)$. $\square$

Lemma 9.10. *Let $\Gamma \in \mathcal{D}^e_{n,m}(\mathcal{L})$. Let A be a subset of $V(\Gamma)$ such that $\sharp A \geq 3$. If some trivalent vertex of A belongs to exactly one edge of Γ_A, then $I(\Gamma, A) = 0$.*

Proof: Let b be the mentioned trivalent vertex, and let e be its edge in Γ_A, let $d \in A$ be the other element of e. The group $]0, \infty[$ acts on the map t from A to $T_{c(b)}R$ by moving $t(b)$ on the half-line from $t(d)$ through $t(b)$, by multiplying $(t(b) - t(d))$ by a scalar. When $\sharp A \geq 3$, this action is not trivial on $\check{\mathcal{S}}_A(T_{m(A)}\check{R}, L, \Gamma)$, and $p(\Gamma)$ factors through the corresponding quotient of $[0, 1] \times F(A, L, \Gamma)$ by this action, which is of smaller dimension. $\square$

9.3 Cancellations of non-degenerate faces

From now on, we are going to study cancellations that are no longer individual, and orientations have to be taken seriously into account. Recall that the codimension-one faces are oriented as parts of the boundary of $C(R, L; \Gamma)$, with the outward normal first convention, where $C(R, L; \Gamma)$ is oriented by an orientation of L and an order on $V(\Gamma)$. The relations between an orientation of $V(\Gamma)$, which orients $C(R, L; \Gamma)$, a vertex-orientation of Γ and an edge-orientation of the set $H(\Gamma)$ of half-edges of Γ are explained in Lemma 7.1, Corollary 7.2 and Remark 7.5. Fortunately, in order to compare similar orientations, we do not have to fix everything.

Lemma 9.11. *Let $\Gamma \in \mathcal{D}^e_{n,m}(\mathcal{L})$. Let A be a subset of $V(\Gamma)$ such that at least one element of A belongs to exactly two edges of Γ_A. Let $\mathcal{E}(\Gamma, A)$ denote the set of graphs of $\mathcal{D}^e_{n,m}(\mathcal{L})$ that are isomorphic to Γ by an isomorphism that is only allowed to change the labels and the orientations of the edges of Γ_A. Such an isomorphism preserves A, and*

$$\sum_{\tilde{\Gamma} \mid \tilde{\Gamma} \in \mathcal{E}(\Gamma, A)} \zeta_{\tilde{\Gamma}} I(\tilde{\Gamma}, A) = 0.$$

Proof: Let us first check that the isomorphisms of the statement preserve A. The vertices of the elements of $\mathcal{D}^e_{n,m}(\mathcal{L})$ are not numbered. A vertex is characterized by the half-edges that contain it. Therefore, the isomorphisms of the statement preserve the vertices of $V(\Gamma) \setminus A$, so they preserve A setwise. These isomorphisms also preserve the vertices that have adjacent edges outside $E(\Gamma_A)$ pointwise. The isomorphisms described below actually induce the identity map on $V(\Gamma)$.

Among the vertices of A that belong to exactly two edges of Γ_A and one edge $j_E^{-1}(k)$ of Γ outside Γ_A, choose the vertices such that k is minimal. If there is one such vertex, then call this vertex v_m. Otherwise, there are two choices, and v_m is the vertex that belongs to the first half-edge of $j_E^{-1}(k)$.

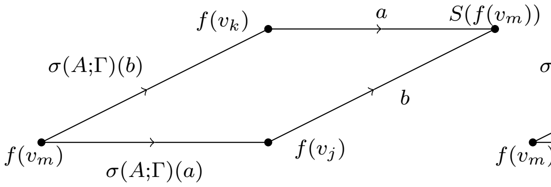
We first describe an orientation-reversing diffeomorphism of the complement of a codimension-three submanifold of $F(A, L, \Gamma)$. Let v_j and v_k denote the (possibly equal) two other vertices of the two edges of Γ_A that contain v_m. Consider the transformation S of the space $\overline{\mathcal{S}}_A(T_{c(A)}R)$ of non-constant maps f from A to $T_{c(A)}R$ up to translations and dilations, that maps f to $S(f)$, where

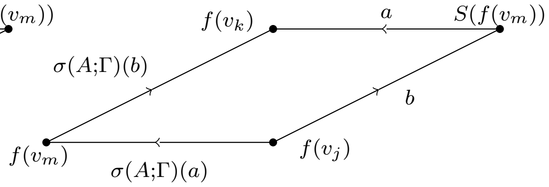
$$\begin{aligned} S(f)(v_\ell) &= f(v_\ell) \text{ if } v_\ell \neq v_m, \text{ and,} \\ S(f)(v_m) &= f(v_j) + f(v_k) - f(v_m). \end{aligned}$$

This is an orientation-reversing involution of $\overline{\mathcal{S}}_A(T_{c(A)}R)$. The set of elements of $\check{\mathcal{S}}_A(T_{c(A)}R)$ whose image under S is not in $\check{\mathcal{S}}_A(T_{c(A)}R)$ is a codimension-three submanifold of $\check{\mathcal{S}}_A(T_{c(A)}R)$. The fibered product of S by the identity of the base $B(A, L, \Gamma)$ is an orientation-reversing smooth involution outside a codimension-three submanifold F_S of $F(A, L, \Gamma)$. It is still denoted by S, as its product by $\mathbf{1}_{[0,1]}$ is, too.

Now, let $\sigma(A; \Gamma)(\tilde{\Gamma})$ be obtained from $(\tilde{\Gamma} \in \mathcal{E}(\Gamma, A))$ by exchanging the labels of the two edges of Γ_A that contain v_m, and by reversing their orientations if (and only if) they both start or end at v_m. Then, as the following pictures show,

$$p(\tilde{\Gamma}) \circ S = p(\sigma(A; \Gamma)(\tilde{\Gamma})).$$





Recall

$$I(\Gamma, A) = \int_{[0,1] \times F(A,L,\Gamma)} p(\Gamma)^*(\Omega_{E(\Gamma)})[\Gamma],$$

with the map $p(\Gamma) = \left(\mathbf{1}_{[0,1]} \times \prod_{e\in E(\Gamma)} \tilde{p}_e\right) : [0,1] \times C(R,L;\Gamma) \to [0,1] \times \prod_{e\in E(\Gamma)} C_e(R)$ and $\Omega_{E(\Gamma)} = \bigwedge_{e\in E(\Gamma)} p^*_{j_E(e)}(\tilde{\omega}(j_E(e)))$.

$$\begin{aligned} I(\tilde{\Gamma},A) &= \int_{[0,1]\times(F(A,L,\tilde{\Gamma})\setminus F_S)} p(\tilde{\Gamma})^*(\Omega_{E(\Gamma)})[\tilde{\Gamma}] \\ &= -\int_{[0,1]\times(F(A,L,\tilde{\Gamma})\setminus F_S)} S^*\left(p(\tilde{\Gamma})^*(\Omega_{E(\Gamma)})\right)[\tilde{\Gamma}] \\ &= -\int_{[0,1]\times(F(A,L,\tilde{\Gamma})\setminus F_S)} (p(\tilde{\Gamma})\circ S)^*(\Omega_{E(\Gamma)})[\tilde{\Gamma}] \\ &= -\int_{[0,1]\times(F(A,L,\tilde{\Gamma})\setminus F_S)} p(\sigma(A;\Gamma)(\tilde{\Gamma}))^*(\Omega_{E(\Gamma)})[\tilde{\Gamma}] \\ &= -I(\sigma(A;\Gamma)(\tilde{\Gamma}),A) \end{aligned}$$

since $[\tilde{\Gamma}] = [\sigma(A;\Gamma)(\tilde{\Gamma})]$. Now, $\sigma(A;\Gamma)$ defines an involution of $\mathcal{E}(\Gamma,A)$, and it is easy to conclude:

$$\sum_{\tilde{\Gamma}\in\mathcal{E}(\Gamma,A)} I(\tilde{\Gamma},A) = \sum_{\tilde{\Gamma}\in\mathcal{E}(\Gamma,A)} I(\sigma(A;\Gamma)(\tilde{\Gamma}),A) = -\sum_{\tilde{\Gamma}\in\mathcal{E}(\Gamma,A)} I(\tilde{\Gamma},A) = 0.$$

□

The symmetry used in the above proof was observed by Kontsevich in [Kon94].

Lemma 9.12. *Let $\Gamma \in \mathcal{D}^e_n(\mathcal{L})$. If A is a subset of $V(\Gamma)$ such that Γ_A is a connected diagram of even degree without univalent vertices, and if $\Gamma^{eo}(A)$ denotes the graph obtained from Γ by reversing all the orientations of the edges of Γ_A, then*

$$I(\Gamma,A) + I(\Gamma^{eo}(A),A) = 0.$$

When (Γ,A) is an element of the set $\mathcal{D}^{e,F}_{n,m}(\mathcal{L})$ defined in Proposition 9.2, let $\Gamma^{eo,rev}(A)$ denote the graph obtained from Γ by reversing all the orientations of the edges of Γ_A and by reversing the order of the univalent vertices of Γ_A on $\mathcal{L}_A$. Recall

$$I'(\Gamma,A) = \int_{[0,1]\times F(A,L,\Gamma)} \bigwedge_{e\in E(\Gamma)} p^*_e(\tilde{\omega}(j_E(e)))\frac{1}{2}\left([\Gamma] - (-1)^{\sharp E(\Gamma_A)}[\Gamma^{rev}(A)]\right)$$

from Proposition 9.7. Then

$$I(\Gamma,A) + I(\Gamma^{eo,rev}(A),A) = I'(\Gamma,A) + I'(\Gamma^{eo,rev}(A),A).$$

PROOF: Set $\overline{\Gamma}(A) = \Gamma^{\mathrm{eo,rev}}(A)$ in both cases. The opposite of the identity map of $T_{c(A)}\check{R}$ induces a diffeomorphism from the fiber of $F(A,L,\Gamma)$ to the fiber of $F(A,L,\overline{\Gamma}(A))$, which induces a diffeomorphism from $F(A,L,\Gamma)$ to $F(A,L,\overline{\Gamma}(A))$ over the identity map of the base. Denote by $\mathcal{S}$ the product of this diffeomorphism by $\mathbf{1}_{[0,1]}$, and let us discuss orientations carefully, to see when this diffeomorphism preserves the orientation.

Order and orient the vertices of Γ, so that the corresponding orientation of $H(\Gamma)$, as in Remark 7.5, is that induced by the edge-orientation of Γ. There is a natural bijection from $V(\Gamma)$ to $V(\overline{\Gamma}(A))$. This bijection is the identity on $V(\Gamma)\setminus A$, which is the set of vertices of $\Gamma\setminus\Gamma_A$, which is unaffected by the modifications. When A is not a pair of vertices in a θ-component, a vertex of Γ_A is characterized by the labels of the edges that contain it, such a vertex is mapped to the vertex of $\overline{\Gamma}(A)$ with the same set of labels of adjacent edges. When A is a pair of vertices in a θ-component, the vertex at which an edge of Γ_A labeled by i begins, is sent to the vertex of $\overline{\Gamma}(A)$, at which the edge of $\overline{\Gamma}(A)$ labeled by i ends.

Order the vertices of $\overline{\Gamma}(A)$ like the vertices of Γ if $\sharp E(\Gamma_A)$ is even, and permute two vertices if $\sharp E(\Gamma_A)$ is odd. Orient $C(R,L;\Gamma)$ and $C(R,L;\overline{\Gamma}(A))$ with respect to the above orders of $V(\Gamma)$ and $V(\overline{\Gamma}(A))$, using the orientations of R and L. Then $\mathcal{S}$ reverses the orientation if and only if $\sharp E(\Gamma_A)$ is even, since $\sharp V(\Gamma_A)$ is even.

Orient the vertices of $\overline{\Gamma}(A)$ like the vertices of Γ. So the orientation of $H(\overline{\Gamma}(A))$, associated to that vertex-orientation and to the above order of vertices, is that induced by the edge-orientation of $\overline{\Gamma}(A)$. When orientations are fixed as above, set $I_0(\Gamma,A)=\int_{[0,1]\times F(A,L,\Gamma)} p(\Gamma)^*(\Omega_{E(\Gamma)})$ so that $I(\Gamma,A)=I_0(\Gamma,A)[\Gamma]$. Let e be an edge of Γ. Forget its orientation, so that e is also an edge of $\overline{\Gamma}(A)$, and $p_{j_E(e)}\circ p(\overline{\Gamma}(A))$ restricts to $[0,1]\times F(A,L,\overline{\Gamma}(A))$ as $p_{j_E(e)}\circ p(\Gamma)\circ\mathcal{S}^{-1}$.

$$\begin{aligned}
I_0(\overline{\Gamma}(A)),A) &= \textstyle\int_{[0,1]\times F(A,L,\overline{\Gamma}(A))} p(\overline{\Gamma}(A))^*(\Omega_{E(\overline{\Gamma}(A))})\\
&= \textstyle\int_{[0,1]\times F(A,L,\overline{\Gamma}(A))} \bigwedge_{e\in E(\Gamma)} \left(p_{j_E(e)}\circ p(\overline{\Gamma}(A))\right)^*(\tilde{\omega}(j_E(e)))\\
&= \textstyle\int_{[0,1]\times F(A,L,\overline{\Gamma}(A))} (\mathcal{S}^{-1})^*\left(\bigwedge_{e\in E(\Gamma)} \left(p_{j_E(e)}\circ p(\Gamma)\right)^*(\tilde{\omega}(j_E(e)))\right)\\
&= (-1)^{\sharp E(\Gamma_A)+1}\textstyle\int_{[0,1]\times F(A,L,\Gamma)} \bigwedge_{e\in E(\Gamma)} \left(p_{j_E(e)}\circ p(\Gamma)\right)^*(\tilde{\omega}(j_E(e)))\\
&= (-1)^{\sharp E(\Gamma_A)+1} I_0(\Gamma,A).
\end{aligned}$$

In particular, if Γ_A is a connected diagram of even degree without univalent vertices, then $\sharp E(\Gamma_A)$ is even, and $[\Gamma]=[\overline{\Gamma}(A)]=[\Gamma^{\mathrm{eo}}(A)]$, so $I(\Gamma,A)=-I(\Gamma^{\mathrm{eo}}(A),A)$. Otherwise,

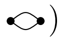
$$\begin{aligned}
I(\Gamma,A)+I(\overline{\Gamma}(A),A) &= \tfrac{1}{2}\left(I_0(\Gamma,A)-(-1)^{\sharp E(\Gamma_A)}I_0(\overline{\Gamma}(A)),A)\right)[\Gamma]\\
&\quad +\tfrac{1}{2}\left(I_0(\overline{\Gamma}(A)),A)-(-1)^{\sharp E(\Gamma_A)}I_0(\Gamma,A)\right)[\Gamma^{\mathrm{rev}}(A))]\\
&= I'(\Gamma,A)+I'(\overline{\Gamma}(A),A).
\end{aligned}$$

$\square$

Lemmas 9.9, 9.10 and 9.11 allow us to get rid of the pairs $(A;\Gamma)$ with $\sharp A\geq 3$ such that

- at least one element of A does not have all its adjacent edges in $E(\Gamma_A)$, or
- Γ_A is disconnected.

Therefore, according to Lemma 9.8, and to Lemma 9.12, which rules out the case where Γ_A is an even degree connected component of Γ, without univalent vertices (where $\sharp A \equiv 0 \bmod 4$), we are left with

- the pairs (Γ, A) of the statement of Proposition 9.2, for which Γ_A is a connected component of Γ (which may be an edge between two univalent vertices), and
- the following pairs, for which $\sharp A = 2$ (since Lemma 9.9 rules out the disconnected Γ_A with a trivalent vertex, and Lemma 9.11 rules out $\Gamma_A =$ )
  - Γ_A is an edge between two trivalent vertices,
  - Γ_A is an edge between a trivalent vertex and a univalent one,
  - Γ_A is a pair of isolated consecutive univalent vertices.

The following lemma allows us to get rid of the case where Γ_A is an edge between two trivalent vertices using the Jacobi relation.

Lemma 9.13. *The contributions to $(Z_{n,m}(1)-Z_{n,m}(0))$ of the faces $F(A,L,\Gamma)$ for which Γ_A is an edge between two trivalent vertices, cancel. More precisely, let $\Gamma \in \mathcal{D}^e_{n,m}(\mathcal{L})$. Let A be a subset of $V(\Gamma)$ such that Γ_A is an edge $e(\ell)$ with label ℓ. Let Γ/Γ_A be the labelled edge-oriented graph obtained from Γ by contracting Γ_A to one point, as in Figure 9.1. (The labels of the edges of Γ/Γ_A belong to $\underline{3n} \setminus \{\ell\}$, Γ/Γ_A has one four-valent vertex and its other vertices are univalent or trivalent.) Let $\mathcal{E}(\Gamma;A)$ be the subset of $\mathcal{D}^e_{n,m}(\mathcal{L})$ that contains the graphs $\tilde{\Gamma}$ equipped with a pair A of vertices joined by an edge $e(\ell)$ with label ℓ such that $\tilde{\Gamma}/\tilde{\Gamma}_A$ is equal to Γ/Γ_A. Then*

$$\sum_{\tilde{\Gamma} \mid \tilde{\Gamma} \in \mathcal{E}(\Gamma;A)} \zeta_{\tilde{\Gamma}} I(\tilde{\Gamma}, A) = 0.$$

The contributions to $\left(\check{Z}_{n,3n-2}(1) - \check{Z}_{n,3n-2}(0)\right)$ of these types of faces cancel in the same way.

Proof: Let us prove that there are 6 graphs in $\mathcal{E}(\Gamma;A)$. Let a, b, c, d be the four half-edges of Γ/Γ_A that contain its four-valent vertex. In $\tilde{\Gamma}$, Edge $e(\ell)$ goes from a vertex $v(\ell,1)$ to a vertex $v(\ell,2)$. Vertex $v(\ell,1)$ is adjacent to the

Figure 9.1: The graph Γ, its bold subgraph Γ_A, and Γ/Γ_A

first half-edge of $e(\ell)$ and to two half-edges of $\{a,b,c,d\}$. The unordered pair of $\{a,b,c,d\}$ adjacent to $v(\ell,1)$ determines $\tilde{\Gamma}$ as an element of $\mathcal{D}^e_{n,m}(\mathcal{L})$ and there are 6 graphs in $\mathcal{E}(\Gamma;A)$ labelled by the pairs of elements of $\{a,b,c,d\}$. They are $\Gamma=\Gamma_{ab}$, Γ_{ac}, Γ_{ad}, Γ_{bc}, Γ_{bd} and Γ_{cd}.

The face $F(A,L,\Gamma)$ is fibered over $B(A,L,\Gamma)$ with fiber $\check{\mathcal{S}}_A(T_{c(v(\ell,1))}R) =_\tau S^2$, which contains the direction of the vector from $c(v(\ell,1))$ to $c(v(\ell,2))$. Consistently order the vertices of the $\Gamma_{..}$ starting with $v(\ell,1)$, $v(\ell,2)$ (the other vertices are in natural correspondences for different $\Gamma_{..}$). Use these orders to orient the configuration spaces $C(R,L;\Gamma_{..})$.

The oriented face $F(A,L,\Gamma_{..})$ and the map

$$p(\Gamma_{..})\colon [0,1]\times (F(A,L,\Gamma_{..})\subset C(R,L;\Gamma_{..})) \longrightarrow [0,1]\times \prod_{e\in E(\Gamma_{..})} C_2(R)^e$$

are the same for all the elements $\Gamma_{..}$ of $\mathcal{E}(\Gamma;A)$. Therefore the

$$I_0(\Gamma_{..},A)=\int_{[0,1]\times F(A,L,\Gamma_{..})} p(\Gamma_{..})^*(\Omega_{E(\Gamma_{..})})$$

are the same for all the elements $\Gamma_{..}$ of $\mathcal{E}(\Gamma;A)$ (for our consistent orders of the vertices), and the sum of the statement is

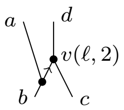
$$\sum_{\tilde{\Gamma}|\tilde{\Gamma}\in\mathcal{E}(\Gamma;A)} \zeta_{\tilde{\Gamma}} I_0(\tilde{\Gamma},A)[\tilde{\Gamma}] = \zeta_\Gamma I_0(\Gamma,A) \sum_{\tilde{\Gamma}|\tilde{\Gamma}\in\mathcal{E}(\Gamma;A)} [\tilde{\Gamma}].$$

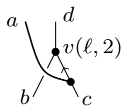
Let e_1 be the first half-edge of $e(\ell)$, and let e_2 be the other half-edge of $e(\ell)$. Equip $\Gamma=\Gamma_{ab}$ with a vertex-orientation, which is represented by (a,b,e_1) at $v(\ell,1)$, and (c,d,e_2) at $v(\ell,2)$, and which is consistent with its given edge-orientation (i.e. such that the edge-orientation of $H(\Gamma)$ is equivalent to its vertex-orientation, with respect to the above order of vertices). The orientation of $H(\Gamma)$ is represented by $(a,b,e_1,c,d,e_2,\dots)$. It induces the edge-orientation of $H(\Gamma)$, which is the same for all the elements of $\mathcal{E}(\Gamma;A)$.

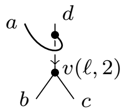
Thus, permuting the letters b,c,d cyclically gives rise to two other graphs (Γ_{ac} and Γ_{ad}) in $\mathcal{E}(\Gamma;A)$ equipped with suitable vertex-orientations, which are respectively represented by

$$(a, c, e_1) \text{ at } v(\ell, 1) \text{ and } (d, b, e_2) \text{ at } v(\ell, 2), \text{ or}$$
$$(a, d, e_1) \text{ at } v(\ell, 1) \text{ and } (b, c, e_2) \text{ at } v(\ell, 2).$$

The three other elements of $\mathcal{E}(\Gamma; A)$ with their suitable vertex-orientation are obtained from the three previous ones by exchanging the ordered pair before e_1 with the ordered pair before e_2. This amounts to exchanging the vertices $v(\ell, 1)$ and $v(\ell, 2)$ in the picture, and does not change the unlabelled vertex-oriented graph. The first three graphs can be represented by three graphs identical outside the pictured disk:



,


and


Then the sum $\sum_{\tilde{\Gamma} | \tilde{\Gamma} \in \mathcal{E}(\Gamma; A)} [\tilde{\Gamma}]$ is zero thanks to the Jacobi relation (or IHX). $\square$

Now, we get rid of the remaining faces with the help of the *STU* relation.

Lemma 9.14. *The contributions to $(Z_n(1) - Z_n(0))$ or to $(\check{Z}_{n,3n-2}(1) - \check{Z}_{n,3n-2}(0))$ of the faces $F(A, L, \Gamma)$ such that*

- *Γ_A is an edge between a trivalent vertex and a univalent vertex, or,*
- *Γ_A is a pair of consecutive univalent vertices, and Γ_A is not an edge of Γ,*

cancel. More precisely, let $\Gamma \in \mathcal{D}^e_{n,m}(\mathcal{L})$, let A be a pair of consecutive univalent vertices of Γ on a component of $\mathcal{L}$, and assume that Γ_A is not an edge of Γ. Let Γ/Γ_A be the labelled edge-oriented graph obtained from Γ by contracting Γ_A to one point. (The labels of the edges of Γ/Γ_A belong to $\underline{m}$, Γ/Γ_A has one bivalent vertex injected on $\mathcal{L}$.) Let $\mathcal{E}(\Gamma/\Gamma_A)$ be the subset of $\mathcal{D}^e_{n,m}(\mathcal{L})$ that contains the graphs $\tilde{\Gamma}$ equipped with a pair A of vertices that are either

- *two consecutive univalent vertices, or,*
- *a univalent vertex and a trivalent vertex connected by an edge,*

such that Γ/Γ_A is equal to $\tilde{\Gamma}/\tilde{\Gamma}_A$. If $m = 3n$ or if $m = 3n - 2$, then

$$\sum_{\tilde{\Gamma} | \tilde{\Gamma} \in \mathcal{E}(\Gamma; A)} \zeta_{\tilde{\Gamma}} I(\tilde{\Gamma}, A) = 0.$$

Proof: Note that the face $F(A, L, \Gamma)$ has two connected components if the only univalent vertices of Γ on the component of $\mathcal{L}$ of the univalent vertices of A are the two vertices of A. The two connected components correspond to the two possible linear orders of A at the collapse.

Below, we consider these connected components as two different faces, and a face corresponds to a subset A equipped with a linear order compatible with i_Γ. In particular, the graph and its face are determined by the labelled edge-oriented graph Γ/Γ_A obtained from Γ by contracting A to one point, together with a linear order of the two half-edges of the bivalent vertex. Let $k \in \underline{m} \setminus j_E(E(\Gamma))$. Define Γ_k^+ (resp. Γ_k^-) to be the graph in $\mathcal{D}^e_{n,m}(\mathcal{L})$ with an edge $e(k)$ such that $j_E(e(k)) = k$, which goes from a univalent vertex u to a trivalent vertex t (resp. from a trivalent vertex t to a univalent vertex u) forming a pair $A = \{u, t\}$ such that $\Gamma_k^+/\Gamma_{k,A}^+$ (resp. $\Gamma_k^-/\Gamma_{k,A}^-$) coincides with Γ/Γ_A.

Order the sets of vertices of the $\Gamma_k^\pm$ by putting the vertices of A first, with respect to the order induced by the edge orientation (source first), and so that the orders of the remaining vertices are the same for all $\Gamma_k^\pm$. For $(\tilde{\Gamma}, A) \in \mathcal{E}(\Gamma/\Gamma_A)$ such that A is an ordered pair of univalent vertices of $\tilde{\Gamma}$, order $V(\tilde{\Gamma})$ by putting the vertices of A first with respect to the linear order induced by the collapse, and next the other ones with the same order as for the $\Gamma_k^\pm$.

Let $\phi \colon \mathbb{R}^3 \to \check{R}$ be an orientation-preserving diffeomorphism onto a neighborhood of the image of A in a configuration of $F(A, L, \Gamma)$. There exists $\rho_\phi \colon \mathbb{R}^3 \to GL^+(\mathbb{R}^3)$ such that $T_x\phi\left(\rho_\phi(x)\left(\vec{v}\right)\right) = \tau(\phi(x), \vec{v})$, for any $\vec{v} \in (\mathbb{R}^3 = T_x\mathbb{R}^3)$. Then for Γ_k^+, the configuration space is locally diffeomorphic to $\mathcal{L} \times \check{R} \times \dots$, where $\mathcal{L}$ contains the position $c(u) = \phi(x)$ of the univalent vertex u of A, and $\check{R}$ contains the position $c(t) = \phi(x + \lambda\rho_\phi(x)(\vec{v}))$ of the trivalent vertex t of A, for a small positive λ, which plays the role of an inward normal near the collapse, with $\vec{v} \in S^2$, which is equal to $p_\tau \circ p_{e(k)}(c)$, when λ reaches 0. The face is diffeomorphic to $S^2 \times \mathcal{L} \times \dots$, where the projection onto the factor S^2 is $p_\tau \circ p_{e(k)}$, and the dots contain the coordinates of the remaining vertices, which are the same for all the considered diagrams.

For Γ_k^-, the configuration space is locally diffeomorphic to $\check{R} \times \mathcal{L} \times \dots$, where $\mathcal{L}$ contains the position $c(u) = \phi(x)$ of the univalent vertex u of A, and $\check{R}$ contains the position $c(t) = \phi(x - \lambda\rho_\phi(x)(\vec{v}))$ of the trivalent vertex t of A, for a small positive λ, which still plays the role of an inward normal near the collapse, with $\vec{v} \in S^2$, which is $p_\tau \circ p_{e(k)}(c)$, when λ reaches 0. The face is again diffeomorphic to $S^2 \times \mathcal{L} \times \dots$, where the projection onto the factor S^2 is $p_\tau \circ p_{e(k)}$, and the dots contain the coordinates of the remaining vertices, which are the same as for the Γ_k^+.

For $(\tilde{\Gamma}, A) \in \mathcal{E}(\Gamma/\Gamma_A)$ such that A is an ordered pair of univalent vertices of $\tilde{\Gamma}$, the configuration space is locally diffeomorphic to $\mathcal{L}\times\mathcal{L}\times\dots$, where the first $\mathcal{L}$ contains the position $c(u_1)=\phi(x)$ of the first univalent vertex u_1 of A, and the second $\mathcal{L}$ contains the position $c(u_2)=\phi(x+\lambda\rho_\phi(x)(\vec{t}))$ of the vertex u_2 that follows u_1 along $\mathcal{L}$, for a small positive λ, which still plays the role of an inward normal near the collapse, where p_τ maps the oriented unit tangent vector to L at $c(u_1)$ to $\vec{t}\in S^2$, when λ reaches 0. The face is diffeomorphic to $\mathcal{L}\times\dots$ in the same notation as before. So the previous faces are the products by S^2 of this one, and $p_\tau\circ p_{e(k)}$ is the projection to the factor S^2. Since the other p_e do not depend on this factor S^2, $\int_{[0,1]\times F(A,L,\Gamma_k^+)} p(\Gamma_k^+)^*(\Omega_{E(\Gamma_k^+)})$ is equal to

$$\int_{(t,c)\in[0,1]\times F(A,L,\Gamma)}\left(\int_{\{t\}\times\check{\mathcal{S}}_{e(k)}(T_{c(A)}\check{R})}\tilde{\omega}(k)\right)p(\Gamma)^*(\Omega_{E(\Gamma)}),$$

where $\{t\}\times\check{\mathcal{S}}_{e(k)}(T_{c(A)}\check{R})$ is the factor S^2 above. Furthermore,

$$\left(\int_{\{t\}\times\check{\mathcal{S}}_{e(k)}(T_{c(A)}\check{R})}\tilde{\omega}(k)\right)=1$$

since the integral of the closed form $\tilde{\omega}(k)$ over any representative of the homology class of the fiber of the unit tangent bundle of $\check{R}$ in $[0,1]\times\partial C_2(R)$ is 1.

This argument, which also works for Γ_k^-, implies that all the integrals $I_0(\tilde{\Gamma},A)=\int_{[0,1]\times F(A,L,\tilde{\Gamma})}p(\tilde{\Gamma})^*(\Omega_{E(\tilde{\Gamma})})$ coincide for all the $(\tilde{\Gamma},A)\in\mathcal{E}(\Gamma/\Gamma_A)$ equipped with orders of their vertices as above, and it suffices to prove that

$$\zeta_\Gamma([\Gamma]+[\Gamma'])+\sum_{k\in\underline{m}\setminus j_E(E(\Gamma))}\zeta_{\Gamma_k^+}([\Gamma_k^+]+[\Gamma_k^-])=0,$$

where Γ' is the graph obtained from Γ by permuting the order of the two univalent vertices on $\mathcal{L}$, and where all the graphs $\tilde{\Gamma}$ are vertex-oriented so that the vertex-orientation of $H(\tilde{\Gamma})$ induced by the fixed order of the vertices coincides with the edge-orientation of $H(\tilde{\Gamma})$ (as in Remark 7.5), for $m=3n-2$ and for $m=3n$.

Let a and b denote the half-edges of Γ that contain the vertices of A. Assume, without loss of generality, that the vertex of b follows the vertex of a on $\mathcal{L}$ for Γ (near the connected face). Let $o_V(H(\Gamma)\setminus\{a,b\})$ be an order of $H(\Gamma)\setminus\{a,b\}$, such that the order $(a,b,o_V(H(\Gamma)\setminus\{a,b\}))$ (i.e. (a,b) followed by the elements $H(\Gamma)\setminus\{a,b\}$ ordered by $o_V(H(\Gamma)\setminus\{a,b\})$) induces the edge-orientation of $H(\Gamma)$.

Let f (resp. s) denote the first (resp. second) half-edge of $e(k)$ in $\Gamma_k^\pm$. Then $(f,s,a,b,o_V(H(\Gamma)\setminus\{a,b\}))$ induces the edge-orientation of $H(\Gamma_k^\pm)$.

Equip the trivalent vertex of A in $\Gamma_k^{\pm}$ with the vertex-orientation $((f \text{ or } s), a, b)$, which corresponds to the picture , and equip the other vertices of $\Gamma_k^{\pm}$ with the same vertex-orientation as that in Γ. Then the vertex-orientation of $H(\Gamma_k^+)$ is induced by $(f, s, a, b, o_V(H(\Gamma) \setminus \{a, b\}))$ and coincides with its edge-orientation. Similarly, the vertex-orientation of $H(\Gamma_k^-)$ is induced by $(f, a, b, s, o_V(H(\Gamma) \setminus \{a, b\}))$ and coincides with its edge-orientation. Thus for any k, $[\Gamma_k^+] = [\Gamma_k^-]$, and $[\Gamma_k^+]$ is independent of k.

Note that $[\Gamma]$ looks like locally, and coincides with $[\Gamma_k^+]$ outside the pictured part, but Γ' must be equipped with the opposite vertex-orientation and $(-[\Gamma'])$ looks like .

Thus, we are left with the proof that

$$\zeta_\Gamma(\left[\quad\right] - \left[\quad\right]) + 2(m - \sharp E(\Gamma))\zeta_{\Gamma_k^+}\left[\quad\right] = 0$$

if $m \in \{3n-2, 3n\}$. With the expression of the ζ_Γ in Notation 7.16 and Notation 7.29, this equality is equivalent to the STU relation, when $m \neq \sharp E(\Gamma)$. In particular, it is equivalent to the STU relation when $m = 3n$. When $m = 3n-2$, if $m = \sharp E(\Gamma)$, then Γ has exactly 2 univalent vertices, so $([\Gamma] + [\Gamma']) = 0$, and the equality is still true. $\square$

Proposition 9.2 is now proved. Proposition 9.7 follows from Proposition 9.2 and Lemma 9.12. $\square$

Proof of Theorem 7.32: We follow the face cancellations in the proof of Proposition 9.2, to study the effect of changing $\omega_{S^2}(i) = \tilde{\omega}_{S^2|\{0\}\times S^2}$ to $\omega'_{S^2}(i) = \tilde{\omega}_{S^2|\{1\}\times S^2}$, for a closed two-form $\tilde{\omega}_{S^2}$ on $[0,1] \times S^2$ as in Lemma 9.1, for some $i \in \underline{3}$, and consequently using the form $\omega(i) = p^*_{[0,1]\times S^2}(\tilde{\omega}_{S^2})$ on $[0,1] \times C_2(S^3)$. Here, the involved graphs have no looped edges, 4 vertices, at most 3 edges, and hence at most one trivalent vertex. They are , and . The only cancellation that requires an additional argument is the cancellation of the faces, for which Γ is isomorphic to and A is a pair of univalent vertices of Γ. (The cancellation of Lemma 9.14 would involve .) In this case, the open face $F(A, L, \Gamma)$ is the configuration space of 2 vertices on the knot (one of them stands for the two vertices of A) and a trivalent vertex in $\mathbb{R}^3$, the integral $\int_{[0,1]\times F(A,L,\Gamma)} \bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$ is the pull-back of a 6-form on $[0,1] \times (S^2)^3$, by a map whose image is in the codimension 2 subspace of $[0,1] \times (S^2)^3$ in which two S^2-coordinates coincide, since the two edges that contain the vertices of A have the same direction. Therefore $I(\Gamma, A) = 0$

for these faces, and $w_C\left(\check{Z}_{2,3}(\mathbb{R}^3, K, (p^*_{S^2}(\omega_{S^2}(i)))_{i\in\underline{3}})\right)$ is independent of the chosen $\omega_{S^2}(i)$. Conclude with the arguments of Remark 7.31. $\square$

Chapter 10

First properties of Z and anomalies

10.1 Some properties of $Z(\check{R}, L, \tau)$

Lemma 9.5 allows us to set

$$Z_n(\check{R}, L, \tau) = Z_n(\check{R}, L, (\omega(i)))$$

for any collection $(\omega(i))$ of homogeneous propagating forms of $(C_2(R), \tau)$, under the assumptions of Theorem 7.20. We do not know yet how $Z_n(\check{R}, L, \tau)$ varies when τ varies inside its homotopy class when $L \neq \emptyset$, but the naturality of the construction of Z_n implies the following proposition.

Proposition 10.1. *Let ψ be an orientation-preserving diffeomorphism from the $\mathbb{Q}$–sphere R of Section 3.2 to $\psi(R)$. Use the restriction of ψ to the ball $\mathring{B}_{1,\infty}$ of the beginning of Section 3.2 as an identification of $\mathring{B}_{1,\infty}$ with a neighborhood of $\psi(\infty)$ in $\psi(R)$. Define $\psi_*(\tau) = T\psi \circ \tau \circ (\psi^{-1} \times \mathbf{1}_{\mathbb{R}^3})$. Then*

$$Z_n(\psi(\check{R}), \psi(L), \psi_*(\tau)) = Z_n(\check{R}, L, \tau)$$

for all $n \in \mathbb{N}$, where $p_1(\psi_*(\tau)) = p_1(\tau)$.

Proof: The diffeomorphism ψ induces natural diffeomorphisms ψ_* from $C_2(R)$ to $C_2(\psi(R))$ and from the $\check{C}(R, L; \Gamma)$ to the $\check{C}(\psi(R), \psi(L); \Gamma)$. If ω is a homogeneous propagating form of $(C_2(R), \tau)$, then $(\psi_*^{-1})^*(\omega)$ is a homogeneous propagating form of $(C_2(\psi(R)), \psi_*(\tau))$ since the restriction of $(\psi_*^{-1})^*(\omega)$ to $U\psi(\check{R})$ is $(T\psi^{-1})^*(p_\tau^*(\omega_{S^2})) = (p_\tau \circ T\psi^{-1})^*(\omega_{S^2}) = p_{\psi_*(\tau)}^*(\omega_{S^2})$. For any Jacobi diagram Γ on the source of L,

$$\begin{aligned} I\left(\psi(R), \psi(L), \Gamma, (\psi_*^{-1})^*(\omega)\right) &= \int_{\check{C}(\psi(R),\psi(L);\Gamma)} \bigwedge_{e\in E(\Gamma)} p_e^*((\psi_*^{-1})^*(\omega)) \\ &= \int_{\check{C}(\psi(R),\psi(L);\Gamma)} (\psi_*^{-1})^* \left(\bigwedge_{e\in E(\Gamma)} p_e^*(\omega)\right) \\ &= I\left(R, L, \Gamma, \omega\right), \end{aligned}$$

where Γ is equipped with an implicit orientation $o(\Gamma)$. Therefore, for all $n \in \mathbb{N}$, $Z_n(\psi(\check{R}), \psi(L), \psi_*(\tau))) = Z_n(\check{R}, L, \tau)$. $\square$

We study some other properties of $Z_n(\check{R}, L, \tau)$.

Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Thanks to Lemma 9.6, $Z_n(\check{R}, \emptyset, \tau)$ depends only on the homotopy class of τ for any integer n. Set $Z_n(R, \tau) = Z_n(\check{R}, \emptyset, \tau)$, and $Z(R, \tau) = (Z_n(R, \tau))_{n \in \mathbb{N}}$. Using Notation 7.16, let $z_n(R, \tau) = p^c(Z_n(R, \tau))$ be the connected part of $Z_n(R, \tau)$, and set $z(R, \tau) = (z_n(R, \tau))_{n \in \mathbb{N}}$.

We give a direct elementary proof of the following proposition, which could also be proved in the same way as Corollary 10.4 below.

Proposition 10.2. *For any propagating form ω of $C_2(R)$,*

$$Z(R, \omega) = \exp\left(z(R, \omega)\right).$$

In particular, for any asymptotically standard parallelization τ of R,

$$Z(R, \tau) = \exp\left(z(R, \tau)\right).$$

Proof: Let Γ be a trivalent Jacobi diagram whose components are isomorphic to some Γ_i for $i = 1, \dots, r$ and such that Γ has k_i connected components isomorphic to Γ_i. Then

$$I(R, \emptyset, \Gamma, (\omega)_{i \in \underline{3n}})[\Gamma] = \prod_{i=1}^{r} \left(I(R, \emptyset, \Gamma_i, (\omega)_{i \in \underline{3\deg(\Gamma_i)}})^{k_i} [\Gamma_i]^{k_i} \right)$$

and $\sharp\mathrm{Aut}(\Gamma) = \prod_{i=1}^{r} \left(k_i! (\mathrm{Aut}(\Gamma_i))^{k_i}\right)$. Conclude with Proposition 7.26. $\square$

Recall the coproduct maps Δ_n defined in Section 6.5.

Proposition 10.3.

$$\Delta_n(Z_n(R, L, \tau)) = \sum_{i=0}^{n} Z_i(R, L, \tau) \otimes Z_{n-i}(R, L, \tau)$$

Proof: Let $T_i = Z_i(R, L, \tau) \otimes Z_{n-i}(R, L, \tau)$.

$$T_i = \sum \frac{1}{\sharp\mathrm{Aut}(\Gamma')} I(\Gamma', (\omega)_{j \in \underline{3i}}) \frac{1}{\sharp\mathrm{Aut}(\Gamma'')} I(\Gamma'', (\omega)_{j \in \underline{3(n-i)}})[\Gamma'] \otimes [\Gamma''],$$

where the sum runs over the pairs (Γ', Γ'') in $\mathcal{D}_i^u(\mathcal{L}(L)) \times \mathcal{D}_{n-i}^u(\mathcal{L}(L))$. Use Remark 7.27 to view the summands as a measure of configurations of graphs $\Gamma' \sqcup \Gamma''$ (which may correspond to several elements of $\mathcal{D}_n^u(\mathcal{L}(L))$) together with a choice of an embedded subgraph Γ'. $\square$

Corollary 10.4. *If L has one component, recall the projection p^c given by Corollary 6.37 from $\check{\mathcal{A}}(S^1)$ to the space $\check{\mathcal{A}}^c(S^1)$ of its primitive elements. Set $\check{z}(R, L, \tau) = p^c\left(\check{Z}(R, L, \tau)\right)$. Then*

$$\check{Z}(R, L, \tau) = \exp(\check{z}(R, L, \tau)).$$

PROOF: This is a direct consequence of Lemma 6.34, Proposition 10.3 and Theorem 6.38. □

10.2 On the anomaly β

We now study how $z_n(R, \tau)$, which is defined before Proposition 10.2, depends on τ.

Definition 10.5. Let $\rho\colon (B^3, \partial B^3) \to (SO(3), 1)$ be the map of Definition 4.5, which induces the double covering map of $SO(3)$. Extend it to $\mathbb{R}^3$ by considering B^3 as the unit ball of $\mathbb{R}^3$ and by letting ρ map $(\mathbb{R}^3 \setminus B^3)$ to 1. Consider the parallelization $\tau_s \circ \psi_{\mathbb{R}}(\rho)$ such that $\psi_{\mathbb{R}}(\rho)(x, v) = (x, \rho(x)(v))$. Set

$$\beta_n = z_n(S^3, \tau_s \circ \psi_{\mathbb{R}}(\rho)).$$

Proposition 10.6. *Let $(\check{R}, \tau_0)$ be an asymptotic rational homology $\mathbb{R}^3$, and let τ_1 be a parallelization of $\check{R}$ that coincides with τ_0 outside B_R. Then for any integer n,*

$$z_n(R, \tau_1) - z_n(R, \tau_0) = \frac{p_1(\tau_1) - p_1(\tau_0)}{4}\beta_n.$$

Proposition 10.6 is an easy consequence of the following proposition 10.7. Proposition 10.7 below looks more complicated but it is very useful since it offers more practical definitions of the *anomaly*

$$\beta = (\beta_n)_{n \in \mathbb{N}}$$

when applied to $(\check{R}, \tau_0, \tau_1) = (\mathbb{R}^3, \tau_s, \tau_s \circ \psi_{\mathbb{R}}(\rho))$ (and to the case where $\tilde{\omega}_{S^2}(i)$ is the pull-back of $\omega_{0,S^2}(i)$, under the natural projection from $[0, 1] \times S^2$ to S^2).

Proposition 10.7. *Let $(\check{R}, \tau_0)$ be an asymptotic rational homology $\mathbb{R}^3$, and let τ_1 be a parallelization of $\check{R}$ that coincides with τ_0 outside B_R. For $i \in \underline{3n}$, let $\omega_{0,S^2}(i)$ and $\omega_{1,S^2}(i)$ be two volume-one forms on S^2. Then there exists a closed two form $\tilde{\omega}_{S^2}(i)$ on $[0, 1] \times S^2$ such that the restriction of $\tilde{\omega}_{S^2}(i)$ to $\{t\} \times S^2$ is ω_{t,S^2} for $t \in \{0, 1\}$. For any such forms $\tilde{\omega}_{S^2}(i)$, there exist closed 2-forms $\tilde{\omega}(i)$ on $[0, 1] \times U\check{R}$ such that*

- *the restriction of* $\tilde{\omega}(i)$ *to* $\{t\} \times U\check{R}$ *is* $p_{\tau_t}^*(\omega_{t,S^2}(i))$ *for* $t \in \{0,1\}$,
- *the restriction of* $\tilde{\omega}(i)$ *to* $[0,1] \times (U(\check{R} \setminus B_R))$ *is* $(\mathbf{1}_{[0,1]} \times p_{\tau_0})^*(\tilde{\omega}_{S^2}(i))$.

Then

$$\begin{aligned} z_n(R,\tau_1) - z_n(R,\tau_0) &= z_n\left([0,1] \times UB_R; (\tilde{\omega}(i))_{i \in \underline{3n}}\right) \\ &= \sum_{\Gamma \in \mathcal{D}_n^c} \zeta_\Gamma \int_{[0,1]\times \check{\mathcal{S}}_{V(\Gamma)}(TB_R)} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e)))[\Gamma] \\ &= \frac{p_1(\tau_1) - p_1(\tau_0)}{4} \beta_n \end{aligned}$$

and $\beta_n = 0$ *if* n *is even. (Recall that the orientation of* $\check{\mathcal{S}}_{V(\Gamma)}(T\check{R})$ *is defined in Lemma 9.3.)*

Proof: The existence of $\tilde{\omega}_{S^2}(i)$ comes from Lemma 9.1. In order to prove the existence of $\tilde{\omega}(i)$, which is defined on $\partial([0,1]\times S^2 \times B_R)$ by the conditions, we need to extend it to $[0,1] \times S^2 \times B_R$. The obstruction belongs to

$$H^3([0,1] \times S^2 \times B_R, \partial([0,1] \times S^2 \times B_R)) \cong H_3([0,1] \times S^2 \times B_R),$$

which is trivial. So $\tilde{\omega}(i)$ extends. In order to prove that the first equality is a consequence of Corollary 9.4, extend the forms $\tilde{\omega}(i)$ of the statement to $[0,1] \times C_2(R)$ as forms that satisfy the conditions in Proposition 9.2. First extend the $\tilde{\omega}(i)$ to $[0,1]\times(\partial C_2(R)\setminus UB_R)$ as $(\mathbf{1}_{[0,1]}\times p_{\tau_0})^*(\tilde{\omega}_{S^2}(i))$. Next extend the restriction of $\tilde{\omega}(i)$ to $\{0\}\times\partial C_2(R)$ (resp. to $\{1\}\times\partial C_2(R)$) on $\{0\}\times C_2(R)$ (resp. on $\{1\} \times C_2(R)$) as a propagating form of $(C_2(R),\tau_0)$) (resp. of $(C_2(R),\tau_1)$)) as in Section 3.3. Thus $\tilde{\omega}(i)$ is defined consistently on $\partial([0,1] \times C_2(R))$, and it extends as a closed form that satisfies the assumptions in Proposition 9.2 as in Lemma 9.1. Corollary 9.4, Lemma 9.6 and Lemma 9.12 yield

$$z_n(R,\tau_1) - z_n(R,\tau_0) = \sum_{\Gamma \in \mathcal{D}_n^c} \frac{1}{(3n)!2^{3n}} \int_{[0,1]\times \check{\mathcal{S}}_{V(\Gamma)}(TB_R)} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e)))[\Gamma],$$

which is zero if n is even. So everything is proved when n is even. Assume that n is odd.

There exists a map $g\colon (\check{R}, \check{R} \setminus B_R) \to (SO(3),1)$ such that $\tau_1 = \tau_0 \circ \psi_{\mathbb{R}}(g)$. Using τ_0 to identify $\check{\mathcal{S}}_{V(\Gamma)}(T\check{R})$ with $\check{R} \times \check{\mathcal{S}}_{V(\Gamma)}(\mathbb{R}^3)$, makes clear that $(z_n(R,\tau_0 \circ \psi_{\mathbb{R}}(g)) - z_n(R,\tau_0))$ does not depend on τ_0. For any $g\colon (\check{R}, \check{R} \setminus B_R) \to (SO(3),1)$, set $z_n'(g) = z_n(R,\tau_0 \circ \psi_{\mathbb{R}}(g)) - z_n(R,\tau_0)$. Then z_n' is a homomorphism from $[(B_R,\partial B_R),(SO(3),1]$ to the vector space $\mathcal{A}_n^c(\emptyset)$ over $\mathbb{R}$. According to Theorem 4.6 and Lemma 4.7, $z_n'(g) = \frac{\deg(g)}{2} z_n'(\rho_{B_R}(B^3))$. Furthermore, it is easy to see that $z_n'(\rho_{B_R}(B^3))$ is independent of $\check{R}$. Since $z_n(S^3,\tau_s) = 0$ according to Example 7.18, by definition, $z_n'(\rho_{B_R}(B^3)) = \beta_n$, and, according to Theorem 4.6, $p_1(\tau_0 \circ \psi_{\mathbb{R}}(g)) - p_1(\tau_0) = 2\deg(g)$. $\square$

Remark 10.8. The anomaly β is the opposite of the constant ξ defined in [Les04a, Section 1.6].

Corollary 10.9. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$, then*

$$z_n(R, \tau) - \frac{p_1(\tau)}{4}\beta_n$$

is independent of τ. Set $\mathfrak{z}_n(R) = z_n(R, \tau) - \frac{p_1(\tau)}{4}\beta_n$, $\mathfrak{z}(R) = (\mathfrak{z}_n(R))_{n\in\mathbb{N}}$ and

$$\mathcal{Z}(R) = \exp(\mathfrak{z}(R)).$$

Then

$$\mathcal{Z}(R) = Z(R, \tau)\exp\left(-\frac{p_1(\tau)}{4}\beta\right)$$

is the invariant $\mathcal{Z}(R, \emptyset)$ that was announced in Theorem 7.20.

PROOF: See Proposition 10.2. □

Proposition 10.10. $\beta_1 = \frac{1}{12}[\ominus]$.

PROOF: According to Proposition 7.17, $z_1(R, \tau) = \frac{\Theta(R,\tau)}{12}[\ominus]$. According to Proposition 10.6, $z_1(R, \tau_1) - z_1(R, \tau_0) = \frac{p_1(\tau_1)-p_1(\tau_0)}{4}\beta_1$, while Corollary 4.9 implies that

$$\Theta(R, \tau_1) - \Theta(R, \tau_0) = \frac{1}{4}(p_1(\tau_1) - p_1(\tau_0)).$$

□

Corollary 10.11. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$, then $Z_1(R, \tau) = z_1(R, \tau) = \frac{\Theta(R,\tau)}{12}[\ominus]$ and*

$$\mathcal{Z}_1(R) = \mathfrak{z}_1(R) = \frac{\Theta(R)}{12}[\ominus]$$

in $\mathcal{A}_1(\emptyset) = \mathbb{R}[\ominus]$.

PROOF: The first equality is a direct consequence of Proposition 7.17. The second one follows from Corollary 4.9, Corollary 10.9 and Proposition 10.10. □

Remark 10.12. The values of β_{2n+1} are unknown, when $n \geq 1$.

10.3 On the anomaly α

We define the *anomaly*

$$\alpha = (\alpha_n)_{n\in\mathbb{N}},$$

which is sometimes called the *Bott and Taubes anomaly,* below. Let $v \in S^2$. Let D_v denote the linear map

$$\begin{array}{rccc} D_v : & \mathbb{R} & \longrightarrow & \mathbb{R}^3 \\ & 1 & \mapsto & v. \end{array}$$

Let $\mathcal{D}^c_n(\mathbb{R})$ denote the set of degree n, connected, $\underline{3n-2}$–numbered diagrams on $\mathbb{R}$ with at least one univalent vertex, without looped edges. As in Definition 7.6, a degree n diagram $\check{\Gamma}$ is *numbered* if the edges of $\check{\Gamma}$ are oriented, and if $E(\check{\Gamma})$ is equipped with an injection $j_E \colon E(\check{\Gamma}) \hookrightarrow \underline{3n-2}$, which numbers its edges. Let $\check{\Gamma} \in \mathcal{D}^c_n(\mathbb{R})$. Define $\check{C}(D_v;\check{\Gamma})$ as in Section 7.1, where the line D_v of $\mathbb{R}^3$ replaces the link L of $\check{R}$, and $\mathbb{R}$ replaces the source $\mathcal{L}$. Let $\check{Q}(v;\check{\Gamma})$ be the quotient of $\check{C}(D_v;\check{\Gamma})$ by the translations parallel to D_v and by the dilations. Then $\check{Q}(v;\check{\Gamma})$ is the space denoted by $\check{\mathcal{S}}(\mathbb{R}^3, v;\check{\Gamma})$ before Lemma 8.15. Let $Q(v;\check{\Gamma})$ denote the closure of $\check{Q}(v;\check{\Gamma})$ in $\mathcal{S}_{V(\check{\Gamma})}(\mathbb{R}^3)$. According to Lemma 8.15, $Q(v;\check{\Gamma})$, which coincides with $\mathcal{S}(\mathbb{R}^3, v;\check{\Gamma})$, is a compact smooth manifold with ridges.

To each edge e of $\check{\Gamma}$, associate a map p_{e,S^2}, which maps a configuration of $\check{Q}(v;\check{\Gamma})$ to the direction of the vector from the origin of e to its end in S^2. This map extends to $Q(v;\check{\Gamma})$, according to Theorem 8.12.

Now, define $\check{Q}(\check{\Gamma})$ (resp. $Q(\check{\Gamma})$) as the total space of the fibration over S^2 whose fiber over v is $\check{Q}(v;\check{\Gamma})$ (resp. $Q(v;\check{\Gamma})$). The configuration space $\check{Q}(\check{\Gamma})$ and its compactification $Q(\check{\Gamma})$ carry natural smooth structures. The configuration space $\check{Q}(\check{\Gamma})$ is oriented as follows, when a vertex-orientation $o(\check{\Gamma})$ is given. Equip $\check{C}(D_v;\check{\Gamma})$ with its orientation induced by Corollary 7.2, as before. Orient $\check{Q}(v;\check{\Gamma})$ so that $\check{C}(D_v;\check{\Gamma})$ is locally homeomorphic to the oriented product (translation vector z in $\mathbb{R}v$, ratio of homothety $\lambda \in]0,\infty[$) $\times\check{Q}(v;\check{\Gamma})$ and orient $\check{Q}(\check{\Gamma})$ with the (base($= S^2$) $\oplus$ fiber) convention. (This can be summarized by saying that the S^2-coordinates replace (z,λ).)

Proposition 10.13. *For $i \in \underline{3n-2}$, let $\omega(i,S^2)$ be a volume-one form of S^2. Define*

$$I(\check{\Gamma}, o(\check{\Gamma}), \omega(i,S^2)) = \int_{\check{Q}(\check{\Gamma})} \bigwedge_{e\in E(\check{\Gamma})} p^*_{e,S^2}(\omega(j_E(e),S^2)).$$

Set

$$\alpha_n = \frac{1}{2}\sum_{\check{\Gamma}\in\mathcal{D}^c_n(\mathbb{R})} \zeta_{\check{\Gamma}} I(\check{\Gamma}, o(\check{\Gamma}), \omega(i,S^2))[\check{\Gamma}, o(\check{\Gamma})] \quad \in \mathcal{A}(\mathbb{R}),$$

where $\zeta_{\check{\Gamma}} = \frac{(3n-2-\sharp E(\check{\Gamma}))!}{(3n-2)!2^{\sharp E(\check{\Gamma})}}$. *Then* α_n *does not depend on the chosen* $\omega(i, S^2)$, $\alpha_1 = \frac{1}{2}\,[$  $]$ *and* $\alpha_{2n} = 0$ *for all* n.

Proof: Let us first prove that α_n does not depend on the chosen $\omega(i, S^2)$, by proving that its variation vanishes when $\omega(i, S^2)$ is changed to some $\tilde{\omega}(i, 1, S^2)$. According to Lemma 9.1, there exists a closed 2–form $\tilde{\omega}(i, S^2)$ on $[0,1] \times S^2$ whose restriction to $\{0\} \times S^2$ is $\omega(i, S^2) = \tilde{\omega}(i, 0, S^2)$, and whose restriction to $\{1\} \times S^2$ is $\tilde{\omega}(i, 1, S^2)$. According to Stokes's theorem, for any $\check{\Gamma} \in \mathcal{D}^c_n(\mathbb{R})$,

$$I(\check{\Gamma}, o(\check{\Gamma}), (\tilde{\omega}(i, 1, S^2))_{i \in \underline{3n-2}}) - I(\check{\Gamma}, o(\check{\Gamma}), (\tilde{\omega}(i, 0, S^2))_{i \in \underline{3n-2}})$$

$$= \sum_F \int_{[0,1] \times F} \bigwedge_{e \in E(\check{\Gamma})} p^*_{e,S^2}\left(\tilde{\omega}\left(j_E(e), S^2\right)\right),$$

where $p_{e,S^2} \colon [0,1] \times \check{Q}(\check{\Gamma}) \to [0,1] \times S^2$ denotes the product by $\mathbf{1}_{[0,1]}$ of p_{e,S^2}, and the sum runs over the codimension-one faces F of $Q(\check{\Gamma})$. These faces fiber over S^2, and the fibers over $v \in S^2$ are the codimension-one faces $f(A, v; \check{\Gamma})$ for the strict subsets A of $V(\check{\Gamma})$ with cardinality at least 2 whose univalent vertices are consecutive on $\mathbb{R}$ of $Q(v, \check{\Gamma}) = \mathcal{S}(\mathbb{R}^3, v; \check{\Gamma})$ listed in Lemma 8.16. Let $F(A, \check{\Gamma})$ denote the face with fiber $f(A, v; \check{\Gamma})$. Now, it suffices to prove that the contributions of all the $F(A, \check{\Gamma})$ vanish.

When the product of all the p_{e,S^2} factors through a quotient of $[0,1] \times F(A, \check{\Gamma})$ of smaller dimension, the face $F(A, \check{\Gamma})$ does not contribute. This allows us to get rid of

- the faces $F(A, \check{\Gamma})$ for which $\check{\Gamma}_A$ is not connected, and A is not a pair of univalent vertices of $\check{\Gamma}$, as in Lemma 9.9,
- the faces $F(A, \check{\Gamma})$ for which $\sharp A \geq 3$, and $\check{\Gamma}_A$ has a univalent vertex that was trivalent in $\check{\Gamma}$, as in Lemma 9.10.

We also have faces that cancel each other, for graphs that are identical outside their $\check{\Gamma}_A$ part.

- The faces $F(A, \check{\Gamma})$ (which are not already listed) such that $\check{\Gamma}_A$ has at least a bivalent vertex cancel (by pairs) by the parallelogram identification as in Lemma 9.11.
- The faces $F(A, \check{\Gamma})$ for which $\check{\Gamma}_A$ is an edge between two trivalent vertices, cancel by triples, thanks to the Jacobi (or IHX) relation as in Lemma 9.13.

- Similarly, two faces for which A is a pair of (necessarily consecutive in $\mathbb{R}$) univalent vertices of $\check{\Gamma}$, cancel $(3n-2-\sharp E(\Gamma))$ faces $F(\check{\Gamma}', A')$ for which $\check{\Gamma}'_{A'}$ is an edge between a univalent vertex of $\check{\Gamma}$ and a trivalent vertex of $\check{\Gamma}$, thanks to the STU relation (and to Lemma 6.25) as in Lemma 9.14.

Here, there are no faces left, and α_n does not depend on the chosen $\omega(i, S^2)$.

The computation of α_1 is straightforward.

Let us prove that $\alpha_n = 0$ for any even n. Let $\check{\Gamma}$ be a numbered graph and let $\check{\Gamma}^{eo}$ be obtained from $\check{\Gamma}$ by reversing the orientations of the $(\sharp E)$ edges of $\check{\Gamma}$. Consider the map r from $\check{Q}(\check{\Gamma}^{eo})$ to $\check{Q}(\check{\Gamma})$ that composes a configuration by the multiplication by (-1) in $\mathbb{R}^3$. It sends a configuration over $v \in S^2$ to a configuration over $(-v)$, and it is therefore a fibered space map over the orientation-reversing antipode of S^2. Equip $\check{\Gamma}$ and $\check{\Gamma}^{eo}$ with the same vertex-orientation, and with the same orders on their vertex sets. Then our map r is orientation-preserving if and only if $\sharp T(\check{\Gamma}) + 1$ is even. The vertex-orientation of $H(\check{\Gamma})$ and $H(\check{\Gamma}^{eo})$ can be consistent with both the edge-orientations of $H(\check{\Gamma})$ and $H(\check{\Gamma}^{eo})$ if and only if $\sharp E(\check{\Gamma})$ is even. Furthermore for all the edges e of $\check{\Gamma}^{eo}$, $p_{e,S^2,\check{\Gamma}^{eo}} = p_{e,S^2,\check{\Gamma}} \circ r$, then since $\sharp E(\check{\Gamma}) = n + \sharp T(\check{\Gamma})$,

$$I(\check{\Gamma}^{eo}, \omega(i, S^2))[\check{\Gamma}^{eo}] = (-1)^{n+1} I(\check{\Gamma}, \omega(i, S^2))[\check{\Gamma}].$$

□

Remark 10.14. It is known that $\alpha_3 = 0$ [Poi02, Proposition 1.4]. Sylvain Poirier also found that $\alpha_5 = 0$ with the help of a Maple program. Furthermore, according to [Les02, Corollary 1.4], α_{2n+1} is a combination of diagrams with two univalent vertices, and $\mathcal{Z}(S^3, L) = \check{\mathcal{Z}}(S^3, L)$ is obtained from the Kontsevich integral Z^K by inserting d times the plain part of 2α on each degree d connected component of a diagram.

10.4 Dependence on the forms for straight links

In this section, we prove Theorems 7.39 and 7.41. In order to do it, we will prove the following lemma.

Lemma 10.15. *Under the assumptions of Theorem 7.39, set*

$$Z_{n,3n-2}(\check{R}, L, (\omega(i))_{i \in \underline{3n-2}}) = \sum_{\Gamma \in \mathcal{D}^e_{n,3n-2}(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i \in \underline{3n-2}})[\Gamma] \in \mathcal{A}_n(\mathcal{L})$$

is independent of the chosen propagating forms $\omega(i)$ of $(C_2(R), \tau)$.

Lemma 10.16. *Lemma 10.15 implies Theorem 7.39 and Theorem 7.41.*

Let us prove that Lemma 10.15 implies Theorem 7.39. $\mathcal{A}_n(\mathcal{L}) = \mathcal{A}_n(\emptyset) \oplus \mathcal{A}'_n(\mathcal{L})$, where $\mathcal{A}'_n(\mathcal{L})$ is the subspace of $\mathcal{A}_n(\mathcal{L})$ generated by the Jacobi diagrams with at least one univalent vertex. Since we know from Lemma 9.6 that $Z_n(\check{R}, \emptyset, (\omega(i))_{i\in\underline{3n}})$ is independent of the chosen propagating forms $\omega(i)$ of $(C_2(R), \tau)$, we focus on the projection $Z'_n(\check{R}, L, (\omega(i))_{i\in\underline{3n}})$ of

$$Z_n(\check{R}, L, (\omega(i))_{i\in\underline{3n}}) = \sum_{\Gamma\in\mathcal{D}^e_n(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i\in\underline{3n}})[\Gamma] \in \mathcal{A}_n(\mathcal{L})$$

onto $\mathcal{A}'_n(\mathcal{L})$. This sum is a sum over diagrams with at least two univalent vertices, according to Lemma 6.25. Recall Notation 7.29. Lemma 10.15 guarantees that for any subset I of $\underline{3n}$ with cardinality $(3n-2)$,

$$Z_{n,I}(\check{R}, L, (\omega(i))_{i\in I}) = \sum_{\Gamma\in\mathcal{D}^e_{n,I}(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i\in I})[\Gamma] \in \mathcal{A}_n(\mathcal{L})$$

is independent of the chosen $\omega(i)$. Observe that

$$Z'_n(\check{R}, L, (\omega(i))_{i\in\underline{3n}}) = \frac{2((3n-2)!)}{(3n)!} \sum_{I\subset\underline{3n}|\sharp I=3n-2} Z_{n,I}(\check{R}, L, (\omega(i))_{i\in I})$$

since for a numbered graph Γ of $\mathcal{D}^e_n(\mathcal{L})$, the coefficient of $I(R, L, \Gamma, (\omega(i))_{i\in\underline{3n}})$ is $\frac{(3n-\sharp E(\Gamma))!}{(3n)!2^{\sharp E(\Gamma)}}[\Gamma]$ in the left-hand side, and

$$\frac{2((3n-2)!)}{(3n)!} \sum_{I\subset\underline{3n}|\sharp I=3n-2, j_E(E(\Gamma))\subseteq I} \frac{(3n-2-\sharp E(\Gamma))!}{(3n-2)!2^{\sharp E(\Gamma)}}[\Gamma]$$

$$= \frac{2((3n-2)!)}{(3n)!} \frac{(3n-\sharp E(\Gamma))!}{2((3n-2-\sharp E(\Gamma))!)} \frac{(3n-2-\sharp E(\Gamma))!}{(3n-2)!2^{\sharp E(\Gamma)}}[\Gamma]$$

in the right-hand-side. Thus, Lemma 10.15 and Lemma 9.5 imply Theorem 7.39.

Lemma 10.15 also directly implies that

$$\sum_{\Gamma\in\mathcal{D}^e_{n,3n-2}(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i\in\underline{3n-2}})\check{p}([\Gamma])$$

is independent of the chosen $\omega(i)$, so it implies Theorem 7.41 as above. $\square$

Proof of Lemma 10.15: It suffices to prove that, when $L\colon \mathcal{L} \hookrightarrow \check{R}$ is a straight embedding with respect to τ, $Z_{n,3n-2}(\check{R}, L, (\omega(i))_{i\in\underline{3n-2}})$ does not change when some $\omega(i)$ is changed to $\omega(i)+d\eta$ for some one-form η on $C_2(R)$, as in Lemma 4.2, which restricts to $\partial C_2(R)$ as $p_\tau^*(\eta_{S^2})$ for some one-form η_{S^2} on S^2. Assume that the forms $\omega(j)$ restrict to $\partial C_2(R)$ as $p_\tau^*(\omega_{S^2}(j))$. Set $\tilde{\omega}(i,0)=\omega(i)$. Let $p_{C_2}\colon [0,1]\times C_2(R)\to C_2(R)$ and $p_{S^2}\colon [0,1]\times S^2\to S^2$ denote the projections onto the second factor. Define the closed 2-form $\tilde{\omega}_{S^2}(i)$ on $[0,1]\times S^2$ by

$$\tilde{\omega}_{S^2}(i)=p_{S^2}^*(\omega_{S^2}(i))+d(tp_{S^2}^*(\eta_{S^2})),$$

where t is the coordinate on $[0,1]$. Define the closed 2-form $\tilde{\omega}(i)$ on $[0,1]\times C_2(R)$ by

$$\tilde{\omega}(i)=p_{C_2}^*(\omega(i))+d(tp_{C_2}^*(\eta)).$$

For $j\in\underline{3n}\setminus\{i\}$, define $\tilde{\omega}_{S^2}(j)=p_{S^2}^*(\omega_{S^2}(j))$ and $\tilde{\omega}(j)=p_{C_2}^*(\omega(j))$, and for any $j\in\underline{3n}$, let $\tilde{\omega}(j,t)$ denote the restriction of $\tilde{\omega}(j)$ to $\{t\}\times C_2(R)$. Thus it suffices to prove that $Z_{n,3n-2}(1)=Z_{n,3n-2}(0)$, where

$$Z_{n,3n-2}(t)=\sum_{\Gamma\in\mathcal{D}_n^e(\mathcal{L})}\zeta_\Gamma I(R,L,\Gamma,(\tilde{\omega}(i,t))_{i\in\underline{3n-2}})[\Gamma]\in\mathcal{A}_n(\mathcal{L}).$$

Proposition 9.2 expresses $Z_{n,3n-2}(1)-Z_{n,3n-2}(0)$ as a sum over numbered graphs Γ equipped with a connected component Γ_A, with no univalent vertex, or whose univalent vertices form a non-empty set of consecutive vertices in Γ on some component S^1_j of $\mathcal{L}$. The faces for which A has no univalent vertex do not contribute, as in the proof of Lemma 9.6, and we focus on the remaining faces.

Such a $F(A,L,\Gamma)$ may split according to the possible compatible linear orders of the univalent vertices of Γ_A, which are represented by lifts $\check{\Gamma}_A$ of Γ_A on $\mathbb{R}$ as before Lemma 8.15, and we view $Z_{n,3n-2}(1)-Z_{n,3n-2}(0)$ as a sum over pairs $(\Gamma,\check{\Gamma}_A)$ of terms

$$I(\Gamma,\check{\Gamma}_A)=\int_{[0,1]\times F(\check{\Gamma}_A,L,\Gamma)}\bigwedge_{e\in E(\Gamma)}p_e^*(\tilde{\omega}(j_E(e)))[\Gamma]$$

associated to the corresponding face components denoted by $F(\check{\Gamma}_A,L,\Gamma)$.

We study the sum of the contributions $I(\tilde{\Gamma},\check{\Gamma}_A)$ of the pairs of numbered graphs $(\tilde{\Gamma},\check{\Gamma}_A)$ as above, where $\check{\Gamma}_A$ is a fixed numbered graph on $\mathbb{R}$ as above, which is viewed as a subgraph of $\tilde{\Gamma}$, such that the univalent vertices of $\check{\Gamma}_A$ are consecutive on S^1_j in $\tilde{\Gamma}$, with respect to their linear order, and $\tilde{\Gamma}\setminus\check{\Gamma}_A$ is equal to a fixed $\Gamma\setminus\check{\Gamma}_A$, as above. Recall from Proposition 6.22 that, for any such pair $(\tilde{\Gamma},\check{\Gamma}_A)$, if all the vertices of the graph $\tilde{\Gamma}$ inherit their orientations from a fixed vertex-orientation of Γ, then $[\tilde{\Gamma}]=[\check{\Gamma}_A]\sharp_j[\Gamma\setminus\check{\Gamma}_A]$ in $\check{\mathcal{A}}(\mathcal{L})$.

Note that the contributions $I(\tilde{\Gamma}, \check{\Gamma}_A)$ vanish if $i \notin j_E(\check{\Gamma}_A)$ because all the $p_e^*(\tilde{\omega}(j_E(e)))$ for $e \in E(\check{\Gamma}_A)$ factor through the projection onto $F(A, L, \check{\Gamma}_A)$ whose dimension is $(2\sharp E(\check{\Gamma}_A) - 1)$.

If $i \in j_E(\check{\Gamma}_A)$, let $e(i)$ be the edge such that $j_E(e(i)) = i$. The sum of the contributions $I(., \check{\Gamma}_A)$ that involve $\check{\Gamma}_A$ factors through

$$I = \int_{[0,1]\times \cup_{c(v)\in K_j} \check{\mathcal{S}}(T_{c(v)}\check{R}, \vec{t}_{c(v)}; \check{\Gamma}_A)} p_{e(i)}^*(d(tp_{C_2}^*(\eta))) \bigwedge_{e\in E(\check{\Gamma}_A)\setminus e(i)} p_e^*(\tilde{\omega}(j_E(e))),$$

where $\vec{t}_{c(v)}$ denotes the unit tangent vector to K_j at $c(v)$.

Recall that $\check{Q}(\check{\Gamma}_A)$ was defined in Section 10.3, together with natural maps $p_{e,S^2} \colon \check{Q}(\check{\Gamma}_A) \to S^2$. Let p_{e,S^2} also denote $\mathbf{1}_{[0,1]} \times p_{e,S^2} \colon [0,1] \times \check{Q}(\check{\Gamma}_A) \to [0,1] \times S^2$. The form $p_{e(i)}^*(d(tp_{C_2}^*(\eta))) \bigwedge_{e\in E(\check{\Gamma}_A)\setminus e(i)} p_e^*(\tilde{\omega}(j_E(e)))$ is the pull-back of the closed form

$$\Omega = p_{e(i),S^2}^*(d(t\eta_{S^2})) \bigwedge_{e\in E(\check{\Gamma}_A)\setminus e(i)} p_{e,S^2}^*(\tilde{\omega}_{S^2}(j_E(e)))$$

on $[0,1] \times \check{Q}(\check{\Gamma}_A)$ under the projection

$$[0,1] \times \cup_{c(v)\in K_j} \check{\mathcal{S}}(T_{c(v)}\check{R}, \vec{t}_{c(v)}; \check{\Gamma}_A) \to [0,1] \times \check{Q}(\check{\Gamma}_A).$$

The image of this projection is the product by $[0,1]$ of the restriction of the bundle $\check{Q}(\check{\Gamma}_A)$ over $p_\tau(U^+K_j)$, and I is the integral of Ω along this image. Compute the integral by integrating first along the fibers of $\check{Q}(\check{\Gamma}_A)$, next along $[0,1]$. Finally, I is the integral of a one-form along $p_\tau(U^+K_j) \subset S^2$, and it vanishes because K_j is straight. $\square$

10.5 The general variation for homogeneous propagating forms

Set $\mathcal{D}^c(\mathbb{R}) = \cup_{n\in\mathbb{N}} \mathcal{D}_n^c(\mathbb{R})$, where $\mathcal{D}_n^c(\mathbb{R})$ is the set of degree n connected $(3n-2)$-numbered Jacobi diagrams on $\mathbb{R}$ introduced in the beginning of Section 10.3. In this section, we write various sums over numbered diagrams, but all the edges of a diagram are equipped with the same propagating forms. So neither the set in which the edges is numbered nor its cardinality matters, provided that the cardinality is greater than the possible number of edges for a given degree. (See Proposition 7.26 and Remark 7.31.)

Proposition 10.17. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $L = \sqcup_{j=1}^k K_j$ be an embedding of $\mathcal{L} = \sqcup_{j=1}^k S_j^1$ into $\check{R}$. Let $\tilde{\omega}(0)$ and $\tilde{\omega}(1)$ be* *two homogeneous propagating forms of* $C_2(R)$*, and let* $\tilde{\omega}$ *be a closed* 2*-form on* $[0,1]\times\partial C_2(R)$ *whose restriction* $\tilde{\omega}(t)$ *to* $\{t\}\times(\partial C_2(R)\setminus UB_R)$ *is* $p_\tau^*(\omega_{S^2})$*, for any* $t\in[0,1]$*, and whose restriction* $\tilde{\omega}(t)$ *to* $\{t\}\times\partial C_2(R)$ *coincides with the restriction to* $\partial C_2(R)$ *of the given* $\tilde{\omega}(t)$*, for* $t\in\{0,1\}$*. For any* $j\in\underline{k}$*, define* $I_j=\sum_{\Gamma_B\in\mathcal{D}^c(\mathbb{R})}\zeta_{\Gamma_B}I(\Gamma_B,K_j,\tilde{\omega})$*, where*

$$I(\Gamma_B,K_j,\tilde{\omega})=\int_{u\in[0,1]}\int_{w\in K_j}\int_{\check{\mathcal{S}}(T_w\check{R},\vec{t}_w;\Gamma_B)}\bigwedge_{e\in E(\Gamma_B)}p_e^*(\tilde{\omega}(u))[\Gamma_B],$$

and $\vec{t}_w$ *denotes the unit tangent vector to* K_j *at* w*. Define*

$$z(\tilde{\omega})=\sum_{n\in\mathbb{N}}z_n\left([0,1]\times UB_R;\tilde{\omega}\right)$$

as in Corollary 9.4. Then

$$Z(\check{R},L,\tilde{\omega}(1))=\left(\prod_{j=1}^{k}\exp\left(I_j\right)\sharp_j\right)Z\left(\check{R},L,\tilde{\omega}(0)\right)\exp\left(z(\tilde{\omega})\right),$$

where $\sharp_j$ *stands for the insertion on a diagram on* $\mathbb{R}$ *on the component* S^1_j *of* $\mathcal{L}$*.*

Proof: When $L=\emptyset$, this statement follows from Corollary 9.4 and Proposition 10.2 together with Lemma 9.1, which ensures that there exists a closed 2-form $\tilde{\omega}$ on $[0,1]\times C_2(R)$ which extends the 2-form $\tilde{\omega}$ of the statement. Using Notation 7.16, we are left with the proof that

$$\check{Z}(\check{R},L,\tilde{\omega}(1))=\left(\prod_{j=1}^{k}\exp\left(I_j\right)\sharp_j\right)\check{Z}\left(\check{R},L,\tilde{\omega}(0)\right).$$

Let us begin this proof with the proof of the following corollary of Proposition 9.2.

Lemma 10.18. *Under the assumptions of Proposition 10.17, let* $\tilde{\omega}$ *be a closed* 2*-form on* $[0,1]\times C_2(R)$ *which extends the* 2*-form* $\tilde{\omega}$ *of Proposition 10.17. For any* $t\in[0,1]$*, let* $\tilde{\omega}(t)$ *denote the restriction to* $\{t\}\times C_2(R)$ *of* $\tilde{\omega}$*. Set*

$$\check{Z}(t)=\left(\check{Z}_n(R,L,\tilde{\omega}(t))\right)_{n\in\mathbb{N}}.$$

For $\Gamma_B\in\mathcal{D}^c(\mathbb{R})$ *and* $j\in\underline{k}$*, set*

$$\eta(R,L,\Gamma_B,K_j,\tilde{\omega})(u)=\int_{w\in K_j}\int_{\check{\mathcal{S}}(T_w\check{R},\vec{t}_w;\Gamma_B)}\bigwedge_{e\in E(\Gamma_B)}p_e^*(\tilde{\omega}(u))[\Gamma_B],$$

where $\vec{t_w}$ denotes the unit tangent vector to K_j at w, and set

$$\gamma_j(u) = \sum_{\Gamma_B \in \mathcal{D}^c(\mathbb{R})} \zeta_{\Gamma_B} \eta(R, L, \Gamma_B, K_j, \tilde{\omega})(u).$$

Then $\check{Z}(t)$ is differentiable, and

$$\check{Z}'(t)dt = (\sum_{j=1}^{k} \gamma_j(t)\sharp_j)\check{Z}(t).$$

PROOF OF LEMMA 10.18: The variations of $\check{Z}_n(t)$ are given by Proposition 9.2, by sending the diagrams with components without univalent vertices to 0. They involve only faces $F(A, L, \Gamma)$ for which Γ_A is a connected component of Γ, with univalent vertices on one component of $\mathcal{L}$. Again, such a face may split according to the possible compatible linear orders of the univalent vertices of Γ_A, which are represented by lifts $\check{\Gamma}_A$ of Γ_A on $\mathbb{R}$, as before Lemma 8.15. The corresponding face component is denoted by $F(\check{\Gamma}_A, L, \Gamma)$, and we have

$$\check{Z}_n(t) - \check{Z}_n(0) = \sum_{\left\{\begin{array}{c} (\Gamma, A) \mid \Gamma \in \mathcal{D}^e_n(\mathcal{L}), A \subseteq V(\Gamma), \sharp A \geq 2; \\ \text{every component of } \Gamma \text{ has univalent vertices,} \\ \Gamma_A \text{ is a connected component of } \Gamma, \\ \text{the univalent vertices of } \Gamma_A \\ \text{are consecutive on one component of } \Gamma, \\ \check{\Gamma}_A \text{ is a compatible lift of } \Gamma_A \text{ on } \mathbb{R}. \end{array}\right\}} I(\Gamma, \check{\Gamma}_A),$$

where the set of univalent vertices of $\check{\Gamma}_A$ is equipped with the unique linear order induced by Γ if there are univalent vertices of $\Gamma \setminus \Gamma_A$ on the component of Γ_A, and it is equipped with one of the linear orders compatible with Γ otherwise, and

$$I(\Gamma, \check{\Gamma}_A) = \zeta_\Gamma \int_{[0,t] \times F(\check{\Gamma}_A, L, \Gamma)} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega})[\Gamma].$$

Observe that this expression implies that $\check{Z}_n$ (which is valued in a finite-dimensional vector space) is differentiable. (For any smooth compact d-dimensional manifold C and for any smooth $(d+1)$-form ω on $[0,1] \times C$, the function $\left(t \mapsto \int_{[0,t] \times C} \omega\right)$ is differentiable.) Assume that the vertices of Γ_A are on a component K_j of $L(\sqcup_{j=1}^k S_j^1)$. The forms associated to edges of

Γ_A do not depend on the configuration of $(V(\Gamma) \setminus A)$. They are integrated along $[0,1]\times(\cup_{c(v)\in K_j}\check{\mathcal{S}}(T_{c(v)}\check{R},\vec{t}_{c(v)};\check{\Gamma}_A))$, where $\vec{t}_{c(v)}$ denotes the unit tangent vector to K_j at $c(v)$, while the other ones are integrated along $\check{C}(R,L;\Gamma\setminus\Gamma_A)$ at $u \in [0,1]$.

Group the contributions of the pairs $(\Gamma,\check{\Gamma}_A)$ with common $(\Gamma \setminus \Gamma_A, \check{\Gamma}_A)$ to view the global variation $(\check{Z}(t) - \check{Z}(0))$ as

$$\sum_{j=1}^{k}\int_0^t\left(\sum_{\check{\Gamma}_A\in\mathcal{D}^c(\mathbb{R})}\zeta_{\check{\Gamma}_A}\eta(R,L,\check{\Gamma}_A,K_j,\tilde{\omega})(u)\sharp_j\right)\check{Z}(u).$$

Use Proposition 7.26 and Remark 7.27 to check that the coefficients are correct.

Therefore

$$\check{Z}(t)-\check{Z}(0)=\int_0^t\left(\sum_{j=1}^{k}\gamma_j(u)\sharp_j\right)\check{Z}(u).$$

□

Back to the proof of Proposition 10.17, set $I_j(t) = \int_0^t \gamma_j(u)$.

By induction on the degree, it is easy to see that the equation $\check{Z}'(t)dt = (\sum_{j=1}^k \gamma_j(t)\sharp_j)\check{Z}(t)$ of Lemma 10.18 determines $\check{Z}(t)$ as a function of the $I_j(t)$ and $\check{Z}(0)$ whose degree 0 part is 1, and that $\check{Z}(t) = \prod_{j=1}^k \exp(I_j(t))\sharp_j\check{Z}(0)$. □

Let us now apply Lemma 10.18 to study the variation of the quantity $z(R,L,\tau)$ of Corollary 10.4 when τ varies smoothly.

Lemma 10.19. *Let $(\tau(t))_{t\in[0,1]}$ define a smooth homotopy of asymptotically standard parallelizations of $\check{R}$.*

$$\frac{\partial}{\partial t}\check{Z}(\check{R},L,\tau(t))=\left(\sum_{j=1}^{k}\frac{\partial}{\partial t}\left(2\int_{[0,t]\times U^+K_j}p^*_{\tau(.)}(\omega_{S^2})\right)\alpha\sharp_j\right)\check{Z}(\check{R},L,\tau(t)).$$

PROOF: Fix a homogeneous propagating form ω of $(C_2(R),\tau(0))$, and a form $\tilde{\omega}$ on $[0,1]\times C_2(R)$ such that $\tilde{\omega}(t)$ is a homogeneous propagating form ω of $(C_2(R),\tau(t))$ for all $t\in[0,1]$ as in Lemma 9.1. Lemma 10.18 ensures that

$$\frac{\partial}{\partial t}\check{Z}(\check{R},L,\tau(t))dt=(\sum_{j=1}^{k}\gamma_j(t)\sharp_j)\check{Z}(\check{R},L,\tau(t)),$$

where

$$\gamma_j(u)=\sum_{\Gamma_B\in\mathcal{D}^c(\mathbb{R})}\zeta_{\Gamma_B}\eta(R,L,\Gamma_B,K_j,\tilde{\omega})(u),$$

and

$$\eta(R,L,\Gamma_B,K_j,\tilde{\omega})(u) = \int_{w\in K_j}\int_{\check{\mathcal{S}}(T_w\check{R},\vec{t}_w;\Gamma_B)} \bigwedge_{e\in E(\Gamma_B)} p_e^*(p_{\tau(u)}^*(\omega_{S^2}))[\Gamma_B].$$

The restriction of $p_{\tau(.)}$ from $[0,1]\times U^+K_j$ to S^2 induces a map

$$p_{a,\tau,\Gamma_B}\colon [0,1]\times \cup_{w\in K_j}\check{\mathcal{S}}(T_w\check{R},\vec{t}_w;\Gamma_B) \to \check{Q}(\Gamma_B)$$

over $\left(p_{\tau(.)}\colon [0,1]\times U^+K_j \to S^2\right)$, which restricts to the fibers as the identity map, for any $\Gamma_B \in \mathcal{D}^c(\mathbb{R})$. (Recall that $\check{Q}(\Gamma_B)$ was defined in the beginning of Section 10.3.)

$$\int_0^1 \eta(R,L,\Gamma_B,K_j,\tilde{\omega})(u) = \int_{\mathrm{Im}(p_{a,\tau,\Gamma_B})}\left(\bigwedge_{e\in E(\Gamma_B)} p_{e,S^2}^*(\omega_{S^2})\right)[\Gamma_B].$$

Integrating $\left(\bigwedge_{e\in E(\Gamma_B)} p_{e,S^2}^*(\omega_{S^2})\right)[\Gamma_B]$ along the fiber in $\check{Q}(\Gamma_B)$ yields a two-form on S^2, which is homogeneous, because everything is. Thus this form may be expressed as $2\alpha(\Gamma_B)\omega_{S^2}[\Gamma_B]$, where $\alpha(\Gamma_B)\in\mathbb{R}$, and where

$$\sum_{\Gamma_B\in\mathcal{D}^c(\mathbb{R})} \zeta_{\Gamma_B}\alpha(\Gamma_B)[\Gamma_B] = \alpha.$$

Therefore

$$\int_0^t \eta(R,L,\Gamma_B,K_j,\tilde{\omega})(u) = 2\alpha(\Gamma_B)\int_{[0,t]\times U^+K_j} p_{\tau(.)}^*(\omega_{S^2})[\Gamma_B],$$

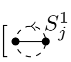
and $\gamma_j(t) = 2\frac{\partial}{\partial t}\left(\int_{[0,t]\times U^+K_j} p_{\tau(.)}^*(\omega_{S^2})\right)\alpha dt.$ $\square$

Corollary 10.20.

$$\prod_{j=1}^k \exp(-I_\theta(K_j,\tau(t))\alpha)\sharp_j\check{Z}(\check{R},L,\tau(t))$$

does not change when τ varies by a smooth homotopy.

Proof: With the notation of Lemma 7.15,

$$\check{Z}_1(\check{R},L,\tau(t)) = \frac{1}{2}\sum_{j=1}^k I_\theta(K_j,\tau(t))\,[$$  $$].$$

Therefore, Lemma 10.19 and Proposition 10.13 imply that $\frac{\partial}{\partial t}I_\theta(K_j,\tau(t)) = 2\frac{\partial}{\partial t}\int_{[0,t]\times U^+K_j} p^*_{\tau(.)}(\omega_{S^2})$. (This can also be checked directly, as an exercise.) Thus,

$$\frac{\partial}{\partial t}\check{Z}(\check{R},L,\tau(t)) = \sum_{j=1}^{k}(\frac{\partial}{\partial t}I_\theta(K_j,\tau(t))\alpha\sharp_j)\check{Z}(\check{R},L,\tau(t)),$$

according to Lemma 10.19. Therefore the derivative of

$$\prod_{j=1}^{k}\exp(-I_\theta(K_j,\tau(t))\alpha)\sharp_j\check{Z}(\check{R},L,\tau(t))$$

with respect to t vanishes. $\square$

Proof of Theorem 7.20: According to the naturality of Proposition 10.1, Lemma 9.5, Proposition 10.7, Corollary 10.9 and Proposition 10.13, it suffices to prove that

$$\prod_{j=1}^{k}(\exp(-I_\theta(K_j,\tau)\alpha)\sharp_j)\,\check{Z}(\check{R},L,\tau)\in\check{\mathcal{A}}(\mathcal{L})$$

is independent of the homotopy class of parallelization τ.

When τ changes in a ball that does not meet the link, the forms can be changed only in the neighborhoods of the unit tangent bundle to this ball. Applying Proposition 10.17, again to $\check{Z}$, where the $p_e^*(\tilde{\omega}(u))$ are independent of u over K_j, proves that

$$\prod_{j=1}^{k}(\exp(-I_\theta(K_j,\tau)\alpha)\sharp_j)\,\check{Z}(\check{R},L,\tau)\in\check{\mathcal{A}}(\mathcal{L})$$

is invariant under the natural action of $\pi_3(SO(3))$ on the homotopy classes of parallelizations.

We now examine the effect of the twist of the parallelization by a map $g\colon (B_R,1)\to(SO(3),1)$. Without loss of generality, assume that there exists $v\in S^2$ such that $p_\tau(U^+K_j)=v$ and g maps K_j to rotations with axis v, for any $j\in\underline{k}$. We want to compare $\check{Z}(\check{R},L,\tau\circ\psi_{\mathbb{R}}(g))$ with $\check{Z}(\check{R},L,\tau)$. There exists a closed form ω on $[0,1]\times UB_R$ that is equal to $p^*_\tau(\omega_{S^2})$ on $\partial([0,1]\times UB_R)\setminus(1\times UB_R)$ and that is equal to $p^*_{\tau\circ\psi_{\mathbb{R}}(g)}(\omega_{S^2})$ on $1\times UB_R$. Extend this form to a closed form Ω on $[0,1]\times C_2(R)$, which restricts to $[0,1]\times(\partial C_2(R)\setminus UB_R)$ as $p^*_\tau(\omega_{S^2})$, and to $1\times\partial C_2(R)$ as $p^*_{\tau\circ\psi_{\mathbb{R}}(g)}(\omega_{S^2})$, as in Lemma 9.1. Let $\Omega(t)$ denote the restriction of Ω to $\{t\}\times C_2(R)$. According to Proposition 10.17,

$$\check{Z}(\check{R}, L, \tau \circ \psi_{\mathbb{R}}(g)) = \prod_{j=1}^{k} (\exp(I_j)\sharp_j)\, \check{Z}(\check{R}, L, \tau),$$

where $I_j = \int_0^1 \gamma_j(u)$, with $\gamma_j(t) = \sum_{\Gamma_B \in \mathcal{D}^c(\mathbb{R})} \zeta_{\Gamma_B} \eta(R, L, \Gamma_B, K_j, \Omega)(t)$ and $\eta(R, L, \Gamma_B, K_j, \Omega)(t) = \int_{w \in K_j} \int_{\check{\mathcal{S}}(T_w \check{R}, \vec{t}_w; \Gamma_B)} \left(\bigwedge_{e \in E(\Gamma_B)} p_e^*(\Omega(t)) \right) [\Gamma_B]$. It suffices to prove that $I_j = (I_\theta(K_j, \tau \circ \psi_{\mathbb{R}}(g)) - I_\theta(K_j, \tau))\, \alpha$. Proposition 10.17 implies that the degree one part $I_{1,j}$ of I_j is

$$\begin{aligned} I_{1,j} &= \check{Z}_1(\check{R}, K_j, \tau \circ \psi_{\mathbb{R}}(g)) - \check{Z}_1(\check{R}, K_j, \tau) \\ &= \tfrac{1}{2}(I_\theta(K_j, \tau \circ \psi_{\mathbb{R}}(g)) - I_\theta(K_j, \tau))\, [\, \circlearrowleft \,]\,. \end{aligned}$$

Let ${}^{\tau}\psi(g^{-1})\colon UB_R \to UB_R$ denote the map induced by $\tau \circ \psi_{\mathbb{R}}(g^{-1}) \circ \tau^{-1}$, recall that $p_\tau = p_{S^2} \circ \tau^{-1}$, so

$$p_{\tau \circ \psi_{\mathbb{R}}(g)} = p_{S^2} \circ \psi_{\mathbb{R}}(g^{-1}) \circ \tau^{-1} = p_\tau \circ {}^{\tau}\psi(g^{-1}).$$

Let $(.-1)\colon [1,2] \to [0,1]$ map x to $x-1$. Set ${}^{\tau}_{-1}\psi(g^{-1}) = ((.-1) \times {}^{\tau}\psi(g^{-1}))$. Extend Ω over $[0,2] \times C_2(R)$ so that Ω restricts to $[1,2] \times UB_R$ as ${}^{\tau}_{-1}\psi(g^{-1})^*(\Omega)$. For any Γ_B, the map ${}^{\tau}_{-1}\psi(g^{-1})$ induces an orientation-preserving diffeomorphism

$${}^{\tau}_{-1}\psi(g^{-1})_*\colon [1,2] \times \cup_{w \in K_j} \check{\mathcal{S}}(T_w \check{R}, \vec{t}_w; \Gamma_B) \to [0,1] \times \cup_{w \in K_j} \check{\mathcal{S}}(T_w \check{R}, \vec{t}_w; \Gamma_B)$$

such that $p_e \circ {}^{\tau}_{-1}\psi(g^{-1})_* = {}^{\tau}_{-1}\psi(g^{-1}) \circ p_e$ for any edge e of Γ_B. Using these diffeomorphisms ${}^{\tau}_{-1}\psi(g^{-1})_*$ to pull back $\left(\bigwedge_{e \in E(\Gamma_B)} p_e^*(\Omega(t)) \right)$ proves that $\gamma_j(t+1) = \gamma_j(t)$. In particular,

$$I_j(2) = \int_0^2 \gamma_j(u) = 2I_j.$$

Set $\check{Z}(2) = \check{Z}(R, L, \tau \circ \psi_{\mathbb{R}}(g)^2)$. Then

$$\check{Z}(2) = \prod_{j=1}^{k} \exp\left(\left(I_\theta(K_j, \tau \circ \psi_{\mathbb{R}}(g)^2) - I_\theta(K_j, \tau) \right) \alpha \right) \sharp_j \check{Z}(\check{R}, L, \tau),$$

since g^2 is homotopic to the trivial map outside a ball (see Lemma 5.2, 2). According to Proposition 10.17,

$$\check{Z}(2) = \prod_{j=1}^{k} (\exp(2I_j)\sharp_j)\, \check{Z}(\check{R}, L, \tau).$$

By induction on the degree, $2I_j = (I_\theta(K_j, \tau \circ \psi_{\mathbb{R}}(g)^2) - I_\theta(K_j, \tau))\alpha$. The degree one part of this equality implies that

$$2(I_\theta(K_j, \tau \circ \psi_{\mathbb{R}}(g)) - I_\theta(K_j, \tau)) = I_\theta(K_j, \tau \circ \psi_{\mathbb{R}}(g)^2) - I_\theta(K_j, \tau).$$

So $I_j = (I_\theta(K_j, \tau \circ \psi_{\mathbb{R}}(g)) - I_\theta(K_j, \tau))\,\alpha$, as wanted. □

10.6 Some more properties of Z

When $\check{R} = \mathbb{R}^3$, then $\mathcal{Z}(S^3, L) = \check{\mathcal{Z}}(S^3, L)$ is the configuration space invariant studied by Altschüler, Freidel [AF97], Dylan Thurston [Thu99], Sylvain Poirier [Poi02] and others, after work of many people including Witten [Wit89], Guadagnini, Martellini and Mintchev [GMM90], Kontsevich [Kon94, Kon93], Bott and Taubes [BT94], Bar-Natan [BN95b] ...

Reversing a link component orientation The following proposition is obvious from the definition of Z.

Proposition 10.21. *Let $L \colon \sqcup_{j=1}^k S_j^1 \to R$ be a link in a $\mathbb{Q}$-sphere R. For a Jacobi diagram Γ on $\sqcup_{j=1}^k S_j^1$, let $U_j(\Gamma)$ denote the set of univalent vertices of Γ mapped to S_j^1. This set is ordered cyclically by S_j^1. When the orientation of the component $L(S_j^1)$ is changed, $\mathcal{Z}(L)$ is modified by reversing the circle S_j^1 (that is reversing the cyclic order of $U_j(\Gamma)$) in classes $[\Gamma]$ of diagrams Γ on $\sqcup_{j=1}^k S_j^1$ and multiplying them by $(-1)^{\sharp U_j(\Gamma)}$ in $\mathcal{A}(\sqcup_{j=1}^k S_j^1)$.*

In other words, we can forget the orientation of the link L and view $Z(L)$ as valued in $\mathcal{A}(\sqcup_{j=1}^k S_j^1)$, where the S_j^1 are not oriented, as in Definitions 6.13 and 6.16.

□

Remark 10.22. The orientation of a component $L(S_j^1)$ is used in two ways. It defines a cyclic order on $U_j(\Gamma)$, and it defines the orientation of the vertices of $U_j(\Gamma)$ as in Definition 6.13. The local orientation of S_j^1 near the image of a vertex orients the corresponding local factor of the configuration space. The cyclic order is encoded in the isotopy class of the injection of U_j into the source S_j^1.

Link component numbering The following proposition is obvious from the definition of $\mathcal{Z}$.

Proposition 10.23. *When the numbering of the components of L is changed, $\mathcal{Z}(L)$ is modified by the corresponding change of numberings of the circles S^1_j in diagram classes of $\mathcal{A}(\sqcup_{j=1}^k S^1_j)$.*

$\square$

For a link $L\colon \mathcal{L}(L) \to R$ in a $\mathbb{Q}$-sphere R, $\mathcal{Z}(R,L)$ is valued in $\mathcal{A}(\mathcal{L}(L))$. The one-manifold $\mathcal{L}(L)$ is a disjoint union of oriented circles, which have been numbered so far. But the numbers may be changed to any decoration that marks the component.

This gives sense to the statement of the following theorem.

Theorem 10.24. *For any two links L_1 and L_2 in rational homology spheres R_1 and R_2,*

$$\mathcal{Z}(R_1 \sharp R_2, L_1 \sqcup L_2) = \mathcal{Z}(R_1, L_1)\mathcal{Z}(R_2, L_2)$$

A generalization of Theorem 10.24 is proved in Section 17.2. See Theorem 17.10 in particular. See also Section 13.3. The proof given in Section 17.2 can be read without reading the intermediate chapters. Theorem 10.24 and Corollary 10.11 yield the following corollary.

Corollary 10.25. *For any two rational homology spheres R_1 and R_2,*

$$\Theta(R_1 \sharp R_2) = \Theta(R_1) + \Theta(R_2).$$

$\square$

Lemma 10.26. *Under the assumptions of Lemma 5.14,*

$$Z_n(-\check{R}, L, \overline{\tau}) = (-1)^n Z_n(\check{R}, L, \tau).$$

Proof: If ω is a homogeneous propagating form of $(C_2(R), \tau)$, then $-\omega$ is a homogeneous propagating form of $(C_2(-R), \overline{\tau})$. Let Γ be a degree n numbered Jacobi diagram. When the orientation of R is reversed, the orientation of $\check{C}(R, L; \Gamma)$ is reversed if and only if $\sharp T(\Gamma)$ is odd. Thus the integrals will be multiplied by $(-1)^{\sharp E(\Gamma) + \sharp T(\Gamma)}$, where $2\sharp E(\Gamma) = 3\sharp T(\Gamma) + \sharp U(\Gamma)$, so $2n = 2(\sharp E(\Gamma) - \sharp T(\Gamma))$. $\square$

Theorem 10.27. *For any link L in a rational homology sphere R,*

$$\mathcal{Z}_n(-R, L) = (-1)^n \mathcal{Z}_n(R, L).$$

Proof: Thanks to Theorem 7.20,

$$\mathcal{Z}(-R, L) = \exp(-\frac{1}{4}p_1(\overline{\tau})\beta) \prod_{j=1}^{k} (\exp(-I_\theta(K_j, \overline{\tau})\alpha)\sharp_j)\, Z(-\check{R}, L, \overline{\tau}),$$

where $I_\theta(K_j, \overline{\tau}) = -I_\theta(K_j, \tau)$ and $p_1(\overline{\tau}) = -p_1(\tau)$ according to Lemma 5.14, so we get the result, since α and β vanish in even degrees. $\square$

Chapter 11

Rationality

In this chapter, we give equivalent definitions of $\mathcal{Z}$ based on algebraic intersections of propagating chains and we prove that $\mathcal{Z}$ and the anomalies α and β are rational.

11.1 From integrals to algebraic intersections

In order to warm up, we first prove the following rationality result, which is due to Sylvain Poirier [Poi02] and Dylan Thurston [Thu99], independently.

Proposition 11.1. *The anomaly α of Section 10.3 is rational:*

$$\alpha \in \check{\mathcal{A}}(\mathbb{R};\mathbb{Q}).$$

For any link $L\colon \mathcal{L} \to \mathbb{R}^3$, $\mathcal{Z}(S^3, L)$ is rational:

$$\mathcal{Z}(S^3, L) \in \mathcal{A}(\mathcal{L};\mathbb{Q}).$$

PROOF: Let us fix n and prove that α_n is in $\check{\mathcal{A}}_n(\mathbb{R};\mathbb{Q})$. For any degree n numbered Jacobi diagram $\check{\Gamma}$ on $\mathbb{R}$, define the smooth map

$$g(\check{\Gamma})\colon Q(\check{\Gamma}) \times (S^2)^{\underline{3n-2}\setminus j_E(E(\check{\Gamma}))} \to (S^2)^{3n-2}$$

as the product $\left(\prod_{e\in E(\check{\Gamma})} p_e\right) \times \mathbf{1}\left((S^2)^{\underline{3n-2}\setminus j_E(\check{\Gamma})}\right)$. Note that a regular point of $g(\check{\Gamma})$ is not in the image of $\partial Q(\check{\Gamma}) \times (S^2)^{\underline{3n-2}\setminus j_E(E(\check{\Gamma}))}$. According to the Morse-Sard theorem [Hir94, Chapter 3, Section 1], the set of regular points with respect to $g(\check{\Gamma})$ is dense (it is even residual as an intersection of *residual* sets –which contain intersections of countable families of dense open sets, by definition– in a complete metric space). Since $Q(\check{\Gamma})$ is compact, so are

$\partial Q(\check{\Gamma})$, the boundary of the domain of $g(\check{\Gamma})$, and the subset of the domain of $g(\check{\Gamma})$ consisting of the points at which the derivative of $g(\check{\Gamma})$ is not surjective. Therefore, the set of regular points with respect to $g(\check{\Gamma})$ is open. Thus the finite intersection over all the $\check{\Gamma} \in \mathcal{D}_n^c(\mathbb{R})$ of the sets of regular points with respect to $g(\check{\Gamma})$ is also open and dense. Let $\prod_{i=1}^{3n-2} B(x_i)$ be a product of open balls of S^2, which is in this intersection. Then for any $\check{\Gamma} \in \mathcal{D}_n^c(\mathbb{R})$, the local degree of $g(\check{\Gamma})$ (which is an integer) is constant over $\prod_{i=1}^{3n-2} B(X_i)$. In particular, if $\omega(i, S^2)$ is a volume-one form of S^2 that is supported on $B(X_i)$ for each $i \in \underline{3n-2}$, then $I(\check{\Gamma}, o(\check{\Gamma}), \omega(i, S^2))$, which is nothing but this integral local degree, is an integer for any $\check{\Gamma}$ in $\mathcal{D}_n^c(\mathbb{R})$. Therefore α_n, which is defined in Proposition 10.13, is in $\check{\mathcal{A}}_n(\mathbb{R};\mathbb{Q})$.

For a fixed n and a given k–component link L of S^3, there exists a similar product $\prod_{i=1}^{3n} B_L(Y_i)$ of open balls of S^2 consisting of points of $(S^2)^{3n}$ that are regular for all maps

$$\left(\prod_{e\in E(\Gamma)} p_{e,S^2}\right) \times \mathbf{1}\left((S^2)^{\underline{3n}\setminus j_E(\Gamma)}\right) : C(S^3, L; \Gamma) \times (S^2)^{\underline{3n}\setminus j_E(\Gamma)} \to (S^2)^{3n}$$

associated to Jacobi diagrams Γ of $\mathcal{D}_n^e(\mathcal{L})$. Then if $\omega(i, S^2)$ is a volume-one form of S^2 that is supported on $B_L(Y_i)$ for each $i \in \underline{3n}$,

$$I(S^3, L, \Gamma, (p_{S^2}^*(\omega(i, S^2))))$$

is an integer for every Γ of $\mathcal{D}_n^e(\mathcal{L})$.

If the link is straight, then Theorem 7.39 implies that $Z_n(\mathbb{R}^3, L, \tau_s)$ is rational. Thus $Z(\mathbb{R}^3, L, \tau_s)$ is rational for any straight link L of $\mathbb{R}^3$. In particular $I_\theta(K, \tau_s)$ is rational for any component K of a straight link L, and Theorem 7.20 together with the rationality of α implies that $\mathcal{Z}(S^3, L)$ is rational. $\square$

With the notation of the above proof, the $P(i) = p_{S^2}^{-1}(y_i) \subset C_2(S^3)$ for $y_i \in B_L(Y_i)$ are propagating chains such that, for any Γ of $\mathcal{D}_n^e(\mathcal{L})$, the intersection over $E(\Gamma)$ of the $p_e^{-1}(P(j_E(e)))$ in $C(S^3, L; \Gamma)$ is transverse, and the integral $I(S^3, L, \Gamma, (p_{S^2}^*(\omega(i, S^2))))$ is nothing but their algebraic intersection.

We are going to use Version 7.39 of Theorem 7.20 to replace the configuration space integrals by algebraic intersections in configuration spaces, and thus to prove the rationality of Z^s for straight links in any rational homology sphere as follows.

Definition 11.2. A smooth map $f \colon B \to A$ is *transverse* to a submanifold C of A along a subset K of B if for any point x of $K \cap f^{-1}(C)$,

$$T_{f(x)}A = T_x f(T_x B) + T_{f(x)} C.$$

When A or B have ridges, we furthermore require this equality to hold when A or B are replaced by all their open faces (of any dimension).

A smooth map $f\colon B \to A$ is *transverse* to a submanifold C of A if it is transverse to C along B. Say that a smooth map $f\colon B \to A$ is *transverse* to a rational chain C of A, which is a multiple of a union of compact smooth embedded submanifolds with boundaries and corners $\cup_{k\in J} C_k$, if f is transverse to C_k for any $k \in J$.

A *rational simplicial chain*, which is a rational combination of simplices in a triangulated smooth manifold is an example of what we call a rational chain. Rational multiples of compact immersion images provide other examples of chains. An immersion image will be represented as a union of embedded manifolds by decomposing the source as a union of compact manifolds with boundaries and corners glued along their boundaries.

Recall from Notation 7.29 that $\mathcal{D}^e_{k,\underline{3n}}(\mathcal{L})$ denotes the set of $\underline{3n}$-numbered degree k Jacobi diagrams with support $\mathcal{L}$ without looped edges, and let $\mathcal{D}^e_{\underline{3n}}(\mathcal{L}) = \cup_{k\in\mathbb{N}} \mathcal{D}^e_{k,\underline{3n}}(\mathcal{L})$. Note that $\mathcal{D}^e_n(\mathcal{L}) = \mathcal{D}^e_{n,\underline{3n}}(\mathcal{L})$ but $\underline{3n}$-numbered Jacobi diagrams may have a degree different from n.

Definition 11.3. Say that a family $(P(i))_{i\in\underline{3n}}$ of propagating chains of $(C_2(R),\tau)$ is *in general* $3n$ *position* with respect to a link $L\colon \mathcal{L} \to \check{R}$ if for any $\Gamma \in \mathcal{D}^e_{\underline{3n}}(\mathcal{L})$, and for any subset E of $E(\Gamma)$, the map

$$p(\Gamma, E) = \prod_{e\in E} p_e\colon C(R, L; \Gamma) \to (C_2(R))^{j_E(E)}$$

is transverse to $\prod_{e\in E} P(j_E(e))$. For such a family $(P(i))_{i\in\underline{3n}}$ in general $3n$ position, the intersection $\cap_{e\in E(\Gamma)} p_e^{-1}(P(j_E(e)))$ consists of a finite number of points x, which sit in the interior of $C(R, L; \Gamma)$, and, for each such x,

- for every edge $e \in E(\Gamma)$, $p_e(x)$ is in the interior of one of the smooth embedded 4-simplices $\Delta_{j_E(e),i}$ with boundaries that constitute the chain $P(j_E(e))$ if $P(j_E(e))$ is a simplicial chain, and, $p_e(x)$ meets the union of smooth embedded 4-manifolds with boundaries that constitute $P(j_E(e))$ in the interior of finitely many of them, in general, and the family $\left(\Delta_{j_E(e),i}\right)_{i\in J(e,x)}$ of met manifolds is indexed by a finite set $J(e, x)$,

- for every map $i\colon E(\Gamma) \to \cup_{e\in E(\Gamma)} J(e, x)$ such that $i(e) \in J(e, x)$, the local maps, from small open neighborhoods of x in $C(R, L; \Gamma)$, to the product over $E(\Gamma)$ of the fibers of the locally trivialized normal bundles to the $\Delta_{j_E(e),i(e)}$, are local diffeomorphisms.

The following lemma will be proved in Section 11.3.

Lemma 11.4. *Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. For any link $L\colon \mathcal{L} \to \check{R}$, for any integer n, there exists a family $(P(i))_{i\in\underline{3n}}$ of propagating chains of $(C_2(R), \tau)$ in general $3n$-position with respect to L.*

For such a family, define $I(\Gamma, o(\Gamma), (P(i))_{i\in\underline{3n}})$ as the algebraic intersection in $(C(R, L; \Gamma), o(\Gamma))$ of the codimension 2 rational chains $p_e^{-1}(P(j_E(e)))$ –cooriented by the coorientation of $P(j_E(e))$ in $C_2(R)$– over the edges e of $E(\Gamma)$.

For any finite set V, equip $C_V(R)$ with a Riemannian metric that is symmetric with respect to permutations of elements of V. Let d denote the associated distance. Our choice of distances will not matter thanks to the following easy lemma.

Lemma 11.5. *On a compact smooth manifold, all the distances associated to Riemannian metrics are equivalent.*

Proof: Let g_1 and g_2 be two Riemannian metrics on the compact manifold M, let $\parallel . \parallel_1$ and $\parallel . \parallel_2$ be the two associated norms on tangent vectors, and let d_1 and d_2 be the two associated distances. View the unit tangent bundle UM of M as the set of unit tangent vectors to M with respect to $\parallel . \parallel_1$. Then the image of UM under the continuous map $\parallel . \parallel_2$ is a compact interval $[a, b]$ with $a > 0$, and we have for any nonzero tangent vector x of M

$$a \leq \frac{\parallel x \parallel_2}{\parallel x \parallel_1} \leq b.$$

Let p and q be two distinct points of M. For any smooth path $\gamma\colon [0, 1] \to M$, such that $\gamma(0) = p$ and $\gamma(1) = q$,

$$d_2(p, q) \leq \int_0^1 \parallel \gamma'(t) \parallel_2 dt \leq b \int_0^1 \parallel \gamma'(t) \parallel_1 dt.$$

Therefore $d_2(p, q) \leq b d_1(p, q)$. Similarly, $d_1(p, q) \leq \frac{d_2(p,q)}{a}$. $\square$

For a subset X of $C_V(R)$ and for $\varepsilon > 0$, set

$$N_\varepsilon(X) = \{x \in C_V(R) \mid d(x, X) < \varepsilon\}.$$

Definition 11.6. For a small positive number η, a closed 2-form $\omega(i)$ on $C_2(R)$ is said to be *η-dual* to $P(i)$, if it is supported in $N_\eta(P(i))$ and if for any 2-dimensional disk D embedded in $C_2(R)$ transverse to $P(i)$ whose boundary sits outside $N_\eta(P(i))$, $\int_D \omega(i) = \langle D, P(i)\rangle_{C_2(R)}$.

The following lemma will be proved in Section 11.4.

Lemma 11.7. *Under the hypotheses of Lemma 11.4, assume that Lemma 11.4 is true. For any $\eta > 0$, there exist propagating forms $\omega(i)$ of $(C_2(R), \tau)$ η-dual to the $P(i)$ of Lemma 11.4. If η is small enough, then*

$$I(\Gamma, o(\Gamma), (P(i))_{i\in\underline{3n}}) = I(\Gamma, o(\Gamma), (\omega(i))_{i\in\underline{3n}})$$

for any $\Gamma \in \mathcal{D}^e_{k,\underline{3n}}(\mathcal{L})$, where $k \leq n$.

Thus $I(\Gamma, o(\Gamma), (\omega(i))_{i\in\underline{3n}})$ is rational, in this case, and we get the following theorem.

Theorem 11.8. *The anomaly β is rational:*

$$\beta \in \mathcal{A}(\emptyset; \mathbb{Q}).$$

Let $(\check{R}, \tau)$ be an asymptotic rational homology $\mathbb{R}^3$. For any link $L\colon \mathcal{L} \to \check{R}$,

$$\mathcal{Z}(R, L) \in \mathcal{A}(\mathcal{L}; \mathbb{Q}).$$

Proof of the theorem assuming Lemmas 11.4 and 11.7: Theorems 7.39 and 7.20, Lemmas 11.4 and 11.7 imply that $Z(S^3, \tau) = Z(\mathbb{R}^3, \emptyset, \tau)$ is rational for any τ. So $z(S^3, \tau) = p^c(Z(S^3, \tau))$ is rational for any τ, too, and, by Definition 10.5, the anomaly β is rational. Therefore, Theorem 7.39, Lemmas 11.4 and 11.7 also imply that $Z(\check{R}, L, \tau)$ is rational for any asymptotic rational homology $\mathbb{R}^3$ $(\check{R}, \tau)$ and for any straight link L with respect to τ. In particular, $I_\theta(K, \tau)$ is rational for any component K of a straight link L. Since $p_1(\tau)$ and the anomalies α and β are rational, Theorem 7.20 now implies that $\mathcal{Z}(R, L)$ is rational. $\square$

11.2 More on general propagating chains

By a Thom theorem [Tho54, Théorème II.27, p. 55], any integral codimension 2 homology class in a manifold can be represented as the class of an embedded closed (oriented) submanifold. We prove a relative version of this result below, following Thom's original proof in this special case of his theorem.

Theorem 11.9. *Let A be a compact smooth (oriented) manifold with boundary, and let C be a smooth codimension 2 closed (oriented) submanifold of ∂A such that the homology class of C vanishes in $H_{dim(A)-3}(A; \mathbb{Z})$, then there exists a compact smooth codimension 2 submanifold B of A transverse to ∂A whose boundary is C.*

Proof: Let us first sketch Thom's proof with his notation. The normal bundle to C in ∂A is an oriented disk bundle. It is the pull-back of a universal disk bundle $A_{SO(2)}$ over a compact classifying space $B_{SO(2)}$ via a map f_C from C to $B_{SO(2)}$. See [MS74, p. 145]. Like Thom [Tho54, p. 28, 29], define the Thom space $M(SO(2))$ of $SO(2)$ to be the space obtained from the total space $A_{SO(2)}$ by identifying its subspace $E_{SO(2)}$ consisting of the points in the boundaries of the fibers D^2 of $A_{SO(2)}$ with a single point a. Regard $B_{SO(2)}$ as the zero section of $A_{SO(2)}$. So $B_{SO(2)}$ sits inside $M(SO(2))$.

This map f_C extends canonically to ∂A. Its extension $f_{\partial A}$ injects the fibers of an open tubular neighborhood of C in ∂A to fibers of $A_{SO(2)}$ and maps the complement of such a neighborhood to a. Thus C is the preimage of $B_{SO(2)}$. In order to prove the theorem, it suffices to extend the map $f_{\partial A}$ to a map f_A from A to $M(SO(2))$ so that, in a neighborhood of any point of $f_A^{-1}(B_{SO(2)})$, the differential of a local projection on the fiber of the normal bundle to $B_{SO(2)}$ – which is isomorphic to the tangent space to a fiber of $A_{SO(2)}$ – composed with f_A is well defined and surjective. Indeed the compact submanifold $B = f_A^{-1}(B_{SO(2)})$ of A, with respect to such an extension, has the wanted properties.

The map $f_{\partial A}$ can be extended as a continuous map, using the fact that $M(SO(2))$ is a $K(\mathbb{Z};2)$ [Tho54, ii), p. 50]. In other words, the only non-trivial homotopy group of $M(SO(2))$ is its π_2, which is isomorphic to $\mathbb{Z}$.

Let us now give some details about the above sketch and show how $(B_{SO(2)}, M(SO(2)))$ can be replaced by $(\mathbb{C}P^N, \mathbb{C}P^{N+1})$ for some large integer N, following [Tho54, ii), p. 50]. View the fiber of a disk bundle as the unit disk of $\mathbb{C}$. The corresponding complex line bundle over C injects into a trivial complex bundle $\mathbb{C}^{N+1} \times C$ as in [MS74, Lemma 5.3, p. 61] for some integer N, by some map $(f_{1,C}, \mathbf{1}(C))$. Therefore, it is the pull-back of the tautological complex line bundle γ_N^1 over $B'_{SO(2)} = \mathbb{C}P^N$ by the map $f'_C \colon C \to \mathbb{C}P^N$ that sends a point x of C to the image of the fiber over x under $f_{1,C}$. The disk bundle $A'_{SO(2)}$ associated to γ_N^1 is diffeomorphic to the normal bundle to $\mathbb{C}P^N$ in $\mathbb{C}P^{N+1}$ by the inverse of the following (orientation-reversing) map:

$$\begin{array}{lcl} \mathbb{C}P^{N+1} \setminus \{[(0,\dots,0,1)]\} & \to & \gamma_N^1 \\ [z_1,\dots,z_{N+1},z] & \mapsto & \left(\frac{\overline{z}}{\sum_{i=1}^{N+1}|z_i|^2}(z_1,\dots,z_{N+1}), [z_1,\dots,z_{N+1}]\right). \end{array}$$

This map also shows that the space $M'(SO(2))$, obtained from $A'_{SO(2)}$ by identifying $E'_{SO(2)} = \partial A'_{SO(2)}$ to a point, is homeomorphic to the whole $\mathbb{C}P^{N+1}$.

The long exact sequence associated to the fibration $S^1 \hookrightarrow S^{2N+3} \to \mathbb{C}P^{N+1}$ implies that $\pi_2(\mathbb{C}P^{N+1}) = \mathbb{Z}[\mathbb{C}P^1]$ and that $\pi_i(\mathbb{C}P^{N+1}))$ is trivial for any $i \in \underline{2N+2} \setminus \{2\}$. See Theorem A.14. Assume that $2N+2$ is bigger than the dimension of A, without loss of generality.

It is not hard to see that $\pi_2(M'(SO(2))) = H_2(M'(SO(2));\mathbb{Z})$ is freely generated by the class of the image $[F]$ of a fiber under the identification of $E'_{SO(2)}$ with the point a. (Indeed, since $B'_{SO(2)}$ is connected, the homology class of $[F]$ is well defined. Since any 2-cycle is homologous to a 2-cycle that is transverse to $B'_{SO(2)}$, any homology class of degree 2 is a multiple of $[F]$, which therefore generates $H_2(M'(SO(2));\mathbb{Z})$.)

Extend f'_C to a map $f_{\partial A}$ valued in $M'(SO(2))$ as before so that $C = f_{\partial A}^{-1}(B'_{SO(2)})$. Recall that any smooth manifold is triangulable [Cai35], [Whi40], fix a triangulation for $(A, \partial A)$ transverse to C. In particular, $C = f_{\partial A}^{-1}(B'_{SO(2)})$ avoids the 1-skeleton. Extend $f_{\partial A}$ skeleton by skeleton starting with the zero and one-skeleta for which there is no obstruction to extending $f_{\partial A}$ to a map valued in $M'(SO(2)) \setminus B'_{SO(2)}$, which is connected. There is no obstruction to extending $f_{\partial A}$, as a map valued in the contractible $M'(SO(2)) \setminus B'_{SO(2)}$, to the two-skeleton of $(A, \partial A)$, but such a map would not necessarily extend to the three-skeleton. For an arbitrary generic extension $f_A^{(2)}$ of $f_{\partial A}$ to the two-skeleton of $(A, \partial A)$ as a map to $M'(SO(2))$, for each 2–cell D of A, the algebraic intersection $c(f_A^{(2)})(D)$ of its image with $B'_{SO(2)}$ defines a 2-cochain $c(f_A^{(2)})$ with $\mathbb{Z}$-coefficients, and $f_A^{(2)}$ extends to the 3–skeleton if and only if this cochain (which is fixed on ∂A and Poincaré dual to C on ∂A) is a cocycle. Thus, in order to prove that $f_{\partial A}$ extends to the 3–skeleton it suffices to prove that the class of $c(f_A^{(2)})_{|\partial A}$ in $H^2(\partial A;\mathbb{Z})$ is in the natural image of $H^2(A;\mathbb{Z})$, or, equivalently, that its image in $H^3(A, \partial A;\mathbb{Z})$, by the boundary map of the long cohomology exact sequence of $(A, \partial A)$, vanishes. This image is represented by a cochain that maps a 3-cell $\mathcal{B}$ of $(A, \partial A)$ to the algebraic intersection of $\partial\mathcal{B}$ and C, which is, up to a fixed sign, the algebraic intersection of $\mathcal{B}$ and C (pushed inside A). Therefore, the class in $H^3(A, \partial A;\mathbb{Z})$ of this relative cocycle is Poincaré dual to the class of C in $H_{\dim(A)-3}(A)$, which vanishes, and $f_{\partial A}$ can be extended to the 3–skeleton. Since the next homotopy groups $\pi_i(M'(SO(2)))$, for $3 \leq i < \dim(A)$, vanish, there is no obstruction to extending $f_{\partial A}$ to the manifold A.

Finally, make f_A smooth, using an approximation theorem [Hir94, Chapter 2, Theorem 2.6] of continuous maps by smooth maps, and make f_A transverse to $B'_{SO(2)}$, with the help of a transversality theorem [Hir94, Chapter 3, Theorem 2.1]. $\square$

Corollary 11.10. *If R is a $\mathbb{Z}$-sphere, for any asymptotically standard parallelization τ of $\check{R}$, for any $X \in S^2$, there exists a 4-dimensional submanifold of $C_2(R)$ that is transverse to the ridges whose boundary is $p_\tau^{-1}(X)$.*

Proof: First extend p_τ as a regular map from a regular neighborhood $N(\partial C_2(R))$ of $\partial C_2(R)$, where $N(\partial C_2(R))$ is a smooth cobordism with ridges embedded in $C_2(R)$ from a smooth manifold $\partial C_2'(R)$ without ridges to $\partial C_2(R)$, and $N(\partial C_2(R))$ is homeomorphic to the product $[0,1]\times\partial C_2(R)$. Then apply Theorem 11.9 to $C_2'(R) = C_2(R)\setminus \mathrm{Int}(N(\partial C_2(R)))$ and to $p^{-1}_{\tau|\partial C_2'(R)}(X)$. $\square$

When R is a $\mathbb{Q}$-sphere, perform the same first step as in the above proof. Take a collar neighborhood of $\partial C_2'(R)$ in $N(\partial C_2(R))$, which is (diffeomorphic to and) identified with $[0,8]\times\partial C_2'(R)$ so that $\partial C_2'(R) = \{0\}\times\partial C_2'(R)$. Assume that p_τ factors through the projection to $\partial C_2'(R)$ on $[0,8]\times\partial C_2'(R)$. There exists a positive integer k such that $kp^{-1}_{\tau|\partial C_2'(R)}(X)$ is null-homologous in $C_2'(R)$. Let $p_k\colon S^2\to S^2$ be a degree k map that does not fix X and such that X is regular and has k preimages. Then $(p_k\circ p_{\tau|\partial C_2'(R)})^{-1}(X)$ bounds a 4-manifold P', properly embedded in $C_2'(R)$, according to Theorem 11.9. For $j\in\underline{k}$, let $\{\gamma_j\colon[0,4]\to S^2\}_{j\in\underline{k}}$ be a collection of smooth injective paths ending at $X=\gamma_j(4)$ whose images do not meet outside X and such that $p_k^{-1}(X)=\{\gamma_j(0)\mid j\in\underline{k}\}$. Also assume that all the derivatives of γ_j vanish at 0 and 4. Consider

$$\begin{array}{llll} p_{[0,8]}\times p_\tau\colon & [0,8]\times\partial C_2'(R) & \to & [0,8]\times S^2\\ & (t,x) & \mapsto & (t,p_\tau(x))\end{array}$$

Then

$$\begin{array}{ll} P = & p^{-1}_{\tau|N(\partial C_2(R))\setminus([0,4[\times\partial C_2'(R))}(X)+\frac{1}{k}P'\\ & +\frac{1}{k}(p_{[0,8]}\times p_\tau)^{-1}_{|[0,4]\times\partial C_2'(R)}\left(\{(t,\gamma_j(t))\mid j\in\underline{k}, t\in[0,4]\}\right)\end{array}$$

is a propagating chain of $(\check{R},\tau)$. See Figure 11.1.

11.3 Existence of transverse propagating chains

In this section, we prove Lemma 11.4.

In order to warm-up, we first prove a weak version of this lemma. The proof is a straightforward adaptation of a Thom proof [Tho54, p. 23, 24, Lemma I.4].

Assume that R is an integer homology 3–sphere, and let $(P(i))_{i\in\underline{3n}}$ be a family of propagating chains of $(C_2(R),\tau)$ for an asymptotically standard parallelization τ of $\check{R}$. Assume that these chains are submanifolds of $C_2(R)$ transverse to $\partial C_2(R)$ as in Corollary 11.10.

Let $N(P(i))$ denote the normal bundle to $P(i)$ embedded in $C_2(R)$ as a tubular neighborhood whose fibers $N_x(P(i))$ over a point $x\in P(i)$ are disks embedded in $C_2(R)$. Let $(K_{i,j})_{j\in J(i)}$ be a finite cover of $P(i)$ by compact

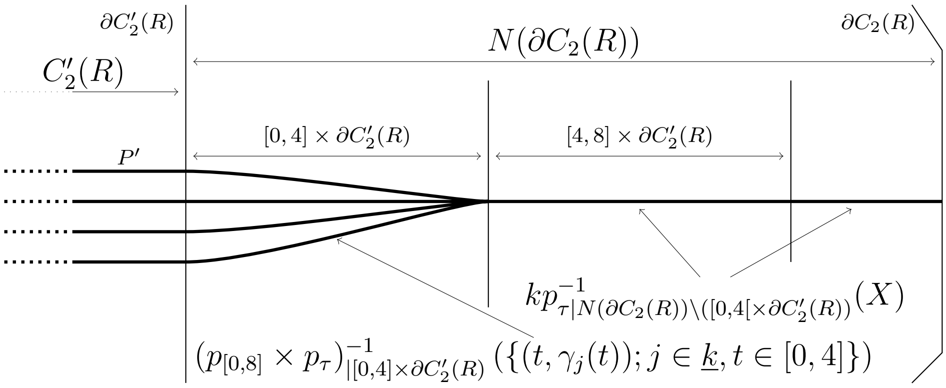


Figure 11.1: A propagating chain of $(\check{R},\tau)$

subspaces $K_{i,j}$ embedded in open subspaces $\mathcal{O}_{i,j}$ of $P(i)$, diffeomorphic to $A_{i,j}$, where $A_{i,j} = \mathbb{R}^4$, $\mathbb{R}^+\times\mathbb{R}^3$ or $(\mathbb{R}^+)^2\times\mathbb{R}^2$, by diffeomorphisms $\phi_{i,j}\colon \mathcal{O}_{i,j}\to A_{i,j}$. Let $\left(\psi_{i,j}\colon N(P(i))_{|\mathcal{O}_{i,j}}\to A_{i,j}\times D^2\right)_{j\in J}$ be associated bundle charts over the $\phi_{i,j}$ such that for any $\{j,k\}\subset J(i)$, for any $x\in\mathcal{O}_{i,j}\cap\mathcal{O}_{i,k}$, the map

$$(v\mapsto p_{D^2}\circ\psi_{i,k}\circ\psi_{i,j}^{-1}(\phi_{i,j}(x),v))$$

is a linear map of $SO(2)$, where p_{D^2} denotes the natural projection onto D^2. Consider the space $\mathcal{H}_i$ of smooth diffeomorphisms of $N(P(i))$, that are isotopic to the identity map, that fix a neighborhood of $\partial N(P(i))$ pointwise and that map any fiber of $N(P(i))$ to itself. Equip this space $\mathcal{H}_i$ with the following distance[1] d. Each fiber is equipped with the distance d_P induced by the norm of $\mathbb{R}^2$. This allows us to define a C^0 distance d_0 between two elements h and k of $\mathcal{H}_i$ by $d_0(h,k)=\sup_{x\in N(P(i))}d_P(h(x),k(x))$. Since $A_{i,j}\times D^2$ is a subset of $\mathbb{R}^6$, the differential of a map $\psi_{i,j}\circ h\circ\psi_{i,j}^{-1}$ for $h\in\mathcal{H}_i$ maps every element x of $A_{i,j}\times D^2$ to a linear map of $\mathbb{R}^6$. The norm $\parallel L\parallel$ of a linear map L of $\mathbb{R}^6$ is defined to be $\parallel L\parallel=\sup_{x\in S^5}\parallel L(x)\parallel$. For h and k in $\mathcal{H}_i$, set

$$d_{(1)}(h,k)=\sup_{j\in J(i),x\in\phi_{i,j}(K_{i,j})\times D^2}\left(\parallel T_x(\psi_{i,j}\circ h\circ\psi_{i,j}^{-1})-T_x(\psi_{i,j}\circ k\circ\psi_{i,j}^{-1})\parallel\right)$$

and

$$d(h,k)=\sup\left(d_0(h,k),d_{(1)}(h,k)\right).$$

[1]This distance induces the strong (or weak, which is the same since $N(P(i))$ is compact) C^1-topology. See [Hir94, Chapter 2, p.35].

Lemma 11.11. *Under the above hypotheses, there is a dense open subset of $\prod_{i=1}^{3n}\mathcal{H}_i$ such that for any (h_i) in this subset, the chains obtained from the $P(i)$ by replacing $P(i)$ by $h_i^{-1}(P(i))$ are in general $3n$ position with respect to L in the sense of Definition 11.3.*

PROOF: We are going to list finitely many sufficient conditions on the (h_i), which guarantee the conclusion, which is that for any graph Γ of $\mathcal{D}^e_{\underline{3n}}(\mathcal{L})$, and for any subset E of $E(\Gamma)$,

$$p(\Gamma,E)=\prod_{e\in E}p_e\colon C(R,L;\Gamma)\to (C_2(R))^{j_E(E)}$$

is transverse to $\prod_{e\in E}h^{-1}_{j_E(e)}P(j_E(e))$. Next we will prove that each of these conditions is realized in an open dense subset of $\prod_{i=1}^{3n}\mathcal{H}_i$.

Extend the elements of $\mathcal{H}_i$ to diffeomorphisms of $C_2(R)$, by the identity map of $C_2(R)\setminus \mathring{N}(P(i))$. The propagating chains obtained from the $P(i)$ by replacing $P(i)$ by $h_i^{-1}(P(i))$ are in general $3n$ position, with respect to L, if and only if, for any triple (Γ,E,ℓ), where $\Gamma\in\mathcal{D}^e_{\underline{3n}}(\mathcal{L})$, $E\subseteq E(\Gamma)$, and ℓ is a map $\ell\colon E\to\cup_{i\in\underline{3n}}J(i)$ such that $\ell(e)\in J(j_E(e))$, the following condition $(*)(\Gamma,E,\ell)$ holds.

$(*)(\Gamma,E,\ell)$: the map $p(\Gamma,E)$ is transverse to $\prod_{e\in E}h^{-1}_{j_E(e)}(P(j_E(e)))$ along $p(\Gamma,E)^{-1}\left(\prod_{e\in E}h^{-1}_{j_E(e)}(K_{j_E(e),\ell(e)})\right)$ (as in Definition 11.2).

In order to prove our lemma, it suffices to prove that, for any of the finitely many (Γ,E,ℓ) as above, the set $\mathcal{H}(\Gamma,E,\ell)$ in which the condition $(*)(\Gamma,E,\ell)$ is realized, is a dense open subset of $\prod_{i=1}^{3n}\mathcal{H}_i$. This condition is equivalent to

$(*)(\Gamma,E,\ell)$: the map $\left(\prod_{e\in E}h_{j_E(e)}\right)\circ p(\Gamma,E)$ is transverse to $\prod_{e\in E}P(j_E(e))$ along $p(\Gamma,E)^{-1}\left(\prod_{e\in E}h^{-1}_{j_E(e)}(K_{j_E(e),\ell(e)})\right)$.

Set

$$C_{E,\ell}=C(R,L;\Gamma)\cap p(\Gamma,E)^{-1}\left(\prod_{e\in E}\psi^{-1}_{j_E(e),\ell(e)}(\phi_{j_E(e),\ell(e)}(K_{j_E(e),\ell(e)})\times D^2)\right).$$

The condition $(*)(\Gamma,E,\ell)$ can equivalently be written as:
$(0)_{e\in E}$ is a regular point of the map

$$\prod_{e\in E}(p_{D^2}\circ\psi_{j_E(e),\ell(e)}\circ h_{j_E(e)}\circ p_e)$$

on $C_{E,\ell}$.

Note that the set of regular values of this map on the compact domain $C_{E,\ell}$ is open. Therefore, if $(h_i)_{i\in\underline{3n}}\in\mathcal{H}(\Gamma,E,\ell)$ and if the $d_0(h_i,h_i')$ are small enough, the preimage of $(0)_{e\in E}$ under the restriction of $\prod_{e\in E} p_{D^2}\circ\psi_{j_E(e),\ell(e)}\circ h'_{j_E(e)}\circ p(\Gamma,E)$ to $C_{E,\ell}$ consists of regular points of $\prod_{e\in E} p_{D^2}\circ\psi_{j_E(e),\ell(e)}\circ h_{j_E(e)}\circ p(\Gamma,E)$, which are regular for $\prod_{e\in E} p_{D^2}\circ\psi_{j_E(e),\ell(e)}\circ h'_{j_E(e)}\circ p(\Gamma,E)$ provided that the $d(h_i,h'_i)$ are small enough. Therefore $\mathcal{H}(\Gamma,E,\ell)$ is open.

In order to prove density, we use explicit deformations of the $h_i\in\mathcal{H}_i$, for a given $(h_i)_{i\in\underline{3n}}\in\prod_{i=1}^{3n}\mathcal{H}_i$. Fix a smooth map $\chi\colon D^2\to[0,1]$, which maps the disk of radius $\frac{1}{2}$ to 1 and the complement of the disk of radius $\frac{3}{4}$ to 0. For each compact $K_{i,j}$, such that $j\in J(i)$, fix a smooth map $\chi_{i,j}\colon A_{i,j}\to[0,1]$, which maps $\phi_{i,j}(K_{i,j})$ to 1, and which vanishes outside a compact of $A_{i,j}$. For $w\in D^2$ define

$$\begin{array}{llll} h_{i,j,w}\colon & A_{i,j}\times D^2 & \to & A_{i,j}\times D^2\\ & (x,v) & \mapsto & (x,v+\chi(v)\chi_{i,j}(x)w). \end{array}$$

Note that $h_{i,j,w}$ is a diffeomorphism as soon as $\parallel w\parallel$ is smaller than a fixed positive number $\eta<\frac{1}{2}$. Extend $\psi_{i,j}^{-1}\circ h_{i,j,w}\circ\psi_{i,j}$ by the identity map outside $N(P(i))_{|\mathcal{O}_{i,j}}$. Note that there exists a constant C such that $d(\psi_{i,j}^{-1}\circ h_{i,j,w}\circ\psi_{i,j}\circ h_i,h_i)\leq C\parallel w\parallel$.

Thus, it suffices to prove that for any ε such that $0<\varepsilon<\eta$, there exists $(w_e)_{e\in E}$ with $\parallel w_e\parallel<\varepsilon$ such that the restriction of $\prod_{e\in E}(\psi^{-1}_{j_E(e),\ell(e)}\circ h_{j_E(e),\ell(e),w_e}\circ\psi_{j_E(e),\ell(e)}\circ h_{j_E(e)})\circ p(\Gamma,E)$ to $(C_{E,\ell})$ is transverse to $\prod_{e\in E}P(j_E(e))$ along $C_{E,\ell}$. Since this happens when $(-w_e)_{e\in E}$ is a regular value of the restriction of $\prod_{e\in E}\left(p_{D^2}\circ\psi_{j_E(e),\ell(e)}\circ h_{j_E(e)}\right)\circ p(\Gamma,E)$ to $C_{E,\ell}$, and since such regular values form a dense set according to the Morse-Sard theorem, the lemma is proved. $\square$

The magic in the Thom proof above is that it proves the density of manifolds in general $3n$ position without bothering to construct a single one.

Lemma 11.11 does not quite prove Lemma 11.4 for two reasons. First, the h_i do not fix the boundary of $\partial C_2(R)$ pointwise, so the perturbations $h_i^{-1}(P(i))$ are no longer propagating chains of $(C_2(R),\tau)$. Second, we have to deal with immersed manifolds (multiplied by an element of $\mathbb{Q}$) rather than embedded ones, when R is not an integer homology sphere.

To deal with this latter issue, we start with immersions f_i of manifolds $\tilde{P}(i)$ to $C_2(R)$ whose images $f_i(\tilde{P}(i))$ represent chains kP as in the end of Section 11.2, and (extended) immersions f_i of the pull-backs $N(\tilde{P}(i))$ of the normal bundles to their images. Our immersions f_i have the properties that the restriction to the preimage of $C'_2(R)$ of each immersion f_i is an embedding, the preimage of $N(\partial C_2(R))$ has k connected components $C_{j,i}$, $j\in\underline{k}$ in $N(\tilde{P}(i))$, and f_i embeds each of these k connected components into $N(\partial C_2(R))$. We will think of the intersection with a preimage of $f_i(\tilde{P}(i))\cap N(\partial C_2(R))$ as the sum of the intersections with the preimages of the $f_i(C_{i,j})$, and argue with covers of $\tilde{P}(i)$ rather than covers of its image. So this latter issue is not a big one (when we don't ask the f_i to be fixed and to coincide with each other on the various $C_{i,j}$ on $N(\partial C_2(R) \setminus [0,8[\times \partial C_2'(R))$. We keep this in mind, and we do not discuss this issue any longer.

The first issue is more serious, we want the boundaries of our propagating chains to be equal to $p_{\tau|\partial C_2(R)}^{-1}(X_i)$ for some $X_i \in S^2$. Recall that p_τ also denotes a regular extension of p_τ on $N(\partial C_2(R))$ and that a collar $[0,8] \times \partial C_2'(R)$ of $\partial C_2'(R)$ in $N(\partial C_2(R))$, where p_τ factors through the natural projection onto $\partial C_2'(R)$, has been fixed. For an interval I included in $[0,8]$, set

$$N_I = I \times \partial C_2'(R).$$

For $a \in [1,8]$, set

$$N_{[a,9]} = N(\partial C_2(R)) \setminus [0,a[\times \partial C_2'(R).$$

We will actually impose that our propagating chains intersect $N_{[7,9]}$ as $p_{\tau|N_{[7,9]}}^{-1}(X_i)$, by modifying our immersions f_i provided by the construction of the end of Section 11.2, only on $N_{[4,7[}$.

We first describe appropriate choices for the X_i, to allow transversality near the boundaries.

Let Γ of $\mathcal{D}_{\underline{3n}}^e(\mathcal{L})$, let E be a subset of $E(\Gamma)$. A condition on $(X_i)_{i\in\underline{3n}}$ is that $(X_i)_{i\in j_E(E)}$ is a regular (for the restriction to any stratum of $C(R,L;\Gamma)$) point of the map

$$\prod_{e\in E} p_\tau \circ p_e$$

from

$$C(\Gamma,E) = C(R,L;\Gamma) \cap \bigcap_{e\in E} p_e^{-1}(N(\partial C_2(R)))$$

to $(S^2)^{j_E(E)}$. According to the Morse-Sard theorem [Hir94, Chapter 3, Theorem 1.3], this condition holds when $(X_i)_{i\in\underline{3n}}$ is in a dense subset of $(S^2)^{3n}$, which is furthermore open since $C(\Gamma,E)$ is compact. Thus this condition holds for any of the finitely many pairs (Γ,E) as above, if $(X_i)_{i\in\underline{3n}}$ belongs to the intersection of the corresponding open dense subsets of $(S^2)^{3n}$, which is still open and dense.

Fix $(X_i)_{i\in\underline{3n}}$ in this open dense subset of $(S^2)^{3n}$. Now, we refer to the proof of Lemma 11.11 and adapt it to produce the wanted family of propagating chains of Lemma 11.4. We fix a finite cover $(K_{i,j})_{j\in J(i)}$ of $\tilde{P}(i) \cap f_i^{-1}\left(C_2'(R) \cup N_{[0,7]}\right)$, which contains a special element $K_{i,0} = \tilde{P}(i) \cap f_i^{-1}\left(N_{[5,7]}\right)$, such that for any $j \in J'(i) = J(i) \setminus \{0\}$, $K_{i,j}$ is a compact subset of $\tilde{P}(i) \cap f_i^{-1}\left(C_2'(R) \cup N_{[0,5]}\right)$.

When $j \in J'(i)$, $K_{i,j}$ is embedded in an open subspace $\mathcal{O}_{i,j}$ of $\tilde{P}(i) \cap f_i^{-1}\left(C_2'(R) \cup N_{[0,6]}\right)$. These $\mathcal{O}_{i,j}$ are diffeomorphic to $\mathbb{R}^4$ via diffeomorphisms $\phi_{i,j} \colon \mathcal{O}_{i,j} \to \mathbb{R}^4$, and we have bundle charts $\left(\psi_{i,j} \colon N(\tilde{P}(i))_{|\mathcal{O}_{i,j}} \to \mathbb{R}^4 \times D^2\right)$, for $j \in J'(i)$, as in the proof of Lemma 11.11.

The bundle $N(K_{i,0})$ is trivialized by p_τ in the following way. Fix a small neighborhood D_i of X_i diffeomorphic to the standard disk D^2 and a diffeomorphism $\psi_{D,i}$ from D_i to D^2. Without loss of generality, assume that $N(K_{i,0}) = f_i^{-1}(p_\tau^{-1}(D_i) \cap N_{[5,7]})$, and identify $N(K_{i,0})$ with $K_{i,0} \times D^2$ so that the projection onto D^2 may be expressed as $p_{D^2} = \psi_{D,i} \circ p_\tau \circ f_i$.

The space $\mathcal{H}_i$ is now the space of smooth diffeomorphisms of $N(\tilde{P}(i))$ that are isotopic to the identity map, that fix a neighborhood of $\partial N(\tilde{P}(i))$ and a neighborhood of $f_i^{-1}\left(f_i(N(\tilde{P}(i))) \cap (N_{[7,9]})\right)$ pointwise and that map any fiber of $N(\tilde{P}(i))$ to itself. The space $\mathcal{H}_i$ is equipped with a distance similar to that described before Lemma 11.11.

We want to prove that the subspace of $\prod_{i \in \underline{3n}} \mathcal{H}_i$ consisting of the $(h_i)_{i \in \underline{3n}}$ such that the $f_i\left(h_i^{-1}(\tilde{P}(i))\right)$ are in general $3n$ position with respect to L, in the sense of Definition 11.3, is open and dense.

It is open as in the proof of Lemma 11.11. [2]

Moreover, for any Γ of $\mathcal{D}^e_{\underline{3n}}(\mathcal{L})$, for any triple (E_X, E_N, E_C) of pairwise disjoint subsets of $E(\Gamma)$, the subset $\mathcal{H}(\Gamma, E_X, E_N, E_C)$ of $\prod_{i \in \underline{3n}} \mathcal{H}_i$ such that the restriction of $p(\Gamma, E_N \cup E_C)$ to $C(R,L;\Gamma) \cap p(\Gamma, E_X)^{-1} \prod_{e \in E_X}(N_{[5,9]} \cap p_\tau^{-1}(X_{j_E(e)}))$ is transverse[3] to

$$\prod_{e \in E_N \cup E_C} \left(f_{j_E(e)}\left(h^{-1}_{j_E(e)}(\tilde{P}(j_E(e)))\right)\right)$$

along

$$\begin{array}{rl} & p(\Gamma, E_X)^{-1} \prod_{e \in E_X}(N_{[5,9]} \cap p_\tau^{-1}(X_{j_E(e)})) \\ \cap & p(\Gamma, E_N)^{-1} \prod_{e \in E_N}(N_{[5,9]}) \\ \cap & p(\Gamma, E_C)^{-1} \prod_{e \in E_C}\left(C_2'(R) \cup N_{[0,5]}\right) \end{array}$$

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[2] If we ask only for transversality of the

$$p(\Gamma, E) = \prod_{e \in E} p_e \colon C(R,L;\Gamma) \to (C_2(R))^{j_E(E)}$$

to $\prod_{e \in E} f_{j_E(e)}\left(h^{-1}_{j_E(e)}(\tilde{P}(j_E(e)))\right)$ along $\prod_{e \in E} p_e^{-1}(C_2'(R) \cup N_{[0,5]})$, density could also be proved as in Lemma 11.11.

[3] Our hypotheses on $(X_i)_{i \in \underline{3n}}$ guarantee that $C(R,L;\Gamma) \cap p(\Gamma, E_X)^{-1} \prod_{e \in E_X}(N_{[5,9]} \cap p_\tau^{-1}(X_{j_E(e)}))$ is a manifold.

is open. The $\mathcal{H}(\Gamma, E_X, \emptyset, E_C)$ are furthermore dense as in the proof of Lemma 11.11.

In order to prove Lemma 11.4, it suffices to prove that for any Γ of $\mathcal{D}^e_{\underline{3n}}(\mathcal{L})$, for any pair (E_N, E_C) of disjoint subsets of $E(\Gamma)$, the subset $\mathcal{H}(\Gamma, \emptyset, E_N, E_C)$ of $\prod_{i\in\underline{3n}}\mathcal{H}_i$ is dense. To do that, we fix $(h_i)_{i\in\underline{3n}} \in \prod_{i\in\underline{3n}}\mathcal{H}_i$ and $\varepsilon \in]0,1[$ and we prove that there exists $(h'_i)_{i\in\underline{3n}} \in \mathcal{H}(\Gamma, \emptyset, E_N, E_C)$ such that $\max_{i\in\underline{3n}}(d(h_i, h'_i)) < \varepsilon$. There exists $\eta \in]0,1[$ such that the restriction of h_i to

$$f_i^{-1}\left(f_i(N(\tilde{P}(i))) \cap N_{[7-2\eta,9]}\right)$$

is the identity map for any $i \in j_E(E_N)$.

For $i \in J_E(E_N)$, our h'_i will be constructed as some $h_{i,\eta,w} \circ h_i$. Let χ_η be a smooth map from $[4,9]$ to $[0,1]$ that maps $[5, 7-2\eta]$ to 1 and that maps the complement of $[5-\eta, 7-\eta]$ to 0. Recall our smooth map $\chi\colon D^2 \to [0,1]$, which maps the disk of radius $\frac{1}{2}$ to 1 and the complement of the disk of radius $\frac{3}{4}$ to 0. For $w \in D^2$ define

$$\begin{array}{llll} h_{\eta,w}\colon & [4,9] \times D^2 & \to & D^2 \\ & (t,v) & \mapsto & v + \chi(v)\chi_\eta(t)w. \end{array}$$

Define $h_{i,\eta,w} \in \mathcal{H}_i$, for w sufficiently small, to coincide with the identity map outside

$$f_i^{-1}\left(f_i(N(\tilde{P}(i))) \cap N_{[5-\eta,7-\eta]}\right),$$

and with the map[4] that sends $(p,v) \in \left(\tilde{P}(i) \cap f_i^{-1}(N_{\{x\}})\right) \times D^2$

$$\left(\subset \left(\tilde{P}(i) \cap f_i^{-1}(N_{[5-\eta,7-\eta]})\right) \times D^2 = N(\tilde{P}(i)) \cap f_i^{-1}(N_{[5-\eta,7-\eta]})\right)$$

to $(p, h_{\eta,w}(x,v))$, for $x \in [5-\eta, 7-\eta]$. There exists $u \in]0,1[$ such that as soon as $\parallel w \parallel < u$, $h_{i,\eta,w}$ is indeed a diffeomorphism and $d(h_{i,\eta,w} \circ h_i, h_i) < \varepsilon$. Fix $(h'_i)_{i\in J_E(E_C)}$ such that $d(h_i, h'_i) < \varepsilon$ and $(h'_i)_{i\in J_E(E_C)} \times (h_i)_{i\notin J_E(E_C)}$ is in the dense open set $\cap_{E_x\subseteq E_N}\mathcal{H}(\Gamma, E_x, \emptyset, E_C)$, (this does not impose anything on $(h_i)_{i\notin J_E(E_C)}$).

After reducing u, we may now assume that as soon as $\parallel w \parallel < u$, for any $E_x \subseteq E_N$, $p(\Gamma, E_C \cup E_x)$ is transverse to

$$\begin{array}{ll} & \prod_{e\in E_C} f_{j_E(e)}\left(\left(h'_{j_E(e)}\right)^{-1}(\tilde{P}(j_E(e)))\right) \\ \times & \prod_{e\in E_x} f_{j_E(e)}\left(\left(h_{j_E(e),\eta,w_{j_E(e)}} \circ h_{j_E(e)}\right)^{-1}(\tilde{P}(j_E(e)))\right) \end{array}$$

[4]For any $x \in [0,9]$, we assume that $f_i\left(\left(\tilde{P}(i) \cap f_i^{-1}(N_{\{x\}})\right) \times D^2\right) \subset N_{\{x\}}$, without loss of generality.

along

$$p(\Gamma, E_C)^{-1}\left(\prod_{e\in E_C}\left(C_2'(R)\cup N_{[0,5]}\right)\right)\cap p(\Gamma, E_x)^{-1}\left(\prod_{e\in E_x}(N_{[7-2\eta,9]})\right)$$

since $h^{-1}_{j_E(e)}\left(\tilde{P}(j_E(e))\right) = p_\tau^{-1}(X_{j_E(e)}))$ on $N_{[7-2\eta,9]}$. Furthermore, $p(\Gamma, E_C \cup E_N)$ is transverse to

$$\begin{aligned} & M\left((h_i')_{i\in J_E(E_C)}, (h_{i,\eta,w_i})_{i\in J_E(E_N)}\right) \\ = \ & \prod_{e\in E_C}\left(f_{j_E(e)}\left(\left(h'_{j_E(e)}\right)^{-1}(\tilde{P}(j_E(e)))\right)\right) \\ \times \ & \prod_{e\in E_N} f_{j_E(e)}\left(\left(h_{j_E(e),\eta,w_{j_E(e)}}\circ h_{j_E(e)}\right)^{-1}(\tilde{P}(j_E(e)))\right)\end{aligned}$$

along

$$p(\Gamma, E_C)^{-1}\left(\prod_{e\in E_C}\left(C_2'(R)\cup N_{[0,5]}\right)\right)\cap p(\Gamma, E_N)^{-1}\left(\prod_{e\in E_N} N_{[5,9]}\right),$$

if and only if, for any subset E_x of E_N, the following condition $(*)(E_x)$ holds.
$(*)(E_x)$: $p(\Gamma, E_C \cup E_N)$ is transverse to $M\left((h_i')_{i\in J_E(E_C)}, (h_{i,\eta,w_i})_{i\in J_E(E_N)}\right)$ along

$$\begin{aligned} & p(\Gamma, E_C)^{-1}\left(\prod_{e\in E_C}\left(C_2'(R)\cup N_{[0,5]}\right)\right) \\ \cap \ & p(\Gamma, E_x)^{-1}\left(\prod_{e\in E_x} N_{[7-2\eta,9]}\right) \\ \cap \ & p(\Gamma, E_N\setminus E_x)^{-1}\left(\prod_{e\in E_N\setminus E_x} N_{[5,7-2\eta]}\right).\end{aligned}$$

Let D_u denote the open disk of $\mathbb{R}^2$ centered at 0 of radius u. Since our former hypotheses guarantee transversality of $p(\Gamma, E_C \cup E_x)$, as soon as the $\parallel w_i \parallel$ are smaller than u, for $i \in E_x$, the condition $(*)(E_x)$ is realized as soon as $(w_i)_{i\in j_E(E_N\setminus E_x)}$ is in an open dense subset of $D_u^{j_E(E_N\setminus E_x)}$. Thus, it is realized as soon as $(w_i)_{i\in j_E(E_N)}$ is in an open dense subset $\mathcal{D}(E_x)$ of $D_u^{j_E(E_N)}$, and we have the wanted transversality when $(w_i)_{i\in j_E(E_N)}$ is in the intersection of the open dense subsets $\mathcal{D}(E_x)$ over the subsets E_x of E_N.

$\square$

11.4 More on forms dual to transverse propagating chains

Though Lemma 11.7 is not surprising, we prove it and we refine it in this section. Its refinement is used in Chapter 17. Recall the notation before

Definition 11.6 and let D_ε (resp. $\mathring{D}_\varepsilon$) denote the closed (resp. open) disk of $\mathbb{C}$ centered at 0 with radius ε.

Lemma 11.12. *Recall that our configuration spaces are equipped with Riemannian metrics. Let $(\check{R},\tau)$ be an asymptotic rational homology $\mathbb{R}^3$. Let $L\colon \mathcal{L} \to \check{R}$ be a link in $\check{R}$. Let $n \in \mathbb{N}$, and let $(P(i))_{i\in\underline{3n}}$ be a family of propagating chains of $(C_2(R),\tau)$ in general $3n$ position with respect to L. For any $\varepsilon > 0$, there exists $\eta > 0$ such that for any $i \in \underline{3n}$, for any $\Gamma \in \mathcal{D}^e_{\underline{3n}}(\mathcal{L})$, for any $e \in E(\Gamma)$ with associated restriction map $p_e\colon C(R,L;\Gamma) \to C_2(R)$,*

$$p_e^{-1}(N_\eta(P(j_E(e)))) \subset N_\varepsilon(p_e^{-1}(P(j_E(e)))).$$

Proof: Of course, it is enough to prove the lemma for a fixed (Γ, e). Set $i = j_E(e)$. The compact $p_e\left(C(R,L;\Gamma) \setminus N_\varepsilon(p_e^{-1}(P(i)))\right)$ does not meet $P(i)$. So there exists $\eta > 0$ such that this compact does not meet $N_\eta(P(i))$ either. Therefore, $p_e^{-1}(N_\eta(P(i))) \subset N_\varepsilon(p_e^{-1}(P(i)))$. $\square$

Lemma 11.13. *Under the hypotheses of Lemma 11.12, let $\Gamma \in \mathcal{D}^e_{\underline{3n}}(\mathcal{L})$. Then the intersection in $C(R,L;\Gamma)$ over the edges e of $E(\Gamma)$ of the codimension 2 rational chains $p_e^{-1}(P(j_E(e)))$ is a finite set $I_S(\Gamma,(P(i))_{i\in\underline{3n}})$.*

For any $\varepsilon > 0$, there exists $\eta > 0$, such that, for any $\Gamma \in \mathcal{D}^e_{\underline{3n}}(\mathcal{L})$,

$$\bigcap_{e\in E(\Gamma)} p_e^{-1}(N_\eta(P(j_E(e)))) \subset N_\varepsilon(I_S(\Gamma,(P(i))_{i\in\underline{3n}})).$$

So, for any family $(\omega(i))_{i\in\underline{3n}}$ of propagating forms of $(C_2(R),\tau)$, η-dual to the $P(i)$, $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$ is supported in $N_\varepsilon(I_S(\Gamma,(P(i))_{i\in\underline{3n}}))$.

Furthermore, if $N_\varepsilon(I_S(\Gamma,(P(i))))$ is a disjoint union over the points x of $I_S(\Gamma,(P(i)))$ of the $N_\varepsilon(x)$, then the integral over $N_\varepsilon(x)$ of $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$ is the rational intersection number of the rational chains $p_e^{-1}(P(j_E(e)))$ at x, which is nothing but the sign of x with respect to an orientation $o(\Gamma)$ of Γ, when all the $P(j_E(e))$ are embedded manifolds with coefficient 1 near $p_e(x)$.

Proof: Again, it suffices to prove the lemma for a fixed $\Gamma \in \mathcal{D}^e_{\underline{3n}}(\mathcal{L})$. We refer to Definition 11.3, which describes the image under $p(\Gamma) = \prod_{e\in E(\Gamma)} p_e$ of an intersection point x.

Fix such an x. For each edge e, $p_e(x)$ sits inside a non-singular open 4-dimensional smooth ball δ_e of a smooth piece $\Delta_{j_E(e),k}$ of $P(j_E(e))$. Consider a tubular neighborhood $N_u(\delta_e)$ whose fibers are disks D_θ orthogonal to δ_e of radius θ. The bundle $N_u(\delta_e)$ is isomorphic to $\delta_e \times D_\theta$, with respect to a trivialization of $N_u(\delta_e)$. Another trivialization would compose the diffeomorphism from $N_u(\delta_e)$ to $\delta_e \times D_\theta$ by a map $(x,v) \mapsto (v, \phi(x)(v))$ for some $\phi\colon \delta_e \to SO(2)$.

The projection $p_e(x)$ might sit simultaneously in different non-singular 4-dimensional smooth parts $\Delta_{j_E(e),k}$ of $P(j_E(e))$. Let $K(e,x)$ be the finite set of components $\Delta_{j_E(e),k}$ of $P(j_E(e))$ such that $p_e(x) \in \Delta_{j_E(e),k}$. We focus on one element of $K(e,x)$ for each e, and next take the sum over all the choices in $\prod_{e\in E(\Gamma)} K(e,x)$ multiplied by the products of the coefficients of the elements of $K(e,x)$ in the rational chains $P(j_E(e))$. Similarly, our forms η-dual to the $P(i)$ are thought of and constructed as linear combinations of forms η-dual to the elements of $K(e,x)$.

Without loss of generality, assume that ε is small enough so that

$$p_e(N_\varepsilon(x)) \subset \delta_e \times D_\theta$$

for any edge e of Γ, and for any $\delta_e = \delta_{e,k}$ associated to an element $\Delta_{j_E(e),k}$ of $K(e,x)$, and so that $p_e(N_\varepsilon(x))$ does not meet the components $\Delta_{j_E(e),k}$ of $P(j_E(e))$ that are not in $K(e,x)$. Reduce ε and choose $\eta < \theta$ small enough so that $p_e(N_\varepsilon(x))$ does not meet the neighborhoods $N_\eta(\Delta_{j_E(e),k})$ of these components, either.

Let $p_{D_\theta}\colon \delta_e \times D_\theta \to D_\theta$ denote the natural projection. Let ω_η be a volume-one form supported on $\mathring{D}_\eta$. Forms η-dual to $P(j_E(e))$ can be constructed by patching forms $(p_{D_\theta})^*(\omega_\eta)$ (multiplied by the coefficients of the $\Delta_{j_E(e),k}$) together, as in Lemma B.4. Conversely, for any form $\omega(j_E(e))$ η-dual to a piece $\Delta_{j_E(e),k}$ of $P(j_E(e))$ that contains δ_e, there exists a one-form α_e on $\delta_e \times D_\theta$, such that $\omega(j_E(e)) = p^*_{D_\theta}(\omega_\eta) + d\alpha_e$ on $\delta_e \times D_\eta$. Then $\int_{\{x\in\delta_e\}\times\partial D_\eta} \alpha_e = \int_{\{x\in\delta_e\}\times D_\eta} \omega(j_E(e)) - p^*_{D_\theta}(\omega_\eta) = 0$. So α_e is exact on $\delta_e \times (D_\theta \setminus D_\eta)$, and α_e can and will be assumed to be supported on $\delta_e \times (D_\theta \setminus D_\eta)$.

In the neighborhood $N_\varepsilon(x)$ of x, $\prod_{e\in E(\Gamma)} p_{D_\theta} \circ p_e$ is a local diffeomorphism around x. Without loss of generality, assume that η and ε are small enough so that

$$\Pi_p = \prod_{e\in E(\Gamma)} p_{D_\theta} \circ p_e \colon N_\varepsilon(x) \to D_\theta^{E(\Gamma)}$$

restricts to a diffeomorphism from $\Pi_p^{-1}\left(D_{2\eta}^{E(\Gamma)}\right)$ to $D_{2\eta}^{E(\Gamma)}$, for each x (and for each choice in $\prod_{e\in E(\Gamma)} K(e,x)$). If $\prod_{e\in E(\Gamma)} K(e,x)$ has one element, and if the coefficient of the element of $K(e,x)$ in $P(j_E(e))$ is 1 for any edge e, then

$$\int_{N_\varepsilon(x)} \bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e))) = \int_{N_\varepsilon(x)} \bigwedge_{e\in E(\Gamma)} p_e^*(p^*_{D_\theta}(\omega_\eta)).$$

Indeed, changing one $\omega(j_E(e))$ to $(p_{D_\theta})^*(\omega_\eta)$ amounts to add the integral obtained by replacing $\omega(j_E(e))$ by $d\alpha_e$. Since all the forms are closed, this latter integral is the integral over $\Pi_p^{-1}(\partial(D_{2\eta}^{E(\Gamma)}))$ of the form obtained by replacing $d\alpha_e$ by α_e, which is zero since the whole form is supported in $\Pi_p^{-1}(D_\eta^{E(\Gamma)})$. Therefore the integral is the sign of the intersection point x with respect to the given orientation and coorientations.

The open neighborhoods $N_\varepsilon(x)$ may be assumed to be disjoint from each other for distinct x. As a consequence, since $C(R,L;\Gamma)$ is compact, the set of intersection points x is finite. Consider the complement $C^c(\eta_0)$ in $C(R,L;\Gamma)$ of the union over the intersection points x of the $N_\varepsilon(x)$. Since $p_{e_1}^{-1}(P(j_E(e_1)))$ does not meet $\bigcap_{e\in E(\Gamma)\setminus\{e_1\}} p_e^{-1}(P(j_E(e)))$ in $C^c(\eta_0)$, there is an $\varepsilon_1 > 0$ such that $\overline{N_{\varepsilon_1}}(p_{e_1}^{-1}(P(j_E(e_1))))$ does not meet $\bigcap_{e\in E(\Gamma)\setminus\{e_1\}} p_e^{-1}(P(j_E(e)))$ either in $C^c(\eta_0)$. Iterating, we find $\varepsilon_2 > 0$ such that

$$C^c(\eta_0) \cap \bigcap_{e\in E(\Gamma)} N_{\varepsilon_2}(p_e^{-1}(P(j_E(e)))) = \emptyset.$$

According to Lemma 11.12, η can be reduced so that $p_e^{-1}(N_\eta(P(i))) \subset N_{\varepsilon_2}(p_e^{-1}(P(i)))$ for any i. Then $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$ is supported where we want it to be. $\square$

Lemma 11.7 follows. $\square$

Theorem 11.8 is now proved. $\square$

11.5 A discrete definition of the anomaly β

In this section, we give a discrete definition of the anomaly β and we mention a few recent results of Kévin Corbineau on β_3.

Lemma 11.14. *Let $n \in \mathbb{N}$. Let $\Gamma \in \mathcal{D}_n^c$. Recall the compactification $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$ of $\check{\mathcal{S}}_{V(\Gamma)}(\mathbb{R}^3)$ from Theorem 8.11. For any edge $e = j_E^{-1}(i)$ of Γ, we have a canonical projection*

$$p_e \colon B^3 \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \to B^3 \times S^2.$$

Let $i \in \underline{3n}$. When Γ is fixed, set $p_i = p_{j_E^{-1}(i)}$. For any $a_i \in S^2$, define the following cooriented chains of $B^3 \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$:

$$A(\Gamma, i, a_i) = p_i^{-1}(B^3 \times \{a_i\}),$$

$$B(\Gamma, i, a_i) = p_i^{-1}\left(\{\cup_{m\in B^3}(m, \rho(m)(a_i))\}\right)$$

and

$$H(\Gamma, i, a_i) = p_i^{-1}(G(a_i)),$$

*where ρ is introduced in Definition 4.5 and the chain $G(a_i)$ of $B^3 \times S^2$ is introduced in Lemma 4.13. The codimension of $A(\Gamma, i, a_i)$ and $B(\Gamma, i, a_i)$ is 2, while the codimension of $H(\Gamma, i, a_i)$ is 1. An element $(a_1, \dots, a_{3n})$ of $(S^2)^{3n}$ is β_n-*admissible *if for $h \in \underline{3n}$ and for any $\Gamma \in \mathcal{D}_n^c$, the intersection of the $A(\Gamma, i, a_i)$ for $i \in \underline{h-1}$, the $B(\Gamma, i, a_i)$ for $i \in \underline{3n} \setminus \underline{h}$ and $H(\Gamma, h, a_h)$ is transverse. Then the sets of elements of $(S^2)^{3n}$ that are β_n-admissible is an open dense subset of $(S^2)^{3n}$.*

PROOF: The principle of the proof is the same as the proof of Proposition 11.1. See also Section 11.3. This lemma is proved in details in [Cor16]. $\square$

Proposition 11.15. *For any β_n-admissible element $(a_1, \dots, a_{3n})$ of $(S^2)^{3n}$,*

$$\beta_n = \sum_{h=1}^{3n} \sum_{\Gamma \in \mathcal{D}_n^c} \frac{1}{(3n)!2^{3n}} I(\Gamma, h)[\Gamma],$$

where

$$I(\Gamma, h)[\Gamma] = \langle \cap_{i=1}^{h-1} A(\Gamma, i, a_i), H(\Gamma, h, a_h), \cap_{i=h+1}^{3n} B(\Gamma, i, a_i) \rangle_{B^3 \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)} [\Gamma]$$

with the orientation of $\check{\mathcal{S}}_{V(\Gamma)}(\mathbb{R}^3)$ of Lemma 9.3.

PROOF: For $a \in S^2$ and $t \in [0,1]$, define the following chain $G(a,t)$ of $[0,1] \times B^3 \times S^2$.

$$G(a,t) = [0,t] \times B^3 \times \{a\} + (\{t\} \times G(a)) + \{(u, m, \rho(m)(a)) \mid u \in [t,1], m \in B^3\}$$

of $[0,1] \times B^3 \times S^2$. Let $(t_i)_{i \in \underline{3n}}$ be a strictly decreasing sequence of $]0,1[$. Let $\Gamma \in \mathcal{D}_n^c$. For $i \in \underline{3n}$, p_i also denotes the canonical projection associated to $e = j_E^{-1}(i)$ from $[0,1] \times B^3 \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$ to $[0,1] \times B^3 \times S^2$.

If $(a_1, \dots, a_{3n})$ is β_n-admissible, then for any $\Gamma \in \mathcal{D}_n^c$, the intersection of the $p_i^{-1}(G(a_i, t_i))$ is transverse, and is equal to

$$\sqcup_{h=1}^{3n} \{t_h\} \times \cap_{i=1}^{h-1} A(\Gamma, i, a_i) \cap H(\Gamma, h, a_h) \cap \cap_{i=h+1}^{3n} B(\Gamma, i, a_i).$$

Indeed, it is clear that the intersection may be expressed as above at the times $t \in \{t_h \mid h \in \underline{3n}\}$. Since $(a_1, \dots, a_{3n})$ is β_n-admissible, this intersection is transverse at these times. So it does not intersect the boundaries of the $H(\Gamma, h, a_h)$. Therefore, there is no intersection on $([0,1] \setminus \{t_h \mid h \in \underline{3n}\}) \times B^3 \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$.

Then for any $\alpha > 0$, there exist closed 2-forms $\tilde{\omega}(i)$ on $[0,1] \times \mathbb{R}^3 \times S^2$, as in Proposition 10.7, applied when $\tau_0 = \tau_s$ and $\tau_1 = \tau_0 \circ \psi_{\mathbb{R}}(\rho)$ on UB^3, such that $\tilde{\omega}(i)$ is α-dual to $G(a_i, t_i)$, for any i.

According to Theorem 4.6, $p_1(\tau_0 \circ \psi_{\mathbb{R}}(\rho)) - p_1(\tau_0) = 2\mathrm{deg}(\rho) = 4$. Therefore, Proposition 10.7 implies

$$\beta_n = \sum_{\Gamma \in \mathcal{D}_n^c} \frac{1}{(3n)!2^{3n}} \int_{[0,1]\times \check{\mathcal{S}}_{V(\Gamma)}(TB_R)} \bigwedge_{e \in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e)))[\Gamma],$$

where $\bigwedge_{e\in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e))) = \bigwedge_{i=1}^{3n} p_i^*(\tilde{\omega}(i))$.

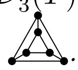
As in Section 11.4, for α small enough, $\int_{[0,1]\times\check{\mathcal{S}}_{V(\Gamma)}(TB^3)} \bigwedge_{i=1}^{3n} p_i^*(\tilde{\omega}(i))$ is the algebraic intersection of the $p_i^{-1}(G(a_i, t_i))$.

For the signs, note that the coorientation of $\{t_h\} \times G(a_h)$ in $[0,1] \times B^3 \times S^2$ is reprensented by the orientation of $[0,1]$, followed by the coorientation of $G(a_h)$ in $B^3 \times S^2$. $\square$

In his Ph. D. thesis [Cor16, Théorème 2.15], Kévin Corbineau obtained the following simplified expression for β_3.

Theorem 11.16. *For $j \in \underline{n}$, set*

$$H_h(\Gamma, j, a_j) = p_j^{-1}(G_h(a_j)),$$

where the chain $G_h(a_j)$ of $B^3 \times S^2$ is introduced in Lemma 4.13. Let $\mathcal{D}_3^c(T)$ be the set of numbered graphs in $\mathcal{D}_3^c$ that are isomorphic to the graph . *For any element β_3-admissible $(a_1, \dots, a_9)$ of $(S^2)^9$,*

$$\beta_3 = \sum_{j=2}^{8} \sum_{\Gamma \in \mathcal{D}_3^c(T)} \frac{1}{(9)!2^9} I_h(\Gamma, j)[\Gamma],$$

where

$$I_h(\Gamma, j)[\Gamma] = \langle \cap_{i=1}^{j-1} A(\Gamma, i, a_i), H_h(\Gamma, j, a_j), \cap_{i=j+1}^{9} B(\Gamma, i, a_i) \rangle_{B^3 \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)}[\Gamma]$$

with the orientation of $\check{\mathcal{S}}_{V(\Gamma)}(\mathbb{R}^3)$ of Lemma 9.3.

The Ph. D. thesis of Kévin Corbineau contains an algorithm to compute β_3, too.

Part III
Functoriality

Recall that D_1 denotes the closed disk of $\mathbb{C}$ centered at 0 with radius 1. In this book, a *rational homology cylinder* (or $\mathbb{Q}$*–cylinder*) is a compact oriented 3-manifold, whose boundary neighborhood is identified with a boundary neighborhood $N(\partial(D_1 \times [0,1]))$ of $D_1 \times [0,1]$, and which has the same rational homology as a point.

Roughly speaking, *q–tangles* are parallelized cobordisms, between limit planar configurations of points up to dilation and translation, in rational homology cylinders. The category of q–tangles and its structures are described precisely in Section 13.1. Framed links in rational homology spheres are particular q–tangles, which are cobordisms between empty configurations.

In this third part of the book, we define a functorial extension to q–tangles of the invariant $\mathcal{Z}^f$ of framed links in $\mathbb{Q}$–spheres defined in Section 7.6, and we prove that it has a lot of useful properties. These properties are listed in Theorem 13.12. They ensure that $\mathcal{Z}^f$ is a functor, which behaves naturally with respect to other structures of the category of q–tangles, such as cabling or duplication. They allow one to reduce the computation of $\mathcal{Z}^f$ for links to the computation for elementary pieces of the links.

In Chapter 12, we introduce particular q–tangles, for which the involved planar configurations are injective in Section 12.1, and,for which the involved planar configurations are corners of the Stasheff polyhedra of Example 8.3 in Section 12.3, and we define $\mathcal{Z}^f$ for these particular q–tangles, without proofs. We also state a functoriality result, a monoidality result and a duplication property, under simple hypotheses, in order to introduce the involved structures, and to motivate their introduction. These results are just particular cases of Theorem 13.12.

In Chapter 13, we state our general Theorem 13.12, and we describe our strategy towards a consistent definition of $\mathcal{Z}^f$ for general q–tangles in Section 13.2. Our proofs involve convergence results, which rely on intricate compactifications of configuration spaces described in Chapter 14. The consistency of our strategy is proved in Chapter 14, Chapter 15, where $\mathcal{Z}^f$ is studied as a holonomy for the q-tangles that are paths in spaces of planar configurations, and Chapter 16, where discretizable versions of $\mathcal{Z}^f$ are introduced. These discretizable versions are used in the proofs of some important properties of $\mathcal{Z}^f$ given in Chapter 17. The proof of Theorem 13.12 will be finished in Chapter 17.

This functoriality part contains a generalization of results of Sylvain Poirier [Poi00] who constructed the functor $\mathcal{Z}^f$ and proved Theorem 13.12 for combinatorial q–tangles of $\mathbb{R}^3$. His results are recalled in Section 12.4.

Chapter 12

A first introduction to the functor $\mathcal{Z}^f$

In Section 12.1, we extend the definition of the invariant $\mathcal{Z}$ of Theorem 7.20 to long tangle representatives as in Figure 1.10. Then we define the framed version $\mathcal{Z}^f$ of $\mathcal{Z}$ and state that it is multiplicative under the allowed vertical compositions, in Section 12.2.

In Section 12.3, we state that $\mathcal{Z}^f$ reaches a limit with nice cabling properties, when some vertical infinite strands of the long tangle representatives approach each other, and we define the restriction of $\mathcal{Z}^f$ to *combinatorial q–tangles*, which are parallelized cobordisms between limit configurations on the real line in rational homology cyclinders. This definition is due to Poirier [Poi00] when the involved rational homology cyclinder is the standard one $D_1 \times [0,1]$. In Section 12.4, we list sufficiently many properties of the Poirier restriction of $\mathcal{Z}^f$ to characterize the restriction of $\mathcal{Z}^f$ to combinatorial q–tangles, in terms of the anomaly α.

12.1 Extension of $\mathcal{Z}$ to long tangles

For a rational homology cylinder $\mathcal{C}$, $\check{R}(\mathcal{C})$ denotes the asymptotically standard $\mathbb{Q}$-homology $\mathbb{R}^3$ obtained by replacing the standard cylinder $\mathcal{C}_0 = D_1 \times [0,1]$ with $\mathcal{C}$, in $\mathbb{R}^3$ viewed as $\mathbb{C} \times \mathbb{R}$, where $\mathbb{C}$ is horizontal and $\mathbb{R}$ is vertical.

Definition 12.1. A *long tangle representative* (or LTR for short) in $\check{R}(\mathcal{C})$ is an embedding $L \colon \mathcal{L} \hookrightarrow \check{R}(\mathcal{C})$ of a one-manifold $\mathcal{L}$, as in Figure 1.10, such that

- $L(\mathcal{L})$ intersects the closure $\check{\mathcal{C}}_0^c$ of the complement of $\mathcal{C}_0$ in $\mathbb{R}^3$ as

$$L(\mathcal{L}) \cap \check{\mathcal{C}}_0^c = (c^-(B^-) \times]-\infty, 0]) \cup (c^+(B^+) \times [1, \infty[)$$

for two finite sets B^- and B^+ and two injective maps $c^-\colon B^- \hookrightarrow \mathrm{Int}(D_1)$ and $c^+\colon B^+ \hookrightarrow \mathrm{Int}(D_1)$, which are called the *bottom configuration* and the *top configuration* of L, respectively, and

- $L(\mathcal{L}) \cap \mathcal{C}$ is a compact one-manifold whose unoriented boundary is $(c^-(B^-) \times \{0\}) \cup (c^+(B^+) \times \{1\})$.

For a $\underline{3n}$-numbered degree n Jacobi diagram with support $\mathcal{L}$ without looped edges, let $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$ be its configuration space defined as[1] in Section 7.1. The univalent vertices on a *strand*, which is the image under L of an open connected component of $\mathcal{L}$ (diffeomorphic to $\mathbb{R}$), move along this whole long component, as in Figure 12.1.

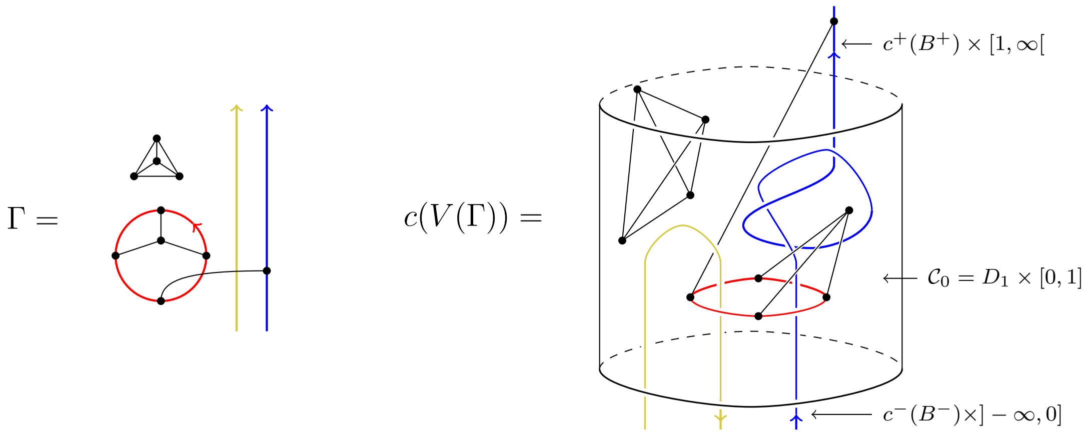


Figure 12.1: A (black) Jacobi diagram Γ on the source of an LTR L, and a configuration c of $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$

For any $i \in \underline{3n}$, let $\omega(i)$ be a propagating form of $(C_2(R(\mathcal{C})), \tau)$. Let $o(\Gamma)$ be a vertex-orientation of Γ. As in Section 7.2, define

$$I(\mathcal{C}, L, \Gamma, o(\Gamma), (\omega(i))_{i\in\underline{3n}}) = \int_{(\check{C}(\check{R}(\mathcal{C}),L;\Gamma),o(\Gamma))} \bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e))),$$

where $(\check{C}(\check{R}(\mathcal{C}), L; \Gamma), o(\Gamma))$ denotes the manifold $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$, equipped with the orientation induced by the vertex-orientation $o(\Gamma)$ and by the edge-orientation of Γ, as in Corollary 7.2, and

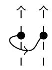
$$I(\mathcal{C}, L, \Gamma, (\omega(i))_{i\in\underline{3n}})[\Gamma] = I(\mathcal{C}, L, \Gamma, o(\Gamma), (\omega(i))_{i\in\underline{3n}})[\Gamma, o(\Gamma)].$$

Theorem 12.2. *The above integral is convergent.*

[1]The only difference is that $\mathcal{L}$ is not necessarily a disjoint union of circles.

This theorem is proved in Section 14.2. See Lemma 14.17. Again, its proof involves appropriate compactifications $C(\check{R}(\mathcal{C}), L; \Gamma)$ of the configuration spaces $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$. The compactifications are more complicated in this case. Their study is performed in Chapter 14.

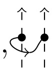
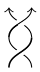
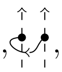
As an example, let us compute $I(\mathcal{C}, L, \Gamma, o(\Gamma), (\omega(i))_{i \in \underline{3n}})$ when

- $\mathcal{C} = \mathcal{C}_0 = D_1 \times [0,1]$,
- L is an LTR  whose bottom and top configurations coincide and map $B^- = B^+$ to $\{-\frac{1}{2}, \frac{1}{2}\}$, and $L(\mathcal{L}) \cap \mathcal{C}_0$ projects to $\mathbb{R}^2$ as ,
- $(\Gamma, o(\Gamma))$ is the vertex-oriented diagram  whose chord is oriented and numbered,
- the propagating forms $\omega(i)$ pull back through $p_{S^2} \colon C_2(S^3) \to S^2$,

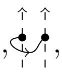
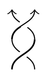
and prove the following lemma.

Lemma 12.3.

$$I\left(\mathcal{C}_0, \text{[braid]}, \text{[diagram]}, (p^*_{S^2}(\omega_{i,S}))_{i \in \underline{3}}\right) = I\left(\mathcal{C}_0, \text{[braid]}, \text{[diagram]}, (p^*_{S^2}(\omega_{i,S}))_{i \in \underline{3}}\right) = 1$$

for any arbitrary numbering of the edge of the involved Jacobi diagram and for any choice of volume-one forms $\omega_{i,S}$ of S^2.

Proof: Let us compute $I\left(\text{[braid]}, \text{[diagram]}, \left(p^*_{S^2}(\omega_{i,S})\right)_{i \in \underline{3}}\right)$  . The configuration space $(\check{C}(\mathbb{R}^3,$  $;\Gamma), o(\Gamma))$ is naturally diffeomorphic to $]-\infty, \infty[\times]-\infty, \infty[$, where the first factor parametrizes the height of the vertex on the left strand oriented from bottom to top and the second one parametrizes the height of the vertex on the right strand.

The map p_{S^2} maps $]-\infty, 0]^2$ and $[1, \infty[^2$ to the vertical circle through the horizontal real direction. Therefore, the integral of $p^*_{S^2}(\omega_{i,S})$ vanishes there, and the integral of $p^*_{S^2}(\omega_{i,S})$ over $(\check{C}(\mathbb{R}^3,$  $;\Gamma), o(\Gamma))$ is the integral of $p^*_{S^2}(\omega_{i,S})$ over $]-\infty, \infty[^2 \setminus (]-\infty, 0[^2 \cup]1, \infty[^2)$ or over $[-\infty, \infty]^2 \setminus ([-\infty, 0[^2 \cup]1, \infty]^2)$, to which $p^*_{S^2}(\omega_{i,S})$ extends naturally. The boundary of this domain, which is drawn in Figure 12.2, is mapped to the vertical half

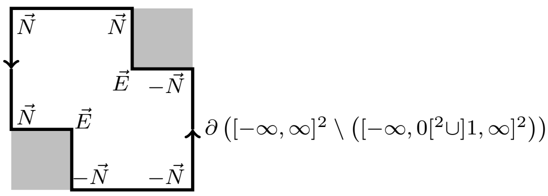


Figure 12.2: Images of boundary points of $[-\infty,\infty]^2 \setminus ([-\infty,0[^2 \cup]1,\infty]^2)$ under p_{S^2}

circle between the two vertical directions north $\vec{N}$ and south $(-\vec{N})$ through the horizontal east direction $\vec{E}$ towards the right.

Thus, $p_{S^2}\left(\partial C(\mathbb{R}^3, \quad ;\Gamma)\right)$ is algebraically trivial (as in the beginning of Section 7.5), and the differential degree of p_{S^2} is constant on the set of regular values of p_{S^2}, according to Lemma 2.3. It can be computed as in Section 1.2.3 at the vector that points towards the reader, at which it is equal to one, and

$$I\left(\quad , \quad , \left(p^*_{S^2}\left(\omega_{i,S}\right)\right)_{i\in\underline{3}}\right) = 1$$

for any arbitrary numbering of the edge of $\Gamma = $ and for any choice of volume-one forms $\omega_{i,S}$ of S^2. Similarly, for the opposite orientation of the edge of Γ,

$$I\left(\quad , \quad , \left(p^*_{S^2}\left(\omega_{i,S}\right)\right)_{i\in\underline{3}}\right) = 1.$$

$\square$

Definition 12.4. A *parallelization* of $\mathcal{C}$ is a parallelization of $\check{R}(\mathcal{C})$ that agrees with the standard parallelization of $\mathbb{R}^3$ outside $\mathcal{C}$. A *parallelized rational homology cylinder* $(\mathcal{C},\tau)$ is a rational homology cylinder equipped with such a parallelization.

The following lemma is another important example of computations.

Lemma 12.5. *Let $K\colon \mathbb{R} \hookrightarrow \check{R}(\mathcal{C})$ be a component of L. Let τ be a parallelization of $\mathcal{C}$ (which is standard near $\partial\mathcal{C}$ by Definition 12.4). For any $i \in \underline{3}$, let $\omega(i)$ and $\omega'(i)$ be propagating forms of $(C_2(R(\mathcal{C})),\tau)$, which restrict to*

$\partial C_2(R(\mathcal{C}))$ as $p_\tau^*(\omega(i)_{S^2})$ and $p_\tau^*(\omega'(i)_{S^2})$, respectively. Let $\eta(i)_{S^2}$ be a one-form on S^2 such that $\omega'(i)_{S^2} = \omega(i)_{S^2} + d\eta(i)_{S^2}$. Then when K goes from bottom to top or from top to bottom

$$I({}_k\hat{\xi}^K, (\omega'(i))_{i\in\underline{3}}) - I({}_k\hat{\xi}^K, (\omega(i))_{i\in\underline{3}}) = \int_{U^+K} p_\tau^*(\eta(k)_{S^2}) = \int_{p_\tau(U^+K)} \eta(k)_{S^2}.$$

When K goes from bottom to bottom (resp. from top to top), let $S(K)$ be the half-circle from $-\vec{N}$ to $\vec{N}$ (resp. from $\vec{N}$ to $-\vec{N}$) through the horizontal direction from the initial vertical half-line of K (the first encountered one) to the final one, then

$$I({}_k\hat{\xi}^K, (\omega'(i))_{i\in\underline{3}}) - I({}_k\hat{\xi}^K, (\omega(i))_{i\in\underline{3}}) = \int_{p_\tau(U^+K)\cup S(K)} \eta(k)_{S^2}.$$

In particular, $I({}_k\hat{\xi}^K, (\omega(i))_{i\in\underline{3}})$ depends only on $\omega(k)_{S^2}$, it is also denoted by $I(\hat{\xi}^K, \omega(k)_{S^2})$.

Proof: In any case, the configuration space $\check{C}(\check{R}(\mathcal{C}), L; \hat{\xi}^{\mathrm{K}})$ is identified naturally with the interior of a triangle, as in the left part of Figure 12.3. When K goes from bottom to top, p_τ extends smoothly to the triangle, as a map which sends the horizontal side and the vertical side to $\vec{N}$. Conclude as in Lemma 7.13 and Lemma 7.15. The case in which K goes from top to bottom is similar.

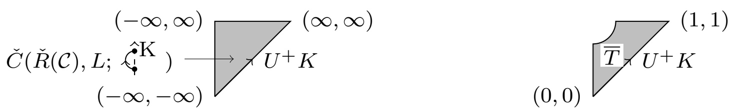
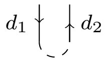


Figure 12.3: Compactifications of configuration spaces for the proof of Lemma 12.5

Let us study the case in which K goes from top to top. Let $d_1 = -\{z_1\} \times [1,\infty]$ and $d_2 = \{z_2\} \times [1,\infty]$ denote the vertical half-lines of K above $\mathcal{C}$, where d_1 is before d_2. View K as a path composition $d_1(K \cap (D_1 \times [0,1]))d_2$ and parametrize

$K =$  by

$$\begin{array}{rlcl} m\colon &]0,1[& \to & K \\ & t \in]0,1/3] & \mapsto & (z_1, 1/(3t)) \\ & t \in [2/3,1[& \mapsto & (z_2, 1/(3(1-t))). \end{array}$$

Let $T_0 = \{(t_1,t_2) \in]0,1[^2 | \ t_1 < t_2\}$. We study the integral of $\omega'(k) - \omega(k) = \mathrm{d}\eta(k)$ over $\check{C}(\check{R}(\mathcal{C}), L; \hat{\xi}^K) = m^2(T_0)$. View T_0 as

$$T = \left\{(t_1,t_2,\alpha) \in]0,1[^2 \times]-\frac{\pi}{2},\frac{\pi}{2}[\,|\ t_1 < t_2, \tan(\alpha) = \frac{1/(3(1-t_2)) - 1/(3t_1)}{|z_2 - z_1|}\right\}$$

Note that when $(t_1, t_2) \in]0, 1/3[\times]2/3, 1[$, and when $(t_1, t_2, \alpha) \in T$,

$$p_\tau(m(t_1), m(t_2)) = \cos\alpha \frac{z_2 - z_1}{|z_2 - z_1|} + \sin\alpha \vec{N} \in S(K).$$

Let $\overline{T}$ be the closure of T in $[0,1]^2 \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. This closure, drawn in the right part of Figure 12.3, is a smooth blow-up of $\{(t_1, t_2) \in [0,1]^2 \mid t_1 \leq t_2\}$ at $(0, 1)$ (with corners), which lifts as $(0,1) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ in $\overline{T}$. The map m^2 extends as a smooth map valued in $C_2(R(\mathcal{C}))$ on $\overline{T}$. Its composition with p_τ sends the vertical side of $\overline{T}$ to $-\vec{N}$, its horizontal side to $\vec{N}$, and the blown up upper-left corner to $S(K)$. The integral of $\mathrm{d}\eta(k)$ over $\check{C}(\check{R}(\mathcal{C}), L;\ \hat{\Game}^K\ )$ is the integral of $d\eta(k)$ over $m^2\left(\overline{T}\right)$. So it is the integral of $\eta(k)$ over $m^2(\partial\overline{T})$, where $\eta(k)$ can be assumed to be equal to $p_\tau^*(\eta(k)_{S^2})$, as in Lemma 4.2. Furthermore, $p_\tau \circ m^2$ restricts to $\partial\overline{T}$ as a degree one map onto $p_\tau(U^+K) \cup S(K)$. So the stated conclusion follows. The case in which K goes from bottom to bottom can be treated similarly. $\square$

Definition 12.6. For a long component $K \colon \mathbb{R} \hookrightarrow \check{R}(\mathcal{C})$ of a tangle in a parallelized $\mathbb{Q}$–cylinder $(\mathcal{C}, \tau)$, with the notation of Lemma 12.5, define

$$I_\theta(K, \tau) = 2I\left(\ \hat{\Game}^{\mathrm{K}}, \omega_{S^2}\right).$$

The factor 2 in the definition of I_θ in Definition 12.6 may seem unnatural. It allows to get homogeneous formulas in Theorem 12.7 below. Theorem 12.7 generalizes Theorem 7.20 to long tangle representatives. It will be proved in Section 14.3.

Theorem 12.7. *Let $\mathcal{C}$ be a rational homology cylinder equipped with a parallelization τ (standard on $\partial\mathcal{C}$). Let $L \colon \mathcal{L} \hookrightarrow R(\mathcal{C})$ be a long tangle representative in $R(\mathcal{C})$. Let $n \in \mathbb{N}$. For any $i \in \underline{3n}$, let $\omega(i)$ be a homogeneous propagating form of $(C_2(R(\mathcal{C})), \tau)$. Set*

$$Z_n(\mathcal{C}, L, (\omega(i))) = \sum_{\Gamma \in \mathcal{D}_n^e(\mathcal{L})} \zeta_\Gamma I(\mathcal{C}, L, \Gamma, (\omega(i))_{i \in \underline{3n}})[\Gamma] \in \mathcal{A}_n(\mathcal{L}),$$

where $\zeta_\Gamma = \frac{(3n - \sharp E(\Gamma))!}{(3n)! 2^{\sharp E(\Gamma)}}$, with Definition 7.6 for $\mathcal{D}_n^e(\mathcal{L})$. Then $Z_n(\mathcal{C}, L, (\omega(i)))$ is independent of the chosen $\omega(i)$, it depends only on $(\mathcal{C}, L(\mathcal{L}) \cap \mathcal{C})$ up to diffeomorphisms that fix $\partial\mathcal{C}$ (and $L \cap \partial\mathcal{C}$), pointwise, on $p_1(\tau)$, and on the $I_\theta(K_j, \tau)$, for the components K_j, $j \in \underline{k}$, of L. It is denoted by $Z_n(\mathcal{C}, L, \tau)$. Set

$$Z(\mathcal{C}, L, \tau) = (Z_n(\mathcal{C}, L, \tau))_{n \in \mathbb{N}} \in \mathcal{A}(\mathcal{L})$$

and recall the anomalies $\alpha \in \check{\mathcal{A}}(S^1;\mathbb{R})$ *and* $\beta \in \mathcal{A}(\emptyset;\mathbb{R})$ *from Sections 10.3 and 10.2. Then the expression*[2]

$$\exp(-\frac{1}{4}p_1(\tau)\beta)\prod_{j=1}^{k}(\exp(-I_\theta(K_j,\tau)\alpha)\sharp_j)\,Z(\mathcal{C},L,\tau)$$

depends only on the boundary preserving diffeomorphism class of $(\mathcal{C},L)$. *It is denoted by* $\mathcal{Z}(\mathcal{C},L)$.

12.2 Definition of $\mathcal{Z}^f$ for framed tangles

Parallels of knots are defined in Definition 5.34.

Definition 12.8. A *parallel* $K_\parallel$ of an embedding of a long component $K\colon \mathbb{R} \hookrightarrow \check{R}(\mathcal{C})$ parametrized so that $K(\mathbb{R}) \cap \mathcal{C} = K([0,1])$, is the image of an embedding $K_\parallel\colon \mathbb{R} \hookrightarrow \check{R}(\mathcal{C})$ such that there exists an embedding

$$k\colon [-1,1]\times\mathbb{R} \to R(\mathcal{C}) \setminus (L(\mathcal{L})\setminus K(\mathbb{R}))$$

such that $K = k_{|\{0\}\times\mathbb{R}}$,$K_\parallel = k_{|\{1\}\times\mathbb{R}}$ and $k(u,t) = K(t) + u\varepsilon(t)(1,0,0)$ for any $(u,t) \in [-1,1]\times(\mathbb{R}\setminus]0,1[)$, for a small continuous function $\varepsilon\colon \mathbb{R}\setminus]0,1[\to \mathbb{R}\setminus\{0\}$ such that $\varepsilon(0)\varepsilon(1)$ is positive for components that go from bottom to top or from top to bottom and negative for components that go from bottom to bottom or from top to top. (We push in one of the two horizontal real directions in a way consistent with the orientation.) Parallels are considered up to isotopies that stay within these parallels and up to the exchange of $k_{|\{1\}\times\mathbb{R}}$ and $k_{|\{-1\}\times\mathbb{R}}$. A component K of an LTR is *framed* if K is equipped with such a class of parallels, which is called a *framing* of K. An LTR is *framed* if all its components are. The *self-linking number* of a circle component K in a framed LTR is the linking number $lk(K,K_\parallel)$ of K and its parallel $K_\parallel$.

For a long component knot equipped with a parallel, we define its *self-linking number* $lk(K,K_\parallel)$ in Definitions 12.9 and 12.10 below, and, in Definition 13.4, which covers the remaining cases (when K goes from bottom to bottom or from top to top and when $K(1)-K(0)$ is not in the direction of the real line).

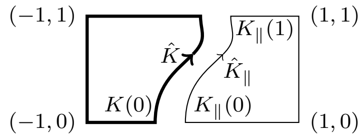
[2]Again, the subscript j of $\sharp_j$ indicates that $\exp(-I_\theta(K_j,\tau)\alpha)$ is inserted on the component of K_j of the source $\mathcal{L}$ of L.

Definition 12.9. When K goes from bottom to top, and when $\varepsilon(0)$ is positive, let $[K_\parallel(1),(1,1)]$ be the straight segment from $K_\parallel(1)$ to $(1,1)\in D_1\times\{1\}$, define $[K(1),(-1,1)]$, $[(1,0),K_\parallel(0)]$ and $[(-1,0),K(0)]$ similarly, and note that they are pairwise disjoint. Define the topological circle embeddings

$$\begin{aligned}\hat{K} &= K\cup[K(1),(-1,1)]\cup-(\{-1\}\times[0,1])\cup[(-1,0),K(0)]\\ \hat{K}_\parallel &= K_\parallel\cup[K_\parallel(1),(1,1)]\cup-(\{1\}\times[0,1])\cup[(1,0),K_\parallel(0)],\end{aligned}$$

as in the figure below, and set

$$lk(K,K_\parallel)=lk(\hat{K},\hat{K}_\parallel).$$



When $\varepsilon(0)$ is negative, the above definition determines $lk(K_\parallel,K)$, set

$$lk(K,K_\parallel)=lk(K_\parallel,K).$$

So $lk(K,K_\parallel)$ is defined when K goes from bottom to top. When K goes from top to bottom, $(-K)$ goes from bottom to top, and $-K_\parallel$ is a parallel of $(-K)$, set $lk(K,K_\parallel)=lk(-K,-K_\parallel)$.

Definition 12.10. When K goes from bottom to bottom or from top to top, and when $K(1)-K(0)$ is equal to $(v,0,0)$ for $v\in\mathbb{R}\setminus\{0\}$, we define the self-linking number $lk(K,K_\parallel)$ as follows. Let us first assume that $v>0$ and that $\varepsilon(0)>0$, as in Figure 12.4. In this case, define topological circle embeddings $\hat{K}_\parallel = K_\parallel([0,1])\cup[K_\parallel(1),K_\parallel(0)]$, where $[K_\parallel(1),K_\parallel(0)]$ is the straight segment from $K_\parallel(1)$ to $K_\parallel(0)$, in $D_1\times\{0\}$ or in $D_1\times\{1\}$, and $\hat{K}=K([0,1])\cup\gamma([0,1])$ for an arbitrary path γ from $\gamma(0)=K(1)$ to $\gamma(1)=K(0)$ such that $\gamma(]0,1[)\subset\check{R}(\mathcal{C})\setminus\mathcal{C}$ as in Figure 12.4 and set

$$lk(K,K_\parallel)=lk(\hat{K},\hat{K}_\parallel).$$

In the other cases, in which K goes from bottom to bottom or from top to top, and $K(1)-K(0)$ is equal to $(v,0,0)$ for $v\in\mathbb{R}\setminus\{0\}$, $lk(K,K_\parallel)$ is defined so that we have again $lk(K,K_\parallel)=lk(K_\parallel,K)=lk(-K,-K_\parallel)$.

Example 12.11. Let K be a framed component of an LTR in $\mathbb{R}^3$, with a regular projection on $\mathbb{R}\times\mathbb{R}$, and such that $K(1)-K(0)$ is equal to $(v,0,0)$ for $v\in\mathbb{R}\setminus\{0\}$ if K goes from bottom to bottom or from top to top. The *self-linking number* of K is its *writhe*, which is the algebraic number of its crossings, that is the number of positive crossings minus the number of negative crossings in its regular projection. (Check it as an exercise.)

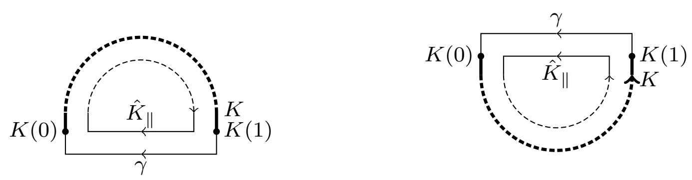


Figure 12.4: Pictures of γ and $\hat{K}_{\|}$

Definition 12.12. When a long tangle representative $L = (K_j)_{j \in \underline{k}}$ is framed by some $L_{\|} = (K_{j\|})_{j \in \underline{k}}$, with the notation of Theorem 12.7, set

$$\mathcal{Z}^f(\mathcal{C}, (L, L_{\|})) = \prod_{j=1}^{k} \left(\exp(lk(K_j, K_{j\|})\alpha)\sharp_j\right) \mathcal{Z}(\mathcal{C}, L).$$

General relations between I_θ and self-linking numbers will be given in Section 16.3. See Proposition 16.10.

A *tangle representative* is a pair $(\mathcal{C}, L(\mathcal{L}) \cap \mathcal{C})$ for a rational homology cylinder $\mathcal{C}$ and a long tangle representative $L \colon \mathcal{L} \hookrightarrow R(\mathcal{C})$ as in Definition 12.1. Tangle representatives and LTR are in natural one-to-one correspondence, and $\mathcal{Z}$ is also viewed as a function of tangle representatives.

A tangle representative $(\mathcal{C}_1, L_1)$ is *right-composable* by a tangle representative $(\mathcal{C}_2, L_2)$ when the top configuration of $(\mathcal{C}_1, L_1)$ coincides with the bottom configuration of $(\mathcal{C}_2, L_2)$. In this case, the *product*

$$(\mathcal{C}_1\mathcal{C}_2, L_1L_2) = (\mathcal{C}_1, L_1)(\mathcal{C}_2, L_2)$$

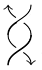
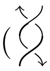
is obtained by stacking $(\mathcal{C}_2, L_2)$ above $(\mathcal{C}_1, L_1)$, after affine reparametrizations of $D_1 \times [0,1]$, which becomes $D_1 \times [0,1/2]$ for $(\mathcal{C}_1, L_1)$ and $D_1 \times [1/2,1]$ for $(\mathcal{C}_2, L_2)$.

The product of two framed tangle representatives is framed naturally. We will prove the following functoriality theorem for $\mathcal{Z}^f$ in Section 17.2.

Theorem 12.13. $\mathcal{Z}^f$ *is* functorial: *For two framed tangle representatives $L_1 = (\mathcal{C}_1, L_1)$ and $L_2 = (\mathcal{C}_2, L_2)$ such that the top configuration of L_1 coincides with the bottom configuration of L_2,*

$$\mathcal{Z}^f(L_1L_2) = \mathcal{Z}^f\left(\begin{array}{|c|}\hline L_2 \\ \hline L_1 \\ \hline\end{array}\right) = \begin{array}{|c|}\hline \mathcal{Z}^f(L_2) \\ \hline \mathcal{Z}^f(L_1) \\ \hline\end{array} = \mathcal{Z}^f(L_1)\mathcal{Z}^f(L_2).$$

The product $\mathcal{Z}^f(L_1)\mathcal{Z}^f(L_2)$ is the natural product of Section 6.4. When applied to the case in which $\mathcal{L}_1$ and $\mathcal{L}_2$ are empty, the above theorem implies that the invariant Z of $\mathbb{Q}$–spheres is multiplicative under connected sum.

Remark 12.14. The multiplicativity of Theorem 12.13 does not hold for the unframed version $\mathcal{Z}$ of $\mathcal{Z}^f$. Indeed, as an unframed tangle, the vertical product $\cap$ is equal to $\cap$, but it can be proved as an exercise, which uses Lemma 12.3 and the behaviour of $\mathcal{Z}^f$ under component orientation reversals described in Theorem 13.12, that $\mathcal{Z}($$)\mathcal{Z}(\cap) \neq \mathcal{Z}(\cap)$. (Here, the image of any of the involved boundary planar two-point configurations is $\{-\frac{1}{2}, \frac{1}{2}\} \subset \mathbb{C}$.)

It will follow from Proposition 15.18 that $\mathcal{Z}$ is invariant under any isotopy of a tangle representative in a rational homology cylinder during which the bottom configuration and the top configuration are constant up to translation and dilation.

Definition 12.15. In this book, a *tangle* is an equivalence class of tangle representatives under the equivalence relation that identifies two representatives if and only if they can be obtained from one another by a diffeomorphism h from the pair $(\mathcal{C}, L)$ to another such $(\mathcal{C}, L')$,

- which fixes a neighborhood of $\partial(D_1 \times [0,1])$ setwise,
- which fixes a neighborhood of $(\partial D_1) \times [0,1]$ pointwise, and
- such that $h(c^-(B^-)\times\{0\}) \subset D_1\times\{0\}$ coincides with $c^-(B^-)\times\{0\}$ up to translation and dilation (as a planar configuration of $\check{\mathcal{S}}_{c^-(B^-)}(\mathbb{C}\times\{0\})$), and, the classes of $h(c^+(B^+) \times \{1\}) \subset D_1 \times \{1\}$ and $c^+(B^+) \times \{1\}$ in $\check{\mathcal{S}}_{c^+(B^+)}(\mathbb{C} \times \{1\})$ coincide, too,

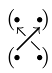
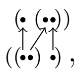
and which is isotopic to the identity map through such diffeomorphisms.

12.3 Defining $\mathcal{Z}^f$ for combinatorial q–tangles

The tangles of Definition 12.15 can be framed by parallels of their components as above, to become *framed tangles*, and $\mathcal{Z}^f$ is an invariant of these framed tangles, which are framed cobordisms (up to isotopy) in $\mathbb{Q}$-cylinders from an injective configuration $c^-(B^-) \in \check{\mathcal{S}}_{c^-(B^-)}(\mathbb{C})$ of points in $\mathbb{C}$, up to dilation and translation, to another planar configuration $c^+(B^+) \in \check{\mathcal{S}}_{c^+(B^+)}(\mathbb{C})$. In Section 16.5, we extend $\mathcal{Z}^f$ to framed cobordisms between limit configurations in the compactifications $\mathcal{S}_{c(B)}(\mathbb{C})$ of the spaces $\check{\mathcal{S}}_{c(B)}(\mathbb{C})$ introduced in Subsection 8.1.2 and studied in Section 8.3. The category of these limit cobordisms called *q–tangles* is equipped with interesting cabling operations described in Section 13.1, under which $\mathcal{Z}^f$ behaves nicely, as stated in Theorem 13.12. In this section, we define $\mathcal{Z}^f$ for particular q–tangles, called *combinatorial q–tangles*, defined below, and we give two examples of cabling properties for these combinatorial q–tangles, in order to motivate and introduce our more general presentation of q–tangles in Section 13.1.

Definition 12.16. A *combinatorial q–tangle* is a framed tangle representative whose bottom and top configurations are on the real line, up to isotopies of $\mathcal{C}$ which globally preserve the intersection of the bottom disk $D_1 \times \{0\}$ with $\mathbb{R} \times \{0\}$ and the intersection of the top disk $D_1 \times \{1\}$ with $\mathbb{R} \times \{1\}$, equipped with non-associative words w^- and w^+ in the letter $\bullet$ (as in Example 8.3) such that the letters of w^- (resp. w^+) are in canonical one-to-one correspondence with the elements of the source B^- (resp. B^+) of the bottom (resp. top) configuration. The non-associative words w^- and w^+ are called the bottom and top configurations of the combinatorial q–tangle, respectively.

Examples 12.17. Combinatorial q–tangles in the standard cylinder are unambiguously represented by one of their regular projections to $\mathbb{R} \times [0, 1]$ such that the parallels of their components are parallel in the figures, together with their bottom and top configurations. Examples of these combinatorial q–tangles include

, ,  and .

Recall from Example 8.3 that the involved non-associative words are corners of $\mathcal{S}_{<,\underline{k}}(\mathbb{R}) \subset \mathcal{S}_{\underline{k}}(\mathbb{C})$. A combinatorial q–tangle L from a bottom word w^- to a top word w^+ is thought of as the limit, when t tends to 0, of framed tangles $L(t)$ whose bottom and top configurations are the configurations $w^-(t)$ and $w^+(t)$, defined in Example 8.3, respectively, in the isotopy class of L with respect to the isotopies of Definition 12.16.

In Theorem 13.8 and Remark 13.11, following Poirier [Poi00], we prove that $\lim_{t\to 0} \mathcal{Z}^f(L(t))$ exists and that the formula

$$\mathcal{Z}^f(L) = \lim_{t\to 0} \mathcal{Z}^f(L(t))$$

defines an isotopy invariant $\mathcal{Z}^f$ of these (framed) combinatorial q–tangles.

Let us compute $\mathcal{Z}^f$ for the combinatorial q–tangle $\uparrow\uparrow$ as a first example.

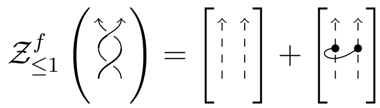
Lemma 12.18.

$$\mathcal{Z}^f\left(\uparrow\uparrow\right) = 1 = [\uparrow\uparrow]$$

Proof: By definition, the left-hand side is the limit, when t tends to 0, of the evaluation of Z of the LTR whose image is $\{0, t\} \times \mathbb{R}$. There is an action of $\mathbb{R}$ by vertical translation on the involved configuration spaces, and the integrated forms factor through the quotients by this action of $\mathbb{R}$ of the configuration spaces, whose dimensions are smaller (by one) than the degrees of the integrated forms. So the integrals vanish for all non-empty diagrams. $\square$

As a second example, we compute $\mathcal{Z}^f_{\leq 1}$, which is the truncation of $\mathcal{Z}^f$ in degrees lower than 2, for the combinatorial q-tangle .

Lemma 12.19.

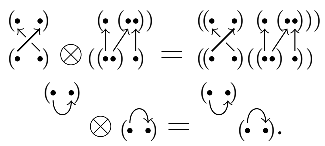


PROOF: Since $\mathcal{Z}$ is invariant under the isotopies that preserve the bottom and the top configurations up to translation and dilation, the contributions of the non-empty diagrams whose univalent vertices are on one strand of the tangle vanish. So we are left with the contribution of the numbered graphs with one vertex on each strand, which is treated by Lemma 12.3. $\square$

The obtained invariant $\mathcal{Z}^f$ is still multiplicative under vertical composition as in Theorem 12.13, and we can now define other interesting operations.

For two combinatorial q–tangles $L_1 = (\mathcal{C}_1, L_1)$, from w_1^- to w_1^+, and $L_2 = (\mathcal{C}_2, L_2)$, from w_2^- to w_2^+, define the product $L_1 \otimes L_2$, from the bottom configuration $w_1^- w_2^-$ to the top configuration $w_1^+ w_2^+$, by shrinking $\mathcal{C}_1$ and $\mathcal{C}_2$ to make them respectively replace the products by $[0,1]$ of the horizontal disks with radius $\frac{1}{4}$ and respective centers $-\frac{1}{2}$ and $\frac{1}{2}$.

Examples 12.20.



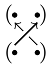
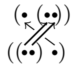
The following theorem can be easily deduced from the cabling property and the functoriality property of Theorem 13.12.

Theorem 12.21. $\mathcal{Z}^f$ *is* monoidal: *For two combinatorial q–tangles L_1 and L_2,*

$$\mathcal{Z}^f(L_1 \otimes L_2) = \mathcal{Z}^f\left(\boxed{L_1 \mid L_2}\right) = \boxed{\mathcal{Z}^f(L_1) \mid \mathcal{Z}^f(L_2)} = \mathcal{Z}^f(L_1) \otimes \mathcal{Z}^f(L_2),$$

where $\mathcal{Z}^f(L_1) \otimes \mathcal{Z}^f(L_2)$ denotes the image of $\mathcal{Z}^f(L_1) \otimes \mathcal{Z}^f(L_2)$ under the natural product from $\mathcal{A}(\mathcal{L}_1) \otimes_{\mathbb{R}} \mathcal{A}(\mathcal{L}_2)$ to $\mathcal{A}(\mathcal{L}_1 \sqcup \mathcal{L}_2)$ induced by the disjoint union of diagrams.

We can also *double* a component K according to its parallelization in a combinatorial q–tangle L. This operation replaces a component with two parallel components, and, if this component has boundary points, it replaces the corresponding letters in the non-associative words with $(\bullet\bullet)$. The obtained combinatorial q–tangle is denoted by $L(2 \times K)$. For example,

 $(2 \times \nearrow) =$ 

The following duplication property for $\mathcal{Z}^f$ is a part of Theorem 13.12, which is proved in Section 17.4.

Theorem 12.22. *Let K be a component of a combinatorial q–tangle L. Recall Notation 6.28. Then*

$$\mathcal{Z}^f(L(2 \times K)) = \pi(2 \times K)^* \mathcal{Z}^f(L).$$

These results are some particular cases of the properties of $\mathcal{Z}^f$, which are listed in Theorem 13.12 and proved in Chapter 17.

12.4 Good monoidal functors for combinatorial q–tangles

Recall the notation of Definition 6.19. In this section, all combinatorial q–tangles are combinatorial q–tangles in $\mathbb{R}^3$ or in the standard cylinder, and $\mathbb{K} = \mathbb{R}$. In [Poi00], Sylvain Poirier extended the natural projection $\check{\mathcal{Z}}^f(\mathbb{R}^3, .)$ of $\mathcal{Z}^f(\mathbb{R}^3, .)$ in $\check{\mathcal{A}}(.)$, from framed links of $\mathbb{R}^3$ to these combinatorial q–tangles, in an elegant way, and he proved that his extension Z^l is a *good monoidal functor* with respect to the definition below. Good monoidal functors on the category of combinatorial q–tangles (in $\mathbb{R}^3$) are characterized in [Les02]. In this section, we review these results of [Poi00] and of [Les02]. The quoted results of Poirier will be reproved (with much more details) and generalized in this book, while the proofs of the results of [Les02] will not be reproduced in this book, since they do not involve analysis on configuration spaces. For this section, coefficients for spaces of Jacobi diagrams are in $\mathbb{C}$.

Definition 12.23. A *good monoidal functor* from the category of combinatorial q–tangles (in $\mathbb{R}^3$) to the category of spaces of Feynman diagrams is a map Y that maps a combinatorial q–tangle L with source $\mathcal{L}$ to an element of $\check{\mathcal{A}}(\mathcal{L}; \mathbb{C})$ (w.r.t. Definition 6.19) such that:

- For any combinatorial q-tangle L, the degree zero part $Y_0(L)$ of $Y(L)$ is 1, which is the class of the empty diagram.

- Y is *functorial:* For two combinatorial q–tangles L_1 and L_2 such that the top configuration $\mathrm{top}(L_1)$ of L_1 coincides with the bottom configuration $\mathrm{bot}(L_2)$ of L_2,

$$Y(L_1L_2) = Y\left(\begin{array}{|c|}\hline L_2 \\ \hline L_1 \\ \hline\end{array}\right) = \begin{array}{|c|}\hline Y(L_2) \\ \hline Y(L_1) \\ \hline\end{array} = Y(L_1)Y(L_2).$$

- Y is *monoidal:* For two combinatorial q–tangles L_1 and L_2,

$$Y(L_1 \otimes L_2) = Y\left(\begin{array}{|c|c|}\hline L_1 & L_2 \\ \hline\end{array}\right) = \begin{array}{|c|c|}\hline Y(L_1) & Y(L_2) \\ \hline\end{array} = Y(L_1) \otimes Y(L_2).$$

- Y is compatible with the deletion of a component: If L' is a subtangle of L with source $\mathcal{L}'$, then $Y(L')$ is obtained from $Y(L)$ by forgetting $\mathcal{L} \setminus \mathcal{L}'$ and all the diagrams with univalent vertices on $\mathcal{L} \setminus \mathcal{L}'$.

- Y is compatible with the duplication of a *regular* component, which is a component that can be represented without horizontal tangent vectors: For such a component K of a combinatorial q–tangle L,

$$\mathcal{Z}^f(L(2 \times K)) = \pi(2 \times K)^* \mathcal{Z}^f(L)$$

  with respect to Notation 6.28.

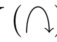
- Y is invariant under the 180-degree rotation around a vertical axis through the real line.

- Let $s_{\frac{1}{2}}$ be the orthogonal symmetry with respect to the horizontal plane at height $\frac{1}{2}$. Let $\sigma_{\frac{1}{2}}$ be the linear endomorphism of the topological vector space $\check{\mathcal{A}}(S^1)$ such that $\sigma_{\frac{1}{2}}([\Gamma]) = (-1)^d[\Gamma]$, for any element $[\Gamma]$ of $\check{\mathcal{A}}_d(S^1)$. For any framed knot $K(S^1)$, $Y(K) \in \check{\mathcal{A}}(S^1; \mathbb{R})$ and

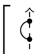
$$Y \circ s_{\frac{1}{2}}(K) = \sigma_{\frac{1}{2}} \circ Y(K).$$

- Y behaves as in Proposition 10.21 with respect to changes of component orientations. It can be defined as an invariant of unoriented tangles valued in a space of diagrams whose support is the unoriented source of the tangle, as in Definitions 6.13 and 6.16.

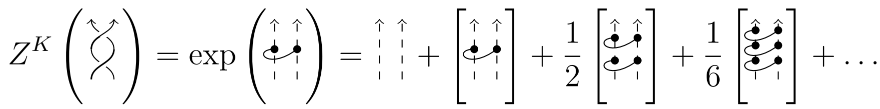
- The degree one part a_1^Y of the element $a^Y \in \check{\mathcal{A}}([0,1])$ such that $a_0^Y = 0$ and

$Y\Big($  $\Big) = \exp(a^Y)Y\big($  $\big)$

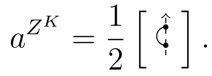
  is

$a_1^Y = \frac{1}{2}\Big[$  $\Big].$

The first example of such a good monoidal functor was constructed from the Kontsevich integral of links in $\mathbb{R}^3$, which is described in [CD01], by Thang Lê and Jun Murakami in [LM96]. See also [Les99]. We will denote it by Z^K and we call it the *Kontsevich integral* of combinatorial q–tangles. The Kontsevich integral furthermore satisfies that



This easily implies that the element a^{Z^K} of the above definition is



So it vanishes in all degrees greater than one.

In [Poi00], Sylvain Poirier extended $\check{\mathcal{Z}}^f(\mathbb{R}^3, .)$ from framed links of $\mathbb{R}^3$ to combinatorial q–tangles of $\mathbb{R}^3$ and he proved that his extension Z^l satisfies the above properties with

$$a^{Z^l} = \alpha,$$

where α is the anomaly of Section 10.3.

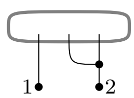
Remark 12.24. The published version [Poi02] of [Poi00] does not contain the cited important results of [Poi00], which will be generalized and proved with much more details in the present book.

Definition 12.25. Say that an element $\gamma = (\gamma_n)_{n \in \mathbf{N}}$ in $\check{\mathcal{A}}([0,1])$ is a *two-leg element* if, for any $n \in \mathbf{N}$, γ_n is a combination of diagrams with two univalent vertices.

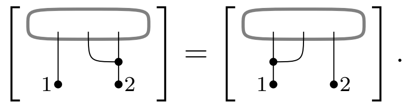
Forgetting $[0,1]$ from such a two-leg element gives rise to a unique series γ^s of diagrams with two distinguished univalent vertices v_1 and v_2, such that γ^s is symmetric with respect to the exchange of v_1 and v_2, according to the following lemma due to Pierre Vogel. See [Vog11, Corollary 4.2].

Lemma 12.26 (Vogel)**.** *Two-leg Jacobi diagrams are symmetric with respect to the exchange of their two legs in a diagram space quotiented by the AS and Jacobi relations.*

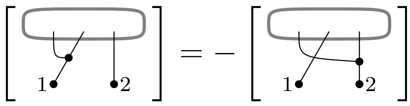
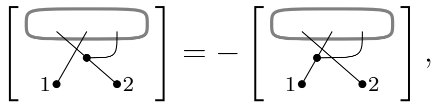
Proof: Since a chord is obviously symmetric, we can restrict ourselves to a two-leg diagram with at least one trivalent vertex and whose two univalent vertices are numbered by 1 and 2, respectively. We draw it as



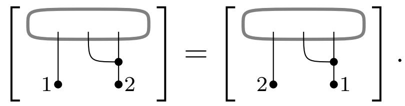
where the trivalent part inside the disk bounded by the thick gray topological circle is not represented. Applying Lemma 6.23, when the annulus is a neighborhood of the thick topological circle that contains the pictured trivalent vertex, proves that



Similarly,

 and 

so



□

Definition 12.27. Let γ be a two-leg element of $\check{\mathcal{A}}([0,1])$, recall that γ^s is the series obtained from γ by erasing $[0,1]$. For a chord diagram Γ, define $\Psi(\gamma)(\Gamma)$ by replacing each chord with γ^s. As it is proved in [Les02, Lemmas 6.1 and 6.2], $\Psi(\gamma)$ is a well defined morphism of topological vector spaces from $\check{\mathcal{A}}(C)$ to $\check{\mathcal{A}}(C)$ for any one-manifold C, and $\Psi(\gamma)$ is an isomorphism as soon as $\gamma_1 \neq 0$.

The following theorem is Theorem 1.3 in [Les02].

Theorem 12.28. *If Y is a good monoidal functor as above, then a^Y is a two-leg element of $\check{\mathcal{A}}([0,1])$, such that for any integer i, $a^Y_{2i} = 0$, and, for any framed link L,*

$$Y(L) = \Psi(2a^Y)(Z^K(L)),$$

where Z^K denotes the Kontsevich integral of framed links (denoted by $\hat{Z}_f$ in [LM96], and by Z in [Les99]).

The following corollary is a particular case of [Les02, Corollary 1.4].

Corollary 12.29. *The anomaly α is a two-leg element of $\check{\mathcal{A}}([0,1])$, and, for any framed link L of $\mathbb{R}^3$,*

$$\check{\mathcal{Z}}^f(\mathbb{R}^3, L) = \Psi(2\alpha)(Z^K(L)).$$

Chapter 13

More on the functor $\mathcal{Z}^f$

In this chapter, we state our general Theorem 13.12, which ensures that $\mathcal{Z}^f$ is a functor, which behaves naturally with respect to various structures of the category of q–tangles, such as cabling or duplication. We first describe the category of q–tangles in Section 13.1 before stating Theorem 13.12 in Section 13.3.

The main steps of the generalization of the construction of $\mathcal{Z}^f$ to q–tangles are described in Section 13.2. The details of these steps will be given in Chapters 14, 15 and 16, and the proof of Theorem 13.12 will be finished in Chapter 17.

13.1 Tangles and q–tangles

Recall that a tangle representative is a pair $(\mathcal{C}, L(\mathcal{L}) \cap \mathcal{C})$ for a rational homology cylinder $\mathcal{C}$ and a long tangle representative $L \colon \mathcal{L} \hookrightarrow R(\mathcal{C})$ as in Definition 12.1.

Definition 13.1. In this book, a *braid representative* is a tangle representative $T(\tilde{\gamma})$ of the standard cylinder $D_1 \times [0,1]$, whose components called *strands* may be expressed as

$$\{(\tilde{\gamma}_b(t), t) \mid t \in [0,1]\},$$

for an element b of a finite set B, which labels the strands, where

$$\tilde{\gamma}_b \colon [0,1] \to \mathring{D}_1$$

is a path such that for any t, and for any pair (b, b') of distinct elements of B, $\tilde{\gamma}_b(t) \neq \tilde{\gamma}_{b'}(t)$, as in Figure 13.1. Such a braid representative can be

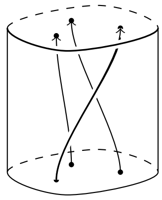


Figure 13.1: A braid representative with three strands

viewed naturally as a path $\tilde{\gamma}\colon [0,1] \to \check{C}_B[\mathring{D}_1]$, where $\check{C}_B[\mathring{D}_1]$ is the space of injections of B into $\mathring{D}_1$, defined in the beginning of Section 8.6.

In this book, a *braid* (resp. a *q–braid*) is a homotopy class of paths $\gamma\colon [0,1] \to \check{\mathcal{S}}_B(\mathbb{C})$ (resp. of paths $\gamma\colon [0,1] \to \mathcal{S}_B(\mathbb{C})$) for some finite set B, where $\mathcal{S}_B(\mathbb{C})$ is the compactification of $\check{\mathcal{S}}_B(\mathbb{C})$ described in Theorem 8.11. A braid γ induces the tangle $T(\gamma)$, which is also called a braid as above. The path $\overline{\gamma}$ is the path such that $\overline{\gamma}(t) = \gamma(1-t)$. A braid is naturally framed by the parallels obtained by pushing it in the direction of the real line of $\mathbb{C}$.

Tangles, as in Definition 12.15, can be multiplied if they have representatives that can be, that is if the top configuration of the first tangle agrees with the bottom configuration of the second one, up to dilation and translation. The product is associative. Framed tangles multiply vertically to give rise to framed tangles.

A *q–tangle* is a framed tangle whose bottom and top configurations are allowed to be limit configurations in some $\mathcal{S}_{B^-}(\mathbb{C})$ and in some $\mathcal{S}_{B^+}(\mathbb{C})$. More precisely, a q–tangle is represented by a product

$$T(\gamma^-)(\mathcal{C}, L)T(\gamma^+),$$

as in Figure 13.2, where γ^- and γ^+ are q–braids, $(\mathcal{C}, L)$ is a framed tangle whose bottom configuration is $\gamma^-(1)$ and whose top configuration is $\gamma^+(0)$, the strands of $T(\gamma^-)$ and $T(\gamma^+)$ get their orientations from the orientation of L. For consistency, we allow braids with 0 or 1 strand, and we agree that $\mathcal{S}_\emptyset(\mathbb{C})$ and $\mathcal{S}_{\{b\}}(\mathbb{C})$ each have one element, which is the unique configuration of one point in $\mathbb{C}$ up to translation in the latter case. Note that the restriction of a q–tangle to one of its components is a framed tangle since configurations of at most two points are always injective. The components of a q–tangle representative are framed since braids are.

Now, q–tangles are classes of these representatives under the equivalence relation that identifies $T(\gamma^-)(\mathcal{C}, L)T(\gamma^+)$ with $T(\gamma^{-\prime})(\mathcal{C}', L')T(\gamma^{+\prime})$ if and only if $\gamma^-(0) = \gamma^{-\prime}(0)$, $\gamma^+(1) = \gamma^{+\prime}(1)$, and the framed tangles $(\mathcal{C}, L)$ and $T(\alpha)(\mathcal{C}', L')T(\beta)$ represent the same framed tangle for any braids α and β such that

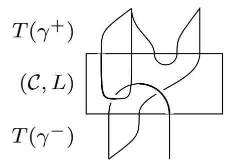


Figure 13.2: A q-tangle representative

- the composition $T(\alpha)(\mathcal{C}', L')T(\beta)$ is well defined,
- the path α of $\check{\mathcal{S}}_{B^-}(\mathbb{C})$ is homotopic to $\overline{\gamma^-}\gamma^{-\prime}$ in $\mathcal{S}_{B^-}(\mathbb{C})$ and
- the path β of $\check{\mathcal{S}}_{B^+}(\mathbb{C})$ is homotopic to $\gamma^{+\prime}\overline{\gamma^+}$ in $\mathcal{S}_{B^+}(\mathbb{C})$ (by homotopies that fix the boundary points),

The *source* of a q-tangle (represented by) $T(\gamma^-)(\mathcal{C}, L)T(\gamma^+)$ is (identified with) the source $\mathcal{L}$ of $L\colon \mathcal{L} \hookrightarrow \mathcal{C}$, its *bottom configuration* is $\gamma^-(0)$ and its *top configuration* is $\gamma^+(1)$.

Example 13.2. The combinatorial q–tangles of Definition 12.16 are examples of q–tangles. Since the external pair of parentheses in a *combinatorial configuration*, which is a non-associative word in $\bullet$, is always present, it may be omitted from the notation. Similarly, the only possible two-point combinatorial configuration $(\bullet\bullet)$ may also be omitted from the notation in combinatorial q–tangles. So the examples of Example 12.17 may be represented by:

$$\nwarrow\!\!\!\!\nearrow, \quad {}^{\bullet\,(\bullet\bullet)}_{(\bullet\bullet)\,\bullet}\uparrow\!\nearrow\!\uparrow, \quad \cup \text{ and } \cap.$$

When the boundary of a q–tangle is empty, the q–tangle is a framed link in $\check{R}(\mathcal{C})$. Conversely, any asymptotically standard $\mathbb{Q}$-homology $\mathbb{R}^3$ equipped with a framed link may be obtained in this way, up to diffeomorphism.

Notation 13.3. In addition to the (vertical) product, which extends to q–tangles naturally, q–tangles support a *cabling* operation, of a component K of a q–tangle T_m by a q–tangle T_i, which produces a q–tangle $T_m(T_i/K)$. This operation roughly consists in replacing the strand K in T_m by a tangle T_i, with respect to the framing of K, as in Figure 13.3.

For example, when T_i is the trivial braid $|_1|_2$ with two strands and when T_m is a combinatorial q–tangle, $T_m(T_i/K)$ is the tangle $T_m(2 \times K)$ described before Theorem 12.22. As another example, the product $L_1 \otimes L_2$ described before Theorem 12.22 can be written as $(L_1\otimes\ |_2)(L_2/\ |_2)$, where $L_1\otimes\ |_2 = (|_1|_2)(L_1/\ |_1)$.

Let us describe the cabling operation in general in (lengthy) details. A *semi-pure q–tangle* is a tangle with identical bottom and top configuration (up to dilation and translation). A *pure q–tangle* is a semi-pure tangle whose interval components connect a point of the bottom configuration to the corresponding point of the top configuration. The cabling operation that produces $T_m(T_i/K)$ is defined for any pair (T_m, T_i) of q–tangles equipped with an interval component K of T_m that goes from bottom to top. It is also defined for any pair (T_m, T_i) of q–tangles equipped with a framed circle component K of T_m provided that T_i is semi-pure.

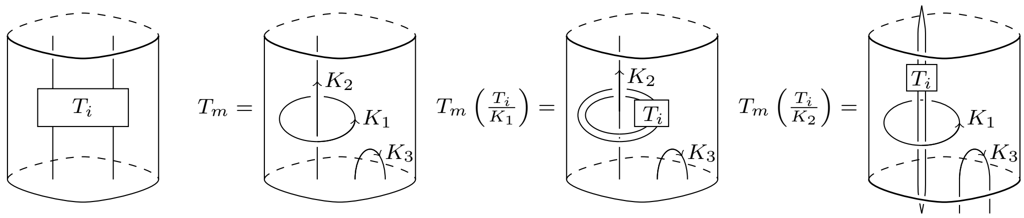


Figure 13.3: Examples of cablings

We begin with the details in the case for which K is a circle, because they are lighter. When K is a closed component, and when T_i is a framed tangle $(\mathcal{C}_i, L_i)$ with identical injective bottom and top configuration, pick a tubular neighborhood $D^2 \times K$ of K that is trivialized with respect to the parallelization of K and that does not meet the other components of T_m. Write this neighborhood as $D^2 \times [0,1]/(0 \sim 1)$ and replace it with $(\mathcal{C}_i, L_i)$ using the identification of $N(\partial(D_1 \times [0,1]))$ with a neighborhood of $\partial(D_1 \times [0,1])$ in order to obtain $T_m(T_i/K)$. Note that when γ is a braid such that $\gamma(1)$ is the bottom configuration of T_i,

$$T_m\left(\frac{T_i}{K}\right) = T_m\left(\frac{T(\gamma)T_iT(\overline{\gamma})}{K}\right).$$

Any semi-pure q–tangle T_q can be written as $T(\gamma)(\mathcal{C}_i, L_i)T(\overline{\gamma})$, for some q–braid γ and some framed tangle $(\mathcal{C}_i, L_i)$. For such a tangle, set $T_m(T_q/K) = T_m((\mathcal{C}_i, L_i)/K)$. It is easy to check that this definition is consistent.

Let us now define *cabling* or *duplication* for configurations. Let B and B_i be non-empty finite sets, let $b \in B$ and let $B(B_i/b) = (B \setminus \{b\}) \cup B_i$. Let c_m be an element of $\mathcal{S}_B(\mathbb{C})$, and let c_i be an element of $\mathcal{S}_{B_i}(\mathbb{C})$. We define the configuration $c_m(c_i/b)$ of $\mathcal{S}_{B(B_i/b)}(\mathbb{C})$ obtained by letting c_i replace b. Up to translation there is only one configuration of a set of one element. So $c_m(c_i/b)$ is this unique configuration if $\sharp B(B_i/b) = 1$. If $\sharp B_i = 1$, $c_m(c_i/b) = c_m$ and if $\sharp B = 1$, $c_m(c_i/b) = c_i$ (with natural identifications). Assume $\sharp B_i \geq 2$ and

$\sharp B \geq 2$. When c_m and c_i are both combinatorial configurations, it makes natural sense to let c_i replace b in order to produce $c_m(c_i/b)$. In general, recall Definition 8.23 and define the Δ-parenthesization $\tau(c_m)(\tau(c_i)/b)$ of $B(B_i/b)$ from the respective Δ-parenthesizations $\tau(c_m)$ and $\tau(c_i)$ of c_m and c_i by the following one-to-one correspondence

$$\begin{array}{llll}\phi\colon & \tau(c_m) \sqcup \tau(c_i) & \to & \tau(c_m)(\tau(c_i)/b) \\ & A & \mapsto & \begin{cases} A & \text{if } A \in \tau_i \text{ or } (A \in \tau_m \text{ and } b \notin A) \\ A(B_i/b) & \text{if } A \in \tau_m \text{ and } b \in A. \end{cases}\end{array}$$

With the notation of Theorem 8.26, the configuration $c_m(c_i/b)$ is the configuration of $\mathcal{S}_{B(B_i/b),\tau(c_m)(\tau(c_i)/b)}(\mathbb{C})$ that restricts to B_i as c_i and to $B(\{b'\}/b)$ as c_m, for any element b' of B_i.

Let us assume that K is an interval component that goes from bottom to top of a q–tangle

$$T_m = T(\gamma_m^-)(\mathcal{C}_m, L_m)T(\gamma_m^+)$$

and let us define $T_m(T_i/K)$ for a q–tangle $T_i = T(\gamma_i^-)(\mathcal{C}_i, L_i)T(\gamma_i^+)$. Let B_i^-, (resp. B_i^+, B^-, B^+) denote the set of strand indices of γ_i^- (resp. γ_i^+, γ_m^-, γ_m^+) and let b_K^- (resp. b_K^+) denote the strand index of K in B^- (resp. B^+). Let c_m^- (resp. c_i^-) denote the bottom configuration of T_m (resp. T_i). Let c_m^+ (resp. c_i^+) denote the top configuration of T_m (resp. T_i). Assume that $\gamma_m^-(]0,1]) \subset \check{\mathcal{S}}_{B^-}(\mathbb{C})$, $\gamma_i^-(]0,1]) \subset \check{\mathcal{S}}_{B_i^-}(\mathbb{C})$, $\gamma_m^+([0,1[) \subset \check{\mathcal{S}}_{B^+}(\mathbb{C})$ and that $\gamma_i^+([0,1[) \subset \check{\mathcal{S}}_{B_i^+}(\mathbb{C})$. Let I_K denote the intersection of K with $\mathcal{C}_m$, identify I_K with $[0,1]$, let $D^{(i)}$ be a copy of the disk D^2, and let $D^{(i)} \times [0,1]$ be a tubular neighborhood of I_K in $\mathcal{C}_m$ that does not meet the other components of L_m and that meets $\partial\mathcal{C}_m$ along $D^{(i)} \times \partial[0,1]$ inside $D_1 \times \partial[0,1]$, such that $(\{\pm 1\} \times [0,1]) \subset (\partial D^{(i)} \times [0,1])$ is the given parallel of I_K. Replace $D^{(i)} \times [0,1]$ with $(\mathcal{C}_i, L_i)$ in order to get a tangle $(\mathcal{C}_m, L_m)\left((\mathcal{C}_i, L_i)/I_K\right)$.

Let $\gamma_m^-(\gamma_i^-/K)$ be the path composition $\gamma_m^-(c_i^-/K)\left(\gamma_m^-(1)(\gamma_i^-/K)\right)$ of the paths $\gamma_m^-(c_i^-/K)$ and $\gamma_m^-(1)(\gamma_i^-/K)$ in $\mathcal{S}_{B^-(B_i^-/b_K^-)}(\mathbb{C})$, where $\gamma_m^-(c_i^-/K)(t) = \gamma_m^-(t)(c_i^-/b_K^-)$ for any $t \in [0,1]$, and the restriction to $]0,1]$ of $\gamma_m^-(1)(\gamma_i^-/K)$ is represented by a map from $]0,1]$ to $\check{C}_{B^-(B_i^-/b_K^-)}[\mathring{D}_1]$,

- which maps 1 to the bottom configuration of $(\mathcal{C}_m, L_m)\left((\mathcal{C}_i, L_i)/I_K\right)$,
- whose restriction to $B^- \setminus \{b_K^-\}$ is constant[1], and,
- whose restriction to B_i^- is a lift of $\gamma^-_{i|]0,1]}$ in $\check{C}_{B_i^-}[p_{\mathbb{C}}(D^{(i)} \times \{0\})]$, such that $\gamma_m^-(1)(\gamma_i^-/K)$ is composable by $\gamma_m^-(c_i^-/K)$ on its left.

[1]It is thus located in $\check{C}_{B^-\setminus\{b_K^-\}}[\mathring{D}_1 \setminus p_{\mathbb{C}}(D^{(i)} \times \{0\})]$.

The tangle $T_m(T_i/K)$ is defined as

$$T(\gamma_m^-(\gamma_i^-/K))\,((\mathcal{C}_m, L_m)\,((\mathcal{C}_i, L_i)/I_K))\,T(\gamma_m^+(\gamma_i^+/K)),$$

where the definition of $T(\gamma_m^+(\gamma_i^+/K))$, which is very similar to the definition of $T(\gamma_m^-(\gamma_i^-/K))$ follows (and can be skipped...).

The path $\gamma_m^+(\gamma_i^+/K)$ is the path composition $\left(\gamma_m^+(0)(\gamma_i^+/K)\right)\gamma_m^+(c_i^+/K)$ in $\mathcal{S}_{B^+(B_i^+/b_K^+)}(\mathbb{C})$, where $\gamma_m^+(c_i^+/K)(t) = \gamma_m^+(t)(c_i^+/b_K^+)$ for any $t \in [0,1]$, and the restriction to $[0,1[$ of $\gamma_m^+(0)(\gamma_i^+/K)$ is represented by a map from $[0,1[$ to $\check{C}_{B^+(B_i^+/b_K^+)}[\mathring{D}_1]$, which maps 0 to the top configuration of $(\mathcal{C}_m, L_m)\,((\mathcal{C}_i, L_i)/I_K)$, whose restriction to $B^+ \setminus \{b_K^+\}$ is constant, and whose restriction to B_i^+ is a lift of $\gamma^+_{i|[0,1[}$ in $\check{C}_{B_i^+}[p_{\mathbb{C}}(D^{(i)} \times \{1\})]$, such that $\gamma_m^+(0)(\gamma_i^+/K)$ is composable by $\gamma_m^+(c_i^+/K)$ on its right.

A particular case of cablings is the case when the inserted q–tangle T_i is just the q–tangle $y \times [0,1]$ associated with the constant path of $\mathcal{S}_{B_i}(\mathbb{C})$ that maps $[0,1]$ to a configuration y of $\mathcal{S}_{B_i}(\mathbb{C})$. (Formally, this q–tangle is represented by $T(\gamma)(\gamma(1) \times [0,1])T(\overline{\gamma})$ for some path γ of $\mathcal{S}_{B_i}(\mathbb{C})$ such that $\gamma(0) = y$ and $\gamma(1) \in \check{\mathcal{S}}_{B_i}(\mathbb{C})$.) Set

$$T_m(y \times K) = T_m((y \times [0,1])/K).$$

If K is a closed component, then $T_m(y \times K)$ depends only on $\sharp B_i$, it is denoted by $T_m((\sharp B_i) \times K)$. If K is an interval component, and if y is the unique configuration of $\mathcal{S}_{\{1,2\}}(\mathbb{R})$, then $T_m(y \times K)$ is again denoted by $T_m(2 \times K)$. These special cablings are called *duplications*. Our functor $\mathcal{Z}^f$ will behave well under all cablings.

We end this section by completing Definition 12.10 of the self-linking number for a framed q–tangle component, which goes from bottom to bottom or from top to top.

Definition 13.4. The *self-linking number* $lk(K, K_\parallel)$ of a framed component $K = K([0,1])$ of a q–tangle that goes from bottom to bottom (resp. from top to top) is defined as follows. The self-linking number depends only on the component, so there is no loss of generality in representing K by a tangle representative with injective bottom (resp. top) configuration whose ends are at distance bigger than 2ε for a small positive ε. Also assume that $K_\parallel(0) = K(0) + (\varepsilon, 0, 0)$ and that $K_\parallel(1) = K(1) - (\varepsilon, 0, 0)$. (There is no loss of generality in this assumption either, it suffices to choose a parallel that satisfies this assumption to define the self-linking number. Recall Definition 12.8.) Let $\hat{K} = K([0,1]) \cup \gamma([0,1])$ for an arbitrary path γ from $\gamma(0) = K(1)$ to $\gamma(1) = K(0)$ such that $\gamma(]0,1[) \subset \check{R}(\mathcal{C}) \setminus \mathcal{C}$.

Let $[K(1), K(0)]$ denote the straight segment from $K(1)$ to $K(0)$ in D_1. Let $\alpha_1\colon [0,1] \to D_1$ be an arc from $K_{\|}(1)$ to a point a_1 inside $[K(1), K(0)]$ such that $\alpha_1(t) = K(1) - \varepsilon \exp(2i\pi\theta_1 t)$ for some real number θ_1, and let $\alpha_0\colon [0,1] \to D_1$ be an arc from $K_{\|}(0)$ to a point a_0 inside $[K(1), K(0)]$ such that $\alpha_0(t) = K(0) + \varepsilon \exp(2i\pi\theta_0 t)$ for some $\theta_0 \in \mathbb{R}$.

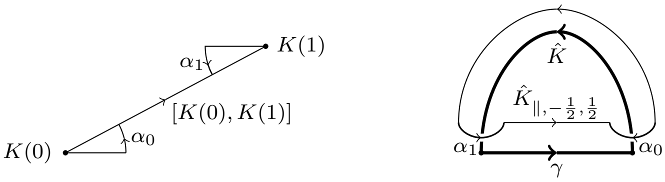


Figure 13.4: A general picture of α_0 and α_1 and a picture of $\hat{K}_{\|}$ when $\theta_0 = -\frac{1}{2}$ and $\theta_1 = \frac{1}{2}$, when K goes from bottom to bottom

If K goes from bottom to bottom, define

$$\hat{K}_{\|,\theta_0,\theta_1} = K_{\|}([0,1]) \cup ((\alpha_1 \cup [a_1, a_0] \cup \overline{\alpha}_0) \times \{0\})$$

and set

$$lk(K, K_{\|}) = lk(\hat{K}, \hat{K}_{\|,\theta_0,\theta_1}) + \theta_1 + \theta_0.$$

If K goes from top to top, define

$$\hat{K}_{\|,\theta_0,\theta_1} = K_{\|}([0,1]) \cup ((\alpha_1 \cup [a_1, a_0] \cup \overline{\alpha}_0) \times \{1\})$$

and set

$$lk(K, K_{\|}) = lk(\hat{K}, \hat{K}_{\|,\theta_0,\theta_1}) - (\theta_1 + \theta_0).$$

Note that these definitions do not depend on the chosen θ_1 and θ_0, which are well-determined mod $\mathbb{Z}$. The angles $2\pi\theta_1$ and $2\pi\theta_0$ are both congruent to the angle from the oriented real line to $\overrightarrow{K(0)K(1)}$ mod 2π. Therefore, $(lk(K, K_{\|}) - 2\theta_1) \in \mathbb{Q}$ when K goes from bottom to bottom and $(lk(K, K_{\|}) + 2\theta_1) \in \mathbb{Q}$ when K goes from top to top.

When $\overrightarrow{K(0)K(1)}$ directs and orients the real line, we can choose $\theta_0 = \theta_1 = 0$ and this definition coincides with Definition 12.10. When $\overrightarrow{K(1)K(0)}$ directs and orients the real line, we can choose $\theta_1 = \frac{1}{2} = -\theta_0$ so that $\hat{K}_{\|}$ is simply as in Figure 13.4 and the present definition is again consistent with Definition 12.10.

Lemma 13.5. *The self-linking number does not depend on the orientations of the components.*

PROOF: This is easy to see for closed components, and this is part of the definition for components that go from bottom to top or from top to bottom. When K goes from top to top, let K' stand for $(-K)$, and let $K'_{\|}$ be the parallel obtained from $K_{\|}$ by a rotation of angle π around K (and by reversing the orientation). Choose the corresponding angles θ'_0 and θ'_1 as $\theta'_0 = \theta_1 - \frac{1}{2}$ and $\theta'_1 = \theta_0 + \frac{1}{2}$. So $\hat{K}'_{\|,\theta'_0,\theta'_1}$ is isotopic to $(-\hat{K}_{\|,\theta_0,\theta_1})$ in the complement of $\hat{K}$. See Figure 13.5. $\square$

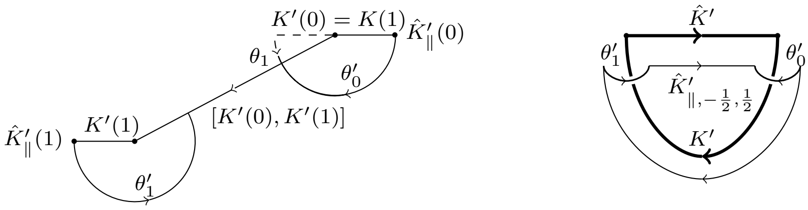


Figure 13.5: A general picture of θ'_0 and θ'_1 and a picture of $\hat{K}'_{\|}$ when $\theta'_0 = -\frac{1}{2}$ and $\theta'_1 = \frac{1}{2}$, when K goes from top to top

In Proposition 16.10, the real-valued self-linking numbers $lk(K, K_{\|})$ are proved to coincide with $I_\theta(K, \tau)$ for the interval components K of the straight tangles of Definition 16.9. The proof of Proposition 16.10 relies only on the beginning of Section 16.3.

13.2 Definition of $\mathcal{Z}^f$ for all q–tangles

Recall Definition 12.12 of the extension of $\mathcal{Z}^f$ for q–tangles whose bottom and top configurations are injective. In this section, we extend the definition of $\mathcal{Z}^f$ to all q–tangles.

In Chapter 15, we will prove the following particular case of the functoriality property stated in Theorem 12.13. It is a direct corollary of Proposition 15.18.

Proposition 13.6. *Let $(\mathcal{C}_1, L_1)$ and $(\mathcal{C}_2, L_2)$ be two framed tangle representatives such that the bottom of L_2 coincides with the top of L_1, if one of them is a braid γ, then*

$$\mathcal{Z}^f(\mathcal{C}_1\mathcal{C}_2, (L_1L_2)_{\|}) = \mathcal{Z}^f(\mathcal{C}_1, L_1, L_{1\|})\mathcal{Z}^f(\mathcal{C}_2, L_2, L_{2\|}).$$

The following lemma, which allows us to consider the bottom and top configurations of tangles up to translation and dilation, is also a direct corollary of Proposition 15.18.

Lemma 13.7. *Let $\gamma\colon [0,1] \to \check{C}_B[\mathring{D}_1]$, and let $p_{CS} \circ \gamma\colon [0,1] \to \check{\mathcal{S}}_B(\mathbb{C})$ be its composition by the natural projection $p_{CS}\colon \check{C}_B[\mathring{D}_1] \to \check{\mathcal{S}}_B(\mathbb{C})$ (which mods out by translations and dilations). Then $\mathcal{Z}(\gamma)$ and $\mathcal{Z}^f(\gamma)$ depend only on $p_{CS} \circ \gamma$.*

Under the assumptions of the lemma, we set $\mathcal{Z}(p_{CS} \circ \gamma) = \mathcal{Z}(\gamma)$ and $\mathcal{Z}^f(p_{CS} \circ \gamma) = \mathcal{Z}^f(\gamma)$. Recall that $\mathcal{Z}$ and $\mathcal{Z}^f$ coincide for braids. We extend the definition of $\mathcal{Z}$ to piecewise smooth paths of $\check{\mathcal{S}}_B(\mathbb{C})$ so that $\mathcal{Z}$ is multiplicative with respect to path composition of smooth paths. Thus the following theorem, which is essentially due to Poirier [Poi00], allows us to extend $\mathcal{Z}^f$ to q–tangles.

Theorem 13.8. *Let $p_{CS} \circ \gamma\colon [0,1] \to \mathcal{S}_B(\mathbb{C})$ be a path whose restriction to $]0,1[$ is the projection of some $\gamma\colon]0,1[\to \check{C}_B[\mathring{D}_1]$ that can be described by a collection of piecewise polynomial[2] continuous maps $(\gamma_b\colon [0,1] \to \mathring{D}_1)_{b \in B}$. Then $\lim_{\varepsilon \to 0} \mathcal{Z}(p_{CS} \circ \gamma_{|[\varepsilon, 1-\varepsilon]})$ makes sense, and it depends only on the homotopy class of $p_{CS} \circ \gamma$, relatively to its boundary. It is denoted by $\mathcal{Z}(p_{CS} \circ \gamma)$ or $\mathcal{Z}^f(p_{CS} \circ \gamma)$.*

Theorem 13.8 and Theorem 16.32, which generalizes its homotopy invariance part, will be proved in Section 16.5.

Proposition 13.9. *Any q–braid $\gamma\colon [0,1] \to \mathcal{S}_B(\mathbb{C})$ is homotopic to a q–braid $p_{CS} \circ \tilde{\gamma}$ as in the statement of Theorem 13.8, relatively to its boundary. Setting $\mathcal{Z}(\gamma) = \mathcal{Z}(p_{CS} \circ \tilde{\gamma})$ extends the definition of $\mathcal{Z}$ to all q–braids, consistently. Furthermore, $\mathcal{Z}$ is multiplicative with respect to the q–braid composition: For two composable paths γ_1 and γ_2 of $\mathcal{S}_B(\mathbb{C})$,*

$$\mathcal{Z}(\gamma_1 \gamma_2) = \mathcal{Z}(\gamma_1)\mathcal{Z}(\gamma_2).$$

PROOF: Let us first exhibit a q–braid $p_{CS} \circ \tilde{\gamma}$ homotopic to a given q–braid $\gamma\colon [0,1] \to \mathcal{S}_B(\mathbb{C})$, with the wanted properties. Define a path $\tilde{\gamma}_1\colon [0,1/3] \to C_B[\mathring{D}_1]$, such that $p_{CS} \circ \tilde{\gamma}_1(0) = \gamma(0)$, $\tilde{\gamma}_1(]0,1/3]) \subset \check{C}_B[\mathring{D}_1]$, and $\tilde{\gamma}_1$ is a path obtained by replacing all the parameters μ_A in the charts of Lemma 8.25 by εt for $t \in [0,1/3]$ for some small $\varepsilon > 0$, so $\tilde{\gamma}_1$ is described by a collection of polynomial maps $(\tilde{\gamma}_{1,b}\colon [0,1/3] \to \mathring{D}_1)_{b \in B}$. Similarly define a polynomial path $\tilde{\gamma}_3\colon [2/3,1] \to C_B[\mathring{D}_1]$ such that $p_{CS} \circ \tilde{\gamma}_3(1) = \gamma(1)$ and $\tilde{\gamma}_3([2/3,1[) \subset \check{C}_B[\mathring{D}_1]$. Define a path $\tilde{\gamma}_2'\colon [1/3,2/3] \to C_B[\mathring{D}_1]$, such that $\tilde{\gamma}_2'(1/3) = \tilde{\gamma}_1(1/3)$, $\tilde{\gamma}_2'(2/3) = \tilde{\gamma}_3(2/3)$ and $p_{CS} \circ \tilde{\gamma}_2'$ is a path composition $(p_{CS} \circ \overline{\tilde{\gamma}_1})\gamma(p_{CS} \circ \overline{\tilde{\gamma}_3})$. This path $\tilde{\gamma}_2'$ of $C_B[\mathring{D}_1]$ is homotopic to a path of the interior $\check{C}_B[\mathring{D}_1]$ of the manifold $C_B[D_1]$ with ridges, and it is homotopic to a polynomial path $\tilde{\gamma}_2$

[2]Every γ_b is polynomial over a finite number of intervals that covers $[0,1]$.

in $\check{C}_B[\mathring{D}_1]$. Now, the path composition $\tilde{\gamma} = \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3$ satisfies the hypotheses of Theorem 13.8, and $p_{CS} \circ \tilde{\gamma}$ is homotopic to γ, relatively to its boundary. Furthermore, for any other path $\tilde{\gamma}'$ that satisfies these properties, $p_{CS} \circ \tilde{\gamma}$ and $p_{CS} \circ \tilde{\gamma}'$ are homotopic relatively to the boundary, so the definition of $\mathcal{Z}(\gamma)$ is consistent.

In order to prove the multiplicativity, pick a piecewise polynomial path $\tilde{\gamma}_1\colon [0,1] \to C_B[\mathring{D}_1]$, such that γ_1 and $p_{CS} \circ \tilde{\gamma}_1$ are homotopic, relatively to the boundary, and $\tilde{\gamma}_1(]0,1[) \subset \check{C}_B[\mathring{D}_1]$, and pick a piecewise polynomial path $\tilde{\gamma}_2\colon [0,1] \to C_B[\mathring{D}_1]$, such that, for any $t \in [0,1/2]$, $\tilde{\gamma}_2(t) = \tilde{\gamma}_1(1-t)$, γ_2 and $p_{CS} \circ \tilde{\gamma}_2$ are homotopic, relatively to the boundary, and $\tilde{\gamma}_2(]0,1[) \subset \check{C}_B[\mathring{D}_1]$. Thus,

$$\mathcal{Z}(\gamma_1\gamma_2) = \lim_{\varepsilon\to 0} \mathcal{Z}\left(p_{CS} \circ \left(\tilde{\gamma}_{1|[\varepsilon,1/2]}\tilde{\gamma}_{2|[1/2,1-\varepsilon]}\right)\right),$$

while

$$\begin{aligned}\mathcal{Z}(\gamma_1)\mathcal{Z}(\gamma_2) &= \lim_{\varepsilon\to 0} \mathcal{Z}(p_{CS} \circ \tilde{\gamma}_{1|[\varepsilon,1-\varepsilon]})\mathcal{Z}(p_{CS} \circ \tilde{\gamma}_{2|[\varepsilon,1-\varepsilon]}) \\ &= \lim_{\varepsilon\to 0} \mathcal{Z}(p_{CS} \circ \tilde{\gamma}_{1|[\varepsilon,1-\varepsilon]} p_{CS} \circ \tilde{\gamma}_{2|[\varepsilon,1-\varepsilon]}) \\ &= \lim_{\varepsilon\to 0} \mathcal{Z}\left(p_{CS} \circ \left(\tilde{\gamma}_{1|[\varepsilon,1/2]}\tilde{\gamma}_{2|[1/2,1-\varepsilon]}\right)\right),\end{aligned}$$

where the second equality comes from Proposition 13.6. □

Definition 13.10. Proposition 13.6, Theorem 13.8 and Proposition 13.9 allow us to extend $\mathcal{Z}^f$ unambiguously to q–tangles

$$T(\gamma^-)(\mathcal{C}, L)T(\gamma^+),$$

for which γ^- and γ^+ are q-braids, $(\mathcal{C}, L)$ is a framed tangle, whose bottom configuration is $\gamma^-(1)$ and whose top configuration is $\gamma^+(0)$, and the strands of $T(\gamma^-)$ and $T(\gamma^+)$ get their orientations from the orientation of L, by setting

$$\mathcal{Z}^f(T(\gamma^-)(\mathcal{C}, L)T(\gamma^+)) = \mathcal{Z}(\gamma^-)\mathcal{Z}^f(\mathcal{C}, L)\mathcal{Z}(\gamma^+),$$

with respect to Definition 12.12 of $\mathcal{Z}^f(\mathcal{C}, L)$.

Remark 13.11. Let $(\mathcal{C}, L_q)$ be a q–tangle from a bottom limit configuration $c^- \in \mathcal{S}_{B^-}(\mathbb{C})$ to a top configuration $c^+ \in \mathcal{S}_{B^+}(\mathbb{C})$. These limit configurations are initial points $\gamma^\pm(0) = c^\pm$ of polynomial paths $\gamma^\pm$ of $\mathcal{S}_{B^\pm}(\mathbb{C})$, such that $\gamma^\pm(]0,1]) \subset \check{\mathcal{S}}_{B^\pm}(\mathbb{C})$ (as in the proof of Proposition 13.9). This allows us to regard $c^\pm$ as a limit of injective configurations, and to view L as a limit of framed tangles between injective configurations. Indeed, $c^\pm$ the limit of $\gamma^\pm(t)$ when t tends to 0, and L_q can be viewed as the limit of

$$L_{q,\varepsilon} = T(\overline{\gamma^-_{|[0,\varepsilon]}})L_qT(\gamma^+_{|[0,\varepsilon]})$$

when ε tends to 0. Then $\mathcal{Z}^f(\mathcal{C}, L_q)$ can be defined alternatively by

$$\mathcal{Z}^f(\mathcal{C}, L_q) = \lim_{\varepsilon \to 0} \mathcal{Z}^f(L_{q,\varepsilon}).$$

Indeed, the above consistent definition implies that

$$\mathcal{Z}^f(\mathcal{C}, L_q) = \mathcal{Z}(\gamma^-_{|[0,\varepsilon]})\mathcal{Z}^f(L_{q,\varepsilon})\mathcal{Z}(\overline{\gamma^+_{|[0,\varepsilon]}}),$$

while Theorem 13.8 and Proposition 13.9 imply that $\lim_{\varepsilon\to 0} \mathcal{Z}(\gamma^-_{|[0,\varepsilon]}) = \mathbf{1}$ and that $\lim_{\varepsilon\to 0} \mathcal{Z}(\overline{\gamma^+_{|[0,\varepsilon]}}) = \mathbf{1}$.

Variants of $\mathcal{Z}^f$ in the spirit of Theorem 7.39 will be constructed in Section 16.2. These variants will allow us to prove Theorem 13.12 in Chapter 17.

Proving Theorem 13.12 will require lengthy studies of compactifications of configuration spaces, which are not manifolds with boundaries. These studies are useful to get all the nice and natural properties of $\mathcal{Z}^f$, which are stated in Theorem 13.12. Another approach to obtain invariants of tangles and avoid our complicated configuration spaces was proposed by Koytcheff, Munson and Volić in [KMV13].

In the next three chapters, we show the details of the construction of $\mathcal{Z}^f$ for q-tangles, following the outline of this section.

13.3 Main properties of the functor$\mathcal{Z}^f$

In this section, we state the main properties of the functor $\mathcal{Z}^f = (\mathcal{Z}^f_n)_{n\in\mathbb{N}}$, whose construction is outlined in the previous section.

Theorem 13.12. *The invariant $\mathcal{Z}^f$ of q–tangles satisfies the following properties.*

- $\mathcal{Z}^f$ *coincides with the invariant $\mathcal{Z}^f$ of Section 7.6 for framed links in $\mathbb{Q}$-spheres.*
- $\mathcal{Z}^f$ *coincides with the Poirier functor Z^l of [Poi00] for combinatorial q–tangles in $\mathbb{R}^3$.*
- Naturality and component orientations: *If L is a q–tangle with source $\mathcal{L}$, then $\mathcal{Z}^f(L) = \left(\mathcal{Z}^f_k(L)\right)_{k\in\mathbb{N}}$ is valued in $\mathcal{A}(\mathcal{L}) = \mathcal{A}(\mathcal{L};\mathbb{R})$, and $\mathcal{Z}^f_0(L)$ is the class of the empty diagram. If L' is a subtangle of L with source $\mathcal{L}'$, then $\mathcal{Z}^f(L')$ is obtained from $\mathcal{Z}^f(L)$ by mapping all the diagrams with univalent vertices on $\mathcal{L}\setminus\mathcal{L}'$ to zero and by forgetting $\mathcal{L}\setminus\mathcal{L}'$.*

If $\mathcal{L}$ and the components of L are unoriented, then $\mathcal{Z}^f(L)$ is valued in the space $\mathcal{A}(\mathcal{L})$ of Definition 6.16, as in Proposition 10.21. Otherwise, component orientation reversals affect $\mathcal{Z}^f$ as in Proposition 10.21.

- Framing dependence: *For a q–tangle $L = (\mathcal{C}, \sqcup_{j=1}^k K_j, \sqcup_{j=1}^k K_{j\|})$,*

$$\prod_{j=1}^{k} \left(\exp\left(-lk(K_j, K_{j\|})\alpha\right) \sharp_j\right) \mathcal{Z}^f(L)$$

is independent of the framing of L, it is denoted by $\mathcal{Z}(\mathcal{C}, \sqcup_{j=1}^k K_j)$.

- Functoriality: *For two q-tangles L_1 and L_2 such that the bottom configuration of L_2 coincides with the top configuration of L_1,*

$$\mathcal{Z}^f(L_1 L_2) = \mathcal{Z}^f(L_1)\mathcal{Z}^f(L_2),$$

with products obtained by stacking above in natural ways on both sides.

- First duplication property: *Let K be a component of a q–tangle L, then*

$$\mathcal{Z}^f(L(2 \times K)) = \pi(2 \times K)^* \mathcal{Z}^f(L)$$

with respect to Notation 6.28.

- Second duplication property: *Let B be a finite set, let y be an element of $\mathcal{S}_B(\mathbb{C})$. Let K be a component that goes from bottom to top of a q–tangle L. Then*

$$\mathcal{Z}^f(L(y \times K)) = \pi(B \times K)^* \mathcal{Z}^f(L).$$

- Orientation: *Let $s_{\frac{1}{2}}$ be the orthogonal hyperplane symmetry that fixes the horizontal plane at height $\frac{1}{2}$. Extend $s_{\frac{1}{2}}$ from $\partial\mathcal{C}$ to an orientation reversing diffeomorphism $s_{\frac{1}{2}}$ of $\mathcal{C}$. For all $n \in \mathbb{N}$,*

$$\mathcal{Z}_n^f(s_{\frac{1}{2}}(\mathcal{C}), s_{\frac{1}{2}} \circ L) = (-1)^n \mathcal{Z}_n^f(\mathcal{C}, L),$$

where the parallels of interval components $s_{\frac{1}{2}}(K)$ of $s_{\frac{1}{2}}(L)$ are defined so that $lk_{s_{\frac{1}{2}}(\mathcal{C})}\left(s_{\frac{1}{2}}(K), s_{\frac{1}{2}}(K)_{\|}\right) = -lk_{\mathcal{C}}\left(K, K_{\|}\right)$.

- Symmetry: *Let ρ be a rotation of $\mathbb{R}^3$ that preserves the standard homology cylinder $D_1 \times [0,1]$ (setwise). Let L be a q–tangle of a rational homology cylinder $\mathcal{C}$. Extend ρ from $\partial\mathcal{C}$ to an orientation-preserving* *diffeomorphism* ρ *of* $\mathcal{C}$. *If the angle of* ρ *is different from* 0 *and* π, *assume that the interval components of* L *go from bottom to top or from top to bottom. Then*

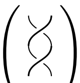
$$\mathcal{Z}^f(\rho(\mathcal{C}), \rho \circ L) = \mathcal{Z}^f(\mathcal{C}, L),$$

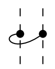
where the parallels of interval components $\rho(K)$ *of* $\rho(L)$ *are defined so that* $lk_{\rho(\mathcal{C})}\left(\rho(K), \rho(K)_{\parallel}\right) = lk_{\mathcal{C}}\left(K, K_{\parallel}\right)$. *(This applies when* ρ *is a diffeomorphism of* $\mathcal{C}$ *that restricts to* $\partial\mathcal{C}$ *as the identity map.)*

- Cabling property: *Let* B *be a finite set with cardinality greater than* 1. *Let* $y \in \mathcal{S}_B(\mathbb{C})$ *and let* $y \times [0,1]$ *denote the corresponding q-braid, and let* K *be a strand of* $y \times [0,1]$. *Let* L *be a q–tangle with source* $\mathcal{L}$. *Then* $\mathcal{Z}^f((y \times [0,1])\,(L/K))$ *is obtained from* $\mathcal{Z}^f(L)$ *by the natural injection from* $\mathcal{A}(\mathcal{L})$ *to* $\mathcal{A}\left(\sqcup_{b \in B}\mathbb{R}^{\{b\}}\left(\frac{\mathcal{L}}{K}\right)\right)$.

- *The expansion* $\mathcal{Z}^f_{\leq 1}$ *up to degree* 1 *of* $\mathcal{Z}^f$ *satisfies*

$$\mathcal{Z}^f_{\leq 1}\left(\quad \right) = 1 + \left[\quad \right],$$





where the endpoints of the tangle are supposed to lie on $\mathbb{R} \times \{0,1\}$.

- *For any integer* n, *if* L *is a q–tangle with source* $\mathcal{L}$, *then*

$$\Delta_n\left(\mathcal{Z}^f_n(L)\right) = \sum_{i=0}^{n} \mathcal{Z}^f_i(L) \otimes \mathcal{Z}^f_{n-i}(L)$$

with respect to the coproduct maps Δ_n *of Section 6.5.*

The definition of $\mathcal{Z}^f$ in Section 13.2 obviously extends the definition of $\mathcal{Z}^f$ for tangles with empty boundary. Note that the naturality property in Theorem 13.12 easily follows from the definition (which will be justified later). The behaviour of $Z(\mathcal{C}, L, \tau)$ with respect to the coproduct can be observed from the definition, as in the proof of Proposition 10.3. Since the correction factors are group-like according to Lemmas 6.34 and 6.35, Definition 10.5 and Proposition 10.13, the compatibility between the various products and the coproduct ensures that $\mathcal{Z}^f$ behaves as stated in Theorem 13.12, with respect to the coproduct, for framed tangles between injective configurations. Then Remark 13.11 ensures that this also holds for general q–tangles.

Corollary 13.13. *If L has at most one component, let p^c be the projection given by Corollary 6.37 from $\mathcal{A}(\mathcal{L})$ to the space $\mathcal{A}^c(\mathcal{L})$ of its primitive elements. Set*

$$\mathfrak{z}^f(\mathcal{C}, L) = p^c\left(\mathcal{Z}^f(\mathcal{C}, L)\right).$$

Then

$$\mathcal{Z}^f(\mathcal{C}, L) = \exp\left(\mathfrak{z}^f(\mathcal{C}, L)\right).$$

PROOF: In these cases, Lemma 6.34 guarantees that $\mathcal{A}(\mathcal{L})$ is a Hopf algebra and Theorem 13.12 implies that $\mathcal{Z}^f(\mathcal{C}, L)$ is group-like. Conclude with Theorem 6.38. $\square$

The proof of Theorem 13.12 will be finished at the end of Section 17.4. The multiplicativity of $\mathcal{Z}$ under connected sum of Theorem 10.24 is a direct corollary of the functoriality of $\mathcal{Z}^f$ in the above statement. The functoriality also implies that $\mathcal{Z}$ and $\mathcal{Z}^f$ map tangles consisting of vertical segments in the standard cylinder to 1. Consider such a trivial braid consisting of the two vertical segments $\{-\frac{1}{2}\} \times [0,1]$ and $\{\frac{1}{2}\} \times [0,1]$. Cable $\{-\frac{1}{2}\} \times [0,1]$ by a q–tangle $(\mathcal{C}_1, L_1)$ and cable $\{\frac{1}{2}\} \times [0,1]$ by a q–tangle $(\mathcal{C}_2, L_2)$. Call the resulting q–tangle $(\mathcal{C}_1 \otimes \mathcal{C}_2, L_1 \otimes L_2)$. (Formally, this tangle may be expressed as $\left((\{-\frac{1}{2}, \frac{1}{2}\} \times [0,1])\left(\frac{(\mathcal{C}_1, L_1)}{\{-\frac{1}{2}\}\times[0,1]}\right)\right)\left(\frac{(\mathcal{C}_2, L_2)}{\{\frac{1}{2}\}\times[0,1]}\right)$.)

Corollary 13.14. *The functor $\mathcal{Z}^f$ satisfies the following* monoidality property *with respect to the above structure.*

$$\mathcal{Z}^f(\mathcal{C}_1 \otimes \mathcal{C}_2, L_1 \otimes L_2) = \mathcal{Z}^f(\mathcal{C}_1, L_1) \otimes \mathcal{Z}^f(\mathcal{C}_2, L_2),$$

where the product $\otimes$ of the right-hand side is simply induced by the disjoint union of diagrams.

PROOF: This is a consequence of the cabling property and the functoriality in Theorem 13.12. $\square$

More generally, Theorem 13.12 implies that the Poirier functor Z^l is a good monoidal functor. The multiplicativity of $\mathcal{Z}$ under connected sum of Theorem 10.24 is also a consequence of Corollary 13.14.

The first duplication property may be iterated. Note that $\pi(r \times K)^*$ is nothing but the composition of $(r-1)$ $\pi(2 \times K)^*$. Also note that iterating duplications $\frac{(\bullet\bullet)}{\bullet}$ for configurations produces elements in the 0-dimensional strata of some $\mathcal{S}_B(\mathbb{R})$ discussed in Example 8.3. For example,

$\cup\,(2 \times \cup) =$  and  $\left(2 \times\right.$  $\left.\right) =$ .

So
$$\mathcal{Z}^f\left({}^{((\bullet\bullet)\bullet)\ (\bullet(\bullet\bullet))}\right) = \pi\left(3\times\cup\right)^* \mathcal{Z}^f\left(\cup\right).$$

The second duplication property together with the behaviour of $\mathcal{Z}^f$ under component orientation reversal yields a similar duplication property for a component K that goes from top to bottom. The behaviour of $\mathcal{Z}^f$ under orientation change, the functoriality and the duplication properties allow us to generalize the cabling property to cablings of components K that go from bottom to top or from top to bottom in arbitrary q–tangles, by arbitrary q–tangles. The cabling property may be similarly generalized to components K that go from top to top or from bottom to bottom, cabled by q–tangles in a rational homology cylinder, whose bottom or top configurations are combinatorial configurations (as in Example 8.3). In both cases, the insertion of the non-trivial part T_i can be performed near an end of K, so that the result is a vertical composition of a tangle obtained by cabling a strand in a trivial vertical braid with T_i, and a possibly iterated duplication of the tangle T_m, as in Figure 13.6.

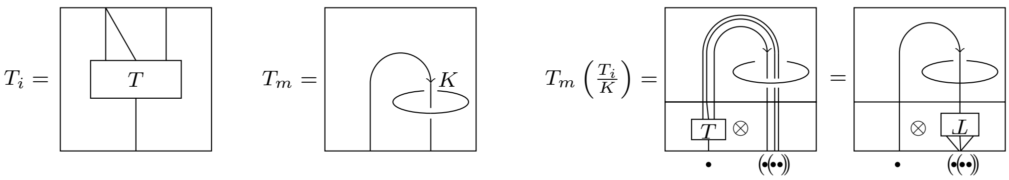


Figure 13.6: Cabling a component that goes from bottom to bottom, in two different ways

The behaviour of $\mathcal{Z}^f$ when a component K of a link $L : \mathcal{L} \longrightarrow \check{R}(\mathcal{C}_m)$ is cabled by a semi-pure q–tangle $(\mathcal{C}, T_i)$ can be described as follows.

- cut the source of K to replace it with a copy of $\mathbb{R}$ using Proposition 6.26,
- duplicate the corresponding strand and $\mathcal{Z}^f(R(\mathcal{C}_m), L)$, accordingly, as in the duplication property above,
- multiply the obtained element by $\mathcal{Z}^f(\mathcal{C}, T_i)$ that is concatenate the diagrams, naturally,
- finally, close the source of $L(T/K)$.

This follows easily from Theorem 13.12, by viewing (R, L) as a vertical composition of two tangles, where the bottom one is just a cup $\cup$, which is a trivial strand going from top to top in a standard cylinder. The process is illustrated in Figure 13.7. The possible conjugation of $\mathcal{Z}^f(\mathcal{C}, T_i)$ does not affect the result, thanks to Lemma 6.30.

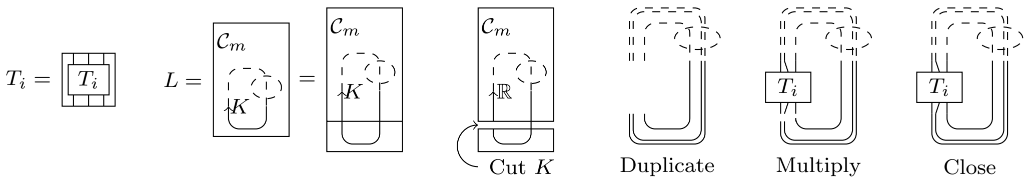


Figure 13.7: Cabling a link component with a semi-pure tangle, step by step

The first cutting step would not be legitimate if L had interval components. So this cabling property does not generalize to this case.

For a q-braid $\gamma\colon [0,1] \to \mathcal{S}_B(\mathbb{C})$, $\mathcal{Z}^f(\gamma) = \mathcal{Z}(\gamma)$ stands for $\mathcal{Z}^f(T(\gamma)) = \mathcal{Z}^f(D_1 \times [0,1], T(\gamma))$. The following proposition, which leads to interesting cablings, is a corollary of Theorem 13.12.

Proposition 13.15. *Let q be a positive integer. Let $\gamma_{1,q}$ be a braid represented by the map*

$$\begin{array}{ccc} [0,1] \times \underline{q} & \to & \mathbb{C} \\ (t,k) & \mapsto & \frac{1}{2}\exp\left(\frac{2i\pi(k+\chi(t))}{q}\right) \end{array}$$

for a surjective map $\chi\colon [0,1] \to [0,1]$ with nonnegative derivative, which is constant in neighborhoods of 0 *and* 1*, as in Figure 13.1 of $\gamma_{1,3}$. Then*

$$\mathcal{Z}^f(\gamma_{1,q}) = \exp\left(\pi\left(q \times \uparrow\right)^*(\frac{1}{q}\alpha)\right)\left(\exp(-\frac{1}{q}\alpha) \otimes \cdots \otimes \exp(-\frac{1}{q}\alpha)\right)$$

with the notation after Lemma 6.27, where $\alpha \in \check{\mathcal{A}}\left(\uparrow\right)$.

Proof: Let $\tilde{\gamma}_{1,1}$ be the trivial one-strand braid K in the standard cylinder equipped with its parallel $K_\parallel$ such that $lk(K, K_\parallel) = 1$. According to the framing dependence property in Theorem 13.12, $\mathcal{Z}^f(\tilde{\gamma}_{1,1}) = \exp(\alpha)$. Let $\tilde{\gamma}_{q,q}$ be the q–tangle obtained by cabling $\tilde{\gamma}_{1,1}$ as in the second duplication property, by replacing the one-point configuration with the planar configuration of $\mathbb{C}$ consisting of the q points $\frac{1}{2}\exp\left(\frac{2i\pi k}{q}\right)$, for $k \in \underline{q}$. This duplication operation equips each strand K_k of $\tilde{\gamma}_{q,q}$ with a parallel $K_{k\parallel,1}$ such that $lk(K_k, K_{k\parallel,1}) = 1$. The q–tangle $\tilde{\gamma}_{q,q}$ coincides with $\gamma_{1,q}^q$ except for the framing since the standard framing of $\gamma_{1,q}^q$ equips K_k with a parallel $K_{k\|}$ such that $lk(K_k, K_{k\|}) = 0$. According to the second duplication property,

$$\check{Z}^f(\tilde{\gamma}_{q,q}) = \exp\left(\pi\left(q \times \uparrow\right)^*(\alpha)\right),$$

whereas

$$\check{Z}^f(\gamma_{1,q}^q) = \check{Z}^f(\tilde{\gamma}_{q,q})\left(\exp(-\alpha) \otimes \cdots \otimes \exp(-\alpha)\right).$$

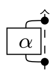
By the invariance of $\check{Z}^f$ under rotation, $\check{Z}^f(\gamma_{1,q})$ is invariant under cyclic permutation of the strands. So, by functoriality, $\check{Z}^f(\gamma_{1,q}^q) = \check{Z}^f(\gamma_{1,q})^q$. The result follows by unicity of a q^{th}-root of $\check{Z}^f(\gamma_{1,q}^q)$ with 1 as degree 0 part. $\square$

The Kontsevich integral of the trivial knot O has been computed by Dror Bar-Natan, Thang Lê and Dylan Thurston in [BNLT03]. Thus, Corollary 12.29 allows one to express $\check{Z}^f$ for the unknot and for the torus knots as a function of the anomaly α. Note that the symmetry properties imply that $\check{Z}^f(\cap)$ vanishes in odd degrees, and that $\check{Z}^f(\cap) = \check{Z}^f(\cup) = \sqrt{\check{Z}^f(O)}$, where we implicitly use the natural isomorphism of Proposition 6.22.

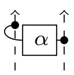
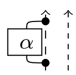
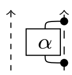
Lemma 13.16.

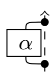
$$\check{Z}^f\left(\right.$$  $$\left.\right) = \exp\left(\Psi(2\alpha)\left(\right.\right.$$  $$\left.\left.\right)\right).$$

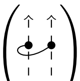
Proof: This lemma can be deduced from Theorem 12.28. Below, as an exercise, we alternatively deduce it from Proposition 13.15 and Theorem 13.12, assuming that α is a two-leg element of $\check{\mathcal{A}}\left(\uparrow\right)$, but without assuming Theorem 12.28. Since α is a two-leg element, we picture it as

$$\alpha =$$  .

So, using Lemma 12.26,

$$\pi\left(2 \times \uparrow\right)^*(\alpha) = 2$$  $+$  $+$  .

Since  can be slid along its interval,

$$\exp\left(\pi\left(2 \times \uparrow\right)^*(\alpha)\right) = \exp\left(\Psi(2\alpha)\left(\right.\right.$$  $$\left.\left.\right)\right)\left(\exp(\alpha) \otimes \exp(\alpha)\right).$$

$\square$

Chapter 14

Invariance of $\mathcal{Z}^f$ for long tangles

In this chapter, we study appropriate compactifications of the configuration space $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$ associated to a long tangle representative $L\colon \mathcal{L} \hookrightarrow R(\mathcal{C})$ and to a Jacobi diagram Γ with support $\mathcal{L}$. These compactifications $C_L = C(R(\mathcal{C}), L; \Gamma)$ and $C^f_L = C^f(R(\mathcal{C}), L; \Gamma)$ are introduced in Definition 14.15. They allow us to prove Theorem 12.2, which ensures that the integrals involved in the extension of $\mathcal{Z}^f$ to long tangles converge, in Section 14.2, and to prove Theorem 12.7, which ensures the topological invariance of this extension of $\mathcal{Z}^f$, in Section 14.3. Our compactifications are locally diffeomorphic to products of smooth manifolds by singular subspaces of $\mathbb{R}^n$ associated to trees. We study these singular subspaces in Section 14.1 and we show how Stokes' theorem applies for them, in this preliminary section, which is independent of the rest of the book. The local structure of our compactifications and their codimension-one faces are described in Theorem 14.16.

14.1 Singular models associated to trees

Definitions 14.1. In this book, an *oriented tree* is a tree $\mathcal{T}$ as in Figure 14.1 whose edges are oriented so that

- there is exactly one vertex without outgoing edges –This vertex $T(\mathcal{T})$ is called the *top* of $\mathcal{T}$, it is also simply denoted by T, when $\mathcal{T}$ is fixed.–
- the edges of $\mathcal{T}$ are oriented towards $T(\mathcal{T})$. In other words, for any vertex V of $\mathcal{T}$ different from $T(\mathcal{T})$, the orientation of the edges in the injective path $[V, T(\mathcal{T})]$, from V to $\mathcal{T}$, is induced by the orientation of $[V, T(\mathcal{T})]$.

Let $\mathcal{T}$ be such an oriented tree. A univalent vertex of $\mathcal{T}$ with one outgoing edge is called a *leaf* of $\mathcal{T}$. The set of leaves of $\mathcal{T}$ is denoted by $L(\mathcal{T})$. A *node* of $\mathcal{T}$ is a vertex with at least two ingoing edges. A *branch* of $\mathcal{T}$ is an oriented injective path of oriented edges that goes from a leaf ℓ to a node N or to the top T. Such a branch, which is denoted by $[\ell, N]$ or by $[\ell, T]$, is viewed as the subset of the set $E(\mathcal{T})$ of edges of $\mathcal{T}$ between ℓ and N, or between ℓ and T. The edge adjacent to a leaf ℓ is denoted by $\mathbf{e}(\ell)$. For any two vertices N_1, N_2 on the same branch $[\ell, T]$, such that N_2 is closer to T than N_1, $[N_1, N_2]$ is the set of edges between N_1 and N_2. These edges may contain N_1 or N_2 as an endpoint. $]N_1, N_2]$ (resp. $[N_1, N_2[$) denotes the set of edges of $[N_1, N_2]$ that do no contain N_1 (resp. N_2) as an endpoint. For example, the set $[\ell, N] \setminus \{\mathbf{e}(\ell)\}$ is denoted by $]\ell, N]$. Similarly, for any two edges e_1, e_2 on the same branch $[\ell, T]$, such that e_1 is closer to ℓ than T, $[e_1, e_2]$ is the set of edges between e_1 and e_2, including e_1 and e_2, and $]e_1, e_2]$ (resp. $[e_1, e_2[$) denotes the set $[e_1, e_2] \setminus \{e_1\}$ (resp. $[e_1, e_2] \setminus \{e_2\}$). We also mix edges and vertices in this notation. For example, $]e_1, T]$ is the set of edges between e_1 and T different from e_1 that may contain T. The edges are ordered naturally on such an interval of edges. The first one is the first one that we meet, when we follow the orientation of the interval induced by the tree orientation.

For two leaves ℓ_1 and ℓ_2 of $\mathcal{T}$, $N(\ell_1, \ell_2)$ denotes the node of $\mathcal{T}$ such that $[\ell_1, T] \cap [\ell_2, T] = [N(\ell_1, \ell_2), T]$. For a subset $\mathcal{E}$ of $E(\mathcal{T})$, $L(\mathcal{E}) = L(\mathcal{E}, \mathcal{T})$ denotes the set of leaves ℓ of $\mathcal{T}$ such that $[\ell, T]$ contains at least one edge of $\mathcal{E}$, and, for $\ell \in L(\mathcal{E})$, $\mathbf{e}(\mathcal{E}, \ell)$ is the closest edge to ℓ in $[\ell, T] \cap \mathcal{E}$.

Examples 14.2. In Figure 14.1, $L(\mathcal{T}) = \{\ell_0, \ell_1, \ldots, \ell_5\}$. When $\mathcal{E} = \{e_6, e_{13}\}$, $L(\mathcal{E}) = L(\mathcal{T})$, $\mathbf{e}(\mathcal{E}, \ell_0) = \mathbf{e}(\mathcal{E}, \ell_3) = \mathbf{e}(\mathcal{E}, \ell_4) = e_6$ and $\mathbf{e}(\mathcal{E}, \ell_5) = e_{13}$.

This section is devoted to the study of the following space $X(\mathcal{T})$ associated to an oriented tree $\mathcal{T}$.

Definition 14.3. For $\left((u_e)_{e \in E(\mathcal{T})}\right) \in [0, \infty[^{E(\mathcal{T})}$ and for a branch $[\ell, N]$ of $\mathcal{T}$, define

$$U\left([\ell, N]\right) = \prod_{e \in [\ell, N]} u_e.$$

Define $X(\mathcal{T})$ to be the set of the elements $\left((u_e)_{e \in E(\mathcal{T})}\right)$ of $[0, \infty[^{E(\mathcal{T})}$ such that, for any two leaves ℓ_1 and ℓ_2 of $\mathcal{T}$, the equality

$$*(\ell_1, \ell_2) \ : \ U\left([\ell_1, N(\ell_1, \ell_2)]\right) = U\left([\ell_2, N(\ell_1, \ell_2)]\right)$$

holds. Set $\mathring{X}(\mathcal{T}) = X(\mathcal{T}) \cap]0, \infty[^{E(\mathcal{T})}$.

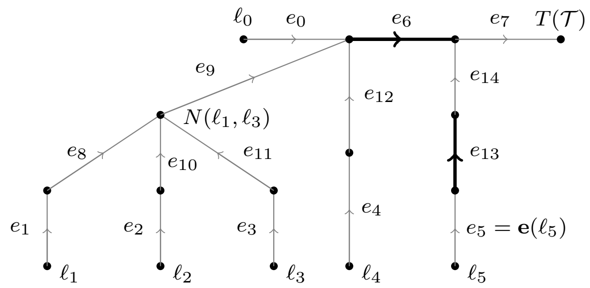


Figure 14.1: A tree $\mathcal{T}$ with a bold codimension-one system of edges

Remarks 14.4. Note that for any two branches $[\ell_1, N]$ and $[\ell_2, N]$ of $\mathcal{T}$ that end at the same vertex N, if $\left((u_e)_{e\in E(\mathcal{T})}\right) \in X(\mathcal{T})$, then $U([\ell_1, N]) = U([\ell_2, N])$.

Let ℓ_0 be a leaf of $\mathcal{T}$. Then $\mathring{X}(\mathcal{T})$ is the set of the elements $\left((u_e)_{e\in E(\mathcal{T})}\right)$ of $]0,\infty[^{E(\mathcal{T})}$, such that for any leaf ℓ of $L(\mathcal{T}) \setminus \{\ell_0\}$, $U([\ell, T]) = U([\ell_0, T])$. Indeed, when no variable is zero, the equations $*(\ell_1, \ell_2)$ of Definition 14.3 are equivalent to $U([\ell_1, T]) = U([\ell_2, T])$.

Definition 14.5. A *reducing system* of edges in an oriented tree $\mathcal{T}$ is a set $\mathcal{E}_r$ of edges such that $L(\mathcal{E}_r) = L(\mathcal{T}) \setminus \{\ell_0\}$ for some leaf ℓ_0, which is then denoted by $\ell_0(\mathcal{E}_r)$, and $\mathbf{e}(\mathcal{E}_r, .)$ is a bijection from $L(\mathcal{E}_r)$ to $\mathcal{E}_r$. A *maximal free system* of edges of $\mathcal{T}$ is the complement $E(\mathcal{T}) \setminus \mathcal{E}_r$ of a reducing system $\mathcal{E}_r$ of edges of $\mathcal{T}$ in $E(\mathcal{T})$.

Examples 14.6. For example, for every leaf ℓ_0 of $\mathcal{T}$, the set of edges adjacent to the leaves of $L(\mathcal{T}) \setminus \{\ell_0\}$ is a reducing system of edges. In Figure 14.1, $\{e_1, e_{10}, e_9, e_{12}, e_{13}\}$ is also a reducing system of edges.

Lemma 14.7. *For every reducing system $\mathcal{E}_r$ of edges of $\mathcal{T}$, for any edge e_1 of $\mathcal{E}_r$, there exists a unique pair (ℓ, ℓ') of leaves of $\mathcal{T}$ such that $e_1 = \mathbf{e}(\mathcal{E}_r, \ell)$, $e_1 \in [\ell, N(\ell, \ell')]$, and e_1 is the only element of $\mathcal{E}_r$ in $[\ell, N(\ell, \ell')] \cup [\ell', N(\ell, \ell')]$, so the equation $*(\ell, \ell')$, which may be written as*

$$u_{e_1} = \frac{U([\ell', N(\ell, \ell')])}{\prod_{e\in[\ell, N(\ell,\ell')]\setminus\{e_1\}} u_e}$$

when $x = \left((u_e)_{e\in E(\mathcal{T})}\right) \in \mathring{X}(\mathcal{T})$, *determines* u_{e_1} *in terms of the variables associated to* $E(\mathcal{T}) \setminus \mathcal{E}_r$ *in* $\mathring{X}(\mathcal{T})$, *and we have*

$$\frac{du_{e_1}}{u_{e_1}} = \sum_{e\in[\ell',N(\ell,\ell')]} \frac{du_e}{u_e} - \sum_{e\in[\ell,N(\ell,\ell')]\setminus\{e_1\}} \frac{du_e}{u_e}$$

in $\mathring{X}(\mathcal{T})$.

PROOF: If there is an edge of $\mathcal{E}_r$ after e_1 on $[\ell, T]$, let $e_2 = \mathbf{e}(\mathcal{E}_r, \ell')$ be the second edge of $\mathcal{E}_r$ on $[\ell, T]$. This defines ℓ'. Otherwise, set $\ell' = \ell_0(\mathcal{E}_r)$. See Figure 14.2. This proves the existence of ℓ' such that e_1 is the only element of $\mathcal{E}_r$ in $[\ell, N(\ell, \ell')] \cup [\ell', N(\ell, \ell')]$. For such an $\ell' \neq \ell_0(\mathcal{E}_r)$, $\mathbf{e}(\mathcal{E}_r, \ell')$ must be the second edge of $\mathcal{E}_r$ on $[\ell, T]$. Thus the condition of the statement determines ℓ'. $\square$

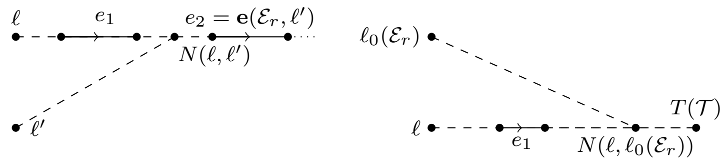


Figure 14.2: The two cases in Lemma 14.7

Lemma 14.8. *The set* $\mathring{X}(\mathcal{T})$ *is a smooth manifold of dimension*

$$d(\mathcal{T}) = \sharp E(\mathcal{T}) - \sharp L(\mathcal{T}) + 1.$$

For every maximal free system $\mathcal{E}_b$ *of edges of* $\mathcal{T}$, $\mathring{X}(\mathcal{T})$ *is freely parametrized by the (variables of) the edges of* $\mathcal{E}_b$. *If* $\mathcal{E}$ *is a subset of cardinality* $d(\mathcal{T})$, *the form* $\wedge_{e\in\mathcal{E}} du_e$ *is a non-vanishing volume form on* $\mathring{X}(\mathcal{T})$ *if and only if* $\mathcal{E}$ *is a maximal free system of edges of* $\mathcal{T}$.

PROOF: Let $\mathcal{E}_b$ be a maximal free system of edges of $\mathcal{T}$ and let $\mathcal{E}_r$ be its complement. An edge e of $\mathcal{E}_r$ may be expressed as $\mathbf{e}(\mathcal{E}_r, \ell_e)$ for a unique ℓ_e of $L(\mathcal{E}_r)$ and any associated variable u_e can be expressed in terms of the variables associated to $\mathcal{E}_b$, as in Lemma 14.7. Furthermore, any $\left((u_e)_{e\in E(\mathcal{T})}\right) \in]0, \infty[^{E(\mathcal{T})}$ such that

$$U([\ell_e, T]) = U([\ell_0(\mathcal{E}_r), T]),$$

for any e in $\mathcal{E}_r$, is in $\overset{\circ}{X}(\mathcal{T})$. The equation that defines u_{e_1} in Lemma 14.7, for $e_1 \in \mathcal{E}_r$, implies $*(\ell_{e_1}, \ell')$, where $\ell' = \ell_0(\mathcal{E}_r)$, if e_1 is the only edge of $\mathcal{E}_r$ in $[\ell_{e_1}, T]$. Otherwise, let e_1, e_2, ..., e_k denote the edges of $\mathcal{E}_r$ on $[\ell_{e_1}, T]$, where $e_i = \mathbf{e}(\mathcal{E}_r, \ell_{e_i})$ denotes the i^{th} edge that we meet from ℓ_{e_1} to T. Applying Lemma 14.7 to e_i defines u_{e_i} by an equation, which is equivalent to $*(\ell_{e_i}, \ell_{e_{i+1}})$ if $i < k$, and to $*(\ell_{e_k}, \ell_0(\mathcal{E}_r))$, if $i = k$. These equations together imply $*(\ell_{e_1}, \ell_0(\mathcal{E}_r))$, for any ℓ_{e_1} of $L(\mathcal{T})$.

Therefore, any $x = \Big((u_e)_{e \in E(\mathcal{T})}\Big) \in]0, \infty[^{E(\mathcal{T})}$, such that the equations that define the u_e for $e \in \mathcal{E}_r$ are satisfied, is in $\overset{\circ}{X}(\mathcal{T})$, which is therefore freely parametrized by the (variables of) the edges of $\mathcal{E}_b$, and $\overset{\circ}{X}(\mathcal{T})$ is a smooth submanifold of $\mathbb{R}^{E(\mathcal{T})}$ of dimension

$$d(\mathcal{T}) = \sharp E(\mathcal{T}) - \sharp L(\mathcal{T}) + 1.$$

Let $\mathcal{E}$ be a subset of cardinality $d(\mathcal{T})$, such that the form $\wedge_{e \in \mathcal{E}} du_e$ is a non-vanishing volume form on $\overset{\circ}{X}(\mathcal{T})$. Set $\mathcal{E}^c = E(\mathcal{T}) \setminus \mathcal{E}$. The map $\mathbf{e}(\mathcal{E}^c, .)\colon L(\mathcal{E}^c) \to \mathcal{E}^c$ is injective. Indeed, if $\mathbf{e}(\mathcal{E}^c, \ell_1) = \mathbf{e}(\mathcal{E}^c, \ell_2)$, for two leaves ℓ_1 and ℓ_2, then $[\ell_1, N(\ell_1, \ell_2)] \cup [\ell_2, N(\ell_1, \ell_2)] \subset \mathcal{E}$ and the relation $U([\ell_1, N(\ell_1, \ell_2)]) = U([\ell_2, N(\ell_1, \ell_2)])$ gives rise to a nontrivial relation between the forms du_e for $e \in [\ell_1, N(\ell_1, \ell_2)] \cup [\ell_2, N(\ell_1, \ell_2)]$. If $L \setminus L(\mathcal{E}^c)$ has two distinct elements ℓ_1 and ℓ_2, then we similarly have a non-trivial linear relation between coordinate forms du_e of $[\ell_1, N(\ell_1, \ell_2)] \cup [\ell_2, N(\ell_1, \ell_2)]$. Therefore the cardinality of $L(\mathcal{E}^c)$ is at least $\sharp L(\mathcal{T}) - 1$, which is the cardinality of $E(\mathcal{T}) \setminus \mathcal{E}$. Thus, $\mathbf{e}(\mathcal{E}^c, .)$ is a bijection. $\square$

Definition 14.9. A *codimension-one system* of edges in an oriented tree $\mathcal{T}$ is a set $\mathcal{E}$ of edges such that there is (exactly) one edge of $\mathcal{E}$ in any path from a leaf to the top of $\mathcal{T}$.

Examples 14.10. An example of a codimension-one system of edges of $\mathcal{T}$ is the set of edges that start at leaves. In Figure 14.1, $\mathcal{E} = \{e_6, e_{13}\}$, $\{e_7\}$ and $\{e_0, e_9, e_{12}, e_5\}$ are other codimension-one systems of edges of $\mathcal{T}$.

Lemma 14.11. *For any codimension-one system $\mathcal{E}_1$ of $X(\mathcal{T})$ and any edge $e_0 \in \mathcal{E}_1$, $\mathcal{E}_1 \setminus \{e_0\}$ can be completed to a reducing system that does not contain e_0.*

PROOF: For each element e of $\mathcal{E}_1$, choose a leaf ℓ_e such that $\mathbf{e}(\mathcal{E}_1, \ell_e) = e$. Let $L_1 = \{\ell_e, e \in \mathcal{E}_1\}$ be the set of these leaves, and let $L_2 = L(\mathcal{T}) \setminus L_1$. Let $\mathcal{E}_2 = \mathbf{e}(L_2)$ be the set of edges adjacent to the leaves of L_2. Then $\mathcal{E}_1 \cup \mathcal{E}_2 \setminus \{e_0\}$ is a reducing system of $\mathcal{T}$. $\square$

Lemma 14.12. *The closure of $\mathring{X}(\mathcal{T})$ in $[0,\infty[^{E(\mathcal{T})}$ is $X(\mathcal{T})$.*

Let $x = \left((u_e)_{e\in E(\mathcal{T})}\right)$ be an element of $X(\mathcal{T})\setminus\mathring{X}(\mathcal{T})$. Let $\mathcal{E}(x)$ denote the set of edges of $\mathcal{T}$ such that $u_e = 0$, let $\mathcal{E}_1$ be the image of $\mathbf{e}(\mathcal{E}(x), .)\colon L(\mathcal{E}(x)) \to \mathcal{E}(x)$, and let $X_{\mathcal{E}(x)}(\mathcal{T})$ be the set of elements x' of $X(\mathcal{T})$ such that $\mathcal{E}(x') = \mathcal{E}(x)$. Then $\mathcal{E}_1$ is a codimension-one system of edges of $\mathcal{T}$ and $X_{\mathcal{E}(x)}(\mathcal{T})$ is a smooth manifold of dimension $d(\mathcal{T}) - 1 - (\sharp\mathcal{E}(x) - \sharp\mathcal{E}_1)$.

Proof: Let $x = \left((u_e)_{e\in E(\mathcal{T})}\right) \in X(\mathcal{T}) \setminus \mathring{X}(\mathcal{T})$. Let us first prove that the image $\mathcal{E}_1$ of $\mathbf{e}(\mathcal{E}(x), .)\colon L(\mathcal{E}(x)) \to \mathcal{E}(x)$ is a codimension-one system of edges of $\mathcal{T}$. Since $\mathcal{E}(x) \neq \emptyset$, there is a leaf ℓ of $\mathcal{T}$ such that $U([\ell, T]) = 0$. This implies that $U([\ell, T]) = 0$ for all leaves. Thus, $L(\mathcal{E}(x)) = L(\mathcal{T})$. Furthermore no branch $[\ell, T]$ can contain more than one edge of $\mathcal{E}_1$. Otherwise, the first two edges of $\mathcal{E}_1$ on such a branch $[\ell_1, T]$ would be $\mathbf{e}(\mathcal{E}(x), \ell_1)$ and $\mathbf{e}(\mathcal{E}(x), \ell_2)$, respectively, and $\mathbf{e}(\mathcal{E}(x), \ell_1)$ would be the only edge of $\mathcal{E}(x)$ on $[\ell_1, N(\ell_1,\ell_2)] \cup [\ell_2, N(\ell_1,\ell_2)]$. Then we would have $U([\ell_1, N(\ell_1,\ell_2)]) = 0$ and $U([\ell_2, N(\ell_1,\ell_2)]) \neq 0$, and $*(\ell_1,\ell_2)$ would not be satisfied. Now, it suffices to prove the following two assertions.

- x is in the closure of $\mathring{X}(\mathcal{T})$ in $[0,\infty[^{E(\mathcal{T})}$, and
- $X_{\mathcal{E}(x)}(\mathcal{T})$ is a smooth manifold of dimension $d(\mathcal{T}) - 1 - (\sharp\mathcal{E}(x) - \sharp\mathcal{E}_1)$.

Let us first prove them, when $\mathbf{e}(\mathcal{E}(x), .)\colon L(\mathcal{T}) \to \mathcal{E}_1$ is injective. So $\mathbf{e}(\mathcal{E}(x), .)$ is bijection. Define $x(t) = \left((u_e(t))_{e\in E(\mathcal{T})}\right) \in \mathring{X}(\mathcal{T})$ from $x = \left((u_e)_{e\in E(\mathcal{T})}\right)$ for $t \in]0,\infty[$, as follows. Pick $\ell_0 \in L(\mathcal{T})$ and let $e_0 = \mathbf{e}(\mathcal{E}(x), \ell_0)$. Replace all the variables u_e for $e \in \mathcal{E}(x) \setminus \mathcal{E}_1$ with t, replace u_{e_0} with t^k for some positive integer k, and leave the variables associated to the edges of $E(\mathcal{T}) \setminus \mathcal{E}(x)$ (which are not zero) unchanged. For an edge $f = \mathbf{e}(\mathcal{E}_1, \ell_f)$ of $\mathcal{E}_1 \setminus \{e_0\}$, set

$$u_f(t) = \frac{U([\ell_0, T])(t)}{\prod_{e\in[\ell_f,T]\setminus\{f\}} u_e(t)}.$$

Then $u_f(t)$ is equal to $\alpha t^{k+r(f)}$ for some $\alpha > 0$ and some $r(f) \in \mathbb{Z}$. Choose k so that $k + r(f) \geq 1$ for any $f \in \mathcal{E}_1$. Then $x(t)$ tends to x when t tends to zero. Furthermore, since all $u_e(t)$ are nonzero, the defining equations for the $u_f(t)$ are equivalent[1] to the equations $U([\ell, T])(t) = U([\ell_0, T])(t)$, they are satisfied for any $\ell \in L(\mathcal{T})$, so $x(t) \in \mathring{X}(\mathcal{T})$. This proves that x is in the closure of $\mathring{X}(\mathcal{T})$.

[1] Recall that f is the only edge of $\mathcal{E}_1$ on $[\ell_f, T]$.

The defining equations of $X(\mathcal{T})$ are satisfied as soon as the u_e, for $e \in \mathcal{E}_1$, vanish (still under the assumption that $\mathbf{e}(\mathcal{E}(x),.)\colon L(\mathcal{E}(x)) \to \mathcal{E}(x)$ is bijective). Therefore, $X_{\mathcal{E}(x)}(\mathcal{T})$ is a manifold freely parametrized by the variables corresponding to the edges of $E(\mathcal{T}) \setminus \mathcal{E}(x)$. Its dimension is $\sharp E(\mathcal{T}) - \sharp\mathcal{E}(x) - \sharp L(\mathcal{T}) + \sharp\mathcal{E}_1 = d(\mathcal{T}) - 1 - (\sharp\mathcal{E}(x) - \sharp\mathcal{E}_1)$. The two assertions are proved when $\mathbf{e}(\mathcal{E}(x),.)\colon L(\mathcal{E}(x)) \to \mathcal{E}(x)$ is injective.

In general, for each element e of $\mathcal{E}_1$, choose a leaf ℓ_e such that $\mathbf{e}(\mathcal{E}(x),\ell_e) = e$. Let $L_1 = \{\ell_e, e \in \mathcal{E}_1\}$ be the set of these leaves, let $\mathcal{T}_1$ be the subtree of $\mathcal{T}$ such that $E(\mathcal{T}_1) = \cup_{e\in\mathcal{E}_1}[\ell_e, T]$ (so $L(\mathcal{T}_1) = L_1$), and let x_1 be the natural projection of x in $[0,\infty[^{E(\mathcal{T}_1)}$. Note that $\mathcal{E}(x) \subseteq E(\mathcal{T}_1)$, $\mathcal{E}(x) = \mathcal{E}(x_1)$, and the restriction of $\mathbf{e}(\mathcal{E}(x),.)$ to $L(\mathcal{T}_1)$ is the map $\mathbf{e}_{\mathcal{T}_1}(\mathcal{E}(x_1),.)$ associated to $\mathcal{T}_1$, which is injective.

In particular, the first part of the proof expresses x_1 as a limit at 0 of some continuous function $x_1(.)\colon]0,\infty[\to \mathring{X}(\mathcal{T}_1)$, such that $u_e(t)$ is constant for any edge e of $\mathcal{E}(\mathcal{T}_1) \setminus \mathcal{E}(x)$. Define $x(.)\colon]0,\infty[\to]0,\infty[^{E(\mathcal{T})}$, so that the variables $u_e(t)$ for $e \notin \mathcal{E}(x)$ are constant (and different from zero), and the variables $u_e(t)$ for $e \in \mathcal{E}(\mathcal{T}_1)$ are the same for $x(t)$ and $x_1(t)$. Let $L_2 = L(\mathcal{T}) \setminus L_1$. For $\ell_2 \in L_2$, $\mathbf{e}(\mathcal{E}(x),\ell_2) = \mathbf{e}(\mathcal{E}(x),\ell_1(\ell_2))$ for a unique $\ell_1(\ell_2)$ of L_1, the equation $*(\ell_2,\ell_1(\ell_2))$ between non-vanishing constant products holds for $x(t)$ for any $t \in]0,\infty[$, and it implies $U([\ell_2,T])(t) = U([\ell_1(\ell_2),T])(t) = U([\ell_0,T])(t)$, so $x(.)$ is valued in $\mathring{X}(\mathcal{T})$. Since its limit at 0 is x, x is in the closure of $\mathring{X}(\mathcal{T})$.

let $\mathcal{E}_2 = \mathbf{e}(L_2)$ be the set of edges adjacent to the leaves of L_2 and let $\mathcal{E}_3 = E(\mathcal{T}) \setminus (E(\mathcal{T}_1) \cup \mathcal{E}_2)$. Then any element x' of $X_{\mathcal{E}(x)}(\mathcal{T})$ is determined by its projection $x'_1 \in X_{\mathcal{E}(x)}(\mathcal{T}_1)$ and by the free nonzero variables associated to the edges of $\mathcal{E}_3$. More precisely, for an edge $e = \mathbf{e}(\ell_2 \in L_2)$, the equation $*(\ell_2,\ell_1(\ell_2))$ between non-vanishing products determines u'_e as a function of x'_1 and the free nonzero variables associated to the edges of $\mathcal{E}_3$. For elements $\left((u_e)_{e\in E(\mathcal{T})}\right) \in \{0\}^{\mathcal{E}(x)}\times]0,\infty[^{E(\mathcal{T})\setminus\mathcal{E}(x)}$, if the equations $*(\ell_2,\ell_1(\ell_2))$ are satisfied for all $\ell_2 \in L_2$, then all the equations $*(\ell_2,\ell')$ for $\ell_2 \in L_2$ and $\ell' \in L(\mathcal{T})$ are satisfied, as we prove below. Let $\ell_2 \in L_2$ and let $N(\ell_1(\ell_2))$ denote the closest node to $\mathbf{e}(\mathcal{E}(x),\ell_2) = \mathbf{e}(\mathcal{E}(x),\ell_1(\ell_2))$ in $[\ell_1(\ell_2),\mathbf{e}(\mathcal{E}(x),\ell_2)]$. See Figure 14.3. Then $*(\ell_2,\ell_1(\ell_2))$ is equivalent to $U([\ell_2,N(\ell_1(\ell_2))]) = U([\ell_1(\ell_2),N(\ell_1(\ell_2)]])$. In particular, if $\mathbf{e}(\mathcal{E}(x),\ell_2) = \mathbf{e}(\mathcal{E}(x),\ell')$, for $\ell' \in L(\mathcal{T})$, as in Figure 14.3, then $*(\ell',\ell_1(\ell_2))$ is equivalent to $U([\ell',N(\ell_1(\ell_2))]) = U([\ell_1(\ell_2),N(\ell_1(\ell_2)]])$. So $*(\ell_2,\ell_1(\ell_2))$ and $*(\ell',\ell_1(\ell_2))$ imply $*(\ell_2,\ell')$. If $\mathbf{e}(\mathcal{E}(x),\ell_2) \neq \mathbf{e}(\mathcal{E}(x),\ell')$, then $\mathbf{e}(\mathcal{E}(x),\ell_2) \in [\ell_2,N(\ell_2,\ell')]$ and $\mathbf{e}(\mathcal{E}(x),\ell') \in [\ell',N(\ell_2,\ell')]$, so $*(\ell_2,\ell')$ is equivalent to $0=0$, and is satisfied. Therefore $X_{\mathcal{E}(x)}(\mathcal{T})$ is a smooth manifold whose dimension is $\sharp\mathcal{E}_3 + \sharp E(\mathcal{T}_1) - \sharp\mathcal{E}(x) - \sharp L(\mathcal{T}_1) + \sharp\mathcal{E}_1 = \sharp E(\mathcal{T}) - \sharp L_2 - \sharp\mathcal{E}(x) - \sharp L(\mathcal{T}_1) + \sharp\mathcal{E}_1$.

$\square$

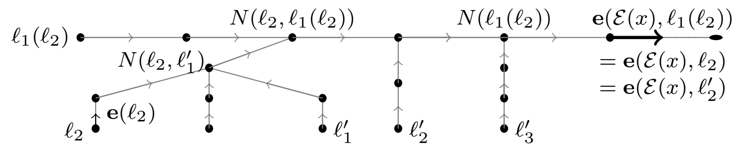


Figure 14.3: Example of leaves ℓ'_i such that $\mathbf{e}(\mathcal{E}(x),\ell'_i)=\mathbf{e}(\mathcal{E}(x),\ell_2)$ for the proof of Lemma 14.12

Lemma 14.13. *The codimension-one faces of $X(\mathcal{T})$ are in one-to-one correspondence with the codimension-one systems of edges of $\mathcal{T}$. In a neighborhood of such an open face, $X(\mathcal{T})$ has the structure of a smooth manifold with boundary.*

PROOF: According to Lemma 14.12 above, if $X_{\mathcal{E}}(\mathcal{T})$ is a non-empty manifold of dimension $d(\mathcal{T})-1$, then $\mathcal{E}$ is a codimension-one system. Let $\mathcal{E}_c$ be a codimension-one system of edges of $\mathcal{T}$. For any edge of $e_0 \in \mathcal{E}_c$, $\mathcal{E}_c \setminus \{e_0\}$ can be completed to a reducing system $\mathcal{E}_r$ that does not contain e_0, as in Lemma 14.11. In particular, $\mathring{X}(\mathcal{T})$ is freely parametrized by the variables associated to the edges of $E(\mathcal{T}) \setminus \mathcal{E}_r$. When all these variables are nonzero, except maybe ($u_{e_0} \in [0,\varepsilon[$), we get a local parametrization, near the locus $u_{e_0}=0$, by $[0,\varepsilon[\times]0,\varepsilon[^{E(\mathcal{T})\setminus(\mathcal{E}_r\cup\{e_0\})}$, where all the variables u_e, such that $e \in \mathcal{E}_c \setminus \{e_0\}$ and $e=\mathbf{e}(\mathcal{E}_r,\ell_e)$, may be expressed as

$$u_e = \frac{\prod_{f\in[\ell_0(\mathcal{E}_r),N(\ell_0(\mathcal{E}_r),\ell_e)]\setminus\{e_0\}} u_f}{\prod_{f\in[\ell_e,N(\ell_0(\mathcal{E}_r),\ell_e)]\setminus\{e\}} u_f} u_{e_0}.$$

$\square$

Lemma 14.14. *Let $\varepsilon \in]0,\infty[$. Let $X^{\varepsilon}(\mathcal{T}) = X(\mathcal{T}) \cap [0,\varepsilon]^{E(\mathcal{T})}$. Assume that $X(\mathcal{T})$ is oriented. For any smooth form Ω on $[0,\varepsilon]^{E(\mathcal{T})}\times[0,1]^n$ of degree $(d(\mathcal{T})+n)$, the integral $\int_{X^{\varepsilon}(\mathcal{T})\times[0,1]^n}\Omega$ of Ω along the interior of $X^{\varepsilon}(\mathcal{T})\times[0,1]^n$ is absolutely convergent. Let ω be a smooth form on $[0,\varepsilon]^{E(\mathcal{T})}\times[0,1]^n$ of degree $(d(\mathcal{T})-1+n)$. Then the integral $\int_{\partial(X^{\varepsilon}(\mathcal{T})\times[0,1]^n)}\omega$ of ω along the interiors of the codimension-one faces of $X^{\varepsilon}(\mathcal{T})\times[0,1]^n$ is absolutely convergent. Furthermore, Stokes' theorem applies to this setting. So we have*

$$\int_{\partial(X^{\varepsilon}(\mathcal{T})\times[0,1]^n)}\omega = \int_{X^{\varepsilon}(\mathcal{T})\times[0,1]^n} d\omega,$$

where the codimension-one faces of $X(\mathcal{T})$ are described in Lemma 14.13.

Proof: For an ordered subset $\mathcal{E}$ of $E(\mathcal{T})$, set $\Omega_{\mathcal{E}} = \wedge_{e\in\mathcal{E}} du_e$, where the factors are ordered with respect to the order of $\mathcal{E}$. Any smooth form Ω on $[0,\varepsilon]^{E(\mathcal{T})} \times [0,1]^n$ of degree $(d(\mathcal{T})+n)$ is a sum of forms $g_{\mathcal{E}}\Omega_{\mathcal{E}} \wedge (\wedge_{i=1}^n dx_i)$, for ordered subsets $\mathcal{E}$ of $E(\mathcal{T})$ of cardinality $d(\mathcal{T})$, and for smooth maps $g_{\mathcal{E}}\colon [0,\varepsilon]^{E(\mathcal{T})} \times [0,1]^n \to \mathbb{R}$, up to forms that vanish identically on the interior of $X^\varepsilon(\mathcal{T}) \times [0,1]^n$. These forms are bounded on $[0,\varepsilon]^{E(\mathcal{T})} \times [0,1]^n$. They are zero on the interior of $X^\varepsilon(\mathcal{T}) \times [0,1]^n$, unless $\mathcal{E}$ is a maximal free system, according to Lemma 14.8. When $\mathcal{E}$ is a maximal free system, Lemma 14.8 ensures that $\mathring{X}^\varepsilon(\mathcal{T})$ is freely parametrized by the variables associated to the edges of $\mathcal{E}$, so the integral of Ω along the interior of $X^\varepsilon(\mathcal{T}) \times [0,1]^n$ is absolutely convergent.

Let $\mathcal{E}_1$ be an ordered subset of $E(\mathcal{T})$ of cardinality $d(\mathcal{T})$, and let g be a smooth function on $[0,\varepsilon]^{E(\mathcal{T})} \times [0,1]^n$. Set $\omega = g\Omega_{\mathcal{E}_1} \wedge (\wedge_{i=2}^n dx_i)$. Then

$$\begin{aligned}\int_{X^\varepsilon(\mathcal{T})\times[0,1]^n} d\omega &= (-1)^{d(\mathcal{T})}\int_{\mathring{X}^\varepsilon(\mathcal{T})\times[0,1]^n} \tfrac{\partial g}{\partial x_1}\Omega_{\mathcal{E}_1} \wedge (\wedge_{i=1}^n dx_i)\\ &= (-1)^{d(\mathcal{T})}\int_{\mathring{X}^\varepsilon(\mathcal{T})\times\partial[0,1]\times[0,1]^{n-1}} \omega\\ &= \int_{\partial(X^\varepsilon(\mathcal{T})\times[0,1]^n)} \omega,\end{aligned}$$

since $\mathring{X}^\varepsilon(\mathcal{T}) \times \partial[0,1] \times [0,1]^{n-1}$ is the only part of $\partial\left(X^\varepsilon(\mathcal{T}) \times [0,1]^n\right)$ where the integral of ω does not vanish.

Let us now consider a form $\omega = g\Omega_{\mathcal{E}_2} \wedge (\wedge_{i=1}^n dx_i)$, for an ordered subset $\mathcal{E}_2$ of $E(\mathcal{T})$ of cardinality $d(\mathcal{T})-1$, and assume that the du_z, for $z \in \mathcal{E}_2$, are linearly independent in the $C^\infty(\mathring{X}^\varepsilon(\mathcal{T});\mathbb{R})$-module $\Omega^1(\mathring{X}^\varepsilon(\mathcal{T}))$ (otherwise both sides of the equality to be proved are zero). The integral of

$$d\omega = \sum_{h\in E(\mathcal{T})} \frac{\partial g}{\partial u_h} du_h \wedge \Omega_{\mathcal{E}_2} \wedge (\wedge_{i=1}^n dx_i)$$

is absolutely convergent over the interior of $X^\varepsilon \times [0,1]^n$.

Let $\mathcal{E}_3$ be the set of edges f such that $du_f \wedge \Omega_{\mathcal{E}_2}$ is not zero on $\mathring{X}^\varepsilon(\mathcal{T})$. According to Lemma 14.8, an edge f is in $\mathcal{E}_3$ if and only if $\mathcal{E}_2 \cup \{f\}$ is a maximal free system of edges of $\mathcal{T}$. Let $p_{\mathcal{E}_2}\colon X^\varepsilon(\mathcal{T}) \to [0,\varepsilon]^{\mathcal{E}_2}$ denote the composition of the inclusion $X^\varepsilon(\mathcal{T}) \hookrightarrow [0,\varepsilon]^{E(\mathcal{T})}$ with the natural projection.

We will prove that there exists $f_2 \in \mathcal{E}_3$, and piecewise smooth functions a and b from $]0,\varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}\left(X^\varepsilon(\mathcal{T})\right)$ to $[0,\varepsilon]$, such that $X^\varepsilon(\mathcal{T}) \cap p_{\mathcal{E}_2}^{-1}\left(]0,\varepsilon]^{\mathcal{E}_2}\right)$ is the set

$$\{x_{f_2}(u_{f_2},U) \mid U \in]0,\varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}\left(X^\varepsilon(\mathcal{T})\right), u_{f_2} \in [a(U),b(U)]\},$$

where $x_{f_2}(u_{f_2},U) \in X^\varepsilon(\mathcal{T})$, and its image under the composition of the inclusion $X^\varepsilon(\mathcal{T}) \hookrightarrow [0,\varepsilon]^{E(\mathcal{T})}$ with the natural projection to $[0,\varepsilon]^{\{f_2\}\cup\mathcal{E}_2}$ is

(u_{f_2}, U), which determines $x_{f_2}(u_{f_2}, U)$, since $\mathcal{E}_2 \cup \{f_2\}$ is a maximal free system of edges of $\mathcal{T}$. Without loss of generality, we will assume that $du_{f_2} \wedge \omega_{\mathcal{E}_2}$ orients $X^\varepsilon(\mathcal{T})$. We will prove that the boundary of $X^\varepsilon(\mathcal{T})$ consists of

- $(d(\mathcal{T})-1)$-dimensional strata where some u_z vanishes, for $z \in \mathcal{E}_2$, along which the integral of ω vanishes,
- (negligible) strata of dimension less than $(d(\mathcal{T})-1)$, and,
- $(d(\mathcal{T})-1)$-dimensional strata of

  $$\partial_b(X^\varepsilon(\mathcal{T})) = \{x_{f_2}(b(U), U) \mid U \in]0, \varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}(X^\varepsilon(\mathcal{T}))\},$$

  and

  $$-\partial_a(X^\varepsilon(\mathcal{T})) = -\{x_{f_2}(a(U), U) \mid U \in]0, \varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}(X^\varepsilon(\mathcal{T}))\},$$

  which behave as standard codimension-one faces of $X^\varepsilon(\mathcal{T})$,

with respect to some natural stratification. Thus

$$\int_{\partial(X^\varepsilon(\mathcal{T})\times[0,1]^n)} \omega = \int_{\partial_b(X^\varepsilon(\mathcal{T}))\times[0,1]^n} \omega - \int_{\partial_a(X^\varepsilon(\mathcal{T}))\times[0,1]^n} \omega.$$

On the other hand,

$$\int_{X^\varepsilon(\mathcal{T})\times[0,1]^n} d\omega = \int_{[0,1]^n} \left(\int_{\mathring{X}^\varepsilon(\mathcal{T})} \frac{\partial g_X}{\partial u_{f_2}} du_{f_2} \wedge \Omega_{\mathcal{E}_2} \right) \wedge_{i=1}^n dx_i,$$

where g_X is the restriction of g to $X^\varepsilon(\mathcal{T}) \times [0,1]^n$, where g_X and the u_h are functions of u_{f_2} and of the u_z, for $z \in \mathcal{E}_2$.

$$\begin{aligned} \int_{\mathring{X}^\varepsilon(\mathcal{T})} \frac{\partial g_X}{\partial u_{f_2}} du_{f_2} \wedge \Omega_{\mathcal{E}_2} &= \int_{U \in]0,\varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}(X^\varepsilon(\mathcal{T}))} \left(\int_{u_{f_2} \in [a(U), b(U)]} \frac{\partial g_X}{\partial u_{f_2}} du_{f_2} \right) \Omega_{\mathcal{E}_2} \\ &= \int_{U \in]0,\varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}(X^\varepsilon(\mathcal{T}))} \left(g_X(b(U)) - g_X(a(U)) \right) \Omega_{\mathcal{E}_2}. \end{aligned}$$

Therefore, the proof will be finished when we have constructed f_2, a and b with the above properties, which we do now.

Let $\mathcal{E}_2^c = E(\mathcal{T}) \setminus \mathcal{E}_2$. The cardinality of $\mathcal{E}_2^c$ is $\sharp L(\mathcal{T})$. The set $L(\mathcal{T}) \setminus L(\mathcal{E}_2^c)$ cannot contain two distinct leaves ℓ_3 and ℓ_4, because Equation $*(\ell_3, \ell_4)$ would imply that $\Omega_{\mathcal{E}_2} = 0$. Similarly, $\mathbf{e}(\mathcal{E}_2^c, \ell)$ cannot map two distinct leaves ℓ_3 and ℓ_4 of $L(\mathcal{E}_2^c)$ to the same element. Therefore, the cardinality of the image of $\mathbf{e}(\mathcal{E}_2^c, \ell)$ is at least $\sharp L(\mathcal{T}) - 1$, and there are two possible cases. Either $L(\mathcal{E}_2^c) = L(\mathcal{T})$ and $\mathbf{e}(\mathcal{E}_2^c, \ell)$ is a bijection, or $L(\mathcal{E}_2^c) = L(\mathcal{T}) \setminus \{\ell_0\}$ for a unique $\ell_0 \in L(\mathcal{T})$.

Let us first study the second case and assume that $L(\mathcal{E}_2^c) \neq L(\mathcal{T})$. Then the map $\mathbf{e}(\mathcal{E}_2^c, .)$ is a bijection from $L(\mathcal{E}_2^c)$ to $\mathcal{E}_2^c \setminus \{f\}$ for a unique element f of $\mathcal{E}_2^c$. There is a leaf ℓ_1 such that $f \in [\ell_1, T]$. Let f_2 be the last edge (the closest to f) of $\mathcal{E}_2^c$ in $[\ell_1, f[$, then $f_2 = \mathbf{e}(\mathcal{E}_2^c, \ell_2)$. If there is an edge of $\mathcal{E}_2^c$ in $]f, T]$, then let $\mathbf{e}(\mathcal{E}_2^c, \ell_3)$ be the first edge of $\mathcal{E}_2^c$ in $]f, T]$ as in Figure 14.4. Otherwise, set $\ell_3 = \ell_0$. Note that f and f_2 are in $\mathcal{E}_3$.

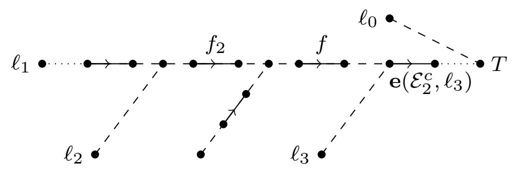


Figure 14.4: The edge f, when $L(\mathcal{E}_2^c) \neq L(\mathcal{T})$, in the proof of Lemma 14.14

The product $u_{f_2}u_f$ is given by the expression

$$u_{f_2}u_f = \frac{U\left([\ell_3, N(\ell_2, \ell_3)]\right)}{\prod_{z \in [\ell_2, N(\ell_2, \ell_3)] \setminus \{f, f_2\}} u_z} = \eta_2(U),$$

in $U \in]0, \varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}\left(X^\varepsilon(\mathcal{T})\right)$. In particular, $u_f = \frac{\eta_2(U)}{u_{f_2}}$ and

$$\frac{\eta_2(U)}{\varepsilon} \leq u_{f_2} \leq \varepsilon.$$

Let e be an edge of $\mathcal{E}_2^c$ different from f. Then e may be expressed as $e = \mathbf{e}(\mathcal{E}_2^c \setminus \{f\}, \ell_e)$ for a unique ℓ_e. As in Lemma 14.7, there is a leaf ℓ' of $\mathcal{T}$ such that e is the only element of $\mathcal{E}_2^c \setminus \{f\}$ in $[\ell_e, N(\ell_e, \ell')] \cup [\ell', N(\ell_e, \ell')]$. In particular, f_2 does not belong to $[\ell_e, N(\ell_e, \ell')] \cup [\ell', N(\ell_e, \ell')]$. Note that f cannot be on $[\ell', N(\ell_e, \ell')]$. If $f \notin [\ell_e, N(\ell_e, \ell')]$, then u_e is a function of $U \in]0, \varepsilon]^{\mathcal{E}_2}$, $e \notin \mathcal{E}_3$, and u_e is different from zero. If $f \in [\ell_e, N(\ell_e, \ell')]$, then $N(\ell_e, \ell_2)$ is on $]f_2, f[$ (equivalently, $f_2 \in [\ell_2, N(\ell_e, \ell_2)]$ and $f \notin [\ell_2, N(\ell_e, \ell_2)]$), as in the left part of Figure 14.5, $e \in \mathcal{E}_3$, f_2 is the unique element of $\mathcal{E}_2^c$ in $[\ell_2, N(\ell_e, \ell_2)]$, and $u_e = \frac{U([\ell_2, N(\ell_e, \ell_2)])}{\prod_{z \in [\ell_e, N(\ell_e, \ell_2)] \setminus \{e\}} u_z}$, so, when $U \in]0, \varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}\left(X^\varepsilon(\mathcal{T})\right)$ is fixed, u_e is a linear function $u_e = \eta_{e,f_2}(U)u_{f_2}$ of u_{f_2}. In particular, $u_{f_2} \leq \frac{\varepsilon}{\eta_{e,f_2}(U)}$. Set $a(U) = \frac{\eta_2(U)}{\varepsilon}$ and $b(U) = \min_{e \in \mathcal{E}_3 \setminus \{f\}} \frac{\varepsilon}{\eta_{e,f_2}(U)}$, where $\eta_{f_2,f_2} = 1$. Since $U \in p_{\mathcal{E}_2}\left(X^\varepsilon(\mathcal{T})\right)$, $a(U) \leq b(U)$. This proves that $X^\varepsilon(\mathcal{T}) \cap p_{\mathcal{E}_2}^{-1}\left(]0, \varepsilon]^{\mathcal{E}_2}\right)$ is the set

$$\{x_{f_2}(u_{f_2}, U) \mid U \in]0, \varepsilon]^{\mathcal{E}_2} \cap p_{\mathcal{E}_2}\left(X^\varepsilon(\mathcal{T})\right), u_{f_2} \in [a(U), b(U)]\}.$$

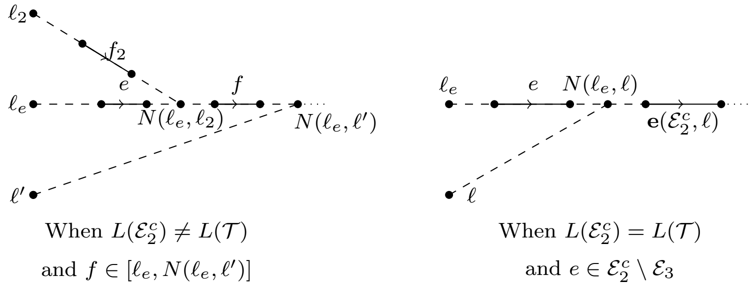


Figure 14.5: More figures for the proof of Lemma 14.14

The boundary part $\partial_a(X^\varepsilon(\mathcal{T}))$, corresponds to $u_f = \varepsilon$, and the boundary part $\partial_b(X^\varepsilon(\mathcal{T}))$, along which $u_e = \varepsilon$, for some $e \in \mathcal{E}_3$, lies in the intersection of the non-singular $\mathring{X}$ with loci $(u_e = \varepsilon)$, for some $e \in \mathcal{E}_3$. The locus $(u_e = \varepsilon)$ is transverse to $\mathring{X}$, for any $e \in \mathcal{E}_3$, since $\mathcal{E}_2 \cup \{e\}$ is a maximal free system, for such an e. So $\mathring{X} \cap (u_e = \varepsilon)$ is a manifold of dimension $(d(\mathcal{T})-1)$ for all $e \in \mathcal{E}_3$. Let e and e' be two distinct edges of $\mathcal{E}_3$. Then $u_e = u_{e'} = \varepsilon$ implies a non-trivial equation among variables u_h associated to edges h of $\mathcal{E}_3$, unless e and e' are both adjacent to leaves and meet at a node, which implies that $u_e = u_{e'}$ on $\mathring{X}$. We may assume that this configuration never occurs, without loss of generality, because the space $X(\mathcal{T})$ is canonically diffeomorphic to the space $X(\mathcal{T}')$ obtained from $\mathcal{T}$ by erasing e'. Except for this special configuration, the loci $\mathring{X} \cap (u_e = u_{e'} = \varepsilon)$ are of dimension less than $(d(\mathcal{T})-1)$, and we have a stratification of $\partial_a(X^\varepsilon(\mathcal{T}))$ and $\partial_b(X^\varepsilon(\mathcal{T}))$, where the $(d(\mathcal{T})-1)$-dimensional strata are the loci of $\partial_a(X^\varepsilon(\mathcal{T}))$ and $\partial_b(X^\varepsilon(\mathcal{T}))$ where $u_e = \varepsilon$ for (exactly) one e of $\mathcal{E}_3$. These strata are smooth open $(d(\mathcal{T})-1)$-manifolds. Therefore, we have the announced Stokes formula for the form $\omega = g\Omega_{\mathcal{E}_2} \wedge (\wedge_{i=1}^n dx_i)$, when $L(\mathcal{E}_2^c) \neq L(\mathcal{T})$.

Let us now assume that $L(\mathcal{E}_2^c) = L(\mathcal{T})$, and recall that $\mathbf{e}(\mathcal{E}_2^c, .)$ is a bijection from $L(\mathcal{T})$ to $\mathcal{E}_2^c$, in that case. The elements of $\mathcal{E}_3$ are the edges e of $\mathcal{E}_2^c$ such that there is no edge of $\mathcal{E}_2^c$ on $]e, T]$. In particular, $\mathcal{E}_3$ is a codimension-one system of edges of $\mathcal{T}$.

Let $e = \mathbf{e}(\mathcal{E}_2^c, \ell_e)$ be an element of $\mathcal{E}_2^c \setminus \mathcal{E}_3$, let ℓ be such that the first edge of $\mathcal{E}_2^c$ on $]e, T]$ is $\mathbf{e}(\mathcal{E}_2^c, \ell)$. Then e is the only element of $\mathcal{E}_2^c$ in $[\ell_e, N(\ell, \ell_e)] \cup [\ell, N(\ell, \ell_e)]$, as in Figure 14.5. So u_e depends only on the fixed variables of $\mathcal{E}_2$ and it is not zero.

If $f \in \mathcal{E}_3$, let $\ell(f)$ denote the unique leaf such that $L(\mathcal{E}_2^c \setminus \{f\}) = L(\mathcal{T}) \setminus \{\ell(f)\}$. The variable u_f is a linear function $u_f = \eta_f(U) u_{f_2}$ of u_{f_2} for one (arbitrary) $f_2 \in \mathcal{E}_3$,

$$\eta_f(U) = \frac{\prod_{z\in[\ell(f_2),T]\setminus\{f_2\}} u_z}{\prod_{z\in[\ell(f),T]\setminus\{f\}} u_z}.$$

Here, $a(U) = 0$ and $b(U) = \min_{f\in\mathcal{E}_3}(\frac{\varepsilon}{\eta_f})$. Thus, $\partial_a(X^\varepsilon(\mathcal{T}))$ is the codimension-one face associated to $\mathcal{E}_3$, as in Lemma 14.13, while $\partial_b(X^\varepsilon(\mathcal{T}))$ may be stratified as in the previous case. So we have the announced Stokes formula, for all forms as in the statement. $\square$

14.2 Configuration spaces of graphs on long tangles

An ∞-*component* of an LTR L is a connected component of the intersection of the image of L with $R(\mathcal{C}) \setminus \mathcal{C}$. When approaching ∞, a univalent vertex v follows such an ∞-component, which may be written as $\{y_\pm(v)\}\times]1,\infty[$ or $\{y_\pm(v)\}\times]-\infty,0[$, where $y_\pm(v)$ denotes the orthogonal projection on $\mathbb{C}$ of the ∞-component of v. The projection $y_\pm(v)$ depends on the considered ∞-component of v, in general. When such a component is fixed, it is simply denoted by $y(v)$, and we speak of the ∞-component of a univalent vertex which is mapped to ∞.

Definition 14.15. Define the two-point compactification of $\mathbb{R} =]-\infty,+\infty[$ to be $[-\infty,+\infty]$. The open manifold $\check{C}(R(\mathcal{C}),L;\Gamma)$ embeds naturally in the product of $C_{V(\Gamma)}(R(\mathcal{C}))$, by the two-point compactifications of the sources of the components of L that go from top to top or from bottom to bottom. Let $C^f(R(\mathcal{C}),L;\Gamma)$ be the closure of $\check{C}(R(\mathcal{C}),L;\Gamma)$ in this product. This closure maps naturally onto the closure $C(R(\mathcal{C}),L;\Gamma)$ of $\check{C}(\check{R}(\mathcal{C}),L;\Gamma)$ in $C_{V(\Gamma)}(R(\mathcal{C}))$. An element of $C^f(R(\mathcal{C}),L;\Gamma)$ is a configuration $c_{V(\Gamma)}$ of $C(R(\mathcal{C}),L;\Gamma)$ equipped with the additional data[2] of an ∞-component for each univalent vertex of $p_b(c_{V(\Gamma)})^{-1}(\infty)$.

In this section, we prove the following theorem.

Theorem 14.16. *Let $L\colon \mathcal{L} \hookrightarrow R(\mathcal{C})$ be a long tangle representative and let Γ be a numbered degree n Jacobi diagram with support $\mathcal{L}$ without looped edges. For any $c^0_{V(\Gamma)}$ in $C^f(R(\mathcal{C}),L;\Gamma)$, there exist*

[2]This piece of data is automatically determined by $c_{V(\Gamma)}$ in most cases. But it may happen that it is not. For example, when Γ has a unique univalent vertex on a strand that goes from top to top, and when this vertex is mapped to ∞, the strand of the vertex may not be determined by the configuration in $C_{V(\Gamma)}(R(\mathcal{C}))$.

- *a manifold W with boundary and ridges,*
- *a small $\varepsilon > 0$,*
- *an oriented tree $\mathcal{T}^0 = \mathcal{T}(c^0_{V(\Gamma)})$ (described in Notation 14.34), and*
- *a smooth map $\varphi\colon [0,\varepsilon[^{E(\mathcal{T}^0)} \times W \to C_{V(\Gamma)}(R(\mathcal{C}))$, whose restriction to $\left(X(\mathcal{T}^0) \cap [0,\varepsilon[^{E(\mathcal{T}^0)}\right) \times W$ is injective,*

such that $\varphi\left([0,\varepsilon[^{E(\mathcal{T}^0)} \times W\right) \cap C(R(\mathcal{C}), L; \Gamma)$ is an open neighborhood of $c^0_{V(\Gamma)}$ in $C^f(R(\mathcal{C}), L; \Gamma)$ (where the ∞-component for each univalent vertex of the $p_b(c_{V(\Gamma)})^{-1}(\infty)$ is the same as the ∞-component for $p_b(c^0_{V(\Gamma)})^{-1}(\infty)$), which is equal to $\varphi\left(\left(X(\mathcal{T}^0) \cap [0,\varepsilon[^{E(\mathcal{T}^0)}\right) \times W\right)$.

Let $\vec{N}$ denote the upward vertical vector. The codimension-one open faces of $C^f(R(\mathcal{C}), L; \Gamma)$ are

- *the faces corresponding to the collapse of a subgraph at one point in $\check{R}(\mathcal{C})$ as before,*
- *the faces corresponding to a set of vertices mapped to ∞, for which the configuration up to dilation at ∞ is injective, and does not map a point to 0, as before,*
- *additional faces called* T-faces *(for which $C(R(\mathcal{C}), L; \Gamma)$ is not transverse to the ridges of $C_{V(\Gamma)}(R(\mathcal{C}))$), where*
  - *a set of vertices $B \sqcup \sqcup_{j \in I} B_j$ is mapped to ∞, for a nonempty set I,*
  - *the corresponding configuration, up to dilation, from $B \sqcup \sqcup_{j \in I} B_j$ to $T_\infty R(\mathcal{C})$, maps each B_j to a nonzero point of the vertical line, and it injects B outside zero and the images of the B_j, which are distinct,*
  - *each subset B_j contains univalent vertices of at least 2 distinct ∞-components,*
  - *for each B_j, the infinitesimal configuration of B_j is an injective configuration of a Jacobi diagram on the lines that extend the half-lines above (resp. below) $\mathcal{C}$, if B_j is mapped to $\lambda\vec{N}$ for some $\lambda > 0$ (resp. for some $\lambda < 0$), up to global translation along these lines. (No inversion is involved, here.)*

Together with Lemma 14.14, Theorem 14.16 implies the following lemma.

Lemma 14.17. *Theorem 12.2 is true.*

Let η be a form of degree $(dim(C_L) - 1)$ of $C_{V(\Gamma)}(R(\mathcal{C}))$. Then $\int_{C_L} d\eta$ is the sum $\sum_F \int_F \eta$, which runs over the codimension-one faces F of C^f_L, oriented as such, and listed in Theorem 14.16.

$\square$

Example 14.18. Let $K\colon]0,1[\hookrightarrow \check{R}(\mathcal{C})$ be a (long) component of L. Assume that $K\colon]0,1[\hookrightarrow \check{R}(\mathcal{C})$ goes from top to top. Let $d_1 = -\{z_1\} \times [1,\infty]$ and $d_2 = \{z_2\} \times [1,\infty]$ denote the vertical half-lines of K above $\mathcal{C}$, where d_1 is before d_2. Let $G = \{(h,k) \in]0,1] \times \mathbb{R}; k + \frac{1}{h} \geq 1\}$. Define the diffeomorphism

$$\begin{array}{llll} g\colon & G & \to & d_1 \times d_2 \\ & (h,k) & \mapsto & ((z_1, \frac{1}{h}), (z_2, k + \frac{1}{h})). \end{array}$$

This diffeomorphism g extends as a continuous map g from $G \cup (\{0\} \times \mathbb{R})$ to $C(R(\mathcal{C}), L;\ \hat{\text{ç}}^K\ )$, which maps $(0,k)$ to the limit $g(0,k)$ at 0 in $C_2(R(\mathcal{C}))$ of the $g(]0,\varepsilon] \times k)$, where the image of $g(0,k)$ under the canonical map from $C_2(R(\mathcal{C}))$ to $R(\mathcal{C})^2$ is (∞,∞), the corresponding configuration in $T_\infty R(\mathcal{C})$ up to dilation is constant, and $p_\tau(g(0,k)) = \frac{(z_2 - z_1, k)}{\|(z_2 - z_1, k)\|}$. The extended g provides a differentiable structure on its image, with the codimension-one face $-g(\{0\} \times \mathbb{R})$ whose image under p_τ is the open half-circle from $\vec{N}$ to $-\vec{N}$ through the direction of $(z_2 - z_1)$. This codimension-one face of $C^f(R(\mathcal{C}), L;\ \hat{\text{ç}}^K\ )$ is an example of a T-face, for which $B \sqcup \sqcup_{j \in I} B_j = B_1$ is the pair of vertices of the graph. Since this codimension-one face sits in a codimension 2 face of $C_2(R(\mathcal{C}))$, $C(R(\mathcal{C}), L;\ \hat{\text{ç}}^K\ )$ is not transverse to the ridges of $C_2(R(\mathcal{C}))$.

Recall that the elements of $C^f_L = C^f(R(\mathcal{C}), L; \Gamma)$ are elements of the closure C_L of $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$ in $C_{V(\Gamma)}(R(\mathcal{C}))$ equipped with the additional data of the ∞-components of the univalent vertices sent to ∞.

First note that the configuration space C_L intersects $p_b^{-1}(\check{R}(\mathcal{C})^{V(\Gamma)})$ as a smooth submanifold as in the case of links. The only difference with the case of links occurs when some univalent vertices approach ∞. Our configuration space is a local product of the space of the restrictions of the configurations to the points near ∞ and the space of the restrictions of the configurations to the other points, which is a smooth manifold with boundary whose structure has been studied in details in Chapter 8.

Recall the orientation-reversing embedding ϕ_∞

$$\begin{array}{llll} \phi_\infty: & \mathbb{R}^3 & \longrightarrow & S^3 \\ & \mu(x \in S^2) & \mapsto & \begin{cases} \infty & \text{if } \mu = 0 \\ \frac{1}{\mu}x & \text{otherwise.} \end{cases} \end{array}$$

According to Corollary 8.43, with the notation of Chapter 8 and especially those of Section 8.7, an element $c_{V(\Gamma)}$ of $C_{V(\Gamma)}(R(\mathcal{C}))$ consists of

- a subset $V = p_b(c_{V(\Gamma)})^{-1}(\infty)$ of $V(\Gamma)$,
- an element $c_{V(\Gamma)\setminus V}$ of $C_{V(\Gamma)\setminus V}[\check{R}(\mathcal{C})]$,
- an ∞-parenthesization $(\mathcal{P}_s = \{V = V(1), V(2), \dots, V(\sigma)\}, \mathcal{P}_d)$ of V,
- for each $V(i) \in \mathcal{P}_s$, an injective configuration

$$T_0\phi_\infty \circ f_i \in \check{\mathcal{S}}(T_\infty R(\mathcal{C}), K^s_d(V(i))),$$

  up to dilation,
- for each $A \in \mathcal{P}_d$, an injective configuration

$$T_{[f_{k(A)}(A)]}\phi_{\infty *} \circ w_A \in \check{S}_{K(A)}(T_{[T_0\phi_\infty(f_{k(A)}(A))]}\mathrm{B}\ell(R(\mathcal{C}), \infty)),$$

  up to dilation and translation (see Proposition 8.42).

Proposition 8.44 describes the restriction maps, naturally. As reminded above, the configuration space $C_{V(\Gamma)}(R(\mathcal{C}))$ has a natural stratification induced by $V = p_b(c)^{-1}(\infty)$, the parenthesization associated to $c_{V(\Gamma)\setminus V}$ (as before Proposition 8.32) and the above ∞-parenthesization of V. Each stratum has a well defined dimension.

Below, we refine this partition, which is induced on $C^f(R(\mathcal{C}), L; \Gamma)$ by the stratification of $C_{V(\Gamma)}(R(\mathcal{C}))$.

Notation 14.19. The sets of $\mathcal{P}$ that contain a univalent vertex are called *univalent*. A *possibly separating set* associated to the above ∞-parenthesization and to the data of the ∞-components of the elements of V is a set $A \in \mathcal{P}_d$ such that

- the kids of A have all their univalent vertices on the same ∞-component and
- A has at least two univalent vertices on different ∞-components.

Let $\mathcal{P}_X$ denote the set of possibly separating sets associated to the above ∞-parenthesization. A set A of $\mathcal{P}_X$ is *separating* (with respect to $c_{V(\Gamma)}$) if it has at least two univalent kids A_1 and A_2 such that $w_A(A_1) - w_A(A_2)$ is not vertical. The set of separating sets of $c_{V(\Gamma)}$ is denoted by $\mathcal{P}_x$.

Recall that $\vec{N}$ denotes the upward unit vertical vector. Let $p_{\mathbb{C}} \colon (\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}) \to \mathbb{C}$ (resp. $p_{\mathbb{R}} \colon \mathbb{C} \times \mathbb{R} \to \mathbb{R}$) denote the orthogonal projection onto the horizontal plane $\mathbb{C}$ (resp. the vertical line $\mathbb{R}$).

We are going to prove the following two propositions.

Proposition 14.20. *The space $C^f_L = C^f(R(\mathcal{C}), L; \Gamma)$ of Definition 14.15 is the space of configurations $c_{V(\Gamma)}$ of $C_{V(\Gamma)}(R(\mathcal{C}))$ as above, equipped with ∞-components for the univalent vertices of $V = p_b(c_{V(\Gamma)})^{-1}(\infty)$, such that the following conditions are satisfied.*

1. *If the configuration $c_{U(\Gamma)\setminus(V\cap U(\Gamma))}$ is injective, then it factors through the restriction to $U(\Gamma) \setminus (V \cap U(\Gamma))$ of a representative of i_Γ that maps the univalent vertices of $V \cap U(\Gamma)$ to their ∞-components further than the elements of $U(\Gamma) \setminus (V \cap U(\Gamma))$. If the configuration $c_{U(\Gamma)\setminus(V\cap U(\Gamma))}$ is not injective, then it factors through a limit of such restrictions. In any case, the possible infinitesimal configurations of vertices of $U(\Gamma) \setminus (V \cap U(\Gamma))$ are locally ordered on the tangent space to their component, as in the case of links (see Sections 8.4 and 8.8).*

2. *The f_i map the elements of $K^s_d(V(i))$ that contain a univalent vertex on an ∞-component $y\times]1,\infty[$ (resp. $y\times]-\infty,0[$) to the half-line $\mathbb{R}^+\vec{N}$ (resp. $\mathbb{R}^+(-\vec{N})$), the order on such a half-line between two elements of $K^s_d(V(i))$ that contain univalent vertices on a common ∞-component is prescribed: namely, if A_1 and A_2 are two elements of $K^s_d(V(i))$ that respectively contain two vertices v_1 and v_2 on the same ∞-component, and if $i_\Gamma(v_1)$ is closer to ∞ than $i_\Gamma(v_2)$, then $f_i(A_1)$ is closer to 0 than $f_i(A_2)$.*

3. *If $v_1 \in A_1$ and $v_2 \in A_2$ are two univalent vertices of distinct kids A_1 and A_2 of an element $A \in \mathcal{P}_d$, fix a normalization of w_A and let $\vec{y} = y(v_2) - y(v_1)$.*

   - *If $\vec{y} = 0$ (that is if v_1 and v_2 are on the same ∞-component), then $w_A(A_2) - w_A(A_1)$ is a nonzero vertical vector, which may be expressed as $\alpha\vec{N}$, where α is positive if the ∞-component is above $\mathcal{C}$ and if v_1 is closer to ∞ than v_2, or if the ∞-component is under $\mathcal{C}$ and if v_2 is closer to ∞ than v_1, and α is negative otherwise.*
   - *If $A \notin \mathcal{P}_X$, then $w_A(A_2) - w_A(A_1)$ may be written as $\alpha\vec{N}$, where the sign of α is determined as above, when v_1 and v_2 are on the same ∞-component.*
   - *If $A \in \mathcal{P}_X$ and if $\vec{y} \neq 0$, then $w_A(A_2) - w_A(A_1)$ may be expressed as $(\alpha\vec{N} + \beta\vec{y})$ for some nonzero pair (α, β) of $\mathbb{R}\times\mathbb{R}^+$, and if $v_3 \in A_3$ is a univalent vertex of another kid A_3 of A, then there exists $\alpha_3 \in \mathbb{R}$ such that $w_A(A_3) - w_A(A_1)$ is equal to $(\alpha_3\vec{N} + \beta\,(y(v_3) - y(v_1)))$, where the sign of α_3 is determined as above, if $y(v_3) = y(v_1)$.*

Proposition 14.21. *The space C_L^f of Proposition 14.20 is partitioned by the data for a configuration $c_{V(\Gamma)}$ of*

- *the set $V = p_b(c_{V(\Gamma)})^{-1}(\infty)$,*
- *the parenthesization $\mathcal{P}(c_{V(\Gamma)|V(\Gamma)\setminus V})$ of $V(\Gamma)\setminus V$ associated to $c_{V(\Gamma)|V(\Gamma)\setminus V}$ (as before Proposition 8.32),*
- *the ∞-parenthesization $(\mathcal{P}_s = \{V = V(1), V(2), \dots, V(\sigma)\}, \mathcal{P}_d)$ (as in Definition 8.37), together with the data of the ∞-components of the univalent vertices that are mapped to ∞, and the set $\mathcal{P}_x$ of separating sets of $\mathcal{P}_d$.*

The part associated with the above data is a smooth submanifold of $C_{V(\Gamma)}(R(\mathcal{C}))$ of dimension

$$\sharp U(\Gamma) + 3\sharp T(\Gamma) - \sharp\mathcal{P}(V(\Gamma)\setminus V; c) - \sharp\mathcal{P}_s - \sharp\mathcal{P}_d + \sharp\mathcal{P}_x.$$

The partition is a stratification of C_L^f.

Remark 14.22. Proposition 14.21 implies that the only codimension-one new parts –which come necessarily from strata for which $\sharp\mathcal{P}_s \geq 1$– come from the parts such that $\sharp\mathcal{P}_s = 1$ and $\mathcal{P}_d = \mathcal{P}_x$. They are the T-faces of Theorem 14.16, for which $\mathcal{P}_s = \{B \sqcup \sqcup_{j\in I} B_j\}$ and $\mathcal{P}_d = \mathcal{P}_X = \mathcal{P}_x = \{B_j \mid j \in I\}$.

The rest of this section is devoted to the proofs of Theorem 14.16 and the above two propositions.

Let $c^0_{V(\Gamma)}$ be a configuration of C_L^f in a stratum as in the statement of Proposition 14.21. Let c^0 (resp. $c^0_{V(\Gamma)|V(\Gamma)\setminus V}$) denote the restriction of $c^0_{V(\Gamma)}$ to $V(= V(c^0))$ (resp to $V(\Gamma)\setminus V$).

It is easy to see that $c^0_{V(\Gamma)}$ has a neighborhood $N_\Gamma(c^0_{V(\Gamma)})$ in C_L^f consisting of configurations, which map

- V to a fixed open neighborhood N_∞ of ∞ in $R(\mathcal{C})$,
- the univalent vertices of V to their ∞-component with respect to c^0,
- $V(\Gamma)\setminus V$ to $R(\mathcal{C})\setminus N_\infty$.

Let $C_V(N_\infty, L, c^0)$ denote the space of restrictions to V of configurations of $C(R(\mathcal{C}), L; \Gamma)$, which map univalent vertices of V to N_∞ and to their ∞-components (determined by c^0) and vertices of $V(\Gamma)\setminus V$ to $R(\mathcal{C})\setminus N_\infty$.

The configuration $c^0_{V(\Gamma)}$ has a neighborhood $N_\Gamma(c^0_{V(\Gamma)})$ in C_L^f, which is a product of $C_V(N_\infty, L, c^0)$ by a smooth submanifold N_2 with boundary of

$C_{V(\Gamma)\setminus V}(R(\mathcal{C}))$. In this product decomposition, the N_2-part contains the restriction of the configurations $c_{V(\Gamma)}$ to $V(\Gamma) \setminus V$. This space has been studied before (see Section 8.6 and Theorem 8.26), and the manifold W of the statement of Theorem 14.16 will be a product $W_V \times N_2$. This allows us to forget about the N_2-part. We focus only on c^0, and on a neighborhood $N(c^0)$ of c^0 in $C_V(R(\mathcal{C}))$.

The configuration c^0 is described by

- an ∞-parenthesization $(\mathcal{P}_s = \{V = V(1), V(2), \dots, V(\sigma)\}, \mathcal{P}_d)$ of V,
- for any $i \in \overline{\sigma}$, an element $f_i^0 \colon K_d^s(V(i)) \to \mathbb{R}^3$ of a manifold W_i^s, which is an open neighborhood of f_i^0 in the manifold consisting of the maps $f_i \colon K^s(V(i)) \to \mathbb{R}^3$ such that
  - $f_i(V(i+1)) = 0$, if $i \neq \sigma$,
  - $\sum_{A \in K_d^s(V(i))} \parallel f_i(A) \parallel^2 = 1$,
  - $\parallel f_i(A) \parallel > \eta$ for any i and for any $A \in K_d^s(V(i))$,
  - $\parallel f_i(A_2) - f_i(A_1) \parallel > \eta$ for any two distinct elements A_1 and A_2 of $K^s(V(i))$,

  for some real number $\eta > 0$, (where f_i represents an injective configuration $T_0\phi_\infty \circ f_i$ up to dilation of $\check{\mathcal{S}}(T_\infty R(\mathcal{C}), K_d^s(V(i))))$ –
- for any element A of $\mathcal{P}_d$, an element $w_A^0 \colon K(A) \to \mathbb{R}^3$ of the manifold $\tilde{W}_A$ consisting of the maps $w_A \colon K(A) \to \mathbb{R}^3$ up to dilation and translation.

Notation 14.23. We choose univalent basepoints for univalent sets of $\mathcal{P}$. We also impose that any $A \in \mathcal{P}_d$ that has (at least) two kids with univalent vertices on different ∞-components, has (at least) two kids with basepoints on different ∞-components. (As usual, our basepoints also satisfy the conditions that for two elements A and B of $\mathcal{P}_d$, such that $B \subset A$, if $b(A) \in B$, then $b(B) = b(A)$.)

We normalize the configurations of $\tilde{W}_A$ as follows in a neighborhood $\tilde{N}_A$ of a given $w_A^0 \in \tilde{W}_A$. Choose a kid $k_n(A)$ such that $|p_{\mathbb{R}}(w_A^0(k_n(A))) - p_{\mathbb{R}}(w_A^0(b(A)))|$ or $|p_{\mathbb{C}}(w_A^0(k_n(A))) - p_{\mathbb{C}}(w_A^0(b(A)))|$ is maximal in the set

$$\{|p_{\mathbb{R}}(w_A^0(k)) - p_{\mathbb{R}}(w_A^0(b(A)))|, |p_{\mathbb{C}}(w_A^0(k)) - p_{\mathbb{C}}(w_A^0(b(A)))| \mid k \in K(A)\}$$

and call it the *normalizing kid* of A. If $|p_{\mathbb{R}}(w_A^0(k_n(A))) - p_{\mathbb{R}}(w_A^0(b(A)))| \geq |p_{\mathbb{C}}(w_A^0(k_n(A))) - p_{\mathbb{C}}(w_A^0(b(A)))|$, say that $k_n(A)$ is *vertically normalizing* or *v-normalizing*, and normalize the configurations w_A in a neighborhood of

w_A^0 by imposing $w_A(b(A)) = 0$ and $|p_{\mathbb{R}}(w_A(k_n(A)))| = 1$. Otherwise, say that $k_n(A)$ is *horizontally normalizing* or *h-normalizing*, and normalize the configurations w_A in a neighborhood of w_A^0 by imposing $w_A(b(A)) = 0$ and $|p_{\mathbb{C}}(w_A(k_n(A)))| = 1$. (These normalizations are compatible with the smooth structure of $C_{V(\Gamma)}(R(\mathcal{C}))$.)

In our neighborhood $\tilde{N}_A$, we also impose that $\parallel w_A(k) - w_A^0(k) \parallel < \varepsilon$, for a small $\varepsilon \in]0,1[$. So $\tilde{N}_A$ is diffeomorphic to the product W_A of the product, over the non-normalizing kids k of A that do not contain $b(A)$, of the open balls $\mathring{B}(w_A^0(k),\varepsilon)$ of radius ε centered at $w_A^0(k)$, by the set of elements $w_A(k_n(A))$ of $\mathring{B}(w_A^0(k_n(A)),\varepsilon)$ such that $|p_{\mathbb{R}}(w_A(k_n(A)))| = 1$ (resp. such that $|p_{\mathbb{C}}(w_A(k_n(A)))| = 1$) if $k_n(A)$ is v-normalizing (resp. if $k_n(A)$ is h-normalizing). We also impose that $\parallel w_A(B_2) - w_A(B_1) \parallel > \eta$ for any two distinct kids B_1 and B_2 of A in our normalized neighborhood W_A. Note that $\parallel w_A(k) \parallel < 3$ for any $k \in K(A)$ in this neighborhood.

All the considered maps f_i, w_A are also considered as maps from V to $\mathbb{R}^3$, which are constant on the elements of $K^s(V(i))$ and $K(A)$, respectively, and which respectively map $(V \setminus V(i)) \cup V(i)^s$ and $V \setminus A$ to 0. (Recall Definition 8.37.)

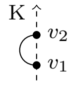
For $A \in \mathcal{P}_d$, recall from Definition 8.40 that $k(A)$ as the maximal integer among the integers k such that $A \subseteq V(k)$. We use a chart ψ of $C_V(R(\mathcal{C}))$ of a neighborhood $N(c^0)$ of c^0 in $C_V(R(\mathcal{C}))$, which maps

$$((u_i)_{i\in\underline{\sigma}}, (\mu_A)_{A\in\mathcal{P}_d}, (f_i)_{i\in\underline{\sigma}}, (w_A)_{A\in\mathcal{P}_d}) \in [0,\varepsilon[^{\sigma} \times [0,\varepsilon[^{\mathcal{P}_d} \times \prod_{i\in\underline{\sigma}} W_i^s \times \prod_{A\in\mathcal{P}_d} W_A$$

to a configuration $c = \psi\left((u_i),(\mu_A),(f_i),(w_A)\right) \in C_V(R(\mathcal{C}))$, such that, when the u_i and the μ_A do not vanish, c is the following injective configuration:

$$c = \phi_\infty \circ \left(\sum_{V(k)\in\mathcal{P}_s} U_k \left(f_k + \sum_{C\in\mathcal{P}_d | k(C)=k} \left(\prod_{D\in\mathcal{P}_d | C\subseteq D} \mu_D \right) w_C \right)\right),$$

where $U_k = \prod_{i=1}^k u_i$, as in Proposition 8.41 (except for our different choices of normalizations, which do not change the structure). In this chart, $c^0 = \psi\left((u_i^0 = 0),(\mu_A^0 = 0),(f_i^0),(w_A^0)\right)$.

Example 14.24. In the special case of Example 14.18, $\Gamma =$ , consider configurations that map v_1 to $-\{z_1\} \times [1,\infty]$, and v_2 to $\{z_2\} \times [1,\infty]$. When $V = \{v_1, v_2\}$ and $\mathcal{P} = \mathcal{P}_s = \mathcal{P}_d = \{V\}$, set $f = f_1 = f_\sigma$, $u = u_1$, $\mu = \mu_V$, $w = w_V$, $w(v_1) = 0$ and $\parallel f(v_1) \parallel = 1$. So $c = \phi_\infty \circ (u\,(f + \mu w))$, $c(v_1) = \phi_\infty(uf(v_1)) = \frac{1}{u}\frac{f(v_1)}{\|f(v_1)\|^2} = \frac{1}{u}f(v_1)$, $c(v_2) = \phi_\infty(u(f(v_1) + \mu w(v_2)))$, where $f^0(v_1) = \vec{N}$.

Back to the general proof of our theorem 14.16, Propositions 14.20 and 14.21, we will often reduce ε and reduce the spaces W_k^s and W_A to smaller manifolds, which are neighborhoods of f_k^0 and w_A^0 in the initial manifolds W_k^s and W_A. In particular, we assume that the image $N(c^0)$ of ψ is in $C_V(N_\infty)$, and we set

$$N_\Gamma(c^0) = N(c^0) \cap C_V(N_\infty, L, c^0).$$

The intersection of $N_\Gamma(c^0)$ with $\check{C}_V(R(\mathcal{C}))$ is determined by the conditions that univalent vertices belong to their ∞-components, and that their order on the ∞-components is prescribed by the isotopy class of injections from $U(\Gamma)$ to $\mathcal{L}$. These conditions are closed conditions which still hold in $N_\Gamma(c^0)$.

The *first condition* is that the basepoints $b(A)$ of the univalent elements A of $K_d^s(V(k))$ are sent to their ∞-components, for any $k \in \underline{\sigma}$. We examine what this condition imposes on the f_k and prove the following two lemmas.

Lemma 14.25. *For any $k \in \underline{\sigma}$, for any univalent element A of $K_d^s(V(k))$, $f_k^0(b(A)) = \pm \| f_k^0(b(A)) \| \vec{N}$, where $\| f_k^0(b(A)) \| \geq \eta$, and where the $\pm$ sign is $+$ if A has a univalent vertex above $\mathcal{C}$ and $-$ otherwise.*

PROOF: For an element A of $K_d^s(V(k))$, and for a configuration

$$c = \psi\left((u_i), (\mu_A), (f_i), (w_A)\right) \in C_V(R(\mathcal{C}))$$

such that $U_k = \prod_{i=1}^k u_i \neq 0$,

$$c(b(A)) = \frac{1}{U_k} \frac{f_k(b(A))}{\| f_k(b(A)) \|^2}.$$

So the condition $p_{\mathbb{C}}(c(b(A))) = y(b(A))$ is equivalent to the closed condition

$$p_{\mathbb{C}}(f_k(b(A))) = U_k \| f_k(b(A)) \|^2 y(b(A)), \tag{14.1}$$

where

$$\| f_k(b(A)) \|^2 = \| p_{\mathbb{C}}(f_k(b(A))) \|^2 + \| p_{\mathbb{R}}(f_k(b(A))) \|^2 \leq 1.$$

So $\| p_{\mathbb{C}}(f_k(b(A))) \| = O(U_k)$ (that means that there exists $C \in \mathbb{R}^{*+}$ such that $\| p_{\mathbb{C}}(f_k(b(A))) \| \leq C U_k$). In particular, since $U_k = 0$ for c^0, $p_{\mathbb{C}}(f_k^0(b(A))) = 0$ and $f_k^0(b(A)) = \pm \| f_k^0(b(A)) \| \vec{N}$, where $\| f_k^0(b(A)) \| \geq \eta$. Lemma 14.25 follows easily, since the sign of $p_{\mathbb{R}}(f_k(b(A)))$ is constant on $N_\Gamma(c^0)$. $\square$

In particular, if $V(\sigma) \in \mathcal{P}_d$, then $f_\sigma^0(V(\sigma)) = \vec{N}$ if $V(\sigma)$ has a univalent vertex above $\mathcal{C}$, and $f_\sigma^0(V(\sigma)) = -\vec{N}$ if $V(\sigma)$ has a univalent vertex under $\mathcal{C}$.

Lemma 14.26. *Let v_1 and v_2 be two vertices on some ∞-component of L. Assume that v_1 is closer to ∞ than v_2 and that $v_1 \in A_1$ and $v_2 \in A_2$, for two different kids A_1 and A_2 of $V(k)$, where $k \in \underline{\sigma}$. Then*

$$\| f_k^0(A_1) \| < \| f_k^0(A_2) \|. \tag{14.2}$$

Proof: The configuration c^0 is a limit at 0 of a family $c(t)$ of injective configurations, indexed by $t \in]0, \varepsilon[$,

$$c = c(t) = \psi\left((u_i), (\mu_A), (f_i), (w_A)\right)$$

for which $\| c(v_1) \| > \| c(v_2) \|$. Therefore, $\| f_k^0(A_1) \| \leq \| f_k^0(A_2) \|$. Since $f_k^0(A_1) \neq f_k^0(A_2)$, the result follows. $\square$

In particular, $|p_{\mathbb{R}}(f_k^0(A_1))| \leq |p_{\mathbb{R}}(f_k^0(A_2))| + \eta$. We possibly reduce W_k^s by imposing $|p_{\mathbb{R}}(f_k(A)) - p_{\mathbb{R}}(f_k^0(A))| < \varepsilon$ for some positive ϵ such that $\varepsilon < \frac{\eta}{2}$. This condition ensures that the univalent vertices $b(A)$, for the elements A of $K_d^s(V(k))$, are well ordered on any ∞-component.

Lemma 14.27. *Let $k \in \underline{\sigma}$ be such that $V(k) \notin \mathcal{P}_d$. Recall $U_k = \prod_{i=1}^k u_i$. Let $\ell(k) \in \mathbb{N}$ be the number of univalent non-special kids of $V(k)$, which are $A_{k,1}$, $A_{k,2}$, $\dots$, $A_{k,\ell(k)}$. Let $K_{td}^s(V(k))$ be the set of non-univalent non-special kids D of $V(k)$. For $D \in K_{td}^s(V(k))$, let $\mathring{B}_D = \mathring{B}(f_k^0(D), \varepsilon)$ be the open ball of center $f_k^0(D)$ and of radius ε in $\mathbb{R}^3$.*

- *If $\ell(k) \geq 1$, set*

$$W_k^L = \prod_{i=1}^{\ell(k)-1}]p_{\mathbb{R}}(f_k^0(A_{k,i})) - \varepsilon, p_{\mathbb{R}}(f_k^0(A_{k,i})) + \varepsilon[\times \prod_{D \in K_{td}^s(V(k))} \mathring{B}_D.$$

Up to reducing ε, there is a smooth injective map

$$\phi_k \colon [0, \varepsilon^k[\times W_k^L \to W_k^s$$

such that, up to reducing $N_\Gamma(c^0)$ (to a smaller open neighborhood of c^0 in $C_V(N_\infty, L, c^0)$)), all elements

$$c = \psi\left((u_i)_{i \in \underline{\sigma}}, (\mu_A)_{A \in \mathcal{P}_d}, (f_i)_{i \in \underline{\sigma}}, (w_A)_{A \in \mathcal{P}_d}\right)$$

of $N_\Gamma(c^0)$ satisfy the condition

$$f_k = \phi_k(\prod_{i=1}^k u_i, (p_{\mathbb{R}}(f_k(A_{k,i})))_{i \in \underline{\ell(k)-1}}, (f_k(D))_{D \in K_{td}^s(V(k))}),$$

which is equivalent to

$$p_{\mathbb{C}}(c(b(A_{k,i}))) = y(b(A_{k,i})),$$

for any $i \in \underline{\ell(k)}$, *when* $c \in \check{C}_V(R(\mathcal{C}))$, *and implies Equation 14.1 in any case.*

- *If* $\ell(k) = 0$, *set* $W_k^L = W_k^s$.

If $V(\sigma) \in \mathcal{P}_d$, *and if* $V(\sigma)$ *is not univalent, also set* $W_\sigma^L = W_\sigma^s$. *If* $V(\sigma) \in \mathcal{P}_d$, *and if* $V(\sigma)$ *is univalent, then all elements* c *of* $N_\Gamma(c^0)$ *as above satisfy the condition*

$$p_{\mathbb{C}}(f_\sigma(V(\sigma))) = U_\sigma y(b(V(\sigma))),$$

which is equivalent to $p_{\mathbb{C}}(c(b(V(\sigma)))) = y(b(V(\sigma)))$, *when* $c \in \check{C}_V(R(\mathcal{C}))$; *these elements* c *are also such that*

$$p_{\mathbb{R}}(f_\sigma(V(\sigma))) = \sqrt{1 - U_\sigma^2 |y(b(V(\sigma)))|^2} p_{\mathbb{R}}(f_\sigma^0(V(\sigma))).$$

In this case, set $W_\sigma^L = \{*_\sigma\}$.

Let N_1 *denote the subspace of* $N(c^0)$, *where the first condition (stated before Lemma 14.25) is satisfied. Then* N_1 *is a smooth manifold parametrized by*

$$[0, \varepsilon[^{\sigma} \times [0, \varepsilon[^{\mathcal{P}_d} \times \prod_{k \in \underline{\sigma}} W_k^L \times \prod_{A \in \mathcal{P}_d} W_A.$$

Proof: The proof of Lemma 14.25 proves that for $i \in \underline{\ell(k) - 1}$, $p_{\mathbb{C}}(f_k(A_{k,i}))$ is an implicit function of $U_k = \prod_{i=1}^k u_i$ and $p_{\mathbb{R}}(f_k(A_{k,i}))$, which is close to $\pm \| f_k^0(A_{k,i}) \| \neq 0$ on N_1 and $N_\Gamma(c^0)$. This implicit function is determined by Equation 14.1.

Then the condition that $\sum_{A \in K_d^s(V(k))} \| f_k(A) \|^2 = 1$ in W_k^s determines $\| f_k(A_{k,\ell(k)}) \| \neq 0$ as a function of U_k, $(p_{\mathbb{R}}(f_k(A_{k,i})))_{i \in \underline{\ell(k)-1}}$ and of the $f_k(D)$ for $D \in K_{td}^s(V(k))$. Now, Equation 14.1 determines $p_{\mathbb{C}}(f_k(A_{k,\ell(k)}))$, which in turn determines $p_{\mathbb{R}}(f_k(A_{k,\ell(k)}))$. This is how the map ϕ_k of the statement is constructed. It is easy to check that ϕ_k has the wanted properties and that N_1 is parametrized naturally, as announced, using the maps ϕ_k. $\square$

In Example 14.24, $p_{\mathbb{C}}(f(v_1)) = uy(v_1)$ and $p_{\mathbb{R}}(f(v_1)) = \sqrt{1 - u^2 \| y(v_1) \|^2}$. So $f(v_1)$ is just a smooth function of the small parameter u.

We now restrict to the submanifold N_1 of $N(c^0)$ of Lemma 14.27 and take care of the univalent basepoints of the kids of elements of $\mathcal{P}_d$ in the following lemmas.

Lemma 14.28. *For $X \in S^2$, let $s(X)$ denote the orthogonal symmetry that reverses the line $\mathbb{R}X$ and preserves the plane orthogonal to X pointwise. Let A be an element of $\mathcal{P}_d$. Recall that $k(A)$ is the maximal integer k such that $A \subseteq V(k)$. The restriction of c^0 to A maps A to*

$$X_A^0 = \frac{f^0_{k(A)}(A)}{\parallel f^0_{k(A)}(A) \parallel} \in \partial B\!\!\!/(R(\mathcal{C}), \infty)$$

and it is represented by $s(X_A^0) \circ w_A^0$, up to translation and dilation, as a configuration of the ambient $\mathbb{R}^3$ outside $\mathcal{C}$. If p and q are univalent vertices in two different kids of A, if they belong to an ∞-component K^+ and if p is closer to ∞ than q, then there exists $\alpha^0 \in \mathbb{R}$ such that $|\alpha^0| > \eta$, and $w_A^0(q) - w_A^0(p) = \alpha^0 \vec{N}$, where $\alpha^0 > 0$ if K^+ is above $\mathcal{C}$, and $\alpha^0 < 0$ otherwise.

We introduce some notation before the proof.

Notation 14.29. For an element A of $\mathcal{P}_d$ such that $k(A) = k$, and a configuration $c = \psi\left((u_i), (\mu_A), (f_i), (w_B)\right) \in N(c^0)$ set

$$M_A = \prod_{D \in \mathcal{P}_d | A \subseteq D} \mu_D,$$

and, for any element q of A,

$$\tilde{f}_k(q) = f_k(A) + \sum_{C \in \mathcal{P}_d | q \in C} M_C w_C(q),$$

so that $c(q) = \frac{1}{U_k} \frac{\tilde{f}_k(q)}{\parallel \tilde{f}_k(q) \parallel^2}$ when $U_k \neq 0$. Both M_A and $\tilde{f}_k$ depend on the configuration c.

Proof of Lemma 14.28: The configuration $c^0_{V(\Gamma)}$ is a limit at 0 of a family $c(t)$ of injective configurations of $N_\Gamma(c^0)$, indexed by $t \in]0, \varepsilon[$,

$$c = c(t) = \psi\left((u_i), (\mu_B), (f_i), (w_B)\right),$$

where the u_i and the μ_B are positive. Set $k = k(A)$. Let $p = b(A)$ be the basepoint of A and let q be the basepoint of a kid of A that does not contain p. Since $\tilde{f}_k(q) - \tilde{f}_k(p) = M_A w_A(q)$,

$$\parallel \tilde{f}_k(q) \parallel^2 = \parallel \tilde{f}_k(p) \parallel^2 + 2M_A \langle w_A(q), \tilde{f}_k(p) \rangle + M_A^2 \parallel w_A(q) \parallel^2.$$

Then

$$\begin{aligned} c(q) - c(p) &= \frac{\|\tilde{f}_k(p)\|^2 \tilde{f}_k(q) - \|\tilde{f}_k(q)\|^2 \tilde{f}_k(p)}{U_k \|\tilde{f}_k(q)\|^2 \|\tilde{f}_k(p)\|^2} \\ &= \frac{M_A \|\tilde{f}_k(p)\|^2 w_A(q)}{U_k \|\tilde{f}_k(q)\|^2 \|\tilde{f}_k(p)\|^2} - \frac{2M_A \langle w_A(q), \tilde{f}_k(p) \rangle + M_A^2 \|w_A(q)\|^2}{U_k \|\tilde{f}_k(q)\|^2 \|\tilde{f}_k(p)\|^2} \tilde{f}_k(p). \end{aligned}$$

When the μ_B tend to 0 and when $\tilde{f}_k$ tends to f_k^0, $\tilde{f}_k(p)$ and $\tilde{f}_k(q)$ tend to $f_k^0(A)$, so that

$$\frac{U_k \| \tilde{f}_k(q) \|^2}{M_A}(c(q) - c(p))$$

tends to

$$w_A(q) - 2\langle w_A(q), X_A^0 \rangle X_A^0 = s(X_A^0)(w_A(q)),$$

and w_A^0 is the limit of the $s(X_A^0) \circ c_{|A}$, up to dilation and translation.

If A contains a univalent vertex of an ∞-component above $\mathcal{C}$, then $X_A^0 = \vec{N}$, according to Lemma 14.25. In this case, if a and q are univalent vertices in two different kids of A, if they belong to an ∞-component K^+, and if a is closer to ∞ than q, and $c(q) - c(a) = -\alpha(t)\vec{N}$ for some positive $\alpha(t)$ for any $t > 0$. So $(w_A^0(q) - w_A^0(a))$, which is defined up to dilation, is a positive multiple of $\vec{N}$. $\square$

Notation 14.30. Let $\mathcal{P}_{\overline{X}}(= \mathcal{P}_{\overline{X}}(c^0))$ denote the set of elements of $\mathcal{P}_d(= \mathcal{P}_d(c^0))$ that contain or are equal to an element of $\mathcal{P}_X(= \mathcal{P}_X(c^0))$. Recall that the elements of $\mathcal{P}_{\overline{X}}$ have at least two kids with basepoints on different ∞-components.

Lemma 14.31. *Let $A \in \mathcal{P}_d$ be such that $k(A) = k$. Let c be as in Notation 14.29. For any univalent kid B of A such that $y(b(B)) - y(b(A)) = 0$, if $c \in N_\Gamma(c^0)$, we have*

$$p_{\mathbb{C}}(w_A(B)) = U_k \left(2\langle \tilde{f}_k(b(A)), w_A(B)\rangle + M_A \| w_A(B) \|^2\right) y(b(B)). \quad (14.3)$$

Furthermore, as soon as $p_{\mathbb{C}}(c(b(A))) = y(b(A))$ and $c \in \check{C}_V(R(\mathcal{C}))$, Equation 14.3 implies $p_{\mathbb{C}}(c(b(B))) = y(b(B))$ for such a B.

When $A \in \mathcal{P}_{\overline{X}}$, if $c \in N_\Gamma(c^0)$, then there exists $\lambda_A = \lambda_A(c) \in \mathbb{R}^+$ such that

- *for any univalent kid B of A,*

$$\begin{aligned} p_{\mathbb{C}}(w_A(B)) = \ & \lambda_A \| \tilde{f}_k(b(A)) \|^2 (y(b(B)) - y(b(A))) \\ & + U_k \left(2\langle \tilde{f}_k(b(A)), w_A(B)\rangle + M_A \| w_A(B) \|^2\right) y(b(B)) \end{aligned} \quad (14.4)$$

- *λ_A is continuous on $N_\Gamma(c^0)$,*
- *$\lambda_A M_A = U_k$*

- *for any three elements A, B and D in $\mathcal{P}_d$ such that A and B are in $\mathcal{P}_{\overline{X}}$, and $A \cup B \subseteq D$,*

$$\lambda_A \prod_{C \in \mathcal{P}_d | A \subseteq C \subset D} \mu_C = \lambda_B \prod_{C \in \mathcal{P}_d | B \subseteq C \subset D} \mu_C$$

- *For $M \in \mathcal{P}_X$, set $\tilde{\lambda}_M = \lambda_M \mu_M$. For any $A \in \mathcal{P}_{\overline{X}} \setminus \mathcal{P}_X$, there exists $M \in \mathcal{P}_X$ such that $M \subset A$ and*

$$\lambda_A = \tilde{\lambda}_M \prod_{D \in \mathcal{P}_d | M \subset D \subset A} \mu_D.$$

- *For a univalent kid B of $A \in \mathcal{P}_{\overline{X}}$, as soon as $p_{\mathbb{C}}(c(b(A))) = y(b(A))$ and $c \in \check{C}_V(R(\mathcal{C}))$, Equation 14.4 implies $p_{\mathbb{C}}(c(b(B))) = y(b(B))$.*

Proof: Consider a univalent kid B of A and assume that $p_{\mathbb{C}}(c(b(A))) = y(b(A))$, which is equivalent to

$$p_{\mathbb{C}}(\tilde{f}_k(b(A))) = U_k \| \tilde{f}_k(b(A)) \|^2 y(b(A)),$$

where $\tilde{f}_k(b(A)) = f_k(A) + \sum_{C \in \mathcal{P}_d | A \subset C} M_C w_C(A)$, and $\tilde{f}_k(B) = \tilde{f}_k(b(A)) + M_A w_A(B)$. So the condition $p_{\mathbb{C}}(c(b(B))) = y(b(B))$ may be written as

$$p_{\mathbb{C}}\left(\tilde{f}_k(b(A)) + M_A w_A(B)\right) = U_k \| \tilde{f}_k(B) \|^2 y(b(B)),$$

which is equivalent to

$$\begin{aligned} M_A p_{\mathbb{C}}(w_A(B)) = & \quad U_k \| \tilde{f}_k(b(A)) \|^2 (y(b(B)) - y(b(A))) \\ & + \quad U_k \left(\| \tilde{f}_k(B) \|^2 - \| \tilde{f}_k(b(A)) \|^2 \right) y(b(B)), \end{aligned} \tag{14.5}$$

where

$$\| \tilde{f}_k(B) \|^2 - \| \tilde{f}_k(b(A)) \|^2 = M_A \left(2\langle \tilde{f}_k(b(A)), w_A(B) \rangle + M_A \| w_A(B) \|^2 \right).$$

When $y(b(B)) - y(b(A)) = 0$ and $M_A \neq 0$, Equation 14.5 simplifies to Equation 14.3, which also holds in the closure C_L. It also shows that $p_{\mathbb{C}}(w_A^0(b(B))) = 0$. So $|p_{\mathbb{R}}(w_A^0(b(B)))| \geq \eta$.

When $y(b(B)) - y(b(A)) \neq 0$ and $M_A \neq 0$, Equation 14.5 is equivalent to

$$\begin{aligned} p_{\mathbb{C}}(w_A(B)) - & \quad U_k \left(2\langle \tilde{f}_k(b(A)), w_A(B) \rangle + M_A \| w_A(B) \|^2 \right) y(b(B)) \\ & = \tfrac{U_k}{M_A} \| \tilde{f}_k(b(A)) \|^2 (y(b(B)) - y(b(A))) \end{aligned} \tag{14.6}$$

and it tells that the left-hand side is colinear to $\| \tilde{f}_k(b(A)) \|^2 (y(b(B)) - y(b(A)))$ when $M_A \neq 0$, and that the scalar product of these two vectors is non-negative. This remains true in the closure C_L and this uniquely defines $\lambda_A = \lambda_A(B,c)$, such that Equation 14.4 holds for B. Furthermore, $\lambda_A(B,c)$ is continuous on C_L, $\lambda_A(B,c)$ tends to $\frac{\|p_{\mathbb{C}}(w_A(B))\|}{\|\tilde{f}_k(b(A))\|^2 \|y(b(B))-y(b(A))\|}$ when U_k tends to 0, and $\lambda_A(B,c) = \frac{U_k}{M_A}$ when $M_A \neq 0$. In particular, if B' is another univalent kid of A such that $y(b(B')) - y(b(A)) \neq 0$, and if $M_A \neq 0$, then $\lambda_A(B,c) = \lambda_A(B',c)$, and this remains true in the closure C_L when $M_A = 0$. The other properties of the parameters λ_A are obvious when the parameters μ_D do no vanish. So they hold in C_L and Lemma 14.31 is proved. (Note that the set $\mathcal{P}_X$ of possibly separating sets is the subset of $\mathcal{P}_{\overline{X}}$ consisting of its sets that are minimal with respect to the inclusion.) $\square$

Lemma 14.32. *The configuration*

$$c^0 = \psi\left((0)_{k\in\underline{\sigma}}, (0)_{A\in\mathcal{P}_d}, (f^0_k)_{k\in\underline{\sigma}}, (w^0_A)_{A\in\mathcal{P}_d}\right)$$

of $C_V(N_\infty, L, c^0)$ is such that

- $p_{\mathbb{C}} \circ w^0_A(B) = 0$ *for any univalent kid B of A, if $A \in \mathcal{P}_d \setminus \mathcal{P}_X$,*
- *for any $A \in \mathcal{P}_X$, there exists $\lambda^0_A \geq 0$ such that*

$$p_{\mathbb{C}} \circ w^0_A(B) = \lambda^0_A \| f^0_{k(A)}(A) \|^2 (y(b(B)) - y(b(A)))$$

for any univalent kid B of A.

Proof: Lemma 14.31 implies that $\lambda^0_A = 0$ when $A \in \mathcal{P}_{\overline{X}} \setminus \mathcal{P}_X$. $\square$

Example 14.33. Let us go back to Example 14.24, $\| y(v_2) - y(v_1) \| \neq 0$. When $A = V$ and $B = \{v_2\}$, $\tilde{f}(v_2) = f(v_1) + \mu w(v_2)$. When c is injective, Equation 14.5 is equivalent to

$$\begin{aligned} p_{\mathbb{C}}(w(v_2)) = \ & \lambda \| \tilde{f}(v_1) \|^2 (y(v_2) - y(v_1)) \\ & + u\left(2\langle \tilde{f}(v_1), w(v_2)\rangle + \mu \| w(v_2) \|^2\right) y(v_2), \end{aligned}$$

where $\lambda = \lambda_V = \frac{u}{\mu}$. So $p_{\mathbb{C}}(w^0(v_2)) = \lambda^0(y(v_2) - y(v_1))$.

Let us now define the oriented tree $\mathcal{T}^0$ (as in Definition 14.1) of the statement of Theorem 14.16.

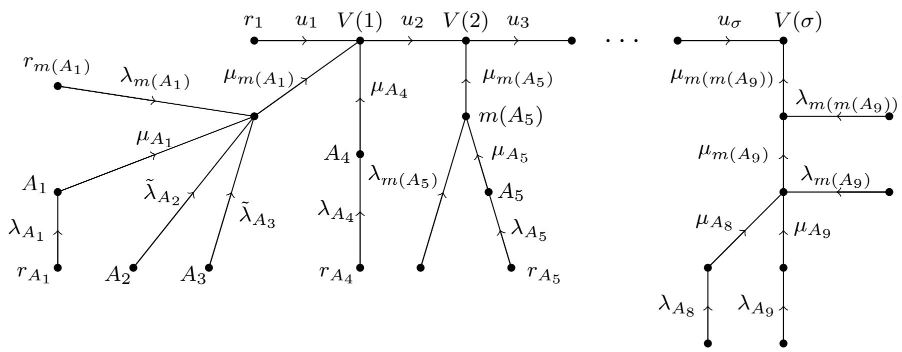


Figure 14.6: A tree $\mathcal{T}^0 = \mathcal{T}(c^0)$

Notation 14.34. The set $E(\mathcal{T}^0)$ of edges of $\mathcal{T}^0$ is in one-to-one correspondence with $\{u_i\}_{i\in\underline{\sigma}}\cup\{\mu_B\}_{B\in\mathcal{P}_{\overline{X}}\setminus\mathcal{P}_x}\cup\{\lambda_A\}_{A\in\mathcal{P}_{\overline{X}}\setminus\mathcal{P}_x}\cup\{\tilde{\lambda}_A=\lambda_A\mu_A\}_{A\in\mathcal{P}_x}$ and the edges are labeled by these variables. So $E(\mathcal{T}^0)$ is in one-to-one correspondence with the disjoint union of $\underline{\sigma}$, $\mathcal{P}_x$, and two disjoint copies of $\mathcal{P}_{\overline{X}}\setminus\mathcal{P}_x$. (Recall that $\mathcal{P}_{\overline{X}}$ is the set of elements of $\mathcal{P}_d$ that contain or are equal to an element of $\mathcal{P}_X$.)

The set of vertices of $\mathcal{T}(\mathcal{P})$ is in one-to-one correspondence with the disjoint union

$$\mathcal{P}_{\overline{X}}\sqcup\mathcal{P}_s\sqcup\{r_1\}\sqcup\{r_A \mid A\in\mathcal{P}_{\overline{X}}\setminus\mathcal{P}_x\}.$$

Its elements label the vertices. When $V(\sigma)\in\mathcal{P}_d$, it labels two vertices, one as an element of $\mathcal{P}_d$, and the other one as an element of $\mathcal{P}_s$.

The edge labeled by u_i starts[3] at the vertex labeled by $V(i-1)$, if $i>1$, and at the univalent vertex labeled by r_1 if $i=1$. It ends at the vertex labeled by $V(i)$, viewed as an element of $\mathcal{P}_s$.

For $A\in\mathcal{P}_{\overline{X}}\setminus\mathcal{P}_x$, the edge labeled by λ_A starts from the univalent vertex labeled by r_A, and it goes to the vertex labeled by A, viewed as an element of $\mathcal{P}_d$, at which the edge labeled by μ_A starts.

For $A\in\mathcal{P}_{\overline{X}}\setminus\mathcal{P}_x$ (resp. for $A\in\mathcal{P}_x$), the edge labeled by μ_A (resp. by $\tilde{\lambda}_A$) starts at the vertex A, viewed as an element of $\mathcal{P}_d$. It ends at the mother $m(A)$ of A (which is $V(\sigma)$, seen as an element of $\mathcal{P}_s$, when $A=V(\sigma)\in\mathcal{P}_d$). See Figure 14.6 for an example of a tree $\mathcal{T}^0$.

[3]unlike in Figure 8.6

According to Lemma 14.31, the variables of

$$\{u_i\}_{i\in\underline{\sigma}} \cup \{\mu_B\}_{B\in\mathcal{P}_{\overline{X}}\setminus\mathcal{P}_x} \cup \{\lambda_A\}_{A\in\mathcal{P}_{\overline{X}}\setminus\mathcal{P}_x} \cup \{\tilde{\lambda}_A = \lambda_A\mu_A\}_{A\in\mathcal{P}_x}$$

satisfy the equations that define $X(\mathcal{T}^0)$.

Lemma 14.35. *The dimension of $X(\mathcal{T}^0)$ is $d(\mathcal{T}^0) = \sharp\mathcal{P}_s + \sharp\mathcal{P}_{\overline{X}} - \sharp\mathcal{P}_x$.*

PROOF: According to Lemma 14.8, $d(\mathcal{T}^0) = \sharp E(\mathcal{T}^0) - \sharp L(\mathcal{T}^0) + 1$. The set of leaves different from r_1 is in one-to-one correspondence with $\mathcal{P}_{\overline{X}}$. $\square$

Lemma 14.36. *Let $c^0 = \psi\left((0),(0),(f_i^0),(w_B^0)\right)$ be a configuration of $C_V(R(\mathcal{C}))$ that satisfies the equations of Lemmas 14.27 and 14.31. Then c^0 belongs to $C_V(N_\infty, L, c^0)$.*

Let A be a univalent element of $\mathcal{P}_d$. Let $K_u(A)$ denote the set of univalent kids of A that do not contain $b(A)$, and let $K_t(A)$ denote the set of non-univalent kids of A that do not contain $b(A)$. Recall Notation 14.23. For any element D of $K_t(A)$, let B'_D denote the open ball $\mathring{B}(w_A^0(D),\varepsilon)$ of radius ε with center $w_A^0(D)$ in $\mathbb{R}^3$ if D is not normalizing, let B'_D denote the set of elements $w_A(D)$ of $\mathring{B}(w_A^0(D),\varepsilon)$ such that $|p_{\mathbb{R}}(w_A(D))| = 1$ (resp. such that $|p_{\mathbb{C}}(w_A(D))| = 1$) if D is v-normalizing (resp. if D is h-normalizing). For any element D of $K_u(A)$, let J'_D denote the interval $]p_{\mathbb{R}}(w_A^0(D)) - \varepsilon, p_{\mathbb{R}}(w_A^0(D)) + \varepsilon[$ if D is not v-normalizing, and $J'_D = \{p_{\mathbb{R}}(w_A^0(D))\}$ if D is v-normalizing. Set

$$W_A^L = \prod_{D\in K_u(A)} J'_D \times \prod_{D\in K_t(A)} B'_D.$$

When A is a non-univalent element of $\mathcal{P}_d$, set $W_A^L = W_A$. Recall that W_k^L has been introduced in Lemma 14.27.

Assume that $2\varepsilon < \lambda_A^0$ for all $A \in \mathcal{P}_x$. Let $\mathcal{P}_{x,hn}$ be the set of elements A of $\mathcal{P}_x$ such that $k_n(A)$ is a univalent horizontally normalizing kid. Set

$$W = [0,\varepsilon[^{\mathcal{P}_d\setminus\mathcal{P}_{\overline{X}}} \times \left(\prod_{k\in\underline{\sigma}} W_k^L\right) \times \left(\prod_{A\in\mathcal{P}_d} W_A^L\right) \times \prod_{A\in\mathcal{P}_x\setminus\mathcal{P}_{x,hn}}]\lambda_A^0 - \varepsilon, \lambda_A^0 + \varepsilon[.$$

With the notation 14.34, there exists a smooth map from $[0,\varepsilon[^{E(\mathcal{T}^0)}\times W$ to $C_V(R(\mathcal{C}))$ which restricts to $\left([0,\varepsilon[^{E(\mathcal{T}^0)}\cap X(\mathcal{T}^0)\right)\times W \times$ as a continuous injective map φ, whose image is an open neighborhood $N_\Gamma(c^0)$ of c^0 in $C_V(N_\infty, L, c^0)$.

PROOF: Here and in Notation 14.34, the parameters u_i and μ_B of $E(\mathcal{T}^0)$ are the initial parameters of c in the chart ψ of $C_V(R(\mathcal{C}))$, and the parameters λ_A are defined in Lemma 14.31. Lemma 14.31 implies that the parameters λ_A satisfy the equations of $X(\mathcal{T}^0)$ for configurations in $C_V(N_\infty, L, c^0)$. The factor of $[0,\varepsilon[^{\mathcal{P}_d \setminus \mathcal{P}_{\overline{X}}}$ of W contain the parameters μ_B for $B \in \mathcal{P}_d \setminus \mathcal{P}_{\overline{X}}$.

Lemma 14.27 shows how to express the parameter f_k of $\psi^{-1}(c)$, for configurations c in $C_V(N_\infty, L, c^0)$, as a smooth function of $W \times [0,\varepsilon[^{\underline{\sigma}}$, where the factor $[0,\varepsilon[^{\underline{\sigma}}$ contains the u_i.

We now construct the w_A as smooth maps from $[0,\varepsilon[^{E(\mathcal{T}^0)} \times W$ to $\check{\mathcal{S}}_{K(A)}(\mathbb{R}^3)$, in order to finish constructing a smooth map

$$\varphi \colon [0,\varepsilon[^{E(\mathcal{T}^0)} \times W \to C_{V(\Gamma)}(R(\mathcal{C})).$$

The coordinates of the non-univalent kids of A and the vertical coordinates of the univalent kids of A are part of W, and we do not change them. We only need to determine the horizontal coordinate $p_{\mathbb{C}}(w_A(B))$ for any univalent kid B of A as a smooth map.

Note that λ_A is a free parameter in $[0,\varepsilon[^{E(\mathcal{T}^0)} \times W$ when $A \in \mathcal{P}_{\overline{X}} \setminus \mathcal{P}_{x,hn}$.

Let A be a univalent element of $\mathcal{P}_d$. Set $k = k(A)$. Assume by induction that w_C and μ_C have already been constructed as smooth functions of $[0,\varepsilon[^{E(\mathcal{T}^0)} \times W$ for any $C \in \mathcal{P}_d$ that contains A, so that $\tilde{f}_k(b(A))$ is a smooth function of $[0,\varepsilon[^{E(\mathcal{T}^0)} \times W$ defined by the expression from Notation 14.29:

$$\tilde{f}_k(b(A)) = f_k(A) + \sum_{C \in \mathcal{P}_d | A \subset C} M_C w_C(A). \tag{14.7}$$

Equation 14.3 – or Equation 14.4 if $A \in \mathcal{P}_{\overline{X}}$ – determines $p_{\mathbb{C}}(w_A(B))$ as a smooth implicit function of, $\tilde{f}_k(b(A))$, the u_i, the μ_C for the $C \in \mathcal{P}_d$ such that $A \subseteq C$ – and λ_A if $A \in \mathcal{P}_{\overline{X}}$ –. The parameter μ_A is among the parameters unless $A \in \mathcal{P}_x$. In that case, if $A \notin \mathcal{P}_{x,hn}$, then μ_A is determined as $\frac{\tilde{\lambda}_A}{\lambda_A}$. So μ_A is this smooth function of our parameters. If $A \in \mathcal{P}_{x,hn}$, then $p_{\mathbb{C}}(w_C(k_n(A)))$ is still a smooth implicit function of $\tilde{f}_k(b(A))$, the u_i and the μ_C for the $C \in \mathcal{P}_d$ such that $A \subseteq C$, and λ_A, and the normalizing condition $|p_{\mathbb{C}}(w_A(k_n(A)))| = 1$ determines λ_A as a smooth implicit fonction of the given parameters.

We have constructed a smooth function $\varphi \colon [0,\varepsilon[^{E(\mathcal{T}^0)} \times W \to C_{V(\Gamma)}(R(\mathcal{C}))$. Set

$$\mathring{W} =]0,\varepsilon[^{\mathcal{P}_d \setminus \mathcal{P}_{\overline{X}}} \times \times \left(\prod_{k \in \underline{\sigma}} W_k^L\right) \times \left(\prod_{A \in \mathcal{P}_d} W_A^L\right) \times \prod_{A \in \mathcal{P}_x \setminus \mathcal{P}_{x,hn}}]\lambda_A^0 - \varepsilon, \lambda_A^0 + \varepsilon[.$$

Lemma 14.31 implies that φ maps $\left(]0,\varepsilon[^{E(\mathcal{T}^0)\cap X(\mathcal{T}^0)}\right)\times \mathring{W}$ to $C_V(N_\infty, L, c^0)$ (up to reducing ε). Since the closure of $\mathring{X}(\mathcal{T}^0)$ in $[0,\infty[^{E(\mathcal{T}^0)}$ is $X(\mathcal{T}^0)$, according to Lemma 14.12, $\varphi\left(\left([0,\varepsilon[^{E(\mathcal{T}^0)\cap X(\mathcal{T}^0)}\right)\times W\right) \subseteq C_V(N_\infty, L, c^0)$, too. In particular, any c^0 that satisfies the equations of Lemmas 14.27 and 14.31 is in $C_V(N_\infty, L, c^0)$.

Lemma 14.31 also implies that $N_\Gamma(c^0) \subseteq \varphi\left(\left([0,\varepsilon[^{E(\mathcal{T}^0)\cap X(\mathcal{T}^0)}\right)\times W\right)$ for an open neighborhood $N_\Gamma(c^0)$ of c^0 in $C_V(N_\infty, L, c^0)$. The injectivity of the restriction of φ to $\left(\left([0,\varepsilon[^{E(\mathcal{T}^0)\cap X(\mathcal{T}^0)}\right)\times W\right)$ comes from the fact that the parameters λ_A for $A \in \mathcal{P}_x$ are well defined by Equation 14.4, and the equations of Lemma 14.31 determine the other ones[4]. $\square$

Proof of Proposition 14.20: Lemmas 14.25, 14.26, 14.28 and 14.32 show that a configuration $c^0_{V(\Gamma)}$ of C_L must satisfy the conditions of the statement of Proposition 14.20. According to Lemma 14.36, if a configuration $c^0_{V(\Gamma)}$ satisfies the conditions of the statement of Proposition 14.20, its restriction c^0 to $V = p_b(c_{V(\Gamma)})^{-1}(\infty)$ is in $C_V(N_\infty, L, c^0)$, so $c^0_{V(\Gamma)}$ is in C_L. $\square$

Proof of Proposition 14.21: The strata of Proposition 14.21 are smooth submanifolds of $C_{V(\Gamma)}(R(\mathcal{C}))$, since they correspond to the locus where all the variables associated to $\mathcal{T}^0$ are zero, where the parameters λ_A are either zero or not zero on the whole stratum, in the charts of Lemma 14.36. Let us assume $V = V(\Gamma)$. Then the codimension of the stratum of c^0 is $d(\mathcal{T}^0) + \sharp(\mathcal{P}_d \setminus \mathcal{P}_{\overline{X}})$, which is $\sharp\mathcal{P}_s + \sharp\mathcal{P}_d - \sharp\mathcal{P}_x$, according to Lemma 14.35. $\square$

Lemma 14.37. *The codimension-one faces of $C^f(R(\mathcal{C}), L;\Gamma)$ are those listed in Theorem 14.16. In a neighborhood of these faces, $C^f(R(\mathcal{C}), L;\Gamma)$ has the structure of a smooth manifold with boundary.*

Let $s = s(\vec{N})$ be the orthogonal symmetry that leaves the horizontal plane unchanged and reverses the vertical real line. A configuration $c^0_{V(\Gamma)|V} = c^0 = \left(T_0\phi_\infty \circ f_1^0, \left(T_{[f_{k(A)}(A)]}\phi_{\infty *} \circ w_A^0\right)_{A\in\mathcal{P}_X}, (\lambda_A^0)_{A\in\mathcal{P}_X}\right)$ of a T-face is the limit at $t=0$ of a family of injective configurations $c(t)_{t\in]0,\varepsilon[}$ on the vertical parts of the tangle, far above or far below, such that $c(t)_{|A} = s\circ w_A^0$ up to dilation and translation, for any $A \in \mathcal{P}_X(=\mathcal{P}_d = \mathcal{P}_x)$. In particular, for an edge

[4] We can remove these other parameters λ_C, for $C \in \mathcal{P}_{\overline{X}}\setminus\mathcal{P}_x$, from our parametrization for the statement, but it was more convenient to keep them for the proof, since the definition of the λ_A, for $A \in \mathcal{P}_{x,hn}$, involves the λ_C for the C that contain A, and these λ_C are functions of λ_A.

$e=(v_1,v_2)$ whose vertices are in A,

$$p_\tau \circ p_e(c^0) = \frac{s\circ w_A^0(v_2) - s\circ w_A^0(v_1)}{\parallel s\circ w_A^0(v_2) - s\circ w_A^0(v_1) \parallel}.$$

For an edge $e=(v_1,v_2)$ whose vertices are in different kids of V,

$$\begin{aligned} p_\tau \circ p_e(c^0) &= \tfrac{\phi_\infty\circ f_1^0(v_2)-\phi_\infty\circ f_1^0(v_1)}{\|\phi_\infty\circ f_1^0(v_2)-\phi_\infty\circ f_1^0(v_1)\|} \\ &= \tfrac{\|f_1^0(v_1)\|^2 f_1^0(v_2)-\|f_1^0(v_2)\|^2 f_1^0(v_1)}{\|\|f_1^0(v_1)\|^2 f_1^0(v_2)-\|f_1^0(v_2)\|^2 f_1^0(v_1)\|}. \end{aligned}$$

For an edge $e=(v_1,v_2)$ such that $v_1\in V$ and $v_2 \notin V$, $p_\tau\circ p_e(c^0) = -\frac{f_1^0(v_1)}{\|f_1^0(v_1)\|}$ and for an edge $e=(v_1,v_2)$ such that $v_2\in V$ and $v_1\notin V$, $p_\tau\circ p_e(c^0)=\frac{f_1^0(v_2)}{\|f_1^0(v_2)\|}$.

Proof: Let $\partial_\infty(C^f(R(\mathcal{C}),L;\Gamma))$ be the subspace of $C^f(R(\mathcal{C}),L;\Gamma)$ consisting of the configurations as above that map at least a univalent vertex to ∞. Outside this subspace, $C^f(R(\mathcal{C}),L;\Gamma)=C(R(\mathcal{C}),L;\Gamma)$, and $C^f(R(\mathcal{C}),L;\Gamma)$ has the structure of a smooth manifold with ridges. Recall from Remark 14.22 that the only codimension-one new parts are the T-faces of Theorem 14.16, where $\mathcal{P}_s=\{B\sqcup\sqcup_{j\in I}B_j\}$ and $\mathcal{P}_d=\mathcal{P}_X=\mathcal{P}_x=\{B_j \mid j\in I\}$. Lemmas 14.36 and 14.13 imply that these strata arise as codimension-one faces of $C^f(R(\mathcal{C}),L;\Gamma)$, along which $C^f(R(\mathcal{C}),L;\Gamma)$ is a smooth manifold with boundary.

According to Lemma 14.25, for any univalent vertex of $B\sqcup\sqcup_{j\in I}B_j$,

$$f_1^0(b) = \pm \parallel f_1^0(b) \parallel \vec{N},$$

while Lemma 14.28 implies that the restriction of c^0 to any A of $\mathcal{P}_X=\{B_j \mid j\in I\}$ is represented by $s\circ w_A^0$, up to translation and dilation, as a configuration of $\mathbb{R}^3$. Furthermore, according to Lemma 14.32,

$$p_{\mathbb{C}}(w_A^0(b)) = \lambda_A^0 \parallel f_1^0(b(A)) \parallel^2 (y(b)-y(b(A)))$$

for any $A\in\mathcal{P}_X$, and for any $b\in A$. Therefore, the configuration c^0 is the limit of the following family $c(t)$ of configurations, indexed by $t\in]0,\varepsilon[$, where
$c(t)(b)=\frac{1}{t}\frac{f_1^0(b)}{\|f_1^0(b)\|^2}$ for any trivalent vertex of B,
$c(t)(b)=(y(b),0)+\frac{1}{t}\frac{f_1^0(b)}{\|f_1^0(b)\|^2}$ for any univalent vertex of B,
$c(t)(b)=(y(b(A)),0)+\frac{1}{t}\frac{f_1^0(b(A))}{\|f_1^0(b(A))\|^2}+\frac{s\circ w_A^0(b)}{\lambda_A^0\|f_1^0(b(A))\|^2}$, for any vertex b of an element A of $\mathcal{P}_X=\{B_j \mid j\in I\}$. So c^0 is the limit at $t=0$ of the family of injective configurations $c(t)_{t\in]0,\varepsilon[}$, and $c(t)_{|A}=s\circ w_A^0$ up to dilation and translation, for any $A\in\mathcal{P}_X$. $\square$

Theorem 14.16 is now proved. $\square$

14.3 Variations of integrals on configuration spaces of long tangles

In this section, we prove Theorem 12.7.

Lemma 14.38. *For any two propagating forms ω and ω' of $C_2(R)$ (as in Definition 3.11) that coincide on $\partial C_2(R)$, there exists a one-form η of $C_2(R)$ that vanishes on $\partial C_2(R)$ such that $\omega' = \omega + d\eta$. In particular, for any two homogeneous propagating forms ω and ω' of $C_2(R)$ as in Definition 3.13 that coincide on UB_R, there exists a one-form η of $C_2(R)$ that vanishes on $\partial C_2(R)$ such that $\omega' = \omega + d\eta$.*

PROOF: Exercise. See the proof of Lemma 4.2. $\square$

Lemma 14.39. *The element $Z_n(\mathcal{C}, L, (\omega(i)))$ of $\mathcal{A}_n(\mathcal{L})$ is independent of the chosen homogeneous propagating forms $\omega(i)$ of $(C_2(R(\mathcal{C})), \tau)$, under the assumptions of Theorem 12.7.*

More generally, if the $\omega(i)$ are only assumed to be homogeneous propagating forms of $C_2(R(\mathcal{C}))$, $Z_n(\mathcal{C}, L, (\omega(i)))$ depends only on $(\mathcal{C}, L \cap \mathcal{C}, \tau)$ and on the restrictions of the $\omega(i)$ to $U\mathcal{C}$.

PROOF: By Lemma 14.38, it suffices to prove that Z does not vary when $\omega(i)$ is changed to $\omega(i) + d\eta$ for a one-form η on $C_2(R(\mathcal{C}))$ that vanishes on $\partial C_2(R(\mathcal{C}))$. Let $\Omega_\Gamma = \bigwedge_{e \in E(\Gamma)} p_e^*(\omega(j_E(e)))$ and let $\tilde{\Omega}_\Gamma$ be obtained from Ω_Γ by replacing $\omega(i)$ by η. The variations of the integrals $\int_{(C(R(\mathcal{C}),L;\Gamma),o(\Gamma))} \Omega_\Gamma$ are computed with Stokes' theorem, as the sum over the codimension-one faces F of $C(R(\mathcal{C}), L; \Gamma)$ of $\int_F \tilde{\Omega}_\Gamma$, as allowed by Lemma 14.17.

These faces are the faces listed in Theorem 14.16. The arguments of Lemmas 9.9, 9.10, 9.11, 9.13, 9.14 allow us to get rid of all the faces, except for the faces in which some vertices are at ∞, and for the faces $F(\check{\Gamma}_A, L, \Gamma)$ in which $\check{\Gamma}_A$ is a connected diagram on $\mathbb{R}$ (these faces are components of $F(A, L, \Gamma)$ as in the proof of Lemma 10.18) and Γ is a diagram that contains $\check{\Gamma}_A$ as a subdiagram on a component $\mathcal{L}_j$ of L. The contribution of the faces $F(\check{\Gamma}_A, L, \Gamma)$ is zero when $i \notin j_E(E(\check{\Gamma}_A))$ for dimension reasons, and it is zero when $i \in j_E(E(\check{\Gamma}_A))$ because η vanishes on $\partial C_2(R(\mathcal{C}))$, and we are left with the faces for which some vertices are at ∞. Let F be such a face. Let V be the set of vertices mapped to ∞ in F, let E_∞ be the set of edges between elements of V and let E_m denote the set of edges with one end in V. When $i \in j_E(E_\infty \cup E_m)$, the contribution vanishes because η vanishes on $\partial C_2(R(\mathcal{C}))$.

Assume $i \notin j_E(E_\infty \cup E_m)$. The face F is diffeomorphic to a product by

$$\check{C}_{V(\Gamma)\setminus V}(\check{R}(\mathcal{C}), L; \Gamma),$$

whose dimension is

$$3\sharp(T(\Gamma) \cap (V(\Gamma) \setminus V)) + \sharp(U(\Gamma) \cap (V(\Gamma) \setminus V)),$$

of a space C_V of dimension

$$3\sharp(T(\Gamma) \cap V) + \sharp(U(\Gamma) \cap V) - 1,$$

along which $\bigwedge_{e \in E_\infty \cup E_m} p_e^*(\omega(j_E(e)))$ has to be integrated. Since the degree of this form, which is $2\sharp(E_\infty \cup E_m)$, is larger than the dimension of C_V as a count of half-edges shows, the faces for which some vertices are at ∞ (including the T-faces), do not contribute either. □

Proposition 14.40. *Let $L\colon \mathcal{L} \hookrightarrow R(\mathcal{C})$ denote a long tangle representative in a rational homology cylinder. Let τ denote an asymptotically standard parallelization of $R(\mathcal{C})$. Let ω_0 and ω_1 be two homogeneous propagating forms of $C_2(R(\mathcal{C}))$ (as in Definitions 3.11 and 3.13). Let $\tilde{\omega}$ be a closed 2-form on $[0,1] \times \partial C_2(R(\mathcal{C}))$ whose restriction $\tilde{\omega}(t)$ to $\{t\} \times \left(\partial C_2(R(\mathcal{C})) \setminus UB_{R(\mathcal{C})}\right)$ is $p_\tau^*(\omega_{S^2})$, for any $t \in [0,1]$, and such that the restriction of ω_i to $\partial C_2(R(\mathcal{C}))$ is $\tilde{\omega}(i)$, for $i \in \{0,1\}$. For any component K_j of $L = \sqcup_{j=1}^k K_j$, define $I_j = \sum_{\Gamma_B \in \mathcal{D}^c(\mathbb{R})} \zeta_{\Gamma_B} I(\Gamma_B, K_j, \tilde{\omega})$, where*

$$I(\Gamma_B, K_j, \tilde{\omega}) = \int_{u \in [0,1]} \int_{w \in K_j \cap B_{R(\mathcal{C})}} \int_{\check{\mathcal{S}}(T_w \check{R}(\mathcal{C}), \vec{t}_w; \Gamma_B)} \bigwedge_{e \in E(\Gamma_B)} p_e^*(\tilde{\omega}(u))[\Gamma_B]$$

and $\vec{t}_w$ denotes the unit tangent vector to K_j at w.
(The notation $\check{\mathcal{S}}(T_w \check{R}(\mathcal{C}), \vec{t}_w; \Gamma_B)$ is introduced before Lemma 8.15 and $\mathcal{D}^c(\mathbb{R})$ is introduced at the beginning of Section 10.5.) Define

$$z(\tilde{\omega}) = \sum_{n \in \mathbb{N}} z_n\left([0,1] \times UB_{R(\mathcal{C})}; \tilde{\omega}\right)$$

as in Corollary 9.4. Then

$$Z(\mathcal{C}, L, \omega_1) = \left(\prod_{j=1}^k \exp\left(I_j\right) \sharp_j\right) Z\left(\mathcal{C}, L, \omega_0\right) \exp\left(z(\tilde{\omega})\right).$$

Proof: According to Proposition 10.17, this statement holds when L is a link. Using Notation 7.16, it suffices to prove that

$$\check{Z}(\mathcal{C}, L, \omega_1) = \left(\prod_{j=1}^k \exp\left(I_j\right) \sharp_j\right) \check{Z}\left(\mathcal{C}, L, \omega_0\right).$$

As in the proof of Lemma 14.39 above, the only faces that contribute to the variation of $\check{Z}(\mathcal{C}, L, \omega_t)$ are the faces $F(\check{\Gamma}_A, L, \Gamma)$ for which $\check{\Gamma}_A$ is a connected diagram on $\mathbb{R}$, and Γ is a diagram that contains $\check{\Gamma}_A$, as a subdiagram on a component $\mathcal{L}_j$ of L. Their contribution yields the result as in the proof of Proposition 10.17. $\square$

Lemma 14.41. *Recall Definition 12.6 of $I_\theta(K,\tau)$ for a long component $K\colon \mathbb{R} \hookrightarrow \check{R}(\mathcal{C})$ of a tangle in a parallelized $\mathbb{Q}$–cylinder $(\mathcal{C},\tau)$. Let $\mathcal{C}$ be a $\mathbb{Q}$–cylinder and let $(\tau_t)_{t\in[0,1]}$ be a smooth homotopy of parallelizations of $\mathcal{C}$ (among parallelizations that are standard near the boundary of $\mathcal{C}$). For any component K of a tangle in $\mathcal{C}$,*

$$I_\theta(K,\tau_u) - I_\theta(K,\tau_0) = 2\int_{\cup_{t\in[0,u]} p_{\tau_t}(U^+K)} \omega_{S^2}.$$

Proof: When K is closed, I_θ is defined in Lemma 7.15, and the lemma follows from Proposition 10.13 and Lemma 10.19. Lemma 9.1 implies the existence of a closed 2-form ω on $[0,1] \times C_2(R(\mathcal{C}))$ that restricts to $\{t\} \times C_2(R(\mathcal{C}))$ as a homogeneous propagating form of $(C_2(R(\mathcal{C})), \tau_t)$ for all $t \in [0,1]$. The integral of this form on $\partial\left([0,u] \times C(R(\mathcal{C}), K; \hat{\updownarrow}^{\mathrm{K}})\right)$ is zero, and it is half the difference between the two sides of the equality to be proved when K is a long component. $\square$

Proof of Theorem 12.7: Let ω be a homogeneous propagating form of $(C_2(R(\mathcal{C})), \tau)$. Let us study the variation of $\check{Z}(\mathcal{C}, L, \tau) = \left(\check{Z}_n(\mathcal{C}, L, \omega)\right)_{n\in\mathbb{N}}$ when τ varies inside its homotopy class.

Let $(\tau(t))_{t\in[0,1]}$ be a smooth homotopy of parallelizations of $\mathcal{C}$ standard near the boundary of $\mathcal{C}$. Set

$$\check{Z}(t) = \check{Z}(\mathcal{C}, L, \tau(t)).$$

$$\frac{\partial}{\partial t}\check{Z}(t) = \sum_{j=1}^{k} \left(\frac{\partial}{\partial t}\left(2\int_{\cup_{u\in[0,t]} p_{\tau(u)}(U^+K_j)} \omega\right)\alpha\sharp_j\right)\check{Z}(t)$$

as in Lemma 10.19. Lemma 14.41 implies that

$$I_\theta(K_j, \tau(t)) - I_\theta(K_j, \tau(0)) = 2\int_{\cup_{u\in[0,t]} p_{\tau(u)}(U^+K_j)} \omega$$

for any j, conclude as in Corollary 10.20 that

$$\prod_{j=1}^{k} \left(\exp(-I_\theta(K_j, \tau(t))\alpha)\sharp_j\right)\check{Z}(t)$$

is constant, and note that $\check{Z}_1(\mathcal{C}, K_j, \tau(t)) = \frac{1}{2} I_\theta(K_j, \tau(t))\,[\quad]$, for an interval component K_j.

Proposition 14.40 and Lemma 9.1 imply that changing the trivialization τ in a ball B_τ that does not meet the tangle does not change $\check{Z}$ (where the form ω^∂ of Lemma 9.1 is easily assumed to pull back through the projection of $[0,1] \times (\partial C_2(R(\mathcal{C})) \setminus U(B_\tau))$ onto $\partial C_2(R(\mathcal{C})) \setminus U(B_\tau)$ on $[0,1] \times (\partial C_2(R(\mathcal{C})) \setminus U(B_\tau)))$. Then the proof of Theorem 12.7 can be concluded like the proof of Theorem 7.20 at the end of Section 10.5, with the following additional argument for the strands that go from bottom to bottom or from top to top. In the proof of Theorem 7.20, we assumed that $p_\tau(U^+K_j) = v$ for some $v \in S^2$ and that g maps K_j to rotations with axis v, for any $j \in \underline{k}$, in order to ensure that ${}^\tau\psi(g^{-1})$ induces a diffeomorphism of $\cup_{w \in K_j} \check{\mathcal{S}}(T_w\check{R}, \vec{t}_w; \check{\Gamma}_B)$. Without loss of generality, we instead assume that $v = \vec{N}$, and that $p_\tau(U^+K_j) = \pm\vec{N}$, for all components K_j of L, except possibly in a neighborhood of the boundary of $\mathcal{C}$, which is mapped to 1 by g, (so that $p_\tau(U^+K_j)$ can move from $\vec{N}$ to $-\vec{N}$, in that neighborhood). $\square$

Chapter 15

The invariant Z as a holonomy for braids

In this chapter, we interpret the extension of $\mathcal{Z}^f$ to long tangles of the previous chapter as a holonomy for long braids, and we study it as such.

Recall the compactification $\mathcal{S}_V(T)$ of the space $\check{\mathcal{S}}_V(T)$ of injective maps from a finite set V to a vector space T up to translation and dilation, from Theorem 8.11. Let B be a finite set of cardinality at least 2. Let Γ be a Jacobi diagram on a disjoint union of lines $\mathbb{R}_b$ indexed by elements b of B. Let $p_B \colon U(\Gamma) \to B$ be the natural map induced by i_Γ. We assume that p_B is onto. Let $U_b = U_b(\Gamma) = p_B^{-1}(b)$ be the set of univalent vertices of Γ that are sent to $\mathbb{R}_b$ by i_Γ. Let $\check{\mathcal{V}}(\Gamma) \subset \check{\mathcal{S}}_{V(\Gamma)}(\mathbb{R}^3)$ be the quotient by the translations and the dilations of the space of injective maps c from $V(\Gamma)$ to $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ that map $U_b(\Gamma)$ to a vertical line $y(c,b) \times \mathbb{R}$, with respect to the order induced by i_Γ, for each $b \in B$, so that the planar configuration $y(c,.) \colon B \to \mathbb{C}$ is injective. Let $\mathcal{V}(\Gamma)$ denote the closure of the image of $\check{\mathcal{V}}(\Gamma)$ in $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \times \mathcal{S}_B(\mathbb{C})$ under the map $(c \mapsto (c, y(c,.)))$.

15.1 On the structure of $\mathcal{V}(\Gamma)$

In this section, we investigate the structure of $\mathcal{V}(\Gamma)$, as we did in Section 14.2 for $C(R(\mathcal{C}), L; \Gamma)$.

Lemma 15.1. *An element (c,y) of $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \times \mathcal{S}_B(\mathbb{C})$ is in $\mathcal{V}(\Gamma)$ if and only if*

- *for any $b \in B$, for any $(v_1, v_2) \in U_b^2$, the restriction of c to $\{v_1, v_2\}$ is vertical, and its direction is that prescribed by i_Γ,*

- *for any pair* (b_1, b_2) *of distinct elements of* B, $\forall (v_1, v_2) \in U_{b_1} \times U_{b_2}$, *there exists* $\beta \in \mathbb{R}^+$ *such that the restriction* $c_{|\{v_1,v_2\}}$ *of* c *to* $\{v_1, v_2\}$ *satisfies*

$$\left(p_{\mathbb{C}}\left(c_{|\{v_1,v_2\}}(v_2) - c_{|\{v_1,v_2\}}(v_1)\right)\right) = \beta(y_{|\{b_1,b_2\}}(b_2) - y_{|\{b_1,b_2\}}(b_1))$$

(where β *is determined up to a multiplication by a positive number),*

- *for any triple* (b_1, b_2, b_3) *of distinct elements of* B, $\forall (v_1, v_2, v_3) \in U_{b_1} \times U_{b_2} \times U_{b_3}$, *there exists* $\beta \in \mathbb{R}^+$ *such that*

$$\begin{aligned}&\left(p_{\mathbb{C}}\left(c_{|\{v_1,v_2,v_3\}}(v_2) - c_{|\{v_1,v_2,v_3\}}(v_1)\right), p_{\mathbb{C}}\left(c_{|\{v_1,v_2,v_3\}}(v_3) - c_{|\{v_1,v_2,v_3\}}(v_1)\right)\right)\\ &\quad = \beta\left(y_{|\{b_1,b_2,b_3\}}(b_2) - y_{|\{b_1,b_2,b_3\}}(b_1), y_{|\{b_1,b_2,b_3\}}(b_3) - y_{|\{b_1,b_2,b_3\}}(b_1)\right)\end{aligned}$$

in $\mathbb{C}^2$.

It is easy to see that an element of $\mathcal{V}(\Gamma)$ must satisfy these conditions, since they are closed and satisfied on $\check{\mathcal{V}}(\Gamma)$. The converse will be proved after Lemma 15.5.

Let (c^0, y^0) satisfy the conditions of Lemma 15.1. We are going to study how a neighborhood $N(c^0, y^0)$ of (c^0, y^0) in $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \times \mathcal{S}_B(\mathbb{C})$ intersects $\check{\mathcal{V}}(\Gamma)$.

Notation 15.2. As in Theorem 8.26, the configuration c^0 in $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$ is described by a Δ-parenthesization $\mathcal{P}$ of $V = V(\Gamma)$ (as in Definition 8.23) and

$$\left(c^0_Z \in \check{\mathcal{S}}_{K(Z)}(\mathbb{R}^3)\right)_{Z \in \mathcal{P}}.$$

The configurations c in a neighborhood of c^0 in $\mathcal{S}_V(\mathbb{R}^3)$ may be expressed as

$$\begin{aligned} c((\mu_Z), (c_Z)) &= \sum_{Z \in \mathcal{P}} \left(\prod_{Y \in \mathcal{P} | Z \subseteq Y \subset V} \mu_Y\right) c_Z \\ &= c_V + \sum_{Z \in D(V)} \mu_Z \left(c_Z + \sum_{Y \in D(Z)} \mu_Y \left(c_Y + \dots\right)\right). \end{aligned}$$

for $((\mu_Z)_{Z \in \mathcal{P} \setminus \{V\}}, (c_Z)_{Z \in \mathcal{P}}) \in ([0, \varepsilon[)^{\mathcal{P} \setminus \{V\}} \times \prod_{Z \in \mathcal{P}} W_Z$, as in Lemma 8.25.

Similarly, the configuration y^0 in $\mathcal{S}_B(\mathbb{C})$ may be written as

$$\left(y^0_D \in \check{\mathcal{S}}_{K(D)}(\mathbb{C})\right)_{D \in \mathcal{P}}$$

for a Δ-parenthesization $\mathcal{P}_B$ of B, and the configurations y in a neighborhood of y^0 in $\mathcal{S}_B(\mathbb{C})$ may be expressed as

$$\begin{aligned} y((u_D), (y_D)) &= \sum_{D \in \mathcal{P}_B} \left(\prod_{E \in \mathcal{P}_B | D \subseteq E \subset B} u_E\right) y_D \\ &= y_B + \sum_{D \in D(B)} u_D \left(y_D + \sum_{E \in D(D)} u_E \left(y_E + \dots\right)\right). \end{aligned}$$

for $((u_D)_{D\in\mathcal{P}_B\setminus\{B\}},(y_D)_{D\in\mathcal{P}_B}) \in ([0,\varepsilon[)^{\mathcal{P}_B\setminus\{B\}} \times \prod_{D\in\mathcal{P}_B} N_D$.

We normalize the y_D by choosing basepoints $b(D)$ for the $D \in \mathcal{P}_B$ and imposing $y_D(b(D)) = 0$ and $\sum_{E\in K(D)} \parallel y_D(E) \parallel^2 = 1$.

For a set Y of $\mathcal{P}$, we also choose a basepoint $b(Y)$, which is univalent if there is a univalent vertex in Y. We choose a kid $k_n(Y)$ such that $|p_{\mathbb{R}}(c^0_Y(k_n(Y))) - p_{\mathbb{R}}(c^0_Y(b(Y)))|$ or $|p_{\mathbb{C}}(c^0_Y(k_n(Y))) - p_{\mathbb{C}}(c^0_Y(b(Y)))|$ is maximal in the set

$$\{|p_{\mathbb{R}}(c^0_Y(k)) - p_{\mathbb{R}}(c^0_Y(b(Y)))|, |p_{\mathbb{C}}(c^0_Y(k)) - p_{\mathbb{C}}(c^0_Y(b(Y)))| \mid k \in K(Y)\}$$

and we call it the *normalizing kid* of Y. If $|p_{\mathbb{R}}(c^0_Y(k_n(Y))) - p_{\mathbb{R}}(c^0_Y(b(Y)))| \geq |p_{\mathbb{C}}(c^0_Y(k_n(Y))) - p_{\mathbb{R}}(c^0_Y(b(Y)))|$, we say that $k_n(Y)$ is *vertically normalizing* or *v-normalizing*, and we normalize the configurations c_Y in a neighborhood of c^0_Y by imposing $c_Y(b(Y)) = 0$ and $|p_{\mathbb{R}}(c_Y(k_n(Y)))| = 1$. Otherwise, we say that $k_n(Y)$ is *horizontally normalizing* or *h-normalizing*, and we normalize the configurations c_Y in a neighborhood of c^0_Y by imposing $c_Y(b(Y)) = 0$ and $|p_{\mathbb{C}}(c_Y(k_n(Y)))| = 1$. (These normalizations are compatible with the smooth structure of $\mathcal{S}_V(\mathbb{R}^3)$.)

In our neighborhood, we also impose that $\parallel c_Y(k) - c^0_Y(k) \parallel < \varepsilon$, for any kid k of Y, for a small $\varepsilon \in]0,1[$. So the manifold W_Y is diffeomorphic to the product of the product, over the non-normalizing kids k of Y that do not contain $b(Y)$, of balls $\mathring{B}(c^0_Y(k),\varepsilon)$, by the set of elements $c_Y(k_n(Y))$ of $\mathring{B}(c^0_Y(k_n(Y)),\varepsilon)$, such that $|p_{\mathbb{R}}(c_Y(k_n(Y)))| = 1$ (resp. such that $|p_{\mathbb{C}}(c_Y(k_n(Y)))| = 1$) if k is v-normalizing (resp. if k is h-normalizing). Note that $|p_{\mathbb{C}}(c_Y(k))| < 2$ for any $k \in K(Y)$.

For a set Y of $\mathcal{P}$, let $B(Y) \subseteq B$ be the set of (labels of) the connected components of its univalent vertices, and let $\hat{B}(Y)$ denote the smallest element of $\mathcal{P}_B$ such that $B(Y) \subseteq \hat{B}(Y)$, if $\sharp B(Y) \geq 2$. If $\sharp B(Y) = 1$, set $\hat{B}(Y) = B(Y)$. If $B(Y) \neq \emptyset$, the set Y is called *univalent*.

For $D \in \mathcal{P}_B$, define the set $\mathcal{P}_{X,D}$ of elements Y of $\mathcal{P}$ such that $\hat{B}(Y) = D$, and $\hat{B}(Z) \neq D$ for every daughter Z of Y. Note that any element Y of $\mathcal{P}_{X,D}$ has at least two kids Y_a and Y_b such that $B(Y_a) \neq D$ and $B(Y_b) \neq D$. Set

$$\mathcal{P}'_X = \cup_{D\in\mathcal{P}_B}\mathcal{P}_{X,D}.$$

Let $\widehat{\mathcal{P}'}_X$ be the subset of $\mathcal{P}$ consisting of the univalent sets Z of $\mathcal{P}$ such that $\sharp B(Z) \geq 2$.

Lemma 15.3. *Let (c^0, y^0) be a configuration parametrized as above, which satisfies the conditions of Lemma 15.1. For any $Y \in \mathcal{P} \setminus \mathcal{P}'_X$, for any two univalent kids Y_1 and Y_2 of Y, $p_{\mathbb{C}}(c^0_Y(Y_2) - c^0_Y(Y_1)) = 0$. For any $Y \in \mathcal{P}_{X,D}$,* *there exists a unique*[1] $\lambda^0(Y) \in \mathbb{R}^+$ *such that for any two univalent kids* Y_1 *and* Y_2 *of* Y *such that* $B(Y_1) \subset D_1$*, and* $B(Y_2) \subset D_2$*, where* D_1 *and* D_2 *are two kids of* D*,*

$$p_{\mathbb{C}}(c^0_Y(Y_2) - c^0_Y(Y_1)) = \lambda^0(Y)(y^0_D(D_2) - y^0_D(D_1)).$$

Conversely, if an element (c^0, y^0) *satisfies the above properties and the first condition of Lemma 15.1, then it satisfies the conditions of Lemma 15.1.*

PROOF: Let $Y \in \mathcal{P}$. The first condition of Lemma 15.1 ensures that when v and v' belong to $Y \cap U_b$,

$$p_{\mathbb{C}} \circ c^0_Y(v) = p_{\mathbb{C}} \circ c^0_Y(v') = (p_{\mathbb{C}} \circ c^0_Y)(b \in B).$$

The second and the third condition ensure that there exists a unique $\lambda^0(Y) \in \mathbb{R}^+$ such that $p_{\mathbb{C}} \circ c^0_Y$ viewed as a map on $B(Y)$ coincides with $\lambda^0(Y) y^0_{\hat{B}(Y)|B(Y)}$ up to translation, when $Y \in \widehat{\mathcal{P}'}_X$. Since $p_{\mathbb{C}} \circ c^0_Y$ is constant on $B(Z)$ for any kid Z of Y, if there exists such a kid Z such that $Z \in \widehat{\mathcal{P}'}_X$ and $\hat{B}(Z) = \hat{B}(Y)$, then $y^0_{\hat{B}(Y)|B(Z)}$ is not constant and $\lambda^0(Y)$ must be equal to zero.

The last assertion is an easy exercise. $\square$

In order to finish the proof of Lemma 15.1, we are going to prove that the configurations that satisfy the conditions of its statement are in $\mathcal{V}(\Gamma)$. We take a closer look at the structure of $\mathcal{V}(\Gamma)$.

For a univalent $Y \in \mathcal{P}$, define $d(Y) \in B$ such that $b(Y) \in U_{d(Y)}$. Also assume that $p_B(b(V))$ is the basepoint b_0 of B. As always, our basepoints satisfy that if $Z \subset Y$ and if $b(Y) \in Z$, then $b(Z) = b(Y)$.

For $D \in \mathcal{P}_B$, set $U_D = \prod_{E \in \mathcal{P}_B \setminus \{B\} | D \subseteq E} u_E$. For $d \in D$, set

$$\tilde{y}_D(d) = y_D(d) + \sum_{E \in \mathcal{P}_B | d \in E \subset D} \left(\prod_{F \in \mathcal{P}_B | E \subseteq F \subset D} u_F \right) y_E(d).$$

For $Y \in \mathcal{P}_X$, set $M_Y = \prod_{Z \in \mathcal{P} \setminus \{V\} | Y \subseteq Z} \mu_Z$.

Lemma 15.4. *With the given normalization conditions, and with the above notation, there exist continuous maps* λ *and* $\lambda(Y)$*, for* $Y \in \mathcal{P}'_X$*, from* $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$ *to* $\mathbb{R}^+$ *such that, for any configuration* (c, y) *of* $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$*,*

- $p_{\mathbb{C}} \circ c(v) = \lambda y(d)$ *for any* $d \in B$ *and for any* $v \in U_d$*,*

[1] $\lambda^0(Y)$ depends on the fixed normalizations of c^0_Y and y^0_D.

- *for $D \in \mathcal{P}_B$, if $Y \in \mathcal{P}_{X,D}$, and if Y_a and Y_b are two univalent kids of Y, then*

$$p_{\mathbb{C}}\left(c_Y(Y_b) - c_Y(Y_a)\right) = \lambda(Y)\left(\tilde{y}_D(d(Y_b)) - \tilde{y}_D(d(Y_a))\right),$$

$\lambda(Y)(c^0) = \lambda^0(Y)$ *with the notation of Lemma 15.3, and*

$$*(Y): \quad \lambda(Y)M_Y = \lambda U_D,$$

where $D = \hat{B}(Y)$.

When $V \in \mathcal{P}'_X$, $*(V)$ is equivalent to $\lambda(V) = \lambda$. When $V \notin \mathcal{P}'_X$, we also use both notations $\lambda(V)$ and λ for λ, depending on the context.

Proof: In $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$, where $\prod_{D \in \mathcal{P}_B \setminus \{B\}} u_D \times \prod_{W \in \mathcal{P} \setminus \{V\}} \mu_W \neq 0$, there exists some $\lambda > 0$ such that $p_{\mathbb{C}} \circ c(v) = \lambda y(b)$ for any $b \in U_b$ and for any $v \in U_b$. This map λ, starting from $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$, can be extended continuously on $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$, as follows. Let $b \in B$ be such that b and b_0 belong to different kids of B and let $v \in U_b$. Then $p_{\mathbb{C}}\left(c(v)\right) = \lambda y(b)$ on $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$ and $y(b)$ does not vanish on $N(c^0, y^0)$. The closed condition that $p_{\mathbb{C}}\left(c(v)\right)$ and $y(b)$ are colinear and that their scalar product is nonnegative is satisfied on $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$. It allows us to define $\lambda(v)$ such that $p_{\mathbb{C}}\left(c(v)\right) = \lambda(v)y(b)$ on $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$ and $\lambda(v)$ is continuous. Now, since $\lambda(v) = \lambda$ is independent of v as above on $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$, it is also on $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$. Set $\lambda = \lambda(v)$. Then $p_{\mathbb{C}} \circ c(v) = \lambda y(b)$ for any $b \in U_b$ and for any $v \in U_b$ on $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$, this is also true on $N(c^0, y^0) \cap \mathcal{V}(\Gamma)$.

Let $Y \in \mathcal{P}$. Let Y_a and Y_b be two univalent kids of Y. If $c \in N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$, then

$$M_Y\left(p_{\mathbb{C}}\left(c_Y(Y_b) - c_Y(Y_a)\right)\right) = \lambda U_{\hat{B}(Y)}\left(\tilde{y}_{\hat{B}(Y)}(d(Y_b)) - \tilde{y}_{\hat{B}(Y)}(d(Y_a))\right)$$

In particular, $p_{\mathbb{C}}\left(c_Y(Y_b) - c_Y(Y_a)\right)$ and $\left(\tilde{y}_{\hat{B}(Y)}(d(Y_b)) - \tilde{y}_{\hat{B}(Y)}(d(Y_a))\right)$ are colinear, and their scalar product is nonnegative on $N(c^0, y^0) \cap \check{\mathcal{V}}(\Gamma)$. As above, as soon as there exist two kids Y_a and Y_b as above, such that $d(Y_a)$ and $d(Y_b)$ are in two distinct kids of $\hat{B}(Y)$, we can define the continuous function $\lambda(Y)$ such that

$$p_{\mathbb{C}}\left(c_Y(Y_b) - c_Y(Y_a)\right) = \lambda(Y)\left(\tilde{y}_{\hat{B}(Y)}(d(Y_b)) - \tilde{y}_{\hat{B}(Y)}(d(Y_a))\right)$$

for any two univalent kids Y_a and Y_b of Y, and $\lambda(Y)M_Y = \lambda U_{\hat{B}(Y)}$. $\square$

Let $\mathcal{P}'_B$ be the set of elements D of $\mathcal{P}_B$ such that $\mathcal{P}_{X,D} \neq \emptyset$. Let $\widehat{\mathcal{P}'}_B$ be the set of elements D of $\mathcal{P}_B$ that contain, or are equal to, an element of $\mathcal{P}'_B$.

For any collections $(B_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$, $(B_i^+)_{i\in\mathbb{Z}/n\mathbb{Z}}$, $(B_i')_{i\in\mathbb{Z}/n\mathbb{Z}}$ of (non-necessarily distinct) sets of $\widehat{\mathcal{P}'}_B$ such that

$$\left(B_{i-1}^+ \cup B_i\right) \subseteq B_i',$$

for any collections $(Y_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$, $(Y_i^+)_{i\in\mathbb{Z}/n\mathbb{Z}}$, $(Y_i')_{i\in\mathbb{Z}/n\mathbb{Z}}$ such that

$$Y_i \in \mathcal{P}_{X,B_i}\ ,\ Y_i^+ \in \mathcal{P}_{X,B_i^+}\ ,\ Y_i' \in \widehat{\mathcal{P}'}_X \text{ and } \left(Y_i^+ \cup Y_i\right) \subseteq Y_i',$$

$$\prod_{i=1}^n \left(\lambda(Y_i)\frac{M_{Y_i}}{M_{Y_i'}}\right)\prod_{i=1}^n \frac{U_{B_i^+}}{U_{B_{i+1}'}} = \prod_{i=1}^n \frac{U_{B_i}}{U_{B_i'}}\prod_{i=1}^n\left(\lambda(Y_i^+)\frac{M_{Y_i^+}}{M_{Y_i'}}\right)$$

for configurations of $N(c^0,y^0)\cap\check{\mathcal{V}}(\Gamma)$. This equation is equivalent to the following equation, which also holds in $N(c^0,y^0)\cap\mathcal{V}(\Gamma)$.

$$\begin{aligned}&\textstyle\prod_{i=1}^n \left(\lambda(Y_i)\prod_{Z\in\mathcal{P}|Y_i\subseteq Z\subset Y_i'}\mu_Z\right)\prod_{i=1}^n\left(\prod_{E\in\mathcal{P}_B|B_i^+\subseteq E\subset B_{i+1}'}u_E\right)\\ &\quad\textstyle= \prod_{i=1}^n\left(\lambda(Y_i^+)\prod_{Z\in\mathcal{P}|Y_i^+\subseteq Z\subset Y_i'}\mu_Z\right)\prod_{i=1}^n\left(\prod_{E\in\mathcal{P}_B|B_i\subseteq E\subset B_i'}u_E\right)\end{aligned}$$

For $Y\in\mathcal{P}$, let $K_u(Y)$ denote the set of univalent kids of Y that do not contain $b(Y)$, and let $K_t(Y)$ denote the set of non-univalent kids of Y that do not contain $b(Y)$.

We are going to prove the following lemma.

Lemma 15.5. *Let (c^0,y^0) satisfy the conditions of Lemma 15.1 (and hence of Lemma 15.3). There is a neighborhood of (c^0,y^0) in $\mathcal{V}(\Gamma)$ that is parametrized by the following variables*

1. $((u_D)_{D\in\mathcal{P}_B\setminus\{B\}},(y_D)_{D\in\mathcal{P}_B})\in([0,\varepsilon[)^{\mathcal{P}_B\setminus\{B\}}\times\prod_{D\in\mathcal{P}_B}N_D$
2. $(\mu_Z)_{Z\in\mathcal{P}\setminus\{V\}}\in([0,\varepsilon[)^{\mathcal{P}\setminus\{V\}}$
3. *the (c_Z) for non-univalent sets Z of $\mathcal{P}$,*
4. *for univalent sets Z of $\mathcal{P}$, the $c_Z(Y)$ for the kids $Y\in K_t(Z)$ of Z, and the $p_{\mathbb{R}}\circ c_Z(Y)$ for the kids $Y\in K_u(Z)$ of Z,*
5. *the parameter λ, and, for every element Y of $\mathcal{P}'_X$, the parameters $\lambda(Y)$, which are defined in Lemma 15.4,*

which satisfy the constraints

1. *For a vertically normalizing kid Y of an element Z of $\mathcal{P}$, $p_{\mathbb{R}} \circ c_Z(Y) = p_{\mathbb{R}} \circ c^0_Z(Y) = \pm 1$, and for a horizontally normalizing kid Y, $|p_{\mathbb{C}} \circ c_Z(Y)| = 1$.*

2. *$*(Y)$: $\quad M_Y \lambda(Y) = \lambda U_D$ for any element Y of $\mathcal{P}_{X,D}$, where $M_Y = \prod_{Z \in \mathcal{P}\setminus\{V\} | Y \subseteq Z} \mu_Z$ and $U_D = \prod_{E \in \mathcal{P}_B \setminus \{B\} | D \subseteq E} u_E$,*

3. *for any collections $(B_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$, $(B^+_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$, $(B'_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ of (non-necessarily distinct) sets of $\widehat{\mathcal{P}'}_B$ such that*

$$\left(B^+_{i-1} \cup B_i\right) \subseteq B'_i,$$

for any collections $(Y_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$, $(Y^+_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$, $(Y'_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ such that

$$Y_i \in \mathcal{P}_{X,B_i}\ ,\ Y^+_i \in \mathcal{P}_{X,B^+_i}\ ,\ Y'_i \in \widehat{\mathcal{P}'}_X \text{ and } \left(Y^+_i \cup Y_i\right) \subseteq Y'_i,$$

$$\begin{aligned}\prod_{i=1}^n \left(\lambda(Y_i) \prod_{Z\in\mathcal{P}|Y_i\subseteq Z \subset Y'_i} \mu_Z\right) \prod_{i=1}^n \left(\prod_{E\in\mathcal{P}_B|B^+_i\subseteq E\subset B'_{i+1}} u_E\right) \\ = \prod_{i=1}^n \left(\lambda(Y^+_i) \prod_{Z\in\mathcal{P}|Y^+_i\subseteq Z\subset Y'_i} \mu_Z\right) \prod_{i=1}^n \left(\prod_{E\in\mathcal{P}_B|B_i\subseteq E\subset B'_i} u_E\right),\end{aligned} \tag{15.1}$$

4. *for any univalent kid Z_b of an element Z of $\mathcal{P}$ such that $d(Z_b) = d(Z)$,*

$$p_{\mathbb{C}} \circ c_Z(Z_b) = p_{\mathbb{C}} \circ c_Z(b(Z)) = 0,$$

5. *for any univalent element Z of $\mathcal{P}$ such that $\sharp B(Z) \geq 2$, for any maximal[2] element Y of $\mathcal{P}'_X$ such that $Y \subseteq Z$ and $\hat{B}(Z) = \hat{B}(Y)$, for any univalent kid Z_b of Z, $p_{\mathbb{C}} \circ c_Z(Z_b)$ is equal to*

$$p_{\mathbb{C}} \circ c_Z(Z_b) = \frac{M_Y}{M_Z}\lambda(Y)\left(\tilde{y}_{\hat{B}(Z)}(d(Z_b)) - \tilde{y}_{\hat{B}(Z)}(d(Z))\right), \tag{15.2}$$

The analysis that was performed before Lemma 15.5 ensures that all the elements in our neighborhood $N(c^0, y^0)$ of (c^0, y^0) in $\mathcal{S}_V(\mathbb{R}^3) \times \mathcal{S}_B(\mathbb{C})$ that are in $\mathcal{V}(\Gamma)$ are described by the parameters described in the statement and that they satisfy all the equations of the statement. In order to prove Lemma 15.5, we are going to prove that, conversely, the elements described in its statement are in $\mathcal{V}(\Gamma)$. Note that the elements described in this statement such that neither λ, nor the μ_Z nor the u_D vanish, correspond to elements of $\check{\mathcal{S}}_V(\mathbb{R}^3) \times \check{\mathcal{S}}_B(\mathbb{C})$ and are in $\check{\mathcal{V}}(\Gamma)$.

We define a tree $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$ with oriented edges from the tree whose vertices are the elements of $\widehat{\mathcal{P}'}_X$ and whose edges start at an element Z of $\widehat{\mathcal{P}'}_X \setminus \{V\}$, end at its mother $m(Z)$ and are labeled by μ_Z,

[2]Note that the minimal univalent elements Z of $\mathcal{P}$ such that $\sharp B(Z) \geq 2$ are in $\mathcal{P}'_X$.

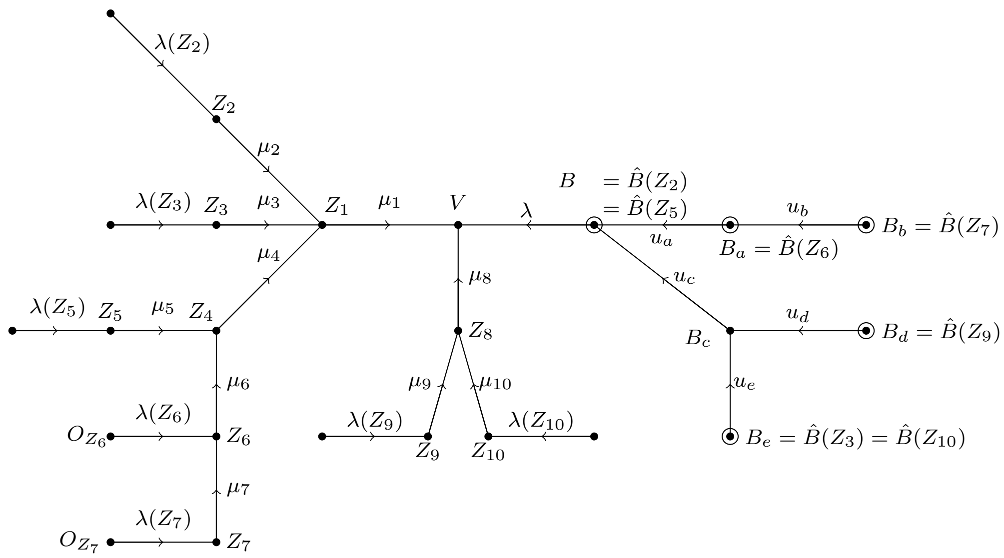


Figure 15.1: The tree $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$ associated to a configuration c

- by gluing an edge, which arrives at the vertex Y, starts at a vertex labeled by O_Y, and which is labeled by $\lambda(Y)$, at the vertex Y, for any Y of $\mathcal{P}'_X$,
- by gluing an edge, which arrives at the vertex V, starts at a vertex labeled by B and is labeled by λ, at the vertex V,
- by gluing to the root B of this edge the subtree of $\mathcal{P}_B$ whose vertices are the elements of $\widehat{\mathcal{P}'}_B$ and whose edges are labeled by the u_D for $D \in \widehat{\mathcal{P}'}_B \setminus \{B\}$, where the edge labeled by u_D starts at D and ends at its mother $m(D)$.

An example is drawn in Figure 15.1. Let $\mathcal{P}'_{X,hn}$ be the set of elements Z of $\mathcal{P}'_X$ such that $k_n(Z)$ is a univalent horizontally normalizing kid.

Proof of Lemma 15.1: We prove that our configuration (c^0, y^0) of Lemma 15.3 and 15.5 is in $\mathcal{V}(\Gamma)$ by exhibiting a family $(c(t), y(t))$ of configurations of $\check{\mathcal{V}}(\Gamma)$, for $t \in]0, \varepsilon[$ tending to 0, which tends to (c^0, y^0). Our family will be described by nonvanishing $u_D(t)$, $\mu_Z(t)$ tending to 0, nonvanishing $\lambda(Y)(t)$ tending to our given $\lambda^0(Y)$, and nonvanishing $\lambda(t)$ tending to our given λ^0, such that the equations

$$*(Y)(t)\colon \quad M_Y(t)\lambda(Y)(t) = \lambda(t) U_D(t)$$

are satisfied for any element Y of $\mathcal{P}_{X,D}$. The parameters $(y_D)_{D\in\mathcal{P}_B}$, $c_Z(Y)$ for non-univalent kids Y of elements Z of $\mathcal{P}$, $p_{\mathbb{R}}(c_Z(Y))$ for univalent kids Y of elements Z of $\mathcal{P}$, $\lambda(Z)$, for $Z \in \mathcal{P}'_X \setminus \mathcal{P}'_{X,hn}$, of the statement of Lemma 15.5 will be defined to be the same as those of (c^0, y^0).

The varying parameters $\lambda(Z)$, for $Z \in \mathcal{P}'_{X,hn}$, will be determined by Equation 15.2, which may be written as

$$p_{\mathbb{C}}(c_Z(k_n(Z))) = \lambda(Z)\left(\tilde{y}_{\hat{B}(Z)}(d(k_n(Z))) - \tilde{y}_{\hat{B}(Z)}(d(Z))\right),$$

and by the condition $|p_{\mathbb{C}}(c_Z(k_n(Z)))| = 1$ at the end of the process below. To summarize, $\lambda(Z)$ will be determined by the equation

$$\lambda(Z)\left|\tilde{y}_{\hat{B}(Z)}(d(k_n(Z))) - \tilde{y}_{\hat{B}(Z)}(d(Z))\right| = 1. \tag{15.3}$$

Set $\mu_Y(t) = t$ for any $Y \in \widehat{\mathcal{P}'}_X \setminus \mathcal{P}'_X$, and $u_D(t) = t$ for any $D \in \widehat{\mathcal{P}'}_B \setminus \mathcal{P}'_B$. Denote $\lambda(t)$ by $u_B(t)$ to make notation homogeneous. Recall that when $V \in \mathcal{P}_{X,B}$, $\lambda(V) = \lambda$.

For $D \in \mathcal{P}'_B$, we are going to define some integer $g(D) \geq 1$ and set $u_D(t) = t^{g(D)}$ when $D \neq B$. We are going to set $u_B(t) = \lambda(t) = t^{g(B)}$ if $\lambda^0 = 0$, and $u_B(t) = \lambda(t)$, where $\lambda(t) = \lambda^0$ if $\lambda^0 \neq 0$ and if[3] $V \notin \mathcal{P}'_{X,hn}$.

Let $Y \in \mathcal{P}'_X$. If Y is maximal in $\mathcal{P}'_X$, set $\lambda'(Y) = 1$. If Y is not maximal in $\mathcal{P}'_X$, let $s(Y)$ be the minimal element of $\mathcal{P}'_X$ that contains Y strictly. If $\lambda^0(s(Y)) = 0$, set $\lambda'(Y)(t) = 1$. If $\lambda^0(s(Y)) \neq 0$, set $\lambda'(Y)(t) = \lambda(s(Y))(t)$. For $Y \in \mathcal{P}'_X$, we are going to define some integer $g(Y) \geq 1$ and set

$$\lambda(Y)(t) = t \quad \text{and} \quad \mu_Y(t) = t^{g(Y)}\lambda'(Y) \quad \text{if } \lambda^0(Y) = 0,$$

$\lambda(Y)(t) = \lambda^0(Y)$ if $\lambda^0(Y) \neq 0$ and if $Y \notin \mathcal{P}'_{X,hn}$,

$$\text{and } \mu_Y(t) = \frac{t^{g(Y)+1}}{\lambda(Y)(t)}\lambda'(Y)(t) \quad \text{if } \lambda^0(Y) \neq 0.$$

(If $Y = V$, just forget about μ_V, which is useless.)

Order the elements of $\mathcal{P}'_B$, by calling them D_1, D_2, $\dots D_{\sharp\mathcal{P}'_B}$ so that D_1 is maximal in $\mathcal{P}'_B$ (with respect to the inclusion) and D_{i+1} is maximal in $\mathcal{P}'_B \setminus \{D_1, D_2, \dots, D_i\}$. Note that $B \in \mathcal{P}'_B$, so $D_1 = B$.

Recall $\lambda(t) = t^{g(B)}$ if $\lambda^0 = 0$ and $\lambda(t) \neq 0$ if $\lambda^0 \neq 0$, and set $g(D) = 1$ for any $D \in \widehat{\mathcal{P}'}_B \setminus \mathcal{P}'_B$.

We are going to define the integers $g(D_i)$ and the integers $g(Y)$ for every $Y \in \mathcal{P}_{X,D_i}$, for $i \in \underline{\sharp\mathcal{P}'_B}$, inductively, so that the following assertion $(*(i))$ holds for every i.

[3]Otherwise, it is just a parameter, which is determined at the end by Equation 15.3.

$(*(i))$: for every $Y \in \mathcal{P}_{X,D_i}$, $M_Y(t)\lambda(Y)(t) = \lambda(t)U_{D_i}(t) = \lambda(t)t^{f(i)}$ for all $t \in]0, \varepsilon[$, where $f(i) = \sum_{D \in \widehat{\mathcal{P}'}_B | D_i \subseteq D \subset B} g(D)$.

Recall $D_1 = B$ and note $f(1) = 0$. If $V \in \mathcal{P}_{X,B}$, set $g(D_1) = 1$. So $(*(1))$ is satisfied. (If $\lambda^0 \neq 0$, then $g(D_1)$ is not used and $(*(1))$ is satisfied.) If $V \notin \mathcal{P}_{X,B}$, then $\lambda^0 = 0$, according to Lemma 15.3. Define $g(D_1)$ to be the maximum over the elements Y of $\mathcal{P}_{X,B}$ of the number of elements of $\mathcal{P}$ that contain Y strictly, plus one. Then define the $g(Y)$ for the elements Y of $\mathcal{P}_{X,B}$ so that $(*(1))$ holds.

Let $k \in \underline{\sharp \mathcal{P}'_B}$.

Assume that we have defined $g(D_i)$ and the $g(Y)$ for all $Y \in \mathcal{P}_{X,D_i}$, for all $i < k$, so that the assertions $(*(i))$ hold for any $i < k$, and let us define the $g(Y)$ for all $Y \in \mathcal{P}_{X,D_k}$ and $g(D_k)$ so that $(*(k))$ holds. For any $Y \in \mathcal{P}_{X,D_k}$, $M_Y(t)\lambda(Y)(t)$ is equal to $\lambda(t)t^{g(Y)+h(Y)}$, and $\lambda(t)U_{D_k}(t)$ is equal to $\lambda(t)t^{g(D_k)+h(D_k)}$. Let $H(k)$ be the maximum over $h(D_k)$ and all the integers $h(Y)$ for $Y \in \mathcal{P}_{X,D_k}$, and set $g(D_k) = H(k) + 1 - h(D_k)$, and $g(Y) = H(k) + 1 - h(Y)$, so that $(*(k))$ holds with $f(k) = H(k) + 1$.

This process is illustrated in Figure 15.2. Its result does not depend on the arbitrary order that respects our condition on the D_i.

$\square$

Proof of Lemma 15.5: The elements of $\mathcal{V}(\Gamma)$ obviously satisfy the equations of the statement of Lemma 15.5. Let us conversely prove that an element parametrized as in this statement belongs to $\mathcal{V}(\Gamma)$ by exhibiting a sequence of configurations of $\check{\mathcal{V}}(\Gamma)$ that approaches it. We will again focus on the parameters μ_Z for $Z \in \widehat{\mathcal{P}'}_X \setminus \{V\}$, u_D for $D \in \widehat{\mathcal{P}'}_B \setminus \{B\}$, $\lambda(Y)$ for $Y \in \mathcal{P}'_X$, and λ. All of these parameters correspond to edges of our tree $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$, the parameter that corresponds to an edge e will be denoted by $\lambda(e)$. (The other parameters of Lemma 15.5 are fixed as in the proof of Lemma 15.1.)

Our family of configurations $c(t)$, tending to the given one c, whose description involves the parameters $\lambda(e)$, will be indexed by a variable t, tending to 0. The corresponding parameters for the family will be denoted by $\lambda(e)(t)$. When $\lambda(e) \neq 0$, set $\lambda(e)(t) = \lambda(e)$ if λ is not among the $\lambda(Z)$, for $Z \in \mathcal{P}'_{X,hn}$. The varying parameters $\lambda(e)(t) = \lambda(Z)(t)$, for $Z \in \mathcal{P}'_{X,hn}$, are treated as free nonzero parameters. They are determined by Equation 15.3 at the end of the process, as in the proof of Lemma 15.1.

Let $\mathcal{P}'_{X,c}$ be the set of elements Y of $\mathcal{P}'_X$ such that $\lambda(Y)M_Y = 0$. If $Y \in \mathcal{P}'_X \setminus \mathcal{P}'_{X,c}$, then $\left(\lambda(Y)M_Y = \lambda U_{\hat{B}(Y)}\right) \neq 0$. Let $\mathcal{P}'_{B,c}$ be the set of elements D of $\mathcal{P}'_B$ such that $\lambda U_D = 0$.

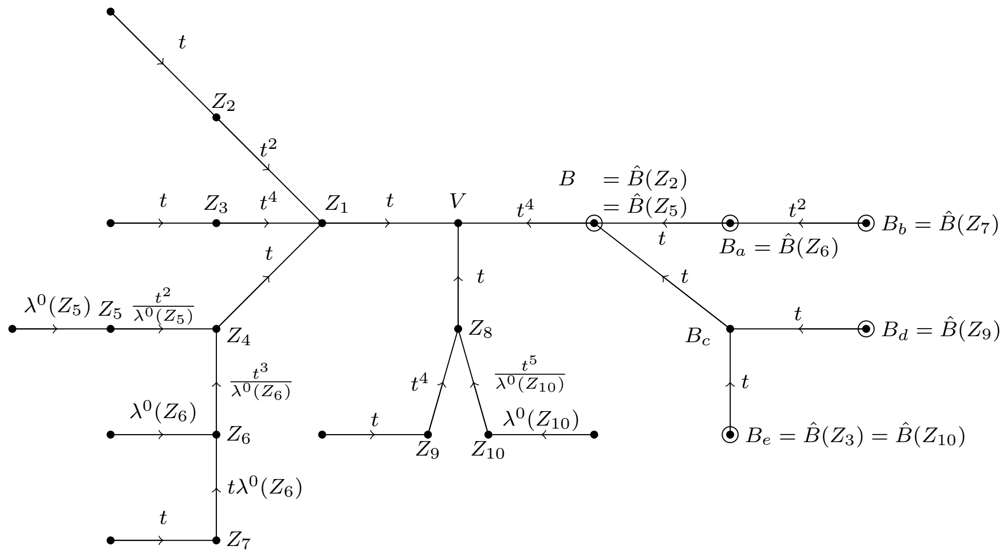


Figure 15.2: Interior configurations tending to our limit configuration c^0, whose associated tree $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$ is as in Figure 15.1, where $\lambda^0(Z_2) = \lambda^0(Z_3) = \lambda^0(Z_7) = \lambda^0(Z_9) = 0$ and the other $\lambda^0(Z_i)$ are not zero.

The main equations

$$*(Y)(t) : \lambda(Y)(t)M_Y(t) = \lambda(t)U_{\hat{B}(Y)}(t)$$

associated to elements of $\mathcal{P}'_X \setminus \mathcal{P}'_{X,c}$ are obviously satisfied.

For $Y \in \mathcal{P}'_X$, recall that O_Y denotes the initial vertex of the edge of $\lambda(Y)$ in the tree $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$. For $Y \in \mathcal{P}'_{X,c}$, define $e(Y)$ to be the closest edge to O_Y between O_Y and the vertex V such that $\lambda(e(Y)) = 0$, in $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$. For $D \in \mathcal{P}'_{B,c}$, define $e(D)$ to be the edge between D and the vertex V that is closest to D such that $\lambda(e(D)) = 0$.

For edges e of $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$ such that $\lambda(e) = 0$ that are not in $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$, set $\lambda(e)(t) = t$. For edges $e \in e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$, set $\lambda(e)(t) = r(e)t^{k(e)}$, where $(r(e), k(e)) \in \mathbb{R}^+ \times \mathbb{N}$, $r(e) \neq 0, k(e) \neq 0$. We are going to show how to define pairs $(r(e), k(e))$, so that the equations $*(Y)(t)$ are satisfied, for all t, and for all Y such that $Y \in \mathcal{P}'_X$. Since they imply the equations 15.1, because all the coefficients $\lambda(e)(t)$ are different from zero, this will conclude the proof.

If $\mathcal{P}'_{B,c} = \emptyset$, then no parameter $\lambda(e)$ vanishes, and there is nothing to prove.

For a set I of edges of $\mathcal{T}(\mathcal{P}, \mathcal{P}_B)$, $\lambda(I)$ denotes the product over the edges e of I of the $\lambda(e)$, and $\lambda_t(I)$ denotes the product over the edges e of I of the $\lambda(e)(t)$. Recall the notation from Definition 14.1.

With this notation, the main equations $*(Y)(t)$ may be written as

$$*(Y)(t) : \lambda_t([O_Y, V]) = \lambda_t([\hat{B}(Y), V]).$$

We also have the obvious sublemma.

Sublemma 15.6. *Condition 15.1 of Lemma 15.5 may be rewritten as follows For any collections $(Y_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$, $(Y_i^+)_{i\in\mathbb{Z}/n\mathbb{Z}}$, $(Y'_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ such that $Y_i \in \mathcal{P}'_X$, $Y_i^+ \in \mathcal{P}'_X$, $Y'_i \in \widehat{\mathcal{P}'}_X$ and $\left(Y_i^+ \cup Y_i\right) \subseteq Y'_i$, and for any collection $(B'_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ of sets of $\widehat{\mathcal{P}'}_B$ such that*

$$\left(\hat{B}(Y_{i-1}^+) \cup \hat{B}(Y_i)\right) \subseteq B'_i,$$

the products of the numbers over the arrows in one direction equals the product of the numbers over the arrows in the opposite direction in the cycle of Figure 15.3.

A typical part of the cycle of Figure 15.3 can be redrawn as in Figure 15.4 to emphasize the different natures of its segments, and to show how the coefficients over the arrows are determined by the sets at their ends.

For $g \in e(\mathcal{P}'_{B,c})$, and for $f \in e(\mathcal{P}'_{X,c})$, let $\mathcal{P}(f, g)$ be the set of elements $Y \in \mathcal{P}'_X$ such that $e(Y) = f$ and $e(\hat{B}(Y)) = g$. Note the following sublemma.

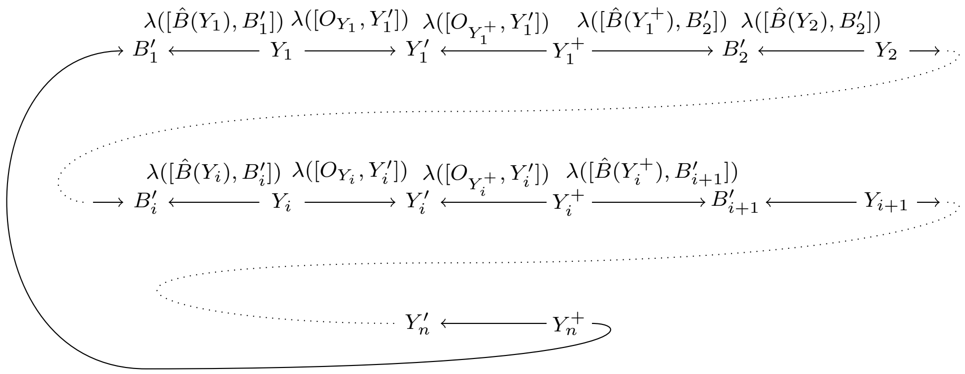


Figure 15.3: The cycle of Condition 15.1 of Lemma 15.5.

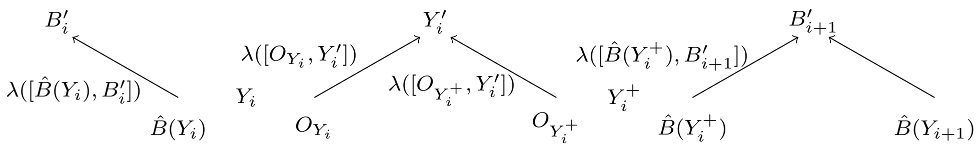


Figure 15.4: Another representation of a part of the cycle of Figure 15.3.

Sublemma 15.7. *Let $g \in e(\mathcal{P}'_{B,c})$, let $f \in e(\mathcal{P}'_{X,c})$, and assume that there exists an element $Y \in \mathcal{P}(f,g)$. Set*

$$c(f,g) = \frac{\lambda([\hat{B}(Y), g[)}{\lambda([O_Y, f[)}.$$

Then $c(f,g)$ is a positive coefficient, which is independent of the chosen element Y of $\mathcal{P}(f,g)$. Equation

$$*_t(f,g) : \lambda_t([f,V]) = c(f,g)\lambda_t([g,V]).$$

must be satisfied for our sequence of configurations $c(t)$ to be in $\check{\mathcal{V}}(\Gamma)$. Furthermore, if Equation $*_t(f,g)$ is satisfied for all t, then for any $Y^+ \in \mathcal{P}(f,g)$, Equation $*(Y^+)(t)$ is satisfied for all t.

Proof: With the given coefficient $c(f,g)$, Equation $*_t(f,g)$ is equivalent to Equation $*(Y)(t)$. For any other element Y^+ of $\mathcal{P}(f,g)$, according to Condition 15.1 of Lemma 15.5 -where $n=1$, Y'_1 is the origin of f, and B'_1 is the origin of g-, we have

$$\lambda([O_Y, f[)\lambda([\hat{B}(Y^+), g[) = \lambda([O_{Y^+}, f[)\lambda([\hat{B}(Y), g[).$$

$\square$

Define the equivalence relation $\sim$ on $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$ to be the relation generated by the equivalences : whenever $g \in e(\mathcal{P}'_{B,c})$ and $f \in e(\mathcal{P}'_{X,c})$, if $\mathcal{P}(f,g) \neq \emptyset$, then $g \sim f$. When $Y \in \mathcal{P}(f,g)$, such a generating elementary equivalence will also be denoted by $g \sim_Y f$ and its inverse will be denoted by $f \sim_Y g$. In this case, set $c(g,f) = c(f,g)^{-1}$.

Sublemma 15.8. *For any cycle*

$$g = g_1 \sim_{Y_1} f_1 \sim_{Y_1^+} g_2 \sim_{Y_2} f_2 \sim_{Y_2^+} g_3 \dots f_n \sim_{Y_n^+} g_{n+1} = g$$

of elementary equivalences for sequences $(g_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ of edges of $e(\mathcal{P}'_{B,c})$ and $(f_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$ of edges of $e(\mathcal{P}'_{X,c})$,

$$\prod_{i=1}^{n} c(g_i, f_i)c(f_i, g_{i+1}) = 1$$

Proof: Apply Condition 15.1 of Lemma 15.5, as stated in Sublemma 15.6, to the above sequences (Y_i) and (Y_i^+), where Y'_i is the origin $o(f_i)$ of f_i, and B'_i is the origin $o(g_i)$ of g_i, and note that Sublemma 15.7 implies that, in each triangle of Figure 15.5, the product of the coefficients over the edges of

$$
\begin{tikzcd}[column sep=small]
o(g_i) \arrow[rrrr, "c(g_i, f_i)"] & & & & o(f_i) \arrow[rrrr, "c(f_i, g_{i+1})"] & & & & o(g_{i+1}) & \\
& \hat{B}(Y_i) \arrow[ul, "\lambda([\hat{B}(Y_i), g_i[)"'] & Y_i & O_{Y_i} \arrow[ur, "\lambda([O_{Y_i}, f_i[)"] & & O_{Y_i^+} \arrow[ul, "\lambda([O_{Y_i^+}, f_i[)"'] & Y_i^+ & \hat{B}(Y_i^+) \arrow[ur, "\lambda([\hat{B}(Y_i^+), g_{i+1}[)"] & & \hat{B}(Y_{i+1}) \arrow[ul]
\end{tikzcd}
$$

Figure 15.5: Definition of $c(g_i, f_i)$ and $c(f_i, g_{i+1})$.

the boundary of the triangle, that are oriented as part of that boundary, is equal to the product of the coefficients over the edges that have the opposite orientation. □

The following sublemma is an easy corollary of the previous one.

Sublemma 15.9. *Let e and e' be two elements of $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$ such that $e \sim e'$. There exists a sequence $(e_i)_{i \in \underline{m}}$ of edges of $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$ such that $e_1 = e$, $e_m = e'$ and $e_i \sim_{Z_i} e_{i+1}$ for any $i \in \underline{m-1}$. For such a sequence, set*

$$c(e, e') = \prod_{i=1}^{m-1} c(e_i, e_{i+1})$$

Then $c(e, e')$ is a positive coefficient, which is independent of the chosen sequences as above.

The equation

$$*_t(e, e') : \lambda_t([e, V]) = c(e, e')\lambda_t([e', V])$$

must be satisfied for our sequence of configurations $c(t)$ to be in $\check{\mathcal{V}}(\Gamma)$. Furthermore, for any three elements e, e_0 and e' of $e(\mathcal{P}'_{B,c})$ that are in the same equivalence class under $\sim$, $*_t(e, e')$ is equivalent to $*_t(e', e)$, and, $*_t(e, e_0)$ and $*_t(e', e_0)$ imply $*_t(e, e')$.

□

The following sublemma allows us to define a partial order on the set of equivalence classes under $\sim$.

Sublemma 15.10. *Let $k \in \mathbb{N}$. It is not possible to find edges $e_1, \dots, e_k$, $e'_1, \dots e'_k$ of $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$ such that $e_j \sim e'_j$ and $e_{j+1} \in]e'_j, V]$, for all $j \in \mathbb{Z}/k\mathbb{Z}$.*

Proof: If there exist edges $e_1, \dots, e_k$, $e'_1, \dots e'_k$ of $e(\mathcal{P}'_{X,c} \cup \mathcal{P}'_{B,c})$ such that $e_j \sim e'_j$ and $e_{j+1} \in]e'_j, V]$, for all $j \in \mathbb{Z}/k\mathbb{Z}$, we construct a cycle as in Figure 15.3, as follows.

An elementary equivalence $g \sim_Y f$, where $Y \in \mathcal{P}(f,g)$ is represented by a path $E(g,f)$

$$o(g) \xleftarrow{\lambda([\hat{B}(Y),g[)} Y \xrightarrow{\lambda([O_Y,f[)} o(f)$$

where $o(g)$ is the set associated to the initial point of g, $o(f)$ is the initial point of f is equal to O_Y, and $o(f)$ is the set associated to the initial point of f, otherwise. The inverse equivalence $f \sim_Y g$ is represented similarly, by the following path $E(f,g)$ $o(f) \xleftarrow{\lambda([O_Y,f[)} Y \xrightarrow{\lambda([\hat{B}(Y),g[)} o(g)$. Again, the coefficients over the arrows are determined by the labels of the ends. In these cases, they do not vanish, and we just picture the paths $E(g,f)$ and $E(f,g)$ as $o(g) \hookleftarrow Y \hookrightarrow o(f)$ and $o(f) \hookleftarrow Y \hookrightarrow o(g)$, respectively. Note that according to our assumptions, no e_j can start at some O_Y.

If $e_j \neq e'_j$ for all $j \in \mathbb{Z}/k\mathbb{Z}$, then our cycle of arrows as above is obtained by assembling the following paths of arrows $A(e_j, e_{j+1})$ from $o(e_j)$ to $o(e_{j+1})$. The path $A(e_j, e_{j+1})$ is obtained from a sequence $E(e_j, e'_j)$ of paths $E(e,e')$ of arrows associated with elementary equivalences that do not[4] involve intermediate arrows ending at some O_Y, by replacing the last arrow $X_j \hookrightarrow o(e'_j)$, which ends at $o(e'_j)$, with an arrow $X_j \to o(e_{j+1})$ that ends at $o(e_{j+1})$. The coefficient of this arrow is obtained from the coefficient of $X_j \hookrightarrow o(e'_j)$ by multiplication by $\lambda([e'_j, e_{j+1}[)$, which vanishes in our case, because it has a factor $\lambda(e'_j)$. According to the recalled criterion, the product of the coefficients over the arrows in the direction of our cycle must be equal to the product of the coefficients over the arrows in the opposite direction. The first product is zero because of its factors $\lambda(e'_j)$. The second one is nonzero because it only contains nonzero factors associated with equivalences. Therefore, the lemma is proved when $e_j \neq e'_j$ for any $j \in \mathbb{Z}/k\mathbb{Z}$. This case is ruled out.

We cannot have $e_j = e'_j$ for all $j \in \mathbb{Z}/k\mathbb{Z}$.

Up to permuting our indices cyclically, it suffices to rule out the case in which $k \geq 2$, and there exists $r \geq 1$ such that $e_r \neq e'_r$, $e_j = e'_j$ for all j such that $r+1 \leq j \leq k$, and $e_1 \neq e'_1$. Then we define a path $A(e_r, e_1)$ by replacing the last arrow $X_r \hookrightarrow o(e'_r)$ in $E(e_r, e'_r)$ with $X_r \to o(e_1)$, and multiplying the corresponding coefficient by $\lambda([e'_r, e_1[) = \prod_{j=r}^{k} \lambda([e'_j, e_{j+1}[)$. Similarly define paths $A(e_s, e_t)$, for all pair (s,t) of integers such that $s < t < k$, $e_s \neq e'_s$, $e_t \neq e'_t$ and $e_j = e'_j$ for all j such that $s+1 \leq j \leq t-1$. Define the cycle, which yields the contradiction, by composing these paths, which include our

[4]Intermediate equivalences in a chain of equivalences from e_j to e'_j involving an edge $f(Y)$ ending at some O_Y can be avoided, because we would necessarily have a segment $o(e(\hat{B}(Y))) \hookleftarrow Y \hookrightarrow O_Y \hookleftarrow Y \hookrightarrow o(e(\hat{B}(Y)))$, which can be simplified.

former paths $A(e_j,e_{j+1})$ for which $e_j\neq e'_j$ and $e_{j+1}\neq e'_{j+1}$. $\square$

Sublemma 15.10 allows us to define the partial order $\preceq$ on the set $E(\sim)$ of equivalence classes of the relation $\sim$ on $e(\mathcal{P}'_{X,c}\cup\mathcal{P}'_{B,c})$ such that two equivalence classes $\overline{e}$ and $\overline{f}$ of $E(\sim)$ satisfy $\overline{f}\preceq\overline{e}$ if and only if there exist a positive integer $k\geq 2$ and two sequences $(e_i)_{i\in\underline{k}\setminus\{1\}}$ and $(e'_i)_{i\in\underline{k-1}}$ such that $e'_1\in\overline{e}$, $e_k\in\overline{f}$, $e_j\sim e'_j$ for all $j\in\underline{k-1}\setminus\{1\}$, and $e_{j+1}\in[e'_j,V]$ (i.e. e_{j+1} is between the initial point of e'_j and V) for all $j\in\underline{k-1}$.

Fix an arbitrary total order on $E(\sim)$ that is compatible with the above partial order by writing

$$E(\sim)=\{\overline{g_i}\mid i\in\underline{m}\}$$

so that for any $(i,j)\in\underline{m}^2$ such that $\overline{g_j}\preceq\overline{g_i}$, $j\leq i$.

We will pick one representative $g_i\in e(\mathcal{P}'_{B,c})$ in each equivalence class $\overline{g_i}$ of $E(\sim)$, and define $\lambda(g_i)(t)=r(g_i)t^{k(g_i)}$, for $i=1,\dots,m$, inductively.

For $e\in\overline{g_i}$, the equation

$$*_t(e,g_i):\lambda_t([e,V])=c(e,g_i)\lambda_t([g_i,V])$$

that must be satisfied determines $\lambda(e)(t)$ as a function of $\lambda(g_i)(t)$ and of the $\lambda(e')(t)=r(e')t^{k(e')}$ for edges e' that belong to $\cup_{j=1}^{i-1}\overline{g_j}$. (Recall that we fixed the coefficients of the other edges.) More precisely, $k(e)-k(g_i)$ is a degree one polynomial in the variables $k(e')$ for edges e' that belong to $\cup_{j=1}^{i-1}\overline{g_j}$, which are already defined by induction. In particular, it suffices to choose $k(g_i)$ to be a sufficiently large integer, and to fix $r(g_i)=1$, so that the $\lambda(e)(t)$ are uniquely determined in such a way that the equations $*_t(e,g_i)$ are satisfied with $k(e)>0$ for any $e\in\overline{g_i}$.

Thus, once the induction is achieved, according to Sublemmas 15.7 and 15.9, Equation $*(Y)(t)$ is satisfied for all t for any $Y\in\mathcal{P}$. This shows that the elements described in the statement of Lemma 15.5 are in $\mathcal{V}(\Gamma)$ and finishes the proof of Lemma 15.5. $\square$

Let $\mathcal{P}_u$ denote the set of univalents elements of $\mathcal{P}$, which are the elements of $\mathcal{P}$ that contain at least one univalent vertex.

For a configuration $y\in\mathcal{S}_B(\mathbb{C})$, and a Jacobi diagram Γ on a disjoint union of lines $\mathbb{R}_b$ indexed by elements b of B, let $\mathcal{V}(y,\Gamma)$ denote the preimage of y under

$$p_{\mathcal{S}_B}\colon\mathcal{V}(\Gamma)\to\mathcal{S}_B(\mathbb{C}).$$

Lemma 15.11. *With respect to the notation introduced before Lemma 15.3 and in Lemma 15.3, let $\mathcal{P}_x$ be the subset of $\mathcal{P}'_X$ consisting of the elements Y such that $\lambda^0(Y)\neq 0$. The dimension of the stratum of c^0 in the fiber $\mathcal{V}(y^0,\Gamma)$ over the configuration $y^0\in\mathcal{S}_B(\mathbb{C})$ is $\sharp U(\Gamma)+3\sharp T(\Gamma)-\sharp\mathcal{P}+\sharp\mathcal{P}_x-1$.*

PROOF: For any $Y \in \mathcal{P} \setminus \mathcal{P}_u$, the configuration c_Y is defined up to global translation and dilation. (With our normalizations, the quotient by translations are replaced by the fact that we send basepoints to zero, and the quotient by dilation is replaced by our horizontal or vertical normalization condition.) For any $Y \in \mathcal{P}_u \setminus \mathcal{P}_x$, the configuration c_Y, whose restriction to the set of univalent vertices is vertical, is defined up to global vertical translation and dilation. For any $Y \in \mathcal{P}_x$, we still have these two codimension-one normalization conditions on the configuration c_Y, when $\lambda(Y)$ is fixed. But varying the parameter $\lambda(Y)$ adds one to the dimension.

Say that a subset of $V(\Gamma)$ is *trivalent* if it contains no univalent vertex. Write the set $K(A)$ of kids of an element A of $\mathcal{P}$ as the union of its set $K_u^0(A)$ of univalent kids and its set $K_t^0(A)$ of trivalent kids (including the kid of the basepoint).

$$K(A) = K_u^0(A) \sqcup K_t^0(A).$$

Then the dimension of the involved space of maps up to dilation and (possibly vertical) translation from $K(A)$ to $\mathbb{R}^3$ is

- $3\sharp K_t^0(A) - 4$ if $A \in \mathcal{P} \setminus \mathcal{P}_u$,
- $\sharp K_u^0(A) + 3\sharp K_t^0(A) - 2$ if $A \in \mathcal{P}_u \setminus \mathcal{P}_x$,
- $\sharp K_u^0(A) + 3\sharp K_t^0(A) - 1$ if $A \in \mathcal{P}_x$.

Let $\mathcal{P}_{ext}$ denote the union of $\mathcal{P}$ with the set of singletons of elements of V. So $\mathcal{P}_{ext}$ contains all the kids of elements of $\mathcal{P}$, the only element of $\mathcal{P}_{ext}$ that is not a kid is V. Let $\mathcal{P}_{ext,u}$ denote the set of univalent sets of $\mathcal{P}_{ext}$, and let $\mathcal{P}_{ext,t}$ denote the set of trivalent sets of $\mathcal{P}_{ext}$. The sum over $A \in \mathcal{P}$ of the above dimensions is equal to

$$3\sharp\mathcal{P}_{ext,t} + \sharp\mathcal{P}_{ext,u} - 1 - 3\sharp(\mathcal{P} \setminus \mathcal{P}_u) - \sharp\mathcal{P}_u - \sharp(\mathcal{P} \setminus \mathcal{P}_x),$$

where $\sharp\mathcal{P}_{ext,t} = \sharp T(\Gamma) + \sharp(\mathcal{P} \setminus \mathcal{P}_u)$, and $\sharp\mathcal{P}_{ext,u} = \sharp U(\Gamma) + \sharp\mathcal{P}_u$. $\square$

Lemma 15.5 simplifies when $y^0 \in \check{\mathcal{S}}_B(\mathbb{C})$. It allows us to describe the structure of $\mathcal{V}(\Gamma)$ over $\check{\mathcal{S}}_B(\mathbb{C})$, in the following lemma.

Lemma 15.12. *Let $y^0 \in \check{\mathcal{S}}_B(\mathbb{C})$. For any $(c^0, y^0) \in \mathcal{V}(\Gamma)$, there exist*

- *a manifold W with boundary and ridges,*
- *an open neighborhood $N(y^0)$ of y^0 in $\check{\mathcal{S}}_B(\mathbb{C})$,*
- *a small $\varepsilon > 0$,*
- *an oriented tree $\mathcal{T}_{\mathcal{V}}(c^0)$, and*

- *a smooth map* $\varphi_1\colon [0,\varepsilon[^{E(\mathcal{T}_{\mathcal{V}}(c^0))}\times W\times N(y^0)\to \mathcal{S}_B(\mathbb{C})$,

such that the product of φ_1 *by the natural projection* $p_{N(y^0)}\colon [0,\varepsilon[^{E(\mathcal{T}_{\mathcal{V}}(c^0))}\times W\times N(y^0)\to N(y^0)$ *has the following properties.*

$$\varphi_1\times p_{N(y^0)}\colon [0,\varepsilon[^{E(\mathcal{T}_{\mathcal{V}}(c^0))}\times W\times N(y^0)\to \mathcal{S}_B(\mathbb{C})\times N(y^0)$$

restricts to $\left(X(\mathcal{T}_{\mathcal{V}}(c^0))\cap[0,\varepsilon[^{E(\mathcal{T}_{\mathcal{V}}(c^0))}\right)\times W\times N(y^0)$ *as a bijection onto an open neighborhood of* (c^0,y^0) *in* $\mathcal{V}(\Gamma)$, *where* $X(\mathcal{T}_{\mathcal{V}}(c^0))$ *was introduced in Definition 14.3.*

PROOF: According to Lemma 15.1, (c^0,y^0) has a neighborhood parametrized as in Lemma 15.5. In such a parametrization, no u_D is involved since $y^0=y^0_V\in\check{\mathcal{S}}_B(\mathbb{C})$. Also note that $\mathcal{P}'_X=\mathcal{P}_{X,B}$. Define the tree $\mathcal{T}_{\mathcal{V}}(c^0)$ associated to (c^0,y^0) to be the tree obtained from $\mathcal{T}(\mathcal{P},\mathcal{P}_B)$ by replacing the pairs of edges labeled by $\lambda(Y)$ and μ_Y with a single edge labeled by $\tilde{\lambda}(Y)=\lambda(Y)\mu_Y$, for $Y\in\mathcal{P}_x$. (In this tree, the left B-part in Figure 15.1 is reduced to one edge labeled by λ, and the edges labeled by $\tilde{\lambda}(Y)$, for $Y\in\mathcal{P}_x$, or by μ_Y, for $Y\in\mathcal{P}'_X\setminus\mathcal{P}_x$, start at univalent vertices.) The lemma is an easy consequence of Lemma 15.5. □

Lemma 15.13. *Let* $y\in\check{\mathcal{S}}_B(\mathbb{C})$. *A codimension-one open face of* $\mathcal{V}(y,\Gamma)$ *is a stratum* $(\mathcal{P},\mathcal{P}_x)$ *as in Lemma 15.11 such that:*

- *Either* $V(\Gamma)\in\mathcal{P}_x$, $\lambda\neq 0$ *and* $\mathcal{P}=\{V(\Gamma),C\}$,
- *or* $V(\Gamma)\notin\mathcal{P}_x$, $\lambda=0$ *and* $\mathcal{P}=\{V(\Gamma)\}\cup\mathcal{P}_x$ *(where* $\mathcal{P}_x$ *can be empty).*

PROOF: According to Lemma 15.11, the faces of the $\partial\mathcal{V}(y,\Gamma)$ with maximal dimension are such that $\sharp\mathcal{P}=\sharp\mathcal{P}_x+1$, where $V(\Gamma)\in\mathcal{P}$. □

Lemmas 14.13 and 15.12 guarantee that $\mathcal{V}(y,\Gamma)$ behaves as a codimension-one face of a manifold with boundary along a face as in Lemma 15.13.

15.2 A one-form on $\check{\mathcal{S}}_B(\mathbb{C})$

For a finite set B of cardinality at least 2, a configuration $y\in\mathcal{S}_B(\mathbb{C})$, and a Jacobi diagram Γ on a disjoint union of lines $\mathbb{R}_b$ indexed by elements b of B, such that $p_B\colon U(\Gamma)\to B$ is onto, recall that $\mathcal{V}(y,\Gamma)$ is the preimage of y under $p_{\mathcal{S}_B}\colon \mathcal{V}(\Gamma)\to\mathcal{S}_B(\mathbb{C})$. When $y\in\check{\mathcal{S}}_B(\mathbb{C})$, let $\check{\mathcal{V}}(y,\Gamma)$ denote the quotient by vertical translations of $\check{C}(S^3,y^0\times\mathbb{R};\Gamma)$ for a representative $y^0\in\check{C}_B[D_1]$ of y. Note that $\check{\mathcal{V}}(y,\Gamma)$ is an open T-face of $C(R(\mathcal{C}),L;\Gamma)$ for tangles whose top configuration is y, as in Theorem 14.16, where the set B of Theorem 14.16 is empty, $I = \{j\}$, $\mathcal{P}_s = \mathcal{P}_d = \mathcal{P}_x = \{V(\Gamma)\}$ with the notation of 14.19. Assume that Γ is equipped with a vertex-orientation $o(\Gamma)$ as in Definition 6.13 and with an edge-orientation $o_E(\Gamma)$ of $H(\Gamma)$ as before Lemma 7.1. The space $\check{\mathcal{V}}(y,\Gamma)$ is a smooth manifold of dimension $\sharp U(\Gamma) + 3\sharp T(\Gamma) - 1$. It is oriented as the part of the boundary of $C(R(\mathcal{C}), L; \Gamma)$ that is the T-face in which all univalent vertices of Γ tend to ∞ above $\mathcal{C}$, for a tangle L whose top configuration is y. Note that $\check{\mathcal{V}}(y,\Gamma)$ is therefore oriented as the part of the boundary of $(-C(R(\mathcal{C}), L; \Gamma))$ that is (minus) the T-face in which all univalent vertices of Γ tend to ∞ below $\mathcal{C}$, for a tangle L whose bottom configuration is y. The orientation of $\check{\mathcal{V}}(y,\Gamma)$ depends on $o(\Gamma)$ and on $o_E(\Gamma)$, but it does not depend on the global orientations[5] of the lines $\mathbb{R}_b$, which are oriented, only locally, near the images of the univalent vertices of Γ by $o(\Gamma)$ as in Definition 6.13.

Below, we define a one-form $\eta_\Gamma = \eta_{\Gamma, o(\Gamma)}$ on $\check{\mathcal{S}}_B(\mathbb{C})$ to be the integral of $\bigwedge_{e \in E(\Gamma)} p^*_{e,S^2}(\omega_{S^2})$ along the interiors $\check{\mathcal{V}}(y,\Gamma)$ of the compact fibers $\mathcal{V}(y,\Gamma)$. We agree that the integral along the fiber of $dx \wedge \omega$ for a volume form ω of the fiber is $\left(\int_{\text{fiber}} \omega\right) dx$.

Proposition 15.14. *The integral of $\bigwedge_{e \in E(\Gamma)} p^*_{e,S^2}(\omega_{S^2})$ along the interior $\check{\mathcal{V}}(y,\Gamma)$ of the fiber $\mathcal{V}(y,\Gamma)$ is absolutely convergent, and it defines[6] a smooth one-form η_Γ on $\check{\mathcal{S}}_B(\mathbb{C})$. The definition of η_Γ extends naturally[7] to diagrams Γ on $\sqcup_{b \in B} \mathbb{R}_b$ such that $p_B \colon U(\Gamma) \to B$ is not onto.*

Let $\gamma \colon [0,1] \to \check{\mathcal{S}}_B(\mathbb{C})$ be a smooth map. Orient $p^{-1}_{\check{\mathcal{S}}_B}(\gamma([0,1]))$ as[8] the local product $[0,1] \times$ fiber. Then the integral

$$\int_{[0,1]} \gamma^*(\eta_\Gamma) = \int_{p^{-1}_{\check{\mathcal{S}}_B}(\gamma([0,1]))} \bigwedge_{e \in E(\Gamma)} p^*_{e,S^2}(\omega_{S^2})$$

is absolutely convergent. The map $\left(t \mapsto \int_{[0,t]} \gamma^*(\eta_\Gamma)\right)$ is differentiable, and

$$\frac{\partial}{\partial t}\left(\int_{[0,t]} \gamma^*(\eta_\Gamma)\right)(u) = \eta_\Gamma(\gamma(u), \frac{\partial}{\partial t}\gamma_u).$$

[5] The reader who prefers working with oriented strands can assume that the lines $\mathbb{R}_b$ are oriented from bottom to top and consider braids L instead of tangles L, above, for the moment, but since we will need to allow various orientations later for our strands $\mathbb{R}_b$, it is better to work with unoriented strands as much as possible.

[6] Again, η_Γ depends on the arbitrary vertex-orientation $o(\Gamma)$ of Γ, but the product $\eta_\Gamma[\Gamma]$ is independent of $o(\Gamma)$.

[7] In general, η_Γ pulls back through $\check{\mathcal{S}}_{p_B(U(\Gamma))}(\mathbb{C})$. So $\eta_\Gamma = 0$ if $\sharp p_B(U(\Gamma)) < 2$.

[8] Note that it amounts to say that the $[0,1]$ factor replaces the upward vertical translation parameter of the quotient $\check{\mathcal{V}}(y,\Gamma)$, as far as orientations are concerned.

Proof: In the proof, we assume that p_B is onto, without loss of generality. Lemmas 15.12 and 14.14 imply that the integral $\int_{p_{\mathcal{S}_B}^{-1}(\gamma([0,1]))} \bigwedge_{e\in E(\Gamma)} p_{e,S^2}^*(\omega_{S^2})$ is absolutely convergent. Let us prove that the integral of $\bigwedge_{e\in E(\Gamma)} p_{e,S^2}^*(\omega_{S^2})$ along the interior $\check{\mathcal{V}}(y,\Gamma)$ of the fiber $\mathcal{V}(y,\Gamma)$ is absolutely convergent, and that it defines a smooth a one-form η_Γ on $\check{\mathcal{S}}_B(\mathbb{C})$.

Let $y^0 \in \check{\mathcal{S}}_B(\mathbb{C})$, let $N(y^0)$ be small neighborhood of y^0 in $\check{\mathcal{S}}_B(\mathbb{C})$ and let $(\zeta_r \colon N(y^0) \to \mathbb{R})_{r\in \underline{2\sharp B-3}}$ be a system of coordinates on $N(y^0)$. These coordinates give rise to associated smooth one-forms $d\zeta_r = dp_{N(y^0)} \circ \zeta_r$ on $\check{\mathcal{V}}(N(y^0),\Gamma) = \check{\mathcal{V}}(\Gamma) \cap p_{\mathcal{S}_B}^{-1}(N(y^0))$. For a local system $(f_1,\dots,f_k)$ of coordinates of the interior of a fiber $\check{\mathcal{V}}(y,\Gamma)$, and a local product structure with the base, we also have associated forms df_i, which depend on the product structure. (Changing this product structure adds some combination of the $d\zeta_r$ and df_j to df_i). We also have an associated volume form of the fiber $\omega_F = df_1 \wedge \dots \wedge df_k$, which also depends on the product structure. A $(k+1)$-form Ω on $\check{\mathcal{V}}(N(y^0),\Gamma)$ may be expressed as $\Omega = \sum_{r=1}^{2\sharp B-3} d\zeta_r \wedge (g_r\omega_F) + \sum_{i=1}^{k} \omega_{i,v}$, where $\omega_{i,v}$ vanishes at the tangent vector ξ_i to a curve of the fiber, whose coordinates f_j for $j \in \underline{k}\setminus\{i\}$ are constant ($\omega_{i,v}$ is expressed as a wedge product of coordinates forms that does not involve df_i, this decomposition is not canonical). In order to check the convergence of the integral of the pull-back Ω of the form $\bigwedge_{e\in E(\Gamma)} p_{e,S^2}^*(\omega_{S^2})$ on $\check{\mathcal{V}}(N(y^0),\Gamma)$ along the fiber, it suffices to cover the fiber by finitely many neighborhoods as above, to express Ω as above with respect to the corresponding charts, and to check that the $g_r\omega_F$ (which are well defined, up to forms whose integrals along the fiber vanish) and their derivatives with respect to the coordinates ζ_j are bounded in each of these neighborhoods. Lemma 15.12 implies that Ω is the restriction to

$$\left(X(\mathcal{T}_{\mathcal{V}}(c^0)) \cap [0,\varepsilon[^{E(\mathcal{T}_{\mathcal{V}}(c^0))}\right) \times W \times N(y^0)$$

of a smooth form on $[0,\varepsilon[^{E(\mathcal{T}_{\mathcal{V}}(c^0))} \times W \times N(y^0)$, locally. The same holds for the $g_r\omega_F$, and all their iterated partial derivatives with respect to the ζ_j. Lemma 14.14 implies that these forms are bounded in a neighborhood of an arbitrary element (c^0,y^0) of $\mathcal{V}(y^0,\Gamma)$ as in Lemma 15.12. Since the fiber $\mathcal{V}(y^0,\Gamma)$ is compact, the integral of $g_r\omega_F$ along the fiber is absolutely convergent, for any $r \in \underline{2\sharp B-3}$. So are the integrals of its iterated partial derivatives with respect to the ζ_j on $\check{\mathcal{V}}(N(y^0),\Gamma)$ for a small neighborhood $N(y^0)$ of y^0 in $\check{\mathcal{S}}_B(\mathbb{C})$. This proves that η_Γ is a well defined smooth one-form on $\check{\mathcal{S}}_B(\mathbb{C})$.

Definition 15.15. For $k \in \mathbb{N} \setminus \{0\}$, set

$$\eta_{k,B} = \sum_{\Gamma \in \mathcal{D}^e_k(\sqcup_{b\in B}\mathbb{R}_b)} \frac{(3k - \sharp E(\Gamma))!}{(3k)!2^{\sharp E(\Gamma)}} \eta_\Gamma[\Gamma] \in \Omega^1(\mathcal{S}_B(\mathbb{C}); \mathcal{A}_k(\sqcup_{b\in B}\mathbb{R}_b)),$$

where $\eta_\Gamma = 0$, if $\sharp p_B(U(\Gamma)) < 2$, or if Γ is not connected, and

$$\eta_B = \sum_{k \in \mathbb{N}\setminus\{0\}} \eta_{k,B}.$$

The form η_B is a one-form on $\check{\mathcal{S}}_B(\mathbb{C})$ with coefficients in the space $\mathcal{A}(\sqcup_{b\in B}\mathbb{R}_b)$ of Jacobi diagrams on $\sqcup_{b\in B}\mathbb{R}_b$, which is treated as an unoriented manifold, as in Definition 6.16. The form η_B will be regarded as a *connection.* For a path $\gamma\colon [a,b] \to \check{C}_B[D_1]$, define the *holonomy* $\mathrm{hol}_\gamma(\eta_B)$ of η_B along γ as

$$\mathrm{hol}_\gamma(\eta_B) = \sum_{r=0}^{\infty} \int_{(t_1,\dots,t_r)\in[a,b]^r | t_1 \le t_2 \le \dots \le t_r} \bigwedge_{i=1}^{r} (\gamma \circ p_i)^*(\eta_B),$$

where $p_i(t_1,\dots,t_r) = t_i$, the wedge product of forms is performed as usual and the diagrams are multiplied from bottom to top (from left to right) with respect to their order of appearance.

The degree 0 part of $\mathrm{hol}_\gamma(\eta_B)$ is the unit $[\emptyset]$ of $\mathcal{A}_k(\sqcup_{b\in B}\mathbb{R}_b)$, and

$$\mathrm{hol}_\gamma(\eta_B) = [\emptyset] + \sum_{r=1}^{\infty} \int_{(t_1,\dots,t_r)\in[a,b]^r | t_1 \le t_2 \le \dots \le t_r} \bigwedge_{i=1}^{r} (\gamma \circ p_i)^*(\eta_B).$$

This holonomy is valued in a space of diagrams on an unoriented source as in Definition 6.16, Proposition 10.21 and Remark 10.22. It satisfies the following properties.

- For an orientation-preserving diffeomorphism $\psi\colon [c,d] \to [a,b]$,
$$\mathrm{hol}_{\gamma\circ\psi}(\eta_B) = \mathrm{hol}_\gamma(\eta_B).$$
- When $\gamma_1\gamma_2$ is the path composition of γ_1 and γ_2,
$$\mathrm{hol}_{\gamma_1\gamma_2}(\eta_B) = \mathrm{hol}_{\gamma_1}(\eta_B)\mathrm{hol}_{\gamma_2}(\eta_B).$$
- $\frac{\partial}{\partial t}\mathrm{hol}_{\gamma_{|[a,t]}}(\eta_B) = \mathrm{hol}_{\gamma_{|[a,t]}}(\eta_B)\eta_B(\gamma'(t))$, and
- $\frac{\partial}{\partial t}\mathrm{hol}_{\gamma_{|[t,b]}}(\eta_B) = -\eta_B(\gamma'(t))\mathrm{hol}_{\gamma_{|[t,b]}}(\eta_B)$.

Lemma 15.16. *Let $(h_t)_{t\in[0,1]}$ be an isotopy of $\check{R}(\mathcal{C})$ such that h_t is the identity map on $(\mathbb{C}\setminus D_1)\times\mathbb{R}$ for any t, and h_t may be expressed as $h_t^-\times\mathbf{1}_{]-\infty,0]}$ (resp. $h_t^+\times\mathbf{1}_{[1,+\infty[}$) on $\mathbb{C}\times]-\infty,0]$ (resp. on $\mathbb{C}\times[1,+\infty[$) for a planar isotopy $(h_t^-)_{t\in[0,1]}$ (resp. $(h_t^+)_{t\in[0,1]}$). Assume $h_0=\mathbf{1}$ and note that h_t preserves $\mathcal{C}$ setwise. Let L be a long tangle representative of $\check{R}(\mathcal{C})$ whose bottom (resp. top) configuration is represented by a map $y^-\colon B^-\to D_1$ (resp. $y^+\colon B^+\to D_1$). Let J_{bb} denote the set of components of L that go from bottom to bottom and let J_{tt} denote the set of components of L that go from top to top. For $K_j\in J_{bb}$, set $\varepsilon(K_j)=-$, and for $K_j\in J_{tt}$, set $\varepsilon(K_j)=+$. For a component K_j of $J_{bb}\cup J_{tt}$, the difference $\left(h_t^{\varepsilon(K_j)}(y^{\varepsilon(K_j)}(K_j(1)))-h_t^{\varepsilon(K_j)}(y^{\varepsilon(K_j)}(K_j(0)))\right)$ is a positive multiple of a complex direction $\exp(i2\pi\theta_j(t))$ for a path $\theta_j\colon[0,1]\to\mathbb{R}$. With the notation of Theorem 12.7, set $\mathcal{Z}(t)=\mathcal{Z}(\mathcal{C},h_t(L))$. Then*

$$\left(\prod_{K_j\in J_{bb}\cup J_{tt}}(\exp(-2\varepsilon(K_j)(\theta_j(t)-\theta_j(0))\alpha)\sharp_j)\right)\mathcal{Z}(t)=$$

$$hol_{h^-_{|[t,0]}\circ y^-}(\eta_{B^-})\mathcal{Z}(0)hol_{h^+_{|[0,t]}\circ y^+}(\eta_{B^+}).$$

PROOF: Let τ be a parallelization of $\mathcal{C}$. Set $L=(K_j)_{j\in\underline{k}}$, and recall that $\mathcal{Z}(t)=\exp(-\frac{1}{4}p_1(\tau)\beta)\left(\prod_{j=1}^k(\exp(-I_\theta(K_j(t),\tau)\alpha)\sharp_j)\right)Z(\mathcal{C},h_t(L),\tau)$.

The algebraic boundary of the chain[9] $\cup_{t\in[t_0,t_1]}C(R(\mathcal{C}),h_t(L);\Gamma)$ is

$$C(R(\mathcal{C}),h_{t_1}(L);\Gamma)-C(R(\mathcal{C}),h_{t_0}(L);\Gamma)-\sum\left(\cup_{t\in[t_0,t_1]}F_t\right),$$

where the sum runs over the codimension-one faces F_t of $C(R(\mathcal{C}),h_t(L);\Gamma)$. Faces cancel as in Section 14.3 except for the anomaly α faces and the faces for which some vertices are at ∞.

The variations due to the anomaly α faces contribute to $\frac{\partial}{\partial t}\check{Z}(\mathcal{C},h_t(L),\tau)$ as

$$\left(\sum_{j=1}^k\frac{\partial}{\partial t}\left(2\int_{[0,t]\times U^+K_j}p_\tau^*(\omega_{S^2})\right)\alpha\sharp_j\right)\check{Z}(\check{R},h_t(L),\tau)$$

as in Lemma 10.19.

When the bottom and top configurations are not fixed, and when $K_j\in J_{bb}\cup J_{tt}$ we have

$$I_\theta(K_j(u),\tau)-I_\theta(K_j(0),\tau)=2\int_{\cup_{t\in[0,u]}p_\tau(U^+K_j(t))\cup S(K_j(t))}\omega_{S^2},$$

[9]This chain is modeled locally by open subsets of $C(R(\mathcal{C}),h_t(L);\Gamma)\times]t-\varepsilon,t+\varepsilon[$ unless the isotopies $h_t^\pm$ are degenerate. See Theorem 14.16 and Lemma 14.13. Stokes' theorem applies thanks to Lemma 14.14.

as in the proof of Lemma 14.41, where $S(K_j(t))$ denotes the half-circle from $\varepsilon(K_j)\vec{N}$ to $-\varepsilon(K_j)\vec{N}$ through $\exp(2i\pi\theta_j(t))$, as in Lemma 12.5. So

$$I_\theta(K_j(t),\tau) - I_\theta(K_j(0),\tau) = 2\int_{\cup_{u\in[0,t]}p_\tau(U^+K_j(u))} \omega_{S^2} - 2\varepsilon(K_j)(\theta_j(t)-\theta_j(0)),$$

and

$$\tilde{\mathcal{Z}}(t) = \left(\prod_{K_j\in J_{bb}\cup J_{tt}} (\exp(-2\varepsilon(K_j)(\theta_j(t)-\theta_j(0))\alpha)\sharp_j)\right)\mathcal{Z}(t)$$

gets no variation from the anomaly α faces, as in Corollary 10.20.

Let F_t be a face of $C(R(\mathcal{C}),h_t(L);\Gamma)$ for which a subset V of $V(\Gamma)$ is mapped to ∞, and let $F=\cup_{t\in[t_0,t_1]}F_t$. Such a face is either a T-face as in Theorem 14.16 and as in Lemma 14.37 or a face $F_\infty(V,L,\Gamma)$ as around Notation 8.17.

In both cases, an element c^0 of the face involves an injective configuration $T_0\phi_\infty\circ f_1^0$ from the kids of V to $(T_\infty R(\mathcal{C})\setminus 0)$ up to dilation. Again, for an edge $e=(v_1,v_2)$ whose vertices are in different kids of V,

$$\begin{aligned} p_\tau\circ p_e(c^0) &= \tfrac{\phi_\infty\circ f_1^0(v_2)-\phi_\infty\circ f_1^0(v_1)}{\|\phi_\infty\circ f_1^0(v_2)-\phi_\infty\circ f_1^0(v_1)\|}\\ &= \tfrac{\|f_1^0(v_1)\|^2 f_1^0(v_2)-\|f_1^0(v_2)\|^2 f_1^0(v_1)}{\|\|f_1^0(v_1)\|^2 f_1^0(v_2)-\|f_1^0(v_2)\|^2 f_1^0(v_1)\|},\end{aligned}$$

for an edge $e=(v_1,v_2)$ such that $v_1\in V$ and $v_2\notin V$, $p_\tau\circ p_e(c^0)=-\frac{f_1^0(v_1)}{\|f_1^0(v_1)\|}$ and for an edge $e=(v_1,v_2)$ such that $v_2\in V$ and $v_1\notin V$, $p_\tau\circ p_e(c^0)=\frac{f_1^0(v_2)}{\|f_1^0(v_2)\|}$.

Let E_∞ be the set of edges between elements of the set V of vertices mapped to ∞ in F, and let E_m denote the set of edges with one end in V. The face F_t is diffeomorphic to a product by $\check{C}_{V(\Gamma)\setminus V}(\check{R}(\mathcal{C}),h_t(L);\Gamma)$, whose dimension is

$$3\sharp(T(\Gamma)\cap(V(\Gamma)\setminus V))+\sharp(U(\Gamma)\cap(V(\Gamma)\setminus V)),$$

of a space $C_{V,t}$ of dimension

$$3\sharp(T(\Gamma)\cap V)+\sharp(U(\Gamma)\cap V)-1 = 2\sharp(E_\infty)+\sharp(E_m)-1,$$

and $\bigwedge_{e\in E_\infty\cup E_m} p_e^*(\omega(j_E(e)))$ has to be integrated along $\cup_{t\in[0,1]}C_{V,t}$, according to the above expression of $p_\tau\circ p_e$ for edges of $E_\infty\cup E_m$. The degree of this form is $2\sharp(E_\infty\cup E_m)$. So the face F cannot contribute unless $E_m=\emptyset$.

Now the expression $p_\tau\circ p_e$ for edges of E_∞ makes also clear that, if f_1^0 is changed to $\phi_\infty\circ T\circ\phi_\infty\circ f_1^0$, for a vertical translation T such that 0 is not in the image of $\phi_\infty\circ T\circ\phi_\infty\circ f_1^0$, then the image under $\prod_{e\in E_\infty} p_e$ is unchanged.

So we have a one-parameter group acting on our face F such that $\prod_{e\in E_\infty} p_e$ factors through this action. Unless V has only one kid, this action is not trivial, and the quotient of the face by this action is of dimension strictly less than the face dimension.

Therefore for the faces that contribute, we have $E_m=\emptyset$, and V has only one kid. Thus, according to Lemma 14.37, we are left with the T-faces of Theorem 14.16 (for which $B=\emptyset$ and I has one element) for which $\mathcal{P}_x=\{V\}$, which yield the derivative $\frac{\partial}{\partial t}\tilde{\mathcal{Z}}=d\tilde{\mathcal{Z}}\left(\frac{\partial}{\partial t}\right)$ where

$$d\tilde{\mathcal{Z}}=-(t\mapsto h_t^-\circ y^-)^*(\eta_{B^-})\tilde{\mathcal{Z}}+\tilde{\mathcal{Z}}(t\mapsto h_t^+\circ y^+)^*(\eta_{B^+}).$$

This proves the equality

$$\tilde{\mathcal{Z}}(t)=\mathrm{hol}_{h^-_{|[t,0]}\circ y^-}(\eta_{B^-})\tilde{\mathcal{Z}}(0)\mathrm{hol}_{h^+_{|[0,t]}\circ y^+}(\eta_{B^+}),$$

and this leads to the formula for $\mathcal{Z}$. □

Corollary 15.17. *Under the hypotheses of Lemma 15.16, let $L_\parallel$ be a parallel of L. Let $h_t(L)_\parallel$ be the parallel of $h_t(L)$ such that, for any component K_j of $J_{bb}\cup J_{tt}$,*

$$lk(h_t(K_j),h_t(K_j)_\parallel)-lk(K_j,K_{j\parallel})=-\varepsilon(K_j)2(\theta_j(t)-\theta_j(0)),$$

and, for any other component K of L, $lk(h_t(K),h_t(K)_\parallel)-lk(K,K_\parallel)$. Use Definition 12.12 to set $\mathcal{Z}^f(t)=\mathcal{Z}^f(\mathcal{C},h_t(L),h_t(L)_\parallel))$, then[10]

$$\mathcal{Z}^f(t)=hol_{h^-_{|[t,0]}\circ y^-}(\eta_{B^-})\mathcal{Z}^f(0)hol_{h^+_{|[0,t]}\circ y^+}(\eta_{B^+}).$$

See Definition 13.4. □

A connection is *flat* if its holonomy along a null-homotopic loop is trivial.

Proposition 15.18. *The connection η_B is flat on $\check{\mathcal{S}}_B(\mathbb{C})$. When $\gamma\colon[0,1]\to\check{C}_B[D_1]$ is smooth with vanishing derivatives at 0 and 1, the image $T(\gamma)$ of the graph of γ in $D_1\times[0,1]$ is a tangle in $D_1\times[0,1]$ and*

$$\mathcal{Z}^f(T(\gamma))=\mathcal{Z}(T(\gamma))=hol_{p_{CS}\circ\gamma}(\eta_B),$$

[10] Again, the holonomies are considered as valued in spaces of diagrams on unoriented sources, where the vertex-orientation of Jacobi diagrams includes local orientations of strands, which can be made consistent with a global orientation induced by L, as in Definition 6.16.

where $p_{CS} \circ \gamma$ is the composition of γ by the natural projection $p_{CS}\colon \check{C}_B[D_1] \to \mathcal{S}_B(\mathbb{C})$. For two framed tangles $(\mathcal{C}_1, L_1)$ and $(\mathcal{C}_2, L_2)$ such that the bottom of L_2 coincides with the top of L_1, if one of them is a braid $T(\gamma)$ as above,

$$\mathcal{Z}^f(\mathcal{C}_1\mathcal{C}_2, L_1L_2) = \mathcal{Z}^f(\mathcal{C}_1, L_1)\mathcal{Z}^f(\mathcal{C}_2, L_2),$$

with products obtained by stacking above in natural ways on both sides, reading from left to right.

PROOF: Applying Lemma 15.16 when L is a trivial braid, and when $h_t^- = h_0^-$ is constant, and $h_t^+ \circ y^+ = \gamma(t)$ shows that $\mathcal{Z}([\gamma]) = \mathrm{hol}_{p_{CS}\circ\gamma}(\eta_B)$. Then the isotopy invariance of $\mathcal{Z}$ shows that η_B is flat on $\check{\mathcal{S}}_B(\mathbb{C})$. Applying Lemma 15.16 when $h_t^- = h_0^-$ is constant and when $\gamma(t) = h_t^+ \circ y^+$ proves that

$$\mathcal{Z}^f(\mathcal{C}, LT(\gamma)) = \mathcal{Z}^f(\mathcal{C}, L)\mathcal{Z}^f(T(\gamma)).$$

So $\mathcal{Z}^f\,(T(\gamma_1)T(\gamma_2)) = \mathcal{Z}^f\,(T(\gamma_1))\,\mathcal{Z}^f\,(T(\gamma_2))$ for braids.

Applying Lemma 15.16 when $h_t^+ = h_0^+$ is constant and when $\gamma(t) = h_{1-t}^- \circ y^-$ proves that $\mathcal{Z}^f(\mathcal{C}_1\mathcal{C}_2, L_1L_2) = \mathcal{Z}^f(\mathcal{C}_1, L_1)\mathcal{Z}^f(\mathcal{C}_2, L_2)$, when $(\mathcal{C}_1, L_1)$ is a braid and $(\mathcal{C}_2, L_2)$ is a framed tangle, too. $\square$

Remark 15.19. In the above proposition, we proved that η_B is flat by proving that its holonomy is $\mathbf{1} = [\emptyset]$ on null-homotopic loops. Flatness of a differentiable connection η is often established by proving that its *curvature* $(d\eta + \eta \wedge \eta)$ vanishes, instead. Let us recall how the curvature vanishing implies the homotopy invariance of the holonomy.

When a loop γ_1 bounds a disk D, which is viewed as the image of a homotopy $\gamma\colon [0,1] \times [0,1] \to D$ that maps $\gamma(u,t)$ to $\gamma_u(t)$ so that γ maps $([0,1] \times \{0,1\}) \cup \{0\} \times [0,1]$ to a point, Stokes' theorem allows us to compute

$$\mathrm{hol}_{\gamma_1}(\eta) = [\emptyset] + \sum_{r=1}^{\infty} \int_{\Delta^{(r)}} \bigwedge_{i=1}^{r} (\gamma_1 \circ p_i)^*(\eta),$$

where $\Delta^{(r)} = \{(t_1, \dots, t_r) \in [0,1]^r \mid t_1 \le t_2 \le \dots \le t_r\}$, by integrating $d\bigwedge_{i=1}^{r}(\gamma_u \circ p_i)^*(\eta)$ over $[0,1] \times \Delta^{(r)}$, as follows.

$$\partial([0,1] \times \Delta^{(r)}) = \big((\partial[0,1]) \times \Delta^{(r)}\big) \cup [0,1] \times \partial(-\Delta^{(r)}),$$

where

$$\partial\Delta^{(r)} = \sum_{j=0}^{r} (-1)^{j+1} F_j(\Delta^{(r)}),$$

with $F_0(\Delta^{(r)}) = \{(0, t_2, \ldots, t_r) \in \Delta^r\}$, $F_r(\Delta^r) = \{(t_1, t_2, \ldots, t_{r-1}, 1) \in \Delta^r\}$, and, for $j \in \underline{r-1}$,

$$F_j(\Delta^r) = \{(t_1, \ldots, t_r) \in \Delta^r \mid t_j = t_{j+1}\}.$$

So

$$\int_{\Delta^{(r)}} \bigwedge_{i=1}^{r} (\gamma_1 \circ p_i)^*(\eta) - \int_{\Delta^{(r)}} \bigwedge_{i=1}^{r} (\gamma_0 \circ p_i)^*(\eta) + \sum_{j=0}^{r} (-1)^j \int_{[0,1] \times F_j(\Delta^{(r)})} \bigwedge_{i=1}^{r} (\gamma \circ p_i)^*(\eta)$$

$$= \int_{[0,1] \times \Delta^{(r)}} d\left(\bigwedge_{i=1}^{r} (\gamma \circ p_i)^*(\eta)\right),$$

where the faces F_0 and F_r do not contribute since γ maps $([0,1] \times \{0,1\})$ to a point and $\int_{\Delta^{(r)}} \bigwedge_{i=1}^{r} (\gamma_0 \circ p_i)^*(\eta)$ vanishes, similarly. Thus

$$\int_{\Delta^{(r)}} \bigwedge_{i=1}^{r} (\gamma_1 \circ p_i)^*(\eta) =$$

$$\begin{array}{ll} & \sum_{j=1}^{r-1} (-1)^{j-1} \int_{[0,1] \times \Delta^{(r-1)}} \bigwedge_{i=1}^{r-1} (\gamma \circ p_i)^*(\eta) \left(\frac{(\gamma \circ p_j)^*(\eta \wedge \eta)}{(\gamma \circ p_j)^*(\eta)}\right) \\ + & \sum_{j=1}^{r} (-1)^{j-1} \int_{[0,1] \times \Delta^{(r)}} \left(\bigwedge_{i=1}^{r} (\gamma \circ p_i)^*(\eta)\right) \left(\frac{(\gamma \circ p_j)^*(d\eta)}{(\gamma \circ p_j)^*(\eta)}\right), \end{array}$$

where the fraction means that the denominator is replaced with the numerator in the preceding expression. So $(\mathrm{hol}_{\gamma_1}(\eta) - [\emptyset])$ is equal to

$$\sum_{r=1}^{\infty} \left(\sum_{j=1}^{r} (-1)^{j-1} \int_{[0,1] \times \Delta^{(r)}} \bigwedge_{i=1}^{r} (\gamma \circ p_i)^*(\eta) \left(\frac{(\gamma \circ p_j)^*(d\eta + \eta \wedge \eta)}{(\gamma \circ p_j)^*(\eta)} \right) \right),$$

which is obviously zero, when $(d\eta + \eta \wedge \eta)$ vanishes.

Corollary 15.20. $d\eta_B + \eta_B \wedge \eta_B = 0.$

Proof: Let us prove that the degree k-part $(d\eta_B + \eta_B \wedge \eta_B)_k$ of $(d\eta_B + \eta_B \wedge \eta_B)$ vanishes, for any $k \in \mathbb{N}$, by induction on the degree k. This is obviously true for $k = 0$. Let us assume that $k > 0$ and that $(d\eta_B + \eta_B \wedge \eta_B)_i$ vanishes for $i < k$. For any disk $D = \gamma([0,1] \times \Delta^{(1)})$ as in Remark 15.19, the degree k part of the holonomy of η_B, along ∂D, which vanishes, is the integral of $(d\eta_B + \eta_B \wedge \eta_B)_k$ along D, according to Remark 15.19. Therefore the integral of $(d\eta_B + \eta_B \wedge \eta_B)_k$ vanishes along any disk D, and $(d\eta_B + \eta_B \wedge \eta_B)_k$ is zero. $\square$

Below, we compute $d\eta_B$ and sketch an alternative proof of the equality of Corollary 15.20.

Lemma 15.21. *The integral of* $\left(-\bigwedge_{e\in E(\Gamma)} p^*_{e,S^2}(\omega_{S^2})\right)$ *along the interiors of the codimension-one faces of* $\mathcal{V}(y,\Gamma)$ *is absolutely convergent and it defines the smooth two-form* $(y\mapsto d\eta_\Gamma(y))$ *on* $\check{\mathcal{S}}_B(\mathbb{C})$*.*

Proof: As in the proof of Proposition 15.14, Lemma 14.14 implies that the integral of $\left(-\bigwedge_{e\in E(\Gamma)} p^*_{e,S^2}(\omega_{S^2})\right)$ along the interiors of the codimension-one faces of $\mathcal{V}(y,\Gamma)$ is absolutely convergent and that this integral defines a smooth two-form on $\check{\mathcal{S}}_B(\mathbb{C})$.

To see that this two-form is $d\eta_\Gamma$, use a chart $\psi\colon \mathbb{R}^s \to N(y)$ of a neighborhood of y in $\check{\mathcal{S}}_B(\mathbb{C})$, with $s = 2\sharp B - 3$, and let ζ_i denote the composition $p_i\circ\psi^{-1}$. So $\eta_\Gamma = \sum_{i=1}^s \eta_i d\zeta_i$ and

$$d\eta_\Gamma = \sum_{(i,j)\in\underline{s}^2 | i<j} \left(\frac{\partial}{\partial\zeta_i}\eta_j - \frac{\partial}{\partial\zeta_j}\eta_i\right) d\zeta_i\wedge d\zeta_j,$$

where

$$\frac{\partial}{\partial\zeta_2}\eta_1 = \lim_{\substack{t_2\to 0\\ t_2\in]0,\infty[}} \frac{1}{t_2}\left(\lim_{\substack{t_1\to 0\\ t_1\in]0,\infty[}}\frac{1}{t_1}\int_{\psi([0,t_1]\times\{(t_2,0,\dots,0)\})-\psi([0,t_1]\times\{(0,0,\dots,0)\})}\eta_\Gamma\right).$$

So

$$\frac{\partial}{\partial\zeta_1}\eta_2 - \frac{\partial}{\partial\zeta_2}\eta_1 = \lim_{\substack{(t_1,t_2)\to 0\\ (t_1,t_2)\in]0,\infty[^2}} \frac{1}{t_1 t_2}\int_{\partial N(t_1,t_2)}\eta_\Gamma,$$

where $N(t_1,t_2) = \psi([0,t_1]\times[0,t_2]\times(0))$ and

$$\int_{\partial N(t_1,t_2)}\eta_\Gamma = \int_{\partial p^{-1}_{\check{\mathcal{S}}_B}(N(t_1,t_2))\setminus\left(\cup_{y\in N(t_1,t_2)}\partial\mathcal{V}(y,\Gamma)\right)} \bigwedge_{e\in E(\Gamma)} p^*_{e,S^2}(\omega_{S^2}).$$

Lemma 15.12 and Lemma 14.14 imply that Stokes' theorem applies to the closed form $\bigwedge_{e\in E(\Gamma)} p^*_{e,S^2}(\omega_{S^2})$, so $\int_{\partial p^{-1}_{\check{\mathcal{S}}_B}(N(t_1,t_2))}\bigwedge_{e\in E(\Gamma)} p^*_{e,S^2}(\omega_{S^2}) = 0$, and

$$\int_{N(t_1,t_2)} d\eta_\Gamma = -\int_{y\in N(t_1,t_2)}\int_{\partial\mathcal{V}(y,\Gamma)}\bigwedge_{e\in E(\Gamma)} p^*_{e,S^2}(\omega_{S^2}).$$

$\square$

Remark 15.22. The flatness condition $d\eta_B + \eta_B\wedge\eta_B = 0$ can be proved from the definition of $d\eta_B$, which can be extracted from Definition 15.15 and Lemma 15.21, as an exercise along the following lines.

Let us use Lemma 15.21 to compute $d\eta_B$. Let $y \in \check{\mathcal{S}}_B(\mathbb{C})$. According to Lemma 15.13, for a connected Jacobi diagram $\Gamma \in \mathcal{D}^e_k(\sqcup_{b\in B}\mathbb{R}_b)$, the faces of the $\partial\mathcal{V}(y,\Gamma)$ with maximal dimension are of two types: Either $V(\Gamma) \in \mathcal{P}_x$ and $\mathcal{P} = \{V(\Gamma), C\}$, or $V(\Gamma) \notin \mathcal{P}_x$.

In the latter case, $\mathcal{P} = \{V(\Gamma)\} \sqcup \mathcal{P}_x$, the configuration of the kids of $V(\Gamma)$ maps the univalent kids of $V(\Gamma)$ to the vertical line through the origin and it is defined up to vertical translation and dilation. Since Γ is connected, if a univalent daughter Y of $V(\Gamma)$ contains a trivalent vertex, then it contains a trivalent vertex that is bivalent, univalent or 0-valent in Γ_Y, and this type of face does not contribute to $d\eta_B$ (as in Sections 9.2 and 9.3). (Note that the univalent daughters of $V(\Gamma)$ are in $\mathcal{P}_x$. So they must have vertices on at least two strands, and Γ_Y cannot be an edge between a univalent vertex and a trivalent one.) For the faces in which all the daughters Y of $V(\Gamma)$ contain only univalent vertices, since Γ_Y does not contain chords, the integrated form is determined by the configuration space of the kids of $V(\Gamma)$, up to dilation and (conjugates of) vertical translations. The dimension of this space is smaller than $3\sharp T(\Gamma) + \sharp U(\Gamma) - 2$, if $\mathcal{P}_x$ is not empty. If $\mathcal{P}_x$ is empty, then the face is independent of the planar configuration y. So it does not contribute to $d\eta_B$ either.

Thus, the only faces of the $\partial\mathcal{V}(y,\Gamma)$ that contribute to $d\eta_B$ are such that $\mathcal{P} = \{V(\Gamma), C\}$, and the univalent vertices of C are on one strand. An analysis similar to that performed in Section 9.3 proves that the only faces that contribute are the STU-faces that involve a collapse of an edge that contains one univalent vertex, and such that the other two diagrams that are involved in the corresponding STU are not connected, so those diagrams are disjoint unions of two components Γ_1 and Γ_2 on $\sqcup_{b\in B}\mathbb{R}_b$ (when the involved diagrams are connected, the corresponding faces cancel by STU). Consider a configuration c_1 of Γ_1 on $\sqcup_{b\in B} y(b) \times \mathbb{R}_b$ and a configuration c_2 of Γ_2 on $\sqcup_{b\in B} y(b) \times \mathbb{R}_b$ for a planar configuration y. View Γ_1 far below Γ_2, and slide it vertically, until it is far above. During this sliding, there will be heights at which one univalent vertex of Γ_1 coincides with one univalent vertex of Γ_2. (For a generic pair (c_1, c_2), there are no heights at which more than one univalent vertex of Γ_1 coincide with one univalent vertex of Γ_2.) Each such collision corresponds to a configuration that contributes in an incomplete STU relation involved in $d\eta_B$. Furthermore the sum of the corresponding graph classes is $([\Gamma_1][\Gamma_2] - [\Gamma_2][\Gamma_1])$. This roughly shows how $d\eta_B = -\eta_B \wedge \eta_B$.

Chapter 16

Discretizable variants of $\mathcal{Z}^f$ and extensions to q–tangles

We introduce and study variants of $\mathcal{Z}^f$, which involve non-homogeneous propagating forms, in Sections 16.1, 16.2 and 16.3. These variants, which allow discrete computations from algebraic intersections as in Chapter 11, will be used in the proofs of important properties of $\mathcal{Z}^f$ in Chapter 17, where we will finish the proof of Theorem 13.12. Section 16.5 is devoted to the extension of $\mathcal{Z}^f$ and its variants to q–tangles. This extension relies on the theory of semi-algebraic sets. We review known facts about semi-algebraic structures, and extract useful lemmas for our purposes, in Section 16.4.

Throughout this chapter, N denotes some (large) fixed integer N, $N \geq 2$, and, for $i \in \underline{3N}$, $\tilde{\omega}(i, S^2) = (\tilde{\omega}(i, t, S^2))_{t\in[0,1]}$ is a closed 2–form on $[0,1] \times S^2$ such that $\tilde{\omega}(i, 0, S^2)$ is a volume-one form of S^2. As in Definition 7.6, for a finite set A, an A-*numbered* Jacobi diagram is a Jacobi diagram Γ whose edges are oriented, equipped with an injection $j_E \colon E(\Gamma) \hookrightarrow A$, which numbers the edges. Let $\mathcal{D}^e_{k,A}(\mathcal{L})$ denote the set of A-numbered degree k Jacobi diagrams with support $\mathcal{L}$ without looped edges.

16.1 Discretizable holonomies

For a finite set B, a $\underline{3N}$-numbered Jacobi diagram Γ on $\sqcup_{b\in B}\mathbb{R}_b$, and an edge e of Γ from a vertex $v(e,1)$ to a vertex $v(e,2)$, there is a map

$$p_{e,S^2} \colon [0,1] \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \to [0,1] \times S^2$$

that maps (t, c) to $(t, p_{S^2}((c(v(e,1)), c(v(e,2)))))$. That provides the form $\bigwedge_{e\in E(\Gamma)} p^*_{e,S^2}(\tilde{\omega}(j_E(e), S^2))$ over $[0,1] \times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$. This form pulls back to provide smooth forms on the smooth strata of $[0,1] \times \mathcal{V}(\Gamma)$. Define the smooth one-form $\eta_\Gamma = \eta_\Gamma((\tilde{\omega}(i,S^2))_{i\in\underline{3N}})$ on $[0,1]\times\check{\mathcal{S}}_B(\mathbb{C})$, so that $\eta_\Gamma(t,y)$ is the integral of $\bigwedge_{e\in E(\Gamma)} p^*_{e,S^2}(\tilde{\omega}(j_E(e),S^2))$ along the interiors $\{t\}\times\check{\mathcal{V}}(y,\Gamma)$ of the fibers $\{t\}\times\mathcal{V}(y,\Gamma)$ of $[0,1]\times\mathcal{V}(\Gamma)$, as in Proposition 15.14.

Definitions 16.1. When A is a subset of $\underline{3N}$ with cardinality $3n$, with $n>0$, set

$$\eta_{B,A} = \sum_{\Gamma\in\mathcal{D}^e_{n,A}(\sqcup_{b\in B}\mathbb{R}_b)} \zeta_\Gamma\eta_\Gamma[\Gamma] \in \Omega^1([0,1]\times\check{\mathcal{S}}_B(\mathbb{C});\mathcal{A}_n(\sqcup_{b\in B}\mathbb{R}_b)),$$

where $\zeta_\Gamma = \frac{(\sharp A-\sharp E(\Gamma))!}{(\sharp A)!2^{\sharp E(\Gamma)}}$. The form

$$\eta_{B,A} = \eta_{B,A}((\tilde{\omega}(i,S^2))_{i\in A})$$

pulls back to a one-form on $[0,1]\times\check{C}_B[D_1]$ still denoted by $\eta_{B,A}$ with coefficients in $\mathcal{A}_n(\sqcup_{b\in B}\mathbb{R}_b)$, which is again viewed as a space of diagrams on an unoriented source as in Definition 6.16, Proposition 10.21 and Remark 10.22. Set $\eta_{B,\emptyset}=0$.

Let $\eta_{B,N}$ denote $\eta_{B,\underline{3N}}$. If $\tilde{\omega}(i,0,S^2)=\omega_{S^2}$, then the restriction of $\eta_{B,N}$ to $\{0\}\times\check{\mathcal{S}}_B(\mathbb{C})$ is the form $\eta_{N,B}$ of Definition 15.15.

For an integer r, and for a set A of cardinality $3n$, let $P_r(A)$ denote the set of r-tuples $(A_1,A_2,\dots,A_r)$, where A_i is a subset of A with a cardinality multiple of 3, the A_i are pairwise disjoint, and their union is A.

Now, for a non-empty subset A of cardinality $3n$ of $\underline{3N}$, for a path $\gamma\colon[0,1]\to[0,1]\times\check{\mathcal{S}}_B(\mathbb{C})$, define the *$A$-holonomy* $\widetilde{\mathrm{hol}}_\gamma(\eta_{B,A})$ of $\eta_{B,A}$ along γ to be

$$\widetilde{\mathrm{hol}}_\gamma(\eta_{B,A}) =$$
$$\sum_{r=0}^\infty \sum_{(A_1,\dots,A_r)\in P_r(A)} \frac{\prod_{i=1}^r(\sharp A_i)!}{(\sharp A)!}\int_{(t_1,\dots,t_r)\in[0,1]^r|t_1\le t_2\le\dots\le t_r} \bigwedge_{i=1}^r(\gamma\circ p_i)^*(\eta_{B,A_i}),$$

where $p_i(t_1,\dots,t_r)=t_i$, and $\widetilde{\mathrm{hol}}_\gamma(\eta_{B,\emptyset})=[\emptyset]$. We have

$$\widetilde{\mathrm{hol}}_{\gamma_1\gamma_2}(\eta_{B,A}) = \sum_{(A_1,A_2)\in P_2(A)}\frac{(\sharp A_1)!(\sharp A_2)!}{(\sharp A)!}\widetilde{\mathrm{hol}}_{\gamma_1}(\eta_{B,A_1})\widetilde{\mathrm{hol}}_{\gamma_2}(\eta_{B,A_2}),$$

$$\frac{\partial}{\partial t}\widetilde{\mathrm{hol}}_{\gamma_{|[0,t]}}(\eta_{B,A}) = \sum_{(A_1,A_2)\in P_2(A)}\frac{(\sharp A_1)!(\sharp A_2)!}{(\sharp A)!}\widetilde{\mathrm{hol}}_{\gamma_{|[0,t]}}(\eta_{B,A_1})\eta_{B,A_2}(\gamma(t),\gamma'(t)),$$

and

$$\frac{\partial}{\partial t}\widetilde{\mathrm{hol}}_{\gamma_{|[t,1]}}(\eta_{B,A}) = -\sum_{(A_1,A_2)\in P_2(A)}\frac{(\sharp A_1)!(\sharp A_2)!}{(\sharp A)!}\eta_{B,A_1}(\gamma(t),\gamma'(t))\widetilde{\mathrm{hol}}_{\gamma_{|[t,1]}}(\eta_{B,A_2}).$$

Behaviour of the coefficients in the above equalities: The contribution of a graph $\Gamma = \Gamma_1 \sqcup \Gamma_2$ equipped with an injection $j_E\colon E(\Gamma) \hookrightarrow A$ comes with a coefficient $\zeta_\Gamma = \frac{(\sharp A - \sharp E(\Gamma))!}{(\sharp A)! 2^{\sharp E(\Gamma)}}$. Let n_i be the degree of Γ_i, for $i \in \{1,2\}$. There are $\frac{(\sharp A - \sharp E(\Gamma))!}{(\sharp A_1 - \sharp E(\Gamma_1))!(\sharp A_2 - \sharp E(\Gamma_2))!}$ partitions of A into $A_1 \sqcup A_2$, where $\sharp A_1 = 3n_1$, $\sharp A_2 = 3n_2$, $j_E(E(\Gamma_1)) \subseteq A_1$, and $j_E(E(\Gamma_2)) \subseteq A_2$. $\square$

Remark 16.2. We fixed the cardinality of the sets A_i of edge indices to be $f_0(n_i) = 3n_i$ for degree n_i graphs. We could have replaced f_0 with any $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(n) \geq 3n-2$ (so that $f(n) \geq \sharp j_E(E(\Gamma))$ for any degree n Jacobi diagram with at least two univalent vertices) and such that $f(n) + f(m) = f(n+m)$, and we would have obtained the same equalities as above. But we had to fix such a map f, to get these equalities.

If $\tilde{\omega}(i, 0, S^2) = \omega_{S^2}$, then for a subset A of $\underline{3N}$ with cardinality $3n$ and for a path $\gamma\colon [0,1] \to \{0\} \times \check{\mathcal{S}}_B(\mathbb{C})$, $\widetilde{\mathrm{hol}}_\gamma(\eta_{B,A})$ is the degree n part of $\mathrm{hol}_\gamma(\eta_B)$.

Notation 16.3. For a finite set A, let $\mathcal{D}^c_{k,A}(\mathbb{R})$ denote the set of connected A-numbered degree k Jacobi diagrams with support $\mathbb{R}$ without looped edges. For a vertex-oriented graph $\check{\Gamma} \in \mathcal{D}^c_{k,A}(\mathbb{R})$, define the two-form $\omega(\check{\Gamma})$ on $[0,1] \times S^2$ as

$$\omega(\check{\Gamma})(t,v) = \int_{\{t\} \times Q(v;\check{\Gamma})} \bigwedge_{e \in E(\check{\Gamma})} p^*_{e,S^2}(\tilde{\omega}(j_E(e), S^2)),$$

where p_{e,S^2} denotes $\mathbf{1}_{[0,1]} \times p_{e,S^2}$ abusively, and $Q(v;\check{\Gamma})$ is defined in Section 10.3. For a subset A of $\underline{3N}$ of cardinality $3k$, define the two-form $\omega(A)$ on $[0,1] \times S^2$ with coefficients in $\check{\mathcal{A}}_k(\mathbb{R})$ as

$$\omega(A) = \omega(A, (\tilde{\omega}(i, S^2))_{i \in \underline{3N}}) = \sum_{\check{\Gamma} \in \mathcal{D}^c_{k,A}(\mathbb{R})} \zeta_{\check{\Gamma}} \omega(\check{\Gamma})[\check{\Gamma}].$$

Here, we view $\check{\mathcal{A}}(\mathbb{R})$ as a space of Jacobi diagrams on the oriented $\mathbb{R}$. So the sets $U(\Gamma)$ of univalent vertices in involved Jacobi diagrams Γ are ordered by i_Γ, and the univalent vertices are oriented by the orientation of $\mathbb{R}$. Let $s_*\colon \check{\mathcal{A}}(\mathbb{R}) \to \check{\mathcal{A}}(\mathbb{R})$ be the map[1] that sends (the class of) a Jacobi diagram Γ on $\mathbb{R}$ to (the class of) the Jacobi diagram $s(\Gamma)$ obtained from Γ by multiplying it by $(-1)^{\sharp U(\Gamma)}$ and by reversing the order of $U(\Gamma)$.

[1]To my knowledge, the map s_* might be the identity map. If it is, real-valued Vassiliev invariants cannot distinguish an oriented knot from that obtained by reversing its orientation. According to the Kuperberg article [Kup96], Vassiliev invariants would fail to distinguish all prime, unoriented knots, in this case.

Lemma 16.4. *Let $\iota\colon [0,1]\times S^2 \to [0,1]\times S^2$ maps (t,v) to $(t,\iota_{S^2}(v))$, where ι_{S^2} is the antipodal map of S^2. With the above notation, the form*

$$\omega(A)=\omega(A,(\tilde{\omega}(i,S^2))_{i\in\underline{3N}})$$

is a closed form of $[0,1]\times S^2$ with coefficients in $\check{\mathcal{A}}_k(\mathbb{R})$ such that $\iota^*(\omega(A)) = -s_*(\omega(A))$ and $\iota^*(\omega(A))=(-1)^k(\omega(A))$.

PROOF: In order to prove that $\omega(A)$ is closed, it suffices to prove that its integral vanishes on the boundary of any 3-ball B of $[0,1]\times S^2$. View

$$Q_k(t,v)=\sum_{\check{\Gamma}\in\mathcal{D}^c_{k,A}(\mathbb{R})}\zeta_{\check{\Gamma}}[\check{\Gamma}]\left(\{t\}\times Q(v;\check{\Gamma})\times (S^2)^{A\setminus j_E(E(\check{\Gamma}))}\right)$$

as a $(6k-2)$-chain with coefficients in $\check{\mathcal{A}}_k(\mathbb{R})$. So the integral $\int_{\partial B}\omega(A)$ is the integral of the pull-back of the closed form $\bigwedge_{a\in A}\tilde{\omega}(a,S^2)$ under a natural map P from $\cup_{(t,v)\in\partial B}Q_k(t,v)$ to $[0,1]\times(S^2)^A$.

The analysis of the boundary of $Q(v;\check{\Gamma})$ in the proof of Proposition 10.13 shows that the codimension-one faces of the boundary of $P(Q_k(t,v))$ can be glued. So the boundary of $P(Q_k(t,v))$ vanishes algebraically.

Therefore, since the cycle $\cup_{(t,v)\in\partial B}P(Q_k(t,v))$ bounds $\cup_{(t,v)\in B}P(Q_k(t,v))$ in $[0,1]\times(S^2)^A$, $\int_{\partial B}\omega(A)$ vanishes, and $\omega(A)$ is closed.

For $\check{\Gamma}\in\mathcal{D}^c_{k,A}(\mathbb{R})$, recall that the class of $s(\check{\Gamma})$ is obtained from the class of $\check{\Gamma}$ by multiplying it by $(-1)^{\sharp U(\check{\Gamma})}$ and by reversing the order of $U(\check{\Gamma})$. Here, we rather consider $\check{\Gamma}$ and $s(\check{\Gamma})$ as diagrams on unoriented sources whose univalent vertices are equipped with matching local orientations of the source at univalent vertices. A configuration of $Q(v;\check{\Gamma})$ is naturally a configuration of $Q(-v;s(\check{\Gamma}))$. Furthermore, the induced identification from $Q(v;\check{\Gamma})$ to $Q(-v;s(\check{\Gamma}))$ reverses the orientation since the local orientations at univalent vertices coincide, the quotients by dilation coincide, but the translations act in opposite directions. Therefore, for any 2-chain Δ of $[0,1]\times S^2$, $\int_\Delta\omega(\check{\Gamma})=-\int_{\iota(\Delta)}\omega(s(\check{\Gamma}))$, where $\iota(\Delta)$ is equipped with the orientation of Δ, so

$$\begin{aligned}\textstyle\sum_{\check{\Gamma}\in\mathcal{D}^c_{k,A}(\mathbb{R})}\zeta_{\check{\Gamma}}\int_\Delta\omega(\check{\Gamma})[\check{\Gamma}] &= -\textstyle\sum_{\check{\Gamma}\in\mathcal{D}^c_{k,A}(\mathbb{R})}\zeta_{s(\check{\Gamma})}\int_{\iota(\Delta)}\omega(s(\check{\Gamma}))[\check{\Gamma}]\\ &= -\textstyle\sum_{\check{\Gamma}\in\mathcal{D}^c_{k,A}(\mathbb{R})}\zeta_{\check{\Gamma}}\int_{\iota(\Delta)}\omega(\check{\Gamma})[s(\check{\Gamma})]\end{aligned}$$

and $\omega(A)=-\iota^*(s_*(\omega(A)))$.

In order to prove that $\iota^*(\omega(A))=(-1)^k(\omega(A))$, use the notation and the arguments of the proof that $\alpha_{2n}=0$ in the proof of Proposition 10.13, in order to prove that for any 2-chain Δ of $[0,1]\times S^2$, and for any $\check{\Gamma}\in\mathcal{D}^c_{k,A}(\mathbb{R})$,

$$\int_\Delta\omega(\check{\Gamma})[\check{\Gamma}]=(-1)^k\int_{\iota(\Delta)}\omega(\check{\Gamma}^{eo})[\check{\Gamma}^{eo}].$$

□

Definition 16.5. For any oriented tangle component K, let U^+K denote the fiber space over K consisting of the tangent vectors to the knot K of $\check{R}(\mathcal{C})$ that orient K, up to dilation, as in Section 7.3. When the ambient manifold is equipped with a parallelization τ, define the one-form $\eta(A, p_\tau(U^+K)) = \eta(A, (\tilde{\omega}(i, S^2))_{i\in\underline{3N}}, p_\tau(U^+K))$ on $[0,1]$ valued in $\check{\mathcal{A}}(\mathbb{R})$ as

$$\eta(A, p_\tau(U^+K))(t) = \eta(A, p_\tau(U^+K), \tau)(t) = \int_{\{t\}\times p_\tau(U^+K)} \omega(A, (\tilde{\omega}(i, S^2))_{i\in\underline{3N}}).$$

More precisely, as in Proposition 15.14,

$$\eta(A, p_\tau(U^+K))(u, \frac{\partial}{\partial t}) = \frac{\partial}{\partial t}\left(\int_{[0,t]\times p_\tau(U^+K)} \omega(A)\right)(u).$$

Define the A-*holonomy* of $\eta(., p_\tau(U^+K))$ along $[a,b] \subseteq [0,1]$ to be

$$\widetilde{\mathrm{hol}}_{[a,b]}(\eta(A, p_\tau(U^+K))) =$$

$$\sum_{r=0}^{\infty} \sum_{(A_1,\dots,A_r)\in P_r(A)} \frac{\prod_{i=1}^r (\sharp A_i)!}{(\sharp A)!} \int_{(t_1,\dots,t_r)\in[a,b]^r | t_1\le t_2\le\dots\le t_r} \bigwedge_{i=1}^r p_i^*(\eta(A_i, p_\tau(U^+K))),$$

as before, where $p_i(t_1,\dots,t_r) = t_i$, and $\widetilde{\mathrm{hol}}_{[a,b]}(\eta(\emptyset, p_\tau(U^+K))) = \mathbf{1} = [\emptyset]$.

Lemma 16.6. $\widetilde{hol}_{[a,b]}(\eta(A, -p_\tau(U^-K))) = s_*\left(\widetilde{hol}_{[a,b]}(\eta(A, p_\tau(U^+K)))\right)$.

PROOF: Lemma 16.4 implies that $\eta(A, -p_\tau(U^-K)) = -\eta(A, p_\tau(U^-K)) = s_*(\eta(A, p_\tau(U^+K)))$. □

16.2 Variants of $\mathcal{Z}^f$ for tangles

We now present alternative definitions of $\mathcal{Z}^f$ involving non-homogeneous propagating forms associated to volume forms $\tilde{\omega}(i, 1, S^2) = \omega_{S^2} + d\eta(i, S^2)$, where ω_{S^2} is the homogeneous volume-one form of S^2.

Let $L\colon \mathcal{L} \hookrightarrow R(\mathcal{C})$ denote a long tangle in $R(\mathcal{C})$ equipped with a parallelization τ that is standard outside $\mathcal{C}$. Recall that we fixed some (large) integer N, $N \geq 2$, and that $\left(\tilde{\omega}(i, S^2) = (\tilde{\omega}(i, t, S^2))_{t\in[0,1]}\right)_{i\in\underline{3N}}$ is a family of a closed 2–forms on $[0,1]\times S^2$ such that $\tilde{\omega}(i, 0, S^2)$ is a volume-one form of S^2. For $i \in \underline{3N}$, let $\tilde{\omega}(i) = (\tilde{\omega}(i,t))_{t\in[0,1]}$ be a closed 2–form on $[0,1]\times C_2(R(\mathcal{C}))$ such that $(\mathbf{1}_{[0,1]} \times p_\tau)^*(\tilde{\omega}(i, S^2)) = \tilde{\omega}(i)$ on $[0,1]\times \partial C_2(R(\mathcal{C}))$.

For a diagram $\Gamma \in \mathcal{D}^e_{k,\underline{3N}}(\mathcal{L})$, define

$$I(\mathcal{C}, L, \Gamma, o(\Gamma), (\tilde{\omega}(i,t))_{i\in\underline{3N}}) = \int_{(\check{C}(\check{R}(\mathcal{C}),L;\Gamma),o(\Gamma))} \bigwedge_{e\in E(\Gamma)} p_e^*(\tilde{\omega}(j_E(e),t)),$$

which converges, according to Theorem 12.2.

Theorem 16.7. *Let $L\colon \mathcal{L} \hookrightarrow R(\mathcal{C})$ denote the long tangle associated to a tangle, in a rational homology cylinder equipped with a parallelization τ. Let $\{K_j\}_{j\in I}$ be the set of components of L. Assume that the bottom (resp. top) configuration of L is represented by a map $y^-\colon B^- \to D_1$ (resp. $y^+\colon B^+ \to D_1$).*

Let $N \in \mathbb{N}$. For $i \in \underline{3N}$, let $\tilde{\omega}(i,S^2) = (\tilde{\omega}(i,t,S^2))_{t\in[0,1]}$ be a closed 2–form on $[0,1]\times S^2$ such that $\tilde{\omega}(i,0,S^2)$ is a volume-one form[2] of S^2, and let $\tilde{\omega}(i) = (\tilde{\omega}(i,t))_{t\in[0,1]}$ be a closed 2–form on $[0,1]\times C_2(R(\mathcal{C}))$ such that $\tilde{\omega}(i) = (\mathbf{1}_{[0,1]}\times p_\tau)^*(\tilde{\omega}(i,S^2))$ on $[0,1]\times \partial C_2(R(\mathcal{C}))$. For a subset A of $\underline{3N}$ with cardinality $3k$, set

$$Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i,t))_{i\in A}) = \sum_{\Gamma\in\mathcal{D}^e_{k,A}(\mathcal{L})} \zeta_\Gamma I(\mathcal{C}, L, \Gamma, (\tilde{\omega}(i,t))_{i\in A})[\Gamma] \in \mathcal{A}_k(\mathcal{L})$$

and

$$Z(\mathcal{C}, L, \tau, A)(t) = Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i,t))_{i\in A}).$$

Then

$$Z(\mathcal{C}, L, \tau, A)(t) = \sum_{\aleph=(A_1,A_2,A_3,(A_{K_j})_{j\in I})\in P_{3+\sharp I}(A)} \beta(\aleph) Z(\aleph, t),$$

where

$$\beta(\aleph) = \frac{(\sharp A_1)!(\sharp A_2)!(\sharp A_3)!\left(\prod_{j\in I}(\sharp A_{K_j})!\right)}{(\sharp A)!}$$

and

$$Z(\aleph,t) = \left(\prod_{j\in I}\widetilde{hol}_{[0,t]}(\eta(A_{K_j}, p_\tau(U^+K_j)))\sharp_j\right) \widetilde{hol}_{[t,0]\times y^-}(\eta_{B^-,A_1}) Z(\mathcal{C}, L, \tau, A_2)(0)\widetilde{hol}_{[0,t]\times y^+}(\eta_{B^+,A_3}).$$

The terms of this formula belong to spaces of diagrams on unoriented one-manifolds as in Definition 6.16, except for the term $\widetilde{hol}_{[0,t]}(\eta(A_{K_j}, p_\tau(U^+K_j)))$, and its action $\sharp_j$ for which we first pick an orientation of the K_j, which we

[2] In this chapter, we apply the theorem only when $\tilde{\omega}(i,0,S^2)$ is the standard homogeneous volume-one form ω_{S^2} on S^2, but this general statement is used in the next chapter.

may forget afterwards[3]*. The formula implies that $Z(\mathcal{C}, L, \tau, A)(t)$ depends only on $(\tilde{\omega}(i,t,S^2))_{i\in A}$, for any t. (It depends also on τ and on the specific embedding L.) It will be denoted by $Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i,t,S^2))_{i\in A})$.*

Proof: Compute $\frac{\partial}{\partial t}Z(A,t) = dZ(A,.)\left(\frac{\partial}{\partial t}\right)$ as in Lemma 10.18 with the help of Proposition 9.2, using the same analysis of faces as in the proof of Lemma 15.16, to find

$$\begin{aligned} dZ(A,.) = \; & \sum_{(A_1,A_2)\in P_2(A)} \tfrac{(\sharp A_1)!(\sharp A_2)!}{(\sharp A)!} \sum_{j\in I} (\eta(A_1, p_\tau(U^+K_j))\sharp_j)\, Z(A_2,t) \\ & + \sum_{(A_1,A_2)\in P_2(A)} \tfrac{(\sharp A_1)!(\sharp A_2)!}{(\sharp A)!} Z(A_1,t)(y^+ \times \{t\})^*(\eta_{B^+,A_2}) \\ & - \sum_{(A_1,A_2)\in P_2(A)} \tfrac{(\sharp A_1)!(\sharp A_2)!}{(\sharp A)!} (y^- \times \{t\})^*(\eta_{B^-,A_1}) Z(A_2,t), \end{aligned}$$

where $\eta(\emptyset, p_\tau(U^+K_j)) = 0$ and $\eta_{B^+,\emptyset} = 0$. This shows that both sides of the equality to be proved vary in the same way, when t varies. Since they take the same value at $t = 0$, the formula is proved. Apply the formula when $\tilde{\omega}(i,0,S^2)$ is the standard form ω_{S^2}, and use Lemma 9.1 together with the isotopy invariance of Z of Theorem 12.7, to see that $Z(\mathcal{C}, L, \tau, A)(t)$ depends only on the $\tilde{\omega}(i,S^2)$, for $i \in A$. So it depends only on the $\tilde{\omega}(i,t,S^2)$. $\square$

Let us introduce some notation in order to rephrase Theorem 16.7. View $Z(\mathcal{C}, L, \tau, .)(t)$ as a map from the set $\mathcal{P}_{(3)}(\underline{3N})$ of subsets of $\underline{3N}$ with cardinality multiple of 3 to $\mathcal{A}_{\leq N}(\mathcal{L}) = \oplus_{k=0}^{N}\mathcal{A}_k(\mathcal{L})$, which maps $\emptyset$ to the class of the empty diagram. Similarly, consider $\widetilde{\mathrm{hol}}_{[0,t]}(\eta(., p_\tau(U^+K_j)))$, $\widetilde{\mathrm{hol}}_{[t,0]\times y^-}(\eta_{B^-,.})$ and $\widetilde{\mathrm{hol}}_{[0,t]\times y^+}(\eta_{B^+,.})$ as maps from $\mathcal{P}_{(3)}(\underline{3N})$ to spaces of diagrams, which map the empty set to the class of the empty diagram. The values of these maps can be multiplied as in the statement of the theorem using the structures of the space of diagrams.

Definition 16.8. For such maps z_1 and z_2 from $P_{(3)}(\underline{3N})$ to spaces of diagrams, define their product $(z_1z_2)_\sqcup$ as the map with domain $\mathcal{P}_{(3)}(\underline{3N})$ such that

$$(z_1z_2)_\sqcup(A) = \sum_{(A_1,A_2)\in P_2(A)} \frac{(\sharp A_1)!(\sharp A_2)!}{(\sharp A)!} z_1(A_1)z_2(A_2)$$

whenever the products $z_1(A_1)z_2(A_2)$ make sense.

This product is associative and $(z_1z_2z_3)_\sqcup$ denotes

$$((z_1z_2)_\sqcup z_3)_\sqcup = (z_1(z_2z_3)_\sqcup)_\sqcup.$$

[3]Both sides are independent of the source orientations in the sense of the last sentence of Proposition 10.21 and Remark 10.22, thanks to Lemma 16.6.

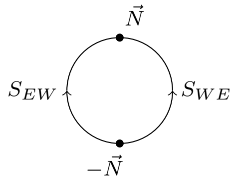


Figure 16.1: The half-circles S_{WE} and S_{EW}.

The maps that send all non-empty elements of $P_{(3)}(\underline{3N})$ to 0, and the empty element to the class of the empty diagram are neutral for this product and denoted by **1**. With this notation, the equality of Theorem 16.7 may be written as

$$Z(\mathcal{C}, L, \tau, .)(t) =$$
$$\left(\left(\prod_{j\in I} \widetilde{\mathrm{hol}}_{[0,t]}(\eta(., U_j^+))\sharp_j\right) \widetilde{\mathrm{hol}}_{[t,0]\times y^-}(\eta_{B^-,.})Z(\mathcal{C}, L, \tau, .)(0)\widetilde{\mathrm{hol}}_{[0,t]\times y^+}(\eta_{B^+,.})\right)_{\sqcup},$$

where $U_j^+ = p_\tau(U^+K_j)$.

16.3 Straight tangles

Definition 16.9. Recall that $\vec{N}$ denotes the vertical unit vector. A tangle $L\colon \mathcal{L} \hookrightarrow \mathcal{C}$ is *straight* with respect to τ if

- $p_\tau(U^+K) \subset \{-\vec{N}, \vec{N}\}$ for closed components K and for components K that go from bottom to top or from top to bottom,
- for any interval component $\mathcal{L}_j$, p_τ maps the unit tangent vectors to $L(\mathcal{L}_j)$ to the vertical half great circle S_{WE} from $-\vec{N}$ to $\vec{N}$ that contains the west-east direction (to the right), or to the vertical half great circle S_{EW} from $-\vec{N}$ to $\vec{N}$ that contains the east-west direction (to the left). See Figure 16.1.

Orient S_{WE} and S_{EW} from $-\vec{N}$ to $\vec{N}$. Straight tangles with respect to τ get the following framing induced by τ. For any $k \in K$, $p_\tau(U_k^+K)$ is an element v_k of the vertical circle $S_{WE}\cup(-S_{EW})$, which intersects the horizontal plane $\mathbb{C}$ in the real line. Let $\rho_{i,\pi/2}(v_k)$ be the image of v_k under the rotation with axis i (($i \in \mathbb{C}$) points toward the sheet) and with angle $\pi/2$. Then $K_\parallel$ is the parallel of K obtained by pushing K in the direction of the section

$\big(k \mapsto \tau(\rho_{i,\pi/2}(v_k))\big)$ of the unit normal bundle of K. (This is consistent with the conventions of Definition 12.8.)

The following proposition generalizes Lemma 7.35 to interval components.

Proposition 16.10. *Let K be a component of a straight q–tangle in a parallelized homology cylinder $(\mathcal{C},\tau)$. Then*

$$I_\theta(K,\tau) = I_\theta(-K,\tau) = lk(K, K_\parallel)$$

with the notation of Definitions 12.9 and 13.4 for $lk(K, K_\parallel)$ and of Definition 12.6 and Lemma 7.15 for I_θ.

In order to prove Proposition 16.10, we describe some propagating forms, which will also be useful in the next chapter.

Notation 16.11. For an interval I of $\mathbb{R}$ that contains $[0,1]$ and for a real number $x \in [1,+\infty[$, $R_{x,I}(\mathcal{C})$ (resp. $R^c_{x,I}(\mathcal{C})$) denotes the part that replaces $D_x \times I$ (resp. the closure of its complement) in $R(\mathcal{C})$.

Let $\chi_{\mathcal{C}}$ be a smooth map from $\check{R}(\mathcal{C})$ to $[0,1]$ that maps $\mathcal{C} = R_{1,[0,1]}(\mathcal{C})$ to 1 and $R^c_{2,[-1,2]}(\mathcal{C})$ to 0. Define

$$\begin{array}{llll} \pi_{\mathcal{C}}\colon & \check{R}(\mathcal{C}) & \to & \mathbb{R}^3 \\ & x & \mapsto & (1-\chi_{\mathcal{C}}(x))x, \end{array}$$

where $0x = 0$, and

$$\begin{array}{llll} p\colon & (\check{R}(\mathcal{C}))^2 \setminus \Big(\check{R}_{2,[-1,2]}(\mathcal{C})^2 \cup \Delta(\check{R}^c_{2,[-1,2]}(\mathcal{C})^2)\Big) & \to & S^2 \\ & (x,y) & \mapsto & \frac{\pi_{\mathcal{C}}(y)-\pi_{\mathcal{C}}(x)}{\|\pi_{\mathcal{C}}(y)-\pi_{\mathcal{C}}(x)\|}. \end{array}$$

The map p extends to

$$D(p) = C_2(R(\mathcal{C})) \setminus \mathring{C}_2(R_{2,[-1,2]}(\mathcal{C})).$$

When a parallelization τ of $\mathcal{C}$ is given, the corresponding extension of p to

$$D(p_\tau) = D(p) \cup UR_{2,[-1,2]}(\mathcal{C})$$

is denoted by p_τ.

Lemma 16.12. *For any 2-form $\omega(S^2)$ such that $\int_{S^2}\omega(S^2) = 1$, and for any parallelization τ of $\mathcal{C}$ that is standard on the boundary, there exists a propagating form ω of $(C_2(\check{R}(\mathcal{C})),\tau)$ that restricts to $D(p_\tau)$ as $p_\tau^*(\omega(S^2))$. For any $X \in S^2$, there exists a propagating chain F of $(C_2(\check{R}(\mathcal{C})),\tau)$ that restricts to $D(p_\tau)$ as $p_\tau^{-1}(X)$.*

PROOF: We again need an extension to the interior of $C_2(R_{2,[-1,2]}(\mathcal{C}))$ of a closed 2-form defined on the boundary. Since this space is a 6-manifold with ridges, which has the same homology as S^2, the form extends as a closed form. $\square$

PROOF OF PROPOSITION 16.10: Lemmas 7.33 and 7.35 leave us with the case of interval components K, where $I_\theta(K,\tau) = 2I(\hat{\zeta}^{\mathrm{K}}, \omega_{S^2})$, and

$$I_\theta(-K,\tau) = I_\theta(K,\tau).$$

First assume that K goes from bottom to top. In this case, Lemma 12.5 guarantees that $I(\hat{\zeta}^{\mathrm{K}}, \omega(S^2))$ is independent of the chosen volume-one form $\omega(S^2)$ of S^2. Choose a volume-one form $\omega_{0,0}$ of S^2, which is ε-dual to $p_\tau^{-1}(i)$ for the complex horizontal direction i and for a small positive number ε. Set $\omega_0(S^2) = \frac{\omega_{0,0}-\iota^*(\omega_{0,0})}{2}$, and choose a propagating form ω_0 that restricts to $D(p_\tau)$ as $p_\tau^*(\omega_0(S^2))$, as in Lemma 16.12. (Recall Definition 11.6.) Extend the intersections of K and $K_\parallel$ with the complement of the ball $B(100)$ of radius 100 in $\mathbb{R}^3$ to large knots C and $C_\parallel$ such that

- $(C, C_\parallel)$ is isotopic to the pair $(\hat{K}, \hat{K}_\parallel)$ of Definition 12.9 of $lk(K, K_\parallel)$. So $lk(K,K_\parallel) = lk(C, C_\parallel)$ and,
- p_τ maps $(C \times C_\parallel) \cap ((\mathbb{R}^3)^2 \setminus B(100)^2)$ outside the support of $\omega_0(S^2)$ so that $\int_{C\times C_\parallel} \omega_0 = \int_{K \times K_\parallel} \omega_0$.

Our choice of ω_0 also allows us to let $K_\parallel$ approach and replace K without changing the rational integral $\int_{K\times K_\parallel} \omega_0$. So $\int_{K\times K_\parallel} \omega_0 = \int_{K\times K \setminus \Delta} \omega_0$. Then

$$\begin{aligned} lk(K,K_\parallel) &= \int_{C\times C_\parallel} \omega_0 = \int_{K\times K_\parallel} \omega_0 = \int_{K\times K\setminus\Delta} \omega_0 \\ &= I\left(\hat{\zeta}^{\mathrm{K}}, \omega_0(S^2)\right) + I\left(\hat{\zeta}^{\mathrm{K}}, \omega_0(S^2)\right) \\ &= I\left(\hat{\zeta}^{\mathrm{K}}, \omega_0(S^2)\right) + I\left(\hat{\zeta}^{\mathrm{K}}, -\iota^*(\omega_0(S^2))\right) = 2I\left(\hat{\zeta}^{\mathrm{K}}, \omega_0(S^2)\right). \end{aligned}$$

This proves Proposition 16.10 when K goes from bottom to top. Since $I_\theta(-K,\tau) = I_\theta(K,\tau)$, Definition 12.9 allows us to deduce it when K goes from top to bottom.

Let us now assume that K goes from bottom to bottom (resp. from top to top). Lemma 13.5 reduces the proof to the case in which $p_\tau(U^+K) \subseteq S_{WE}$.

Let $S(K)$ be the half-circle from $-\vec{N}$ to $\vec{N}$ (resp. from $\vec{N}$ to $-\vec{N}$) through the direction of $[K(0), K(1)]$. Lemma 12.5 implies that $I(\hat{\zeta}^{\mathrm{K}}, \omega(S^2))$ depends only on the integrals of the restriction of $\omega(S^2)$ to the components of $S^2 \setminus (p_\tau(U^+K) \cup S(K))$, with the notation of Lemma 12.5. In particular, when $S(K) = \pm S_{WE}$, (i.e. when the segment $[K(0), K(1)]$ is directed and oriented as the real line, as in Figure 12.4), it does not depend on $\omega(S^2)$, and we conclude as above.

Otherwise, $S^2 \setminus (p_\tau(U^+K) \cup S(K))$ has two connected components A_1 and A_2 such that $\partial \overline{A}_1 = p_\tau(U^+K) \cup S(K) = -\partial \overline{A}_2$. For $j \in \underline{2}$, let $I_j = I(\hat{\xi}^{\mathrm{K}}, \omega_j(S^2))$, where $\omega_j(S^2)$ is a volume-one form supported on A_j. Then I_2 is a rational number, since it is the intersection[4] of a propagating chain with boundary $p_\tau^{-1}(X)$ for $X \in A_2$ with $C(R(\mathcal{C}, K; \hat{\xi}^{\mathrm{K}}))$ in $C_2(R(\mathcal{C}))$. According to Lemma 12.5, for any volume-one form $\omega(S^2)$ of S^2,

$$\begin{aligned} I(\hat{\xi}^{\mathrm{K}}, \omega(S^2)) &= I_1 + \int_{A_1}(\omega(S^2) - \omega_1(S^2)) = I_1 + \int_{A_1} \omega(S^2) - 1 \\ &= I_2 + \int_{A_1}(\omega(S^2) - \omega_2(S^2)) = I_2 + \int_{A_1} \omega(S^2). \end{aligned}$$

The rational numbers I_1 and I_2 are not changed when (K, τ) moves continuously, so that the angle from the real positive half-line to $\overrightarrow{K(0)K(1)}$ varies in $]0, 2\pi[$, and the trivialization varies accordingly so that K remains straight. Therefore, $I(\hat{\xi}^{\mathrm{K}}, \omega_{S^2})$ varies like the area of A_1, which we compute now. If K goes from bottom to bottom (resp. from top to top) and if the direction of $[K(0), K(1)]$ coincides with the direction of $\exp(2i\pi\theta)$ for $\theta \in]0,1[$, then A_1 is the preimage in S^2 of $\{\lambda \exp(2i\pi\alpha) \mid \lambda \in]0, +\infty[, \alpha \in]0, \theta[\}$ (resp. $\{\lambda \exp(2i\pi\alpha) \mid \lambda \in]0, +\infty[, \alpha \in]\theta, 1[\}$) under the stereographic projection from the South Pole onto the horizontal plane through the North Pole, which is identified with $\mathbb{C}$, and the area of A_1 is θ (resp. $1 - \theta$). Therefore, when K goes from bottom to bottom

$$I_\theta(K, \tau) = I_1 + I_2 + 2\theta - 1$$

and when K goes from top to top, $I_\theta(K, \tau) = I_1 + I_2 + 1 - 2\theta$.

Let us treat the case $\theta = \frac{1}{2}$. Define the angles θ_0 and θ_1 of Definition 13.4 to be $\theta_1 = -\theta_0 = \frac{1}{2}$, to define $\hat{K}_\parallel$ as in Figure 13.4, so $\hat{K}_\parallel$ is isotopic to the parallel $-\widehat{(-K)}_\parallel$, which looks as in Figure 12.4. In this case, the interior of A_1 contains the direction i, we can assume $\omega_1(S^2) = \omega_{0,0}$ and compute $lk(K, K_\parallel) = lk(C, C_\parallel)$ for a pair $(C, C_\parallel)$ of parallel closed curves, which coincides with $(K, -\widehat{(-K)}_\parallel)$ in a big neighborhood of $\mathcal{C}$ and which lies in a vertical plane orthogonal to i outside that big neighborhood. So

$$\begin{aligned} lk(K, K_\parallel) &= lk(C, C_\parallel) = \int_{K \times K \setminus \Delta} \omega_{0,0} = I(\hat{\xi}^{\mathrm{K}}, \omega_{0,0}) + I(\hat{\xi}^{\mathrm{K}}, \omega_{0,0}) \\ &= I(\hat{\xi}^{\mathrm{K}}, \omega_{0,0}) + I(\hat{\xi}^{\mathrm{K}}, -\iota^*(\omega_{0,0})) = I_1 + I_2 = I_\theta(K, \tau). \end{aligned}$$

We now deduce the case in which $\theta \in]0,1[$ from the case $\theta = \frac{1}{2}$. Recall that the rational numbers I_1 and I_2 are unchanged under isotopies which

[4] When $\mathcal{C} = D_1 \times [0,1]$ and $\tau = \tau_s$, p_τ extends to $C_2(R(\mathcal{C}))$ and I_2 is the integral local degree of this extended p_τ at a point of A_2.

make θ vary continuously in $]0,1[$. Define $\theta_1 = \theta$ and $\theta_0 = \theta - 1$, so that the arcs α_0 and α_1 of Definition 13.4 vary continuously as θ varies from 0 to 1. Then the isotopy class of the pair $(\widehat{K(\theta)}, \widehat{K(\theta)}_{\|,\theta_0,\theta_1})$ is unchanged. So, when K goes from bottom to bottom, $lk(K(\theta), K(\theta)_\|) = lk(K(\frac12), K(\frac12)_\|) + 2\theta - 1$, while $I_\theta(K(\theta),\tau) = I_\theta(K(\frac12),\tau) + 2\theta - 1$. When K goes from top to top, $lk(K(\theta), K(\theta)_\|) - I_\theta(K,\tau)$ is fixed similarly, when θ varies, so Proposition 16.10 holds in any case. $\square$

View the anomaly β of Section 10.2 as the map from $P_{(3)}(\underline{3N})$ to $\check{\mathcal{A}}(\mathbb{R})$, which maps any subset of $\underline{3N}$ with cardinality $3k$ to β_k.

With the notation of Definition 16.8, we get the following corollary of Theorem 16.7, Theorem 12.7 and Proposition 16.10.

Theorem 16.13. *Let L be a straight tangle with respect to a parallelization τ. Let $J_{bb} = J_{bb}(L)$ (resp. $J_{tt} = J_{tt}(L)$) denote the set of components of L that go from bottom to bottom (resp. from top to top). For $K \in J_{bb} \cup J_{tt}$, the* orientation of K induced by τ *is the orientation of K such that $p_\tau(U^+K) \subseteq S_{WE}$.*

Under the assumptions of Theorem 16.7,

$$\left(Z(\mathcal{C}, L, \tau, ., (\tilde{\omega}(i,1))_{i\in\underline{3N}}) \exp(-\frac{1}{4}p_1(\tau)\beta(.))\right)_\sqcup$$

depends only on the $\tilde{\omega}(i,1,S^2)$, on the boundary-fixing diffeomorphism class of $(\mathcal{C}, L)$, on the orientations of the components of $J_{bb} \cup J_{tt}$ induced by τ and on the parallel $L_\|$ of L induced by τ.

It is denoted by $\mathcal{Z}^f(\mathcal{C}, L, L_\|, ., (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}})$, and we have

$$\mathcal{Z}^f_{\leq N}(\mathcal{C}, L, L_\|) = \mathcal{Z}^f(\mathcal{C}, L, L_\|, ., (\omega_{S^2})_{i\in\underline{3N}}),$$

where $\mathcal{Z}^f(\mathcal{C}, L, L_\|, .) = \mathcal{Z}^f(\mathcal{C}, L, L_\|, ., (\omega_{S^2})_{i\in\underline{3N}})$ maps any subset of $\underline{3N}$ of cardinality $3k$, to the degree k part $\mathcal{Z}^f_k(\mathcal{C}, L, L_\|)$ of the invariant $\mathcal{Z}^f(\mathcal{C}, L, L_\|)$ of Definition 12.12.

Proof: First note that $\mathcal{Z}^f(\mathcal{C}, L, L_\|, ., (\omega_{S^2})_{i\in\underline{3N}})$ maps any subset of $\underline{3N}$ of cardinality $3k$ to $\mathcal{Z}^f_k(\mathcal{C}, L, L_\|)$ as stated.

Let K be a component of L. If K is a knot, or a component that goes from bottom to top or from top to bottom, then the form $\eta(A_K, p_\tau(U^+K))$ of Definition 16.5 vanishes. Since the components of $J_{bb} \cup J_{tt}$ are equipped with the orientation induced by τ, $\eta(A_K, p_\tau(U^+K))$ is the same for all components K of J_{tt} and it is independent of L, it is denoted by $\eta(A_K, S_{WE})$. Similarly,

$\eta(A_K, p_\tau(U^+K)) = -\eta(A_K, S_{WE})$ for all components K of J_{bb}. Therefore, the factor $\left(\prod_{j\in I} \widetilde{\mathrm{hol}}_{[0,1]}(\eta(A_{K_j}, p_\tau(U^+K_j)))\sharp_j\right)$ in Theorem 16.7 is equal to

$$\left(\prod_{K_j\in J_{bb}} \widetilde{\mathrm{hol}}_{[0,1]}(-\eta(A_{K_j}, S_{WE}))\sharp_j\right)\left(\prod_{K_j\in J_{tt}} \widetilde{\mathrm{hol}}_{[0,1]}(\eta(A_{K_j}, S_{WE}))\sharp_j\right).$$

and $\mathcal{Z}^f(\mathcal{C}, L, L_\parallel, ., (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}})$ is determined by $\mathcal{Z}^f(\mathcal{C}, L, L_\parallel)$ and by the holonomies, which depend only on the $\tilde{\omega}(i,.,S^2)$. □

Definition 16.14. With the notation of Theorem 16.13, and under its assumptions, Theorems 16.7 and 12.7 together with Proposition 16.10 imply that when $L = (K_j)_{j\in\underline{k}}$ and $L_\parallel = (K_{j\parallel})_{j\in\underline{k}}$

$$\left(\prod_{j=1}^{k}\left(\exp(-lk(K_j, K_{j\parallel})\alpha)\sharp_j\right)\mathcal{Z}^f(\mathcal{C}, L, L_\parallel, ., (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}})\right)_\sqcup$$

is independent of the framing of L. It is denoted by $\mathcal{Z}(\mathcal{C}, L, ., (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}})$ and it a priori depends on the orientations of the components of $J_{bb}\cup J_{tt}$

The data of an orientation for the components of $J_{bb}\cup J_{tt}$ is called a $J_{bb,tt}$-*orientation* and L is said to be $J_{bb,tt}$-*oriented*, when it is equipped with such an orientation.

All the involved products are products of Definition 16.8, and $lk(K_j, K_{j\parallel})\alpha$ is considered as a function of subsets of $\underline{3N}$ with cardinality multiple of 3, which depends only on the degree. View the invariant of Theorem 12.7 as such a function $\mathcal{Z}(\mathcal{C}, L)(.)$. Then, according to Theorem 16.7 – applied when when $\tilde{\omega}(i,0,S^2) = \omega_{S^2}-$,

$$\mathcal{Z}(\mathcal{C}, L, ., (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}}) =$$

$$\left(\begin{array}{c}\left(\prod_{K_j\in J_{bb}} \widetilde{\mathrm{hol}}_{[0,1]}(-\eta(., S_{WE}))\sharp_j\right)\left(\prod_{K_j\in J_{tt}} \widetilde{\mathrm{hol}}_{[0,1]}(\eta(., S_{WE}))\sharp_j\right)\\ \widetilde{\mathrm{hol}}_{[1,0]\times y^-}(\eta_{B^-,.})\mathcal{Z}(\mathcal{C}, L)(.)\widetilde{\mathrm{hol}}_{[0,1]\times y^+}(\eta_{B^+,.})\end{array}\right)_\sqcup. \tag{16.1}$$

This allows us to extend the definition of $\mathcal{Z}^f(., ., (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}})$ for $J_{bb,tt}$-oriented framed tangles that are not represented by straight tangles so that the equality

$$\mathcal{Z}^f(\mathcal{C}, L, L_\parallel, ., (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}}) =$$

$$\left(\prod_{j=1}^{k}\left(\exp(lk(K_j, K_{j\parallel})\alpha)\sharp_j\right)\mathcal{Z}(\mathcal{C}, L, ., (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}})\right)_\sqcup$$

holds for all these $J_{bb,tt}$-oriented framed tangles, and

$$\mathcal{Z}^f(\mathcal{C}, L, L_\parallel, ., (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}}) =$$

$$\left(\begin{array}{c} \left(\prod_{K_j\in J_{bb}} \widetilde{\mathrm{hol}}_{[0,1]}(-\eta(., S_{WE}))\sharp_j\right) \left(\prod_{K_j\in J_{tt}} \widetilde{\mathrm{hol}}_{[0,1]}(\eta(., S_{WE}))\sharp_j\right) \\ \widetilde{\mathrm{hol}}_{[1,0]\times y^-}(\eta_{B^-,.})\mathcal{Z}^f(\mathcal{C}, L, L_\parallel)(.)\widetilde{\mathrm{hol}}_{[0,1]\times y^+}(\eta_{B^+,.}) \end{array} \right)_{\sqcup}. \tag{16.2}$$

In order to compute $\mathcal{Z}^f(\mathcal{C}, L, L_\parallel, ., .)$ from the discretizable definition of $Z(\mathcal{C}, L, \tau, ., .)$ in Theorem 16.7, we first represent L as a straight tangle with another induced parallel $L' = (K'_j)_{j\in\underline{k}}$ (but with the same $J_{bb,tt}$-orientation induced by the parallelization), and we correct by setting

$$\mathcal{Z}^f(\mathcal{C}, L, L_\parallel, ., (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}}) =$$

$$\left(\prod_{j=1}^{k} \left(\exp(lk(K_j, K_{j\parallel} - K'_j)\alpha)\sharp_j\right) \mathcal{Z}^f(\mathcal{C}, L, L', ., (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}})\right)_{\sqcup}.$$

Remark 16.15. Definition 16.14 is not canonical, because of the arbitrary choice of S_{WE}, and the defined invariant may not have the usual natural dependence on the component orientations (as in Proposition 10.21). Indeed, Definition 16.14 involves the $J_{bb,tt}$-orientation. So it is not symmetric under orientation reversing of a component. See Remark 16.45 for further explanations.

Lemma 16.16. *Let $(\omega(i))_{i\in\underline{3N}}$ denote a fixed family of propagating forms of $(C_2(R(\mathcal{C})), \tau)$. These propagators may be expressed as $\tilde{\omega}(i,1)$ for forms $\tilde{\omega}(i)$ as in Theorem 16.7 (thanks to Lemma 9.1). Let*

$$Z(\mathcal{C}, L, \tau, A) = Z(\mathcal{C}, L, \tau, A)(1)$$

with the notation of Theorem 16.7. Let h_t be an isotopy of $\check{R}(\mathcal{C})$ that is the identity on $(\mathbb{C}\setminus D_1)\times\mathbb{R}$ for any t, that restricts to $\mathbb{C}\times]-\infty,0]$ (resp. to $\mathbb{C}\times[1,+\infty[$) as an isotopy $h^-_t\times\mathbf{1}_{]-\infty,0]}$ (resp. $h^+_t\times\mathbf{1}_{[1,+\infty[}$), for a planar isotopy h^-_t (resp. h^+_t). Assume $h_0=\mathbf{1}$. Let L be a long tangle of $\check{R}(\mathcal{C})$ whose bottom (resp. top) configuration is represented by a map $y^-\colon B^-\to D_1$ (resp. $y^+\colon B^+\to D_1$). Let $(\tau_t)_{t\in[0,1]}$ be a smooth homotopy of parallelizations of $\check{R}(\mathcal{C})$ standard outside $D_1\times[0,1]$ such that $p_{\tau_t|U^+(L)}$ is constant with respect to t. With the notation of Definition 16.8, for $A\in P_{(3)}(\underline{3N})$, set $Z(t,A) = Z(\mathcal{C}, h_t(L), \tau_t, A)$. Then

$$Z(t,.) = \left(\widetilde{hol}_{h^-_{[t,0]}\circ y^-}(\eta_{B^-,.})Z(0,.)\widetilde{hol}_{h^+_{[0,t]}\circ y^+}(\eta_{B^+,.})\right)_{\sqcup}.$$

Proof: The proof is similar to the proof of Lemma 15.16. $\square$

Note that it implies that $\left(\widetilde{\mathrm{hol}}_{h^-_{[t,0]}\circ y^-}(\eta_{B^-,.})\widetilde{\mathrm{hol}}_{h^-_{[0,t]}\circ y^-}(\eta_{B^-,.})\right)_{\sqcup}$ is neutral for the product of Definition 16.8.

The following proposition can be proved as Proposition 15.18.

Proposition 16.17. *With the notation and assumptions of Theorem 16.7, when*

$$\gamma\colon [0,1] \to \check{C}_B[D_1]$$

is smooth with vanishing derivatives at 0 *and* 1*, deform the standard parallelization of* $\mathbb{R}^3$ *to a homotopic parallelization* τ *such that* $T(\gamma)$ *is straight with respect to* τ *at any time of the homotopy. For any subset* A *of* $\underline{3N}$ *with cardinality* $3k$*,*

$$Z(\mathcal{C}_0 = D_1 \times [0,1], T(\gamma), \tau, A, (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}}) = \widetilde{hol}_{\{1\}\times p_{CS}\circ\gamma}(\eta_{B,A}),$$

where p_{CS} *is the natural projection* $\check{C}_B[D_1] \to \mathcal{S}_B(\mathbb{C})$*. Thus*

$$\begin{aligned}\mathcal{Z}(\gamma, A, (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}}) &= \mathcal{Z}^f(\mathcal{C}_0, T(\gamma), T(\gamma)_{\|}, A, (\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}})\\ &= \widetilde{hol}_{\{1\}\times p_{CS}\circ\gamma}(\eta_{B,A}).\end{aligned}$$

Proposition 15.18 could be fully generalized to this setting, too, but we are going to prove more general functoriality properties in Section 17.2.

Lemma 16.18. *Let* $u \in [0,1]$*. Let* $\gamma\colon [0,1] \to \{u\} \times \check{\mathcal{S}}_B(\mathbb{C})$ *be a smooth path.*

- $\widetilde{hol}_\gamma(\eta_{B,A})$ *depends only on* $\gamma(0)$*,* $\gamma(1)$*, the* $\tilde{\omega}(i,S^2)$ *for* $i \in A$*, and the homotopy class of* γ *relatively to* $\partial\gamma$ *in* $\{u\} \times \check{\mathcal{S}}_B(\mathbb{C})$*.*
- *If* $(\overline{\gamma}\colon t \mapsto \gamma(1-t))$ *denotes the inverse of* γ *with respect to the path composition, then* $\left(\widetilde{hol}_\gamma(\eta_{B,.})\widetilde{hol}_{\overline{\gamma}}(\eta_{B,.})\right)_{\sqcup}$ *is neutral with respect to the product of Definition 16.8.*
- *Let* w*,* t *and* ε *be three elements of* $[0,1[$*, such that* $[w,w+\varepsilon]$ *and* $[t,t+\varepsilon]$ *are subsets of* $[0,1]$*, and let* ℓ *be the boundary of the square* $[w,w+\varepsilon] \times p_{CS}\circ\gamma([t,t+\varepsilon])$ *of* $[0,1]\times\check{\mathcal{S}}_B(\mathbb{C})$*, then* $\widetilde{hol}_\ell(\eta_{B,.})$ *is trivial.*

Proof: The first assertion is a direct consequence of Proposition 16.17. It implies that for any $u \in [0,1]$, for any homotopically trivial loop ℓ of $\{u\} \times \check{\mathcal{S}}_B(\mathbb{C})$, $\widetilde{\mathrm{hol}}_\ell(\eta_{B,.})$ is neutral with respect to the product of Definition 16.8. Since $\gamma\overline{\gamma}$ is such a loop, it implies the second assertion. Now, the third assertion is a direct consequence of Lemma 16.16 and Proposition 16.17. $\square$

16.4 Semi-algebraic structures on some configuration spaces

We would like to extend the definitions of our connections η of Sections 16.2 and 15.2 on $\mathcal{S}_B(\mathbb{C})$, in order to extend the definition of $\mathcal{Z}^f$ to q–tangles. Unfortunately, I do not know whether the connections η extend as differentiable forms on $\mathcal{S}_B(\mathbb{C})$. However, we will be able to extend the definitions of their holonomies and prove that these holonomies along paths make sense (as $\sqrt{t^{-1}}$ may be integrated on $[0,1]$ though $\sqrt{t^{-1}}$ is not defined at 0). In order to do that, we will need to prove that integrals over singular spaces converge absolutely. Our proofs rely on the theory of semi-algebraic sets. We review the results of this theory that we will use, below. Our main reference is [BCR98, Section 1.4 and Chapter 2].

Definition 16.19. [BCR98, Definition 2.1.4] A *semi-algebraic subset* of $\mathbb{R}^n$ is a subset of the form

$$\bigcup_{i=1}^{s}\left(\bigcap_{j=1}^{r_i}\{x \in \mathbb{R}^n \mid f_{i,j}(x) < 0\} \cap \bigcap_{j=r_i+1}^{s_i}\{x \in \mathbb{R}^n \mid f_{i,j}(x) = 0\}\right)$$

for an integer s, $2s$ integers $r_1, \dots, r_s$, $s_1, \dots, s_s$, such that $s_i \geq r_i$ for any $i \in \underline{s}$, and $\sum_{i=1}^{s} s_i$ real polynomials $f_{i,j}$ in the natural coordinates of x. A *semi-algebraic set* is a semi-algebraic subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

The set of semi-algebraic subsets of $\mathbb{R}^n$ is obviously stable under finite union, finite intersection and taking complements. The set of semi-algebraic sets is stable under finite products.

Semi-algebraic sets also satisfy the following deeper properties, which are proved in [BCR98].

Theorem 16.20. *[BCR98, Theorem 2.2.1] Let S be a semi-algebraic subset of $\mathbb{R}^{n+1}$, let $\Pi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection onto the space of the first n coordinates. Then $\Pi(S)$ is a semi algebraic subset of $\mathbb{R}^n$.*

[BCR98, Proposition 2.2.2] The closure and the interior of a semi-algebraic set are semi-algebraic sets.

□

Definition 16.21. [BCR98, Definition 2.2.5] A map from a semi-algebraic subset of $\mathbb{R}^n$ to a semi-algebraic subset of $\mathbb{R}^m$ is *semi-algebraic* if its graph is semi-algebraic in $\mathbb{R}^{n+m}$.

The following proposition [BCR98, Proposition 2.2.7] can be deduced from Theorem 16.20 above, as an exercise.

Proposition 16.22. *Let f be a semi-algebraic map from a semi-algebraic set A to a semi-algebraic subset B of $\mathbb{R}^n$. For any semi-algebraic subset S of A, $f(S)$ is semi-algebraic. For any semi-algebraic subset S of $\mathbb{R}^n$, $f^{-1}(S)$ is semi-algebraic. The composition of two composable semi-algebraic maps is semi-algebraic.*

$\square$

As an example, which will be useful very soon, we prove the following proposition.

Proposition 16.23. *Let V denote a finite set of cardinality at least 2. Let T be a vector space of dimension δ. The manifold $\mathcal{S}_V(T)$ of Theorem 8.11 has a canonical structure of a semi-algebraic set. The restriction maps $\mathcal{S}_V(T) \to \mathcal{S}_A(T)$ are semi-algebraic with respect to these structures.*

Proof: The charts of Lemma 8.9 provide canonical semi-algebraic structures on $\overline{\mathcal{S}}_V(T)$ and $\check{\mathcal{S}}_V(T)$, and the restriction maps from $\check{\mathcal{S}}_V(T)$ to $\check{\mathcal{S}}_A(T)$ are semi-algebraic with respect to these structures. The description of $\mathcal{S}_V(T)$ as the closure of the image of $\check{\mathcal{S}}_V(T)$ in $\prod_{A \in \mathcal{P}_{\geq 2}} \overline{\mathcal{S}}_A(T)$ of Lemma 8.46 makes clear that $\mathcal{S}_A(T)$ has a natural semi-algebraic structure, thanks to Theorem 16.20. $\square$

We also have the following easy lemma

Lemma 16.24. *The space $\check{\mathcal{V}}(\Gamma)$ of Chapter 15 and its compactification $\mathcal{V}(\Gamma)$ carry a natural structure of a semi-algebraic set, for which the projection $p_{\mathcal{S}_B}$ from $\mathcal{V}(\Gamma)$ to $\mathcal{S}_B(\mathbb{C})$ and its projections to the $\mathcal{S}_e(\mathbb{R}^3)$ for ordered pairs e of $V(\Gamma)$ are semi-algebraic maps. For any configuration $y \in \mathcal{S}_B(\mathbb{C})$, the spaces $\check{\mathcal{V}}(y, \Gamma)$ and $\mathcal{V}(y, \Gamma)$ are semi-algebraic.*

$\square$

Lemma 16.25. *Let $f \colon]a, b[^d \to \mathbb{R}$ be a C^1 semi-algebraic map. Then its partial derivatives $\frac{\partial f}{\partial x_i}$ are semi-algebraic functions.*

Proof: As in [BCR98, Proposition 2.9.1], note that the set

$$\{(t, x, f(x), (f(x + tx_i) - f(x))/t) \mid t \in]0, 1], x \in]a, b[^d, x + tx_i \in]a, b[^d\}$$

is semi-algebraic. So are its closure, the locus $(t = 0)$ of this closure, and its projection to the graph of the partial derivative of f with respect to x_i. $\square$

Lemma 16.26. *Let f be a semi-algebraic smooth map from an open hypercube $]0,1[^d$ to $\mathbb{R}^n$. Then the* critical set *of f, which is the subset of $]0,1[^d$ for which f is not a submersion, is semi-algebraic.*

Proof: According to Lemma 16.25, the partial derivatives $\frac{\partial p_j \circ f}{\partial x_i}$ with respect to the factors of $\mathbb{R}^d$ of all the $p_j \circ f$ for the projections p_j on the factors of $\mathbb{R}^n$ are semi-algebraic. It is easy to see that the product and the sum of two real-valued semi-algebraic maps are semi-algebraic. Being in the critical set may be written as: for any subset I of $\underline{d}$ of cardinality n, the determinant $\det \left[\frac{\partial p_j \circ f}{\partial x_i}(x)\right]_{i \in I, j \in \underline{n}}$ is equal to zero. $\square$

An essential property of semi-algebraic sets, which we are going to use, is the following decomposition theorem [BCR98, Proposition 2.9.10].

Theorem 16.27. *Let S be a semi-algebraic subset of $\mathbb{R}^n$. Then, as a set, S is the disjoint union of finitely many smooth semi-algebraic submanifolds, each semi-algebraically diffeomorphic to an open hypercube $]0,1[^d$.*

The *dimension* of a semi-algebraic set is the maximal dimension of a hypercube in a decomposition as above. It is proved in [BCR98, Section 2.8] that it does not depend on the decomposition. It is also proved in [BCR98, Proposition 2.8.13] that if A is a semi-algebraic set of dimension $\dim(A)$, then $\dim(\overline{A} \setminus A) < \dim(A)$, and that the dimension of the image of a semi-algebraic set of dimension d under a semi-algebraic map is smaller or equal than d. See [BCR98, Theorem 2.8.8]

The following lemma is a corollary of Theorem 16.27.

Lemma 16.28. *Let f be a continuous semi-algebraic map from a compact semi-algebraic set A of dimension d to a semi-algebraic smooth manifold B with boundary equipped with a smooth differential form ω of degree d. Assume that the restriction of f to each piece of a decomposition as in Theorem 16.27 is smooth. Then the integrals $\int_\Delta f^*(\omega)$ of $f^*(\omega)$ over the open pieces Δ of dimension d of such a decomposition converge absolutely, and $\int_A f^*(\omega)$ is well defined as the sum of these $\int_\Delta f^*(\omega)$.*

Proof: It suffices to prove the lemma when ω is supported on a subset $[-1,1]^n$ of an open subset of B semi-algebraically diffeomorphic to $B_{k,n} =]-2,1]^k \times]-2,2[^{n-k}$. Indeed, using a partition of unity allows us to write ω as a finite sum of such forms around the compact $f(A)$. This allows us to reduce the proof to the case $B = B_{k,n}$. Now a degree d differential form on $B_{k,n}$ is a sum over the parts J of cardinality d of $\underline{n}$ of pull-backs of degree d forms on $B_J = B_{J,k,n} =]-2,1]^{\underline{k} \cap J} \times]-2,2[^{(\underline{n} \setminus \underline{k}) \cap J}$ multiplied by smooth functions on $B_{k,n}$, which are bounded on their compact supports. This allows us to reduce the proof to the case in which ω is such a pull-back of a form ω_J on B_J, under the projection $p_J\colon B_{k,n} \to B_J$, multiplied by a bounded function g_J on $B_{k,n}$.

Decompose A as in Theorem 16.27. It suffices to prove that the integral of ω over each hypercube H converges absolutely. Let f_J denote $p_J \circ f$. Consider the closure $\overline{H} \subset A$ of the hypercube H in A, set $\partial H = \overline{H} \setminus H$. Then ∂H and its image $f_J(\partial H)$ in B_J are algebraic subsets of B_J of dimension less than d, according to the properties of dimension recalled before the lemma. Therefore $f_J^*(\omega_J)$ vanishes on the dimension d pieces of the intersection of H with the semi-algebraic compact set $f_J^{-1}(f_J(\partial H))$. Let $\Sigma(f_J)$ be the set of critical points of $f_{J|H}$. According to Lemma 16.26, $\Sigma(f_J)$ is semi-algebraic. According to the Morse-Sard theorem [Hir94, Chapter 3, Section 1], $f_J(\Sigma(f_J))$, which is semi-algebraic, is of zero measure. Therefore, its dimension is less than d. Now, $B_J \setminus (f_J(\Sigma(f_J) \cup \partial H) \cup \partial B_J)$ is an open semi-algebraic subset of B_J, which therefore has a finite number of connected components according to Theorem 16.27. On each of these connected components, the local degree of f_J is finite because $\overline{H}$ is compact and the points in the preimage of a regular point are isolated. Our assumptions make this local degree locally constant: For a point y in such a component there exists a small disk $D(y)$ around y whose preimage contains disk neighborhoods of the points of the preimage each of which is mapped diffeomorphically to $D(y)$. Now, the image of $\overline{H}$ minus these open disks is a compact that does not meet y. Therefore there is a smaller disk around y that is not met by this compact.

Then $\int_{\overline{H}} f^*(\omega)$ is nothing but the integral of ω_J weighted by this bounded local degree and by a multiplication by $g_J \circ f$. So it is absolutely convergent.

□

Recall that an *open simplex* in $\mathbb{R}^n$ is a subset of the form $v_1 \dots v_k = \{\sum_{i=1}^k t_i v_i \mid t_i \in]0,1], \sum_{i=1}^k t_i = 1\}$, where $v_1, \dots, v_k$ are affinely independent points in $\mathbb{R}^n$, the faces of $v_1 \dots v_k$ are the simplices $v_{i_1} \dots v_{i_j}$ for subsets $\{i_1, \dots, i_j\} \subset \underline{k}$. A *locally finite simplicial complex* in $\mathbb{R}^n$ is a locally finite collection K of disjoint open simplices such that each face of a simplex of K belongs to K. For such a complex K, $|K|$ denotes the union of the simplices of K.

The following Lojasiewicz triangulation theorem [Loj64, Theorem 1, p. 463, §3] ensures that a compact semi-algebraic set may be viewed as a topological chain (as in Section 2.1.5).

Theorem 16.29. *For any locally finite collection $\{B_i\}$ of semi-algebraic subsets of $\mathbb{R}^n$, there exist a locally finite simplicial complex K of $\mathbb{R}^n$ such that $|K| = \mathbb{R}^n$ and a homeomorphism τ from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that:*

- *for any open simplex σ of K, $\tau(\sigma)$ is an analytic submanifold of $\mathbb{R}^n$, and $\tau_{|\sigma}$ is an analytic isomorphism from σ to $\tau(\sigma)$,*

- *for any open simplex σ of K, and for any B_i of the collection $\{B_i\}$, we have $\tau(\sigma) \subset B_i$ or $\tau(\sigma) \subset \mathbb{R}^n \setminus B_i$.*

16.5 Extending $\mathcal{Z}^f$ to q–tangles

In Chapter 15, the behaviour of $\mathcal{Z}$ on braids, which are paths in $\check{\mathcal{S}}_B(\mathbb{C})$ for some finite set B was discussed. Recall that $\mathcal{Z}$ and $\mathcal{Z}^f$ coincide for braids. In this section, we extend $\mathcal{Z}^f$ and its variants of Section 16.2 to paths of $\mathcal{S}_B(\mathbb{C})$, where B is a finite set. This is already mostly done in [Poi00], where the main ideas come from, but our presentation is different, and it provides additional statements and explanations.

Our extension to paths of $\mathcal{S}_B(\mathbb{C})$ will allow us to define the extension of $\mathcal{Z}^f$ to q–tangles in rational homology cylinders so that Proposition 15.18 is still valid in the setting of q–tangles.

Recall the semi-algebraic subsets $\mathcal{V}(\Gamma)$ and $\mathcal{V}(y,\Gamma)$ of $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \times \mathcal{S}_B(\mathbb{C})$, introduced in Chapter 15, for a $\underline{3N}$-numbered Jacobi diagram Γ on $\sqcup_{b \in B}\mathbb{R}_b$. Both $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$ and $\mathcal{S}_B(\mathbb{C})$ are stratified by Δ-parenthesizations according to Theorem 8.26. Let $\mathcal{P}_B$ be a Δ-parenthesization of B and let $\mathcal{P}$ be a Δ-parenthesization of $V(\Gamma)$, and let

$$\mathcal{V}_{\mathcal{P}_B,\mathcal{P}}(\Gamma) = \mathcal{V}(\Gamma) \cap \left(\mathcal{S}_{V(\Gamma),\mathcal{P}}(\mathbb{R}^3) \times \mathcal{S}_{B,\mathcal{P}_B}(\mathbb{C})\right).$$

An element of $\mathcal{S}_{V(\Gamma),\mathcal{P}}(\mathbb{R}^3)$ is denoted by $(c_Y)_{Y\in\mathcal{P}}$, where $c_Y \in \check{\mathcal{S}}_{K(Y)}(\mathbb{R}^3)$. An element of $\mathcal{S}_{B,\mathcal{P}_B}(\mathbb{C})$ is denoted by $(y_D)_{D\in\mathcal{P}_B}$, where $y_D \in \check{\mathcal{S}}_{K(D)}(\mathbb{C})$. Fix Γ and $\mathcal{P}_B$. Recall the natural map $p_B \colon U(\Gamma) \to B$ induced by i_Γ. Let $c = ((c_Y)_{Y\in\mathcal{P}}, (y_D)_{D\in\mathcal{P}_B}) \in \mathcal{V}_{\mathcal{P}_B,\mathcal{P}}(\Gamma)$, and let Y be in the set $\widehat{\mathcal{P}'}_X$ introduced in Notation 15.2. Then

$$p_{\mathbb{C}} \circ c_{Y|U(\Gamma)\cap Y} = \lambda(Y)\left(y_{\hat{B}(Y)} \circ p_B - y_{\hat{B}(Y)} \circ p_B(b(Y))\right)$$

for some $\lambda(Y) \geq 0$, with the normalizations of Notation 15.2. Recall from Lemma 15.11 that $\mathcal{P}_x(c)$ is the subset of $\mathcal{P}$ consisting of separating sets for c, which are the univalent sets Y of $\mathcal{P}$ such that $\lambda(Y) \neq 0$. For a subset $\mathcal{P}_x$ of $\mathcal{P}_u$, let $\mathcal{V}_{\mathcal{P}_B,\mathcal{P},\mathcal{P}_x}(\Gamma) = \{c \in \mathcal{V}_{\mathcal{P}_B,\mathcal{P}}(\Gamma) \mid \mathcal{P}_x(c) = \mathcal{P}_x\}$. We use the data $(\mathcal{P}_B, \mathcal{P}, \mathcal{P}_x)$ to stratify $\mathcal{V}(\Gamma)$ (or $[0,1]\times\mathcal{V}(\Gamma)$ whose strata will be the products by $[0,1]$ of the strata of $\mathcal{V}(\Gamma)$, by definition). Recall that for any $D \in \mathcal{P}_B$, the elements Y of $\mathcal{P}_x$ such that $\hat{B}(Y) = D$, are minimal with respect to the inclusion among the elements of $\mathcal{P}$ such that $\hat{B}(Y) = D$.

For $y \in \mathcal{S}_{B,\mathcal{P}_B}(\mathbb{C})$, for a Δ-parenthesization $\mathcal{P}$ of $V(\Gamma)$ and for a subset $\mathcal{P}_x$ of $\mathcal{P}$, set

$$\mathcal{V}(y,\Gamma,\mathcal{P},\mathcal{P}_x) = \mathcal{V}(y,\Gamma) \cap \mathcal{V}_{\mathcal{P}_B,\mathcal{P},\mathcal{P}_x}(\Gamma).$$

Recall from Lemma 15.11 that when $\mathcal{V}(y,\Gamma,\mathcal{P},\mathcal{P}_x)$ is not empty, its dimension is $\sharp U(\Gamma)+3\sharp T(\Gamma)-1-\sharp(\mathcal{P}\setminus\mathcal{P}_x)$. In particular, the dimension of $\mathcal{V}(y,\Gamma)$ is at most $2\sharp E(\Gamma)-1$.

Fix the family $(\tilde{\omega}(i,S^2) = (\tilde{\omega}(i,t,S^2))_{t\in[0,1]})_{i\in\underline{3N}}$ of closed 2–forms on $[0,1]\times S^2$, once for all, in this section, and assume $\tilde{\omega}(i,0,S^2)=\omega_{S^2}$ for all i. For an edge e of Γ, recall the map

$$p_{e,S^2}\colon [0,1]\times \mathcal{S}_{V(\Gamma)}(\mathbb{R}^3) \to [0,1]\times S^2,$$

which maps $(t, c \in \check{\mathcal{S}}_{V(\Gamma)}(\mathbb{R}^3))$ to $(t, p_{S^2}((c(v(e,1)), c(v(e,2)))))$. This provides the $(2\sharp E(\Gamma))$-form

$$\Omega_\Gamma = \bigwedge_{e\in E(\Gamma)} p^*_{e,S^2}(\tilde{\omega}(j_E(e),S^2))$$

over $[0,1]\times\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$. This form pulls back to provide smooth forms on the smooth strata of $[0,1]\times\mathcal{V}(\Gamma)$.

Let A denote a subset of $\underline{3N}$ with cardinality $3n$. An *ordered r-component A-numbered Jacobi diagram* $\Gamma^{(r)}$ on $\sqcup_{b\in B}\mathbb{R}_b$ is a degree n A-numbered Jacobi diagram $\Gamma^{(r)}$ on $\sqcup_{b\in B}\mathbb{R}_b$ that has r connected components $\Gamma_1, \dots \Gamma_r$ and such that i_Γ is represented by an injection of $V(\Gamma)$ that maps all univalent vertices of Γ_i before (or below) the univalent vertices of Γ_{i+1} for any $i\in\underline{r-1}$. The data of such an ordered r-component A-numbered Jacobi diagram $\Gamma^{(r)}$ is equivalent to the data of an r-tuple $(\Gamma_1,\dots,\Gamma_r)$ of A-numbered Jacobi diagrams with pairwise disjoint $j_E(\Gamma_i)$ such that the sum of the degrees of the Γ_i is n. Let $\mathcal{D}^{e,r}_{n,A}(\sqcup_{b\in B}\mathbb{R}_b)$ denote[5] the set of these ordered r-component A-numbered Jacobi diagrams $\Gamma^{(r)}$ on $\sqcup_{b\in B}\mathbb{R}_b$.

Such a diagram provides the $(2\sharp E(\Gamma^{(r)}))$-form

$$\Omega_{\Gamma^{(r)}} = \bigwedge_{i=1}^{r} P_i^*(\Omega_{\Gamma_i})$$

on (the smooth strata of) $\prod_{i=1}^r([0,1]\times\mathcal{V}(\Gamma_i))$, where $P_i\colon \prod_{i=1}^r([0,1]\times\mathcal{V}(\Gamma_i)) \to [0,1]\times\mathcal{V}(\Gamma_i)$ is the projection onto the i^{th} factor. This form is also the pullback of a smooth form on $[0,1]^r\times(S^2)^{E(\Gamma^{(r)})}$, by a semi-algebraic map. Let

[5] The notation $\mathcal{D}^{e,r}_{n,A}$ has a redundancy since the cardinality of A is $3n$. We keep the redundancy for consistency, here, because we will use other spaces of numbered Jacobi diagrams, where the degree is not determined by the cardinality of the set of indices, in Chapter 17.

$\Delta^{(r)} = \{(t_1,\dots,t_r) \in [0,1]^r \mid 0 \le t_1 \le t_2 \cdots \le t_r\}$. A semi-algebraic path $\gamma\colon [0,1] \to [0,1] \times \mathcal{S}_B(\mathbb{C})$ induces the semi-algebraic map

$$\begin{array}{rlcl} \gamma^{(r)}\colon & \Delta^{(r)} & \to & ([0,1] \times \mathcal{S}_B(\mathbb{C}))^r \\ & (t_1,\dots,t_r) & \mapsto & (\gamma(t_1),\dots,\gamma(t_r)) \end{array}$$

Consider the product $P_{\Gamma^{(r)}}\colon \prod_{i=1}^r([0,1] \times \mathcal{V}(\Gamma_i)) \to ([0,1] \times \mathcal{S}_B(\mathbb{C}))^r$ of natural projections. Assume that γ is injective. Let $C(\Gamma^{(r)},\gamma) = P_{\Gamma^{(r)}}^{-1}(\gamma^{(r)}(\Delta^{(r)}))$. Then $C(\Gamma^{(r)},\gamma)$ is a semi-algebraic subset of $\prod_{i=1}^r([0,1] \times \mathcal{V}(\Gamma_i))$ of dimension at most $2\sharp E(\Gamma^{(r)})$ whose $2\sharp E(\Gamma^{(r)})$-dimensional strata are oriented canonically, as soon as the Jacobi diagrams Γ_i are: Fix an arbitrary vertex-orientation for the Γ_i. The set $C(\Gamma^{(r)},\gamma)$ is locally oriented as the product of the $C(\Gamma_i^{(1)},\gamma)$ for $i \in \underline{r}$. The parameter t_i replaces the translation parameter in $\mathcal{V}(\gamma(t_i),\Gamma_i)$.

Define the A-holonomy $\widetilde{\mathrm{hol}}_{.}(\eta_{B,A})$ along injective semi-algebraic paths γ of $[0,1] \times \mathcal{S}_B(\mathbb{C})$, with respect to our family $(\tilde{\omega}(i,S^2))_{i\in\underline{3N}}$, to be

$$\widetilde{\mathrm{hol}}_{\gamma}(\eta_{B,A}) = [\emptyset] + \sum_{r=1}^{\infty} \sum_{\Gamma^{(r)} \in \mathcal{D}^{e,r}_{n,A}(\sqcup_{b\in B}\mathbb{R}_b)} \zeta_{\Gamma^{(r)}} \int_{C(\Gamma^{(r)},\gamma)} \Omega_{\Gamma^{(r)}}[\Gamma^{(r)}],$$

where

$$\zeta_{\Gamma^{(r)}} = \frac{(\sharp A - \sharp E(\Gamma^{(r)}))!}{(\sharp A)! 2^{\sharp E(\Gamma^{(r)})}}.$$

(Again, we fix an arbitrary vertex-orientation for the components Γ_i of each $\Gamma^{(r)}$ and $\widetilde{\mathrm{hol}}_{\gamma}(\eta_{B,A})$ is independent of our choices.)

The involved integrals make sense as soon as γ is semi-algebraic, thanks to Lemma 16.28, which also justifies the following lemma.

Lemma 16.30. *For any injective semi-algebraic path γ of $[0,1] \times \mathcal{S}_B(\mathbb{C})$, $\lim_{\varepsilon\to 0} \widetilde{hol}_{\gamma_{|[\varepsilon,1-\varepsilon]}}(\eta_{B,A})$ makes sense and it is equal to $\widetilde{hol}_{\gamma}(\eta_{B,A})$.*

$\square$

Together with the identification $\mathcal{Z}(T(\gamma)) = \mathrm{hol}_{p_{CS}\circ\gamma}(\eta_B)$ for braids provided by Proposition 15.18, Lemma 16.30 implies the convergence part of Theorem 13.8. The above convergent integrals extend Definition 16.1 of $\widetilde{\mathrm{hol}}_{\gamma}(\eta_{B,A})$ for injective semi-algebraic paths in $[0,1] \times \check{\mathcal{S}}_B(\mathbb{C})$. Note the following easy lemma.

Lemma 16.31. *The A-holonomy $\widetilde{hol}_{\gamma}(\eta_{B,.})$, which is valued in $\mathcal{A}_n(\sqcup_{b\in B}\mathbb{R}_b)$, extends naturally to non-injective semi-algebraic paths. This holonomy is multiplicative under path composition, with respect to the product of Definition 16.8.*

□

Recall that $\tilde{\omega}(i,0,S^2)$ is the standard homogeneous volume-one form on S^2. When γ is valued in $\{0\}\times\mathcal{S}_B(\mathbb{C})$, there is no need to number the diagram edges, since $\tilde{\omega}(i,0,S^2)=\omega_{S^2}$ for all i, and we simply define

$$\widetilde{\mathrm{hol}}_\gamma(\eta_B)=\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\Gamma^{(r)}\in\mathcal{D}^{e,r}_{k,\underline{3k}}(\sqcup_{b\in B}\mathbb{R}_b)}\zeta_{\Gamma^{(r)}}\int_{C(\Gamma^{(r)},\gamma)}\Omega_{\Gamma^{(r)}}[\Gamma^{(r)}]\in\mathcal{A}(\sqcup_{b\in B}\mathbb{R}_b).$$

In this case, $\widetilde{\mathrm{hol}}_\gamma(\eta_B)$ is nothing but the Poirier functor Z^l of [Poi00, Section 1.4] applied to γ. The projection in $\mathcal{A}_n(\sqcup_{b\in B}\mathbb{R}_b)$ of $\widetilde{\mathrm{hol}}_\gamma(\eta_B)$, which coincides with the holonomy $\mathrm{hol}_\gamma(\eta_B)$ defined in Section 15.2 when γ is valued in $\{0\}\times\check{\mathcal{S}}_B(\mathbb{C})$, coincides with $\widetilde{\mathrm{hol}}_\gamma(\eta_{B,A})$, when $\sharp A=3n$, in this case of homogeneous forms.

Recall that $\mathcal{S}_B(\mathbb{C})$ is a smooth manifold with ridges, which can also be equipped with a semi-algebraic structure for which the local charts provided in Theorem 8.26 are semi-algebraic maps. In such a trivialized open simply connected subspace, any two points can be connected by a semi-algebraic path. In particular, any two points a and b of $\mathcal{S}_B(\mathbb{C})$ are connected by a semi-algebraic path $\gamma\colon[0,1]\to\mathcal{S}_B(\mathbb{C})$ such that $\gamma(0)=a$ and $\gamma(1)=b$. Furthermore, any path from a to b of $\mathcal{S}_B(\mathbb{C})$ can be C^0-approximated by a homotopic semi-algebraic path. So any homotopy class of paths from a to b has a semi-algebraic representative.

Now Theorem 13.8 is a direct corollary of the following one, which will be proved after Lemma 16.38. This theorem is a mild generalization of [Poi00, Proposition 9.2], thanks to Lemma 16.30.

Theorem 16.32. *Let $\gamma\colon[0,1]\to[0,1]\times\mathcal{S}_B(\mathbb{C})$ be a semi-algebraic path. Then $\widetilde{hol}_\gamma(\eta_{B,A})$ depends only on $\gamma(0)$, $\gamma(1)$, the $\tilde{\omega}(i,S^2)$ for $i\in A$, and the homotopy class of γ relatively to $\partial\gamma$.*

According to Lemmas 16.18 and 16.31, Theorem 16.32 holds for smooth paths γ of $\{u\}\times\check{\mathcal{S}}_B(\mathbb{C})$ and their piecewise smooth compositions. We now prove the following other particular case of Theorem 16.32.

Lemma 16.33. *Let $(\gamma_u(t))_{u\in[0,1]}$ be a semi-algebraic homotopy such that*

- *γ is injective on $[0,1]\times]0,1[$,*
- *$\gamma_u(t)=\gamma(u,t)\in[0,1]\times\check{\mathcal{S}}_B(\mathbb{C})$ for any $(u,t)\in]0,1]\times]0,1[$,*
- *$\gamma_u(0)=\gamma_0(0)$ and $\gamma_u(1)=\gamma_0(1)$ for all $u\in[0,1]$,*

- $\gamma_0(t)$ *is in a fixed stratum of* $[0,1]\times\mathcal{S}_B(\mathbb{C})$ *for* $t\in]0,1[$, *where a* stratum *of* $[0,1]\times\mathcal{S}_B(\mathbb{C})$ *is the product by* $[0,1]$ *of a stratum of* $\mathcal{S}_B(\mathbb{C})$ *associated with a* Δ*-parenthesization.*

Then

$$\widetilde{hol}_{\gamma_0}(\eta_{B,A})=\widetilde{hol}_{\gamma_1}(\eta_{B,A}).$$

Lemma 16.33 is the direct consequence of Lemmas 16.36 and 16.37 below. The proof of Lemma 16.36 uses Lemma 16.35 and the following sublemma.

Sublemma 16.34. *Under the assumptions of Lemma 16.33, let* $\Gamma^{(r)}$ *be an element of* $\mathcal{D}^{e,r}_{n,A}(\sqcup_{b\in B}\mathbb{R}_b)$*, and let*

$$C(\Gamma^{(r)},(\gamma_u))=\cup_{u\in[0,1]}C(\Gamma^{(r)},\gamma_u)$$

be the associated semi-algebraic set of dimension $2\sharp E(\Gamma^{(r)})+1$*. Then the codimension-one boundary of* $C(\Gamma^{(r)},(\gamma_u))$ *is*

$$C(\Gamma^{(r)},\gamma_1)-C(\Gamma^{(r)},\gamma_0)-\cup_{u\in[0,1]}\partial C(\Gamma^{(r)},\gamma_u),$$

where

$$\partial C(\Gamma^{(r)},\gamma_u)=\partial_C C(\Gamma^{(r)},\gamma_u)+\partial_\Delta C(\Gamma^{(r)},\gamma_u)$$

with

$$\partial_C C(\Gamma^{(r)},\gamma_u)=\pm\cup_{(t_1,\dots,t_r)\in\Delta^{(r)}}\partial P^{-1}_{\Gamma^{(r)}}(\gamma_u^{(r)}((t_1,\dots,t_r)))$$

and

$$\partial_\Delta C(\Gamma^{(r)},\gamma_u)=\pm P^{-1}_{\Gamma^{(r)}}(\gamma_u^{(r)}(\partial\Delta^{(r)})).$$

PROOF: When the image of (γ_u) is in $[0,1]\times\check{\mathcal{S}}_B(\mathbb{C})$, it is an immediate consequence of Lemmas 15.12 and 14.13. Let us prove that it is still true for our homotopies (γ_u). The part $C(\Gamma^{(r)},\gamma_1)$ is in the boundary as before, we can ignore the contributions of the extremities of γ_1 since they belong to parts of dimension at most $2\sharp E(\Gamma^{(r)})-1$, thanks to Lemma 15.11. For the part that comes from $\cup_u\partial C(\Gamma^{(r)},\gamma_u)$ in the $2\sharp E(\Gamma)$-dimensional boundary, where $C(\Gamma^{(r)},\gamma_u)=P^{-1}_{\Gamma^{(r)}}(\gamma_u^{(r)}(\Delta^{(r)}))$, we may restrict to $u\in]0,1[$ for dimension reasons, which we do. So this part is in the boundary as before, too

It is easy to see that the part over γ_0 of the codimension-one boundary of $C(\Gamma^{(r)},(\gamma_u))$ is included in $C(\Gamma^{(r)},\gamma_0)$. Let us prove that the corresponding algebraic boundary is indeed $-C(\Gamma^{(r)},\gamma_0)$ when $\gamma_0(]0,1[)$ is in some stratum of $[0,1]\times\partial\mathcal{S}_B(\mathbb{C})$, associated to a parenthesization $\mathcal{P}_B$ of B.

Let $t_i\in]0,1[$. In a neighborhood $[0,\eta[\times N(t_i)$ of $(0,t_i)$ in $[0,1]^2$, $\gamma_u(t)=\gamma(u,t)$ is expressed as

$$\gamma_u(t)=\left((y_D(u,t))_{D\in\mathcal{P}_B},(u_D(u,t))_{D\in\mathcal{P}_B\setminus\{B\}}\right),$$

where $u_D(0,t) = 0$, for all $t \in N(t_i)$.

Let c^0 be a point of the $(2\sharp E(\Gamma_i)-1)$–dimensional open part of $\mathcal{V}(\gamma_0(t_i), \Gamma_i)$, where $\gamma_0'(t_i) \neq 0$. According to Lemma 15.11, $\mathcal{P}(c^0) = \mathcal{P}_x(c^0)$. We express a neighborhood of c^0 in $\cup_{(u,t)\in[0,\eta[\times N(t_i)}\mathcal{V}(\gamma_u(t), \Gamma_i)$, as a product by $[0,\eta[\times N(t_i)$, as follows. We use the above coordinates $y_D(u,t)$, $u_D(u,t)$ of the base, and the parameters c_Z, $p_{\mathbb{R}} \circ c_Z(Y)$, $\lambda(Y)$ listed in the third, fourth and fifth sets of variables of Lemma 15.5. Since $\mathcal{P} = \mathcal{P}_x$, $\lambda(V(\Gamma))(c^0) = \lambda(c^0)$, and $\lambda(c^0) \neq 0$. Let $Y \in \mathcal{P} \setminus \{V(\Gamma)\}$, then $\lambda(Y)(c^0) \neq 0$. Let Y^+ be the smallest set of $\mathcal{P}$ that contains Y, strictly. Set $Y' = Y^+$. Set $B_1 = \hat{B}(Y)$, $B_1' = B_1^+ = \hat{B}(Y^+)$. Equation 15.1 in Lemma 15.5 applied to these sets when $n = 1$ reads $\lambda(Y)\mu_Y = \lambda(Y^+)u_{\hat{B}(Y)}$. It implies that $\mu_Y = \frac{\lambda(Y^+)}{\lambda(Y)}u_{\hat{B}(Y)}$ is determined by the listed parameters, and that all μ_Y are zero over $\gamma_0(N(t_i))$ in our neighborhood of c^0 so that $\mathcal{P} = \mathcal{P}_x = \mathcal{P}(c^0)$ over $\gamma_0(N(t_i))$ in our neighborhood of c^0. In particular, if $Y \subsetneq Y'$, then $\hat{B}(Y) \subsetneq \hat{B}(Y')$, and all equations 15.1 are satisfied, there. Over $\gamma(]0,\eta[\times N(t_i))$, no u_D vanishes so that no μ_Y vanishes either in our neighborhood of c^0, and the equations 15.1 are implied by the equations $*(Y)$, which are implied by the equations $\mu_Y = \frac{\lambda(Y^+)}{\lambda(Y)}u_{\hat{B}(Y)}$ and $\lambda(V(\Gamma)) = \lambda$. So we get a neighborhood of c^0 in $\cup_{(u,t)\in[0,\eta[\times N(t_i)}\mathcal{V}(\gamma_u(t), \Gamma_i)$, which is parametrized, by (u,t) and by the parameters c_Z, $p_{\mathbb{R}} \circ c_Z(Y)$, $\lambda(Y)$ in the fourth and fifth lists of variables of Lemma 15.5. For any horizontally normalizing kid Y of $\mathcal{P}$, remove $\lambda(Y)$ from the parameters, since it is determined by the first and fifth constraints of Lemma 15.5. Use the first constraint to remove other superfluous parameters $p_{\mathbb{R}} \circ c_Z(Y)$, $|p_{\mathbb{C}} \circ c_Z(Y)|$ and get a free system of parameters. Thus, we obtain an open $(2\sharp E(\Gamma_i))$-dimensional neighborhood $\mathcal{O}$ of c^0 in $\cup_{t\in N(t_i)}\mathcal{V}(\gamma_0(t), \Gamma_i)$ and a local open embedding of the product of $[0,\eta[\times\mathcal{O}$ into $\cup_{(u,t)\in[0,\eta[\times N(t_i)}\mathcal{V}(\gamma_u(t), \Gamma_i)$. So $\big(-C(\Gamma^{(r)}, \gamma_0)\big)$ is the algebraic boundary of $C(\Gamma^{(r)}, (\gamma_u))$ over γ_0. □

Lemma 16.35. *For a Jacobi diagram in $\mathcal{D}^e_{n,A}(\sqcup_{b\in B}\mathbb{R}_b)$, for an element (t,y) of $[0,1] \times \check{\mathcal{S}}_B(\mathbb{C})$, $d\eta_\Gamma(t,y)$ is the integral of $\Big(-\bigwedge_{e\in E(\Gamma)} p^*_{e,S^2}(\tilde{\omega}(j_E(e), S^2))\Big)$ along the interiors of the codimension-one faces of $\{t\} \times \mathcal{V}(y,\Gamma)$.*

Proof: See the proof of Lemma 15.21. □

Recall from the beginning of Section 15.2 that the fiber $\mathcal{V}(y,\Gamma)$ is oriented so that the orientation of $\mathcal{V}(y,\Gamma)$ preceded by the upward translation parameter –which replaces the parametrization of the paths along which we integrate– matches the usual orientation of our configuration spaces, induced as in Lemma 7.1.

Lemma 16.36. *Let* $(\gamma_u(t))_{u\in[0,1]}$ *be a semi-algebraic homotopy that satisfies the assumptions of Lemma 16.33, or a smooth homotopy valued in* $[0,1]\times\check{\mathcal{S}}_B(\mathbb{C})$ *such that* $\gamma_u(0)$ *and* $\gamma_u(1)$ *do not depend on* u*. Then*

$$\begin{aligned}&\widetilde{hol}_{\gamma_1}(\eta_{B,A})-\widetilde{hol}_{\gamma_0}(\eta_{B,A})\\ &=\sum_{r=0}^{\infty}\sum_{(A_1,\dots,A_r)\in P_r(A)}\frac{\prod_{i=1}^r(\sharp A_i)!}{(\sharp A)!}\int_{[0,1]\times\Delta^{(r)}}\sum_{i=1}^r\delta(i,A_1,\dots,A_r),\end{aligned}$$

where

$$\delta(i,A_1,\dots,A_r)=(-1)^{i-1}\bigwedge_{j=1}^r(\gamma\circ p_j)^*(\eta_{B,A_j})\left(\frac{(\gamma\circ p_i)^*(d\eta_{B,A_i}+(\eta\wedge\eta)_{B,A_i})}{(\gamma\circ p_i)^*(\eta_{B,A_i})}\right),$$

$$d\eta_{B,A_i}=\sum_{\Gamma_i\in\mathcal{D}^e_{n_i,A_i}(\sqcup_{b\in B}\mathbb{R}_b)}\zeta_{\Gamma_i}d\eta_{\Gamma_i}[\Gamma_i],$$

$$(\eta\wedge\eta)_{B,A_i}=\sum_{(\Gamma,\Gamma')\in\mathcal{D}^{e,2}_{n_i,A_i}(\sqcup_{b\in B}\mathbb{R}_b)}\zeta_{\Gamma\sqcup\Gamma'}\eta_\Gamma\wedge\eta_{\Gamma'}[\Gamma][\Gamma'],$$

and the fraction means that $(\gamma\circ p_i)^*(\eta_{B,A_i})$ *is replaced with* $(\gamma\circ p_i)^*(d\eta_{B,A_i}+(\eta\wedge\eta)_{B,A_i})$.

PROOF: Let $\partial_C C(\Gamma^{(r)},(\gamma_u))=-\cup_{u\in]0,1[}\partial_C C(\Gamma^{(r)},\gamma_u)$ and

$$\partial_\Delta C(\Gamma^{(r)},(\gamma_u))=-\cup_{u\in]0,1[}\partial_\Delta C(\Gamma^{(r)},\gamma_u).$$

Since

$$\partial C(\Gamma^{(r)},(\gamma_u))=C(\Gamma^{(r)},\gamma_1)-C(\Gamma^{(r)},\gamma_0)+\partial_C C(\Gamma^{(r)},(\gamma_u))+\partial_\Delta C(\Gamma^{(r)},(\gamma_u))$$

is a null-homologous cycle, we get that

$$\widetilde{\mathrm{hol}}_{\gamma_1}(\eta_{B,A})-\widetilde{\mathrm{hol}}_{\gamma_0}(\eta_{B,A})=\widetilde{\delta}_{(\gamma_u)}(d\eta_{B,A})+\widetilde{\delta}_{(\gamma_u)}((\eta\wedge\eta)_{B,A}),$$

where

$$\widetilde{\delta}_{(\gamma_u)}(d\eta_{B,A})\stackrel{\mathrm{def}}{=}-\sum_{r=0}^{\infty}\sum_{\Gamma^{(r)}\in\mathcal{D}^{e,r}_{n,A}(\sqcup_{b\in B}\mathbb{R}_b)}\zeta_{\Gamma^{(r)}}\int_{\partial_C C(\Gamma^{(r)},(\gamma_u))}\Omega_{\Gamma^{(r)}}\left[\Gamma^{(r)}\right]$$

and

$$\widetilde{\delta}_{(\gamma_u)}((\eta\wedge\eta)_{B,A})\stackrel{\mathrm{def}}{=}-\sum_{r=0}^{\infty}\sum_{\Gamma^{(r)}\in\mathcal{D}^{e,r}_{n,A}(\sqcup_{b\in B}\mathbb{R}_b)}\zeta_{\Gamma^{(r)}}\int_{\partial_\Delta C(\Gamma^{(r)},(\gamma_u))}\Omega_{\Gamma^{(r)}}\left[\Gamma^{(r)}\right].$$

We now study these terms.

Since $P^{-1}_{\Gamma^{(r)}}(\gamma^{(r)}_u((t_1,\dots,t_r))) = \pm\prod_{i=1}^r\{p_{[0,1]}(\gamma_u(t_i))\}\times\mathcal{V}(p_{\check{\mathcal{S}}_B(\mathbb{C})}(\gamma_u(t_i)),\Gamma_i) \cong \pm\prod_{i=1}^r\mathcal{V}(\gamma_u(t_i),\Gamma_i)$, (forgetting the natural $p_{\check{\mathcal{S}}_B(\mathbb{C})}$)

$$\partial P^{-1}_{\Gamma^{(r)}}(\gamma^{(r)}_u((t_1,\dots,t_r))) = \pm\sum_{i=1}^r \partial\mathcal{V}(\gamma_u(t_i),\Gamma_i)\times\prod_{j\in\underline{n}\setminus\{i\}}\mathcal{V}(\gamma_u(t_j),\Gamma_j).$$

According to Lemma 16.35,

$$\widetilde{\delta}_{(\gamma_u)}(d\eta_{B,A}) = \sum_{r=0}^{\infty}\sum_{(A_1,\dots,A_r)\in P_r(A)}\frac{\prod_{i=1}^r(\sharp A_i)!}{(\sharp A)!}\int_{[0,1]\times\Delta^{(r)}}\sum_{i=1}^r\alpha(i,A_1,\dots,A_r),$$

where

$$\alpha(i,A_1,\dots,A_r) = (-1)^{i-1}\bigwedge_{j=1}^r(\gamma\circ p_j)^*(\eta_{B,A_j})\left(\frac{(\gamma\circ p_i)^*(d\eta_{B,A_i})}{(\gamma\circ p_i)^*(\eta_{B,A_i})}\right)$$

and $d\eta_{B,\emptyset}=0$. (Recall that the space $C(\Gamma^{(r)},\gamma_u)$ is oriented locally as the product of the $C(\Gamma^{(1)}_i,\gamma_u)$ for $i\in\underline{r}$, which are oriented so that the parameter t replaces the translation parameter in $\mathcal{V}(\gamma_u(t),\Gamma_i)$. Thus the boundary ∂_C, along which we integrate, is locally diffeomorphic to

$$\left(-\cup_{u\in]0,1[}\left(\prod_{j=1}^{i-1}C(\Gamma^{(1)}_j,\gamma_u)\right)\times\partial_C C(\Gamma^{(1)}_i,\gamma_u)\times\left(\prod_{j=i+1}^{r}C(\Gamma^{(1)}_j,\gamma_u)\right)\right),$$

where the dimension of $C(\Gamma^{(1)}_j,\gamma_u)$ is even. When rewriting such an integral as an integral over $[0,1]\times\Delta^r$ of one-forms $(\gamma\circ p_j)^*(\eta_{B,A_j})$ and the two-form $(\gamma\circ p_i)^*(d\eta_{B,A_i})$, one must take into account the fact that this two-form will be integrated along the product by $[0,1]$ of the interval parametrized by t_i. This gives rise to the factor $(-1)^{i-1}$.)

Recall $\partial\Delta^{(r)} = \sum_{i=0}^r(-1)^{i+1}F_i(\Delta^{(r)})$, where $F_0(\Delta^{(r)}) = \{(0,t_2,\dots,t_r)\in\Delta^r\}$, $F_r(\Delta^{(r)}) = \{(t_1,t_2,\dots,t_{r-1},1)\in\Delta^{(r)}\}$, and, for $i\in\underline{r-1}$,

$$F_i(\Delta^{(r)}) = \{(t_1,\dots,t_i,t_i,t_{i+1},\dots,t_{r-1})\in\Delta^{(r)}\}.$$

Observe that the faces F_0 and F_r do not contribute to $\widetilde{\delta}_{(\gamma_u)}((\eta\wedge\eta)_{B,A})$. Indeed for F_0, the directions of the edges of Γ_1 are in the image of $\mathcal{V}(\gamma_0(0),\Gamma_1)$, which is $(2\sharp E(\Gamma_1)-1)$-dimensional. The contribution of the faces F_i yields:

$$\widetilde{\delta}_{(\gamma_u)}((\eta\wedge\eta)_{B,A}) =$$

$$\sum_{r=1}^{\infty}\sum_{(A_1,\dots,A_{r-1})\in P_{r-1}(A)}\frac{\prod_{i=1}^{r-1}(\sharp A_i)!}{(\sharp A)!}\int_{[0,1]\times\Delta^{r-1}}\sum_{i=1}^{r-1}\beta(i,A_1,\dots,A_{r-1}),$$

where

$$\beta(i,A_1,\dots,A_{r-1})=(-1)^{i-1}\bigwedge_{j=1}^{r-1}(\gamma\circ p_j)^*(\eta_{B,A_j})\left(\frac{(\gamma\circ p_i)^*((\eta\wedge\eta)_{B,A_i})}{(\gamma\circ p_i)^*(\eta_{B,A_i})}\right).$$

□

Note that $(\eta\wedge\eta)_{B,A_i}=\sum_{(\Gamma,\Gamma')\in\mathcal{D}^{e,2}_{n_i,A_i}(\sqcup_{b\in B}\mathbb{R}_b)}\zeta_{\Gamma\sqcup\Gamma'}\eta_\Gamma\wedge\eta_{\Gamma'}[\Gamma][\Gamma']$ is valued in the space of primitive elements of $\mathcal{A}_{n_i}(\sqcup_{b\in B}\mathbb{R}_b)$ since $\eta_\Gamma\wedge\eta_{\Gamma'}[\Gamma][\Gamma']+\eta_{\Gamma'}\wedge\eta_\Gamma[\Gamma'][\Gamma]=\eta_\Gamma\wedge\eta_{\Gamma'}([\Gamma][\Gamma']-[\Gamma'][\Gamma])$.

Lemma 16.37. *The form $d\eta_{B,A}+(\eta\wedge\eta)_{B,A}$ on $[0,1]\times\check{\mathcal{S}}_B(\mathbb{C})$ vanishes identically for any subset A of $\underline{3N}$ with cardinality $3n$.*

Proof: We proceed by induction on n as in the proof of Corollary 15.20, using Lemma 16.36 and Lemma 16.18, which guarantees that $\widetilde{hol}_{.}(\eta_{B,A})$ vanishes along homotopically trivial loops in $\{u\}\times\check{\mathcal{S}}_B(\mathbb{C})$ and along boundaries of squares $[w,w+\varepsilon]\times p_{CS}\circ\gamma([t,t+\varepsilon])$. (This flatness condition can also be proved directly as an exercise, as in Remark 15.22.) □

Lemma 16.33 is proved. □

We now generalize Lemma 16.33 as follows.

Lemma 16.38. *For any two semi-algebraic paths γ and δ of $[0,1]\times\mathcal{S}_B(\mathbb{C})$, which are homotopic relatively to $\{0,1\}$, and which satisfy $\gamma(]0,1[)\subset\check{\mathcal{S}}_B(\mathbb{C})$ and $\delta(]0,1[)\subset\check{\mathcal{S}}_B(\mathbb{C})$, $\widetilde{hol}_{.}(\eta_{B,A})(\gamma)=\widetilde{hol}_{.}(\eta_{B,A})(\delta)$.*

Proof: According to Lemma 16.18, it suffices to take care of homotopies near the endpoints. Thanks to Lemma 16.31, it suffices to prove that there exist $t,t'>0$ and a path ε from $\gamma(t)$ to $\delta(t')$ in $[0,1]\times\check{\mathcal{S}}_B(\mathbb{C})$ such that $\gamma_{|[0,t]}\varepsilon\overline{\delta_{|[0,t']}}$ is null-homotopic and the holonomy along $\gamma_{|[0,t]}\varepsilon\overline{\delta_{|[0,t']}}$ is one. When t and t' are small enough, $\gamma_{|[0,t]}$ and $\delta_{|[0,t']}$ are valued in a subset equipped with a local semi-algebraic chart as in Theorem 8.26, from which it is easy to construct semi-algebraic interpolations in products of sphere pieces and intervals. Furthermore, there is no loss of generality in assuming that $\gamma_{|[0,t]}\varepsilon$ and $\delta_{|[0,t']}$ meet only at $\gamma(0)$ and $\delta(t')$ and that straight interpolation provides a boundary-fixing semi-algebraic homotopy from $\gamma_{|[0,t]}\varepsilon$ to $\delta_{|[0,t']}$, which satisfies the injectivity hypotheses of Lemma 16.33. (Otherwise, we could use an intermediate $\gamma'_{|[0,t]}$.) Thus Lemma 16.33 allows us to prove that $\widetilde{\mathrm{hol}}_{\gamma_{|[0,t]}\varepsilon}(\eta_{B,A}) = \widetilde{\mathrm{hol}}_{\delta_{|[0,t']}}(\eta_{B,A})$. $\square$

Proof of Theorem 16.32: Lemma 16.38 allows us to define a map $\tilde{h}_A$ induced by $\widetilde{\mathrm{hol}}_{.}(\eta_{B,A})$ from homotopy classes of paths with fixed boundaries of $[0,1] \times \mathcal{S}_B(\mathbb{C})$ to $\mathcal{A}_{\frac{\sharp A}{3}}(\sqcup_{b\in B}\mathbb{R}_b)$ as follows. For a path $\gamma\colon [0,1] \to \mathcal{S}_B(\mathbb{C})$, set $\tilde{h}_A(\gamma) = \widetilde{\mathrm{hol}}_{.}(\eta_{B,A})(\delta)$ for any semi-algebraic path δ of $[0,1]\times\mathcal{S}_B(\mathbb{C})$ that is homotopic to γ relatively to $\{0,1\}$, and such that $\delta(]0,1[) \subset \check{\mathcal{S}}_B(\mathbb{C})$.

Now, it suffices to prove that for any semi-algebraic path $\gamma\colon [0,1] \to [0,1] \times \mathcal{S}_B(\mathbb{C})$, $\widetilde{\mathrm{hol}}_\gamma(\eta_{B,A})$ coincides with $\tilde{h}_A(\gamma)$. Recall that a stratum of $[0,1]\times\mathcal{S}_B(\mathbb{C})$ is the product by $[0,1]$ of a stratum of $\mathcal{S}_B(\mathbb{C})$ associated with a Δ-parenthesization. The preimage under γ of such a stratum of $[0,1]\times\mathcal{S}_B(\mathbb{C})$ is semi-algebraic. So γ is a finite composition of paths whose interiors lie in a fixed stratum of $\mathcal{S}_B(\mathbb{C})$ (according to the Lojasiewicz theorem 16.29), and it suffices to prove that $\widetilde{\mathrm{hol}}_\gamma(\eta_{B,A})$ coincides with $\tilde{h}_A(\gamma)$ for any injective semi-algebraic path γ whose interior lies in a fixed stratum of $\mathcal{S}_B(\mathbb{C})$ and which furthermore sits in a subset equipped with a local semi-algebraic chart as in Theorem 8.26, thanks to Lemma 16.31. Such a path can be deformed by sending the vanishing coordinates of $\gamma(t)$ in the $[0,\varepsilon[$ factors in such a chart to $\varepsilon(\frac{1}{2} - |\frac{1}{2} - t|)u$ for $u \in [0,1]$ giving rise to a semi-algebraic homotopy $(\gamma_u(t))_{u\in[0,1]}$ such that $\tilde{h}_A(\gamma) = \widetilde{\mathrm{hol}}_{\gamma_1}(\eta_{B,A})$, and Lemma 16.33 implies that $\widetilde{\mathrm{hol}}_{\gamma_0}(\eta_{B,A}) = \widetilde{\mathrm{hol}}_{\gamma_1}(\eta_{B,A})$. $\square$

Theorem 16.32 allows us to set the following definition.

Definition 16.39. For any continuous path $\gamma\colon [0,1] \to [0,1] \times \mathcal{S}_B(\mathbb{C})$,

$$\widetilde{\mathrm{hol}}_{.}(\eta_{B,A})(\gamma) = \widetilde{\mathrm{hol}}_{.}(\eta_{B,A})(\delta)$$

for any semi-algebraic path δ of $[0,1]\times\mathcal{S}_B(\mathbb{C})$ that is homotopic to γ relatively to $\{0,1\}$.

We may now generalize [Poi00, Proposition 1.18] for braids.

Proposition 16.40. *Let B and C be two finite sets. Let $b_0 \in B$.*

Let γ_B be a path of $[0,1] \times \mathcal{S}_B(\mathbb{C})$, and let γ_C be a path of $[0,1] \times \mathcal{S}_C(\mathbb{C})$. Let $\gamma_B(\gamma_C/b_0) = \gamma_B(\gamma_C/K_{b_0})$ be the q–braid obtained by cabling the strand K_{b_0} of b_0 in $T(\gamma_B)$ by $T(\gamma_C)$, as in Notation 13.3. Then

$$\widetilde{hol}_{\gamma_B(\gamma_C/b_0)}(\eta_{B\left(\frac{C}{b_0}\right),.}) = \left(\pi(C\times b_0)^*(\widetilde{hol}_{\gamma_B}(\eta_{B,.}))\widetilde{hol}_{\gamma_C}(\eta_{C,.})\right)_{\sqcup},$$

where $\pi(C \times b_0)^*$ denotes the duplication of the strand $\mathbb{R}_{b_0}$ at the level of diagrams as in Notation 6.28, and we use the product of Definition 16.8.

Proof: According to Definition 16.39, we assume that γ_C and γ_B are semi-algebraic, without loss of generality. Since a semi-algebraic path is a path composition of finitely many semi-algebraic paths whose interiors lie in a fixed stratum of $[0,1]\times\mathcal{S}_{.}(\mathbb{C})$ (according to the Lojasiewicz theorem 16.29), Lemma 16.30 and the functoriality of Lemma 16.31 allow us to furthermore assume that the image of the interior of γ_C lies in a fixed stratum of $[0,1]\times\mathcal{S}_C(\mathbb{C})$ and that the image of the interior of γ_B lies in a fixed stratum of $[0,1]\times\mathcal{S}_B(\mathbb{C})$, for the proof. (Recall the commutation lemma 6.30.)

Let $B\left(\frac{C}{b_0}\right)$ be the set obtained from B by replacing b_0 by C. We refer to Lemma 15.11 for the description of the stratification of $\mathcal{V}(\Gamma)$ and the dimensions of the fibers for connected diagrams on $B\left(\frac{C}{b_0}\right)\times\mathbb{R}$. When computing the "holonomy" of $\eta_{B\left(\frac{C}{b_0}\right),.}$ we integrate over products of one-parameter families $\mathcal{V}(y,\Gamma)$ with $y\in\gamma_B(\gamma_C/b_0)(]t_i-\varepsilon,t_i+\varepsilon[)$, locally. We may restrict to the strata of $\mathcal{V}(y,\Gamma)$ of dimension $(2\sharp E(\Gamma)-1)$, which are described in Lemma 15.11.

Consider a connected diagram Γ on $B\left(\frac{C}{b_0}\right)\times\mathbb{R}$ together with a Δ-parenthesization $\mathcal{P}$ of its vertices corresponding to such a stratum of configurations. Since $\mathcal{P}=\mathcal{P}_x$, all elements of $\mathcal{P}$ are univalent. Assume $\mathcal{P}\neq\{V(\Gamma)\}$. Let $\Gamma_{\mathcal{P}}$ be obtained from Γ by identifying all the vertices in a daughter A of $V(\Gamma)$ to a single vertex v_A and by erasing the edges between two elements in A, for each A of $D(V(\Gamma))$. Then $\Gamma_{\mathcal{P}}$ is connected, its vertices v_A move along vertical lines. Let $U(\Gamma_{\mathcal{P}})$ and $T(\Gamma_{\mathcal{P}})$ denote the set of univalent vertices of $\Gamma_{\mathcal{P}}$ distinct from the v_A and the set of trivalent vertices of $\Gamma_{\mathcal{P}}$ distinct from the v_A, respectively. The dimension of the one-parameter family of configurations of the vertices of $\Gamma_{\mathcal{P}}$ up to vertical translation is $\sharp D(V(\Gamma))+\sharp U(\Gamma_{\mathcal{P}})+3\sharp T(\Gamma_{\mathcal{P}})$. The form $\bigwedge_{e\in E(\Gamma_{\mathcal{P}})}p^*_{e,S^2}(\tilde{\omega}(j_E(e),S^2))$ factors through this one-parameter family. Thus a count of half-edges shows that unless the vertices v_A are univalent in $\Gamma_{\mathcal{P}}$, the stratum cannot contribute.

Now assume that all the vertices v_A are univalent in $\Gamma_{\mathcal{P}}$, and consider the edge e_A of $\Gamma_{\mathcal{P}}$, with maximal label, that is adjacent to a vertex v_A such that $A\in D(V(\Gamma))$. The subgraph Γ_A of Γ consisting of the vertices of A and the edges of Γ between two such vertices is connected, it has one bivalent vertex b in Γ_A (which is the end of e_A in $\Gamma\cap A$). Its configurations are considered up to vertical translations. Their contribution is opposite to the configurations of the graph Γ'_A obtained from Γ_A by exchanging the labels and possibly the orientations of the two edges of Γ_A that contain b, as in Lemma 9.11.

Thus in the strata that may contribute, the Δ-parenthesization of Γ is $\{V(\Gamma)\}$. If $p_{B\left(\frac{C}{b_0}\right)}(U(\Gamma))\subset C$, Γ is a diagram on $C\times\mathbb{R}$, which contributes as in $\widetilde{\mathrm{hol}}_{\gamma_C}(\eta_{C,.})$. Otherwise, the projection to the horizontal plane of the vertices of $U(\Gamma)\cap p^{-1}_{B\left(\frac{C}{b_0}\right)}(C)$ is reduced to a point. So all diagrams obtained from these diagrams Γ by changing the map from $U(\Gamma)\cap p^{-1}_{B\left(\frac{C}{b_0}\right)}(C)$ to C arbitrarily, contribute together to $\pi(C\times b_0)^*(\widetilde{\mathrm{hol}}_{\gamma_B}(\eta_{B,.}))$, as wanted, locally. Now, since the two kinds of diagrams commute, thanks to Lemma 6.30, we get the proposition. □

Definition 16.41. Recall Proposition 16.17 and Definition 16.39. For a q–braid (representative) $\gamma\colon [0,1]\to \mathcal{S}_B(\mathbb{C})$, set

$$\mathcal{Z}^f(\gamma,.,\left(\tilde{\omega}(i,1,S^2)\right)_{i\in\underline{3N}}) = \widetilde{\mathrm{hol}}_{\{1\}\times\gamma}(\eta_{B,.}).$$

For a q–tangle

$$T = T(\gamma^-)(\mathcal{C},L,L_{\|})T(\gamma^+),$$

such that γ^- and γ^+ are q-braids, and $(\mathcal{C},L)$ is a $J_{bb,tt}$-oriented framed tangle whose bottom configuration is $\gamma^-(1)$ and whose top configuration is $\gamma^+(0)$, as in Definition 13.1, for $N\in\mathbb{N}$, and for a family $(\omega(i,S^2))_{i\in\underline{3N}}$ of volume-one forms of S^2, set

$$\mathcal{Z}^f(T,.,\left(\omega(i,S^2)\right)_{i\in\underline{3N}}) =$$
$$\left(\mathcal{Z}^f\left(\gamma^-,.,\left(\omega(i,S^2)\right)\right)\mathcal{Z}^f\left(\mathcal{C},L,L_{\|},.,\left(\omega(i,S^2)\right)\right)\mathcal{Z}^f\left(\gamma^+,.,\left(\omega(i,S^2)\right)\right)\right)_{\sqcup},$$

with the notation of Definition 16.8 and Definition 16.14. Lemma 16.31 and the isotopy invariance of Theorems 16.13 and 16.32 ensure that this definition is consistent.

Theorem 16.7 allows us to express[6] the variation of

$$\mathcal{Z}^f(T,.,(\omega(i,S^2))_{i\in\underline{3N}})$$

when $(\omega(i,S^2))_{i\in\underline{3N}}$ varies, for framed straight tangles with injective top and bottom configurations. As a corollary of Theorem 16.32, this expression generalizes to q–tangles. We get the following theorem.

Theorem 16.42. *Let $N\in\mathbb{N}$. For $i\in\underline{3N}$, let $\tilde{\omega}(i,S^2)=(\tilde{\omega}(i,t,S^2))_{t\in[0,1]}$ be a closed 2–form on $[0,1]\times S^2$ such that $\int_{\{0\}\times S^2}\tilde{\omega}(i,0,S^2)=1$. Let T denote a $J_{bb,tt}$-oriented q–tangle. Assume that the bottom (resp. top) configuration of T is an element y^- of $\mathcal{S}_{B^-}(\mathbb{C})$, (resp. y^+ of $\mathcal{S}_{B^+}(\mathbb{C})$).*

Then

$$\mathcal{Z}^f(T,.,(\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}}) =$$

[6] See also the proof of Theorem 16.13.

$$\left(\begin{array}{c}\left(\prod_{K_j\in J_{bb}}\widetilde{hol}_{[0,1]}(-\eta(.,S_{WE}))\sharp_j\right)\left(\prod_{K_j\in J_{tt}}\widetilde{hol}_{[0,1]}(\eta(.,S_{WE}))\sharp_j\right)\\ \widetilde{hol}_{[1,0]\times y^-}(\eta_{B^-,.})\mathcal{Z}^f(T,.,(\tilde{\omega}(i,0,S^2))\widetilde{hol}_{[0,1]\times y^+}(\eta_{B^+,.})\end{array}\right)_{\sqcup}.$$

□

Recall that the forms η and their holonomies, which are introduced in Definitions 16.1 and 16.5, depend on the forms $\tilde{\omega}(i,S^2)$. In the end of this chapter, we will apply Theorem 16.42, only when $\tilde{\omega}(i,0,S^2)=\omega_{S^2}$. So $\mathcal{Z}^f(T,.,(\tilde{\omega}(i,0,S^2))$ is simply $\mathcal{Z}^f(T,.)$.

The following lemma is also easy to prove.

Lemma 16.43. *The behaviour of $\mathcal{Z}^f$ under reversing the orientation of a closed component is the same as that decribed as in Proposition 10.21. Similarly, if K is an oriented component that goes from bottom to top or from top to bottom in a q–tangle T, for a Jacobi diagram Γ on the source $\mathcal{L}$ of T, let $U_K(\Gamma)$ denote the set of univalent vertices of Γ mapped to the source $\mathbb{R}_K$ of K. This set is ordered by the orientation of $\mathbb{R}_K$. When the orientation of K is changed, $\mathcal{Z}(T)$ is modified by reversing the orientation of $\mathbb{R}_K$ (that is reversing the order of $U_K(\Gamma)$) in classes $[\Gamma]$ of diagrams Γ on $\mathcal{L}$, and multiplying these classes by $(-1)^{\sharp U_K(\Gamma)}$ in $\mathcal{A}(\mathcal{L})$, simultaneously.*

In other words, we can forget the orientation of closed components and of components that go from bottom to top or from top to bottom, and regard $Z(L)$ as valued in spaces of diagrams where the sources of these components are not oriented, as in Definitions 6.13 and 6.16. But we a priori[7] *need a $J_{bb,tt}$-orientation.*

□

Lemma 16.44. *Let A be a set of cardinality $3k$. Recall Definition 16.5. The form $\eta(A,S_{WE}\cup(-S_{EW}))$ is zero when k is odd, and $\eta(A,S_{WE}\cup(-S_{EW}))=-s_*(\eta(A,S_{WE}\cup(-S_{EW})))$ if k is even*[8]*. If $\eta(A,S_{WE}\cup(-S_{EW}))$ is zero, then $\mathcal{Z}^f(.,.,(\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}})$ is independent of the $J_{bb,tt}$-orientation of the tangles (as in Proposition 10.21).*

Proof: The circle $S_{WE}\cup(-S_{EW})$ is the great circle ∂D of S^2 that is the boundary of the hemisphere D of S^2 centered at $(-i)$. By Definition 16.5, $\eta(A,\partial D)(t)=\int_{\{t\}\times\partial D}\omega(A)$, where

$$\int_{[0,t]\times\partial D}\omega(A)=\int_{(\partial[0,t])\times D}\omega(A)=\int_{\{t\}\times D}\omega(A)-\int_{\{0\}\times D}\omega(A),$$

[7]See Lemma 16.44 below.

[8]In particular, $\eta(A,S_{WE}\cup(-S_{EW}))$ is also zero when k is even, if s_* is the identity map. But this is unknown to me.

and

$$\int_{\{t\}\times S^2} \omega(A) = \int_{\{t\}\times D} (\omega(A) - \iota^*(\omega(A)) = (1+(-1)^{k+1}) \int_{\{t\}\times D} \omega(A),$$

according to Lemma 16.4. In particular, when k is odd, $\int_{\{t\}\times D} \omega(A) = \frac{1}{2}\int_{\{t\}\times S^2} \omega(A)$ is independent of t, and $\eta(A, \partial D) = 0$. When k is even, Lemma 16.4 implies that $\omega(A) = -s_*(\omega(A))$.

Changing the orientation of a component K of J_{bb} amounts to construct the invariant by imposing the condition $p_\tau(U^+K) \subseteq S_{EW}$ rather than $p_\tau(U^+K) \subseteq S_{WE}$. So this replaces the factor $\widetilde{\mathrm{hol}}_{[0,1]}(-\eta(., S_{WE}))$ associated to K with $\widetilde{\mathrm{hol}}_{[0,1]}(-\eta(., S_{EW}))$ in the formula of Theorem 16.42. See Theorem 16.13 and Definition 16.14. This amounts to multiply by $\widetilde{\mathrm{hol}}_{[0,1]}(\eta(., S_{WE} \cup -S_{EW}))$ on the component of K. $\square$

Remarks 16.45. Similarly, if we had imposed that p_τ maps the unit tangent vectors to components of $J_{bb} \cup J_{tt}$ to the vertical half great circle $S(\theta)$ from $-\vec{N}$ to $\vec{N}$ that contains the complex direction $\exp(2i\pi\theta)$, for $\theta \in]0,1[$, in our definition of straight tangles in Section 16.3, then $S(\theta)$ would replace S_{WE} in the formula of Theorem 16.42, and $\mathcal{Z}^f(T, ., (\tilde{\omega}(i,1,S^2))_{i\in \underline{3N}})$ would have been multiplied by

$$\left(\prod_{K_j\in J_{bb}} \widetilde{\mathrm{hol}}_{[0,1]}\eta(., S_{WE} \cup (-S(\theta)))\sharp_j\right)\left(\prod_{K_j\in J_{tt}} \widetilde{\mathrm{hol}}_{[0,1]}\eta(., S(\theta) \cup (-S_{WE}))\sharp_j\right)$$

In particular, when T is a tangle with only one component that goes from bottom to bottom, $\mathcal{Z}^f(T, ., (\tilde{\omega}(i,1,S^2))_{i\in \underline{3N}})$ would have been multiplied by $\widetilde{\mathrm{hol}}_{[0,1]}\eta(., S_{WE} \cup (-S(\theta)))$.

With the notation of Definition 16.5 and Lemma 16.4,

$$\widetilde{\mathrm{hol}}_{[0,1]}\eta(\underline{3}, S_{WE} \cup (-S(\theta))) = \int_{[0,1]\times(S_{WE}\cup(-S(\theta)))} \omega(\underline{3})$$

$$= \frac{1}{6}\sum_{i=1}^{3} \int_{[0,1]\times(S_{WE}\cup(-S(\theta)))} (\tilde{\omega}(i,S^2) - \iota^*\tilde{\omega}(i,S^2))\,[\,\text{\Large $\circlearrowleft$}\,]$$

would have been added to $\mathcal{Z}^f(T, \underline{3}, (\tilde{\omega}(i,1,S^2))_{i\in\underline{3}})$. Let D be a chain of S^2 bounded by $(S_{WE} \cup (-S(\theta)))$, then

$$\int_{[0,1]\times(S_{WE}\cup(-S(\theta)))} \tilde{\omega}(i,S^2) = \int_{(\partial[0,1])\times D} \tilde{\omega}(i,S^2).$$

In particular, if $\theta \in]0, \frac{1}{2}[$, we can choose D and $\tilde{\omega}(i,1,S^2)$ so that $\tilde{\omega}(i,1,S^2) - \iota^*\tilde{\omega}(i,1,S^2)$ is supported outside $D \cup \iota_{S^2}(D)$ for any i. In this case,

$$\widetilde{\mathrm{hol}}_{[0,1]}(\eta(\underline{3}, S_{WE} \cup (-S(\theta)))) = -\int_{\{0\}\times D} \omega_{S^2} = \theta.$$

Thus, as claimed in Remark 16.15, Definition 16.14 is not canonical.

The above calculation does not rule out the alternative choice of the vertical half great circle $S(\frac{1}{2}) = S_{EW}$ for our definition. This choice would multiply $\mathcal{Z}^f(T,.,(\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}})$ by $\widetilde{\mathrm{hol}}_{[0,1]}(\eta(.,S_{WE} \cup (-S_{EW})))$, which is zero in degree 1, according to Lemma 16.44.

It might be tempting to modify the definition of $\mathcal{Z}^f(T,.,(\tilde{\omega}(i,1,S^2))_{i\in\underline{3N}})$, by multiplying it by

$$\left(\prod_{K_j\in J_{bb}} \widetilde{\mathrm{hol}}_{[0,1]}(\eta(.,S_{WE}))\sharp_j\right)\left(\prod_{K_j\in J_{tt}} \widetilde{\mathrm{hol}}_{[0,1]}(-\eta(.,S_{WE}))\sharp_j\right).$$

Unfortunately, $\widetilde{\mathrm{hol}}_{[0,1]}(\eta(.,S_{WE}))$ depends on the closed 2-forms $\tilde{\omega}(i,S^2)$ of $[0,1]\times S^2$, and not only on the $\tilde{\omega}(i,1,S^2)$. Indeed, assume that all $\tilde{\omega}(i,S^2)$ coincide with each other and change all of them by adding $d\eta_S$, for a one-form η_S of $[0,1]\times S^2$ supported on the product of $[1/4,3/4]$ by a small neighborhood of $\vec{N}$. Then the variation of the degree one part of $2\widetilde{\mathrm{hol}}_{[0,1]}(\eta(.,S_{WE}))$ maps $\underline{3}$ to $\int_{[0,1]\times S_{WE}} d\eta_S - \iota^*(d\eta_S)$, where

$$\int_{[0,1]\times S_{WE}} d\eta_S = \int_{\partial([0,1]\times S_{WE})} \eta_S = -\int_{[0,1]\times\{\vec{N}\}} \eta_S$$

and

$$\begin{aligned}\int_{[0,1]\times S_{WE}} -\iota^*(d\eta_S) = -\int_{[0,1]\times \iota_{S^2}(S_{WE})} d\eta_S &= \int_{\partial([0,1]\times -\iota_{S^2}(S_{WE}))} \eta_S \\ &= -\int_{[0,1]\times\{\vec{N}\}} \eta_S.\end{aligned}$$

(In Theorem 16.7, the factors $\widetilde{\mathrm{hol}}_{[t,0]\times y^-}(\eta_{B^-,.})$ and $\widetilde{\mathrm{hol}}_{[0,t]\times y^+}(\eta_{B^+,.})$ also depend on $\tilde{\omega}(i,S^2)$, and both types of dependences cancel each other.)

Chapter 17

Justifying the properties of $\mathcal{Z}^f$

Recall Definition 13.10 of the invariant $\mathcal{Z}^f$. So far we have succeeded in constructing this invariant $\mathcal{Z}^f$ of q–tangles, invariant under boundary-fixing diffeomorphisms, which generalizes both the invariant $\mathcal{Z}^f$ for framed links in $\mathbb{Q}$-spheres and the Poirier functor Z^l for q–tangles in $\mathbb{R}^3$. The framing dependence of Theorem 13.12 comes from Definition 12.12.

The behaviour of $\mathcal{Z}$ and $\mathcal{Z}^f$ under orientation changes of the components described in the statement of Theorem 13.12 can be justified as in the case of links in rational homology spheres treated in Section 10.6.

Lemma 17.1. *$\mathcal{Z}^f$ behaves as prescribed by Theorem 13.12 under diffeomorphisms $s_{\frac{1}{2}}$ and ρ as in Theorem 13.12.*

PROOF: Let L be a tangle representative as in Theorem 12.7. We proceed as in the proof of Proposition 10.1, except that we need to take care of the facts that, for $\psi = s_{\frac{1}{2}}$ or $\psi = \rho$, $\psi_*(\tau)$ is not asymptotically standard and that $s_{\frac{1}{2}}$ reverses the orientation. Therefore, we use $\tau' = \psi_*(\tau) \circ (\mathbf{1}_{\check{R}(\mathcal{C})} \times \psi_{\mathbb{R}^3}^{-1}) = T\psi \circ \tau \circ (\psi^{-1} \times \psi_{\mathbb{R}^3}^{-1})$, as an asymptotically standard parallelization of $R = R(\mathcal{C})$. So $(\rho_*^{-1})^* p_\tau^*(\omega_{S^2}) = p_{\tau'}^*(\omega_{S^2})$ and $(\sigma_{\frac{1}{2}*}^{-1})^* p_\tau^*(\omega_{S^2}) = -p_{\tau'}^*(\omega_{S^2})$. If ω is a homogeneous propagating form of $(C_2(R), \tau)$, then $(\rho_*^{-1})^*(\omega)$ is a homogeneous propagating form of $(C_2(\rho(R)), \tau')$ and $(-\sigma_{\frac{1}{2}*}^{-1})^*(\omega)$ is a homogeneous propagating form of $(C_2(s_{\frac{1}{2}}(R)), \tau')$. Let us now focus on the case $\psi = s_{\frac{1}{2}}$, since the case $\psi = \rho$ is similar, but simpler. For any Jacobi diagram Γ on the source of L, equipped with an implicit orientation $o(\Gamma)$, compute

$I = I\left(s_{\frac{1}{2}}(R), s_{\frac{1}{2}}(L), \Gamma, (-\sigma_{\frac{1}{2}*}^{-1})^*(\omega)\right)$.

$$\begin{aligned} I &= \int_{\check{C}(s_{\frac{1}{2}}(R),s_{\frac{1}{2}}(L);\Gamma)} \bigwedge_{e\in E(\Gamma)} p_e^*(-(\sigma_{\frac{1}{2}*}^{-1})^*(\omega)) \\ &= (-1)^{\sharp E(\Gamma)} \int_{\check{C}(s_{\frac{1}{2}}(R),s_{\frac{1}{2}}(L);\Gamma)} (\sigma_{\frac{1}{2}*}^{-1})^* \left(\bigwedge_{e\in E(\Gamma)} p_e^*(\omega)\right) \\ &= (-1)^{\sharp E(\Gamma)+\sharp T(\Gamma)} I\left(R, L, \Gamma, \omega\right). \end{aligned}$$

Therefore, for all $n \in \mathbb{N}$, $Z_n(s_{\frac{1}{2}}(\mathcal{C}), s_{\frac{1}{2}}(L), \tau') = (-1)^n Z_n(\mathcal{C}, L, \tau)$. In particular, Corollary 4.9, Propositions 5.15 and 7.17 imply that $p_1(\tau') = -p_1(\tau)$, and for any component K_j of L, $I_\theta(K_j, \tau') = -I_\theta(K_j, \tau)$. So, for all $n \in \mathbb{N}$, $Z_n(s_{\frac{1}{2}}(\mathcal{C}), s_{\frac{1}{2}}(L)) = (-1)^n Z_n(\mathcal{C}, L)$, since the anomalies α and β vanish in even degrees, thanks to Propositions 10.7 and 10.13. Now, recall Proposition 16.10 and note that if a component K is straight with respect to τ, then $s_{\frac{1}{2}}(K)$ is straight with respect to τ'. In particular, the condition

$$lk_{s_{\frac{1}{2}}(\mathcal{C})}\left(s_{\frac{1}{2}}(K), s_{\frac{1}{2}}(K)_{\parallel}\right) = -lk_{\mathcal{C}}\left(K, K_{\parallel}\right)$$

is realizable and natural, and we get the wanted equality for a framed tangle from an injective bottom configuration to an injective top configuration. Thanks to Remark 13.11, it is still true for a q-tangle. □

We are thus left with the proofs of the functoriality, the duplication properties and the cabling property, to finish the proof of Theorem 13.12. These proofs will occupy four sections of this chapter, which will end with a section that describes other properties of $\mathcal{Z}^f$. The corresponding properties of variants of $\mathcal{Z}$ and $\mathcal{Z}^f$ involving non-homogeneous propagating forms will be treated simultaneously, since they are often easier to prove, and since we are going to use them, to prove some of the results for homogeneous propagating forms.

17.1 Transversality and rationality

In this section, we generalize the rationality results of Chapter 11 to the tangle case. The generalization will be useful in the proofs of the properties later.

Let S^2_H denote the subset of S^2 consisting of the vectors whose vertical coordinate is in $]-\frac{1}{2}, \frac{1}{2}[$.

Proposition 17.2. *Let $(\mathcal{C}, \tau)$ be a parallelized rational homology cylinder. Let $L\colon \mathcal{L} \hookrightarrow \mathcal{C}$ be a long tangle of $\check{R}(\mathcal{C})$. Let $N \in \mathbb{N}$. $N \geq 2$. There exist $(X_1, X_2, \dots, X_{3N}) \in (S^2_H)^{3N}$, $M \in \mathbb{R}^+$ and propagating chains $P(i)$ of $(C_2(\check{R}(\mathcal{C})), \tau)$ such that*

- *$P(i)$ intersects the domain $D(p_\tau)$ of Notation 16.11 as $p_\tau^{-1}(X_i)$,*
- *the $P(i) \cap C_2(R_{M,[-M,M]}(\mathcal{C}))$ are in general $3N$-position with respect to L, with the natural generalization of the notion of Definition 11.3 (where $R_{M,[-M,M]}(\mathcal{C})$ replaces R, with Notation 16.11),*
- *the intersections*

$$I_S(\Gamma, (P(i))_{i\in\underline{3N}}) = \bigcap_{e\in E(\Gamma)} p_e^{-1}(P(j_E(e)))$$

in $C(R(\mathcal{C}), L; \Gamma)$ are transverse and located in $C(R_{M,[-M,M]}(\mathcal{C}), L; \Gamma)$, for any $\Gamma \in \mathcal{D}^e_{\underline{3N}}(\mathcal{L}) = \cup_{k\in\mathbb{N}} \mathcal{D}^e_{k,\underline{3N}}(\mathcal{L}) = \cup_{k=0}^{3N} \mathcal{D}^e_{k,\underline{3N}}(\mathcal{L})$,

- *for any $\alpha > 0$, there exists $\beta > 0$, such that, for any $\Gamma \in \mathcal{D}^e_{\underline{3N}}(\mathcal{L})$,*

$$\bigcap_{e\in E(\Gamma)} p_e^{-1}(N_\beta(P(j_E(e)))) \subset N_\alpha(I_S(\Gamma, (P(i))_{i\in\underline{3N}})),$$

(N_α is defined before Definition 11.6.)

- *there exists an open ball B_X around $(X_1, X_2, \dots, X_{3N})$ in $(S^2)^{\underline{3N}}$ such that $B_X \subset (S^2_H)^{\underline{3N}}$ and for any $(Y_1, Y_2, \dots, Y_{3N}) \in B_X$, there exist propagating chains $P(i)(Y_i)$ of $(C_2(\check{R}(\mathcal{C})), \tau)$ that satisfy all the above conditions with respect to Y_i with the same M.*

The set of $(X_1, X_2, \dots, X_{3N}) \in (S^2_H)^{3N}$ such that there exist

$$M(X_1, X_2, \dots, X_{3N}) \in \mathbb{R}^+$$

and propagating chains $P(i)$ of $(C_2(\check{R}(\mathcal{C})), \tau)$ that satisfy the above conditions is dense in $(S^2_H)^{3N}$.

In order to prove the proposition, we begin by producing some

$$(W_1, W_2, \dots, W_{3N}) \in (S^2_H)^{3N}$$

(in a given neighborhood of some $(W_1^0, W_2^0, \dots, W_{3N}^0)$ in $(S^2_H)^{3N}$) and

$$M(W_1, W_2, \dots, W_{3N}) \in \mathbb{R}^+.$$

For a 1-manifold $\mathcal{L}$ and a finite set A, $\mathcal{D}^c_{k,A}(\mathcal{L})$ denotes the set of connected A-numbered degree k Jacobi diagrams with support $\mathcal{L}$ without looped edges.

Lemma 17.3. *Let $N \in \mathbb{N}$. $N \geq 2$. Let $y \colon B \hookrightarrow D_1$ be a planar configuration. For a $\underline{3N}$-numbered Jacobi diagram Γ on $\sqcup_{b\in B}\mathbb{R}_b$, define the semi-algebraic map*

$$g(\Gamma)\colon C(S^3, y(B)\times\mathbb{R};\Gamma)\times (S^2)^{\underline{3N}\setminus j_E(E(\Gamma))} \to (S^2)^{\underline{3N}}$$

as the product $\prod_{e\in E(\Gamma)} p_{e,S^2} \times \mathbf{1}\left((S^2)^{\underline{3N}\setminus j_E(E(\Gamma))}\right)$.

The subset $\mathcal{O}((S^2_H)^{3N}, y)$ of $(S^2_H)^{3N}$ of points that are in the complement of the images of the maps $g(\Gamma)$ for all $\Gamma \in \cup_{k=1}^{3N}\mathcal{D}^c_{k,\underline{3N}}(\sqcup_{b\in B}\mathbb{R}_b)$ is dense and open. [1]

Proof: It suffices to prove that the complement of the image of the map $g(\Gamma)$ is open and dense for any of the finitely many graphs $\Gamma \in \cup_{k=1}^{3N}\mathcal{D}^c_{k,\underline{3N}}(\mathbb{R})$. The dimension of $C(S^3, y(B)\times\mathbb{R};\Gamma)$ is the same as the dimension of $(S^2)^{j_E(E(\Gamma))}$. The quotient of $\check{C}(S^3, y(B)\times\mathbb{R};\Gamma)$ by global vertical translations is also a semi-algebraic set with dimension one less. Thus the image of $g(\Gamma)$ is a compact semi-algebraic subset of $(S^2_H)^{3N}$ of codimension at least one. Its complement is thus an open dense semi-algebraic subset of $(S^2_H)^{3N}$. $\square$

Lemma 17.4. *Under the assumptions of Lemma 17.3, let $L\colon \mathcal{L}\hookrightarrow\mathcal{C}$ be a long tangle of $\check{R}(\mathcal{C})$ whose bottom and top planar configurations are subconfigurations (i.e. restrictions) of $y\colon B\hookrightarrow D_1$, where y maps some point of B to 0. Let $B(W_1, W_2, \dots, W_{3N})$ be a ball centered at $(W_1, W_2, \dots, W_{3N})$ of radius $24N\varepsilon^{\frac{1}{12N}}$ of $(S^2_H)^{3N}$ (equipped with the distance coming from the Euclidean norm of $(\mathbb{R}^3)^{\underline{3N}}$) which sits in the subset $\mathcal{O}((S^2_H)^{3N}, y)$ of Lemma 17.3, where $\varepsilon \in]0, \frac{1}{20^{12N}}[$. Let $B(W_i,\varepsilon)$ be the ball of radius ε in S^2, centered at W_i. For any $Y_i \in B(W_i,\varepsilon)$, let $P(Y_i)$ be a propagating chain of $(C_2(R(\mathcal{C})),\tau)$ that coincides with $p_\tau^{-1}(Y_i)$ on the domain $D(p_\tau)$ of Notation 16.11. Then for any $\Gamma\in\cup_{k=1}^{3N}\mathcal{D}^c_{k,\underline{3N}}(\mathcal{L})$,*

$$\bigcap_{e\in E(\Gamma)} p_e^{-1}(\cup_{Y_{j_E(e)}\in B(W_{j_E(e)},\varepsilon)} P(Y_{j_E(e)})) \subset C(R_{\frac{1}{\sqrt{\varepsilon}},[-\frac{1}{\sqrt{\varepsilon}},\frac{1}{\sqrt{\varepsilon}}]}(\mathcal{C}), L;\Gamma),$$

and the support of $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$ is included in $C(R_{\frac{1}{\sqrt{\varepsilon}},[-\frac{1}{\sqrt{\varepsilon}},\frac{1}{\sqrt{\varepsilon}}]}(\mathcal{C}), L;\Gamma)$, for any family $(\omega_i)_{i\in\underline{3N}}$ of closed propagating forms ω_i of $(C_2(R(\mathcal{C})),\tau)$ that restrict to $D(p_\tau)$ as $p_\tau^*(\omega_{i,S^2})$, for 2-forms ω_{i,S^2} supported in $B(W_i,\varepsilon)$.

Proof: Fix a connected $\underline{3N}$-numbered Jacobi diagram Γ on the source $\mathcal{L}$ of L. Note that Γ has at most $6N$ vertices. Define a sequence $\alpha_1,\dots,\alpha_{6N}$ by

$$\alpha_k = \varepsilon^{\frac{1-k}{12N}}.$$

[1] Though we are going to study $\mathcal{Z}^f$ up to degree N, higher degree diagrams will occur in our proofs. See the proof of Lemma 17.4.

Since $\varepsilon < \frac{1}{20^{12N}}$, $\varepsilon^{-\frac{1}{12N}} > 20$. So $\alpha_2 > 20$.

Define an open covering of $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$ associated with colorings of the vertices by colors blue and k, with $k \in \underline{6N}$, such that

- blue vertices and vertices of color $k \geq 2$ do not belong to $R_{2,[-1,2]}(\mathcal{C})$, with Notation 16.11,
- vertices of color 1 belong to $\mathring{R}_{3,[-2,3]}(\mathcal{C})$,
- any vertex of color 2 is connected by an edge of Γ to a vertex of color 1, and is at a distance (with respect to the Euclidean norm of $\mathbb{R}^3$) smaller than $5\alpha_2$ from $(0,0) \in \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$,
- for $k \geq 2$, $k \leq 6N - 1$, any vertex of color $(k+1)$ is connected to a vertex of color k by an edge of length[2] smaller than $5\alpha_{k+1}$,
- when there is an edge of Γ between a blue vertex and a vertex colored by 1, the distance between the blue vertex and $(0,0)$ is greater than $3\alpha_2$ (with respect to the Euclidean distance of $\mathbb{R}^3$),
- when there is an edge of Γ between a blue vertex and a vertex colored by k for $2 \leq k \leq 6N - 1$, the distance between the two vertices is greater than $3\alpha_{k+1}$. (Since Γ has at most $6N$ vertices, if there is a blue vertex, no vertex can be colored by $6N$.)

The subset $U(\mathbf{c})$ of $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$, consisting of the configurations that satisfy the above conditions with respect to a coloring $\mathbf{c}$ of the vertices, is open. Let us prove that $\check{C}(\check{R}(\mathcal{C}), L; \Gamma)$ is covered by these sets. For a configuration c, color its vertices that are in $\mathring{R}_{3,[-2,3]}(\mathcal{C})$ with 1. Then color all the still uncolored vertices v that are connected (by an edge of Γ) to a vertex of color 1 and such that $d(v,(0,0)) < 5\alpha_2$ with 2. Continue by coloring all the possible uncolored vertices that are connected to a vertex of color 2 by an edge of length smaller than $5\alpha_3$ by 3, and so on, in order to end up with a coloring, which obviously satisfies the above conditions, by coloring blue the uncolored vertices.

Note that the distance between a vertex colored by $k \geq 2$ and the point $(0,0) \in \mathbb{R}^3$ is less than

$$5\sum_{i=2}^{k} \alpha_i = 5\frac{\alpha_{k+1} - \alpha_2}{\varepsilon^{-\frac{1}{12N}} - 1} < 5\frac{\alpha_{k+1}}{\frac{5}{6}\varepsilon^{-\frac{1}{12N}}} \leq 6\alpha_k \leq 6\alpha_{6N},$$

[2]This edge length makes sense since vertices of color $k \geq 2$ belong to $R^c_{2,[-1,2]}(\mathcal{C}) \subset \mathbb{R}^3$.

where, since $\varepsilon^{-\frac{1}{12N}} > 20$,

$$6\alpha_{6N} = 6(\varepsilon^{-\frac{1}{12N}})^{6N-1} \leq (\varepsilon^{-\frac{1}{12N}})^{6N} = \frac{1}{\sqrt{\varepsilon}}.$$

Thus the vertices colored by some k are in $R_{\frac{1}{\sqrt{\varepsilon}},[-\frac{1}{\sqrt{\varepsilon}},\frac{1}{\sqrt{\varepsilon}}]}(\mathcal{C})$.

Let us prove that an open set $U(\mathbf{c})$, associated to a coloring $\mathbf{c}$ for which the color blue appears, cannot intersect $\bigcap_{e\in E(\Gamma)} p_e^{-1}(\cup_{Y_{j_E(e)}\in B(W_{j_E(e)},\varepsilon)} P(Y_{j_E(e)}))$. Fix such a coloring $\mathbf{c}$ and remove from Γ all the edges that do not contain a blue vertex (without removing their ends). Let Γ_b be a connected component with at least one blue vertex of the obtained graph. It has blue vertices, which are trivalent or univalent (in Γ and Γ_b), the blue univalent vertices lie on $y\times\mathbb{R}$. Color its other vertices colored by some k red. Red vertices may have 1, 2, or 3 adjacent edges in Γ_b. Let Γ_b' be the uni-trivalent graph obtained by blowing up Γ_b at its red vertices by replacing such a vertex by a red univalent vertex for each adjacent edge. Color the edges between blue vertices blue, and the edges between a blue vertex and a red one purple. With a configuration of $U(\mathbf{c})$ in $\bigcap_{e\in E(\Gamma)} p_e^{-1}(\cup_{Y_{j_E(e)}\in B(W_{j_E(e)},\varepsilon)} P(Y_{j_E(e)}))$ associate the configuration of Γ_b' obtained by sending all the red vertices to $\mathbf{o}=(0,0)$, leaving the positions of the blue vertices unchanged. Thus

- the direction of a blue edge numbered by i is at a distance less than ε from W_i,
- the direction of a purple edge numbered by p is at a distance less than $(\varepsilon + 4\varepsilon^{\frac{1}{12N}})$ from W_p.

Let us justify the second assertion. Let $\mathbf{b}$ denote the blue vertex of the purple edge numbered by p and let $\mathbf{r}$ be its red vertex in the configured graph Γ. Assume that $\mathbf{r}$ is colored by k with $k\geq 2$. Then

$$d(\mathbf{b},\mathbf{r}) > 3\alpha_{k+1},\ (\mathbf{o},\mathbf{b})\in D(p_\tau),\ (\mathbf{r},\mathbf{b})\in D(p_\tau) \text{ and } d(\mathbf{o},\mathbf{r}) < 6\alpha_k.$$

Since the configuration consisting of $\mathbf{b}$ and $\mathbf{r}$ in Γ is in some $P(Y_p)$,

$$\left|\pm\frac{1}{\parallel \overrightarrow{\mathbf{rb}} \parallel}\overrightarrow{\mathbf{rb}} - W_p\right| \leq \varepsilon.$$

Note the following easy sublemma.

Sublemma 17.5. *Let a and h denote two vectors of $\mathbb{R}^n$ such that a and $a+h$ are different from 0. Then*

$$\left|\left|\frac{1}{\parallel a+h \parallel}(a+h) - \frac{1}{\parallel a \parallel}a\right|\right| \leq 2\frac{\parallel h \parallel}{\parallel a \parallel}.$$

Proof: The left-hand side can be written as $\left\|\frac{1}{\|a\|}h+\left(\frac{1}{\|a+h\|}-\frac{1}{\|a\|}\right)(a+h)\right\|$ so that it is less or equal than $\left(\frac{\|h\|}{\|a\|}+\frac{|\|a\|-\|a+h\||}{\|a\|}\right)$.

$\square$

Apply this sublemma with $a=\overrightarrow{\mathbf{rb}}$ and $h=\overrightarrow{\mathbf{or}}$ and get

$$\left\|\frac{1}{\|\overrightarrow{\mathbf{ob}}\|}\overrightarrow{\mathbf{ob}}-\frac{1}{\|\overrightarrow{\mathbf{rb}}\|}\overrightarrow{\mathbf{rb}}\right\|\leq\frac{2\|\overrightarrow{\mathbf{or}}\|}{\|\overrightarrow{\mathbf{rb}}\|}<\frac{4\alpha_k}{\alpha_{k+1}}\leq 4\varepsilon^{\frac{1}{12N}}.$$

If $\mathbf{r}$ is colored by 1, then $d(\mathbf{b},\mathbf{o})>3\alpha_2$. When the configuration consisting of $\mathbf{b}$ and $\mathbf{r}$ in Γ is in some $P(Y_p)$, there exists an $\mathbf{s}$ in $D_3\times[-2,3]$ such that the direction of $\pm\overrightarrow{\mathbf{sb}}$ is at a distance less than ε from W_p. Here, $d(\mathbf{o},\mathbf{s})<5\alpha_1$, so the direction of $\pm\overrightarrow{\mathbf{ob}}$ is still at a distance less than $(\varepsilon+4\varepsilon^{\frac{1}{12N}})$ from W_p. Indeed Sublemma 17.5 applied with $a=\overrightarrow{\mathbf{ob}}$ and $h=\overrightarrow{\mathbf{so}}$ yields

$$\left\|\frac{1}{\|\overrightarrow{\mathbf{ob}}\|}\overrightarrow{\mathbf{ob}}-\frac{1}{\|\overrightarrow{\mathbf{sb}}\|}\overrightarrow{\mathbf{sb}}\right\|\leq\frac{2\|\overrightarrow{\mathbf{os}}\|}{\|\overrightarrow{\mathbf{ob}}\|}\leq\frac{10\alpha_1}{3\alpha_2}<4\varepsilon^{\frac{1}{12N}}.$$

Therefore the directions of the edges numbered by i of the configured graph Γ'_b are at a distance less than $(\varepsilon+4\varepsilon^{\frac{1}{12N}}\leq 8\varepsilon^{\frac{1}{12N}})$ from the W_i. But the directions of these edges cannot be in the image of $\prod_{e\in E(\Gamma'_b)}p_{e,S^2}$ according to our conditions. Indeed, together with $(W_i)_{i\in\underline{3N}\setminus j_E(\Gamma'_b)}$, they form a $3N$-tuple that is at a distance less than $3N\times(8\varepsilon^{\frac{1}{12N}})$ from $(W_i)_{i\in\underline{3N}}$.

Therefore $\bigcap_{e\in E(\Gamma)}p_e^{-1}(\cup_{Y_{j_E(e)}\in B(W_{j_E(e)},\varepsilon)}P(Y_{j_E(e)}))$ does not intersect the open subsets of the coverings that use the blue color.

It is now easy to conclude. $\square$

Proof of Proposition 17.2: Fix an ε as in Lemma 17.4. For a diagram Γ of $\mathcal{D}^e_{\underline{3N}}(\mathcal{L})$ and for a subset E of $E(\Gamma)$, the map

$$q(\Gamma,E)=\prod_{e\in E}p_\tau\circ p_e\colon C(R_{\frac{1}{\varepsilon},[-\frac{1}{\varepsilon},\frac{1}{\varepsilon}]}(\mathcal{C}),L;\Gamma)\cap\bigcap_{e\in E}p_e^{-1}(D(p_\tau))\to(S^2)^E$$

has an open dense set of regular values. The product of this set by $(S^2)^{\underline{3N}\setminus E}$ is also open and dense and so is the intersection $\mathcal{I}_q$ over all such pairs (Γ,E). Thus there exists $(X_1,X_2,\dots,X_{3N})$ in this intersection and $\alpha\in\,]0,\varepsilon]$ such that $\prod_{i=1}^{3N}B(X_i,\alpha)\subset\mathcal{I}_q\cap\prod_{i=1}^{3N}B(W_i,\varepsilon)$ for the $B(W_i,\varepsilon)$ of Lemma 17.4. When $(Y_1,Y_2,\dots,Y_{3N})\in\prod_{i=1}^{3N}B(X_i,\alpha)$, let $P(Y_i)$ be a propagator of $(C_2(\check{R}(\mathcal{C})),\tau)$, which restricts to $D(p_\tau)$ as $p_\tau^{-1}(Y_i)$, for each $i\in\underline{3N}$. See Lemma 16.12. Then the $P(Y_i)$ can be put in general $3N$ position as in Section 11.3, by changing them only on $\mathring{C}_2(R_{2,[-1,2]}(\mathcal{C}))$, since they satisfy the general position conditions on the boundaries. Thus Proposition 17.2 is true with $M = \frac{1}{\sqrt{\varepsilon}}$. The existence for a given $\alpha > 0$ of a $\beta > 0$ such that, for any $\Gamma \in \mathcal{D}^e_{\underline{3N}}(\mathcal{L})$,

$$\bigcap_{e \in E(\Gamma)} p_e^{-1}(N_\beta(P(j_E(e)))) \subset N_\alpha(I_S(\Gamma, (P(i))_{i \in \underline{3N}}))$$

can be proved as in the end of the proof of Lemma 11.13. □

Corollary 17.6. *For forms ω_i β-dual (as in Definition 11.6) to the $P(i)$ of Proposition 17.2, for any subset A of $\underline{3N}$ with cardinality $3k$,*

$$Z(\mathcal{C}, L, \tau, A, (\omega_i)_{i \in \underline{3N}}),$$

which is defined in Theorem 16.7, is rational.

PROOF: As in Lemma 11.7, the involved configuration space integrals can be computed as algebraic intersections of rational preimages of the $P(i)$. □

17.2 Functoriality

In this chapter, we prove the functoriality of $\mathcal{Z}^f$, which implies the multiplicativity of $\mathcal{Z}$ under connected sum. The reader who is interested only by the latter proof, can read the proof by replacing the set B of strands with $\{0\}$, and by viewing $\check{R}(\mathcal{C}_j)$ as an asymptotically standard $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, which is identified with $\mathbb{C} \times \mathbb{R}$ outside $D_1 \times [0, 1]$, where D_1 is the unit disk of $\mathbb{C}$, for $j \in \underline{2}$.

Proposition 17.7. *Let $N \in \mathbb{N}$. Let $(\mathcal{C}_1, L_1, \tau_1)$ and $(\mathcal{C}_2, L_2, \tau_2)$ be two composable tangles. There exist volume-one forms $\omega(i, S^2)$ of S^2 for $i \in \underline{3N}$, such that for any subset A of $\underline{3N}$ with cardinality $3n$, with the notation of Theorem 16.7,*

$$Z(\mathcal{C}_1\mathcal{C}_2, L_1L_2, \tau_1\tau_2, A, (\omega(i, S^2))_{i \in A}) =$$

$$\sum_{(A_1, A_2) \in P_2(A)} \frac{(\sharp A_1)!(\sharp A_2)!}{(\sharp A)!} Z(\mathcal{C}_1, L_1, \tau_1, A_1, (\omega(i, S^2)))Z(\mathcal{C}_2, L_2, \tau_2, A_2, (\omega(i, S^2))).$$

PROOF: Let $y \colon B \hookrightarrow D_1$ be a planar configuration whose image contains 0, and the images of the bottom and top configurations of L_1 and L_2. For $i \in \underline{3N}$, let $\omega(i, S^2)$ be a volume-one form on S^2, which is supported on a disk of S^2_H of radius ε centered at X_i, where $B(X_1, X_2, \ldots, X_{3N})$ is a ball centered at $(X_1, X_2, \ldots, X_{3N})$ of radius $24N\varepsilon^{\frac{1}{12N}}$ of $(S^2_H)^{3N}$ that sits in the subset $\mathcal{O}((S^2_H)^{3N}, y)$ of Lemma 17.3, with $\varepsilon \in]0, \frac{1}{20^{12N}}[$ as in Lemma 17.4, and define $\omega_1(i)$ on $C_2(R(\mathcal{C}_1))$, which extends $p^*_{\tau_1}(\omega(i, S^2))$, as in Lemma 16.12. Perform a global homothety $m_{1,\varepsilon}$ of $\check{R}(\mathcal{C})$ of ratio ε, where $D_1 \times [0,1]$ is consequently changed to $D_\varepsilon \times [0,\varepsilon]$. (This homothety is actually performed on $\mathbb{R}^3 \setminus \mathrm{Int}(D_1) \times]0,1[$ and extended as a homeomorphism to $\check{R}(\mathcal{C})$.) Call $(\varepsilon\mathcal{C}_1, \varepsilon L_1)$ the intersection of the image of the long tangle $(\mathcal{C}_1, L_1)$ by the homothety, with the part that replaces $D_1 \times [0,1]$, which is now standard outside $D_\varepsilon \times [0,\varepsilon]$. Use forms $(m^{-1}_{1,\varepsilon})^*(\omega_1(i))$ for $(\varepsilon\mathcal{C}_1, \varepsilon L_1)$. So

$$Z(\varepsilon\mathcal{C}_1, \varepsilon L_1, m_{1,\varepsilon *}(\tau_1), A_1, (\omega(i, S^2))) = Z(\mathcal{C}_1, L_1, \tau_1, A_1, (\omega(i, S^2)))$$

for any subset A_1 of $\underline{3N}$ with cardinality multiple of 3, where $m_{1,\varepsilon *}(\tau_1) = Tm_{1,\varepsilon} \circ \tau_1 \circ \left(m^{-1}_{1,\varepsilon} \times m^{-1}_{1,\varepsilon}\right)$. Define $\omega_2(i)$ on $C_2(\check{R}(\mathcal{C}_2))$, so that $\omega_2(i)$ extends $p^*_{\tau_2}(\omega(i, S^2))$, as in Lemma 16.12. Perform the same homothety $m_{1,\varepsilon}$ (though acting on a different space) of $\check{R}(\mathcal{C}_2)$, followed by a vertical translation by $(0, 0, 1-\varepsilon)$, where $D_1 \times [0,1]$ is consequently changed to $D_\varepsilon \times [1-\varepsilon, 1]$. Call $m_{2,\varepsilon}$ this composition, and call $(\varepsilon\mathcal{C}_2, \varepsilon L_2)$ the intersection of the image of the long tangle $(\mathcal{C}_2, L_2)$ by $m_{2,\varepsilon}$ with the part that replaces $D_1 \times [0,1]$, which is now standard outside $D_\varepsilon \times [1-\varepsilon, 1]$. Use forms $(m^{-1}_{2,\varepsilon})^*(\omega_2(i))$ for $(\varepsilon\mathcal{C}_2, \varepsilon L_2)$. So

$$Z(\varepsilon\mathcal{C}_2, \varepsilon L_2, m_{2,\varepsilon *}(\tau_2), A_2, (\omega(i, S^2))) = Z(\mathcal{C}_2, L_2, \tau_2, A_2, (\omega(i, S^2)))$$

for any subset A_2 of $\underline{3N}$ with cardinality multiple of 3, where $m_{2,\varepsilon *}(\tau_2) = Tm_{2,\varepsilon} \circ \tau_2 \circ \left(m^{-1}_{2,\varepsilon} \times m^{-1}_{1,\varepsilon}\right)$.

Now, let $(\varepsilon\mathcal{C}_1\mathcal{C}_2, \varepsilon L_1 L_2)$ be obtained from $(\varepsilon\mathcal{C}_1, \varepsilon L_1)$ by inserting

$$(\varepsilon\mathcal{C}_2, \varepsilon L_2) \cap R_{\varepsilon,[1-\varepsilon,1]}(\varepsilon\mathcal{C}_2)$$

instead of $D_\varepsilon \times [1-\varepsilon, 1]$. Here and below we use a natural extension of the $R_{.,.}$ notation introduced in 16.11. Define the propagator $\omega(i)$ of $(R(\varepsilon\mathcal{C}_1\mathcal{C}_2), \tau_1\tau_2)$,

- which coincides with $(m^{-1}_{1,\varepsilon})^*(\omega_1(i))$ on $C_2(R(\varepsilon\mathcal{C}_1\mathcal{C}_2) \setminus R_{2\varepsilon,[1-2\varepsilon,1+\varepsilon]}(\varepsilon\mathcal{C}_1\mathcal{C}_2))$,
- which coincides with $(m^{-1}_{2,\varepsilon})^*(\omega_2(i))$ on $C_2(R(\varepsilon\mathcal{C}_1\mathcal{C}_2) \setminus R_{2\varepsilon,[-\varepsilon,2\varepsilon]}(\varepsilon\mathcal{C}_1\mathcal{C}_2))$, and,
- whose support intersects neither

  $$R_{.1,[-.1,.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2) \times R_{.1,[.9,1.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)$$

  nor $R_{.1,[.9,1.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2) \times R_{.1,[-.1,.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)$ (this is consistent since the form $\omega(i, S^2)$ is supported in S^2_H).

Then compute $Z(\mathcal{C}_1\mathcal{C}_2, L_1L_2, \tau_1\tau_2, A, (\omega(i,S^2))_{i\in A})$ as

$$Z(\varepsilon\mathcal{C}_1\mathcal{C}_2, \varepsilon L_1L_2, \tau_1\tau_2, A, (\omega(i,S^2))_{i\in A})$$

with these propagators $\omega(i)$ and prove the following lemma.

Lemma 17.8. *For $j \in \{1,2\}$, for any $\underline{3N}$-numbered Jacobi diagram Γ_j on the source of L_j, of degree at most N, the form $\bigwedge_{e\in E(\Gamma_j)}(p_e m_{j,\varepsilon}^{-1})^*(\omega_j(j_E(e)))$ on $C(R(\varepsilon\mathcal{C}_j), \varepsilon L_j; \Gamma_j)$ is supported on $C_{V(\Gamma_1)}\left(R_{.1,[-.1,.1]}(\varepsilon\mathcal{C}_1)\right)\cap C(R(\varepsilon\mathcal{C}_1), \varepsilon L_1; \Gamma_1)$ if $j=1$, and on $C_{V(\Gamma_2)}\left(R_{.1,[.9,1.1]}(\varepsilon\mathcal{C}_2)\right)\cap C(R(\varepsilon\mathcal{C}_2), \varepsilon L_2; \Gamma_2)$ if $j=2$.*

For any $\underline{3N}$-numbered Jacobi diagram Γ on the source of L_1L_2 of degree at most N, the form $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$ on $C(R(\varepsilon\mathcal{C}_1\mathcal{C}_2), \varepsilon L_1L_2; \Gamma)$ is supported on

$$\cup_{\{V_1,V_2\}\in P_2(\Gamma)} C_{V_1}\left(R_{.1,[-.1,.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)\right)\times C_{V_2}\left(R_{.1,[.9,1.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)\right),$$

where $P_2(\Gamma)$ denotes the set of partitions $\{V_1,V_2\}$ of $V(\Gamma)$ into two disjoint subsets V_1 and V_2 such that no edge of Γ has one vertex in V_1 and the other one in V_2.

Assuming Lemma 17.8, the proof of Proposition 17.7 can be concluded as follows. Lemma 17.8 implies that both sides of the equality to be proved are sums over pairs (Γ_1,Γ_2) of A-numbered diagrams such that Γ_1 is a diagram on the source of L_1, Γ_2 is a diagram on the source of L_2, $j_E(E(\Gamma_1))\cap j_E(E(\Gamma_2)) = \emptyset$ of terms

$$I(\mathcal{C}_1, L_1, \Gamma_1, (\omega_1(i))_{i\in A})I(\mathcal{C}_2, L_2, \Gamma_2, (\omega_2(i))_{i\in A})[\Gamma_1][\Gamma_2]$$

and it suffices to identify the coefficients, that is to prove the following lemma.

Lemma 17.9.

$$\zeta_{\Gamma_1\cup\Gamma_2} = \sum_{\substack{(A_1,A_2)\mid A_1\subseteq A, A_2=(A\setminus A_1),\\ j_E(E(\Gamma_1))\subseteq A_1, j_E(E(\Gamma_2))\subseteq A_2,\\ \sharp A_1=3\deg(\Gamma_1), \sharp A_2=3\deg(\Gamma_2)}} \frac{(\sharp A_1)!(\sharp A_2)!}{(\sharp A)!}\zeta_{\Gamma_1}\zeta_{\Gamma_2}$$

Proof: $\zeta_{\Gamma_1\cup\Gamma_2} = \frac{(\sharp A-\sharp E(\Gamma_1)-\sharp E(\Gamma_2))!}{(\sharp A)!2^{\sharp E(\Gamma_1\cup\Gamma_2)}}$, and the number of pairs (A_1,A_2) in the sum is $\frac{(\sharp A-\sharp E(\Gamma_1)-\sharp E(\Gamma_2))!}{(\sharp A_1-\sharp E(\Gamma_1))!(\sharp A_2-\sharp E(\Gamma_2))!}$. □

This finishes the proof of Proposition 17.7 up to the proof of Lemma 17.8, which follows. □

Proof of Lemma 17.8: The first assertion follows from Lemma 17.4. Let us focus on the second one. Fix a $\underline{3N}$-numbered Jacobi diagram Γ of degree at most N on the source of L_1L_2. For $i \in \underline{6N}$, let $\beta_i = \varepsilon\alpha_i$ with the sequence $\alpha_i = \varepsilon^{\frac{1-i}{12N}}$ of the proof of Lemma 17.4. Define an open covering of $\check{C}(\check{R}(\varepsilon\mathcal{C}_1\mathcal{C}_2), \varepsilon\mathcal{C}_1\mathcal{C}_2; \Gamma)$ associated with colorings of the vertices by colors $(1,k)$, $(2,k)$ and blue, where $k \in \underline{6N}$, such that

- blue vertices and vertices of color (j,k) with $j \in \underline{2}$, and $k \geq 2$ do not belong to $R_{2\varepsilon,[-\varepsilon,2\varepsilon]}(\varepsilon\mathcal{C}_1\mathcal{C}_2) \cup R_{2\varepsilon,[1-2\varepsilon,1+\varepsilon]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)$,
- vertices of color $(1,1)$ belong to $\mathring{R}_{3\varepsilon,[-2\varepsilon,3\varepsilon]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)$,
- vertices of color $(2,1)$ belong to $\mathring{R}_{3\varepsilon,[1-3\varepsilon,1+2\varepsilon]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)$,
- for $j \in \underline{2}$, any vertex of color $(j,2)$ is connected by an edge of Γ to a vertex of color $(j,1)$, and is at a distance (with respect to the Euclidean norm of $\mathbb{R}^3$) smaller than $5\beta_2$ from $(0, j-1)$,
- for $j \in \underline{2}$, and for $k \geq 2$, $k \leq 6N-1$, any vertex of color $(j,k+1)$ is connected to a vertex of color (j,k) by an edge of length smaller than $5\beta_{k+1}$,
- when there is an edge of Γ between a blue vertex and a vertex colored by $(j,1)$ for $j \in \underline{2}$, the distance between the blue vertex and $(0,j-1)$ is greater than $3\beta_2$,
- when there is an edge of Γ between a blue vertex and a vertex colored by (j,k) for $j \in \underline{2}$ and $2 \leq k \leq 6N-1$, the distance between the two vertices is greater than $3\beta_{k+1}$.

The subset $U(\mathbf{c})$ of $\check{C}(\check{R}(\varepsilon\mathcal{C}_1\mathcal{C}_2), \varepsilon\mathcal{C}_1\mathcal{C}_2; \Gamma)$ consisting of the configurations that satisfy the above conditions with respect to a coloring $\mathbf{c}$ of the vertices is open, and $\check{C}(\check{R}(\varepsilon\mathcal{C}_1\mathcal{C}_2), \varepsilon\mathcal{C}_1\mathcal{C}_2; \Gamma)$ is covered by these sets as in the proof of Lemma 17.4. The only additional thing to notice is that a vertex could not be simultaneously colored by $(1,k)$ and by $(2,k')$, since a vertex colored by (j,k) is at a distance less than

$$6\beta_k \leq \sqrt{\varepsilon} \leq \frac{1}{20^6}$$

from $(0,j-1)$. In particular, the vertices colored by $(1,k)$ are in

$$R_{.1,[-.1,.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2),$$

and the vertices colored by $(2,k)$ are in $R_{.1,[.9,1.1]}(\varepsilon\mathcal{C}_1\mathcal{C}_2)$.

So the form $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$ vanishes on open sets corresponding to colorings for which a vertex $(1,k)$ is connected to a vertex $(2,k')$ (by some edge of Γ), according to the conditions before Lemma 17.8.

As in the proof of Lemma 17.4, we prove that our form vanishes on open sets $U(\mathbf{c})$ associated with colorings for which the color blue appears. Fix such a coloring $\mathbf{c}$ and remove from Γ all the edges that do not contain a blue vertex. Let Γ_b be a connected component of this graph with at least one blue vertex. It has blue vertices, which are trivalent or univalent in Γ and Γ_b, the blue univalent vertices lie on $\varepsilon y \times \mathbb{R}$. Its other vertices are either colored by some $(1,k)$ in which case we color them yellow or by some $(2,k)$ in which case we color them red. Red and yellow vertices may have 1, 2, or 3 adjacent edges in Γ_b. Let Γ_b' be the uni-trivalent graph obtained by blowing up Γ_b at its yellow and red vertices by replacing such a vertex by a univalent vertex of the same color for each adjacent edge. Color the edges between blue vertices blue, the edges between a blue vertex and a yellow one green, and the edges between a blue vertex and a red one purple. With a configuration of $U(\mathbf{c})$ in the support of $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$, associate the configuration of Γ_b' obtained by sending all the yellow vertices to $\mathbf{o}$ and all the red ones to $(0,1)$, leaving the positions of the blue vertices unchanged. Thus

- the direction of a blue edge numbered by i is in the support of $\omega(i,S^2)$ at a distance less than ε from X_i,
- the direction of a green edge numbered by g is at a distance less than $(\varepsilon + 4\varepsilon^{\frac{1}{12N}})$ from X_g, (as in the proof of Lemma 17.4),
- the direction of a purple edge numbered by p is at a distance less than $(\varepsilon + 4\varepsilon^{\frac{1}{12N}})$ from X_p.

But the directions of the edges of Γ_b' cannot be in the image of $\prod_{e\in E(\Gamma)} p_{e,S^2}$ according to our conditions in the beginning of the proof of Proposition 17.7 (the ε rescaling of y does not change the image). Therefore the support of $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$ does not intersect the open subsets of the covering that use the blue color.

It is now easy to conclude.

□

For an integer N, $Z_{\leq N}$ denotes the truncation of Z valued in $\mathcal{A}_{\leq N}(\mathcal{L}) = \prod_{j=0}^{N} \mathcal{A}_j(\mathcal{L})$.

Theorem 17.10. *Let $(\mathcal{C}_1, L_1, \tau_1)$ and $(\mathcal{C}_2, L_2, \tau_2)$ be two composable tangles in parallelized rational homology cylinders. For any $N \in \mathbb{N}$ and for any* *family of volume-one forms* $(\omega_{i,S^2})_{i\in\underline{3N}}$*, with the notation of Theorem 16.7 and Definition 16.8,*

$$Z(\mathcal{C}_1\mathcal{C}_2, L_1L_2, \tau_1\tau_2, ., (\omega_{i,S^2})_{i\in\underline{3N}}) =$$

$$(Z(\mathcal{C}_1, L_1, \tau_1, ., (\omega_{i,S^2}))Z(\mathcal{C}_2, L_2, \tau_2, ., (\omega_{i,S^2})))_{\sqcup}.$$

For any two composable $J_{bb,tt}$*-oriented q–tangles* T_1 *and* T_2*,*

$$\mathcal{Z}^f(T_1T_2, ., (\omega_{i,S^2})_{i\in\underline{3N}}) = \left(\mathcal{Z}^f(T_1, ., (\omega_{i,S^2}))\mathcal{Z}^f(T_2, ., (\omega_{i,S^2}))\right)_{\sqcup},$$

with the notation of Definition 16.41, and

$$\mathcal{Z}^f(T_1)\mathcal{Z}^f(T_2) = \mathcal{Z}^f(T_1T_2).$$

Proof: Let us prove the first assertion. Apply Theorem 16.7, with

$$\tilde{\omega}(i, 1, S^2) = \omega_{i,S^2} \quad \text{and} \quad \tilde{\omega}(i, 0, S^2) = \omega(i, S^2)$$

with the form $\omega(i, S^2)$ of Proposition 17.7. Thus

$$Z_{\leq N}(\mathcal{C}_1\mathcal{C}_2, L_1L_2, \tau_1\tau_2, ., (\omega_{i,S^2})) =$$

$$\left(\begin{array}{l}\left(\prod_{j\in I} \widetilde{\mathrm{hol}}_{[0,1]}(\eta(., p_\tau(U^+K_j)))\sharp_j\right) \\ \widetilde{\mathrm{hol}}_{[1,0]\times y_1^-}(\eta_{B^-,.})Z(\mathcal{C}_1\mathcal{C}_2, L_1L_2, \tau_1\tau_2, ., (\omega(i, S^2)))\widetilde{\mathrm{hol}}_{[0,1]\times y_2^+}(\eta_{B^+,.}).\end{array}\right)_{\sqcup},$$

where y_i^- (resp. y_i^+) represents the bottom (resp. top) configuration of L_i, $y_2^- = y_1^+$, and

$$Z(\mathcal{C}_1\mathcal{C}_2, L_1L_2, \tau_1\tau_2, ., (\omega(i, S^2))) =$$

$$\left(Z(\mathcal{C}_1, L_1, \tau_1, ., (\omega(i, S^2)))Z(\mathcal{C}_2, L_2, \tau_2, ., (\omega(i, S^2)))\right)_{\sqcup}.$$

A neutral factor $\left(\widetilde{\mathrm{hol}}_{[0,1]\times y_1^+}(\eta_{B_1^+,.})\widetilde{\mathrm{hol}}_{[1,0]\times y_2^-}(\eta_{B_2^-,.})\right)_{\sqcup}$ can be inserted in the middle. So the first equality of the statement becomes clear, up to the behaviour of the factors $\widetilde{\mathrm{hol}}_{[0,1]}(\eta(., p_\tau(U^+K_j)))$ of Definition 16.5. For these factors, note that a component K of L_1L_2 consists of a bunch of components K_k from L_1 and L_2, for k in a finite set E, and that $\eta(A, p_\tau(U^+K))$ is the sum of the corresponding $\eta(A, p_\tau(U^+K_k))$.

Now $\widetilde{\mathrm{hol}}_{[0,1]}(\eta(., p_\tau(U^+K)))$ is valued in the commutative algebra $\check{\mathcal{A}}(\mathbb{R})$ and we check the following lemma.

Lemma 17.11. *We have*

$$\widetilde{hol}_{[0,1]}(\sum_{k\in E} \eta(., p_\tau(U^+K_k))) = \left(\prod_{k\in E} \widetilde{hol}_{[0,1]}(\eta(., p_\tau(U^+K_k)))\right)_{\sqcup}.$$

Proof: It suffices to prove the lemma when $E = \{1,2\}$. Recall that $\widetilde{\mathrm{hol}}_{[0,1]}(\sum_{j=1}^2 \eta(.,p_\tau(U^+K_j)))$ is equal to

$$\sum_{r=0}^{\infty} \sum_{(A_1,\dots,A_r)\in P_r(A)} \frac{\prod_{i=1}^r(\sharp A_i)!}{(\sharp A)!} I(A_1,\dots,A_r),$$

where $I(A_1,\dots,A_r)$, which is defined to be

$$\int_{(t_1,\dots,t_r)\in[0,1]^r | t_1\leq t_2\leq\cdots\leq t_r} \bigwedge_{i=1}^r p_i^*(\eta(A_i,p_\tau(U^+K_1)) + \eta(A_i,p_\tau(U^+K_2))),$$

is a sum over maps $f\colon \{1,\dots,r\} \to \{1,2\}$, which decompose A into the disjoint union of $A_{f,1} = \cup_{i\in f^{-1}(1)} A_i$ and $A_{f,2} = \cup_{i\in f^{-1}(2)} A_i$. On the other hand, the terms $I_1((A_i)_{i\in f^{-1}(1)})I_2((A_i)_{i\in f^{-1}(2)})$, where $I_j((A_i)_{i\in f^{-1}(j)}) =$

$$\int_{(t_i)_{i\in f^{-1}(j)}\in[0,1]^{f^{-1}(j)} | t_i\leq t_k \text{ when } i\leq k} \bigwedge_{i\in f^{-1}(j)} p_i^*(\eta(A_i,p_\tau(U^+K_j))),$$

split according to the relative orders of the involved t_i in I_1 and I_2.

The contribution to $\widetilde{\mathrm{hol}}_{[0,1]}(\sum_{j=1}^2 \eta(.,p_\tau(U^+K_j)))$ of the terms such that $\{A_i \mid i \in \underline{r}\}$ is fixed as an unordered set, f is fixed as a map from this unordered set to $\{1,2\}$, and the partial order induced by the numbering in $\underline{r}$ is fixed over the sets $\{A_i \mid i \in f^{-1}(j)\}$ for $j = 1,2$, is also a sum over the possible total orders on the $\{A_i \mid i \in \underline{r}\}$ that induce the given orders on the two subsets.

We can easily identify the involved coefficients to prove that

$$\widetilde{\mathrm{hol}}_{[0,1]}(\sum_{j=1}^2 \eta(.,p_\tau(U^+K_j))) = \left(\prod_{j=1}^2 \widetilde{\mathrm{hol}}_{[0,1]}(\eta(.,p_\tau(U^+K_j)))\right)_{\sqcup}.$$

$\square$

This finishes the proof of the first assertion of Theorem 17.10.

Assume that L_1 and L_2 are oriented in a compatible way. Since $p_1(\tau_1\tau_2) = p_1(\tau_1)+p_1(\tau_2)$, the first assertion, applied to straight tangles with the induced parallelization, and the associativity of the product $()_\sqcup$ show that if L_1 and L_2 are framed by parallels $L_{1\parallel}$ and $L_{2\parallel}$ induced by parallelizations τ_k such that $p_{\tau_k}(U^+L_k) \subset S_{WE}$, for $k \in \underline{2}$, then

$$\mathcal{Z}^f(\mathcal{C}_1\mathcal{C}_2, L_1L_2, (L_1L_2)_\parallel, ., (\omega_{i,S^2})_{i\in\underline{3N}}) =$$

$$\left(\mathcal{Z}^f(\mathcal{C}_1, L_1, L_{1\parallel}, ., (\omega_{i,S^2}))\mathcal{Z}^f(\mathcal{C}_2, L_2, L_{2\parallel}, ., (\omega_{i,S^2}))\right)_\sqcup$$

This generalizes to any pair $((L_1, L_{1\|}), (L_2, L_{2\|}))$ of parallelized $J_{bb,tt}$-oriented tangles with the invariant $\mathcal{Z}^f$ of framed tangles of Definition 16.14, as follows. When $(L_1, L_{1\|})$ is not representable as a straight tangle with respect to a parallelization, then $(L_1, L_{1\|+1})$ is, where $L_{1\|+1}$ is the parallel of L_1 such that $(L_{1\|+1} - L_{1\|})$ is homologous to a positive meridian of L_1 in a tubular neighborhood of L_1 deprived of L_1. Thus the known behaviour of $\mathcal{Z}^f$ under such a framing change yields the second equality of the statement when T_1 and T_2 have injective bottom and top configurations. According to Definitions 16.41 and 16.39, the second equality is also true when T_1 and T_2 are q–braids, thanks to the multiplicativity of $\widetilde{\mathrm{hol}}_{.}(.)$ with respect to the product of Definition 16.8 in Lemma 16.31. Thus the general definition 16.41 of $\mathcal{Z}^f$ implies the second equality for general q–tangles. The third equality is a direct consequence of the second one when $\omega_{i,S^2} = \omega_{S^2}$ for all i. $\square$

17.3 Insertion of a tangle in a trivial q–braid.

In this section, we prove the following result, which is the cabling property of Theorem 13.12 generalized to all variants of the invariant $\mathcal{Z}^f$ of Definition 16.41.

Proposition 17.12. *Let B be a finite set with cardinality greater than 1. Let $y \in \mathcal{S}_B(\mathbb{C})$, let $y \times [0,1]$ denote the corresponding q-braid, and let K be a strand of $y \times [0,1]$. Let L be a q–tangle with source $\mathcal{L}$. Then $\mathcal{Z}^f((y \times [0,1])(L/K))$ is obtained from $\mathcal{Z}^f(L)$ by the natural injection from $\mathcal{A}(\mathcal{L})$ to $\mathcal{A}\left(\mathbb{R}^B\left(\frac{\mathcal{L}}{K}\right)\right)$.*

Furthermore, if L is $J_{bb,tt}$-oriented, for any $N \in \mathbb{N}$, for any subset A of $\underline{3N}$ whose cardinality is a multiple of 3 and for any family of volume-one forms $(\omega_{i,S^2})_{i\in\underline{3N}}$, $\mathcal{Z}^f((y \times [0,1])(L/K), A, (\omega_{i,S^2})_{i\in\underline{3N}})$ is obtained from $\mathcal{Z}^f(L, A, (\omega_{i,S^2})_{i\in\underline{3N}})$ by the natural injection from $\mathcal{A}(\mathcal{L})$ to $\mathcal{A}\left(\mathbb{R}^B\left(\frac{\mathcal{L}}{K}\right)\right)$.

Lemma 17.13. *Under the hypotheses of Proposition 17.12, when $y \in \check{\mathcal{S}}_B(\mathbb{C})$, when L is a $J_{bb,tt}$-oriented framed tangle represented by*

$$L \colon \mathcal{L} \hookrightarrow \mathcal{C}$$

(with injective bottom and top configurations), for any $N \in \mathbb{N}$, there exists a family $(\omega(i,S^2))_{i\in\underline{3N}}$ of volume-one forms of S^2 such that for any subset A of $\underline{3N}$ whose cardinality is a multiple of 3,

$$\mathcal{Z}^f((y \times [0,1])(L/K), A, (\omega(i,S^2))_{i\in\underline{3N}})$$

is obtained from $\mathcal{Z}^f(L, A, (\omega(i,S^2))_{i\in\underline{3N}})$ by the natural injection from $\mathcal{A}(\mathcal{L})$ to $\mathcal{A}\left(\mathbb{R}^B\left(\frac{\mathcal{L}}{K}\right)\right)$.

PROOF: Without loss of generality, translate and rescale y so that $K = \{0\} \times [0,1]$. Let $\eta \in]0,1[$ be the distance between K and the other strands of $y \times [0,1]$. Because of the known variation of $\mathcal{Z}^f$ under framing changes, there is no loss of generality in assuming that L is straight with respect to a parallelization τ, which we do. Let $y_1 \colon B \hookrightarrow D_1$ be a planar configuration whose image contains y and the images of the bottom and top configurations of L. For $i \in \underline{3N}$, let $\omega(i,S^2)$ be a volume-one form on S^2 that is supported on a disk of S^2_H of radius ε centered at X_i, where $B(X_1, X_2, \dots, X_{3N})$ is a ball centered at $(X_1, X_2, \dots, X_{3N})$ of radius $24N\varepsilon^{\frac{1}{12N}}$ of $(S^2_H)^{3N}$ that sits in the subset $\mathcal{O}((S^2_H)^{3N}, y_1)$ of Lemma 17.3, with $\varepsilon \in]0, \frac{1}{20^{12N}}[$ as in Lemma 17.4, and define $\omega_1(i)$ on $C_2(R(\mathcal{C}))$, so that $\omega_1(i)$ coincides with $p_\tau^*(\omega(i,S^2))$ on $D(p_\tau)$, as in Lemma 16.12. Perform a global homothety $m_{1,\eta\varepsilon}$ of $\check{R}(\mathcal{C})$, of ratio $\eta\varepsilon$, where $D_1 \times [0,1]$ is consequently changed to $D_{\eta\varepsilon} \times [0,\eta\varepsilon]$. Call $(\eta\varepsilon\mathcal{C}, \eta\varepsilon L)$ the intersection of the image of the long tangle $(\mathcal{C}, L)$ by this homothety with the part that replaces $D_1 \times [0,1]$, which is now standard outside $D_{\eta\varepsilon} \times [0,1]$. Use forms $(m^{-1}_{1,\eta\varepsilon})^*(\omega_1(i))$ for $(\eta\varepsilon\mathcal{C}, \eta\varepsilon L)$. So

$$\mathcal{Z}^f(\eta\varepsilon\mathcal{C}, \eta\varepsilon L, m_{1,\eta\varepsilon *}(\tau), ., (\omega(i,S^2))) = \mathcal{Z}^f(\mathcal{C}, L, \tau, ., (\omega(i,S^2))).$$

Let $(y \times [0,1])\,(\eta\varepsilon L/K)$ be the tangle obtained from $(y \times [0,1])$ by letting $R_{\eta\varepsilon,[0,1]}(\eta\varepsilon\mathcal{C})$ replace $D_{\eta\varepsilon} \times [0,1]$. Graphs that do not involve vertices of $R_{2\eta\varepsilon,[-\eta\varepsilon,2\eta\varepsilon]}(\eta\varepsilon\mathcal{C})$ cannot contribute to

$$\mathcal{Z}^f(\eta\varepsilon\mathcal{C}, (y \times [0,1])\,(\eta\varepsilon L/K)\,, A, (\omega(i,S^2))_{i\in\underline{3N}}).$$

As in the proof of Lemma 17.4, the only contributing graphs are located in $\check{R}_{\eta\sqrt{\varepsilon},[-\eta\sqrt{\varepsilon},\eta\sqrt{\varepsilon}]}(\eta\varepsilon\mathcal{C})$.

We conclude that $\mathcal{Z}^f(\eta\varepsilon\mathcal{C}, (y \times [0,1])\,(\eta\varepsilon L/K)\,, A, (\omega(i,S^2))_{i\in\underline{3N}})$ is obtained from $\mathcal{Z}^f(\mathcal{C}, L, A, (\omega(i,S^2))_{i\in\underline{3N}})$ by the natural injection from $\mathcal{A}(\mathcal{L})$ to $\mathcal{A}\left(\mathbb{R}^B\left(\frac{\mathcal{L}}{K}\right)\right)$. Since this is also true when η is replaced by a smaller η', this is also true when $\eta\varepsilon L$ is replaced by a legal composition $\gamma^-(\eta\varepsilon L)\gamma^+$ for braids γ^- and γ^+ with constant projections in $\mathcal{S}_{B^\pm}(\ CC)$, which respectively go from $\eta' y^-$ to ηy^- and from ηy^+ to $\eta' y^+$ (up to adjusting the parallelizations), thanks to the isotopy invariance of $\mathcal{Z}^f$. Therefore, this is also true at the limit, when η' tends to zero, thanks to Lemma 16.30. See Definition 16.41. $\square$

Corollary 17.14. *Proposition 17.12 is true when $y \in \check{\mathcal{S}}_B(\mathbb{C})$.*

PROOF: Recall Theorem 16.42, which expresses the variation of

$$\mathcal{Z}^f((y \times [0,1])\,(L/K)\,, A, (\omega_{i,S^2})_{i\in\underline{3N}})$$

when $(\omega_{i,S^2})_{i\in\underline{3N}}$ varies, for q–tangles. This variation is given by the insertion of factors on components that go from bottom to bottom or from top to top, which are identical in both sides of the implicit equality to be proved, and D-holonomies for the bottom and top configurations, for $D \subseteq A$. The D-holonomies satisfy the duplication property of Proposition 16.40. The D-holonomies of the bottom and top configurations of L contribute in the same way to both sides of the equality. The D-holonomies of the bottom and top configurations of $y \times [0,1]$ are inverse to each other. After the insertion they are duplicated both at the top and at the bottom on possibly different number of strands. Let B^+ (resp. B^-) be the set of upper (resp. lower) ∞-components of L. Lemma 6.23 ensures that for any diagram Γ on $\mathcal{L}$ and for any duplication $\pi(B^+ \times K^+)^*$ (resp. $\pi(B^- \times K^-)^*$) of the upper part K^+ (resp. lower part K^-) of the long strand of K by $B^+ \times [1,+\infty[$ (resp. $B^- \times]-\infty,0]$) of a diagram Γ' on $B \times \mathbb{R}$, we have that

$$\Gamma\pi(B^+ \times K^+)^*(\Gamma') = \pi(B^- \times K^-)^*(\Gamma')\Gamma$$

in $\mathcal{A}\left(\mathbb{R}^B\left(\frac{\mathcal{L}}{K}\right)\right)$. Thus the "holonomies" $\pi(B^- \times K^-)^*\widetilde{\mathrm{hol}}_{[1,0]\times y}(\eta_{B,.})$ and $\pi(B^+ \times K^+)^*\widetilde{\mathrm{hol}}_{[0,1]\times y}(\eta_{B,.})$ cancel on each side, and Proposition 17.12 is true when $y \in \check{\mathcal{S}}_B(\mathbb{C})$ as soon as the bottom and top configurations of L may be represented by injective configurations. When $L = T(\gamma^-)(\mathcal{C}, L, L_{\|})T(\gamma^+)$ is a general q–tangle and γ^- and γ^+ are paths of configurations, $(y \times [0,1])\left(\frac{L}{K}\right)$ is equal to

$$(y \times [0,1])\left(\frac{T(\gamma^-)}{K}\right)(y \times [0,1])\left(\frac{(\mathcal{C}, L, L_{\|})}{K}\right)(y \times [0,1])\left(\frac{T(\gamma^+)}{K}\right)$$

and the result follows using the Functoriality theorem 17.10 and the cabling theorem for q–braids (Proposition 16.40). $\square$

End of proof of Proposition 17.12: To treat the case in which y is a limit configuration, pick a path $\gamma\colon [0,1] \to \mathcal{S}_B(\mathbb{C})$ such that $\gamma(1) = y$ and $\gamma([0,1[) \subset \check{\mathcal{S}}_B(\mathbb{C})$, view $y \times [0,1]$ as the path composition $\overline{\gamma}\gamma(0)\gamma$ where $\gamma(0)$ is thought of as a constant map. If y^- and y^+ denote the bottom and top configurations of L, respectively, and if K^- and K^+ denote the strand of K in $\overline{\gamma}$ and in γ, respectively, then $(y \times [0,1])\left(\frac{L}{K}\right)$ may be expressed as

$$T(\overline{\gamma})\left(\frac{y^- \times [0,1]}{K^-}\right)(\gamma(0) \times [0,1])\left(\frac{L}{K}\right)T(\gamma)\left(\frac{y^+ \times [0,1]}{K^+}\right).$$

Use the functoriality theorem 17.10, Corollary 17.14, the cabling theorem for q–braids (Proposition 16.40) and the commutation argument in the above proof to conclude. $\square$

17.4 Duplication property

We are about to show how $\mathcal{Z}^f$ and all its variants behave under a general parallel duplication of a component that goes from bottom to top in a tangle.

Proposition 17.15. *Let K be a component that goes from bottom to top, or from top to bottom, in a q–tangle L, in a rational homology cylinder $\mathcal{C}$. Let y be an element of $\mathcal{S}_B(\mathbb{C})$ for a finite set B. Let $L(y \times K)$ be the tangle obtained by duplicating K as in Section 13.1. Then*

$$\mathcal{Z}^f(L(y \times K)) = \pi(B \times K)^* \mathcal{Z}^f(L),$$

and, for any $N \in \mathbb{N}$, for any subset A of $\underline{3N}$ whose cardinality is a multiple of 3, and for any family $(\omega_{i,S^2})_{i \in \underline{3N}}$ of volume-one forms of S^2,

$$\mathcal{Z}^f(\mathcal{C}, L(y \times K), A, (\omega_{i,S^2})_{i \in \underline{3N}}) = \pi(B \times K)^* \mathcal{Z}^f(\mathcal{C}, L, A, (\omega_{i,S^2})_{i \in \underline{3N}})$$

for any $J_{bb,tt}$-orientation of L, with the natural extension of Notation 6.28.

In order to prove this proposition, we are going to prove the following lemmas.

Lemma 17.16. *Let $L \colon \mathcal{L} \hookrightarrow \mathcal{C}$ be a straight tangle in a parallelized rational homology cylinder $(\mathcal{C}, \tau)$. Let K be a component of L that goes from bottom to top. Let y be an element of $\check{\mathcal{S}}_B(\mathbb{C})$ for a finite set B. Let $N \in \mathbb{N}$. There exists a family of volume-one forms $(\omega(i, S^2))_{i \in \underline{3N}}$ such that for any subset A of $\underline{3N}$ whose cardinality is a multiple of 3,*

$$Z(\mathcal{C}, L(y \times K), \tau, A, (\omega(i, S^2))_{i \in \underline{3N}}) = \pi(B \times K)^* Z(\mathcal{C}, L, \tau, A, (\omega(i, S^2))_{i \in \underline{3N}}).$$

Lemma 17.17. *Lemma 17.16 implies Proposition 17.15.*

Proof: The known behaviour of Z under strand orientation changes, for components that go from bottom to top of Lemma 16.43, allows us to reduce the proof to the case in which K goes from bottom to top. Lemma 17.16, Theorem 16.42 and Proposition 16.40 imply that Proposition 17.15 is true when L is a straight tangle (with injective bottom and top configurations) and when $y \in \check{\mathcal{S}}_B(\mathbb{C})$. Then the duplication property for braids of Proposition 16.40 and the functoriality imply that Proposition 17.15 is true if $y \in \check{\mathcal{S}}_B(\mathbb{C})$ for any $J_{bb,tt}$-oriented q–tangle L for which $(K, K_{\parallel})$ can be represented by a straight knot with respect to a parallelization τ of $\mathcal{C}$ and its associated parallel. Therefore Proposition 17.15 is also true if $y \in \mathcal{S}_B(\mathbb{C})$ by iterating the duplication process as soon as $(K, K_{\parallel})$ is representable by a straight knot. In particular, it is true when K is a strand of a trivial braid whose framing has been changed so that $lk(K, K_{\parallel}) = 2$. (Recall Lemma 7.38 and Proposition 16.10.) Thanks to the functoriality of $\mathcal{Z}^f$, since an element whose degree 0 part is 1 is determined by its square, Proposition 17.15 is also true when K is a strand of a trivial braid whose framing has been changed so that $lk(K, K_{\parallel}) = 1$. If our general $(K, K_{\parallel})$ is not representable, then Proposition 17.15 is true when L is composed by a trivial braid such that the framing of the strand I that extends K is changed so that $lk(I, I_{\parallel}) = -1$. So it is also true for L. $\square$

Let us prove Lemma 17.16. Choose a tubular neighborhood

$$N_{\eta_0}(K) = D_{\eta_0} \times \mathbb{R}_K$$

of $K = \{0\} \times \mathbb{R}_K$ for some η_0 such that $0 < 10\eta_0 < 1$, where D_{η_0} denotes the disk of radius η_0 centered at 0 in $\mathbb{C}$. Assume that the trivialization τ maps $(d \in D_{\eta_0}, k \in \mathbb{R}_K, e_1 = \vec{N})$ to an oriented tangent vector to $d \times K$, and that τ maps $(d, k, (e_2, e_3))$ to the standard frame $(1, i)$ of $D_{\eta_0}(\times k) \subset \mathbb{C}$. Pick a representative y of y in $\check{C}_B[D_{\frac{1}{2}}]$. For $\eta \in]0, \eta_0]$, let $L(\eta^2 y \times K)$ denote the tangle obtained by replacing $\{0\} \times \mathbb{R}_K$ with $\eta^2 y \times \mathbb{R}_K$ in $D_{\eta_0} \times \mathbb{R}_K$.

Let us first reduce the proof of Lemma 17.16 to the proof of the following lemma.

Lemma 17.18. *There exist $\eta_1 \in]0, \eta_0]$ and volume 1 forms $(\omega(i, S^2))$ of S^2 for $i \in \underline{3N}$ such that for any $\eta \in]0, \eta_1]$ and for any subset A of $\underline{3N}$ whose cardinality is a multiple of 3,*

$$Z(\mathcal{C}, L(\eta^2 y \times K), \tau, A, (\omega(i, S^2))_{i \in \underline{3N}}) = \pi(B \times K)^* Z(\mathcal{C}, L, \tau, A, (\omega(i, S^2))_{i \in \underline{3N}}).$$

Lemma 17.19. *Lemma 17.18 implies Lemma 17.16.*

PROOF: The consistent definition 16.41 and Lemma 16.30 allow us to write

$$\begin{aligned}
&\mathcal{Z}^f(\mathcal{C}, L(y \times K), A, (\omega(i, S^2))_{i \in \underline{3N}}) \\
&= \lim_{\eta \to 0} \mathcal{Z}^f(\mathcal{C}, L(\eta^2 y \times K), A, (\omega(i, S^2))_{i \in \underline{3N}}) \\
&= \pi(B \times K)^* \mathcal{Z}^f(\mathcal{C}, L, A, (\omega(i, S^2))_{i \in \underline{3N}}).
\end{aligned}$$

$\square$

A *special Jacobi $\underline{3N}$–diagram* on $B \times \mathbb{R}$ is a connected graph Γ_s with univalent vertices, trivalent vertices, and one bivalent vertex, without looped edges, equipped with an injection j_E from its set of edges $E(\Gamma_s)$ into $\underline{3N}$ and with an isotopy class of injections j_{Γ_s} from its set $U(\Gamma_s)$ of univalent vertices into $B \times \mathbb{R}$. The space of these diagrams is denoted by $\mathcal{D}^{e,\mathbf{special}}_{\underline{3N}}(B \times \mathbb{R})$. If Γ_s has univalent vertices on at least two strands, then the space of configurations of such a graph with respect to y is the space $\check{\mathcal{V}}(y,\Gamma_s)$ of injections of the set $V(\Gamma_s)$ of vertices of Γ_s into $\mathbb{C}\times\mathbb{R}$ whose restriction to $U(\Gamma_s)$ is the composition of $y\times 1_{\mathbb{R}}$ with an injection from $U(\Gamma_s)$ into $B\times\mathbb{R}$ in the isotopy class $[j_{\Gamma_s}]$, up to vertical translation, for our representative $y\in\check{C}_B[D_{\frac{1}{2}}]$. This space $\check{\mathcal{V}}(y,\Gamma_s)$ is similar to former spaces $\check{\mathcal{V}}(y,\Gamma)$ of Section 15.2 and is compactified as in Chapter 15. See also Lemma 16.24. Its compactification is its closure in $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$. If Γ_s has no univalent vertices or univalent vertices on one strand, then configurations are also considered up to dilation, and the compactification is again the closure in $\mathcal{S}_{V(\Gamma)}(\mathbb{R}^3)$. The configurations of $\check{\mathcal{V}}(y,\Gamma_s)$ are normalized so that a vertex of Γ_s is sent to $D_{1/2}\times\{0\}$. (The need of such diagrams will be clear in the statement of Lemma 17.23.)

Lemma 17.20. *The set $\mathcal{O}(L,y)$ of points $(X_i)_{i\in\underline{3N}}$ of $(S^2_H)^{3N}$ that are regular with respect to*

- *the maps $g(\Gamma)$ of Lemma 17.3 associated to $\underline{3N}$-numbered Jacobi diagrams Γ on $B_\infty\times\mathbb{R}$ and to a configuration y_∞ of a finite set B_∞ that contains the configuration $y\colon B\hookrightarrow D_1$, the bottom configuration y_- of L, the top configuration y_+ of L, and 0,*
- *similar maps $g(\Gamma_s)$ associated to special $\underline{3N}$-numbered Jacobi diagrams Γ_s (as above with one bivalent vertex) on $\sqcup_{b\in B}\mathbb{R}_b$ and to the configuration y,*

is a dense open subset of $(S^2_H)^{3N}$.

PROOF: The arguments of Lemma 17.3 allow us to prove that the images of the above maps $g(\Gamma)$ and of the maps $g(\Gamma_s)$, when the univalent vertices of Γ_s are on one strand or when Γ_s has no univalent vertices, are compact semi-algebraic subsets of $(S^2_H)^{3N}$ of codimension at least one. The complement of the union of these images is therefore open and dense.

For a special Γ_s with univalent vertices on at least 2 strands, the images under $g(\Gamma_s)$ of the boundary of the configuration space, and of the parts where $g(\Gamma_s)$ is not a submersion, are also compact semi-algebraic subsets of $(S^2_H)^{3N}$, of codimension at least one. $\square$

Lemma 17.21. *Let $(X_i)_{i\in\underline{3N}}$ be the center of a tiny ball of radius $\varepsilon_0>0$ in the set $\mathcal{O}(L,y)$ defined in Lemma 17.20. There exists $\varepsilon_1\in\left]0,\frac{\varepsilon_0}{\sqrt{3N}}\right[$ and $M_1\in]1,+\infty[$ such that for any family $(\omega(i,S^2))_{i\in\underline{3N}}$ of volume 1 forms of S^2 supported inside disks $D_{i,\varepsilon}$ of S^2 centered at X_i, with radius $\varepsilon\in]0,\varepsilon_1]$,*

- *for any Jacobi diagram* $\Gamma \in \mathcal{D}^e_{\underline{3N}}(B_\infty \times \mathbb{R})$*, and for any special Jacobi diagram* $\Gamma \in \mathcal{D}^{e,\mathbf{special}}_{\underline{3N}}(B \times \mathbb{R})$ *with univalent vertices on at most one strand, the support of* $\wedge_{e \in E(\Gamma)}(p_{S^2} \circ p_e)^*(\omega(j_E(e), S^2))$ *in* $C(\mathbb{R}^3, y_\infty \times K; \Gamma)$ *is empty.*

- *for any special Jacobi diagram* $\Gamma \in \mathcal{D}^{e,\mathbf{special}}_{\underline{3N}}(B \times \mathbb{R})$*, the support of*

$$\wedge_{e \in E(\Gamma)}(p_{S^2} \circ p_e)^*(\omega(j_E(e), S^2))$$

in the space $\check{\mathcal{V}}(y, \Gamma)$ *introduced before Lemma 17.20 is contained in disjoint open subsets where* $\prod_{e \in E(\Gamma)}(p_{S^2} \circ p_e)$ *is a diffeomorphism onto* $\prod_{e \in E(\Gamma)} \mathring{D}_{j_E(e),\varepsilon}$*, and where the distance of the images of a vertex under two configurations is at most* $M_1\varepsilon$*, and the images of the vertices under the configurations of the support are contained in* $D_{M_1} \times [-M_1, M_1]$*.*

PROOF: The definition of $\mathcal{O}(L, y)$ in Lemma 17.20 and the hypotheses on $(X_i)_{i \in \underline{3N}}$ in Lemma 17.22 guarantee that the first assertion is satisfied for all $\varepsilon_1 \in]0, \frac{\varepsilon_0}{\sqrt{3N}}[$. Proceed as in Section 11.4, to reduce the support to disjoint subsets, where $\prod_{e \in E(\Gamma)}(p_{S^2} \circ p_e)$ is a diffeomorphism onto $\prod_{e \in E(\Gamma)} \mathring{D}_{j_E(e),\varepsilon_1}$. (It is simpler, here.) Then there exists an M such that the distance of the images of a vertex under two configurations is at most $M\varepsilon$ on the preimage of $\prod_{e \in E(\Gamma)} \mathring{D}_{j_E(e),\varepsilon}$ under such a diffeomorphism, for any $\varepsilon \in]0, \varepsilon_1]$. $\square$

Choose a Riemannian metric on $\check{R}(\mathcal{C})$, which coincides with the standard metric of $\mathbb{R}^3$ outside $\check{R}_{1,[0,1]}(\mathcal{C})$, and assume that this Riemannian metric restricts as the natural product metric on $N_{\eta_0}(K)$, locally. (Reduce η_0 if it is necessary.)

Let $C_{2,\leq 10\eta_0}(N_{\eta_0}(K))$ denote the closure in $C_2(R(\mathcal{C}))$ of the space of pairs of points $(x_1, x_2) \in N_{\eta_0}(K)^2$ at a distance less than $10\eta_0$ from each other. Naturally extend p_τ to $C_{2,\leq 10\eta_0}(N_{\eta_0}(K))$ by viewing $D_{\eta_0} \times \mathbb{R}_K$ as a subspace of $\mathbb{R}^3$, locally, with the usual formula $p_\tau = \frac{1}{\|x_2 - x_1\|}(x_2 - x_1)$.

Let $\Gamma \in \mathcal{D}^e_{\underline{3N}}(\mathcal{L})$. Let $U_K(\Gamma) = j_\Gamma^{-1}(\mathbb{R}_K)$, where $\mathbb{R}_K$ is viewed as the source of K. For $\eta \in]0, \eta_0]$, let $C(R(\mathcal{C}), L, \eta; \Gamma)$ be the configuration space obtained from $C(R(\mathcal{C}), L; \Gamma)$ by replacing the condition that the restriction of the configurations to univalent vertices on $\mathbb{R}_K$ factors through K and through the restriction to $U_K(\Gamma)$ of a representative j_Γ of the given isotopy class of injections from $U(\Gamma)$ to $\mathcal{L}$ with the condition that $U_K(\Gamma)$ is mapped to the interior of $N_\eta(K)$. (In other words, the conditions on the restriction of a configuration c to $U(\Gamma)$ now only impose that $c(U_K(\Gamma)) \subset \mathring{N}_\eta(K)$, and that $c_{|U(\Gamma) \setminus U_K(\Gamma)}$ may be expressed as $L \circ j_{\Gamma|U(\Gamma) \setminus U_K(\Gamma)}$ for some j_Γ in the given isotopy class of injections.) There is a natural projection p_K from this configuration space $C(R(\mathcal{C}), L, \eta; \Gamma)$ to $\mathring{D}_\eta^{U_K(\Gamma)}$, and $C(R(\mathcal{C}), L; \Gamma)$ is contained in the preimage $p_K^{-1}((0)^{U_K(\Gamma)})$ in $C(R(\mathcal{C}),L,\eta;\Gamma)$. (This preimage also contains $C(R(\mathcal{C}),L;\tilde{\Gamma})$ for Jacobi diagrams $\tilde{\Gamma}$ that differ from Γ because the linear order of the vertices of $U_K(\Gamma)$ is not induced by j_Γ.)

Lemma 17.22. *Let $(X_i)_{i\in\underline{3N}}$ and ε_1 be as in Lemma 17.21. There exist propagating chains $P(i)$ of $(C_2(R(\mathcal{C})),\tau)$ and ε_1–dual propagating forms $\omega(i)$ of $(C_2(R(\mathcal{C})),\tau)$, for $i\in\underline{3N}$, and $\eta_2\in]0,\eta_0]$ such that,*

- *$P(i)\cap D(p_\tau)=p_\tau^{-1}(X_i)\cap D(p_\tau)$ for any $i\in\underline{3N}$,*
- *$P(i)\cap C_{2,\leq 10\eta_2}(N_{\eta_2}(K))=p_\tau^{-1}(X_i)\cap C_{2,\leq 10\eta_2}(N_{\eta_2}(K))$, for the above natural extension of p_τ on $C_{2,\leq 10\eta_2}(N_{\eta_2}(K))$,*
- *the $P(i)$ are in general $3N$-position with respect to $(\check{R}(\mathcal{C}),L,\tau)$, (again as in Definition 11.3)*
- *$\omega(i)$ restricts to $D(p_\tau)\cup C_{2,\leq 10\eta_2}(N_{\eta_2}(K))$ as $p_\tau^*(\omega(i,S^2))$ for some form $\omega(i,S^2)$ as in Lemma 17.21.*

Let $\Gamma\in\mathcal{D}^e_{\underline{3N}}(\mathcal{L})$.

For $(\omega(i))$ as above, and for $\eta\in]0,\eta_2]$, let $Supp(\Gamma,\eta;(\omega(i)))$ denote the support of $\bigwedge_{e\in E(\Gamma)}p_e^*(\omega(j_E(e)))$ in $C(R(\mathcal{C}),L,\eta;\Gamma)$.

For $(P(i))$ as above, for any configuration c in the discrete set

$$I_S(L,\Gamma,(P(i)))=C(R(\mathcal{C}),L;\Gamma)\bigcap\cap_{e\in E(\Gamma)}p_e^{-1}(P(j_E(e))),$$

for any edge e of $E(\Gamma)$, $p_e(c)$ is in the interior of a 4-cell of $P(j_E(e))$, and $C_2(R(\mathcal{C}))$ is diffeomorphic to $D_e\times P(j_E(e))$ near $p_e(c)$, where $D_e\subset D_{\varepsilon_1}$ is the local fiber of a tubular neighborhood of $P(j_E(e))$. Let p_{c,D_e} be the associated projection onto D_e in a neighborhood of c in $C(R(\mathcal{C}),L,\eta;\Gamma)$. Without loss of generality, assume that $\omega(j_E(e))$ may be expressed as $p_{c,D_e}^*(\omega_\varepsilon)$ locally, for a 2-form ω_ε supported on $D_\varepsilon\subset D_e$, such that $\int_{D_\varepsilon}\omega_\varepsilon$ is the rational coefficient of the above 4–cell of $P(j_E(e))$.

With that notation, there exist $\varepsilon\in]0,\varepsilon_1]$, propagating chains $P(i)$, ε–dual propagating forms $\omega(i)$, as above, $\eta_3\in]0,\eta_2]$ and $M_2>1$ such that for any $\eta\in]0,\eta_3]$ and for any Jacobi diagram $\Gamma\in\mathcal{D}^e_{\underline{3N}}(\mathcal{L})$,

- *$Supp(\Gamma,\eta;(\omega(i)))$ is contained in disjoint submanifolds, indexed by the configurations c of $I_S(\Gamma,(P(i)))$, and which contain those, where $p_K\times\left(p_E=\prod_{e\in E(\Gamma)}p_{c,D_e}\right)$ is a diffeomorphism onto $\mathring{D}_\eta^{U_K(\Gamma)}\times\mathring{D}_\varepsilon^{E(\Gamma)}$, and where the distance of the images of a vertex under two configurations of $(p_K\times p_E)^{-1}\left(\mathring{D}_\eta^{U_K(\Gamma)}\times\{W\}\right)$ is at most $M_2\eta$, for any $W\in\mathring{D}_\varepsilon^{E(\Gamma)}$,*

- *the involved configurations map vertices of* $V(\Gamma) \setminus U_K(\Gamma)$ *at a distance greater than* $9M_2\eta$ *from* K,

- *they map two distinct vertices of* $U_K(\Gamma)$ *at a distance greater than* $9M_2\eta$ *from each other.*

PROOF: The existence of the $P(i)$ in general $3N$-position with prescribed behaviour near the boundary can be proved as in Section 11.3. Fix a graph $\Gamma \in \mathcal{D}^e_{\underline{3N}}(\mathcal{L})$. Once the $P(i)$ are in general $3N$-position, for such a given graph Γ, $I_S(\Gamma, (P(i)))$ consists of a finite number of isolated intersection points at which trivalent vertices cannot be on K, since this would correspond to a degenerate configuration for a graph, for which the trivalent vertex on K is replaced with 3 univalent vertices.

The existence of a family $(\omega(i))_{i\in\underline{3N}}$ of propagating forms of $R(\mathcal{C})$, ε_1-dual to $P(i)$, which may be expressed as $p_\tau^*(\omega(i,S^2))$ on $D(p_\tau) \cup C_{2,\leq 10\eta_2}(N_{\eta_2}(K))$, with respect to a family $(\omega(i,S^2))_{i\in\underline{3N}}$ of 2-forms of volume 1 supported inside a disk $D_{i,\varepsilon}$, as in Lemma 17.21, can be proved as in Section 11.4.

Let $c \in I_S(\Gamma, (P(i)))$. Transversality implies that the restriction to a neighborhood of c in $C(R(\mathcal{C}), L; \Gamma)$ of p_E is a submersion, with the notation of the statement. This implies that the restriction of $p_K \times p_E$ to a neighborhood of c in $C(R(\mathcal{C}), L, \eta; \Gamma)$ is a submersion, too, so that it is a local diffeomorphism in a neighborhood of c in $C(R(\mathcal{C}), L, \eta; \Gamma)$. After reducing η and ε, we obtain a neighborhood $N(c)$ of c in $C(R(\mathcal{C}), L, \eta; \Gamma)$ such that $p_K \times p_E$ is a diffeomorphism from $N(c)$ onto $\mathring{D}_\eta^{U_K(\Gamma)} \times \mathring{D}_\varepsilon^{E(\Gamma)}$ for all $c \in I_S(\Gamma, (P(i)))$. The compact intersection of $C(R(\mathcal{C}), L; \Gamma)$ with the complement of $\cup_{c\in I_S(\Gamma,(P(i)))} N(c)$ is mapped outside $\prod_{e\in E(\Gamma)} P(j_E(e))$ by $\prod_{e\in E(\Gamma)} p_e$. Therefore, its image avoids a neighborhood of $\prod P(j_E(e))$. Reducing ε allows us to assume that it avoids the compact closure of the support of $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))$. We may now reduce η so that $C(R(\mathcal{C}), L, \eta; \Gamma)$ does not meet this compact closure outside $\cup_{c\in I_S(\Gamma,(P(i)))} N(c)$. This can be achieved for all the finitely many considered Γ, simultaneously.

Now, the Jacobians of the corresponding inverse local diffeomorphisms (viewed as maps from $D_\eta^{U_K(\Gamma)} \times \prod_{e\in E(\Gamma)} D_\varepsilon$ to $R(\mathcal{C})^{V(\Gamma)} \setminus$ diag) are bounded (after reducing ε and η if necessary).

So we get a M_2 for which the distance of the images of a vertex under two configurations of any $(p_K \times p_E)^{-1}\left(\mathring{D}_\eta^{U_K(\Gamma)} \times \{W\}\right)$ in a connected component of $\mathrm{Supp}(\Gamma, \eta; (\omega(i)))$ is at most $M_2\eta$, for all the finitely many considered Γ. It is easy to reduce η_3 so that the last two conditions are satisfied, with our given M_2. $\square$

Set $\mathcal{D}^e_{\leq N,\underline{3N}}(.) = \cup_{k=0}^N \mathcal{D}^e_{k,\underline{3N}}(.)$.

For $\Gamma \in \mathcal{D}^e_{\leq N,\underline{3N}}(\mathcal{L}(B \times \mathbb{R}_K))$, let $I_S(L(\eta^2 y \times K), \Gamma, (P(i))_{i\in\underline{3N}})$ denote the set of configurations c of $C(R(\mathcal{C}), L(\eta^2 y \times K); \Gamma) \bigcap \bigcap_{e\in E(\Gamma)} p_e^{-1}(P(j_E(e)))$ with respect to the propagating chains $P(i)$.

The following crucial lemma justifies the introduction of special Jacobi diagrams.

Lemma 17.23. *Let η_1 be the minimum in the set $\{\eta_3, \frac{1}{(2N-1)8NM_1}, \frac{1}{100NM_2}\}$ of positive numbers introduced in Lemmas 17.21 and 17.22. For any family $(P(i))_{i\in\underline{3N}}$ of propagating chains as in Lemma 17.22, for any*

$$\Gamma_B \in \mathcal{D}^e_{\leq N,\underline{3N}}(\mathcal{L}(B \times \mathbb{R}_K)),$$

for any configuration c_{η_1} of the set $I_S(L(\eta_1^2 y \times K), \Gamma_B, (P(i))_{i\in\underline{3N}})$, there exists a continuous map

$$\begin{array}{lll}]0,\eta_1] & \to & C_V(\Gamma_B)(R(\mathcal{C})) \\ \eta & \mapsto & c_\eta \end{array}$$

such that, for any $\eta \in]0,\eta_1]$, $c_\eta \in I_S(L(\eta^2 y \times K), \Gamma_B, (P(i))_{i\in\underline{3N}})$ and the graph Γ_B configured by c_η is the union of

- *(small red) special Jacobi diagrams Γ_s on $B \times \mathbb{R}_K$ of diameter less than 10η configured on $\eta^2 y \times K$ in $N_\eta(K)$ (with univalent vertices on at least two strands of $B \times \mathbb{R}_K$), and*
- *a unitrivalent (blue and purple) graph Γ on the source $\mathcal{L}$ of L configured so that its univalent vertices are*
  - *either univalent vertices of Γ_B on $(L \setminus K) \cup (\eta^2 y \times K)$,*
  - *or trivalent vertices of Γ_B attached to a bivalent vertex of a (small red) special graph Γ_s, as in Figure 17.1.*

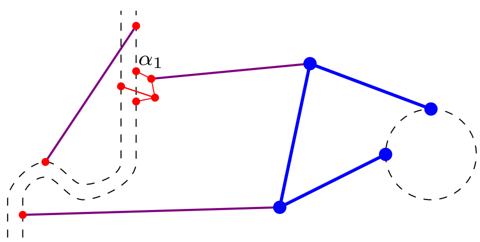


Figure 17.1: A configured diagram Γ_B with its small red vertices, its thin red edges, its big blue vertices and its thick blue edges

Furthermore, the configuration c_η arises as a transverse intersection point, and the intersections that involve at least one (red) special Jacobi diagram cancel algebraically. For a fixed η, the remaining configurations are in natural one-to-one correspondence –independent of η– with triples (Γ, f, c'), where $\Gamma \in \mathcal{D}^e_{\leq N,\underline{3N}}(\mathcal{L})$, $U_K(\Gamma)$ is the set of univalent vertices of Γ_B on $B \times \mathbb{R}_K$, $f \in B^{U_K(\Gamma)}$ and $c' \in I_S(L, \Gamma, (P(i))_{i\in\underline{3N}})$. The inverse of this natural one-to-one correspondence maps (Γ, f, c') to a pair (Γ_f, c_η), where

- *Γ_f is a Jacobi diagram on $\mathcal{L}(B \times \mathbb{R}_K)$ obtained from Γ by changing the (isotopy class of the) injection from $U_K(\Gamma)$ to $\mathbb{R}_K$ to the injection from $U_K(\Gamma)$ to $B \times \mathbb{R}_K$ that maps a vertex u to $f(u) \times \mathbb{R}_K$ so that the order of vertices on each strand of $B \times \mathbb{R}_K$ is induced by their former order on $\mathbb{R}_K$,*
- *$c_\eta \in I_S(L(\eta^2 y \times K), \Gamma_f, (P(i))_{i\in\underline{3N}})$,*
- *$d(c_\eta(v), c'(v))$ is smaller than η for all vertices v of Γ,*

and the sign of the algebraic intersection at c_η is the same as the sign of the algebraic intersection at c', with respect to consistent vertex-orientations of Γ and Γ_f.

Furthermore, for any $\eta \in]0, \eta_1]$, the support $Supp(L(\eta^2 y \times K), \Gamma_B; (\omega(i)))$ of $\bigwedge_{e\in E(\Gamma_B)} p_e^*(\omega(j_E(e)))$ in $C(R(\mathcal{C}), L(\eta^2 y \times K); \Gamma_B)$ consists of disjoint neighborhoods $N_{L(\eta^2 y\times K)}(c_\eta)$ of configurations c_η as above, where projections p_{c_η, D_e} as in the statement of Lemma 17.22 make sense, and such that the restriction of $\prod_{e\in E(\Gamma_B)} p_{c_\eta,D_e}$ to $N_{L(\eta^2 y\times K)}(c_\eta)$ is a diffeomorphism onto $\mathring{D}_\varepsilon^{E(\Gamma_B)}$.

PROOF: Let $\Gamma_B \in \mathcal{D}^e_{\leq N,\underline{3N}}(\mathcal{L}(B \times \mathbb{R}_K))$. Instead of starting with a configuration $c_{\eta_1} \in I_S(L(\eta_1^2 y \times K), \Gamma_B, (P(i))_{i\in\underline{3N}})$, as in the statement, we consider a configuration c in $\mathrm{Supp}(L(\eta_4^2 y \times K), \Gamma_B; (\omega(i)))$, for some $\eta_4 \in]0, \eta_1]$. View Γ_B as a graph configured by c. So its vertices become elements of $R(\mathcal{C})$. Color the vertices of Γ_B in $N_{\eta_1{}^2}(K)$ with $(\mathrm{red}, 1)$. Next color by $(\mathrm{red}, 2)$ its uncolored vertices that are at a distance less than $4{\eta_1}^2 M_1$ from a vertex colored by $(\mathrm{red}, 1)$. For $k \geq 2$, inductively color the still uncolored vertices that are at a distance less than $4{\eta_1}^2 M_1$ from a vertex v colored by (red, k), by $(\mathrm{red}, k + 1)$.

Let

$$\begin{array}{rlcl} r\colon & \check{R}(\mathcal{C}) & \to & [0, \eta_1] \\ & (z_D, t) \in D_{\eta_1} \times \mathbb{R}_K & \mapsto & |z_D| \\ & x \in R(\mathcal{C}) \setminus (\mathring{D}_{\eta_1} \times \mathbb{R}_K) & \mapsto & \eta_1. \end{array}$$

Note that a vertex v colored by (red, k) with $k \geq 2$ satisfies ${\eta_1}^2 \leq r(v) \leq 4k{\eta_1}^2 M_1$, by induction, so $r(v) \leq 8N{\eta_1}^2 M_1 \leq \eta_1$.

Color the vertices that are still uncolored after this algorithm blue. Color the edges between two blue vertices blue. Color the edges between a red vertex and a blue one purple. Also color the edges between red vertices at a distance greater than $8N{\eta_1}^2M_1$ purple. Color the remaining edges between two red vertices red.

Remove the red edges between red vertices, and the red vertices that do not belong to a purple edge from Γ_B. Blow up the obtained graph $\tilde{\Gamma}$ at red vertices that belong to at least two purple edges, so that these red vertices become univalent. A red vertex that belongs to r purple edges is transformed into r red vertices during this process. Let Γ' be the obtained configured uni-trivalent graph with blue and purple edges. Its red vertices are in $N_{\eta_1}(K)$. Let $U_K(\Gamma')$ denote the set of red vertices of Γ'. The restriction of c to $V(\Gamma')$ is in $C(R(\mathcal{C}), L, \eta_1; \Gamma') \bigcap \mathrm{Supp}\left(\bigwedge_{e \in E(\Gamma')} p_e^*(\omega(j_E(e)))\right)$. So, according to Lemma 17.22, it is in one of the disjoint submanifolds where $p_K \times p_E$ restricts as a diffeomorphism onto $\mathring{D}_{\eta_1}^{U_K(\Gamma')} \times \mathring{D}_{\varepsilon}^{E(\Gamma')}$, with $p_E = \prod_{e \in E(\Gamma')} p_{c,D_e}$, and where the distance of the images of a vertex under two configurations of $(p_K \times p_E)^{-1}\left(\mathring{D}_{\eta}^{U_K(\Gamma')} \times \{W\}\right)$ is at most $M_2\eta$, for any $W \in \mathring{D}_{\varepsilon}^{E(\Gamma')}$. Set $W_0 = p_E(c)$. (When $c \in I_S(L(\eta_4^2 y \times K), \Gamma_B, (P(i)))$, $W_0 = 0 = (0)_{e \in E(\Gamma')}$.) Then $c' = (p_K \times p_E)^{-1}((0)_{U_K(\Gamma')}, W_0)$ is a configuration of a graph Γ on $\mathcal{L}$, obtained from Γ' by adding the data of an isotopy of injections of $U_K(\Gamma')$ into $\mathbb{R}_K$, where $U_K(\Gamma) = U_K(\Gamma')$. According to Lemma 17.22, no collision of vertices of Γ can occur. So the red vertices of Γ' were univalent in $\tilde{\Gamma}$ (there was no need to blow them up) and $\tilde{\Gamma} = \Gamma'$.

Furthermore, c' maps two red vertices at a distance at least $9M_2\eta_1$ from each other. In particular, two red vertices of $\tilde{\Gamma}$ are at a distance at least $7M_2\eta_1$ from each other, with respect to c.

The univalent vertices of $\tilde{\Gamma}$ are either univalent vertices of Γ_B, which are sent to $\eta_4^2 y \times K$ or to $L \setminus K$ by c, or trivalent vertices of Γ_B, which belong to a bivalent vertex of a red subgraph of Γ_B. Let Γ_R be the subgraph of Γ_B consisting of its red vertices and of its red edges.

Let $\Gamma_{R,1}$ be a connected component of Γ_R such that $\Gamma_{R,1}$ is not reduced to a univalent vertex. Since two vertices of $\Gamma_{R,1}$ are at a distance at most $(2N-1)8N{\eta_1}^2M_1 \leq \eta_1$ from each other, there is a most one red vertex of $\tilde{\Gamma}$ in $\Gamma_{R,1}$, and c sends $\Gamma_{R,1}$ to a part of $N_{\eta_1}(K)$ that is identified with a part of $\mathbb{R}^3$ of diameter less than $10\eta_1$. So such a $\Gamma_{R,1}$ configured by c may be viewed as a graph with straight edges, directed by the X_i if $c \in I_S(L(\eta_4^2 y \times K), \Gamma_B, (P(i)))$, and by $\tilde{W}_i \in D_{i,\varepsilon}$, in general. In particular, Lemma 17.21 implies that $\Gamma_{R,1}$ must be a configured special Jacobi diagram in $D_{{\eta_1}^2M_1} \times [x - {\eta_1}^2M_1, x + {\eta_1}^2M_1]$. Its configuration $\eta_4^2 c_{R,1}$ is determined up to translation along $\mathbb{R}_K$. The projection of the bivalent vertex α_1 of $\Gamma_{R,1}$ to

$D_{\eta_1{}^2 M_1}$ is $p_{\mathbb{C}}(\eta_4{}^2 c_{R,1}(\alpha_1))$.

Let $\mathcal{A}$ denote the set of bivalent vertices of Γ_R. Write the corresponding configured special Jacobi diagrams $(\Gamma_{R,\alpha}, c_{R,\alpha})_{\alpha\in\mathcal{A}}$. Let $U_K(\Gamma_B) = j_{\Gamma_B}^{-1}(B \times \mathbb{R}_K)$. Note the natural inclusions $U_K(\Gamma) \subseteq U_K(\Gamma_B) \sqcup \mathcal{A}$ and $\mathcal{A} \subseteq U_K(\Gamma)$. Let $f_{\Gamma_B} \colon U_K(\Gamma_B) \to B$ be the map that sends $u \in j_{\Gamma_B}^{-1}(\{b\} \times \mathbb{R}_K)$ to b. Let f denote the restriction of f_{Γ_B} to $U_K(\Gamma) \setminus \mathcal{A}$.

So far, our analysis allows us to associate

$$\Phi(\Gamma_B, c) = (\Gamma, c', \mathcal{A} \subset U_K(\Gamma), f \colon U_K(\Gamma) \setminus \mathcal{A} \to B, (\Gamma_{R,\alpha}, c_{R,\alpha})_{\alpha\in\mathcal{A}})$$

to our configured graph (Γ_B, c) as above, where

- $\Gamma \in \mathcal{D}^e_{\leq N,\underline{3N}}(\mathcal{L})$,
- $c' \in \mathrm{Supp}(L, \Gamma; (\omega(i))) = C(R(\mathcal{C}), L; \Gamma) \bigcap \mathrm{Supp}\left(\bigwedge_{e\in E(\Gamma)} p_e^*(\omega(j_E(e)))\right)$, $(p_E(c') = W_0)$;

  when $c \in I_S(L(\eta_4^2 y \times K), \Gamma_B, (P(i)))$, $c' \in I_S(L, \Gamma, (P(i)))$.)
- the $\Gamma_{R,\alpha}$ are special Jacobi diagrams on $B \times \mathbb{R}_K$ whose disjoint union is numbered in $\underline{3N} \setminus j_E(\Gamma)$,
- the $c_{R,\alpha}$ are configurations of these diagrams with respect to $y \times \mathbb{R}$ in $\bigcap_{e\in E(\Gamma_{R,\alpha})}(p_{S^2} \circ p_e^{-1})(\tilde{W}_{j_E(e)})$, (or in $\bigcap_{e\in E(\Gamma_{R,\alpha})}(p_{S^2} \circ p_e^{-1})(X_{j_E(e)})$ when $c \in I_S(L(\eta_4^2 y \times K), \Gamma_B, (P(i)))$).

Let us now show how to reconstruct (Γ_B, c) from the data

$$(\Gamma, c', \mathcal{A} \subset U_K(\Gamma), f \colon U_K(\Gamma) \setminus \mathcal{A} \to B, (\Gamma_{R,\alpha}, c_{R,\alpha})_{\alpha\in\mathcal{A}})\,.$$

The graph Γ_B is obtained from Γ, the $\Gamma_{R,\alpha}$ by gluing Γ and the $\Gamma_{R,\alpha}$ at the vertices α. The components of its univalent vertices are determined by the components of the univalent vertices of $\Gamma_{R,\alpha}$ and of Γ outside $\mathcal{A}$, and by f. Their order on a strand of $B \times \mathbb{R}_K$ is the restriction to the vertices of such a strand of the order induced on $U_K(\Gamma_B)$ by letting the ordered set of the univalent vertices of $\Gamma_{R,\alpha}$ replace α, for every $\alpha \in \mathcal{A}$, in the ordered set the univalent vertices of Γ.

Below, we construct a smooth embedding

$$\mathbf{c} \colon]0, \eta_1] \times \mathring{D}_\varepsilon^{E(\Gamma_B)} \hookrightarrow C_{V(\Gamma_B)}(R(\mathcal{C}))$$

such that $c(\eta, W_B) \in C(R(\mathcal{C}), L(\eta^2 y \times K); \Gamma_B)$ for any $(\eta, W_B) \in]0, \eta_1] \times \mathring{D}_\varepsilon^{E(\Gamma_B)} \hookrightarrow C_{V(\Gamma_B)}(R(\mathcal{C}))$ and our configuration c is equal to $\mathbf{c}(\eta_4, W_{B,0})$.

Lemma 17.21 guarantees that $p_{E,\alpha} = \prod_{e \in E(\Gamma_{R,\alpha})}(p_{S^2} \circ p_e)$ is a diffeomorphism from a neighborhood of $c_{R,\alpha}$ in the configuration space of configurations of $\Gamma_{R,\alpha}$ on $y \times \mathbb{R}_K$ up to translation along $\mathbb{R}_K$ onto $\prod_{e \in E(\Gamma_{R,\alpha})} \mathring{D}_{j_E(e),\varepsilon}$, for every $\alpha \in \mathcal{A}$. Identify $D_{j_E(e),\varepsilon}$ with $D_\varepsilon = \phi_e(D_{j_E(e),\varepsilon})$, implicitly, and set $W_{\alpha,0}(= (\phi_e \circ p_{S^2} \circ p_e(c_{R,\alpha}))_{e \in E(\Gamma_{R,\alpha})}) = p_{E,\alpha}(c_{R,\alpha})$. Lemma 17.22 provides a neighborhood of c' in $C(R(\mathcal{C}), L, \eta_1; \Gamma)$, where $p_K \times p_E$ is a diffeomorphism onto $\mathring{D}_{\eta_1}^{U_K(\Gamma)} \times \mathring{D}_\varepsilon^{E(\Gamma)}$. Recall that $p_E(c) = W_0$. These diffeomorphisms assemble to form a diffeomorphism Ψ_η from a neighborhood of $c(\eta, W)$ in $C(R(\mathcal{C}), L(\eta^2 y \times K); \Gamma_B)$ to $\mathring{D}_\varepsilon^{E(\Gamma_B)}$, whose inverse $\Psi_\eta^{-1} = \mathbf{c}(\eta, .)$ is described below. Write $W_B = (W_e)_{e \in E(\Gamma_B)} = ((W_\alpha)_{\alpha \in \mathcal{A}}, W)$, where $W_\alpha = (W_e)_{e \in E(\Gamma_{R,\alpha})}$ and $W = (W_e)_{e \in E(\Gamma)}$. $\Psi_\eta^{-1}(W)$ is constructed from representatives $\eta^2 c_{R,\alpha}(\alpha)$ (where the vertical translation parameter is fixed) of the $\eta^2 p_{E,\alpha}^{-1}(W_\alpha)$ by assembling them with

$$c_1(\eta, W) = (p_K \times p_E)^{-1}\left(\left((\eta^2 y(f(u)))_{u \in U_K(\Gamma)\setminus\mathcal{A}}, (p_{\mathbb{C}}(\eta^2 c_{R,\alpha}(\alpha)))_{\alpha \in \mathcal{A}}\right), W\right)$$

so that the height (projection onto $\mathbb{R}_K$) $p_{\mathbb{R}}(\eta^2 c_{R,\alpha}(\alpha))$ of α in $\eta^2 c_{R,\alpha}(\alpha)$ coincides with $p_{\mathbb{R}}(c_1(\eta, W)(\alpha))$.

This ensures that $\mathbf{c}(\eta, W_B) = \Psi_\eta^{-1}(W_B)$ arises as a transverse intersection, for any W_B. This is true when $W_B = 0$, in particular, so it is true for the configuration c_{η_1} of the statement, which may be expressed as $\mathbf{c}(\eta_1, 0)$. The family c_η is the continuous family $\Psi_\eta^{-1}(0)$, in this case. The sign of the corresponding intersection is the sign of the Jacobian of Ψ_η. Since we started with an arbitrary configuration in $\mathrm{Supp}(L(\eta_4^2 y \times K), \Gamma_B; (\omega(i)))$ for some $\eta_4 \in]0, \eta_1]$, the above arguments also prove the final assertion of the lemma. Let us finally focus on the claimed algebraic cancellation. From now on, $W = 0$, and it is not mentioned any longer.

Assume that $\mathcal{A} \neq \emptyset$. Every $\alpha \in \mathcal{A}$ is contained in one edge $e(\alpha)$ of Γ. Choose α_0 in $\mathcal{A}$ such that $j_E(e(\alpha_0))$ is minimal. Let $s(\Gamma_{R,\alpha_0})$ be obtained from Γ_{R,α_0} by exchanging the labels of the two edges e_1 and e_2 of Γ_{R,α_0} that contain α_0, and by reversing their orientations if they both come from α_0 or go to α_0, as in Lemma 9.11, let $s_{\alpha_0}(\Gamma_B)$ be obtained from Γ_B by performing the same changes. Let $s(c_{R,\alpha_0})$ be obtained from c_{R,α_0} by changing the position of $c_{R,\alpha_0}(\alpha_0)$ by a central symmetry with respect to the middle of the two other ends of e_1 and e_2. The intersection point associated to the configured graph

$$(s_{\alpha_0}(\Gamma_B), c_2) = \Phi^{-1}\left(\Gamma, c', \mathcal{A}, f, (\Gamma_{R,\alpha}, c_{R,\alpha})_{\alpha \in \mathcal{A}\setminus\{\alpha_0\}}, s(\Gamma_{R,\alpha_0}), s(c_{R,\alpha_0}), 0\right)$$

and the intersection point associated to (Γ_B, c) cancel algebraically, as in Lemma 9.11. (Our process defines an involution on the configured graphs

(Γ_B, c) such that $\mathcal{A} \neq \emptyset$ such that the image of a configured graph and a configured graph cancel.)

Therefore, the configured graphs (Γ_B, c) that contribute to the intersection are the graphs for which $\mathcal{A} = \emptyset$. They are obtained from some $\Gamma \in \mathcal{D}^e_{\leq N,\underline{3N}}(\mathcal{L})$ some c' and some $f\colon U_K(\Gamma) \to B$ as in the statement.

$\square$

PROOF OF LEMMA 17.18: Lemma 17.23 implies that

- for any $\Gamma \in \mathcal{D}^e_{\leq N,\underline{3N}}(\mathcal{L})$, $I(\mathcal{C}, L, \Gamma, o(\Gamma), (\omega(i))_{i\in\underline{3N}})$ is the algebraic intersection $I(\mathcal{C}, L, \Gamma, o(\Gamma), (P(i))_{i\in\underline{3N}})$ of the preimages of the propagating chains $P(i)$ in $C(R(\mathcal{C}), L; \Gamma)$ with respect to Γ,
- for any $\eta \in]0, \eta_1]$ and for any $\Gamma_B \in \mathcal{D}^e_{\leq N,\underline{3N}}(\mathcal{L}(B \times \mathbb{R}_K))$,

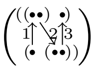
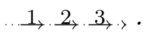
  $$I(\mathcal{C}, L(\eta^2 y{\times}K), \Gamma_B, o(\Gamma_B), (\omega(i))) = I(\mathcal{C}, L(\eta^2 y{\times}K), \Gamma_B, o(\Gamma_B), (P(i))),$$

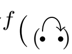
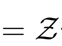
  and,
- for any $\eta \in]0, \eta_1]$ and for any subset A of $\underline{3N}$ with cardinality $3k$,

  $$\sum_{\Gamma\in\mathcal{D}^e_{k,A}(\mathcal{L})} \zeta_\Gamma I(\mathcal{C}, L, \Gamma, (P(i))_{i\in A})\pi(B \times K)^*([\Gamma]) =$$
  $$\sum_{\Gamma_B\in\mathcal{D}^e_{k,A}(\mathcal{L}(B\times\mathbb{R}_K))} \zeta_{\Gamma_B} I(\mathcal{C}, L(\eta^2 y \times K), \Gamma_B, (P(i))_{i\in A})[\Gamma_B].$$

$\square$

Now, both the second duplication property and the first duplication property of Theorem 13.12 for components that go from bottom to top or from top to bottom are proved. Below, we prove the first duplication property, more generally, in the doubling case.

Lemma 17.24. *Let ν be the element of $\mathcal{A}(\mathbb{R})$ be the element obtained from* $\mathcal{Z}^f$  $\in \mathcal{A}(\uparrow_1\uparrow_2\uparrow_3)$ *by inserting* $\uparrow_1\uparrow_2\uparrow_3$ *in* $\mathbb{R}$ *as indicated by the picture* . *Then*

$\mathcal{Z}^f($$) = \mathcal{Z}^f($$) = \nu^{-\frac{1}{2}},$

where $\nu^{-\frac{1}{2}}$ is the unique element of $\mathcal{A}(\mathbb{R})$ whose degree 0 part is 1 such that $(\nu^{-\frac{1}{2}})^2\nu = 1$.

Proof: Use Theorem 13.12 except for the first duplication property, which is about to be proved. By symmetry, $\mathcal{Z}^f(\underset{(\bullet\,\bullet)}{\frown}) = \mathcal{Z}^f(\overset{(\bullet\,\bullet)}{\smile})$. By isotopy invariance $\mathcal{Z}^f\left(\mathrm{N}\right) = \mathcal{Z}^f\left(\uparrow\right) = 1$. By functoriality,

$$\mathcal{Z}^f\left(\mathrm{N}\right) = \mathcal{Z}^f\left(\mathrm{N}\right) = \mathcal{Z}^f\left(\underset{\uparrow}{\bullet}\,\overset{(\bullet\,\bullet))}{\smile}\right)\mathcal{Z}^f\left(\begin{smallmatrix}((\bullet\bullet)\ \bullet)\\ 1\uparrow\searrow 2\uparrow 3\\ (\bullet\ (\bullet\bullet))\end{smallmatrix}\right)\mathcal{Z}^f\left(\underset{((\bullet\,\bullet)}{\frown}\,\underset{\bullet)}{\uparrow}\right).$$

The cabling property implies that $\mathcal{Z}^f\left(\underset{\uparrow}{\bullet}\,\overset{(\bullet\,\bullet))}{\smile}\right)$ is obtained from $\mathcal{Z}^f\left(\overset{(\bullet\,\bullet)}{\smile}\right)$ by the map induced by the natural injection from $\overset{(\bullet\,\bullet)}{\smile}$ to $\underset{\uparrow}{\bullet}\,\overset{(\bullet\,\bullet))}{\smile}$. $\mathcal{Z}^f\left(\underset{((\bullet\,\bullet)}{\frown}\,\underset{\bullet)}{\uparrow}\right)$ is obtained similarly from $\mathcal{Z}^f\left(\underset{(\bullet\,\bullet)}{\frown}\right)$. Since the insertions of $\mathcal{Z}^f\left(\overset{(\bullet\,\bullet)}{\smile}\right)$ and $\mathcal{Z}^f\left(\underset{(\bullet\,\bullet)}{\frown}\right)$ can be performed at arbitrary places according to Proposition 6.22,

$$\mathcal{Z}^f(\underset{(\bullet\,\bullet)}{\frown})\mathcal{Z}^f(\overset{(\bullet\,\bullet)}{\smile})\nu = 1$$

in the algebra $\mathcal{A}(\mathbb{R})$, where elements, whose degree 0 part is the class of the empty diagram, have a unique inverse and a unique square root whose degree 0 part is the class of the empty diagram. $\square$

Lemma 17.25. *The first duplication property of Theorem 13.12 is true when K is the unique component of the tangle $\underset{(\bullet\,\bullet)}{\frown}$ (resp. $\overset{(\bullet\,\bullet)}{\smile}$). In other words,*

$$\mathcal{Z}^f\left(2\times\overset{(\bullet\,\bullet)}{\smile}\right) = \pi(2\times\mathbb{R})^*\left(\mathcal{Z}^f(\overset{(\bullet\,\bullet)}{\smile})\right)$$

and $\mathcal{Z}^f\left(2\times\underset{(\bullet\,\bullet)}{\frown}\right) = \pi(2\times\mathbb{R})^*\left(\mathcal{Z}^f(\underset{(\bullet\,\bullet)}{\frown})\right)$.

Proof: Again $\mathcal{Z}^f\left(2\times\overset{(\bullet\,\bullet)}{\smile}\right) = \mathcal{Z}^f\left(2\times\underset{(\bullet\,\bullet)}{\frown}\right)$ by symmetry. Thus $\mathcal{Z}^f\left(2\times\underset{(\bullet\,\bullet)}{\frown}\right)$ can be computed from $\mathcal{Z}^f\left(2\times\mathrm{N}\right)$ as $\mathcal{Z}^f\left(\underset{(\bullet\,\bullet)}{\frown}\right)$ is computed from $\mathcal{Z}^f\left(\mathrm{N}\right)$ in the proof of Lemma 17.24. Indeed the boxes $\mathcal{Z}^f\left(2\times\underset{(\bullet\,\bullet)}{\frown}\right)$ and $\mathcal{Z}^f\left(2\times\overset{(\bullet\,\bullet)}{\smile}\right)$ can slide across the duplicated strands of

$$\begin{aligned}\pi(2\times\uparrow_1\uparrow_2\uparrow_3)^*\left(\mathcal{Z}^f\left(\begin{smallmatrix}((\bullet\bullet)\ \bullet)\\ 1\uparrow\searrow 2\uparrow 3\\ (\bullet\ (\bullet\bullet))\end{smallmatrix}\right)\right) &= \mathcal{Z}^f\left(\begin{smallmatrix}((\bullet\bullet)\ \bullet)\\ 1\uparrow\searrow 2\uparrow 3\\ (\bullet\ (\bullet\bullet))\end{smallmatrix}(2\times\uparrow_1)(2\times\uparrow_2)(2\times\uparrow_3)\right)\\ &= \pi(2\times\uparrow_1)^*\pi(2\times\uparrow_2)^*\pi(2\times\uparrow_3)^*\left(\mathcal{Z}^f\left(\begin{smallmatrix}((\bullet\bullet)\ \bullet)\\ 1\uparrow\searrow 2\uparrow 3\\ (\bullet\ (\bullet\bullet))\end{smallmatrix}\right)\right),\end{aligned}$$

according to Lemma 6.30, so that we get

$$\mathcal{Z}^f\left(2\times\underset{(\bullet\,\bullet)}{\frown}\right)^2\pi(2\times\mathbb{R})^*(\nu) = 1$$

in the algebra $\mathcal{A}(\mathbb{R}\sqcup\mathbb{R})$. Since $\pi(2\times\mathbb{R})^*$ is an algebra morphism from $\mathcal{A}(\mathbb{R})$ to $\mathcal{A}(\mathbb{R}\sqcup\mathbb{R})$, Lemma 17.24 implies that

$$\pi(2\times\mathbb{R})^*\left(\mathcal{Z}^f(\underset{(\bullet\,\bullet)}{\frown})^2\right)\pi(2\times\mathbb{R})^*(\nu) = 1$$

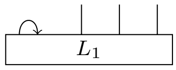
in the algebra $\mathcal{A}(\mathbb{R} \sqcup \mathbb{R})$. Since the multiplication by an element whose degree 0 part is 1 is injective, and since an element whose degree 0 part is 1 is determined by its square, we conclude that $\mathcal{Z}^f\left(2 \times \right) = \pi(2 \times \mathbb{R})^*\left(\mathcal{Z}^f(\,)\right)$ as wanted. $\square$

Proposition 17.26. *Let K be a component of a q–tangle L in a rational homology cylinder $\mathcal{C}$. Let $L(2 \times K)$ be the tangle obtained by duplicating K as in Section 13.1. Then*

$$\mathcal{Z}^f(L(2 \times K)) = \pi(2 \times K)^* \mathcal{Z}^f(L).$$

PROOF: A tangle L, with a strand K that goes from bottom to bottom, can be written as a composition

$$L_1 L_2 =$$



of some tangle L_1, with a cabling L_2 of a trivial braid by the replacement of a strand by , where K is the concatenation of one strand of L_1 that goes from bottom to top, , and another strand of L_1, which goes from top to bottom. The statement for such a pair follows from Lemma 17.25, the cabling property Proposition 17.12 and Proposition 17.15, using functoriality. The case in which K goes from top to top can be treated similarly, by sending below. So the proposition is proved. $\square$

Theorem 13.12 is now proved. $\square$

Remark 17.27. Theorem 16.42 and Proposition 16.40 do not allow us to prove that for any $N \in \mathbb{N}$, for any subset A of $\underline{3N}$ whose cardinality is a multiple of 3, for any family of volume-one forms $(\omega(i, S^2))_{i \in \underline{3N}}$,

$$\mathcal{Z}^f(\mathcal{C}, L(2 \times K), A, (\omega(i, S^2))_{i \in \underline{3N}}) = \pi(2 \times K)^* \mathcal{Z}^f(\mathcal{C}, L, A, (\omega(i, S^2))_{i \in \underline{3N}})$$

for a $J_{bb,tt}$-oriented q–tangle L. Indeed, there is no reason to believe that $\pi(2 \times \mathbb{R})^*\left(\widetilde{\mathrm{hol}}_{[0,t]}(\eta(., S_{WE}))\right)$ is the product of twice $\widetilde{\mathrm{hol}}_{[0,t]}(\eta(., S_{WE}))$ on the two strands of $2 \times \mathbb{R}$. Unfortunately, as noticed in Remark 16.45, we do not know how to get rid of our non-canonical normalization of $\mathcal{Z}^f(\mathcal{C}, L, A, (\omega(i, S^2))_{i \in \underline{3N}})$ and of the corresponding factors $\widetilde{\mathrm{hol}}_{[0,t]}(\eta(., S_{WE}))$, which might not behave well under duplication.

17.5 Behaviour of $\mathcal{Z}^f$ with respect to the coproduct

The behaviour of $\mathcal{Z}^f$ with respect to the coproduct described in Theorem 13.12 is justified after the statement of Theorem 13.12. Let us see how this behaviour generalizes to the variants of $\mathcal{Z}^f$.

Proposition 17.28. *For any q–tangle L in $\mathcal{C}$ in a rational homology cylinder $\mathcal{C}$, for any $N \in \mathbb{N}$, for any subset A of cardinality $3k$ of $\underline{3N}$, and for any family $(\omega(i,S^2))_{i\in\underline{3N}}$ of volume-one forms of S^2,*

$$\Delta_k\left(\mathcal{Z}^f(\mathcal{C},L,A,(\omega(i,S^2))_{i\in\underline{3N}})\right) =$$

$$\sum_{i=0}^{k}\sum_{\substack{A_1\subset A,\\ \sharp A_1=3i,\\ A_2=A\setminus A_1}}\frac{\sharp A_1!\sharp A_2!}{\sharp A!}\mathcal{Z}^f(\mathcal{C},L,A_1,(\omega(i,S^2)))\otimes\mathcal{Z}^f(\mathcal{C},L,A_2,(\omega(i,S^2)))$$

with the coproduct maps Δ_n defined in Section 6.5.

PROOF: First observe that the statement is true for q–tangles that can be represented as straight tangles with respect to a parallelization τ such that $p_1(\tau)=0$, by Theorem 16.13. The coefficients are treated as in Lemma 17.9. The compatibility between product and coproduct implies the following lemma.

Lemma 17.29. *Say that a map F from the set $\mathcal{P}_{(3)}(\underline{3N})$ of subsets of $\underline{3N}$ whose cardinalities are multiple of 3, to a space of Jacobi diagrams is* group-like *if for any element B of $\mathcal{P}_{(3)}(\underline{3N})$,*

$$\Delta(F(B)) = \sum_{(B_1,B_2)\in P_2(B)}\frac{(\sharp B_1)!(\sharp B_2)!}{(\sharp B)!}F(B_1)\otimes F(B_2).$$

Let F and G be two group-like maps from $\mathcal{P}_{(3)}(\underline{3N})$ to spaces $\mathcal{A}_F$ and $\mathcal{A}_G$ of Jacobi diagrams, such that there is a product from $\mathcal{A}_F\times\mathcal{A}_G$ to a space of Jacobi diagrams $\mathcal{A}_{FG}$, and the product $(FG)_\sqcup$ of Definition 16.8 makes sense. Then $(FG)_\sqcup$ is group-like, too.

PROOF: Let $A\in\mathcal{P}_{(3)}(\underline{3N})$.

$$\Delta((FG)_\sqcup(A)) = \sum_{(B,C)\in P_2(A)}\frac{(\sharp B)!(\sharp C)!}{(\sharp A)!}\Delta(F(B))\Delta(G(C))$$

$$= \sum_{(B_1,B_2,C_1,C_2)\in P_4(A)} \frac{(\sharp B_1)!(\sharp B_2)!(\sharp C_1)!(\sharp C_2)!}{(\sharp A)!}(F(B_1)G(C_1)\otimes F(B_2)G(C_2)),$$

where

$$(FG)_{\sqcup}(A_1) = \sum_{(B_1,C_1)\in P_2(A_1)} \frac{(\sharp B_1)!(\sharp C_1)!}{(\sharp A_1)!}F(B_1)G(C_1),$$

so

$$\Delta((FG)_{\sqcup}(A)) = \sum_{(A_1,A_2)\in P_2(A)} \frac{(\sharp A_1)!(\sharp A_2)!}{(\sharp A)!}(FG)_{\sqcup}(A_1)\otimes (FG)_{\sqcup}(A_2).$$

□

Say that a map F from $\mathcal{P}_{(3)}(\underline{3N})$ to a space $\mathcal{A}_F$ of Jacobi diagrams is *cardinality-determined* if it maps any element A of $\mathcal{P}_{(3)}(\underline{3N})$ to a degree $\frac{\sharp A}{3}$ element $F_{\frac{\sharp A}{3}}$ which depends only on the cardinality of A. The truncation $F_{\leq N} = (F_k)_{k\in\mathbb{N}|k\leq N}$ until degree N of any element $(F_k)_{k\in\mathbb{N}}$ of $\mathcal{A}_F$ can be viewed as such a cardinality-determined map. Note that such a truncation of a group-like element is group-like in the sense of the previous lemma.

Let T be a trivial q-braid (represented by a constant path) except for the framing of one of its strands K, which is $lk(K,K_{\parallel}) = 1$ instead of 0. According to Definition 16.14, $\mathcal{Z}^f(D_1\times[0,1],T,.,(\omega(i,S^2))) = \exp(\alpha)\sharp_K[\emptyset]$. In particular, this expression does not depend on $(\omega(i,S^2))_{i\in\underline{3N}}$, and $\mathcal{Z}^f(D_1\times[0,1],T,.,(\omega(i,S^2)))$ is group-like. Similarly, $\exp_{\leq N}\left(-\frac{1}{4}p_1(\tau)\beta(.)\right)_{\sqcup}$ is group-like. Therefore, Lemma 17.29 allows us to conclude the proof of Proposition 17.28 for framed tangles with injective bottom and top configurations. Use Lemma 16.30 to conclude for general q–tangles. □

Proposition 17.30. *Let $(\omega(i,S^2))_{i\in\underline{3N}}$ be a family of volume-one forms of S^2. Let $\mathcal{C}$ be a rational homology cylinder. Let L be a $J_{bb,tt}$-oriented q–tangle of $\mathcal{C}$. Let $\check{Z}(\mathcal{C},L,.,(\omega(i,S^2)))$ denote the projection of $\mathcal{Z}^f(\mathcal{C},L,.,(\omega(i,S^2)))$ on $\check{\mathcal{A}}(\mathcal{L})$. Then*

$$\mathcal{Z}^f(\mathcal{C},L,.,(\omega(i,S^2))_{i\in\underline{3N}}) = \left(\check{Z}(\mathcal{C},L,.,(\omega(i,S^2))_{i\in\underline{3N}})\mathcal{Z}^f_{\leq N}(\mathcal{C},\emptyset)\right)_{\sqcup}.$$

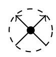
PROOF: According to Theorem 16.42, $\mathcal{Z}^f(\mathcal{C},\emptyset,.,(\omega(i,S^2))_{i\in\underline{3N}})$ does not depend on $(\omega(i,S^2))_{i\in\underline{3N}}$. □

17.6 A proof of universality

In this section, we apply the properties of $\mathcal{Z}^f$ to a prove a generalization of the universality theorem 6.9.

Define a *singular tangle representative with* n *double points* to be an oriented manifold of dimension 1 immersed in $\mathcal{C}$ whose boundary sits in the interior of $D_1 \times \{0,1\}$, and that meets a neighborhood $N(\partial(D_1 \times [0,1]))$ as vertical segments, such that the only singular points of the immersion are n double points  for which the directions of the two meeting branches generate a *tangent plane*. Define a *singular tangle with* n *double points* to be an equivalence class of such representatives under the equivalence relation defined as in the non-singular case in Definition 12.15, by adding the adjective singular.

Extend the invariant $\mathcal{Z}$ of unframed singular tangles by the local rule

$\mathcal{Z}($  $) = \mathcal{Z}($  $) - \mathcal{Z}($  $).$

This local rule relates the invariants $\mathcal{Z}$ of three singular tangles that coincide outside the represented ball and that are as in the pictures in this ball.

Define the *chord diagram* $\Gamma_C(L)$ associated to an unframed singular tangle L with n double points, to be the diagram on the source $\mathcal{L}$ of L with $2n$ vertices, which are univalent, located at the preimages of the double points, and with n edges, one between each pair of preimages of a double point. These edges are called *chords*. (The chords are attached on the left-hand side of the oriented source $\mathcal{L}$, when orientations of univalent vertices are needed, as in Definition 6.16.)

Notation 17.31. Denote the images of $\mathcal{Z}$ and of $\check{p} \circ \mathcal{Z} = \check{\mathcal{Z}}$ under the quotient by the $1T$-relation by $\overline{\mathcal{Z}}$ and $\overline{\check{\mathcal{Z}}}$, respectively. ($\overline{\mathcal{Z}}(L) \in \mathcal{A}(\mathcal{L})/(1T)$ and $\overline{\check{\mathcal{Z}}}(L) \in \check{\mathcal{A}}(\mathcal{L})/(1T)$.)

In this section, we prove the following theorem, which is a generalization of Theorem 6.9 from knots to links and tangles.

Theorem 17.32. *Let n be a natural integer. For any unframed singular tangle L with n double points in a rational homology cylinder $\mathcal{C}$, the expansion $\overline{\mathcal{Z}}_{\leq n-1}(L)$ up to degree $n-1$ of $\overline{\mathcal{Z}}(L)$ vanishes and its expansion $\overline{\mathcal{Z}}_{\leq n}(L)$ up to degree n is equal to*

$$\overline{\mathcal{Z}}_{\leq n}(L) = [\Gamma_C(L)]$$

in $\mathcal{A}(\mathcal{L})/(1T)$, so $\overline{\check{\mathcal{Z}}}_{\leq n}(L) = [\Gamma_C(L)]$ in $\check{\mathcal{A}}(\mathcal{L})/(1T)$.

Theorem 6.9 and its proof, whose easiest part is presented in Section 6.2, generalize to k-component oriented links, with numbered components, to produce the following corollary to Theorem 17.32. The isomorphism of the corollary was first shown by Bar-Natan and Kontsevich [BN95a].

Corollary 17.33. *With the notation of Section 6.1, for any integer k, $\overline{\check{\mathcal{Z}}}_{\leq n}$ induces an isomorphism from $\frac{\mathcal{F}_n(\mathcal{K}_k;\mathbb{Q})}{\mathcal{F}_{n+1}(\mathcal{K}_k;\mathbb{Q})}$ to $\check{\mathcal{A}}(\sqcup_{i=1}^k (S^1)_i)/(1T)$, where $(S^1)_i$ is the copy of S^1 associated with the i^{th} component of a link.*

□

In order to prove Theorem 17.32, we first define framed singular tangles and extend $\mathcal{Z}^f$ to these tangles. A *parallelization* of a singular tangle is an isotopy class of parallels as in the nonsingular case, with the same restrictions near the boundary, where the parallel of a neighborhood of a double point is on one side of the tangent plane of the double point . Recall that there are two ways of desingularizing , the *positive* one for which is replaced by , and the *negative* one for which is replaced by . In particular every desingularization of such a singular tangle gets a natural parallelization from the parallelization of the singular tangle. Locally, the parallel of each branch is well defined. When the branches of double points involve different components, the self-linking number of a component of a desingularization does not depend on the desingularization.

In general, we may define the self-linking number of a component of a singular framed tangle as before, where the *components* of a singular framed tangle are in natural one-to-one correspondence with the components of its source. When desingularizing at a double point c for which both branches belong to a component K_j, the self-linking number $lk((K_j, K_{j\|}) \subset L(c,+))$ (resp. $lk((K_j, K_{j\|}) \subset L(c,-))$) of K_j in the positive (resp. negative) desingularization $L(c,+)$ (resp. $L(c,-)$) at c is related to the self-linking number $lk((K_j, K_{j\|}) \subset L)$ by the relation

$$lk((K_j, K_{j\|}) \subset L(c,+)) = lk((K_j, K_{j\|}) \subset L) + 1$$

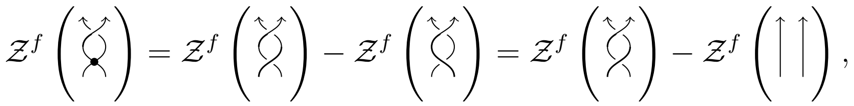
(resp $lk((K_j, K_{j\|}) \subset L(c,-)) = lk((K_j, K_{j\|}) \subset L) - 1$).

We extend $\mathcal{Z}^f$ to (framed) singular q–tangles, formally, by the formula

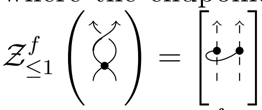
$$\mathcal{Z}^f(\quad) = \mathcal{Z}^f(\quad) - \mathcal{Z}^f(\quad),$$

where the parallels of the three tangles are supposed to be behind, and to match on the boundary of the ball.

As an example,



where the endpoints of the tangles are supposed to lie on the real line. So



, according to Lemmas 12.18 and 12.19.

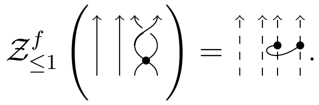
Note that $\mathcal{Z}^f$ is now a functor on the category of singular q–tangles, which satisfies the cabling property and the duplication properties of Theorem 13.12 provided that the components involved in a double point are not duplicated.

Proposition 17.34. *Let n be a natural integer. For any singular q–tangle L with n double points,*

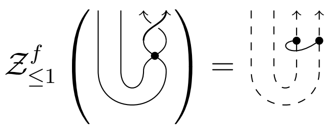
$$\mathcal{Z}^f_{\leq n}(L) = [\Gamma_C(L)] .$$

Proof: In the proof below, we evaluate the lowest degree part that does not vanish in $\mathcal{Z}^f(L)$ for various singular q–tangles. Note that this part is unchanged when such a singular q–tangle is multiplied by a non-singular q–tangle (except for the modification of the sources) since the lowest degree part that does not vanish for a non-singular q-tangle is the class of the empty diagram. In particular, the lowest degree non-vanishing part does not depend on the bottom and top configurations of our q–tangles, which will not be specified.

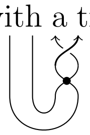
Applying the cabling property of $\mathcal{Z}^f$ to the following cable of a trivial braid with three strands makes clear that

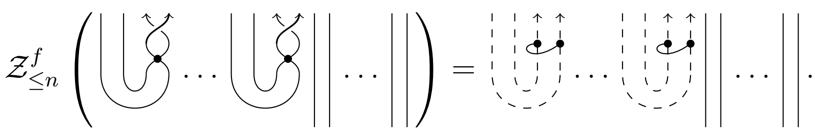


Now, by functoriality,



as wanted for such a q–tangle.

Starting with a trivial braid, and cabling some of its strands by replacing n of them by  successively, we find that



(Here, we first apply the cabling property to the first strand of a trivial braid $\mathbf{1}_{n+k}$, and regard the resulting tangle T_1 as a product $\mathbf{1}_{n+k-1}T_1$, where $\mathbf{1}_{n+k-1}$ is a trivial braid. We next apply the cabling property to the first strand of $\mathbf{1}_{n+k-1}$, and we keep going to find the obtained formula, with the help of the functoriality property.)

Since every singular q–tangle with n double points can be written as a product of a q-tangle as above and a non-singular q–tangle, by moving the double points below, the proposition follows. □

We are ready to deduce Theorem 17.32 from Proposition 17.34.

PROOF OF THEOREM 17.32: Assume that the singular q–tangle L with n double points has k components K_i for $i = 1, \ldots, k$.

The case $n = 0$ is obvious. Assume $n = 1$. Let s_i denote the self-linking number of K_i in L. Let L^+ be the positive desingularization of L , and let L^- be the negative desingularization of L. Let s_i^+ (resp. s_i^-) denote the self-linking number of the i^{th} component in L^+ (resp. in L^-). Recall that $\overline{\mathcal{Z}}$ is the image of $\mathcal{Z}$ under the quotient by the $1T$-relation, and that

$$\mathcal{Z}(L^+) = \prod_{j=1}^{k} \left(\exp(-s_j^+\alpha)\sharp_j\right) \mathcal{Z}^f(L^+).$$

If the two strands involved in the double point belong to two distinct components, then $s_j^+ = s_j = s_j^-$ for any $j \in \underline{k}$, and

$$\mathcal{Z}(\times) - \mathcal{Z}(\times) = \prod_{j=1}^{k} (\exp(-s_j\alpha)\sharp_j) \left(\mathcal{Z}^f(\times) - \mathcal{Z}^f(\times)\right).$$

So the result follows since the lowest degree part of any $\exp(-s_j\alpha)$ is the class of the empty diagram.

If the two strands involved in the double point belong to the same component K_i, then

$$\begin{aligned}\mathcal{Z}(\times) - \mathcal{Z}(\times) &= \textstyle\prod_{j=1}^k \left(\exp(-s_j^+\alpha)\sharp_j\right) \left(\mathcal{Z}^f(\times) - \mathcal{Z}^f(\times)\right) \\ &+ \left(\textstyle\prod_{j=1}^k \left(\exp(-s_j^+\alpha)\sharp_j\right) - \prod_{j=1}^k \left(\exp(-s_j^-\alpha)\sharp_j\right)\right) \mathcal{Z}^f(\times),\end{aligned}$$

where $s_i^+ = s_i^- + 2$, and the $\left(\prod_{j=1}^k \left(\exp(-s_j^+\alpha)\sharp_j\right) - \prod_{j=1}^k \left(\exp(-s_j^-\alpha)\sharp_j\right)\right)$ "factor" begins with its degree one part, which is $-2\alpha_1 = -$ []. So

$$\overline{\mathcal{Z}}_{\leq 1}(\times) - \overline{\mathcal{Z}}_{\leq 1}(\times) = \left[\ \right]$$ 

as wanted. Note that this equality would be wrong if $\overline{\mathcal{Z}}$ was replaced by $\mathcal{Z}$, that is, without moding out by $1T$.

Let us now conclude the proof by induction on n. Assume that the result is known for singular q–tangles with less than n double points. Let C denote the set of double points of L. For $i \in \underline{k}$, let C_i denote the set of double points for which both branches belong to K_i. For a subset I of C, L_I denotes the q–tangle obtained from L by performing negative desingularizations on double points of I and positive ones on double points of $C \setminus I$, and $s_{j,I}$ denotes the self-linking number of the component $K_{j,I}$ in L_I.

Then $\overline{\mathcal{Z}}(L)$ is equal to

$$\begin{array}{rl} & \sum_{I\subseteq C}(-1)^{\sharp I}\overline{\mathcal{Z}}(L_I) \\ = & \sum_{I\subseteq C}(-1)^{\sharp I}\prod_{j=1}^{k}\left(\exp(-s_{j,I}\alpha)\sharp_j\right)\mathcal{Z}^f(L_I) \bmod 1\mathrm{T} \\ = & \sum_{I\subseteq C}(-1)^{\sharp I}\left(\prod_{j=1}^{k}\left(\exp(-s_{j,I}\alpha)\sharp_j\right)-\prod_{j=1}^{k}\left(\exp(-s_{j,\emptyset}\alpha)\sharp_j\right)\right)\mathcal{Z}^f(L_I) \\ & +\prod_{j=1}^{k}\left(\exp(-s_{j,\emptyset}\alpha)\sharp_j\right)\sum_{I\subseteq C}(-1)^{\sharp I}\mathcal{Z}^f(L_I) \bmod 1\mathrm{T}. \end{array}$$

The lowest degree term of the last line is $[\Gamma_C(L)]$. So it suffices to prove that the degree of the first line of the last right-hand side is greater than n (mod 1T). This sum can be rewritten as

$$T_2=\sum_{I\subseteq C}(-1)^{\sharp I}\left(1-\prod_{j=1}^{k}\left(\exp\left(\left(s_{j,I}-s_{j,\emptyset}\right)\alpha\right)\sharp_j\right)\right)\mathcal{Z}(L_I) \bmod 1\mathrm{T},$$

where $s_{j,I}-s_{j,\emptyset}=-2\sharp(I\cap C_j)=\sum_{c\in I\cap C_j}(-2)$, so T_2 can be rewritten as

$$T_2=\sum_{I\subseteq C}(-1)^{\sharp I}\left(1-\prod_{j=1}^{k}\left(\prod_{c\in I\cap C_j}\exp\left(-2\alpha_c\right)\sharp_j\right)\right)\mathcal{Z}(L_I) \bmod 1\mathrm{T},$$

where α_c is a copy of α.

$$\left(1-\prod_{j=1}^{k}\left(\prod_{c\in I\cap C_j}\exp\left(-2\alpha_c\right)\sharp_j\right)\right)=\sum_{D|J(D)\subseteq I}\beta_{D,J(D)}(D)[D]$$

is a sum over disjoint unions D of positive degree diagrams to be inserted to various components of the source $\mathcal{L}$ of L, since the sum has no degree zero term. Each of the connected components of D is of degree at least 3 – when working mod 1T, since $\alpha_1=\alpha_2=0$ –, and is associated to some double point, in the above expression. In particular, to each term D of degree at most n in this sum, we associate the non-empty set $J(D)$ of double points

c of $\cup_{j=1}^{k} C_j$ with positive degree parts of α_c occuring in D. The same term $\beta_{D,J(D)}(D)[D]$ appears in all subsets I of C that contain $J(D)$, so

$$T_2 = \sum_D \left(\sum_{I \subseteq C | J(D) \subseteq I} (-1)^{\sharp I} \overline{\mathcal{Z}}(L_I) \right) \beta_{D,J(D)}(D)[D],$$

where $(-1)^{\sharp J(D)} \sum_{I \subseteq C | J(D) \subseteq I} (-1)^{\sharp I} \overline{\mathcal{Z}}_{\leq (n - \sharp J(D))}(L_I)$ is the class of the chord diagram of the singular q–tangle with $(n - \sharp J(D))$ double points, obtained from L by desingularizing the double points of $J(D)$ in a negative way, by induction. Its degree is $n - \sharp J(D)$, while the degree of D is at least $3 \sharp J(D)$. Therefore the parts of T_2 of degree at most n vanish. $\square$

Part IV
Universality

Chapter 18

The main universality statements and their corollaries

18.1 Universality with respect to Lagrangian-preserving surgeries

Definition 18.1. An *integer (resp. rational) homology handlebody* of genus g is a compact oriented 3-manifold A that has the same integral (resp. rational) homology as the usual solid handlebody H_g of Figure 1.1. The *Lagrangian* $\mathcal{L}_A$ of a compact 3-manifold A is the kernel of the map induced by the inclusion from $H_1(\partial A;\mathbb{Q})$ to $H_1(A;\mathbb{Q})$.

Exercise 18.2. Show that if A is a rational homology handlebody of genus g, then ∂A is a connected genus g surface. (See Appendix A.1, where some basic properties of homology are recalled.)

In Figure 1.1, the Lagrangian of H_g is freely generated by the classes of the curves a_i.

Definition 18.3. An *integral (resp. rational) Lagrangian-Preserving (or LP) surgery* (A'/A) is the replacement of an integer (resp. rational) homology handlebody A, embedded in the interior of a 3-manifold M, by another such A', whose boundary $\partial A'$ is identified with ∂A by an orientation-preserving diffeomorphism which sends $\mathcal{L}_A$ to $\mathcal{L}_{A'}$. The manifold $M(A'/A)$ obtained by such an LP-surgery is

$$M(A'/A) = (M \setminus \mathrm{Int}(A)) \cup_{\partial A} A'.$$

(This only defines the topological structure of $M(A'/A)$, but $M(A'/A)$ is equipped with its unique smooth structure.)

An interesting example of an integral Lagrangian-preserving surgery is presented in Subsection 18.3.1. The Matveev Borromean surgery of Section 18.4 is another example of integral LP surgery.

Lemma 18.4. *If (A'/A) is an integral (resp. rational) LP-surgery, then the homology of $M(A'/A)$ with $\mathbb{Z}$-coefficients (resp. with $\mathbb{Q}$-coefficients) is canonically isomorphic to $H_*(M;\mathbb{Z})$ (resp. to $H_*(M;\mathbb{Q})$). If M is a $\mathbb{Q}$-sphere, if (A'/A) is a rational LP-surgery, and if (J,K) is a two-component link of $M \setminus A$, then the linking number of J and K in M and the linking number of J and K in $M(A'/A)$ coincide.*

PROOF: Exercise. □

Let (A'/A) be a rational LP surgery in a punctured rational homology sphere $\check{R}$. The Mayer-Vietoris long exact sequence (see Theorem A.11) shows that the canonical morphism

$$\partial_{MV} \colon H_2(A \cup_{\partial A} -A'; \mathbb{Q}) \to \mathcal{L}_A,$$

which maps the class of a closed surface in the closed 3–manifold $(A \cup_{\partial A} -A')$ to the boundary of its intersection with A, is an isomorphism. This isomorphism carries the algebraic triple intersection of surfaces to a trilinear antisymmetric form $\mathcal{I}_{AA'}$ on $\mathcal{L}_A$.

$$\mathcal{I}_{AA'}(a_i, a_j, a_k) = \langle \partial_{MV}^{-1}(a_i), \partial_{MV}^{-1}(a_j), \partial_{MV}^{-1}(a_k) \rangle_{A \cup -A'}$$

Let $(a_1, a_2, \ldots, a_g)$ be a basis of $\mathcal{L}_A$, and let $z_1, \ldots, z_g$ be (curves that represent) homology classes of ∂A, such that the system $(z_1, z_2, \ldots, z_g)$ is dual to $(a_1, a_2, \ldots, a_g)$ with respect to $\langle , \rangle_{\partial A}$:

$$\langle a_i, z_j \rangle_{\partial A} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Note that $(z_1, \ldots, z_g)$ is a basis of $H_1(A;\mathbb{Q})$.

Represent $\mathcal{I}_{AA'}$ by the following combination $T(\mathcal{I}_{AA'})$ of tripods whose three univalent vertices form an ordered set:

$$T(\mathcal{I}_{AA'}) = \sum_{\{\{i,j,k\} \subseteq \{1,2,\ldots,g\} \mid i<j<k\}} \mathcal{I}_{AA'}(a_i, a_j, a_k) \bullet\!\!<\begin{smallmatrix} z_j \\ z_j \\ z_i \end{smallmatrix}$$

where the tripods are considered up to the relations

$$\bullet\!\!<\begin{smallmatrix} z \\ y \\ x \end{smallmatrix} = \bullet\!\!<\begin{smallmatrix} x \\ z \\ y \end{smallmatrix} = -\bullet\!\!<\begin{smallmatrix} z \\ x \\ y \end{smallmatrix} = -\bullet\!\!<\begin{smallmatrix} z \\ -y \\ x \end{smallmatrix}.$$

Let G be a graph with $2k$ oriented trivalent vertices and with univalent vertices. Assume that the univalent vertices of G are decorated with disjoint curves of a $\mathbb{Q}$–sphere $\check{R}$. Let $P(G)$ be the set of partitions of the set of univalent vertices of G into disjoint pairs.

For $p \in P(G)$, identifying the two vertices of each pair provides a vertex-oriented trivalent diagram. This yields a *trivalent Jacobi diagram* Γ_p, which is a Jacobi diagram on $\emptyset$. Multiplying it by the product $\ell(p)$, over the disjoint pairs of p, of the linking numbers of the curves corresponding to the two vertices in a pair yields an element $[\ell(p)\Gamma_p]$ of $\mathcal{A}_k(\emptyset)$.

Define
$$\langle\langle G \rangle\rangle = \sum_{p \in P(G)} [\ell(p)\Gamma_p].$$

The contraction $\langle\langle . \rangle\rangle$ is linearly extended to linear combinations of graphs. The disjoint union of combinations of graphs is bilinear.

The universality theorem with respect to Lagrangian-preserving surgeries is the following one. It was proved in [Les04b] for the invariant $\mathcal{Z}$ of rational homology spheres. The statement below is more general since it applies to the invariant $\mathcal{Z}$ of Theorem 12.7, which satisfies the properties stated in Theorem 13.12. Nevertheless, its proof, which will be reproduced in this book, is identical to the proof of the preprint [Les04b], which has never been submitted for publication, except for a few improvements in the redaction.

Theorem 18.5. *Let L be a q–tangle representative in a rational homology cylinder $\mathcal{C}$. Let $x \in \mathbb{N}$. Let $\coprod_{i=1}^{x} A^{(i)}$ be a disjoint union of rational homology handlebodies embedded in $\mathcal{C} \setminus L$. Let $(A^{(i)\prime}/A^{(i)})$ be rational LP surgeries in $\mathcal{C}$. Set $X = [\mathcal{C}, L; (A^{(i)\prime}/A^{(i)})_{i \in \underline{x}}]$ and*
$$\mathcal{Z}_n(X) = \sum_{I \subseteq \underline{x}} (-1)^{x+\sharp I} \mathcal{Z}_n\left(\mathcal{C}_I, L\right),$$
where $\mathcal{C}_I = \mathcal{C}\left((A^{(i)\prime}/A^{(i)})_{i \in I}\right)$ is the rational homology cylinder obtained from $\mathcal{C}$ by performing the LP-surgeries that replace $A^{(i)}$ with $A^{(i)\prime}$ for $i \in I$. If $2n < x$, then $\mathcal{Z}_n(X)$ vanishes, and, if $2n = x$, then
$$\mathcal{Z}_n(X) = \left[\langle\langle \bigsqcup_{i \in \underline{x}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right) \rangle\rangle\right].$$

Before proving Theorem 18.5, we will discuss some of its consequences and variants. In Section 18.3, we show that it yields a direct proof of a surgery formula for the Theta invariant, as in [Les09, Section 9]. The article [Les09] presents many other surgery formulae implied by Theorem 18.5, which are not reproduced in this book.

In Section 18.4, we show how Theorem 18.5 implies that $\mathcal{Z}$ restricts to a universal finite type invariant of integer homology 3-spheres. In Section 18.5, we review the Moussard classification of finite type invariants of rational homology 3-spheres [Mou12], and we show how $\mathcal{Z}$ can be augmented to provide a universal finite type invariant of rational homology 3-spheres, too, following an idea of Gwénaël Massuyeau. In Section 18.6, we show how Theorem 18.5 also implies that the invariant $\frac{1}{6}\Theta$ is the Casson-Walker invariant. Assuming this identification, the surgery formula of Section 18.3 is nothing but a consequence of the Casson-Walker surgery formula of [Wal92], and Section 18.3 can be skipped by a reader who does not need examples, at first.

We sketch the proof of Theorem 18.5 in Section 18.7. The details of the proof will be completed in the following two chapters.

Theorem 18.5, the universality theorems 6.9 and 17.32 for knots or tangles can be put together in Theorem 18.34, which generalizes all of them.

In Section 18.8, we show how the main ingredients of the proof of Theorem 18.5 also lead to Theorem 18.37, which allows us to compute the degree 2 part of $\check{\mathcal{Z}}$, for any null-homologous knot, in Theorem 18.41, with the help of the contents of Section 18.3.

Section 18.2 below gives some background about Dehn surgeries, which will be used in Sections 18.3 and 18.4.

18.2 On Dehn surgeries

In this section, we give the definition of the manifold $R_{(K;p/q)}$ obtained by p/q–surgery on a $\mathbb{Q}$–sphere R along a knot K, we introduce the lens spaces $L(p,q)$, and we give some examples of surgeries on links, which will be used in Section 18.4.

The *exterior* of a knot in a 3-manifold is the closure of the complement of a tubular neighborhood of the knot. The manifold obtained by *Dehn surgery*, on a knot K in a 3-manifold, with respect to a non-separating simple closed curve μ of the boundary $\partial N(K)$ of a tubular neighborhood $N(K)$ of K, is the union $E(K) \cup_{\partial N(K)} T$ of the exterior $E(K)$ of K and a solid torus T, where $E(K)$ and T are glued along $\partial N(K)$ by an orientation-reversing homeomorphism from ∂T to $\partial N(K)$ that maps a meridian of T to μ. The result is next smoothed in a standard way. (Since the gluing of T can be achieved by gluing a meridian disk of T along μ, thickening it and gluing a 3-dimensional ball[1] to the resulting boundary, this surgery operation is well defined.)

[1] The operation of gluing a 3-ball B^3 along S^2 is well defined, because any homeomorphism f from S^2 to S^2 extends to B^3 as the homeomorphism that maps tx to $tf(x)$, for $t \in [0,1]$ and $x \in S^2$.

Example 18.6. The reader can check that the manifold obtained by Dehn surgery on the unknot U of S^3, with respect to its meridian $m(U)$ is $S^2 \times S^1$.

Let K be a knot in a rational homology sphere. If K is null-homologous, then K has a unique parallel $\ell(K)$ such that $lk(K, \ell(K)) = 0$. This parallel is called the *preferred longitude* of K. Let μ be a simple closed curve in the boundary $\partial N(K)$ of a tubular neighborhood of K such that μ does not separate $\partial N(K)$. The class of the curve μ in $H_1(\partial N(K))$ may be expressed as $pm(K) + q\ell(K)$, where $m(K)$ is the meridian of K. The coefficient of the Dehn surgery along K, with respect to μ, is $\frac{p}{q}$, and we refer to this Dehn surgery as the p/q–surgery on K. This coefficient $\frac{p}{q}$ may be expressed as $\frac{lk(K,\mu)}{\langle m(K),\mu\rangle_{\partial N(K)}}$. The p/q–surgery along a non-necessarily null-homologous knot K in a rational homology 3-sphere R is the Dehn surgery with respect to a non-separating simple closed curve μ of $\partial N(K)$ such that $\frac{lk(K,\mu)}{\langle m(K),\mu\rangle_{\partial N(K)}} = \frac{p}{q}$. Let $R_{(K;p/q)}$ denote the result of p/q–Dehn surgery on R along K. As shown in Example 18.6, $S^3_{(U;0)} = S^2 \times S^1$.

According to a theorem independently proved by Lickorish [Lic62] and Wallace [Wal60] in 1960, every closed oriented 3-manifold can be obtained from S^3 by surgery along a link of S^3 whose components are equipped with integers (surgeries are performed simultaneously along all the components of the link). In [Rou85], Rourke gave a quick and elementary proof of this result.

Examples 18.7. The reader can check that the manifold obtained by Dehn surgery on the trivial link of S^3 with g components, which are equipped with the coefficient 0, is the connected sum of g copies of $S^2 \times S^1$, and that this connected sum is homeomorphic to the manifold $H_g \cup_{\mathbf{1}_{\partial H_g}} (-H_g)$.

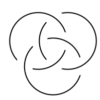


Figure 18.1: Borromean link

As a more difficult exercise, the reader can prove that the manifold obtained by Dehn surgery on the *Borromean link* of S^3, represented in Figure 18.1, whose components are equipped with the coefficient 0, is diffeomorphic to $(S^1)^3$. A hint can be found in [Thu78, Example 13.1.5].

Let p and q be two coprime integers, $p > 0$. View S^3 as the unit sphere of $\mathbb{C}^2$. The *lens space* $L(p, q)$ is the quotient by the action of $\mathbb{Z}/p\mathbb{Z}$, where the generator [1] of $\mathbb{Z}/p\mathbb{Z}$ acts on a unit vector (z_1, z_2) of $\mathbb{C}^2$ by mapping it to $\left(\exp\left(\frac{2i\pi}{p}\right) z_1, \exp\left(\frac{2i\pi q}{p}\right) z_2\right)$.

Let us study more surgeries along *unknots* or *trivial knots*, which are knots that bound an embedded disk, and prove the following well-known lemma.

Lemma 18.8. *Let k be an integer, and let U be a trivial knot. Then $S^3_{(U;1/k)} \cong S^3$. More generally, for any pair (a,b) of coprime integers such that $a > 0$, $S^3_{(U;a/(b+ka))}$ is diffeomorphic to $S^3_{(U;a/b)}$, and*

$$S^3_{(U;a/b)} = L(a, -b).$$

If U is a trivial knot of a 3-manifold M, then $M_{(U;a/b)} = M\sharp L(a,-b)$.

PROOF: The exterior E of the unknot U in S^3 is a solid torus whose meridian $m(E)$ is the preferred longitude $\ell(U)$ of U. The meridian $m(U)$ is the preferred longitude $\ell(E)$ of E and performing $\frac{a}{b}$-surgery along U on S^3 amounts to gluing a solid torus with meridian

$$\mu = am(U) + b\ell(U) = bm(E) + a\ell(E)$$

to E, where $\langle m(E), \mu\rangle_{\partial E} = a$ and $\langle \mu, \ell(E)\rangle_{\partial E} = b$. The manifold $S^3_{(U;a/b)}$ is the union of two solid tori E and T glued by a homeomorphism from $(-\partial T)$ to ∂E that maps the meridian of T to a curve μ as above.

Let $k \in \mathbb{Z}$, $(\ell(E) - km(E))$ is another parallel of E. This shows that for any coprime integers a and b, $S^3_{(U;a/(b+ka))}$ is diffeomorphic to $S^3_{(U;a/b)}$. In particular, $S^3_{(U;1/k)} \cong S^3$.

For a trivial knot U in a 3–manifold M, $M_{(U;a/b)}$ is the connected sum of $S^3_{(U;a/b)}$ and M. (The connected sum replaces a ball in the interior of the above solid torus E with the exterior of a ball which contains U in M.)

Below, S^3 will be viewed as the sphere of $\mathbb{C}^2$ with radius $\sqrt{2}$. The action of $\mathbb{Z}/p\mathbb{Z}$ on S^3, which defines the lens space $L(p,q)$, preserves the solid torus $|z_1| \leq |z_2|$, and the solid torus $|z_1| \geq |z_2|$. Let E be the quotient of the second torus

$$E = \frac{\left\{\left(\exp\left(\frac{2i\pi t}{p}\right)\sqrt{2-|z_2|^2}, z_2\right) \mid t \in [0,1], z_2 \in \mathbb{C}, |z_2| \leq 1\right\}}{\left((\sqrt{2-|z_2|^2}, z_2\right) \sim \left(\exp\left(\frac{2i\pi}{p}\right)\sqrt{2-|z_2|^2}, z_2\exp\left(\frac{2i\pi q}{p}\right)\right)}.$$

The meridian of the solid torus E is

$$m(E) = \{(1, \exp(2i\pi u)) \mid u \in [0,1]\}$$

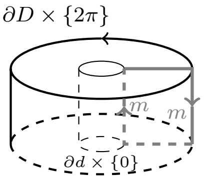


Figure 18.2: The gray image m of $m(U)$ in $D \times [0, 2\pi]$

and the possible (homology classes of) longitudes of E are all the $\ell(E) + km(E)$ for $k \in \mathbb{Z}$, where

$$\ell(E) = \left\{ \left(\exp\left(\frac{2i\pi t}{p}\right), \exp\left(\frac{2i\pi tq}{p}\right) \right) \mid t \in [0,1] \right\}.$$

Note that the boundary of E is oriented as $(-S^1) \times S^1$. The quotient of the solid torus $|z_1| \leq |z_2|$ is also a solid torus whose meridian is

$$\begin{aligned} \mu &= \{(\exp(2i\pi s), 1) \mid s \in [0,1]\} \\ &= \cup_{j=0}^{p-1} \left\{ \left(\exp\left(\tfrac{2i\pi(t+j)}{p}\right), 1 \right) \mid t \in [0,1] \right\} \\ &= \cup_{j=0}^{p-1} \left\{ \left(\exp\left(\tfrac{2i\pi t}{p}\right), \exp\left(-\tfrac{2i\pi qj}{p}\right) \right) \mid t \in [0,1] \right\}. \end{aligned}$$

So $\langle m(E), \mu \rangle_{\partial E} = p$ and $\langle \mu, \ell(E) \rangle_{\partial E} = -q$. (There are q pairs (t, j) with $t \in [0, 1[$ and $j \in \{0, 1, \dots, p-1\}$ such that $(t + j) \in \frac{p}{q}\mathbb{Z}$.) $\square$

Remark 18.9. A homeomorphism from M to $M_{(U;\frac{1}{k})}$ can also be described directly, as follows.

Let D be a disk bounded by U, and let d be a smaller disk inside D. The disk D is parametrized by the disk of radius 2 in $\mathbb{C}$ and d is parametrized by the unit disk. The exterior E of U is homeomorphic to $\left(M \setminus (\mathring{D} \times]0, 2\pi[)\right) \cup_{d \times \{0, 2\pi\}} (d \times [0, 2\pi])$, by a homeomorphism that maps the meridian of U to

$$(\{1\} \times [0, 2\pi]) \cup ([1,2] \times \{2\pi\}) \cup (-\{2\} \times [0, 2\pi]) \cup (-[1,2] \times \{0\}).$$

The homeomorphism of E that restricts to $\left(M \setminus (\mathring{D} \times [0, 2\pi])\right)$ as the identity map and that maps $(z, \theta) \in d \times [0, 2\pi]$ to $(z \exp(ik\theta), \theta)$ maps the above meridian $m(U)$ to a curve homologous to $m(U) + k\ell(U)$. So this homeomorphism extends to provide a homeomorphism from M to $M_{(U;\frac{1}{k})}$.

As another example, we prove the following standard lemma.

Lemma 18.10. *Let m be a meridian of a knot K in a 3-manifold A. Equip m with its preferred longitude*[2] *$\ell(m)$ and equip K with a curve μ that is parallel to K. Then the Dehn surgery on $((K,\mu),(m,\ell(m))$ does not change the 3-manifold A.*

PROOF: Before and after the surgery, the two involved tori can be glued together along an annulus, whose core is a meridian of one of the knots and a longitude of the other one, to form a tubular neighborhood of K. $\square$

The reader is referred to the Rolfsen book [Rol90, Chapter 9, G, H] for many other examples of surgeries.

18.3 Direct proof of a surgery formula for Θ

In this section, we apply Theorem 18.5 to compute $\Theta(R_{(K;p/q)}) - \Theta(R) + \Theta(L(p,q))$ for any null-homologous knot K, in Proposition 18.11. In order to prove Proposition 18.11, we describe a special LP surgery, which was introduced in [Les09, Section 9], and which will also be used in order to compute the degree 2 part of $\check{\mathcal{Z}}$ for a null-homologous knot in Theorem 18.41.

A *Seifert surface* of a null-homologous knot K, in a 3–manifold M, is a compact connected oriented surface Σ, embedded in M, such that the boundary $\partial\Sigma$ of Σ is K. A *symplectic basis* for the H_1 of such a Seifert surface is a basis $(x_1,y_1,\dots,x_g,y_g)$, as in Figure 18.3, where $\langle x_i,y_i\rangle_\Sigma = 1$, for $i \in \underline{g}$.

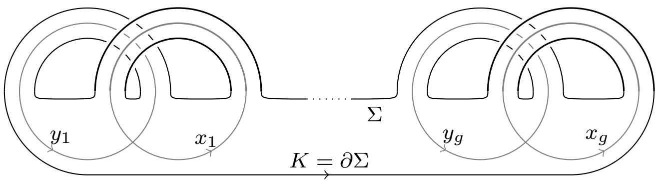


Figure 18.3: Symplectic basis of a Seifert surface

Proposition 18.11. *Let K be a null-homologous knot in a rational homology sphere R. Let Σ be a Seifert surface of K in R, and let $(x_1,y_1,\dots,x_g,y_g)$ be a symplectic basis of Σ. For a curve c of Σ, let c^+ denote its push-off in the direction of the positive normal to Σ. Set*

$$a_2(\Sigma) = \sum_{(i,j)\in\underline{g}^2} \left(lk(x_i,x_j^+)lk(y_i,y_j^+) - lk(x_i,y_j^+)lk(y_i,x_j^+)\right).$$

[2]It is equivalent to equip m with the coefficient 0, since m lies in a ball.

Then

$$\Theta(R_{(K;p/q)}) = \Theta(R) - \Theta(L(p,q)) + 6\frac{q}{p}a_2(\Sigma).$$

This proposition will be proved in Subsection 18.3.2 in these words. In particular, it implies that $a_2(\Sigma)$ is an invariant of K. This invariant will be denoted by $a_2(K)$. It is equal to $\frac{1}{2}\Delta''_K(1)$, where Δ_K is the Alexander polynomial of K, normalized so that $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = 1$.

Definition 18.12. Here is a possible quick definition of the *Alexander polynomial* Δ_K of the null-homologous knot K. Rewrite the symplectic basis $(x_i, y_i)_{i\in\underline{g}}$ as the basis $(z_j)_{j\in\underline{2g}}$ such that $z_{2i-1} = x_i$ and $z_{2i} = y_i$ for $i \in \underline{g}$. Let $V = \left[lk(z_j, z_k^+)\right]_{(j,k)\in\underline{2g}^2}$ denote the associated Seifert matrix, and let ${}^tV$ denote its transpose, then $\Delta_K(t) = \det\left(t^{1/2}V - t^{-1/2}\,{}^tV\right)$.

See [Ale28] or [Les96, Chapter 2] for other definitions of the Alexander polynomial, which will be mentioned later, but not used in this book any longer.

Remark 18.13. Proposition 18.11 is also a consequence of the identification of Θ with $6\lambda_{CW}$ in Theorem 18.30, which is proved independently in Section 18.6, and of the Casson-Walker surgery formula of Walker which is proved in [Wal92, Theorem 5.1].

18.3.1 A Lagrangian-preserving surgery associated to a Seifert surface

Definition 18.14. Let $c(S^1)$ be a curve embedded in the interior of an oriented surface F and let $c(S^1) \times [-1,1]$ be a collar neighborhood of $c(S^1)$ in F. A *right-handed (resp. left-handed) Dehn twist* about the curve $c(S^1)$ is a homeomorphism of F that coincides with the identity map of F outside $c(S^1) \times [-1,1]$ and that maps $(c(z),t) \in c(S^1) \times [-1,1]$ to $(c(z\exp(if(t))),t)$ for $f(t) = \pi(t+1)$ (resp. for $f(t) = -\pi(t+1)$).

Let Σ be a Seifert surface of a knot K in a manifold M. Consider an annular neighborhood $[-3,0] \times K$ of $(\{0\} \times K =)K = \partial\Sigma$ in Σ, a small disk D inside $]-2,-1[\times K$, and a small open disk d in the interior of D. Let $F = \Sigma \setminus d$. Let h_F be the composition of the two left-handed Dehn twists on F along $c = \partial D$ and $K_2 = \{-2\} \times K$ with the right-handed one along $K_1 = \{-1\} \times K$. See Figure 18.4.

View F as $F \times \{0\}$ in the boundary of a handlebody $A_F = F \times [-1,0]$ of M. Extend h_F to a homeomorphism h_A of ∂A_F by defining it to be the identity map outside $F \times \{0\}$.

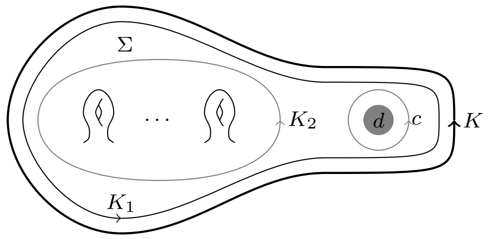


Figure 18.4: K, Σ, F, c, K_1 and K_2

Let A'_F be a copy of A_F. Identify $\partial A'_F$ with ∂A_F with

$$h_A \colon \partial A'_F \to \partial A_F.$$

Define the *surgery associated to* Σ to be the surgery (A'_F/A_F) associated with $(A_F, A'_F; h_A)$. If j denotes the embedding from ∂A_F to M. This surgery replaces

$$M = A_F \cup_j \left(M \setminus \mathring{A}_F\right)$$

with

$$M_F = A'_F \cup_{j \circ h_A} \left(M \setminus \mathring{A}_F\right).$$

Proposition 18.15. *With the above notation, the surgery (A'_F/A_F) associated to Σ is a Lagrangian-preserving surgery with the following properties. There is a homeomorphism from M_F to M,*

- *which extends the identity map of*

$$M \setminus ([-3, 0] \times K \times [-1, 0]),$$

- *which transforms a curve passing through $d \times [-1, 0]$ by a band sum with K,*
- *which transforms a 0–framed meridian m of K passing through $d \times [-1, 0]$, viewed as a curve of $M \setminus \mathring{A}_F$ (which may be expressed as $h_A^{-1}(m)$ in A'_F) to a 0–framed copy of K isotopic to the framed curve $h_A^{-1}(m)$ of Figure 18.5 (with the framing induced by ∂A_F).*

Proof: Observe that $h_{A|(F \times \{-1,0\}) \cup (\partial\Sigma \times [-1,0])}$ extends to $\Sigma \times [-1, 0]$ as

$$\begin{array}{llll} h \colon & \Sigma \times [-1, 0] & \to & \Sigma \times [-1, 0] \\ & (\sigma, t) & \mapsto & h(\sigma, t) = (h_t(\sigma), t), \end{array}$$

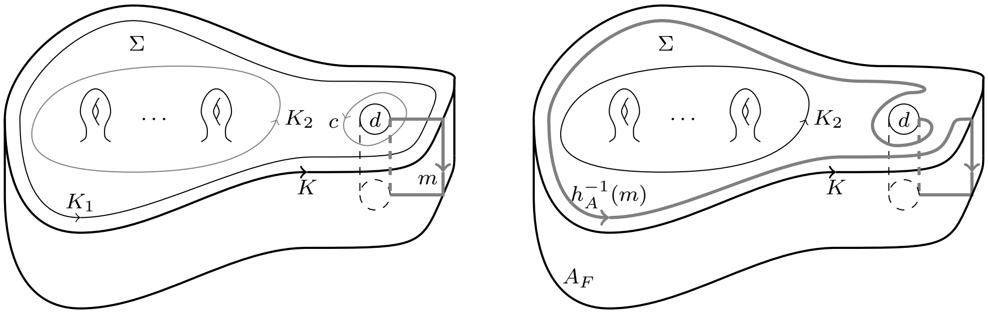


Figure 18.5: m and $h_A^{-1}(m)$

where h_0 is the extension of h_F by the identity map on d, (this extension is isotopic to the identity map of Σ,) h_{-1} is the identity map of Σ, h_t coincides with the identity map outside $[-5/2,-1/2] \times K(S^1)$, and h_t is defined as follows on $[-5/2,-1/2] \times K(S^1)$.

• When $t \leq -1/2$, then h_t coincides with the identity map h_{-1} outside the disk D, whose elements are written as $D(z \in \mathbb{C})$, with $|z| \leq 1$. The elements of d are the $D(z)$ for $|z| < 1/2$. On D, h_t describes the isotopy between the identity map and the left-handed Dehn twist along ∂D located on $\{D(z) \mid 1/2 \leq |z| \leq 1\}$.

$$\begin{array}{lll} h_t(z \in D) & = z\exp\left(i\pi(2t+2)4(|z|-1)\right) & \text{if } |z| \geq 1/2 \\ h_t(z \in D) & = z\exp\left(-2i\pi(2t+2)\right) & \text{if } |z| \leq 1/2. \end{array}$$

• When $t \geq -1/2$, then h_t describes the following isotopy between the left-handed Dehn twist $h_{-1/2}$ along ∂D and the composition of $h_{-1/2}$ with the left-handed Dehn twist along K_2 and the right-handed Dehn twist along K_1, where the first twist is supported on $[-5/2,-2] \times K(S^1)$ and the second one is supported on $[-1,-1/2] \times K(S^1)$,

$$\begin{array}{ll} h_t(u,K(z)) & = (u, K\left(z\exp\left(i(2t+1)(4\pi(u+5/2))\right)\right)) \\ & \qquad\qquad\qquad\qquad \text{if } -5/2 \leq u \leq -2 \\ h_t(u,K(z)) & = h_{-1/2}\left(u, K\left(z\exp\left(i(2t+1)(2\pi)\right)\right)\right) \\ & \qquad\qquad\qquad\qquad \text{if } -2 \leq u \leq -1 \\ h_t(u,K(z)) & = (u, K\left(z\exp\left(i(2t+1)(4\pi(-u-1/2))\right)\right)) \\ & \qquad\qquad\qquad\qquad \text{if } -1 \leq u \leq -1/2. \end{array}$$

Now, M_F is naturally homeomorphic to

$$\left(A'_F \cup_{h_{|\partial A'_F \setminus (\partial d \times [-1,0])}} (M \setminus \mathrm{Int}(\Sigma \times [-1,0]))\right) \cup_{\partial(d \times [-1,0])} (d \times [-1,0]),$$

and hence to

$$(\Sigma \times [-1,0]) \cup_{h_{|\partial(\Sigma\times[-1,0])}} (M \setminus \mathrm{Int}(\Sigma \times [-1,0]))\,,$$

which is mapped homeomorphically to M by the identity map outside $\Sigma \times [-1,0]$ and by h on $\Sigma\times[-1,0]$. Therefore, we indeed have a homeomorphism from M_F to M, which is the identity map outside $[-3,0]\times K\times[-1,0]$, and which maps $d\times[-1,0]$ to a cylinder, which runs along K after being negatively twisted. In particular, looking at the action of the homeomorphism on a framed arc $x\times[-1,0]$, where x is on the boundary of d, shows that the meridian m –viewed as a curve of $M\setminus \mathring{A}_F$– with its framing induced by the boundary of A_F is mapped to a curve isotopic to $h_A^{-1}(m)$ in a tubular neighborhood of K with the framing induced by the boundary of A_F. (In order to see this, thicken the meridian m as a band $m\times[0,1]$ in ∂A_F, part of which lies in the vertical boundary $\partial d\times[-1,0]$ of $d\times[-1,0]$. The part in $\partial d\times[-1,-1/2]$ is mapped by h to a rectangle, which lies in $\partial d\times[-1,0]$. The image under h of the part in $\partial d\times[-1/2,0]$ together with a small additional piece of the thickened meridian can be isotoped in a tubular neighborhood of K to a band on ∂A_F, which is first vertical in $\partial d\times[-1/2,0]$, and which next runs along K.)

Now, $H_1(\partial A_F)$ is generated by the generators of $H_1(\Sigma)\times\{0\}$, the generators of $H_1(\Sigma)\times\{-1\}$, and the homology classes of $c=\partial D$ and m. Among them, only the class of m could be affected by h_A, and it is not. Therefore h_A acts trivially on $H_1(\partial A_F)$, and the defined surgery is an LP–surgery. $\square$

Let $\Sigma\times[-1,2]$ be an extension of the previous neighborhood of Σ, and let $B_F = F\times[1,2]$. Define the homeomorphism h_B of ∂B_F to be the identity map anywhere except on $F\times\{1\}$, where it coincides with the homeomorphism h_F of F, with the obvious identification.

Let B'_F be a copy of B_F. Identify $\partial B'_F$ with ∂B_F with

$$h_B\colon \partial B'_F \to \partial B_F.$$

Define the *inverse surgery associated to* Σ to be the surgery associated to (B_F, B'_F) (or $(B_F, B'_F; h_B)$). Note that the previous study can be used for this surgery by using the central symmetry of $[-1,2]$.

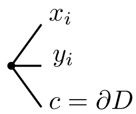
Then, we have the following obvious lemma, which justifies the terminology.

Lemma 18.16. *With the above notation, performing both surgeries* (B'_F/B_F) *and* (A'_F/A_F) *affects neither* M *nor the curves in the complement of* $F\times[-1,2]$ *(up to isotopy), while performing* (A'_F/A_F) *(resp.* (B'_F/B_F)*) changes a* 0*–framed meridian of* K *passing through* $d\times[-1,2]$ *into a* 0*–framed copy of* K *(resp.* $(-K)$*).*

□

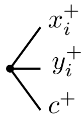
Lemma 18.17. *Let $(x_i, y_i)_{i=1,\dots,g}$ be a symplectic basis of Σ, then the tripod combination $T(\mathcal{I}_{A_FA'_F})$ associated to the surgery (A'_F/A_F) is:*

$$T(\mathcal{I}_{A_FA'_F}) = \sum_{i=1}^{g}$$



For a curve γ of F, let γ^+ denote $\gamma \times \{1\}$. The tripod combination $T(\mathcal{I}_{B_FB'_F})$ associated to the surgery (B'_F/B_F) is:

$$T(\mathcal{I}_{B_FB'_F}) = -\sum_{i=1}^{g}$$



Proof: For a curve γ of F, γ^- denotes $\gamma \times \{-1\}$. In order to compute the intersection form of $(A_F \cup -A'_F)$, use the basis $\left(m, (x_i - x_i^-)_{i \in \underline{g}}, (y_i - y_i^-)_{i \in \underline{g}}\right)$ of the Lagrangian of A_F. Its dual basis is $\left(c, (y_i)_{i \in \underline{g}}, (-x_i)_{i \in \underline{g}}\right)$. Note that the only curve of the Lagrangian basis that is modified by h_A is m, and that $h_A^{-1}(m)$ may be expressed as a path composition $mc^{-1}K_2$. The isomorphism ∂_{MV}^{-1} from $\mathcal{L}_{A_F}$ to $H_2(A_F \cup -A'_F)$ satisfies:

$$\begin{array}{ll} \partial_{MV}^{-1}(x_i - x_i^-) & = S(x_i) = -(x_i \times [-1,0]) \cup (x_i \times [-1,0] \subset A'_F) \\ \partial_{MV}^{-1}(y_i - y_i^-) & = S(y_i) = -(y_i \times [-1,0]) \cup (y_i \times [-1,0] \subset A'_F) \\ \partial_{MV}^{-1}(m) & = S_A(m) = D_m - ((\Sigma \setminus (]-2,0] \times K)) \cup (D_m \subset A'_F)), \end{array}$$

where D_m is a disk of A_F bounded by m, and the given expression of $\partial_{MV}^{-1}(m)$ must be completed in $\partial A_F \cap ([-2,0] \times K \times [-1,0])$, so the boundary of $\partial_{MV}^{-1}(m)$ vanishes actually, as it does algebraically.

Since x_i intersects only y_i among the curves x_j and y_j for $j \in \underline{g}$, $S(x_i)$ intersects only $S(y_i)$ and $S_A(m)$ in our basis of $H_2(A_F \cup -A'_F)$. The algebraic intersection of $S(x_i)$, $S(y_i)$ and $S_A(m)$ is -1.

For the surgery (B'_F/B_F), use the symmetry $T\colon [-1,2] \to [-1,2]$ such that $T(x) = 1-x$, which induces the symmetry $T_F = 1_F \times T$ of $F \times [-1,2]$, which maps A_F onto B_F. Use the image by T_F of the above basis of $\mathcal{L}_{A_F}$ for $\mathcal{L}_{B_F}$, and its dual basis which is the opposite of the image of the above dual basis. (Since T_F reverses the orientation, the intersection numbers on ∂A_F are multiplied by -1.) Use the images under T_F of the former surfaces. Their triple intersection numbers are the same since their positive normals and the ambient orientation are reversed. □

18.3.2 A direct proof of the Casson surgery formula

In this subsection, we prove Proposition 18.11, assuming Theorem 18.5, which will be proved independently.

Note the following easy well-known lemma.

Lemma 18.18. *The variation of the linking number of two knots J and K in a rational homology 3-sphere R after a p/q–surgery on a knot V in R is given by the following formula.*

$$lk_{R_{(V;p/q)}}(J,K) = lk_R(J,K) - \frac{q}{p} lk_R(V,J) lk_R(V,K).$$

PROOF: The p/q–surgery on V is the surgery with respect to a curve $\mu_V \subset \partial N(V)$. Set $q_V = \langle m(V), \mu_V \rangle_{\partial N(V)}$ and $p_V = lk(V, \mu_V)$. So $\frac{p}{q} = \frac{p_V}{q_V}$. In $H_1(R \setminus (V \cup K); \mathbb{Q})$,

$$J = lk_R(J,K)m(K) + lk_R(V,J)m(V) \text{ and } \mu_V = p_V m(V) + q_V lk_R(V,K)m(K).$$

So, in $H_1(R_{(V;p/q)} \setminus K; \mathbb{Q})$, where μ_V vanishes,

$$J = lk_R(J,K)m(K) - \frac{q}{p} lk_R(V,K) lk_R(V,J) m(K).$$

$\square$

PROOF OF PROPOSITION 18.11 ASSUMING THEOREM 18.5: Recall that K bounds a Seifert surface Σ in a rational homology sphere R. Let $\Sigma \times [-1,2]$ be a collar of Σ in R and let $(A'/A) = (A'_F/A_F)$ and $(B'/B) = (B'_F/B_F)$ be the LP–surgeries of Subsection 18.3.1. Let U be a meridian of K passing through $d \times [-1,2]$, such that performing one of the two surgeries transforms U into $\pm K$ and performing both or none of them leaves U unchanged. Then

$$\begin{aligned} \mathcal{Z}_1\left(\left[R_{(U;p/q)}, \emptyset; A'/A, B'/B\right]\right) &= 2\mathcal{Z}_1\left(R_{(U;p/q)}\right) - 2\mathcal{Z}_1\left(R_{(K;p/q)}\right) \\ &= \left[\langle\langle T\left(\mathcal{I}_{AA'}\right) \sqcup T\left(\mathcal{I}_{BB'}\right)\rangle\rangle_{R_{(U;p/q)}}\right]. \end{aligned}$$

According to Lemma 18.17, the tripods associated to the surgery (A, A') and to the surgery (B, B'), are $\sum_{i=1}^{g} \bullet\!\!<\!\begin{smallmatrix} x_i \\ y_i \\ c \end{smallmatrix}$ and $\sum_{j=1}^{g} \begin{smallmatrix} x_j^+ \\ y_j^+ \\ c^+ \end{smallmatrix}\!>\!\!\bullet$, respectively. The only curve that links c algebraically in $R_{(U;p/q)_{i \in N}}$, among those appearing in all the tripods, is c^+ with a linking number $-q/p$, according to Lemma 18.18. Therefore, these two must be paired together with this coefficient, and

$$\langle\langle \bullet\!\!<\!\begin{smallmatrix} x_i & x_j^+ \\ y_i & y_j^+ \\ c & c^+ \end{smallmatrix}\!>\!\!\bullet \rangle\rangle = -\frac{q}{p}\left(lk(x_i, x_j^+) lk(y_i, y_j^+) - lk(x_i, y_j^+) lk(y_i, x_j^+)\right)[\ominus].$$

So

$$\left[\langle\langle T\left(\mathcal{I}_{AA'}\right) \sqcup T\left(\mathcal{I}_{BB'}\right)\rangle\rangle_{R_{(U;p/q)}}\right] = -\frac{q}{p}a_2(\Sigma)\left[\ominus\right]$$

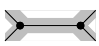
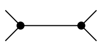
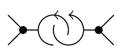
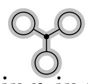
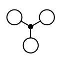
and

$$\mathcal{Z}_1\left(R_{(K;p/q)}\right) = \mathcal{Z}_1\left(R_{(U;p/q)}\right) + \frac{q}{2p}a_2(\Sigma)\left[\ominus\right].$$

Recall that $\mathcal{Z}_1(R) = \frac{1}{12}\Theta(R)\left[\ominus\right]$ according to Corollary 10.11, where $\left[\ominus\right] \neq 0$ in $\mathcal{A}(\emptyset)$. Also recall that Θ is additive under connected sum, according to Corollary 10.25 and that $\Theta(L(p,-q)) = -\Theta(L(p,q))$ according to Proposition 5.15. The result follows, thanks to Lemma 18.8. $\square$

18.4 Finite type invariants of $\mathbb{Z}$-spheres

In this section, we state the fundamental theorem of finite type invariants for integer homology 3-spheres due to Thang Le [Le97] and we show how Theorem 18.5 may be used in its proof. This shows in what sense Theorem 18.5 implies that $\mathcal{Z}$ restricts to a *universal* finite type invariant of integer homology 3-spheres. In order to do this, we first follow Goussarov [GGP01] and Habiro [Hab00] and construct surjective maps from $\mathcal{A}_n(\emptyset)$ to $\mathcal{F}_{2n}(\mathcal{M})/\mathcal{F}_{2n+1}(\mathcal{M})$.

Mapping $\mathcal{A}_n(\emptyset)$ to $\mathcal{F}_{2n}(\mathcal{M})/\mathcal{F}_{2n+1}(\mathcal{M})$ Let Γ be a degree n trivalent Jacobi diagram whose vertices are numbered in $\underline{2n}$. Let $\Sigma(\Gamma)$ be an oriented surface that contains Γ in its interior and such that $\Sigma(\Gamma)$ is a regular neighborhood of Γ in $\Sigma(\Gamma)$. Equip Γ with its vertex-orientation induced by the orientation of $\Sigma(\Gamma)$. Embed $\Sigma(\Gamma)$ in a ball inside $\mathbb{R}^3$. Replace neighborhoods  of the edges  by neighborhoods  of . Thus $\Sigma(\Gamma)$ is transformed into a collection of disjoint oriented surfaces $\Sigma(Y)$ , one for each trivalent vertex. The graph  equipped with its framing induced by $\Sigma(Y)$ is called a Y*-graph*. Thickening the $\Sigma(Y)$ transforms each of them into a standard genus 3 handlebody H with three handles with meridians m_j and longitudes ℓ_j, such that $\langle m_i, \ell_j\rangle_{\partial H} = \delta_{ij}$ and the longitudes ℓ_j are on $\Sigma(Y)$ as in Figure 18.6.

The Matveev *Borromean surgery* on H is the Dehn surgery on the 6-component link inside H with respect to the parallels of its components that are parallel in Figure 18.7. It is studied in [Mat87].

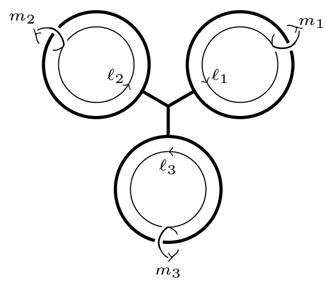


Figure 18.6: Meridians and longitudes of the standard genus 3 handlebody H

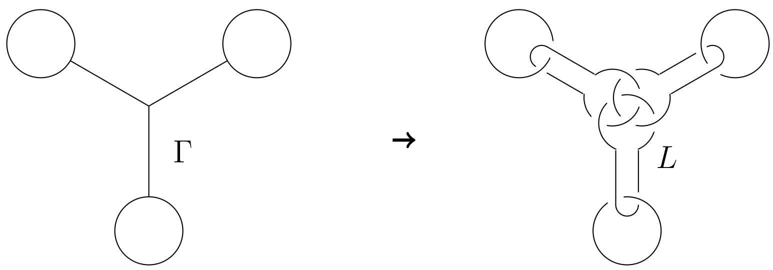


Figure 18.7: Y-graph and associated LP-surgery

Lemma 18.19. *The Matveev Borromean surgery changes the handlebody H to an integer homology handlebody H' with the same boundary and with the same Lagrangian as H. The manifold $H \cup_\partial (-H')$ is diffeomorphic to $(S^1)^3$, and*

$$T(\mathcal{I}_{HH'}) = \pm \begin{smallmatrix}\ell_3\\ \ell_2\\ \ell_1\end{smallmatrix} \quad .$$

PROOF: It is easy to see that the Lagrangian of H' is the same as the Lagrangian of H. As in Example 18.7, $H \cup_\partial (-H)$ is obtained from S^3 by surgery on three 0-framed meridians of three handles of H, where H is supposed to be embedded in S^3 in a standard way. So $H' \cup_\partial (-H)$ is obtained by surgery on the zero-framed nine-component link obtained from the six-component link L_6 of Figure 18.7 by adding a meridian for each outermost component of L_6. Lemma 18.10 implies that $H' \cup_\partial (-H)$ is obtained by surgery on the zero-framed Borromean link. So, according to Example 18.7, $H' \cup_\partial (-H)$ is diffeomorphic to $(S^1)^3$. Easy homological computations imply that H' is an integer homology handlebody. $\square$

With the notation of Section 6.1, define $\psi_n(\Gamma)$ to be the class of

$$[S^3; (A^{(i)\prime}/A^{(i)})_{i\in\underline{2n}}] = \sum_{I\subseteq\underline{2n}} (-1)^{\sharp I} S^3\left((A^{(i)\prime}/A^{(i)})_{i\in I}\right)$$

in $\frac{\mathcal{F}_{2n}(\mathcal{M})}{\mathcal{F}_{2n+1}(\mathcal{M})}$, where the $(A^{(i)\prime}/A^{(i)})$ are the Borromean surgeries associated to the Y-graphs corresponding to the vertices of Γ. The coefficient field[3] $\mathbb{K}$ of in Section 6.1 for $\mathcal{F}_{2n}(\mathcal{M})$ is $\mathbb{R}$, from now on.

In [GGP01, Theorem 4.13, Section 4], Garoufalidis, Goussarov and Polyak proved the following theorem.

Theorem 18.20 (Garoufalidis, Goussarov, Polyak)**.** *Let $n \in \mathbb{N}$. For a degree n trivalent Jacobi diagram Γ, the element $\psi_n(\Gamma)$ of $\frac{\mathcal{F}_{2n}(\mathcal{M})}{\mathcal{F}_{2n+1}(\mathcal{M})}$ constructed above depends only on the class of Γ in $\mathcal{A}_n(\emptyset)$, and the map*

$$\psi_n \colon \mathcal{A}_n(\emptyset) \to \frac{\mathcal{F}_{2n}(\mathcal{M})}{\mathcal{F}_{2n+1}(\mathcal{M})}$$

is surjective. Furthermore $\frac{\mathcal{F}_{2n+1}(\mathcal{M})}{\mathcal{F}_{2n+2}(\mathcal{M})} = \{0\}$.

Assuming the above theorem, the following *Lê fundamental theorem on finite type invariants of $\mathbb{Z}$-spheres* becomes a corollary of Theorem 18.5.

Theorem 18.21 (Lê)**.** *There exists a family $\left(Y_n \colon \mathcal{F}_0(\mathcal{M}) \to \mathcal{A}_n(\emptyset)\right)_{n\in\mathbb{N}}$ of linear maps such that*

- $Y_n(\mathcal{F}_{2n+1}(\mathcal{M})) = 0$,
- *the restriction $\overline{Y}_n$ to $\frac{\mathcal{F}_{2n}(\mathcal{M})}{\mathcal{F}_{2n+1}(\mathcal{M})}$ of the morphism induced by Y_n on $\frac{\mathcal{F}_0(\mathcal{M})}{\mathcal{F}_{2n+1}(\mathcal{M})}$ to $\mathcal{A}_n(\emptyset)$ is a left inverse to ψ_n.*

In particular, for any $n \in \mathbb{N}$, $\frac{\mathcal{F}_{2n}(\mathcal{M})}{\mathcal{F}_{2n+1}(\mathcal{M})} \cong \mathcal{A}_n(\emptyset)$ and $\frac{\mathcal{I}_{2n}(\mathcal{M})}{\mathcal{I}_{2n-1}(\mathcal{M})} \cong \mathcal{A}_n^*(\emptyset)$.

An invariant Y that satisfies the properties in the statement of Theorem 18.21 above is called a *universal finite type invariant of $\mathbb{Z}$–spheres*. In order to prove Theorem 18.21, Lê proved that the Lê-Murakami-Ohtsuki invariant $Z^{LMO} = (Z_n^{LMO})_{n\in\mathbb{N}}$ of [LMO98] is a universal finite type invariant of $\mathbb{Z}$–spheres in [Le97].

As a corollary of Theorem 18.5, we get the following Kuperberg and Thurston theorem [KT99].

[3] For the statements that involve only invariants $\mathcal{Z}$ valued in space of Jacobi diagrams with rational coefficients (when no interval components are involved) we can fix the coefficient field to be $\mathbb{Q}$, if we also restrict the coefficient field of our related spaces of Jacobi diagrams to be $\mathbb{Q}$.

Theorem 18.22 (Kuperberg, Thurston)**.** *The restriction of $\mathcal{Z}$ to $\mathbb{Z}$-spheres is a universal finite type invariant of $\mathbb{Z}$–spheres.*

PROOF: Theorem 18.5 ensures that $\mathcal{Z}_n(\mathcal{F}_{2n+1}(\mathcal{M})) = 0$. So the proof is reduced to the proof of the following lemma.

Lemma 18.23. *For any trivalent Jacobi diagram Γ, $\mathcal{Z}_n \circ \psi_n([\Gamma]) = [\Gamma]$.*

PROOF: Let us show how this lemma follows from Theorem 18.5. Number the vertices of Γ in $\underline{2n}$. Call $(A^{(i)\prime}/A^{(i)})$ the Borromean surgery associated to the vertex i, and embed the tripods

$$T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right) = \varepsilon \begin{array}{l} \diagup\, \ell_3^{(i)} \\ \bullet\!\!- \ell_2^{(i)} \\ \diagdown\, \ell_1^{(i)} \end{array}$$

for a fixed $\varepsilon = \pm 1$ into the graph Γ, naturally, so that the half-edge of $\ell_k^{(i)}$ is on the half-edge that gave rise to the *leaf* -which is the looped edge- of $\ell_k^{(i)}$ in the Y-graph associated to i. Thus the only partition into pairs of the half-edges of $\bigsqcup_{i\in\underline{2n}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right)$ that may produce a nonzero contribution, in the contraction

$$\langle\langle \bigsqcup_{i\in\underline{2n}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right)\rangle\rangle_n,$$

pairs a half-edge associated to a leaf of some $\ell_k^{(i)}$ with the half-edge of the only leaf that links $\ell_k^{(i)}$, which is the other half-edge of the same edge. Therefore

$$\left[\langle\langle \bigsqcup_{i\in\underline{2n}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right)\rangle\rangle_n\right] = [\Gamma].$$

□

□

Remark 18.24. In the original work of Le [Le97] and in the article [GGP01] of Garoufalidis, Goussarov and Polyak, the main filtration used for the space of $\mathbb{Z}$-spheres is defined from Borromean surgeries rather than from integral LP-surgeries. It is proved in [AL05] that the two filtrations coincide. It is also proved in [AL05] that a universal finite type invariant of $\mathbb{Z}$-spheres automatically satisfies the more general formula of Theorem 18.5 for any $X = [\check{R}; (A^{(i)\prime}/A^{(i)})_{i\in\underline{x}}]$, such that R is a $\mathbb{Z}$-sphere, and the $(A^{(i)\prime}/A^{(i)})$ are integral LP-surgeries in $\check{R}$. Other filtrations of the space of $\mathbb{Z}$-spheres are considered and compared in [GGP01] including the original Ohtsuki filtration using surgeries on algebraically split links defined in the Ohtsuki introduction of finite invariants of $\mathbb{Z}$-spheres [Oht96], which gives rise to the same notion of real-valued finite-type invariants.

18.5 Finite type invariants of $\mathbb{Q}$-spheres

For a $\mathbb{Q}$-sphere R, the cardinality of $H_1(R;\mathbb{Z})$ is the product over the prime numbers p of $p^{\nu_p(R)}$, where $\nu_p(R)$ is called the *p–valuation of the order of* $H_1(R;\mathbb{Z})$. In [Mou12, Proposition 1.9], Delphine Moussard proved that ν_p is a degree 1 invariant of $\mathbb{Q}$-spheres with respect to $\mathcal{O}_{\mathcal{L}}^{\mathbb{Q}}$, which is defined in Section 6.1. She also proved [Mou12, Corollary 1.10] that the degree 1 invariants of $\mathbb{Q}$-spheres with respect to $\mathcal{O}_{\mathcal{L}}^{\mathbb{Q}}$ are (possibly infinite) linear combinations of the invariants ν_p and of a constant map.

Define an *augmented trivalent Jacobi diagram* to be the disjoint union of a trivalent Jacobi diagram and a finite number of isolated 0–valent vertices equipped with prime numbers. The *degree* of such a diagram is half the number of its vertices. It is a half integer. For a half integer h, let $\mathcal{A}_h^{\mathrm{aug}}$ denote the quotient of the $\mathbb{Q}$-vector space generated by degree h augmented trivalent Jacobi diagrams, by the Jacobi relation and the antisymmetry relation. The product induced by the disjoint union turns $\mathcal{A}^{\mathrm{aug}} = \prod_{h\in\frac{1}{2}\mathbb{N}} \mathcal{A}_h^{\mathrm{aug}}$ to a graded algebra. In [Mou12], Delphine Moussard proved that for any integer n,

$$\frac{\mathcal{F}_n(\mathcal{M}_Q)}{\mathcal{F}_{n+1}(\mathcal{M}_Q)} \cong \mathcal{A}_{n/2}^{\mathrm{aug}},$$

using the configuration space integral Z_{KKT} described in [KT99] and [Les04a], and the splitting formulae of [Les04b], which are stated in Theorem 18.5. See [Mou12, Theorem 1.7]. The invariant Z_{KKT} is the restriction to $\mathbb{Q}$-spheres of the invariant $\mathcal{Z}$ described in this book.

The maps ψ_n of Section 18.4 can be generalized to canonical maps

$$\psi_h \colon \mathcal{A}_h^{\mathrm{aug}} \to \frac{\mathcal{F}_{2h}(\mathcal{M}_Q)}{\mathcal{F}_{2h+1}(\mathcal{M}_Q)}$$

as follows. For any prime number p, let B_p be a rational homology ball such that $|H_1(B_p;\mathbb{Z})| = p$. Let Γ^a be the disjoint union of a degree k trivalent Jacobi diagram Γ and r isolated 0–valent vertices v_j equipped with prime numbers p_j, for $j \in \underline{r}$. Embed Γ^a in $\mathbb{R}^3$. Thicken it, replace Γ with $2k$ genus 3 handlebodies $A^{(i)}$ associated to the vertices of Γ as in Section 18.4, and replace each vertex v_j with a small ball $B(v_j)$ around it so that the $B(v_j)$ and the $A^{(i)}$ form a family of $2k+r$ disjoint rational homology handlebodies. Define $\psi_{k+r/2}(\Gamma^a)$ to be the class of $[S^3; (A^{(i)\prime}/A^{(i)})_{i\in\underline{2k}}, (B_{p_j}/B(v_j))_{j\in\underline{r}}]$ in $\frac{\mathcal{F}_{2k+r}(\mathcal{M}_Q)}{\mathcal{F}_{2k+r+1}(\mathcal{M}_Q)}$, where the $(A^{(i)\prime}/A^{(i)})$ are the Borromean surgeries associated to the Y-graphs corresponding to the vertices of Γ as in Section 18.4.

Lemma 18.25. *The map ψ_h is well defined.*

PROOF: [Mou12, Lemma 6.11] guarantees that if B'_{p_j} is a $\mathbb{Q}$-ball whose $H_1(.;\mathbb{Z})$ has the same cardinality as $H_1(B_{p_j};\mathbb{Z})$,

$$\left(S^3(B'_{p_j}/B^3) - S^3(B_{p_j}/B^3)\right) \in \mathcal{F}_2(\mathcal{M}_Q).$$

This guarantees that $\psi_{r/2+k}(\Gamma^a)$ does not depend on the chosen balls B_{p_j}. Thus, Theorem 18.20 implies that ψ_h is well defined. □

This map ψ_h is canonical. According to the form of the generators of $\frac{\mathcal{F}_{2h}(\mathcal{M}_Q)}{\mathcal{F}_{2h+1}(\mathcal{M}_Q)}$ exhibited in [Mou12, Section 6.2 and Proposition 6.9], ψ_h is surjective.

Let $\mathcal{A}_h^{\mathrm{aug,c}}$ denote the subspace of $\mathcal{A}_h^{\mathrm{aug}}$ generated by connected degree h diagrams. So, if $\mathcal{A}_h^{\mathrm{aug,c}} \neq 0$, then $h \in \mathbb{N}$ or $h = 1/2$. Set $\mathcal{A}^{\mathrm{aug,c}} = \prod_{h\in\frac{1}{2}\mathbb{N}} \mathcal{A}_n^{\mathrm{aug,c}}$. Let z^{aug} denote the $\mathcal{A}^{\mathrm{aug,c}}$-valued invariant z^{aug} of $\mathbb{Q}$-spheres such that, for any $\mathbb{Q}$-sphere R,

- $z_0^{\mathrm{aug}}(R) = 0$,
- $z_{1/2}^{\mathrm{aug}}(R) = \sum_{p \text{ prime}} \nu_p(R)\bullet_p$, and,
- $z_n^{\mathrm{aug}}(R)$ is the natural projection $z_n^{\mathrm{aug}}(R,\emptyset) = p^c(\mathcal{Z}_n(R,\emptyset))$ of $\mathcal{Z}_n(R,\emptyset)$ to the subspace $\mathcal{A}_n^c(\emptyset)$ of $\mathcal{A}_n(\emptyset)$ generated by connected diagrams. (Recall from Notation 7.16 that the projection p^c maps disconnected diagrams to 0.)

Define an $\mathcal{A}^{\mathrm{aug}}$-valued invariant $Z^{\mathrm{aug}} = (Z_n^{\mathrm{aug}})_{n\in\frac{1}{2}\mathbb{N}}$ to be $Z^{\mathrm{aug}} = \exp(z^{\mathrm{aug}})$ for the $\mathcal{A}^{\mathrm{aug,c}}$-valued invariant z^{aug}. (This means that for any $\mathbb{Q}$-sphere R, $Z^{\mathrm{aug}}(R) = \exp(z^{\mathrm{aug}}(R))$.)

As noticed by Gwénaël Massuyeau, the Moussard fundamental theorem for *finite type invariants of* $\mathbb{Q}$*-spheres* can be stated as follows.

Theorem 18.26 (Moussard). *The family* $(Z_h^{aug}\colon \mathcal{F}_0(\mathcal{M}_Q) \to \mathcal{A}_h^{aug})_{h\in\frac{1}{2}\mathbb{N}}$ *of linear maps is such that, for any* $h \in \frac{1}{2}\mathbb{N}$,

- $Z_h^{aug}(\mathcal{F}_{2h+1}(\mathcal{M}_Q)) = 0$,
- Z_h^{aug} *induces a left inverse to* ψ_h *from* $\frac{\mathcal{F}_{2h}(\mathcal{M}_Q)}{\mathcal{F}_{2h+1}(\mathcal{M}_Q)}$ *to* $\mathcal{A}_h^{aug}$.

In particular, for any $n \in \mathbb{N}$, $\frac{\mathcal{F}_n(\mathcal{M}_Q)}{\mathcal{F}_{n+1}(\mathcal{M}_Q)} \cong \mathcal{A}_{n/2}^{aug}$ *and* $\frac{\mathcal{I}_n(\mathcal{M}_Q)}{\mathcal{I}_{n-1}(\mathcal{M}_Q)} \cong (\mathcal{A}_{n/2}^{aug})^*$.

PROOF: Let us prove that, for any disjoint union Γ^a of a degree k trivalent Jacobi diagram Γ and r isolated 0–valent vertices v_j equipped with prime numbers p_j, for $j \in \underline{r}$, for any representative ψ_Γ of $\psi_{k+r/2}(\Gamma^a)$ in $\mathcal{F}_{2k+r}(\mathcal{M}_Q)$,

$$Z_{\leq k+r/2}^{\mathrm{aug}}(\psi_\Gamma) = [\Gamma^a] + \text{higher degree terms}.$$

Fix ψ_Γ and Γ^a as above. Write ψ_Γ as $[S^3; (A^{(i)\prime}/A^{(i)})_{i\in \underline{2k}}, (B_{p_j}/B(v_j))_{j\in\underline{r}}]$. If $r=0$, then $\psi_\Gamma \in \mathcal{F}_{2k+r}(\mathcal{M})$ and $Z^{\mathrm{aug}}(\psi_\Gamma) = \mathcal{Z}(\psi_\Gamma)$. So $Z^{\mathrm{aug}}_k(\psi_\Gamma) = [\Gamma]$, thanks to Lemma 18.23. (All the involved manifolds are $\mathbb{Z}$-spheres.) The general case follows by induction. If $r>0$, let Γ' be obtained from Γ^a by forgetting Vertex v_r and let $\psi_{\Gamma'}$ be obtained from ψ_Γ by forgetting the surgery $(B_{p_r}/B(v_r))$, which is nothing but a connected sum with $S_{p_r} = B_{p_r} \cup_{S^2} B^3$. Since Z^{aug} is multiplicative under connected sum, according to Theorem 10.24,

$$Z^{\mathrm{aug}}(\psi_\Gamma) = Z^{\mathrm{aug}}(\psi_{\Gamma'})(Z^{\mathrm{aug}}(S_{p_r})-1).$$

Then identifying the non-vanishing terms with minimal degree yields

$$Z^{\mathrm{aug}}_{\leq k+r/2}(\psi_\Gamma) = Z^{\mathrm{aug}}_{k+(r-1)/2}(\psi_{\Gamma'})[\bullet_{p_r}],$$

which allows us to conclude the proof. (Since any $\lambda \in \left(\frac{\mathcal{F}_n(\mathcal{M}_Q)}{\mathcal{F}_{n+1}(\mathcal{M}_Q)}\right)^*$ extends to the linear form $\lambda\circ\psi_{n/2}\circ Z^{\mathrm{aug}}_{n/2}$ of $\mathcal{I}_n(\mathcal{M}_Q)$, the natural injection $\frac{\mathcal{I}_n(\mathcal{M}_Q)}{\mathcal{I}_{n-1}(\mathcal{M}_Q)} \hookrightarrow \left(\frac{\mathcal{F}_n(\mathcal{M}_Q)}{\mathcal{F}_{n+1}(\mathcal{M}_Q)}\right)^*$ is surjective.) $\square$

According to Corollary 10.11,

$$\mathcal{Z}_1(R,\emptyset) = \frac{1}{12}\Theta(R)[\ominus].$$

In particular, according to Theorem 18.5, Θ is of degree at most 2 with respect to $\mathcal{O}^{\mathbb{Q}}_{\mathcal{L}}$, and according to Lemma 18.23,

$$\Theta(\psi_1(\ominus)) = 12.$$

The following easy corollary of Theorem 18.26 can be proved as Corollary 6.10.

Corollary 18.27. *For any real valued invariant ν of $\mathbb{Q}$–spheres that is of degree at most 2 with respect to $\mathcal{O}^{\mathbb{Q}}_{\mathcal{L}}$, there exists real numbers a_θ, a_0, a_p, for any prime number p, and, $a_{p,q}$ for any pair (p,q) of prime numbers such that $p\leq q$, such that*

$$\nu(R) = a_0 + \sum_{p \text{ prime}} a_p\nu_p(R) + \sum_{p,q \text{ prime}|p\leq q} a_{p,q}\nu_p(R)\nu_q(R) + a_\theta\Theta.$$

Note that the above infinite sums of the statement do not cause problems since they are actually finite when applied to a $\mathbb{Q}$-sphere R.

According to Proposition 5.15 (or to Theorem 10.27), for any $\mathbb{Q}$-sphere R,

$$\Theta(-R) = -\Theta(R).$$

Theorem 18.28. *Let ν be a real valued invariant of $\mathbb{Q}$–spheres such that*

- *the invariant ν is of degree at most 2 with respect to $\mathcal{O}^{\mathbb{Q}}_{\mathcal{L}}$, and*
- *for any $\mathbb{Q}$–sphere R, $\nu(-R) = -\nu(R)$,*

then there exists a real number x such that $\nu = x\Theta$

Proof: According to Corollary 18.27,

$$(\nu - a_\theta\Theta)(-R) = (\nu - a_\theta\Theta)(R) = -(\nu - a_\theta\Theta)(R)$$

for any $\mathbb{Q}$-sphere R. So $\nu = a_\theta\Theta$. □

Remark 18.29. A similar result was proved in [Les04b, Proposition 6.2] without using the Moussard theorem.

18.6 Identifying Θ with the Casson-Walker invariant

In 1984, Andrew Casson introduced an invariant of $\mathbb{Z}$-spheres, which counts the conjugacy classes of irreducible representations of their fundamental groups using Heegaard splittings. See [AM90, GM92, Mar88]. This invariant lifts the Rohlin μ-invariant of Definition 5.28 from $\frac{\mathbb{Z}}{2\mathbb{Z}}$ to $\mathbb{Z}$. In 1988, Kevin Walker generalized the Casson invariant to $\mathbb{Q}$-spheres in [Wal92]. Here, the Casson-Walker invariant λ_{CW} is normalized as in [AM90, GM92, Mar88] for integer homology 3-spheres, and as $\frac{1}{2}\lambda_W$ for rational homology 3-spheres, where λ_W is the Walker normalisation in [Wal92]. According to [Wal92, Lemma 3.1], for any $\mathbb{Q}$-sphere R,

$$\lambda_{CW}(-R) = -\lambda_{CW}(R).$$

In [Les98], the author proved that the Casson-Walker generalization satisfies the same splitting formulae as $\frac{1}{6}\Theta$. So λ_{CW} is of degree at most 2 with respect to $\mathcal{O}^{\mathbb{Q}}_{\mathcal{L}}$, and

$$\lambda_{CW}(\psi_1(\ominus)) = 2.$$

(This is actually a consequence of [Les98, Theorem 1.3].)

As a direct corollary of Theorem 18.28, we obtain the following theorem, which was first proved by Kuperberg and Thurston in [KT99] for $\mathbb{Z}$–spheres, and which was generalized to $\mathbb{Q}$-spheres in [Les04b, Section 6]. See [Les04b, Theorem 2.6].

Theorem 18.30. $\Theta = 6\lambda_{CW}$.

Proof: Recall that Lemma 18.23 implies that $\Theta(\psi_1(\ominus)) = 12$. □

18.7 Sketch of the proof of Theorem 18.5

The statement of Theorem 18.5 involves a representative of L in $\mathcal{C}$, which is fixed during the proof, so that $\coprod_{i=1}^{x} A^{(i)}$, which is also fixed from now on, is embedded in $\mathcal{C} \setminus L$.

For $I \subseteq \underline{x}$, recall that $\mathcal{C}_I = \mathcal{C}\left((A^{(i)\prime}/A^{(i)})_{i\in I}\right)$. Set $\check{R}_I = \check{R}(\mathcal{C}_I)$, $R_I = R(\mathcal{C}_I)$, $\check{R} = \check{R}_\emptyset = \check{R}(\mathcal{C})$ and $R = R_\emptyset = R(\mathcal{C})$.

For any part X of R_I, $C_2(X)$ denotes the preimage of X^2 under the blowdown map from $C_2(R_I)$ to R_I^2.

In order to prove Theorem 18.5, we are going to compute the $\mathcal{Z}_n(\mathcal{C}_I, L)$ with antisymmetric homogeneous propagating forms ω_I on the $C_2(R_I)$ such that the ω_I coincide with each other as much as possible. (Antisymmetric propagating forms are defined in Definition 7.22.) More precisely, for any subsets I and J of $\underline{x}$, our forms will satisfy

$$\omega_I = \omega_J \text{ on } C_2\left(\left(R \setminus \cup_{i\in I\cup J}\mathrm{Int}(A^{(i)})\right) \cup_{i\in I\cap J} A^{(i)\prime}\right).$$

In order to get such forms, when dealing with integral LP-surgeries, we begin by choosing parallelizations τ_I of the $\check{R}_I$, which are standard outside $D_1 \times [0,1]$ and which coincide as much as possible, i.e. such that

$$\tau_I = \tau_J \text{ on } \left(\left(R \setminus \cup_{i\in I\cup J}\mathrm{Int}(A^{(i)})\right) \cup_{i\in I\cap J} A^{(i)\prime}\right) \times \mathbb{R}^3.$$

Unfortunately, this first step does not always work for rational LP-surgeries. See Section 19.1 and Example 19.4. To remedy this problem, we will make the definition of $\mathcal{Z}$ more flexible by allowing more general propagating forms associated to generalizations of parallelizations, called *pseudo-parallelizations*.

These pseudo-parallelizations are defined in Chapter 19, where they are shown to satisfy the following properties.

- They generalize parallelizations. They are genuine parallelizations outside a link tubular neighborhood, inside which they can be thought of as an average of genuine parallelizations.
- A parallelization defined near the boundary of a rational homology handlebody always extends to this rational homology handlebody as a pseudo-parallelization. (See Lemma 19.6.)
- A pseudo-parallelization $\tilde{\tau}$ of $\check{R}$, where $\check{R}$ is an asymptotic rational homology $\mathbb{R}^3$, induces a homotopy class of special complex trivializations $\tilde{\tau}_{\mathbb{C}}$ of $T\check{R} \otimes_{\mathbb{R}} \mathbb{C}$, which has a Pontrjagin number $p_1(\tilde{\tau}_{\mathbb{C}})$. Outside the link tubular neighborhood considered above, $\tilde{\tau}_{\mathbb{C}}$ is $\tilde{\tau} \otimes_{\mathbb{R}} 1_{\mathbb{C}}$. We set $p_1(\tilde{\tau}) = p_1(\tilde{\tau}_{\mathbb{C}})$. (See Definitions 19.7 and 19.8.)

- The notion of a *homogeneous propagating form* of $(C_2(R),\tilde{\tau})$ is presented in Definition 19.11. This definition allows us to extend the definition of $\mathcal{Z}$ of Theorem 12.7 using pseudo-parallelizations instead of parallelizations as follows. For any long tangle representative

  $$L\colon \mathcal{L} \hookrightarrow R(\mathcal{C})$$

  in a rational homology cylinder equipped with a pseudo-parallelization $\tilde{\tau}$ that restricts to a neighborhood of the image of L as a genuine parallelization, for any $n \in \mathbb{N}$, for any family $(\omega(i))_{i\in\underline{3n}}$ of homogeneous propagating forms of $(C_2(R(\mathcal{C})),\tilde{\tau})$

  $$Z_n(\mathcal{C},L,(\omega(i))) = \sum_{\Gamma\in\mathcal{D}^e_n(\mathcal{L})} \zeta_\Gamma I(\mathcal{C},L,\Gamma,(\omega(i))_{i\in\underline{3n}})[\Gamma] \in \mathcal{A}_n(\mathcal{L})$$

  depends only on $(\mathcal{C},L,p_1(\tilde{\tau}_{\mathbb{C}}))$ and on the $I_\theta(K_j,\tilde{\tau})$, which are defined as in Lemma 7.15 and Definition 12.6, for the components K_j, $j\in\underline{k}$, of L. It is denoted by $Z_n(\mathcal{C},L,\tilde{\tau})$. Set

  $$Z(\mathcal{C},L,\tilde{\tau}) = (Z_n(\mathcal{C},L,\tilde{\tau}))_{n\in\mathbb{N}} \in \mathcal{A}(\mathcal{L}).$$

  Then Theorem 19.13 ensures that

  $$\mathcal{Z}(\mathcal{C},L) = \exp(-\frac{1}{4}p_1(\tilde{\tau}_{\mathbb{C}})\beta)\prod_{j=1}^{k}(\exp(-I_\theta(K_j,\tilde{\tau})\alpha)\sharp_j)\, Z(\mathcal{C},L,\tilde{\tau}).$$

In the general case, we begin by choosing pseudo-parallelizations τ_I of the $\check{R}_I$ that are standard outside $D_1\times[0,1]$ and that coincide as much as possible, i.e. such that

$$\tau_I = \tau_J \text{ on } \left(\left(R\setminus\cup_{i\in I\cup J}\mathrm{Int}(A^{(i)})\right)\cup_{i\in I\cap J}A^{(i)\prime}\right)\times\mathbb{R}^3.$$

A reader who is only interested by the cases for which pseudo-parallelizations are not necessary, as in the applications of Sections 18.3 and 18.4, can skip Chapter 19, and replace the word pseudo-parallelization with parallelization in the rest of the proof below and in Chapter 20.

Set $\tau_\emptyset = \tau$. Then the $p_1(\tau_I)$ are related by the following lemma.

Lemma 18.31. *Set $p(i) = p_1(\tau_{\{i\}}) - p_1(\tau)$. For any subset I of $\underline{x}$,*

$$p_1(\tau_I) = p_1(\tau) + \sum_{i\in I} p(i).$$

PROOF: Proceed by induction on the cardinality of I. The lemma is obviously true if $\sharp I$ is zero or one. Assume $\sharp I \geq 2$. Let $j \in I$. It suffices to prove that $p_1(\tau_I) - p_1(\tau_{I \setminus \{j\}}) = p_1(\tau_{\{j\}}) - p_1(\tau)$. This follows by applying twice the second part of Proposition 5.25, where $M_0 = A^{(j)}$ and $M_1 = A^{(j)\prime}$ and $D = \mathcal{C}$ or $D = \mathcal{C}_{I \setminus \{j\}}$. The first application identifies $\left(p_1(\tau_{\{j\}}) - p_1(\tau)\right)$ with $p_1(\tau_{|A^{(j)}}, \tau_{\{j\}|A^{(j)\prime}})$. The second one yields the conclusion. $\square$

For any $i \in \underline{x}$, fix disjoint simple closed curves $(a^i_j)_{j=1,\dots,g_i}$ and simple closed curves $(z^i_j)_{j=1,\dots,g_i}$ on $\partial A^{(i)}$, such that

$$\mathcal{L}_{A^{(i)}} = \oplus_{j=1}^{g_i} [a^i_j],$$

and

$$\langle a^i_j, z^i_k \rangle_{\partial A^{(i)}} = \delta_{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

Let $[-4,4] \times (\coprod_{i \in N} \partial A^{(i)})$ be a tubular neighborhood of $(\coprod_{i \in N} \partial A^{(i)})$ in $\mathcal{C}$. This neighborhood intersects $A^{(i)}$ as $[-4,0] \times \partial A^{(i)}$. Let $[-4,0] \times \partial A^{(i)}$ be a neighborhood of $\partial A^{(i)\prime} = \partial A^{(i)}$ in $A^{(i)\prime}$. The manifold $\mathcal{C}_{\{i\}} = \mathcal{C}_i$ is obtained from $\mathcal{C}$ by removing $\left(A^{(i)} \setminus (]-4,0] \times \partial A^{(i)})\right)$ and by gluing back $A^{(i)\prime}$ along $(]-4,0] \times \partial A^{(i)})$.

Let $A^{(i)}_I \subset \mathcal{C}_I$, $A^{(i)}_I = A^{(i)}$ if $i \notin I$, $A^{(i)}_I = A^{(i)\prime}$ if $i \in I$. Let $\eta_{[-1,1]}$ be a one-form with compact support in $]-1,1[$ such that $\int_{[-1,1]} \eta_{[-1,1]} = 1$. Let $(a^i_j \times [-1,1])$ be a tubular neighborhood of a^i_j in $\partial A^{(i)}$. Let $\eta(a^i_j)$ be a closed one-form on $A^{(i)}_I$ such that the support of $\eta(a^i_j)$ intersects $[-4,0] \times \partial A^{(i)}$ inside $[-4,0] \times (a^i_j \times [-1,1])$, where $\eta(a^i_j)$ can be written as

$$\eta(a^i_j) = p^*_{[-1,1]}(\eta_{[-1,1]}),$$

with the projection $p_{[-1,1]} \colon [-4,0] \times (a^i_j \times [-1,1]) \to [-1,1]$ to the $[-1,1]$ factor. Note that the forms $\eta(a^i_j)$ on $A^{(i)}$ and $A^{(i)\prime}$ induce a closed one-form still denoted by $\eta(a^i_j)$ on $(A^{(i)} \cup_{\partial A^{(i)}} -A^{(i)\prime})$ that restrict to the previous ones. The form $\eta(a^i_j)$ on $(A^{(i)} \cup_{\partial A^{(i)}} -A^{(i)\prime})$ is Poincaré dual to the homology class $\partial^{-1}_{MV}(a^i_j)$ in $(A^{(i)} \cup_{\partial A^{(i)}} -A^{(i)\prime})$, with the notation introduced before Theorem 18.5.

The following proposition is the key to the proof of Theorem 18.5. Its proof, which is more complicated than I expected, is given in Chapter 20.

Proposition 18.32. *There exist homogeneous antisymmetric propagating forms ω_I of*

$$(C_2(R_I), \tau_I)$$

such that:

- *For any subsets I and J of $\underline{x}$,*

$$\omega_I = \omega_J \text{ on } C_2\left(\left(R \setminus \cup_{i \in I \cup J} Int(A^{(i)})\right) \cup_{i \in I \cap J} A^{(i)\prime}\right),$$

- *For any $(i,k) \in \underline{x}^2$ such that $i \neq k$, on $A_I^{(i)} \times A_I^{(k)}$,*

$$\omega_I = \sum_{\substack{j=1,\dots,g_i \\ \ell=1,\dots,g_k}} lk(z_j^i, z_\ell^k) p_{A_I^{(i)}}^*(\eta(a_j^i)) \wedge p_{A_I^{(k)}}^*(\eta(a_\ell^k)),$$

where $p_{A_I^{(i)}} \colon A_I^{(i)} \times A_I^{(k)} \to A_I^{(i)}$ and $p_{A_I^{(k)}} \colon A_I^{(i)} \times A_I^{(k)} \to A_I^{(k)}$ again denote the natural projections onto the factor corresponding to the subscript.

For a degree n oriented Jacobi diagram Γ without univalent vertices, if x is even, for $G = \bigsqcup_{i \in \underline{x}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right)$, define

$$\langle\langle G \rangle\rangle_\Gamma = \sum_{p \in P(G) | \Gamma_p \text{ isomorphic to } \Gamma} [\ell(p)\Gamma_p],$$

where the sum runs over the p such that Γ_p is isomorphic to Γ as a non-oriented trivalent graph, with the notation introduced before the statement of Theorem 18.5.

Assuming Proposition 18.32, one can prove the following lemma.

Lemma 18.33. *Let $n \in \mathbb{N}$. Let ω_I be forms as in Proposition 18.32. Let Γ be an oriented Jacobi diagram on $\mathcal{L}$. If Γ has less than x trivalent vertices, then*

$$\sum_{I \subseteq \underline{x}} (-1)^{\sharp I} I(R_I, L, \Gamma, o(\Gamma), (\omega_I)) = 0.$$

If $x = 2n$ and if Γ is a degree n trivalent Jacobi diagram, then

$$\sum_{I \subseteq \underline{x}} (-1)^{\sharp I} I(R_I, L, \Gamma, (\omega_I))[\Gamma] = \sharp Aut(\Gamma) \langle\langle \bigsqcup_{i \in \underline{x}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right) \rangle\rangle_\Gamma.$$

Proof: Let

$$\Delta = \sum_{I \subseteq \underline{x}} (-1)^{\sharp I} I(R_I, L, \Gamma, o(\Gamma), (\omega_I))$$

be the quantity that we want to compute for an oriented degree n Jacobi diagram on $\mathcal{L}$.

Number the vertices of Γ in $\underline{2n}$ arbitrarily, in order to view the open configuration space $\check{C}(\check{R}_I, L; \Gamma)$ as a submanifold of $\check{R}_I^{2n}$. The order of the vertices orders the oriented local factors (some of which are tangle components) of $\check{C}(\check{R}_I, L; \Gamma)$. So it orients $\check{C}(\check{R}_I, L; \Gamma)$. Orient the edges of Γ so that the edge-orientation of $H(\Gamma)$ and the vertex-orientation of Γ induce the above orientation of $\check{C}(\check{R}_I, L; \Gamma)$ as in Lemma 7.1.

For any $i \in \underline{x}$, the forms $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega_I)$ over

$$\check{C}(\check{R}_I, L; \Gamma) \cap (\check{R}_I \setminus A_I^{(i)})^{2n}$$

are identical for $I = K$ and $I = K \cup \{i\}$ for any $K \subseteq \underline{x} \setminus \{i\}$. Since their integrals enter the sum Δ with opposite signs, they cancel each other. This argument allows us to first get rid of the contributions of the integrals over the $\check{C}(\check{R}_I, L; \Gamma) \cap (\check{R}_I \setminus A_I^{(1)})^{2n}$ for any $I \subseteq \underline{x}$, next over the $\left(\check{C}(\check{R}_I, L; \Gamma) \setminus (\check{R}_I \setminus A_I^{(1)})^{2n}\right) \cap (\check{R}_I \setminus A_I^{(2)})^{2n}$, ..., and, finally, to get rid of all the contributions of the integrals over the $\check{C}(\check{R}_I, L; \Gamma) \cap (\check{R}_I \setminus A_I^{(i)})^{2n}$ for any $i \in \underline{x}$, and for any $I \subseteq \underline{x}$.

Thus, we are left with the contributions of the integrals over the subsets P_I of

$$\check{C}(\check{R}_I, L; \Gamma) \subset \check{R}_I^{2n}$$

such that:
For any $i \in \underline{x}$, any element of P_I projects onto $A_I^{(i)}$ under at least one of the $(2n)$ projections onto $\check{R}_I$. These subsets P_I are clearly empty if Γ has less than x trivalent vertices, and the lemma is proved in this case.

Assume $x = 2n$ and Γ is trivalent. Then P_I is equal to

$$\cup_{\sigma\in\mathfrak{S}_{2n}} \prod_{i=1}^{2n} A_I^{(\sigma(i))},$$

where $\mathfrak{S}_{2n}$ is the set of permutations of $\underline{2n}$. We get

$$\Delta = \sum_{\sigma\in\mathfrak{S}_{2n}} \Delta_\sigma$$

with

$$\Delta_\sigma = \sum_{I\subseteq\underline{2n}} (-1)^{\sharp I} \int_{\prod_{i=1}^{2n} A_I^{(\sigma(i))}} \bigwedge_{e\in E(\Gamma)} p_e^*(\omega_I).$$

Let us compute Δ_σ. For any $i \in \underline{x}$,

$$p_i : \check{C}(\check{R}_I, L; \Gamma) \longrightarrow \check{R}_I$$

denotes the projection onto the i^{th} factor. When e is an oriented edge from the vertex $x(e) \in V(\Gamma)$ to $y(e) \in V(\Gamma)$,

$$p_e^*(\omega_I)_{|\prod_{i=1}^{2n} A_I^{(\sigma(i))}} =$$

$$\sum_{\substack{j=1,\ldots,g_{\sigma(x(e))}\\ \ell=1,\ldots,g_{\sigma(y(e))}}} lk(z_j^{\sigma(x(e))}, z_\ell^{\sigma(y(e))}) p^*_{x(e)}(\eta(a_j^{\sigma(x(e))})) \wedge p^*_{y(e)}(\eta(a_\ell^{\sigma(y(e))})),$$

where the vertices are regarded as elements of $\underline{2n}$ via the numbering. Recall that $H(\Gamma)$ denotes the set of half-edges of Γ and that $E(\Gamma)$ denotes the set of edges of Γ. For a half-edge c, let $v(c)$ denote the label of the vertex contained in c.

Let F_σ denote the set of maps f from $H(\Gamma)$ to $\mathbb{N}$ such that for any $c \in H(\Gamma)$, $f(c) \in \{1, 2, \ldots, g_{\sigma(v(c))}\}$. For such a map f, $f(x(e))$ (resp. $f(y(e))$) denotes the value of f at the half-edge of e that contains $x(e)$ (resp. $y(e)$).

$$\Delta_\sigma = \sum_{f\in F_\sigma} \left(\prod_{e\in E(\Gamma)} lk(z^{\sigma(x(e))}_{f(x(e))}, z^{\sigma(y(e))}_{f(y(e))}) \right) I(f)$$

with

$$I(f) = \int_{\prod_{i=1}^{2n}(A^{(\sigma(i))}\cup -A^{(\sigma(i))\prime})} \bigwedge_{e\in E(\Gamma)} p^*_{x(e)}(\eta(a^{\sigma(x(e))}_{f(x(e))})) \wedge p^*_{y(e)}(\eta(a^{\sigma(y(e))}_{f(y(e))})).$$

We have

$$I(f) = \prod_{i=1}^{2n} \int_{(A^{(\sigma(i))}\cup -A^{(\sigma(i))\prime})} \bigwedge_{c\in H(\Gamma)|v(c)=i} p_i^*(\eta(a^{\sigma(i)}_{f(c)})),$$

where the factors of the exterior product over the half-edges of $v^{-1}(i)$ are ordered according to the vertex-orientation of i. Now,

$$\int_{A^{(\sigma(i))}\cup(-A^{(\sigma(i))\prime})} \bigwedge_{c\in v^{-1}(i)} \eta(a^{\sigma(i)}_{f(c)}) = \mathcal{I}_{A^{(\sigma(i))}A^{(\sigma(i))\prime}}\left(\bigotimes_{c\in v^{-1}(i)} a^{\sigma(i)}_{f(c)} \right),$$

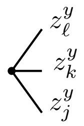
where the factors of the tensor product are ordered according to the vertex-orientation of i, again. Indeed, recall that $\eta\left(a^{\sigma(i)}_{f(c)}\right)$ is a closed form dual to the homology class $\partial^{-1}_{MV}(a^{\sigma(i)}_{f(c)})$ in $A^{(\sigma(i))} \cup (-A^{(\sigma(i))\prime})$, with the notation introduced before Theorem 18.5.

Summarizing,

$$\Delta_\sigma = \sum_{f\in F_\sigma} \left(\left(\prod_{e\in E(\Gamma)} lk(z^{\sigma(x(e))}_{f(x(e))}, z^{\sigma(y(e))}_{f(y(e))}) \right) \left(\prod_{i\in \underline{2n}} \mathcal{I}_{A^{(\sigma(i))}A^{(\sigma(i))\prime}} \left(\bigotimes_{c\in v^{-1}(i)} a^{\sigma(i)}_{f(c)} \right) \right) \right).$$

Now, note that we may restrict the sum to the subset $\tilde{F}_\sigma$ of F_σ consisting of the maps f of F_σ that restrict to $v^{-1}(i)$ as injections for any i.

Finally, Δ is a sum running over all the ways of renumbering the vertices of Γ by elements of $\underline{x}$ (via σ) and of coloring the half-edges c of $v^{-1}(i)$ by three distinct curves $z_{f(c)}^{\sigma(i)}$ via f. In particular, a pair (σ, f) provides a tripod



for any $y \in \underline{x}$ such that $1 \leq j < k < \ell \leq g_y$ and it provides a pairing of the ends of the univalent vertices of the tripods, which gives rise to the graph Γ with a possibly different vertex-orientation. The vertices of the obtained graph are furthermore numbered by the numbering of the vertices of Γ, and its edges are identified with the original edges of Γ.

For a given set of tripods as above, associated to the elements of $\underline{x}$, and a pairing of their univalent vertices, which gives rise to Γ as a non-oriented graph, there are exactly $\sharp\mathrm{Aut}(\Gamma)$ ways of numbering its vertices and edges to get a graph isomorphic to Γ as a numbered graph. So the pairing occurs $\sharp\mathrm{Aut}(\Gamma)$ times. $\square$

Proof of Theorem 18.5 up to the unproved assertions of this section, which are restated precisely at the end of the proof: Lemma 18.33 and Proposition 7.26 easily imply

$$\begin{array}{rcll} \sum_{I\subseteq\underline{x}}(-1)^{\sharp I} Z_n\left(\mathcal{C}_I, L, (\omega_I)\right) & = & 0 & \text{if } 2n < x, \\ & = & \left[\langle\langle \bigsqcup_{i\in\underline{x}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right)\rangle\rangle\right] & \text{if } 2n = x. \end{array}$$

Theorem 19.13 (or Theorem 12.7 if pseudo-parallelizations are not required) implies that

$$\mathcal{Z}\left(\mathcal{C}, L\right) = \exp(-\frac{1}{4}p_1(\tau)\beta)\prod_{j=1}^{k}(\exp(-I_\theta(K_j,\tau)\alpha)\sharp_j)\, Z(\mathcal{C}, L, \tau).$$

Set

$$Y_n = \sum_{I\subseteq\underline{x}}(-1)^{\sharp I}\left(\exp\left(-\frac{1}{4}p_1(\tau_I)\beta\right) Z\right)_n \left(\mathcal{C}_I, L, (\omega_I)\right),$$

where $(.)_n$ stands for the degree n part. Since the framing corrections $\prod_{j=1}^{k}\left(\exp(-I_\theta(K_j,\tau_I)\alpha)\sharp_j\right)$ do not depend on I, it suffices to prove that, if $2n \leq x$, then

$$Y_n = \sum_{I\subseteq\underline{x}}(-1)^{\sharp I} Z_n\left(\mathcal{C}_I, L, (\omega_I)\right).$$

$$\left(\exp\left(-\frac{1}{4}p_1(\tau_I)\beta\right) Z\right)_n (\mathcal{C}_I, L, (\omega_I)) = \begin{array}{l} Z_n(\mathcal{C}_I, L, (\omega_I)) \\ +\sum_{j<n} Z_j(\mathcal{C}_I, L, (\omega_I)) P_{n-j}(I), \end{array}$$

where $P_{n-j}(I)$ stands for an element of $\mathcal{A}_{n-j}(\emptyset)$, which is a linear combination of $m[\Gamma]$, where the m are monomials of degree at most $(n-j)$ in $p_1(\tau)$ and in the $p(i)$ of Lemma 18.31, for degree $(n-j)$ Jacobi diagrams Γ. Furthermore, such an $m[\Gamma]$ appears in $P_{n-j}(I)$, if and only if m is a monomial in the variables $p_1(\tau)$ and $p(i)$ for $i \in I$. Write the sum of the unwanted terms in Y_n, by factoring out the $m[\Gamma]$. Let $K \subseteq \underline{x}$ be the subset of the i such that $p(i)$ appears in m. ($\sharp K \leq n-j$). The factor of $m[\Gamma]$ is

$$\sum_{I | K \subseteq I \subseteq \underline{x}} (-1)^{\sharp I} Z_j(\mathcal{C}_I, L, (\omega_I)).$$

This sum runs over the subsets of $\underline{x} \setminus K$. The cardinality of $\underline{x} \setminus K$ is at least $x+j-n$. Since $2n \leq x$ and $j < n$, $j < n \leq x-n$, hence $2j < x+j-n$, and the beginning of the proof ensures that the above factor of $m[\Gamma]$ is zero. Hence $Y_n = \sum_{I \subseteq \underline{x}} (-1)^{\sharp I} Z_n(\mathcal{C}_I, L, (\omega_I))$. This concludes the reduction of the proof of Theorem 18.5 to the proof of Proposition 18.32, which is given in Chapter 20, and to the proofs that pseudo-parallelizations and associated propagating forms exist and satisfy the announced properties, which are given in Chapter 19. $\square$

18.8 Mixed universality statements

We can mix the statements of Theorem 17.32 and 18.5 to get the following statement, which covers both of them.

Theorem 18.34. *Let y, $z \in \mathbb{N}$. Recall $\underline{y} = \{1, 2, \ldots, y\}$. Set $(\underline{z} + y) = \{y+1, y+2, \ldots, y+z\}$. Let L be a singular q–tangle representative in a rational homology cylinder $\mathcal{C}$, whose double points are numbered by $\underline{y}$ and sitting in balls B_b of desingularizations for $b \in \underline{y}$. For a subset I of $\underline{y}$, let L_I denote the q–tangle obtained from L by performing negative desingularizations on double points of I and positive ones on double points of $\underline{y} \setminus I$ in the balls B_b. Let $\coprod_{i=y+1}^{y+z} A^{(i)}$ be a disjoint union of rational homology handlebodies embedded in $\mathcal{C} \setminus (L \cup_{b=1}^{y} B_b)$. Let $(A^{(i)\prime}/A^{(i)})$ be rational LP surgeries in $\mathcal{C}$.*

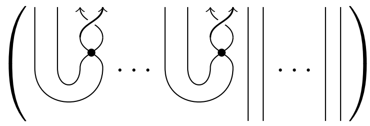
Set $X = [\mathcal{C}, L; (A^{(i)\prime}/A^{(i)})_{i \in \underline{z}+y}]$ and, using Notation 17.31,

$$\overline{\mathcal{Z}}_n(X) = \sum_{I \subseteq \underline{y+z}} (-1)^{\sharp I} \overline{\mathcal{Z}}_n \left(\mathcal{C}\left((A^{(i)\prime}/A^{(i)})_{i \in I \cap (\underline{z}+y)} \right), L_{I \cap \underline{y}} \right).$$

If $2n < 2y+z$*, then* $\overline{\mathcal{Z}}_n(X)$ *vanishes, and, if* $2n = 2y+z$*, then*

$$\overline{\mathcal{Z}}_n(X) = \left[\langle\langle \coprod_{i\in \underline{z}+y} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right)\rangle\rangle_{\check{R}(\mathcal{C})} \sqcup \Gamma_C(L)\right].$$

Proof assuming Theorem 18.5: Write $(\mathcal{C}, L)$ as a product of a tangle L_1 of the form



in the standard rational homology cylinder $D_1 \times [0,1]$, by a non-singular tangle L_2 in $\mathcal{C}$, so that $\coprod_{i=y+1}^{y+z} A^{(i)}$ is in the latter factor, by moving the double points below. Then we deduce from Proposition 17.34 and Theorem 18.5 that

$$\sum_{I\subseteq \underline{y+z}} (-1)^{\sharp I} \mathcal{Z}^f_n\left(\mathcal{C}\left((A^{(i)\prime}/A^{(i)})_{i\in I\cap(\underline{z}+y)}\right), L_{I\cap\underline{y}}\right)$$

satisfies the conclusions of the statement with $\mathcal{Z}^f$ instead of $\overline{\mathcal{Z}}$, by functoriality. Now, the proof of Theorem 18.34 is obtained from the proof[4] of Theorem 17.32 by "multiplying it by $\left[\langle\langle \coprod_{i\in \underline{z}+y} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right)\rangle\rangle_{\check{R}(\mathcal{C})}\right]$". $\square$

In order to prove more interesting mixed universality properties, we are going to say more about the normalization of the propagating forms of Proposition 18.32.

Recall that $[-4,4]\times\partial A^{(i)}$ denotes a regular neighborhood of $\partial A^{(i)}$ embedded in $\mathcal{C}$, that intersects $A^{(i)}$ as $[-4,0]\times \partial A^{(i)}$. All the neighborhoods $[-4,4]\times\partial A^{(i)}$ are disjoint from each other, and disjoint from L. Throughout this paragraph, we will use the corresponding coordinates on the image of this implicit embedding.

For $t\in[-4,4]$, set

$$A^{(i)}_t = \begin{cases} A^{(i)} \cup ([0,t]\times \partial A^{(i)}) & \text{if } t\geq 0 \\ A^{(i)} \setminus (]t,0]\times\partial A^{(i)}) & \text{if } t \leq 0\end{cases}$$

$$\partial A^{(i)}_t = \{t\}\times \partial A^{(i)}.$$

For $i\in\underline{x}$, choose a basepoint p^i in $\partial A^{(i)}$ outside the neighborhoods, $a^i_j\times[-1,1]$ of the a^i_j and outside neighborhoods $z^i_j\times[-1,1]$ the z^i_j. Fix a path $[p^i,q^i]$ from p^i to a point q^i of $\partial\mathcal{C}$ in

$$\mathcal{C}\setminus\left(\mathrm{Int}(A^{(i)})\cup_{k,k\neq i} A^{(k)}_4 \cup L\right)$$

[4]at the end of Section 17.6

so that all the paths $[p^i, q^i]$ are disjoint. Choose a closed 2-form $\omega(p^i)$ on $\left(C_1(R_I) \setminus \mathrm{Int}(A^{(i)})\right)$ such that

- the integral of $\omega(p^i)$ along a closed surface of $(\mathcal{C}_I \setminus \mathrm{Int}(A^{(i)}))$ is its algebraic intersection with $[p^i, q^i]$,
- the support of $\omega(p^i)$ intersects $(\mathcal{C}_I \setminus \mathrm{Int}(A^{(i)}))$ inside a tubular neighborhood of $[p^i, q^i]$ disjoint from
$$\left(\cup_{k,k\neq i} A_4^k\right) \cup \left([0,4] \times \left(\cup_{j=1}^{g_i}((a_j^i \times [-1,1]) \cup (z_j^i \times [-1,1]))\right) \cup L\right).$$
- $\omega(p^i)$ restricts as the usual volume form ω_{S^2} on $\partial C_1(R_I) = S^2$.

For $i \in \underline{x}$, for $j = 1, \ldots, g_i$, the curve $\{4\} \times a_j^i$ bounds a rational chain $\Sigma(a_j^i)$ in $A_4^{(i)}$ and a rational chain $\Sigma'(a_j^i)$ in $A_4^{(i)\prime}$. When viewed as a chain in $\mathcal{C}_I$, such a chain will be denoted by $\Sigma_I(a_j^i)$. $\Sigma_I(a_j^i) = \Sigma(a_j^i)$ if $i \notin I$ and $\Sigma_I(a_j^i) = \Sigma'(a_j^i)$ if $i \in I$. The form $\eta(a_j^i)$, which is supported on $[-4,4] \times a_j^i \times [-1,1]$ in $A_{I,4}^{(i)} \setminus A_{I,-4}^{(i)}$, and which may be expressed as $\eta(a_j^i) = p_{[-1,1]}^*(\eta_{[-1,1]})$ there, extends naturally to $A_{I,4}^{(i)} = \left(A_I^{(i)}\right)_4$, as a closed form dual to the chain $\Sigma_I(a_j^i)$.

For $i \in \underline{x}$, for $j = 1, \ldots, g_i$, z_j^i bounds a rational chain in $\mathcal{C}_I$. Therefore, it cobounds a rational cycle $\Sigma_I(\check{z}_j^i)$ in $\left(\mathcal{C}_I \setminus \mathring{A}_I^{(i)}\right) \setminus (\cup_{i=1}^x [p^i, q^i])$ with a combination of a_ℓ^i with rational coefficients.

$$\partial \Sigma_I(\check{z}_j^i) = z_j^i - \sum_{j=1}^{g_i} lk_e(z_j^i, \{-1\} \times z_\ell^i) a_\ell^i = \check{z}_j^i.$$

Furthermore, $\Sigma_I(\check{z}_j^i)$ may be assumed to intersect $\mathring{A}_I^{(k)}$ as

$$\sum_{m=1}^{g_k} lk(z_j^i, z_m^k) \Sigma_I(a_m^k),$$

for $k \neq i$. There is a closed one form $\eta_I(z_j^i)$ in $\left(\mathcal{C}_I \setminus \mathring{A}_I^{(i)}\right)$, which is supported near $\Sigma_I(\check{z}_j^i)$ and outside the supports of $\omega(p^i)$ and of the other $\omega(p^k)$, and which is dual to $\Sigma_I(\check{z}_j^i)$, such that $\eta_I(z_j^i) = \sum_{m=1}^{g_k} lk(z_j^i, z_m^k) \eta(a_m^k)$ on $\mathring{A}_I^{(k)}$, for $k \neq i$. The integral of $\eta_I(z_j^i)$ along a closed curve of $\left(\mathcal{C}_I \setminus \mathring{A}_I^{(i)}\right)$ is its linking number with z_j^i in $\mathcal{C}_I$.

The proof of the following proposition will be given in Chapter 20.

Proposition 18.35. *The antisymmetric propagating forms ω_I of $(C_2(R_I), \tau_I)$ of Proposition 18.32 can be chosen such that:*

1. *for every $i \in \underline{x}$, the restriction of ω_I to*
$$A_I^{(i)} \times (C_1(R_I) \setminus A_{I,3}^{(i)}) \subset C_2(R_I)$$
equals
$$\sum_{j \in \underline{g_i}} p_1^*(\eta_I(a_j^i)) \wedge p_2^*(\eta_I(z_j^i)) + p_2^*(\omega(p^i)),$$
where p_1 and p_2 denote the first and the second projection of $A_I^{(i)} \times (C_1(R_I) \setminus A_{I,3}^{(i)})$ to $C_1(R_I)$, respectively,

2. *for every i, for any $j \in \{1, 2, \dots, g_i\}$,*
$$\int_{\Sigma_I(a_j^i) \times p^i} \omega_I = 0,$$
where $p^i \in \partial A_I^{(i)}$ and $\partial \Sigma_I(a_j^i) \subset \{4\} \times \partial A_I^{(i)}$.

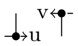
A *two-leg Jacobi diagram* is a unitrivalent Jacobi diagram with two univalent vertices, which are called legs. When these legs are colored by possibly non-compact connected components K_j of a tangle L, a two-leg diagram gives rise to a diagram on the source of the tangle $\mathcal{L}$, by attaching the legs to the corresponding components. The class in $\mathcal{A}(\mathcal{L})$ of this diagram is well defined. Indeed, according to Lemma 6.25, Jacobi diagrams with one univalent vertex vanish in $\mathcal{A}(\mathcal{L})$. Therefore the STU relation guarantees that if the two legs are colored by the same non-compact component, changing their order with respect to the orientation component does not change the diagram class. (See also Lemma 12.26.)

Generalize the contraction of trivalent graphs associated to LP-surgeries of Section 18.1 (before Theorem 18.5) to graphs with legs as follows.

Let $L \colon \mathcal{L} \hookrightarrow \mathcal{C}$ be a long tangle representative in a rational homology cylinder $\mathcal{C}$.

Let G be a graph with oriented trivalent vertices, and with two kinds of univalent vertices, the *decorated ones* and the *legs*, such that the components of the legs of G are nothing but segments from one leg to a decorated univalent vertex. The *legs* are univalent vertices on $\mathcal{L}$. The decorated univalent vertex in a segment of a leg is decorated with the leg component. The other *decorated univalent vertices* of G are decorated with disjoint curves of $\check{R} = \check{R}(\mathcal{C})$ disjoint from the image of L. Such a curve c bounds a compact oriented surface $\Sigma(c)$ in $\mathcal{C}$, and its linking number with a component K_u of L is the algebraic intersection $\langle K_u, \Sigma(c)\rangle$. Let $\check{P}(G)$ be the set of partitions of the set of decorated univalent vertices of G in disjoint pairs, such that no pair contains two vertices of leg segments.

For $p \in \check{P}(G)$, identifying the two decorated vertices of each pair provides a vertex-oriented Jacobi diagram Γ_p on $\mathcal{L}$. Multiplying it by the product $\ell(p)$ over the pairs of p of the linking numbers of the curves that decorate the two vertices yields an element $[\ell(p)\Gamma_p]$ of $\mathcal{A}(\mathcal{L})$.

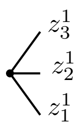
Again define $\langle\langle G\rangle\rangle = \sum_{p\in\check{P}(G)}[\ell(p)\Gamma_p]$ and extend this contraction to linear combinations of graphs, linearly. Assume that the components of L are numbered in $\underline{k}$. To a pair (u,v) of $\underline{k}$, associate the univalent graph $G_{uv} =$  consisting of two distinct segments, the first one with its leg on K_u, and the second one with its leg on K_v. The legs are considered as distinct even if there is an automorphism of $G_{uu} =$  that maps a leg to the other one when K_u is a circle. We think of the leg of the first segment as the first leg of G_{uv}, and the other leg is the second leg of G_{uv}.

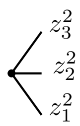
For a finite collection $(A^{(i)\prime}/A^{(i)})_{i\in\underline{2x}}$ of disjoint rational LP surgeries in $\check{R}(\mathcal{C})\setminus L$, define the element

$$\left[\Gamma_{(2)}\left(\mathcal{C}, L; (A^{(i)\prime}/A^{(i)})_{i\in\underline{2x}}\right)\right] = \frac{1}{2}\sum_{(u,v)\in\underline{k}^2}[\langle\langle \coprod_{i\in\underline{2x}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right)\sqcup G_{uv}\rangle\rangle]$$

of $\check{\mathcal{A}}(\mathcal{L})$.

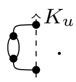
Examples 18.36. Assume that

$$T\left(\mathcal{I}_{A^{(1)}A^{(1)\prime}}\right) =$$ 

and $$T\left(\mathcal{I}_{A^{(2)}A^{(2)\prime}}\right) =$$ 

,

where $lk(z_i^1, z_j^2) = 0$ as soon as $i \neq j$, and $lk(z_i^1, K_u) = lk(z_i^2, K_u) = 0$ if $i \neq 3$. If K_u is an interval, then

$$\langle\langle \coprod_{i\in\underline{2}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right)\sqcup G_{uu}\rangle\rangle = -2lk(z_1^1, z_1^2)lk(z_2^1, z_2^2)lk(z_3^1, K_u)lk(z_3^2, K_u)$$ 

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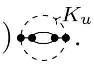
If K_u is a circle, then

$$\langle\langle \coprod_{i\in\underline{2}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right)\sqcup G_{uu}\rangle\rangle = -2lk(z_1^1, z_1^2)lk(z_2^1, z_2^2)lk(z_3^1, K_u)lk(z_3^2, K_u)$$ 

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Theorem 18.37. *Let $L\colon \mathcal{L} \hookrightarrow \mathcal{C}$ be a long tangle representative in a rational homology cylinder $\mathcal{C}$. Let $\coprod_{i=1}^{2x} A^{(i)}$ be a disjoint union of rational homology* *handlebodies embedded in $\mathcal{C} \setminus L$. Let $(A^{(i)\prime}/A^{(i)})$ be rational LP surgeries in $\mathcal{C}$. Set $X = [\mathcal{C}, L; (A^{(i)\prime}/A^{(i)})_{i\in\underline{2x}}]$ and*

$$\check{Z}_n(X) = \sum_{I\subseteq\underline{2x}} (-1)^{\sharp I} \check{Z}_n\left(\mathcal{C}_I, L\right),$$

where $\mathcal{C}_I = \mathcal{C}\left((A^{(i)\prime}/A^{(i)})_{i\in I}\right)$ is the rational homology cylinder obtained from $\mathcal{C}$ by performing the LP-surgeries that replace $A^{(i)}$ by $A^{(i)\prime}$ for $i \in I$. Assume $x \neq 0$. If $n < x+1$, then $\check{Z}_n(X)$ vanishes, and, if $n = x+1$, then

$$\check{Z}_n(X) = \left[\Gamma_{(2)}\left(\mathcal{C}, L; (A^{(i)\prime}/A^{(i)})_{i\in\underline{2x}}\right)\right],$$

where $\check{Z}$ is defined in Proposition 17.30. See also Definition 6.19.

Remark 18.38. Note that this result still holds mod 1T when $x = 0$.

Proof of Theorem 18.37 assuming Proposition 18.35: Recall that $\check{Z}$ takes its values in $\check{\mathcal{A}}(\mathcal{L})$, where the diagrams which have connected trivalent components vanish. Therefore Theorem 18.5 implies the result when $n < x+1$. Assume $n = x+1$. Since the framing correction terms that involve β vanish in $\check{\mathcal{A}}(\mathcal{L})$, and since the other correction terms are the same for all the $\check{Z}\left(\mathcal{C}_I, L\right)$,

$$\check{Z}_{x+1}(X) = \sum_{I\subseteq\underline{2x}} (-1)^{\sharp I} \check{Z}_n\left(\mathcal{C}_I, L, (\omega_I)\right).$$

Let Γ be a Jacobi diagram of degree $x+1$ on $\mathcal{L}$ that contributes to $\check{Z}_{x+1}(X)$. Since its class does not vanish in $\check{\mathcal{A}}_{x+1}(\mathcal{L})$, each component of Γ must contain at least two univalent vertices because one-leg diagrams vanish in $\check{\mathcal{A}}(\mathcal{L})$. Furthermore, as in the proof of Lemma 18.33, the configurations must involve at least one point, which will be a trivalent vertex position in some $A_I^{(i)}$, for each i in $\underline{2x}$. Therefore Γ has at least $2x$ trivalent vertices. Finally, Γ must be a connected two-leg Jacobi diagram of degree $x+1$ on $\mathcal{L}$. Assume that the univalent vertices of Γ are on components K_u and K_v. Order the set of univalent vertices of Γ, the first one is on K_u and the second one is on K_v. Let Γ^U denote the graph Γ equipped with this order. Number the trivalent vertices of Γ in $\underline{2x}$ and orient its edges and its trivalent vertices. So the edge-orientation of $H(\Gamma)$ and its vertex-orientation orient the open configuration space $\check{C}(\check{R}_I, L; \Gamma)$ as an open oriented submanifold of $K_u \times K_v \times \check{R}_I^{2x}$ (as in Lemma 7.1, again).

As in the proof of Lemma 18.33,

$$\Delta(\Gamma) = \sum_{I\subseteq\underline{2x}} (-1)^{\sharp I} I(R_I, L, \Gamma, o(\Gamma), (\omega_I))[\Gamma] = \sum_{\sigma\in\mathfrak{S}_{2x}} \Delta_\sigma[\Gamma],$$

where

$$\Delta_\sigma = \sum_{I \subseteq \underline{2x}} (-1)^{\sharp I} \int_{D(I,\sigma)} \bigwedge_{e \in E(\Gamma)} p_e^*(\omega_I),$$

and

$$D(I,\sigma) = K_u \times K_v \times \prod_{i=1}^{2x} A_I^{(\sigma(i))}.$$

When e is an oriented edge between two trivalent vertices, recall the expression of $p_e^*(\omega_I)$ from Proposition 18.32. Without loss of generality, assume that the legs are the first half-edges of their edges. With the projections $p_i : \check{C}(\check{R}_I, L; \Gamma) \longrightarrow \check{R}_I$, for an edge from a leg $x(e)$ to a trivalent vertex $y(e)$

$$p_e^*(\omega_I)_{|D(I,\sigma)} = \sum_{j \in \underline{g_{\sigma(y(e))}}} p_{x(e)}^*(\eta(z_j^{\sigma(y(e))})) \wedge p_{y(e)}^*(\eta(a_j^{\sigma(y(e))}))$$

according to Proposition 18.35. In this case, the edge e is associated to $x(e)$ and denoted by $e(x(e))$. Let $H_T(\Gamma)$ denote the subset of $H(\Gamma)$ consisting of the half-edges that contain a trivalent vertex, and let $E_T(\Gamma)$ denote the set of edges of Γ between trivalent vertices. Let F_σ denote the set of maps f from $H_T(\Gamma)$ to $\mathbb{N}$ such that for any $c \in H_T(\Gamma)$, $f(c) \in \{1, 2, \dots, g_{\sigma(v(c))}\}$.

$$\Delta_\sigma = \sum_{f \in F_\sigma} \left(\prod_{e \in E_T(\Gamma)} lk(z_{f(x(e))}^{\sigma(x(e))}, z_{f(y(e))}^{\sigma(y(e))}) \right) I(f),$$

where $I(f)$ is equal to

$$\int_{K_u \times K_v} \eta\left(z_{f(y(u))}^{\sigma(y(u))}\right) \wedge \eta\left(z_{f(y(v))}^{\sigma(y(v))}\right) \times \prod_{i=1}^{2x} \int_{(A^{(\sigma(i))} \cup -A^{(\sigma(i))\prime})} \bigwedge_{c \in H(\Gamma) | v(c) = i} p_i^*(\eta(a_{f(c)}^{\sigma(i)})),$$

when $K_u \neq K_v$, or when K_u is a closed component. Recall

$$\int_{K_u} \eta\left(z_{f(y(u))}^{\sigma(y(u))}\right) = lk\left(K_u, z_{f(y(u))}^{\sigma(y(u))}\right).$$

Summarizing, when $K_u \neq K_v$ or when K_u is a closed component,

$$\Delta_\sigma = \sum_{f \in F_\sigma} \left(\left(\prod_{e \in E(\Gamma)} lk(e; f) \right) \left(\prod_{i \in \underline{2x}} \mathcal{I}_{A^{(\sigma(i))} A^{(\sigma(i))\prime}} \left(\bigotimes_{c \in v^{-1}(i)} a_{f(c)}^{\sigma(i)} \right) \right) \right),$$

where $lk(e; f) = lk\left(z_{f(x(e))}^{\sigma(x(e))}, z_{f(y(e))}^{\sigma(y(e))}\right)$ when $e \in E_T(\Gamma)$, and

$$lk(e; f) = lk\left(K_{x(e)}, z_{f(y(e))}^{\sigma(y(e))}\right)$$

when $x(e)$ is univalent.

Finally, $\Delta(\Gamma)$ is a sum, running over all the ways of renumbering the trivalent vertices of Γ by elements of $\underline{2x}$ (via σ), and of coloring the half-edges c of $v^{-1}(i)$ by three distinct curves $z^{\sigma(i)}_{f(c)}$ via f.

In particular, a pair (σ, f) provides a tripod $\prec\!\begin{smallmatrix} z^y_\ell \\ z^y_k \\ z^y_j \end{smallmatrix}$ for any $y \in \underline{2x}$ such that $1 \leq j < k < \ell \leq g_y$ and it provides a pairing of the ends of the univalent vertices of the tripods, and of the legs on K_u and K_v (the first one and the second one when $v = u$), which gives rise to the graph Γ with a possibly different vertex-orientation. The vertices of the obtained graph are furthermore numbered by the numbering of the vertices of Γ, and its edges are identified with the original edges of Γ. The order of $U(\Gamma)$ is induced by the order on the legs of G_{uv}.

Let $\mathrm{Aut}(\Gamma^U)$ denote the set of automorphisms of Γ that fix the univalent vertices of Γ (pointwise). ($\mathrm{Aut}(\Gamma^U) \neq \mathrm{Aut}(\Gamma)$ if and only if u=v, K_u is a closed component and there exists an automorphism of Γ that exchanges its two univalent vertices.)

For a given set of tripods as above, associated to the elements of $\underline{2x}$, and a pairing of their univalent vertices, and of the legs on K_u and K_v (that are distinguished as the first one and the second one when $v = u$), there are exactly $\sharp\mathrm{Aut}(\Gamma^U)$ ways of numbering its vertices and edges, to get a graph isomorphic to Γ^U as a numbered graph, by an isomorphism that fixes the univalent vertices. So the pairing occurs $\sharp\mathrm{Aut}(\Gamma^U)$ times.

For a family G of $2x$ tripods $\prec\!\begin{smallmatrix} z^y_\ell \\ z^y_k \\ z^y_j \end{smallmatrix}$ for any $y \in \underline{2x}$ such that $1 \leq j < k < \ell \leq g_y$, define

$$[\langle\langle G \sqcup G_{uv} \rangle\rangle_{\Gamma^U}] = \sum_{p \in \check{P}(G \sqcup G_{uv}) \mid \Gamma_p \text{ isomorphic to } \Gamma^U} [\ell(p)\Gamma_p],$$

where the sum runs over the p such that Γ_p is isomorphic to Γ^U, as a non-oriented uni-trivalent graph on $\mathcal{L}$, equipped with a fixed order on $U(\Gamma)$, with the notation introduced before the statement of Theorem 18.37, and

$$[\langle\langle G \sqcup G_{uv} \rangle\rangle_{\Gamma}] = \sum_{p \in \check{P}(G \sqcup G_{uv}) \mid \Gamma_p \text{ isomorphic to } \Gamma} [\ell(p)\Gamma_p],$$

where the sum runs over the p such that Γ_p is isomorphic to Γ, as a non-oriented uni-trivalent graph on $\mathcal{L}$. (We forget the order on $U(\Gamma)$.)

Then

$$\Delta(\Gamma) = \sharp\mathrm{Aut}(\Gamma^U)\left[\langle\langle \bigsqcup_{i\in\underline{2x}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right) \sqcup G_{uv}\rangle\rangle_{\Gamma^U}\right].$$

If $u \neq v$, $\Gamma^U = \Gamma$. Assume that $u = v$ and that K_u is a circle. If there exists an automorphism of Γ that exchanges its univalent vertices, then $\sharp\mathrm{Aut}(\Gamma) = 2\sharp\mathrm{Aut}(\Gamma^U)$ and $\langle\langle . \rangle\rangle_{\Gamma^U} = \langle\langle . \rangle\rangle_{\Gamma}$. Otherwise, $\sharp\mathrm{Aut}(\Gamma) = \sharp\mathrm{Aut}(\Gamma^U)$ and $[\langle\langle . \rangle\rangle_{\Gamma}] = 2\,[\langle\langle . \rangle\rangle_{\Gamma^U}]$. So

$$\Delta(\Gamma) = \frac{1}{2}\sharp\mathrm{Aut}(\Gamma)\left[\langle\langle \bigsqcup_{i\in\underline{2x}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right) \sqcup G_{uu}\rangle\rangle_{\Gamma}\right]$$

in any case.

It remains to study the case where the univalent vertices of Γ belong to the same component K_u, which is non-compact. In this case, we compute the sum $\Delta(\Gamma) + \Delta(\Gamma^s)$, where Γ^s is obtained from Γ by exchanging the order of its univalent vertices on K_u.

Again, we find

$$\Delta(\Gamma) + \Delta(\Gamma^s) = \sharp\mathrm{Aut}(\Gamma)\left[\langle\langle \bigsqcup_{i\in\underline{2x}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right) \sqcup G_{uu}\rangle\rangle_{\Gamma}\right],$$

where the contraction $\langle\langle . \rangle\rangle_{\Gamma}$ keeps only the graphs that are isomorphic to Γ (as a graph with an ordered pair of free legs). (Recall $[\Gamma^s] = [\Gamma]$ in $\check{\mathcal{A}}(\mathcal{L})$). If Γ and Γ^s are isomorphic, then $\Delta(\Gamma) = \Delta(\Gamma^s)$ and

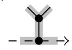
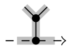
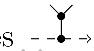
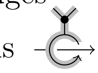
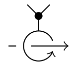
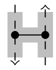
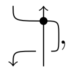
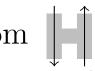
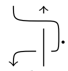
$$\Delta(\Gamma) = \frac{1}{2}\sharp\mathrm{Aut}(\Gamma)\left[\langle\langle \bigsqcup_{i\in\underline{2x}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right) \sqcup G_{uu}\rangle\rangle_{\Gamma}\right].$$

Otherwise, $\sharp\mathrm{Aut}(\Gamma) = \sharp\mathrm{Aut}(\Gamma^s)$ and $[\langle\langle . \sqcup G_{uu}\rangle\rangle_{\Gamma}] = [\langle\langle . \sqcup G_{uu}\rangle\rangle_{\Gamma^s}]$. So in any case,

$$\sum_{\Gamma \text{ as above with 2 univalent vertices on } K_u} \frac{\Delta(\Gamma)}{\sharp\mathrm{Aut}(\Gamma)} = \frac{1}{2}\left[\langle\langle \bigsqcup_{i\in\underline{2x}} T\left(\mathcal{I}_{A^{(i)}A^{(i)\prime}}\right) \sqcup G_{uu}\rangle\rangle\right].$$

□

As in Section 18.4, we can associate an alternate sum of tangles from a framed embedding of a Jacobi diagram on a tangle L into a rational homology cylinder.

Let Γ be such a Jacobi diagram whose trivalent vertices are numbered in $\underline{x}$. Let $\Sigma(\Gamma)$ be an oriented surface that contains Γ in its interior and such that $\Sigma(\Gamma)$ is a regular neighborhood of Γ in $\Sigma(\Gamma)$. Equip Γ with its vertex-orientation induced by the orientation of $\Sigma(\Gamma)$. Embed $\Sigma(\Gamma)$ in $\mathcal{C}$, so that L intersects $\Sigma(\Gamma)$ near univalent vertices as in  –so that L is tangent to $\Sigma(\Gamma)$ at univalent vertices– and L does not meet $\Sigma(\Gamma)$ outside such neighborhoods of the univalent vertices. Note that the embedding of $\Sigma(\Gamma)$ induces a local orientation of L as in Definition 6.16. Replace neighborhoods of the edges between trivalent vertices as in Section 18.4 and replace neighborhoods  of the edges  between a trivalent vertex and a univalent vertex by neighborhoods  of . Replace a chord between two univalent vertices with a crossing change so that  encodes the singular point , associated to the positive crossing change from  to .

Thus $\Sigma(\Gamma)$ transforms L into a singular tangle L^s whose double points are associated with the chords of Γ as above, equipped with a collection of disjoint oriented surfaces $\Sigma(Y)$ , associated to trivalent vertices of Γ. The oriented surfaces $\Sigma(Y)$ are next thickened to become framed genus 3 handlebodies.

Define $\psi(\Sigma(\Gamma))$ to be $[\mathcal{C}, L^s; (A^{(i)\prime}/A^{(i)})_{i\in\underline{x}}]$, where the surgeries $(A^{(i)\prime}/A^{(i)})$ associated to the trivalent vertices of Γ are defined as in Section 18.4, and define $\check{Z}(\psi(\Sigma(\Gamma))) = \sum_{I\subseteq\underline{x}}(-1)^{x+\sharp I}\check{Z}_n\left(\mathcal{C}_I, L^s\right)$, where $\mathcal{C}_I = \mathcal{C}\left((A^{(i)\prime}/A^{(i)})_{i\in I}\right)$.

Corollary 18.39. *Let n be a positive integer. Let Γ be a degree n Jacobi diagram with two univalent vertices. Let $\Sigma(\Gamma)$ be a regular neighborhood of Γ embedded in $\mathcal{C}$ as above. Then $\check{Z}_{\leq n}(\psi(\Sigma(\Gamma))) = [\Gamma]$.*

Proof: This follows from Theorem 18.37 as in the proof of Lemma 18.23. □

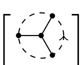
Remark 18.40. This corollary could be true for more general Jacobi diagrams.

To finish this section, we apply Theorem 18.37 with the LP surgeries of Subsection 18.3.1 to compute the degree 2 part of $\check{\mathcal{Z}}$ for a null-homologous knot and to prove Theorem 18.41 below.

Recall the definition of $a_2(K)$ from the statement of Proposition 18.11 and the lines that follow.

Theorem 18.41. *Let K be a null-homologous knot in a rational homology sphere R. Then*

$$\check{\mathcal{Z}}_2(R,K) = \left(\frac{1}{24} - a_2(K)\right)\left[\quad\right].$$

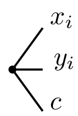


Proof: As in the proof of Proposition 18.11 in Subsection 18.3.2, let K bound a Seifert surface Σ equipped with a symplectic basis $(x_i, y_i)_{i \in \underline{g}}$ in R. Let $\Sigma \times [-1,2]$ be a collar of Σ in R and let $(A'/A) = (A'_F/A_F)$ and $(B'/B) = (B'_F/B_F)$ be the LP–surgeries of Subsection 18.3.1. Let U be a meridian of K passing through $d \times [-1,2]$. According to Proposition 18.15, $(R(A'/A), U)$ is diffeomorphic to (R,K). Similarly, $(R(B'/B), U)$ is diffeomorphic to $(R,-K)$, while $(R(A'/A, B'/B), U)$ is diffeomorphic to (R,U). Then

$$\begin{aligned}\check{\mathcal{Z}}_2\left([R,U;A'/A,B'/B]\right) &= 2\check{\mathcal{Z}}_2\left(R,U\right) - \check{\mathcal{Z}}_2\left(R,K\right) - \check{\mathcal{Z}}_2\left(R,-K\right)\\ &= \left[\Gamma_{(2)}\left(R,U;A'/A,B'/B\right)\right],\end{aligned}$$

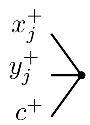
where

$$\left[\Gamma_{(2)}\left(R,U;A'/A,B'/B\right)\right] = \frac{1}{2}[\langle\langle T\left(\mathcal{I}_{AA'}\right) \sqcup T\left(\mathcal{I}_{BB'}\right) \sqcup$$  $$\rangle\rangle] \in \mathcal{A}_2(\mathcal{L}),$$

$$T\left(\mathcal{I}_{AA'}\right) = \sum_{i=1}^{g}$$ 

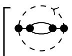
,

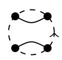
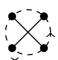
and

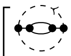
$$T\left(\mathcal{I}_{BB'}\right) = \sum_{j=1}^{g}$$ 

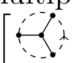
,

according to Lemma 18.17, so

$$\left[\Gamma_{(2)}\left(R,U;A'/A,B'/B\right)\right] = a_2(K)\left[\right.$$  $$\left.\right].$$

Since $\check{\mathcal{A}}_2(S^1)$ is generated by the chord diagrams  and , which are symmetric with respect to the orientation change on S^1, $\check{\mathcal{Z}}_2(R,K) = \check{\mathcal{Z}}_2(R,-K)$, so

$$2\check{\mathcal{Z}}_2\left(R,U\right) - 2\check{\mathcal{Z}}_2\left(R,K\right) = a_2(K)\left[\right.$$  $$\left.\right] = 2a_2(K)\left[\right.$$  $$\left.\right].$$

According to Example 7.21, and the multiplicativity of $\mathcal{Z}$ under connected sum of Theorem 10.24, $\check{\mathcal{Z}}_2\left(R,U\right) = \frac{1}{24}\left[\right.$  $\left.\right]$. □

Remark 18.42. This theorem generalizes a result of Guadagnini, Martellini and Mintchev in [GMM90] for the case $R = S^3$, to any rational homology sphere R. In the case of S^3, the known proof relies on the facts that $\check{\mathcal{Z}}_2$ is of degree 2 and that the space of real-valued knot invariants of degree at most

2 is generated by a_2 and a constant non-zero invariant. This uses the fact that any knot can be unknotted by crossing changes. This is no longer true in general rational homology spheres since crossing changes do not change the homotopy class. Our proof is more direct and our result holds for any null-homologous knot in a rational homology sphere.

Recall the Conway weight system w_C from Example 6.11, the Alexander polynomial of Definition 18.12, and Proposition 6.21. A long knot $\check{K}$ of a $\mathbb{Q}$-sphere $R(\mathcal{C})$ is an embedding of $\mathbb{R}$ into $\check{R}(\mathcal{C})$, whose image intersects the complement of $\mathcal{C}$, as the vertical embedding ($j_{\mathbb{R}}\colon t \mapsto (0,0,t)$) does. Replacing $j_{\mathbb{R}}(\mathbb{R}\setminus]0,1[)$ with an arc from $(0,0,1)$ to $(0,0,0)$, which cobounds an embedded topological disk in $\check{R}\setminus\mathring{\mathcal{C}}$, with an arc of $\partial\mathcal{C}$ with the same ends, provides a knot $\overline{\check{K}}$ well defined, up to isotopy. In [Let20], David Leturcq proved the following theorem, which, together with the properties of the functor $\mathcal{Z}^f$ of the third part of the book, implies Theorem 18.41.

Theorem 18.43 (Leturcq)**.** *For any long knot $\check{K}$ in a rational homology sphere $\check{R}$, such that $\overline{\check{K}}$ is null-homologous, we have the following equality in $\mathbb{R}[[h]]$.*

$$\sum_{n\in\mathbb{N}} w_C\left(\check{\mathcal{Z}}_n(R,\check{K})\right) h^n = \Delta_{\overline{\check{K}}}(\exp(h)).$$

Leturcq's proof of this theorem relies on a direct computation with appropriate propagating forms. In [Let21a], David Leturcq obtains a similar theorem for the Bott-Cattaneo-Rossi invariants of higher dimensional knots [CR05, Let21b]. This theorem in higher dimensions generalizes a theorem of Tadayuki Watanabe [Wat07], who proved it for ribbon knots.

Chapter 19

More flexible definitions of $\mathcal{Z}$ using pseudo-parallelizations

This chapter is devoted to present the *pseudo-parallelizations* introduced in Section 18.7. These generalizations of parallelizations are used in our proof of Theorem 18.5. They were introduced in [Les04b, Section 4.3 and 4.2] and studied in [Les10, Section 10] and [Les13, Sections 7 to 10]. They are defined in Section 19.2.

They will allow us to present more flexible definitions for our invariants $\mathcal{Z}$ and $\check{\mathcal{Z}}$. These more flexible definitions will involve propagating chains or propagating forms associated to pseudo-parallelizations. The generalized definition, which involves homogeneous propagating forms associated to pseudo-parallelizations, of the invariant $\mathcal{Z}$ of Theorem 12.7, which is used in Definition 13.10 of $\mathcal{Z}^f$ for q–tangles, is given in Theorem 19.13. The proof of this theorem will be concluded in Section 19.5. It is based on all the previous sections.

Variants of the definition of $\mathcal{Z}^f$ involving non-homogeneous propagating forms or propagating chains associated to pseudo-parallelizations will be presented in Chapter 21. They will not be used in the proof of Theorem 18.5.

19.1 Why we need pseudo-parallelizations

This section explains why a parallelization of the exterior of a $\mathbb{Q}$-handlebody A does not extend necessarily to A' after a rational LP surgery (A'/A). It justifies why I could not avoid this chapter, and some of its difficulties.

Let A be a compact oriented connected 3-manifold with boundary ∂A. Define the $\mathbb{Z}/2\mathbb{Z}$-*Lagrangian* $\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}}$ of A to be the kernel

$$\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}} = \mathrm{Ker}(H_1(\partial A; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H_1(A; \mathbb{Z}/2\mathbb{Z}))$$

of the map induced by the inclusion map. This is a Lagrangian subspace of $(H_1(\partial A;\mathbb{Z}/2\mathbb{Z});\langle .,.\rangle)$.

Let K be a *framed knot* in an oriented 3-manifold M. It is a knot equipped with a normal nonzero vector field $\vec{N}$, or equivalently with a parallel (up to homotopy). These data induce the direct trivialization τ_K of $TM_{|K}$ (up to homotopy) such that $\tau_K(e_1) = \overrightarrow{TK}$ and $\tau_K(e_2) = \vec{N}$, where K is equipped with an arbitrary orientation and $\overrightarrow{TK}$ is a tangent vector of K which induces the orientation of K. The homotopy class of the trivialization τ_K is well defined and does not depend on the orientation of K.

Lemma 19.1. *Assume that K bounds a possibly non-orientable compact surface Σ in M, and that Σ (or, more precisely, the parallel of K on Σ) induces the given parallelization of K. Let τ be a trivialization of the tangent space of M over Σ. Then the restriction of τ to K is not homotopic to τ_K.*

PROOF: Recall from Section 4.2 that $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$. Let $\mathbb{R}P^2_\pi$ denote the non-orientable submanifold of $SO(3)$ of the rotations of angle π. The homotopy class of a loop of $SO(3)$ is determined by its $\mathbb{Z}/2\mathbb{Z}$-valued algebraic intersection with $\mathbb{R}P^2_\pi$. Let us prove that the homotopy class of the restriction of τ to K is independent of the trivialization τ of TM over Σ. Any other trivialization of TM over Σ may be written as $\tau \circ \psi_{\mathbb{R}}(f)$ for a map f from Σ to $SO(3)$, with the notation of Section 4.2. Assume that f is transverse to $\mathbb{R}P^2_\pi$, without loss of generality. Then $f(K) \cap \mathbb{R}P^2_\pi$ bounds $f(\Sigma) \cap \mathbb{R}P^2_\pi$. So the restriction $f_{|K}$ is null-homotopic, and the homotopy class of the restriction of τ to K is independent of the trivialization τ of TM over Σ. It is also independent of the 3-manifold M that contains Σ. Since the tangent bundle of an oriented 3-manifold over a possibly non-orientable closed surface is trivializable, the homotopy class of the restriction of τ to K is also independent of Σ. Therefore, it is enough to prove the lemma when Σ is a disk, and it is obvious in this case. $\square$

If (K,τ_K) is a framed knot in an oriented 3-manifold M and if τ is a trivialization of the restriction of TM to K, we will say that K is τ-*bounding* if τ is not homotopic to τ_K. (This notion is actually independent of the whole manifold M, it depends only on what happens in a tubular neighborhood of K.)

Definition 19.2. Let α be a smooth map from $[-1,1]$ to $[0,2\pi]$, which maps $[-1,-1+\varepsilon]$ to 0, for some $\varepsilon \in]0,\frac{1}{2}[$, which increases from 0 to 2π on $[-1+\varepsilon, 1-\varepsilon]$, and such that $\alpha(-u)+\alpha(u) = 2\pi$ for any $u \in [-1,1]$.

Let Σ be a surface embedded in an oriented 3-manifold M. Let τ be a trivialization of $TM_{|\Sigma}$. Let γ be a two-sided curve properly embedded in Σ, equipped with a collar $\gamma \times [-1,1]$, in Σ and let $\mathcal{T}_\gamma$ denote the map from $\Sigma \times \mathbb{R}^3$ to itself, which restricts as the identity map of $(\Sigma \setminus (\gamma \times [-1,1])) \times \mathbb{R}^3$, such that

$$\mathcal{T}_\gamma(c \in \gamma, u \in [-1,1]; X \in \mathbb{R}^3) = (c, u, \rho_{\alpha(u)}(X)),$$

where $\rho_{\alpha(u)} = \rho(\alpha(u), (0,0,1))$ denotes the rotation of $\mathbb{R}^3$ with axis directed by $(0,0,1)$ and with angle $\alpha(u)$. Then the *twist of* $\tau_{|\Sigma}$ *across* γ is (the homotopy class of) $\tau_{|\Sigma} \circ \mathcal{T}_\gamma$.

Proposition 19.3. *Let ∂A be a connected oriented compact surface. Let τ be a trivialization of $T(\partial A \times [-2,2])$. Then there exists a unique map*

$$\phi_\tau : H_1(\partial A; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \frac{\mathbb{Z}}{2\mathbb{Z}}$$

such that

1. *when x is a connected curve of $\partial A = \partial A \times \{0\}$, $\phi_\tau(x) = 0$ if and only if x (equipped with its parallelization induced by ∂A) is τ-bounding and,*
2. $$\phi_\tau(x+y) = \phi_\tau(x) + \phi_\tau(y) + \langle x, y \rangle_{\partial A}.$$

The map ϕ_τ satisfies the following properties.

- *If a disjoint union x of curves of ∂A bounds a connected surface Σ, orientable or not, which induces its framing in an oriented 3-manifold M, then $\tau_{|x}$ extends to Σ as a trivialization of $TM_{|\Sigma}$ if and only if $\phi_\tau(x) = 0$.*
- *Let c be curve of ∂A and let $\mathcal{T}_c$ denote the twist across c, then for any $x \in H_1(\partial A; \mathbb{Z}/2\mathbb{Z})$,*

$$\phi_{\tau \circ \mathcal{T}_c}(x) = \phi_\tau(x) + \langle x, c \rangle_{\partial A}.$$

- *When A is a compact oriented connected 3-manifold with boundary ∂A, τ extends as a trivialization over A if and only if $\phi_\tau(\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}}) = \{0\}$.*

PROOF: For a disjoint union $x = \coprod_{i=1}^n x_i$ of connected curves x_i on ∂A, define $\phi_\tau(x) = \sum_{i=1}^n \phi_\tau(x_i)$ in $\mathbb{Z}/2\mathbb{Z}$, where $\phi_\tau(x_i) = 0$ if x_i (equipped with its parallelization induced by ∂A) is τ-bounding, and $\phi_\tau(x_i) = 1$, otherwise. Let us first prove that if such a framed disjoint union x bounds a connected surface Σ, which induces the framing of x in an oriented 3-manifold M, then $\tau_{|x}$ extends to Σ as a trivialization of $TM_{|\Sigma}$ if and only if $\phi_\tau(x) = 0$.

If $\phi_\tau(x) = 0$, group all the curves x_i such that $\phi_\tau(x_i) = 1$ by pairs. Make each such pair bound an annulus which induces the framing. Make each curve x_i such that $\phi_\tau(x_i) = 0$ bound a disk which induces the framing. Let $\hat{\Sigma}$ denote the union of Σ with the above disks and the above annuli. Extend τ to $\hat{\Sigma} \setminus \mathring{\Sigma}$. The restriction to $\hat{\Sigma}$ of the tangent bundle of an oriented 3-manifold M in which $\hat{\Sigma}$ embeds, is independent of M, and it admits a trivialization $\hat{\tau}$ that may be expressed as $\tau \circ \psi_{\mathbb{R}}(f)$ for a map f from $\hat{\Sigma} \setminus \mathring{\Sigma}$ to $SO(3)$. This map f extends to $\hat{\Sigma}$ because the intersection of $f(x)$ and $\mathbb{R}P^2_\pi$ is zero mod 2. (See the proof of Lemma 19.1.) Therefore, $\tau = \hat{\tau} \circ \psi_{\mathbb{R}}(f)^{-1}$ also extends to Σ.

If $\phi_\tau(x) = 1$, assume $\phi_\tau(x_1) = 1$ without loss of generality, group the other curves x_i such that $\phi_\tau(x_i) = 1$ pairwise, make them cobound a disjoint union of annuli, and make the curves x_i such that $\phi_\tau(x_i) = 0$ bound disks, in a framed way as above. Let $\hat{\Sigma}$ denote the union of Σ with these disks and these annuli. The trivialization τ still extends to $\hat{\Sigma} \setminus \mathring{\Sigma}$. If τ extends to Σ, it extends to $\hat{\Sigma}$, where $\partial\hat{\Sigma} = x_1$, and x_1 is τ-bounding, so $\phi_\tau(x_1) = 0$, which is absurd. Therefore, τ does not extend to Σ.

Let us prove that our above definition of $\phi_\tau(x)$ depends only on the class of x in $H_1(\partial A; \mathbb{Z}/2\mathbb{Z})$. Let x be an embedded (possibly non-connected) curve in ∂A and let y be another such in $\partial A \times \{-1\}$ that is homologous to x mod 2. Then there exists a framed (possibly non-orientable) cobordism between x and y in $\partial A \times [-1, 1]$, and it is easy to see that $\phi_\tau(x) = 0$ if and only if $\phi_\tau(y) = 0$.

Let us check that ϕ_τ behaves as predicted under addition. Because we are dealing with elements of $H_1(\partial A; \mathbb{Z}/2\mathbb{Z})$, we can consider representatives x and y of x and y such that every connected component of x intersects at most one component of y and every connected component of y intersects at most one component of x. Next, the known additivity under disjoint union reduces the proof to the case in which x and y are connected and x and y intersect once. Note that both sides of the equality to be proved vary in the same way under trivialization changes. Consider the punctured torus neighborhood of $x \cup y$, and a trivialization τ that restricts to the punctured torus as the direct sum of a trivialization of the torus and the normal vector to ∂A. Then $\phi_\tau(x + y) = \phi_\tau(x) = \phi_\tau(y) = 1$. The last two assertions are left to the reader. $\square$

Example 19.4. For any $\mathbb{Q}$-handlebody A, there exists a Lagrangian subspace $\mathcal{L}^{\mathbb{Z}}$ of $(H_1(\partial A; \mathbb{Z}); \langle ., . \rangle)$, such that $\mathcal{L}_A = \mathcal{L}^{\mathbb{Z}} \otimes \mathbb{Q}$. However, as the following example shows, $\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}}$ is not necessarily equal to $\mathcal{L}^{\mathbb{Z}} \otimes \mathbb{Z}/2\mathbb{Z}$.

Let $\mathcal{M}$ be a Möbius band embedded in the interior of a solid torus $D^2 \times S^1$ so that the core of the solid torus is equal to the core of $\mathcal{M}$. Embed $D^2 \times S^1$

into $S^2 \times S^1 = D^2 \times S^1 \cup_{\partial D^2 \times S^1} (-D^2 \times S^1)$ as the first copy. Orient the knot $\partial\mathcal{M}$ so that $\partial\mathcal{M}$ pierces twice $S^2 \times 1$ positively. Let m be the meridian of $\partial\mathcal{M}$, and let ℓ be the parallel of $\partial\mathcal{M}$ induced by $\mathcal{M}$. Let A be the exterior of the knot $\partial\mathcal{M}$ in $S^2 \times S^1$. Then A is a $\mathbb{Q}$-handlebody such that $\mathcal{L}_A^{\mathbb{Z}} = \mathbb{Z}[2m]$, $\mathcal{L}_A = \mathbb{Q}[m]$, and $\mathcal{L}_A^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z}[\ell]$ (Exercise).

In particular, if A_0 is a solid torus such that $\partial A = \partial A_0$ and $\mathcal{L}_{A_0} = \mathbb{Q}[m]$, the restriction to ∂A of a trivialization τ of A_0 such that $\phi_\tau(\ell) = 1$ does not extend to A.

19.2 Definition of pseudo-parallelizations

Definition 19.5. A *pseudo-parallelization* $\tilde{\tau} = (N(\gamma); \tau_e, \tau_b)$ of an oriented 3-manifold A with possible boundary consists of

- a framed link γ of the interior of A, which will be called *the link of the pseudo-parallelization* $\tilde{\tau}$, equipped with a neighborhood $N(\gamma) = [a,b] \times \gamma \times [-1,1]$, for an interval $[a,b]$ of $\mathbb{R}$ with non-empty interior,
- a parallelization τ_e of A outside $N(\gamma)$,
- a parallelization $\tau_b \colon N(\gamma) \times \mathbb{R}^3 \to TN(\gamma)$ of $N(\gamma)$ such that

$$\tau_b = \begin{cases} \tau_e & \text{over } \partial([a,b] \times \gamma \times [-1,1]) \setminus (\{a\} \times \gamma \times [-1,1]) \\ \tau_e \circ \mathcal{T}_\gamma & \text{over } \{a\} \times \gamma \times [-1,1], \end{cases}$$

where

$$\mathcal{T}_\gamma(t, c \in \gamma, u \in [-1,1]; X \in \mathbb{R}^3) = (t, c, u, \rho_{\alpha(u)}(X)),$$

where $\rho_{\alpha(u)} = \rho(\alpha(u), e_3)$ denotes the rotation of $\mathbb{R}^3$ with axis directed by $e_3 = (0,0,1)$ and with angle $\alpha(u)$, and α is a smooth map from $[-1,1]$ to $[0,2\pi]$ that maps $[-1,-1+\varepsilon]$ to 0, that increases from 0 to 2π on $[-1+\varepsilon, 1-\varepsilon]$, and such that $\alpha(-u) + \alpha(u) = 2\pi$ for any $u \in [-1,1]$, for some $\varepsilon \in]0, \frac{1}{2}[$ such that $\varepsilon < \frac{b-a}{4}$.

Lemma 19.6. *Let A be a compact oriented 3-manifold and let τ be a trivialization of TA defined on a collar $[-4,0] \times \partial A$ of $\partial A (= \{0\} \times \partial A)$. Then there is a pseudo-parallelization of A that extends the restriction of τ to $[-1,0] \times \partial A$.*

Proof: There exists a trivialization τ' of TA on A. After a homotopy of τ around $\{-2\} \times \partial A$, there exists an annulus $\gamma \times [-1,1]$ of $\{-2\} \times \partial A$ such that $\tau = \tau' \circ \mathcal{T}_\gamma$ on $\{-2\} \times \partial A$. Consider the neighborhood $N(\gamma) = [-2,-1] \times \gamma \times [-1,1]$ of γ. Define τ_e to coincide with τ on $([-2,0] \times \partial A) \setminus \mathrm{Int}(N(\gamma))$ and with τ' on $A \setminus (]-2,0] \times \partial A)$. Define τ_b to coincide with τ on $N(\gamma)$. $\square$

Definition 19.7. [Trivialization $\tilde{\tau}_{\mathbb{C}}$ of $TA \otimes_{\mathbb{R}} \mathbb{C}$] Let F_U be a smooth map such that

$$\begin{array}{llll} F_U: & [a,b]\times[-1,1] & \longrightarrow & SU(3) \\ & (t,u) & \mapsto & \left\{\begin{array}{ll} \text{Identity} & \text{if } |u| > 1-\varepsilon \\ \rho_{\alpha(u)} & \text{if } t < a+\varepsilon \\ \text{Identity} & \text{if } t > b-\varepsilon. \end{array}\right. \end{array}$$

F_U extends to $[a,b]\times[-1,1]$ because $\pi_1(SU(3))$ is trivial. Define the trivialization $\tilde{\tau}_{\mathbb{C}}$ of $TA \otimes_{\mathbb{R}} \mathbb{C}$, associated to the pseudo-parallelization $\tilde{\tau}$ of Definition 19.5, as follows.

- On $(A \setminus N(\gamma)) \times \mathbb{C}^3$, $\tilde{\tau}_{\mathbb{C}} = \tau_e \otimes 1_{\mathbb{C}}$,
- Over $[a,b]\times\gamma\times[-1,1]$, $\tilde{\tau}_{\mathbb{C}}(t,c,u;X) = \tau_b(t,c,u;F_U(t,u)^{-1}(X))$.

Since $\pi_2(SU(3))$ is trivial, the homotopy class of $\tilde{\tau}_{\mathbb{C}}$ is well defined.

Definition 19.8. For two compact connected oriented 3-manifolds M_0 and M_1 whose boundaries ∂M_0 and ∂M_1 have collars that are identified by a diffeomorphism, equipped with pseudo-parallelizations τ_0 and τ_1 which agree on the collar neighborhoods of $\partial M_0 = \partial M_1$, and which are genuine parallelizations there, we use the definition of Proposition 5.10 of relative Pontrjagin numbers and define $p_1(\tau_0,\tau_1)$ as $p_1(\tau_{0,\mathbb{C}},\tau_{1,\mathbb{C}})$. Let $\check{R}$ be a rational homology $\mathbb{R}^3$. A pseudo-parallelization $\tilde{\tau}$ of $\check{R}$ is *asymptotically standard* if it coincides with the standard parallelization τ_s of $\mathbb{R}^3$ outside B_R. (Recall Definition 3.6.) For such an asymptotically standard pseudo-parallelization, set $p_1(\tilde{\tau}) = p_1\left((\tau_s)_{|B^3}, \tilde{\tau}_{|B_R}\right)$, again.

Definition 19.9. [Homogeneous boundary form associated to $\tilde{\tau}$] Let $\tilde{\tau} = (N(\gamma);\tau_e,\tau_b)$ be a pseudo-parallelization of a 3-manifold A. Recall ε and the map α from Definition 19.5, and define a smooth map

$$\begin{array}{llll} F: & [a,b]\times[-1,1] & \longrightarrow & SO(3) \\ & (t,u) & \mapsto & \left\{\begin{array}{ll} \text{Identity} & \text{if } |u| > 1-\varepsilon \\ \rho_{\alpha(u)} & \text{if } t < a+\varepsilon \\ \rho_{-\alpha(u)} & \text{if } t > b-\varepsilon. \end{array}\right. \end{array}$$

The map F extends to $[a,b]\times[-1,1]$ because its restriction to the boundary is trivial in $\pi_1(SO(3))$.

Let $p_{\tau_b} = p(\tau_b)$ denote the projection from $UN(\gamma)$ to S^2 induced by τ_b.

$$p_{\tau_b}(\tau_b(t,c,u;X\in S^2)) = X.$$

Define $F(\gamma,\tau_b)$ as follows

$$\begin{array}{rlcl} F(\gamma,\tau_b)\colon & [a,b]\times\gamma\times[-1,1]\times S^2 & \longrightarrow & [a,b]\times\gamma\times[-1,1]\times S^2 \\ & (t,c,u;Y) & \mapsto & (t,c,u;F(t,u)(Y)). \end{array}$$

Define the closed two-form $\omega(\gamma,\tau_b)$ on $U([a,b]\times\gamma\times[-1,1])$ to be

$$\omega(\gamma,\tau_b)=\frac{p\left(\tau_b\circ\mathcal{T}_\gamma^{-1}\right)^*(\omega_{S^2})+p\left(\tau_b\circ F(\gamma,\tau_b)^{-1}\right)^*(\omega_{S^2})}{2}.$$

The *homogeneous boundary form* associated to $(\tilde{\tau},F)$ is the following closed 2-form $\omega(\tilde{\tau},F)$ on UA.

$$\omega(\tilde{\tau},F)=\left\{\begin{array}{ll} p_{\tau_e}^*(\omega_{S^2}) & \text{on } U(A\setminus N(\gamma)) \\ \omega(\gamma,\tau_b) & \text{on } U(N(\gamma)). \end{array}\right.$$

A *homogeneous boundary form* of $(UA,\tilde{\tau})$ is a homogeneous boundary form associated to $(\tilde{\tau},F)$ for some F as above.

The consistency of Definition 19.9 is justified by the following lemma, in the case where β is the constant map with value one. The general case is used in Lemma 21.2.

Lemma 19.10. *Let $(e_1=(1,0,0),e_2=(0,1,0),e_3=(0,0,1))$ denote the standard basis of $\mathbb{R}^3$, and let $v_i:\mathbb{R}^3\longrightarrow\mathbb{R}$ denote the i^{th} coordinate with respect to this basis. Let $\rho_\theta=\rho_{\theta,e_3}$ denote the rotation of $\mathbb{R}^3$ with axis directed by e_3 and with angle θ. Let*

$$\begin{array}{rccl} \mathcal{T}_k: & \mathbb{R}\times S^2 & \longrightarrow & S^2 \\ & (\theta,X) & \mapsto & \rho_{k\theta}(X). \end{array}$$

Let $(\beta\circ v_3)\omega_{S^2}$ denote a volume form on S^2 invariant under the rotations ρ_θ, for some map $\beta\colon[-1,1]\to\mathbb{R}$. Then

$$\mathcal{T}_k^*((\beta\circ v_3)\omega_{S^2})=\mathcal{T}_0^*((\beta\circ v_3)\omega_{S^2})+\frac{k(\beta\circ v_3)}{4\pi}d\theta\wedge dv_3$$

Proof: Recall that ω_{S^2} denotes the homogeneous two-form on S^2 with total area 1. When $X\in S^2$, and when v and w are two tangent vectors of S^2 at X,

$$\omega_{S^2}(v\wedge w)=\frac{1}{4\pi}det(X,v,w),$$

where $X\wedge v\wedge w=det(X,v,w)e_1\wedge e_2\wedge e_3$ in $\bigwedge^3\mathbb{R}^3$.

Since ρ_θ preserves the area in S^2 and leaves v_3 invariant, the restrictions of $\mathcal{T}_k^*((\beta\circ v_3)\omega_{S^2})$ and $\mathcal{T}_0^*((\beta\circ v_3)\omega_{S^2})$ coincide on $\bigwedge^2 T_{(\theta,X)}(\{\theta\}\times S^2)$. Therefore, we are left with the computation of $(\mathcal{T}_k^*((\beta\circ v_3)\omega_{S^2}) - \mathcal{T}_0^*((\beta\circ v_3)\omega_{S^2}))\,(u\wedge v)$ when $u\in T_{(\theta,X)}(\mathbb{R}\times\{X\})$ and $v\in T_{(\theta,X)}(\{\theta\}\times S^2)$, where $\mathcal{T}_0^*((\beta\circ v_3)\omega_{S^2})(u\wedge v)=0$, and

$$\mathcal{T}_k^*((\beta\circ v_3)\omega_{S^2})_{(\theta,X)}(u\wedge v) = \frac{\beta\circ v_3(X)}{4\pi} det(\rho_{k\theta}(X), T_{(\theta,X)}\mathcal{T}_k(u), T_{(\theta,X)}\mathcal{T}_k(v)),$$

by definition. Since $T_{(\theta,X)}\mathcal{T}_k(v)=\rho_{k\theta}(v)$, and since $\rho_{k\theta}$ preserves the volume in $\mathbb{R}^3$,

$$\mathcal{T}_k^*((\beta\circ v_3)\omega_{S^2})_{(\theta,X)}(u\wedge v) = \frac{\beta\circ v_3(X)}{4\pi} det(X, \rho_{-k\theta}(T_{(\theta,X)}\mathcal{T}_k(u)), v).$$

Now, let X_i stand for $v_i(X)$. $T_{(\theta,X)}\mathcal{T}_k(u) = kd\theta(u)\rho_{k\theta+\pi/2}(X_1e_1+X_2e_2)$. Therefore, $\mathcal{T}_k^*(\omega_{S^2})_{(\theta,X)}(u\wedge v) = \frac{kd\theta(u)}{4\pi}det(X,-X_2e_1+X_1e_2,v)$, and,

$$\begin{aligned}\mathcal{T}_k^*(\omega_{S^2})(u\wedge .) &= \tfrac{kd\theta(u)}{4\pi} det\begin{pmatrix} X_1 & -X_2 & dv_1\\ X_2 & X_1 & dv_2\\ X_3 & 0 & dv_3\end{pmatrix}\\ &= \tfrac{kd\theta(u)}{4\pi}\left(-X_3X_1dv_1 - X_3X_2dv_2 + (1-X_3^2)dv_3\right)\\ &= \tfrac{kd\theta(u)}{4\pi}dv_3.\end{aligned}$$

$\square$

Proof of the consistency of Definition 19.9: It suffices to prove that

$$p\left(\tau_b\circ\mathcal{T}_\gamma^{-1}\right)^*(\omega_{S^2}) + p\left(\tau_b\circ\mathcal{T}_\gamma\right)^*(\omega_{S^2}) = 2p\left(\tau_b\right)^*(\omega_{S^2})$$

on $U([b-\varepsilon,b]\times\gamma\times[-1,1])$, where

$$p_{\tau_b\circ\mathcal{T}_\gamma^{\pm1}}(\tau_b(t,c,u;X)) = p_{\tau_b\circ\mathcal{T}_\gamma^{\pm1}}(\tau_b\circ\mathcal{T}_\gamma^{\pm1}(t,c,u;\rho_{\mp\alpha(u)}(X))) = \rho_{\mp\alpha(u)}(X).$$

Let $\tilde{p}_{\tau_b} = p_{[-1,1]}\times p_{\tau_b}\colon U([b-\varepsilon,b]\times\gamma\times[-1,1]) \to [-1,1]\times S^2$, then

$$p\left(\tau_b\circ\mathcal{T}_\gamma^{\pm1}\right) = \mathcal{T}_{\mp1}\circ(\alpha\times\mathrm{Id}_{S^2})\circ\tilde{p}_{\tau_b} \text{ and } p\left(\tau_b\right) = \mathcal{T}_0\circ(\alpha\times\mathrm{Id}_{S^2})\circ\tilde{p}_{\tau_b}.$$

So $p\left(\tau_b\circ\mathcal{T}_\gamma^{\pm1}\right)^*(\omega_{S^2}) = ((\alpha\times\mathrm{Id}_{S^2})\circ\tilde{p}_{\tau_b})^*\left(\mathcal{T}_{\mp1}^*(\omega_{S^2})\right)$.
Thus, Lemma 19.10 implies that Definition 19.9 is consistent.

Definition 19.11. Let $\check{R}$ be a rational homology $\mathbb{R}^3$, equipped with an asymptotically standard pseudo-parallelization $\tilde{\tau}$. A *homogeneous propagating form* of $(C_2(R),\tilde{\tau})$ is a propagating form of $C_2(R)$ (as in Definition 3.11) that coincides with a homogeneous boundary form of $(U\check{R},\tilde{\tau})$ as in Definition 19.9 on $U\check{R}$.

Lemma 19.12. *Such homogeneous propagating forms exist for any $(R, \tilde{\tau})$.*

PROOF: See Section 3.3. $\square$

The main result of this chapter is the following theorem.

Theorem 19.13. *Theorem 7.20 and Theorem 12.7 generalize to the case where τ is a pseudo-parallelization $\tau = (N(\gamma); \tau_e, \tau_b)$ of $R(\mathcal{C})$, such that $N(\gamma)$ does not meet the image of the long tangle representative (or the link) $L\colon \mathcal{L} \hookrightarrow R(\mathcal{C})$, using Definition 19.8 for $p_1(\tau)$.*

In order to prove Theorem 19.13, we prove some preliminary lemmas in the next sections.

19.3 Integration of homogeneous propagating forms along surfaces

Definition 19.14. Let Σ be a compact oriented surface with boundary, and let X be a nowhere vanishing section of the tangent bundle $T\Sigma$ of Σ along the boundary of Σ. The *relative Euler number* $\chi(X;\Sigma)$ is the algebraic intersection in $T\Sigma$ of the graph of a generic extension $\tilde{X}$ of X over Σ as a section of $T\Sigma$ and the graph of the zero section of $T\Sigma$. (Both graphs are naturally oriented by Σ.)

Note that this definition makes sense since all the extensions of X are homotopic relatively to $\partial\Sigma$. This Euler number is an obstruction to extending X over Σ as a nowhere vanishing section of $T\Sigma$. Here are some other well-known properties of this number.

Lemma 19.15. *Let Σ be a compact oriented surface with boundary, and let X be a section of $U\Sigma$ along the boundary of Σ.*

- *If Σ is connected and if $\chi(X;\Sigma) = 0$, then X extends as a nowhere vanishing section of $T\Sigma$.*
- *If X is tangent to the boundary of Σ, then $\chi(X;\Sigma)$ is the Euler characteristic $\chi(\Sigma)$ of Σ.*
- *More generally, let $a^{(1)}, \dots, a^{(k)}$ denote the k connected components of the boundary $\partial\Sigma$ of Σ. For $i = 1, \dots, k$, the unit bundle $U\Sigma_{|a^{(i)}}$ of $T\Sigma_{|a^{(i)}}$ is an S^1-bundle over $a^{(i)}$ with a canonical trivialization induced by $Ta^{(i)}$. Let $d(X, a^{(i)})$ be the degree of the projection on the fiber S^1 of*

this bundle of the section X, with respect to this canonical trivialization. Then

$$\chi(X;\Sigma) = \sum_{i=1}^{k} d(X, a^{(i)}) + \chi(\Sigma).$$

PROOF: First observe all these properties when Σ is a disk. When Σ is connected, there is a disk D that contains all the zeros of an arbitrary generic extension of $\tilde{X}$ of X. If $\chi(X;\Sigma) = 0$, then $\chi(\tilde{X}_{|\partial D}; D) = 0$, and $\tilde{X}_{|\partial D}$ extends to D as a nowhere vanishing section. So X extends to Σ as a nowhere vanishing section.

Let $U^+\partial\Sigma$ denote the unit vector field of $\partial\Sigma$ that is tangent to $\partial\Sigma$ and induces its orientation and let us prove that the following equation $(*(\Sigma))$

$$\chi(U^+\partial\Sigma;\Sigma) = \chi(\Sigma)$$

holds for a general Σ.

For $i = 1, 2$, let Σ_i be a compact oriented surface, and let c_i be a connected component of $\partial\Sigma_i$. Let $\Sigma = \Sigma_1 \cup_{c_1 \sim -c_2} \Sigma_2$. Since the section U^+c_1 is homotopic to $-U^+c_1$ as a section of $U\Sigma$,

$$\chi(U^+\partial\Sigma;\Sigma) = \chi(U^+\partial\Sigma_1;\Sigma_1) + \chi(U^+\partial\Sigma_2;\Sigma_2).$$

Assume that $(*(\Sigma_1))$ is true. Since $\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_2)$, $(*(\Sigma_2))$ is true if and only if $(*(\Sigma))$ is true

Since $S^1 \times S^1$ is parallelizable, $(*(S^1 \times S^1))$ is true. Therefore, $(*(S^1 \times S^1 \setminus \mathring{D}^2))$ is true. The general case follows easily.

The third expression for $\chi(X;\Sigma)$ is an easy consequence of the previous one. $\square$

Lemma 19.16. *Let $e_3 = (0,0,1) \in \mathbb{R}^3$. Let Σ be a compact oriented surface immersed in a 3-manifold M equipped with a parallelization τ, such that $\tau(. \times e_3)$ is a positive normal to Σ along $\partial\Sigma$. Let $s_+(\Sigma) \subset UM$ (resp. $s_-(\Sigma) \subset UM$) be the graph of the section of $UM_{|\Sigma}$ in UM associated with the positive (resp. negative) normal to Σ. Let $s_\tau(\Sigma; e_3)$ be the graph of the section $\tau(\Sigma \times \{e_3\})$. Then*

$$2\left(s_+(\Sigma) - s_\tau(\Sigma; e_3)\right) - \chi\left(\tau(. \times e_2)_{|\partial\Sigma};\Sigma\right) UM_{|*}$$

and

$$2\left(s_-(\Sigma) - s_\tau(\Sigma; -e_3)\right) + \chi\left(\tau(. \times e_2)_{|\partial\Sigma};\Sigma\right) UM_{|*}$$

are cycles, which are null-homologous in $UM_{|\Sigma}$.

PROOF: First note that it suffices to prove that

$$2\left(s_+(\Sigma) - s_\tau(\Sigma; e_3)\right) - \chi\left(\tau(. \times e_2)_{|\partial\Sigma}; \Sigma\right) UM_{|*}$$

is null-homologous in $UM_{|\Sigma}$. Indeed the second cycle (with $s_-(\Sigma)$) is obtained from the first one (with $s_+(\Sigma)$) by applying the involution of $UM_{|\Sigma}$ that changes a vector to the opposite one, because this involution reverses the orientation of a fiber $UM_{|*}$.

Let us prove that the following assertion $(*(\Sigma, \tau))$:
"The cycle $2\left(s_+(\Sigma) - s_\tau(\Sigma; e_3)\right) - \chi\left(\tau(. \times e_2)_{|\partial\Sigma}; \Sigma\right) UM_{|*}$ is null-homologous."
holds for all (Σ, τ) as in the statement.

If $\tau(. \times e_3)$ is a positive normal to Σ over the whole Σ, then $(*(\Sigma, \tau))$ is obviously true.

If the boundary of Σ is empty, embed $\Sigma \times [-1, 1]$ in $\mathbb{R}^3$ as in Figure 19.1. The tangent bundles of $\Sigma \times [-1, 1]$ in $\mathbb{R}^3$ and in M are isomorphic since they are both isomorphic to the direct sum of the tangent bundle of Σ and the trivial normal bundle.

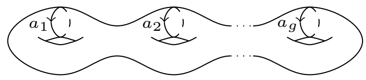


Figure 19.1: The closed oriented surface Σ

Using the trivialization τ_0 of $UM_{|\Sigma}$ induced by the standard parallelization of $\mathbb{R}^3$, the positive normal section to $U(\Sigma)$ is a map from Σ to S^2 that can be homotoped to a constant outside an open disk. Then $([s_+(\Sigma)] - [s_{\tau_0}(\Sigma)]) \in H_2(D^2 \times S^2)$ and $[s_+(\Sigma)] - [s_{\tau_0}(\Sigma)] = c[UM_{|*}]$, where c is the degree of the Gauss map from Σ to S^2 that maps a point to the direction of the positive normal of Σ. It can be computed as the differential degree of this map at a vector that points towards the right-hand side of Figure 19.1, which is clearly $(1 - g)$. This proves $(*(\Sigma, \tau_0))$ when $\partial\Sigma = \emptyset$, for this parallelization τ_0.

Let c be a curve embedded in a surface Σ as in the statement such that $\tau(. \times e_3)$ is a positive normal to Σ along c and such that c separates Σ into two components Σ_1 and Σ_2. Then if $(*(\Sigma_1, \tau_{|\Sigma_1}))$ holds, $(*(\Sigma_2, \tau_{|\Sigma_2}))$ is equivalent to $(*(\Sigma = \Sigma_1 \cup_c \Sigma_2, \tau))$.

For a connected surface Σ with non-empty boundary, the trivialization τ may be expressed as $\tau_1 \circ \psi_{\mathbb{R}}(f)$, for a trivialization τ_1 of TM over Σ such that $\tau_1(. \times e_3)$ is a positive normal to Σ along Σ and for a map f from Σ to $SO(3)$ which maps $\partial\Sigma$ to the circle of $SO(3)$ consisting of the rotations with axis e_3. Any path between two points of this circle in $SO(3)$ is homotopic to a path in this circle since $\pi_1(SO(3))$ is generated by a loop in this circle. In particular, for any connected surface Σ, the trivialization τ can be homotoped so that $\tau(. \times e_3)$ is a positive normal to Σ (over a one-skeleton of Σ and therefore) over the complement of a disk D embedded in the interior of Σ.

If Σ is closed, observe that $\chi\left(\tau(. \times e_2)_{|\partial D}; D\right) = \chi(\Sigma)$ is independent of τ. In particular, since $\pi_2(SO(3)) = 0$, the homotopy class of $\tau_{|D}$ relatively to its boundary depends only on $\chi(\Sigma)$. Since $(*(\Sigma, \tau_0))$ and $(*(\Sigma \setminus \mathring{D}, \tau_{0|\Sigma\setminus\mathring{D}}))$ are true, $(*(D, \tau_{0|D}))$ is true. So is $(*(D, \tau_{|D}))$ for any trivialization τ of D such that such that $\tau(. \times e_3)$ is a positive normal to D along ∂D and $\chi\left(\tau(. \times e_2)_{|\partial D}; D\right) \leq 2$ (note that $\chi\left(\tau(. \times e_2)_{|\partial D}; D\right)$ is even under our assumptions on τ). Finally, patching disks D' equipped with a trivialization τ' such that $\chi\left(\tau'(. \times e_2)_{|\partial D'}; D'\right) = 2$ shows that $(*(D, \tau_{|D}))$ holds for any trivialization τ of D as in the statement.

Now, since $(*(D, \tau_{|D}))$ and $(*(\Sigma \setminus \mathring{D}, \tau_{|\Sigma\setminus\mathring{D}}))$ are true, $(*(\Sigma, \tau))$ is true for any (Σ, τ) as in the statement. $\square$

Definition 19.17. A *homotopy* from a pseudo-parallelization $(N(\gamma); \tau_e, \tau_b)$ to another such $(N(\gamma); \tau'_e, \tau'_b)$ is a homotopy from the pair (τ_e, τ_b) to the pair (τ'_e, τ'_b) such that

$$\tau_b = \begin{cases} \tau_e & \text{over } \partial([a,b] \times \gamma \times [-1,1]) \setminus (\{a\} \times \gamma \times [-1,1]) \\ \tau_e \circ \mathcal{T}_\gamma & \text{over } \{a\} \times \gamma \times [-1,1]. \end{cases}$$

at any time.

Proposition 19.18. *With the notation of Lemma 19.16, let $\tilde{\tau}$ be a pseudo-parallelization that restricts to a neighborhood of $\partial\Sigma$ as a genuine parallelization of M such that $\tilde{\tau}(. \times e_3)$ is the positive normal to Σ along $\partial\Sigma$. Let $\omega(\tilde{\tau})$ be a homogeneous boundary form associated to $\tilde{\tau}$ (as in Definition 19.9). Then*

$$\int_{s_+(\Sigma)} \omega(\tilde{\tau}) = \frac{1}{2}\chi\left(\tilde{\tau}(. \times e_2)_{|\partial\Sigma}; \Sigma\right)$$

and

$$\int_{s_-(\Sigma)} \omega(\tilde{\tau}) = -\frac{1}{2}\chi\left(\tilde{\tau}(. \times e_2)_{|\partial\Sigma}; \Sigma\right).$$

PROOF: When $\tilde{\tau}$ is a genuine parallelization, this is a direct corollary of Lemma 19.16 since

$$\int_{s_{\tilde{\tau}}(\Sigma; \pm e_3)} \omega(\tilde{\tau}) = 0$$

and $\int_{UM_{|*}} \omega(\tilde{\tau}) = 1$.

In general, $\tilde{\tau} = (N(\gamma) = [a,b] \times \gamma \times [-1,1]; \tau_e, \tau_b)$, and $\int_{s_+(\Sigma)} \omega(\tilde{\tau})$ is invariant under an isotopy of Σ that fixes $\partial\Sigma$ since $\omega(\tilde{\tau})$ is closed. It is also invariant under a homotopy of (τ_e, τ_b) as in Definition 19.17 that is fixed in a neighborhood of $\partial\Sigma$. (See Lemma B.2.)

In particular, there is no loss of generality in assuming that Σ meets $N(\gamma)$ along disks $D_c = [a,b] \times \{c\} \times [-1,1]$, and that $\tau_b(. \times e_3)$ is the positive normal to D_c along ∂D_c for these disks, and, thanks to the good behaviour of the two sides of the equality to be proved under gluings along circles that satisfy the boundary conditions, it suffices to prove the proposition when Σ is a meridian disk D_c of γ (with its corners smoothed) such that $\tau_b(. \times e_3)$ is the positive normal to Σ along $\partial\Sigma$.

On $UM_{|D_c}$, $\omega(\tilde{\tau})$ is expressed as

$$\omega(\gamma, \tau_b) = \frac{p\left(\tau_b \circ \mathcal{T}_\gamma^{-1}\right)^* (\omega_{S^2}) + p\left(\tau_b \circ F(\gamma, \tau_b)^{-1}\right)^* (\omega_{S^2})}{2},$$

where $p(\tau_b \circ \mathcal{T}_\gamma^{-1})^*(\omega_{S^2})$ and $p\left(\tau_b \circ F(\gamma, \tau_b)^{-1}\right)^* (\omega_{S^2})$ are propagating forms respectively associated to the parallelizations $\tau_b \circ \mathcal{T}_\gamma^{-1}$ and $(\tau_b \circ F(\gamma, \tau_b)^{-1})$, and $\chi\left(\tau_b \circ \mathcal{T}_\gamma^{-1}(. \times e_2)_{|\partial\Sigma}; \Sigma\right) = \chi\left(\tau_b(. \times e_2)_{|\partial\Sigma}; \Sigma\right)$.

Therefore,

$$\int_{s_+(\Sigma)} \omega(\tilde{\tau}) = \frac{1}{4}\left(\chi\left(\tau_b(. \times e_2)_{|\partial\Sigma}; \Sigma\right) + \chi\left(\left(\tau_b \circ F(\gamma, \tau_b)^{-1}\right)(. \times e_2)_{|\partial\Sigma}; \Sigma\right)\right).$$

Thanks to Lemma 19.15, this average is $\frac{1}{2}\left(\chi\left(\tau_e(. \times e_2)_{|\partial\Sigma}; \Sigma\right)\right)$. This concludes the computation of $\int_{s_+(\Sigma)} \omega(\tilde{\tau})$. The computation of $\int_{s_-(\Sigma)} \omega(\tilde{\tau})$ is similar. □

19.4 Anomalous terms for pseudo-parallelizations

Proposition 19.19. *Let A be a compact 3-manifold equipped with two pseudo-parallelizations τ_0 and τ_1 that coincide with a common genuine parallelization along a regular neighborhood of ∂A. There exists a closed 2-form ω on $[0,1] \times UA$ that restricts*

- *to $\{0\} \times UA$ as a homogeneous boundary form $\omega(\tau_0)$ of (UA, τ_0),*
- *to $\{1\} \times UA$ as a homogeneous boundary form $\omega(\tau_1)$ of (UA, τ_1),*
- *to $[0,1] \times UA_{|\partial A}$ as $p_{UA}^*(\omega(\tau_0))$ with respect to the natural projection $p_{UA}\colon [0,1] \times UA \to UA$.*

Proof: Without loss of generality, assume that A is connected. Set $X = [0,1]\times UA$. Then X is diffeomorphic to $[0,1]\times A\times S^2$ by a diffeomorphism induced by a parallelization τ. The closed two-form ω is defined consistently on ∂X. It suffices to prove that the coboundary map ∂ of the long exact cohomology sequence associated to the pair $(X,\partial X)$ maps the class of $\omega_{|\partial X}$ to 0 in $H^3(X,\partial X)$. Since $H_3(X,\partial X)$ is Poincaré dual to

$$H_3(X)\cong (H_1(A)\otimes H_2(S^2))\oplus (H_3(A)\otimes H_0(S^2)),$$

it is generated by classes of the form $[0,1]\times s_+(\Sigma)$ for surfaces Σ of A such that $\partial\Sigma\subset\partial A$ and for graphs $s_+(\Sigma)$ of sections in UA associated to positive normals of the Σ, and by $[0,1]\times\{a\}\times S^2$ for some $a\in A$, when $\partial A=\emptyset$. The evaluation of $\partial[\omega_{|\partial X}]$ on these classes is the evaluation of $[\omega_{|\partial X}]$ on their boundary, which is clearly zero for $\partial[0,1]\times\{a\}\times S^2$ since

$$\int_{\{(1,a)\}\times S^2}\omega=\int_{\{(0,a)\}\times S^2}\omega=1.$$

Let us conclude the proof by proving that, for a surface Σ as above,

$$\int_{\partial([0,1]\times s_+(\Sigma))}\omega=0.$$

This integral is invariant under the homotopies of (τ_0,τ_1) that fix (τ_0,τ_1) near ∂A. (See Lemma B.2.) Therefore, we can assume that $\tau_0=\tau_1$ in a neighborhood $[-2,0]\times\partial A$ of ∂A in A, that Σ is transverse to $\{-1\}\times\partial A$ and that the positive normal to Σ is $\tau_0(.\times e_3)$ along $\Sigma\cap(\{-1\}\times\partial A)$. Set $A_{-1}=A\setminus(]-1,0]\times\partial A)$ and $\Sigma_{-1}=\Sigma\cap A_{-1}$. Extend ω as $p_{UA}^*(\omega(\tau_0))$ over $[0,1]\times UA_{|[-2,0]\times\partial A}$. Then $\int_{\partial([0,1]\times s_+(\Sigma))}\omega=\int_{\partial([0,1]\times s_+(\Sigma_{-1}))}\omega$. Now, Proposition 19.18 ensures that $\int_{\{0\}\times s_+(\Sigma_{-1})}\omega=\int_{\{1\}\times s_+(\Sigma_{-1})}\omega$. Since $\int_{[0,1]\times\partial s_+(\Sigma_{-1})}\omega=0$, $\int_{\partial([0,1]\times s_+(\Sigma))}\omega=0$. □

Proposition 19.20. *Let A be a compact oriented 3-manifold equipped with three pseudo-parallelizations τ_0, τ_1 and τ_2 that coincide with a common genuine parallelization along a regular neighborhood of ∂A. Let $n\in\mathbb{N}$, and let $\omega(\tau_0)$ and $\omega(\tau_1)$ be homogeneous boundary forms associated of (UA,τ_0) and (UA,τ_1), respectively.*

Under the assumptions of Proposition 19.19, as in Corollary 9.4, set

$$z_n([0,1]\times UA;\omega)=\sum_{\Gamma\in\mathcal{D}_n^c}\zeta_\Gamma\int_{[0,1]\times\check{S}_{V(\Gamma)}(TA)}\bigwedge_{e\in E(\Gamma)}p_e^*(\omega)[\Gamma].$$

If A embeds in a rational homology 3-ball, then $z_n([0,1]\times UA;\omega)$ depends only on the pseudo-parallelizations τ_0 and τ_1. It is denoted by $z_n(A;\tau_0,\tau_1)$ and the following properties are satisfied:

- $z_n(A;\tau_0,\tau_1)=0$, *when n is even.*
- *If B is a compact oriented 3-manifold embedded in the interior of A, if τ_0 and τ_1 coincide on a neighborhood of $A\setminus B$, and if τ_0 restricts to a neighborhood of ∂B as a genuine parallelization, then $z_n(B;\tau_0,\tau_1)=z_n(A;\tau_0,\tau_1)$.*
- *When τ_0 and τ_1 are actual parallelizations,*

$$z_n(A;\tau_0,\tau_1)=\frac{p_1(\tau_0,\tau_1)}{4}\beta_n.$$

- $z_n(A;\tau_0,\tau_2)=z_n(A;\tau_0,\tau_1)+z_n(A;\tau_1,\tau_2)$. *(In particular, $z_n(A;\tau_0,\tau_1)=-z_n(A;\tau_1,\tau_0)$.)*
- *For any orientation-preserving bundle isomorphism Ψ of UA over the identity map of A, $z_n(A;\Psi\circ\tau_0,\Psi\circ\tau_1)=z_n(A;\tau_0,\tau_1)$.*
- *For any orientation-preserving diffeomorphism ψ from A to another compact oriented 3-manifold B,*

$$z_n(B;T\psi\circ\tau_0\circ\left(\psi^{-1}\times 1_{\mathbb{R}^3}\right),T\psi\circ\tau_1\circ\left(\psi^{-1}\times 1_{\mathbb{R}^3}\right))=z_n(A;\tau_0,\tau_1).$$

- *If τ_1' is homotopic to τ_1, relatively to ∂A, in the sense of Definition 19.17, then*

$$z_n(A;\tau_0,\tau_1')=z_n(A;\tau_0,\tau_1).$$

- *For any orientation-preserving diffeomorphism ψ of A isotopic to the identity map of A relatively to ∂A,*

$$z_n(A;\tau_0,T\psi\circ\tau_1\circ\left(\psi^{-1}\times 1_{\mathbb{R}^3}\right))=z_n(A;\tau_0,\tau_1),$$

  where ψ is used to carry the required parametrization of $N(\gamma)$.

PROOF: Lemma 9.12 implies that $z_n(A;\tau_0,\tau_1)=0$, when n is even. Assume that n is odd from now on. Let us first prove that $z_n([0,1]\times UA;\omega)$ does not depend on the closed extension ω when A is a rational homology 3-ball and when τ_0 is standard near ∂A. According to Lemma 19.12, $\omega(\tau_0)$ (resp. $\omega(\tau_1)$) extends to a homogeneous propagating form of $(C_2(S^3(A/B_{S^3})),\tau_0)$ (resp. of $(C_2(S^3(A/B_{S^3})),\tau_1)$). Set $X=[0,1]\times C_2(S^3(A/B_{S^3}))$. The above extensions of $\omega(\tau_0)$ and $\omega(\tau_1)$ together with ω (extended as $p_{\tau_s}^*(\omega_{S^2})$ on $[0,1]\times(\partial C_2(S^3(A/B_{S^3}))\setminus UA)$) make a closed 2-form of ∂X. This form extends as a closed form on X by Lemma 9.1.

Then Corollary 9.4 implies that $z_n([0,1]\times UA;\omega)$ does not depend on ω when A is a rational homology 3-ball and when τ_0 is standard near ∂A.

If A embeds in the interior of such a space B_R, the pseudo-parallelization τ_0 may be extended to B_R as a pseudo-parallelization standard near ∂B_R, according to Lemma 19.6, and the ω of Proposition 19.19 may be extended to $[0,1]\times UB_R$ as $p^*_{UB_R}(\omega(\tau_0))$ on $[0,1]\times U(B_R\setminus A)$. Then $z_n([0,1]\times U(B_R\setminus A);\omega)=0$ since the $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega)$ pull back through $\check{\mathcal{S}}_{V(\Gamma)}(T(B_R\setminus A))$ whose dimension is smaller than the degree $2\sharp E(\Gamma)$ of $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega)$. Therefore $z_n([0,1]\times UA;\omega)=z_n([0,1]\times UB_R;\omega)$ is independent of $\omega_{|[0,1]\times UA}$.

Set $z_n(A;\omega(\tau_0),\omega(\tau_1))=z_n([0,1]\times UA;\omega)$. It is easy to see that

$$z_n(A;\omega(\tau_0),\omega(\tau_2))=z_n(A;\omega(\tau_0),\omega(\tau_1))+z_n(A;\omega(\tau_1),\omega(\tau_2)).$$

Assume that $\tau_0=\tau_1=(N(\gamma);\tau_e,\tau_b)$, and that the forms $\omega(\tau_0)=\omega(\tau_0,F_0)$ and $\omega(\tau_1)=\omega(\tau_0,F_1)$ of Definition 19.9 are obtained from one another by changing the map $F=F_0\colon [a,b]\times[-1,1]\to SO(3)$ to another one F_1, which is homotopic via a homotopy F_t, which induces a homotopy

$$\begin{array}{llll} F_t(\gamma,\tau_b)\colon & [a,b]\times\gamma\times[-1,1]\times S^2 & \longrightarrow & [a,b]\times\gamma\times[-1,1]\times S^2 \\ & (s,c,u;Y) & \mapsto & (s,c,u;F_t(s,u)(Y)). \end{array}$$

Then $z_n(A;\omega(\tau_0),\omega(\tau_1))=z_n(N(\gamma);\omega(\tau_0,F_0),\omega(\tau_0,F_1))$.

Use τ_b to identify $UN(\gamma)$ with $[a,b]\times\gamma\times[-1,1]\times S^2$, and define $\omega(\gamma,\tau_b)$ on $[0,1]\times[a,b]\times\gamma\times[-1,1]\times S^2$ with respect to the formula for $\omega(\gamma,\tau_b)$ in Definition 19.9:

$$\omega(\gamma,\tau_b)=\frac{p\left(\tau_b\circ\mathcal{T}_\gamma^{-1}\right)^*(\omega_{S^2})+p\left(\tau_b\circ F_.(\gamma,\tau_b)^{-1}\right)^*(\omega_{S^2})}{2}$$

This formula does not depend on the coordinate along γ. So ω pulls back through a projection from $[0,1]\times UN(\gamma)$ to $[0,1]\times[a,b]\times[-1,1]\times S^2$, the $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega)$ pull back through $[0,1]\times\check{\mathcal{S}}_{V(\Gamma)}(TN(\gamma)_{|[a,b]\times\{c\}\times[-1,1]})$ and $z_n([0,1]\times UN(\gamma);\omega)$ vanishes.

This proves that $z_n(N(\gamma);\omega(\tau_0,F_0),\omega(\tau_0,F_1))$ vanishes. So we can conclude that $z_n(A;\omega(\tau_0),\omega(\tau_1))$ depends only on τ_0 and τ_1.

Proposition 10.7 implies that $z_n(A;\tau_0,\tau_1)=\frac{p_1(\tau_0,\tau_1)}{4}\beta_n$, as soon as τ_0 and τ_1 are actual parallelizations, and A embeds in a rational homology ball to which τ_0 extends as a genuine parallelization.

For an orientation-preserving bundle isomorphism Ψ of UA over the identity map of A, the pseudo-parallelization

$$\Psi\circ(\tau_0=(N(\gamma);\tau_e,\tau_b))=(N(\gamma);\Psi\circ\tau_e,\Psi\circ\tau_b)$$

makes unambiguous sense. We have the following commutative diagram,

$$\begin{tikzcd} UA \arrow[rrr, bend left, "p(\Psi\circ\tau)"] & & & \\ UA \arrow[u, "\Psi"] \arrow[r, "\tau^{-1}"] \arrow[rrr, bend right, "p(\tau)"'] & A\times S^2 \arrow[ul, "(\Psi\circ\tau)"'] \arrow[rr, "p_{S^2}"] & & S^2 \end{tikzcd}$$

which shows that $p(\tau_b) = p(\Psi \circ \tau_b) \circ \Psi$, so

$$\omega(\Psi \circ \tau_0, F) = \left(\Psi^{-1}\right)^* \omega(\tau_0, F).$$

The form ω on $[0,1] \times UA$ can be pulled back by the orientation-preserving $1_{[0,1]} \times \Psi^{-1}$, similarly. So

$$z_n(A; \Psi \circ \tau_0, \Psi \circ \tau_1) = z_n(A; \tau_0, \tau_1),$$

and $z_n(A;\tau_0,\tau_1) = \frac{p_1(\tau_0,\tau_1)}{4}\beta_n$ as soon as τ_0 and τ_1 are actual parallelizations.

Similarly, for any orientation-preserving diffeomorphism ψ from A to B,

$$z_n(B; T\psi \circ \tau_0 \circ \left(\psi^{-1} \times 1_{\mathbb{R}^3}\right), T\psi \circ \tau_1 \circ \left(\psi^{-1} \times 1_{\mathbb{R}^3}\right)) = z_n(A; \tau_0, \tau_1).$$

If τ_1 is homotopic to τ_0 in the sense of Definition 19.17, then there exists a map $\Psi\colon [0,1] \times UA \to UA$ such that $(t \mapsto \Psi(t,.) \circ \tau_0)$ is a homotopy of pseudo-parallelizations from τ_0 to τ_1. Then

$$z_n(A; \tau_0, \tau_1) = z_n\left([0,1] \times UA; \omega\right),$$

where

$$\omega = \left(\Psi^{-1}\right)^* \left(\omega(\tau_0, F)\right),$$

so ω pulls back through a map from $[0,1] \times UA$ to UA, the $\bigwedge_{e\in E(\Gamma)} p_e^*(\omega)$ pull back through $\check{\mathcal{S}}_{V(\Gamma)}(TA)$, and $z_n([0,1] \times UA; \omega)$ vanishes.

An isotopy $\psi\colon [0,1] \times A \to A$, which maps (t,u) to $\psi_t(u)$, induces a homotopy

$$\begin{array}{llll} \Psi\colon & [0,1] \times UA & \to & UA \\ & (t,u) & \mapsto & \tau_1 \circ (\psi_t \times 1_{S^2}) \circ \tau_1^{-1} \circ T\psi_t^{-1}(u), \end{array}$$

such that

$$\begin{aligned} p\left(T\psi_t \circ \tau_1 \circ \left(\psi_t^{-1} \times 1_{S^2}\right)\right) &= p_{S^2} \circ \tau_1^{-1} \circ \tau_1 \circ \left(T\psi_t \circ \tau_1 \circ \left(\psi_t^{-1} \times 1_{S^2}\right)\right)^{-1} \\ &= p(\tau_1) \circ \Psi(t,.). \end{aligned}$$

So

$$z_n(A; T\psi_0 \circ \tau_1 \circ \left(\psi_0^{-1} \times 1_{\mathbb{R}^3}\right), T\psi_1 \circ \tau_1 \circ \left(\psi_1^{-1} \times 1_{\mathbb{R}^3}\right))$$

$$= z_n\left([0,1] \times UA; (\Psi)^*\left(\omega(\tau_1)\right)\right) = 0,$$

similarly. □

The main result of this section is the following theorem.

Theorem 19.21. *Let A be a compact 3-manifold. Assume that A embeds in a rational homology 3-ball and that it is equipped with two pseudo-parallelizations τ_0 and τ_1 that coincide with a common genuine parallelization along a regular neighborhood of ∂A. With the notation of Proposition 19.20, for any odd natural integer n,*

$$z_n(A; \tau_0, \tau_1) = \frac{p_1(\tau_0, \tau_1)}{4}\beta_n.$$

The proof of this theorem consists in first proving it in many special cases, and next proving that these special cases are sufficient to get a complete proof.

Lemma 19.22. *Let $A = [2,9] \times \gamma \times [-2,2]$ be equipped with a pseudo-parallelization $\tau_0 = (N(\tilde{\gamma}); \tau_e, \tau_b)$, where*

$$N(\tilde{\gamma}) = [3,5] \times \gamma \times [-1,1] \sqcup [6,8] \times \gamma \times [-1,1].$$

There exists a parallelization τ_1 of A that coincides with τ_e in a neighborhood of ∂A. For any such parallelization,

$$z_n(A; \tau_0, \tau_1) = \frac{p_1(\tau_0, \tau_1)}{4}\beta_n.$$

Proof: We first prove the lemma for some chosen pseudo-parallelizations $\tilde{\tau}_0$ and $\tilde{\tau}_1$, which satisfy the assumptions and which behave as "products by γ". For these pseudo-parallelizations, this product behaviour will imply that $p_1(\tilde{\tau}_0, \tilde{\tau}_1) = 0$ and $z_n(A; \tilde{\tau}_0, \tilde{\tau}_1) = 0$.

Define the parallelization τ_A of A by

$$\begin{array}{llll} \tau_A\colon & A \times \mathbb{R}^3 & \to & UA \\ & (s_0, c_0, u_0, e_1) & \mapsto & \frac{d}{ds}(s, c_0, u_0)(s_0, c_0, u_0) \\ & (s_0, c_0, u_0, e_2) & \mapsto & \frac{d}{dc}(s_0, c, u_0)(s_0, c_0, u_0) \\ & (s_0, c_0, u_0, e_3) & \mapsto & \frac{d}{du}(s_0, c_0, u)(s_0, c_0, u_0) \end{array}$$

Define $\tilde{\tau}_e \colon \left(A \setminus \mathring{N}(\tilde{\gamma})\right) \times \mathbb{R}^3 \to U\left(A \setminus \mathring{N}(\tilde{\gamma})\right)$, so that

$$\tilde{\tau}_e = \begin{cases} \tau_A & \text{on } A \setminus ([2,8[\times \gamma \times]-1,1[) \times \mathbb{R}^3 \\ \tau_A \circ \mathcal{T}_\gamma^{-1} & \text{on } [5,6] \times \gamma \times [-1,1] \\ \tau_A \circ \mathcal{T}_\gamma^{-2} & \text{on } [2,3] \times \gamma \times [-1,1], \end{cases}$$

where $\mathcal{T}_\gamma(t, c \in \gamma, u \in [-1,1]; X \in \mathbb{R}^3) = (t, c, u, \rho_{\alpha(u)}(X))$ as in Definition 19.5. Define $\tilde{\tau}_b \colon N(\tilde{\gamma}) \times \mathbb{R}^3 \to UN(\tilde{\gamma})$, so that

$$\tilde{\tau}_b = \begin{cases} \tau_A & \text{on } [6,8] \times \gamma \times [-1,1] \\ \tau_A \circ \mathcal{T}_\gamma^{-1} & \text{on } [3,5] \times \gamma \times [-1,1]. \end{cases}$$

Finally define

$$\begin{array}{llll} \tilde{F}\colon & [3,8] \times [-1,1] & \to & SO(3) \\ & (3,u) & \mapsto & \rho_{-2\alpha(u)} \\ & (8,u) & \mapsto & 1_{SO(3)} \\ & (t,\pm 1) & \mapsto & 1_{SO(3)} \end{array}$$

and define $\tilde{\tau}_1$ such that

$$\tilde{\tau}_1 = \begin{cases} \tau_A & \text{on } A \setminus ([2,8[\times\gamma\times]-1,1[) \times \mathbb{R}^3 \\ \tau_A \circ \mathcal{T}_\gamma^{-2} & \text{on } [2,3] \times \gamma \times [-1,1] \end{cases}$$

and

$$\tilde{\tau}_1(s,c,u,X) = \Big(\tau_A(s,c,u,\tilde{F}(s,u)(X))\Big)$$

when $(s,u) \in [3,8] \times [-1,1]$.

Set $\tilde{\tau}_0 = (N(\tilde{\gamma}); \tilde{\tau}_e, \tilde{\tau}_b)$.

Then $p_1(\tilde{\tau}_0, \tilde{\tau}_1) = 0$. Indeed, the involved trivializations of $T([0,1]\times A)\otimes C$ on $\partial[0,1] \times A$ are obtained from the natural parallelization $T[0,1] \oplus \tau_A$ by composition by a map from $\partial([0,1] \times A) = \gamma \times \partial([0,1] \times [2,9] \times [-2,2])$ to $SU(4)$, which does not depend on the coordinate along γ. So that map extends to $SU(4)$ since $\pi_2(SU(4)) = \{0\}$.

Similarly, $z_n(A; \tilde{\tau}_0, \tilde{\tau}_1) = 0$. Indeed, $z_n(A; \tilde{\tau}_0, \tilde{\tau}_1) = z_n([0,1] \times UA; \omega)$ for a closed two-form ω on $[0,1] \times UA =_{\tau_A} \gamma \times X$ with $X = [0,1] \times S^2 \times [2,9] \times [-2,2]$, where the restriction $\omega_{|\partial}$ of ω to $\partial([0,1] \times UA)$ factors through the projection of $\gamma \times \partial X$ onto ∂X. The involved closed 2-form on ∂X extends as a closed form ω_X to X since $\omega_{|\partial}$ extends to the whole $[0,1] \times UA$, according to Proposition 19.19. Then the extension ω can be chosen as the pull-back of $\omega_{|X}$ under the projection of $\gamma \times X$ onto X. Again, the $\bigwedge_{e \in E(\Gamma)} p_e^*(\omega)$ pull back through a projection onto a space of dimension smaller than the degree of the forms and $z_n([0,1] \times UA; \omega)$ vanishes.

There exists an orientation-preserving bundle isomorphism Ψ of UA over the identity map of A, such that $\tau_0 = \Psi \circ \tilde{\tau}_0$. The parallelization $\tau_1 = \Psi \circ \tilde{\tau}_1$ satisfies the assumptions of the lemma, $z_n(A; \Psi \circ \tilde{\tau}_0, \Psi \circ \tilde{\tau}_1) = z_n(A; \tilde{\tau}_0, \tilde{\tau}_1) = 0$, and $p_1(\Psi \circ \tilde{\tau}_0, \Psi \circ \tilde{\tau}_1) = p_1(\tilde{\tau}_0, \tilde{\tau}_1) = 0$. We conclude for any another parallelization τ_1' that coincides with τ_e near ∂A, because $z_n(A; \tau_1, \tau_1') = \frac{p_1(\tau_1, \tau_1')}{4} \beta_n$. $\square$

Lemma 19.23. *Let A be a compact oriented 3–manifold that embeds in a rational homology 3–ball. Let $[-7,0]\times\partial A$ be a collar neighborhood of A. Let $\gamma\times[-2,2]$ be a disjoint union of annuli in ∂A, and let $N(\gamma)=[-2,-1]\times\gamma\times[-1,1]$. Let $\tau_0=(N(\gamma);\tau_e,\tau_b)$ be a pseudo-parallelization of A that coincides with the restriction of a parallelization τ_1 of A in a neighborhood of ∂A. Then*

$$z_n(A;\tau_0,\tau_1)=\frac{p_1(\tau_0,\tau_1)}{4}\beta_n.$$

PROOF: Recall $A_{-2}=A\setminus(]-2,0]\times\partial A)$. The schema of the proof is given by Figure 19.2.

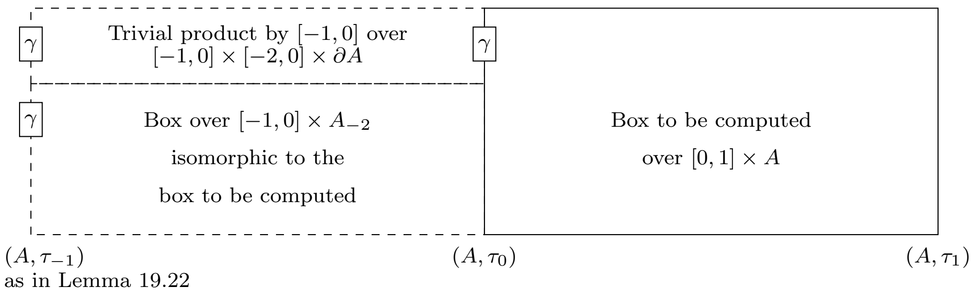


Figure 19.2: Schema of proof for Lemma 19.23

Let $f\colon[-7,0]\to[-7,-2]$ be a diffeomorphism such that $f(t)=t-2$ when $t\geq-3$ and $f(t)=t$ when $t\leq-6$.

Let $\psi\colon A\to A_{-2}$ be a diffeomorphism, which restricts to $A\setminus(]-7,0]\times\partial A)$ as the identity map, and which maps $(t,x)\in[-7,0]\times\partial A$ to $(f(t),x)$.

There exists a bundle isomorphism Φ of UA_{-2} over the identity map of A_{-2} such that $\tau_e|A_{-2}=\Phi\circ T\psi\circ\tau_1\circ(\psi^{-1}\times 1_{\mathbb{R}^3})$.

Let τ_{-1} be the pseudo-parallelization of A that coincides with τ_0 over $[-2,0]\times\partial A$ and with $\Phi\circ T\psi\circ\tau_0\circ(\psi^{-1}\times 1_{\mathbb{R}^3})$ over A_{-2}.

Then, τ_{-1} is a parallelization outside $[-7,0]\times\gamma\times[-2,2]$, and up to reparametrization, $[-7,0]\times\gamma\times[-2,2]$ satisfies the hypotheses of Lemma 19.22. So Lemma 19.22 and Proposition 19.20 ensure that

$$z_n(A;\tau_{-1},\tau_1)=\frac{p_1(\tau_{-1},\tau_1)}{4}\beta_n.$$

In order to conclude, we prove that $z_n(A;\tau_0,\tau_1)=\frac{1}{2}z_n(A;\tau_{-1},\tau_1)$ and that $p_1(\tau_0,\tau_1)=\frac{1}{2}p_1(\tau_{-1},\tau_1)$. The element $z_n(A;\tau_{-1},\tau_0)$ of $\mathcal{A}_n(\emptyset)$ can be written as

$$\begin{aligned}&z_n\left(A_{-2};\Phi\circ T\psi\circ\tau_0\circ(\psi^{-1}\times 1_{\mathbb{R}^3}),\Phi\circ T\psi\circ\tau_1\circ(\psi^{-1}\times 1_{\mathbb{R}^3})\right)\\ =\;&z_n(A;\tau_0,\tau_1).\end{aligned}$$

Similarly, $p_1(\tau_{-1}, \tau_0) = p_1(\tau_0, \tau_1)$. $\square$

Lemma 19.24. *Let Σ be a compact oriented surface. Let γ_0 and γ_1 be two disjoint unions of curves of Σ with respective tubular neighborhoods $\gamma_0 \times [-1,1]$ and $\gamma_1 \times [-1,1]$. Set $A = [0,3] \times \Sigma$, $N(\gamma_0) = [1,2] \times \gamma_0 \times [-1,1]$, and $N(\gamma_1) = [1,2] \times \gamma_1 \times [-1,1]$.*

There exist two pseudo-parallelizations $\tau_0 = (N(\gamma_0); \tau_{e,0}, \tau_{b,0})$ and $\tau_1 = (N(\gamma_1); \tau_{e,1}, \tau_{b,1})$ which coincide near ∂A if and only if γ_0 and γ_1 are homologous mod 2 (i. e. they have the same class in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$). In this case,

$$z_n(A; \tau_0, \tau_1) = \frac{p_1(\tau_0, \tau_1)}{4} \beta_n.$$

Proof: Assume without loss of generality that Σ is connected and that the complement of $(\gamma_0 \times [-1,1]) \cup (\gamma_1 \times [-1,1])$ in Σ is not empty. Let τ be a parallelization of A.

Assume that τ_0 and τ_1 are two pseudo-parallelizations as in the statement, which coincide near ∂A, and let us prove that γ_0 and γ_1 are homologous mod 2.

If the boundary of Σ is empty, choose a disk D of Σ outside $\gamma_0 \times [-1,1] \cup \gamma_1 \times [-1,1]$ and assume, without loss of generality, that $\tau_{e,0}$ and $\tau_{e,1}$ coincide on $[0,3] \times D$. This allows us to assume, without loss of generality, that $\partial\Sigma \neq \emptyset$, by possibly removing the interior of D from Σ. Up to homotopy, for $i \in \{0,1\}$, if τ_i is a pseudo-parallelization as in the statement, then $\tau_{e,i}$ may be expressed as $\tau \circ \psi_{\mathbb{R}}(g_i)$ for some $g_i \colon A \setminus \mathring{N}(\gamma_i) \to SO(3)$, with the notation of Section 4.2, where the restriction of g_i to a meridian curve of γ_i is not homotopic to a constant loop, and the maps g_0 and g_1 coincide near ∂A. Let $c \colon [0,1] \to \Sigma$ be a path such that $c(0)$ and $c(1)$ are in $\partial\Sigma$. Then the restriction of g_i to

$$(\{3\} \times c([0,1])) \cup ((-[0,3]) \times c(1)) \cup (\{0\} \times (-c([0,1]))) \cup ([0,3] \times c(0))$$

is null-homotopic if and only if the mod 2 intersection of c with γ_i is trivial. Therefore γ_0 and γ_1 must be homologous mod 2.

Conversely, if γ_0 and γ_1 are homologous mod 2, define g_0 as the map which maps $A \setminus ([0,2] \times \gamma_0 \times]-1,1[)$ to the constant map with value the unit of $SO(3)$ and which maps $(t,c,u) \in ([0,1] \times \gamma_0 \times [-1,1])$ to $\rho_{-\alpha(u)}$, and define g_1 as a map which maps

$$(A \setminus ([0,1] \times \Sigma)) \setminus ([1,2] \times \gamma_1 \times]-1,1[)$$

to the constant map with value the unit of $SO(3)$, which maps $(t,c,u) \in (\{1\} \times \gamma_1 \times [-1,1])$ to $\rho_{-\alpha(u)}$, and which coincides with g_0 near ∂A. The restriction of g_1 to $[0,1]\times\Sigma$ is a homotopy between the restriction to $\{1\}\times\Sigma$ of g_1 and the restriction to $\{1\}\times\Sigma$ of g_0, which exists since γ_0 and γ_1 are homologous mod 2. (Apply Proposition 19.3 to $A=[0,1]\times\Sigma$ after removing a disk D of Σ outside $\gamma_0\times[-1,1]\cup\gamma_1\times[-1,1]$ if the boundary of Σ is empty.)

Then it is easy to see that there exists $\tau_{b,i}$ such that $\tau_i=(N(\gamma_i);\tau_{e,i}=\tau\circ\psi_{\mathbb{R}}(g_i),\tau_{b,i})$ is a pseudo-parallelization, and τ_0 and τ_1 are two pseudo-parallelizations which coincide near ∂A.

Set $B=[0,6]\times\Sigma$, and $N(\gamma_0^\prime)=[4,5]\times\gamma_0\times[-1,1]$. Extend τ_0 to a pseudo-parallelization $\tau_{0,B}=(N(\gamma_0)\sqcup N(\gamma_0^\prime);\tau_{e,B,0},\tau_{b,B,0})$ of B. Extend τ_1 to a pseudo-parallelization $\tau_{1,B}$ of B that coincides with $\tau_{0,B}$ on $[3,6]\times\Sigma$. Then $z_n(A;\tau_0,\tau_1)=z_n(B;\tau_{0,B},\tau_{1,B})$ and $p_1(\tau_0,\tau_1)=p_1(\tau_{0,B},\tau_{1,B})$.

According to Lemma 19.22, there is a parallelization τ_2 of B that coincides with $\tau_{0,B}$ near ∂B, and $z_n(B;\tau_{0,B},\tau_2)=\frac{p_1(\tau_{0,B},\tau_2)}{4}\beta_n$.

Apply Lemma 19.23 to prove that $z_n(B;\tau_{1,B},\tau_2)=\frac{p_1(\tau_{1,B},\tau_2)}{4}\beta_n$. (In order to apply Lemma 19.23 as it is stated, first rotate $N(\gamma_0)=[1,2]\times\gamma_0\times[-1,1]$ around γ_0 by an isotopy which sends $1\times[-1,1]$ to $2\times[1,-1]$ and apply Proposition 19.20.) This implies that $z_n(B;\tau_{0,B},\tau_{1,B})=\frac{p_1(\tau_{0,B},\tau_{1,B})}{4}$. $\square$

Lemma 19.25. *Let Σ be a compact connected oriented surface with boundary. Let γ, γ_0 and γ_1 be three disjoint unions of curves of Σ with respective tubular neighborhoods $\gamma\times[-1,1]$ $\gamma_0\times[-1,1]$ and $\gamma_1\times[-1,1]$. Assume that $[\gamma_1]=[\gamma_0]+[\gamma]$ in $H_1(\Sigma;\mathbb{Z}/2\mathbb{Z})$. Set $A=[0,6]\times\Sigma$, $N(\gamma)=[4,5]\times\gamma\times[-1,1]$, $N(\gamma_0)=[1,2]\times\gamma_0\times[-1,1]$, and $N(\gamma_1)=[1,2]\times\gamma_1\times[-1,1]$. Let $\tau_0=(N(\gamma)\sqcup N(\gamma_0);\tau_{e,0},\tau_{b,0})$ and $\tau_1=(N(\gamma_1);\tau_{e,1},\tau_{b,1})$ be two pseudo-parallelizations which coincide near ∂A. Then*

$$z_n(A;\tau_0,\tau_1)=\frac{p_1(\tau_0,\tau_1)}{4}\beta_n.$$

Proof: Lemma 19.24 allows us to choose arbitrary representatives of $[\gamma]$ and $[\gamma_0]$, for the proof, without loss of generality. In particular, there is no loss of generality in assuming that γ_0 is connected and that the intersection of γ and γ_0 has no more than one point.

If γ and γ_0 are disjoint, we can perform an isotopy in A to lower γ. So the result is a direct consequence of Lemma 19.24.

Assume that the intersection of γ and γ_0 has one transverse point. Attach two copies Σ and $\Sigma^\prime$ of Σ to a disk D along intervals I and $I^\prime$. Let $\gamma^\prime$, $\gamma_0^\prime$ and $\gamma_1^\prime$ be the respective copies of γ, γ_0 and γ_1, in $\Sigma^\prime$. Let $\tilde{\Sigma}=\Sigma\cup D\cup\Sigma^\prime$ as in Figure 19.3.

Let $B=[0,6]\times\tilde{\Sigma}$ and let $\tau_{B,0}$ be a pseudo-parallelization of B that extends the pseudo-parallelization τ_0 which is used both for A and for $A^\prime=$

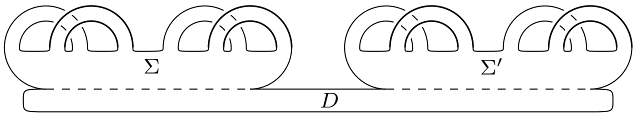


Figure 19.3: $\tilde{\Sigma}$

$[0, 6] \times \Sigma'$. Let $\tau_{B,1}$ be a pseudo-parallelization that coincides with τ_1 on A and on A' and with $\tau_{B,0}$ on $[0, 6] \times D$.

Then $z_n(A; \tau_0, \tau_1) = \frac{1}{2} z_n(B; \tau_{B,0}, \tau_{B,1})$ and $p_1(\tau_0, \tau_1) = \frac{1}{2} p_1(\tau_{B,0}, \tau_{B,1})$. Since the intersection of $(\gamma \cup \gamma')$ and $(\gamma_0 \cup \gamma'_0)$ is zero mod 2, the homology classes of these curves can be represented by curves that do not intersect. So $z_n(B; \tau_{B,0}, \tau_{B,1}) = \frac{p_1(\tau_{B,0}, \tau_{B,1})}{4} \beta_n$. $\square$

Proof of Theorem 19.21: Let us first prove the theorem when A is a rational homology ball, according to the schema of Figure 19.4. Then there exists a parallelization τ_2 of A that coincides with $\tau_0 = (N(\gamma); \tau_e, \tau_b)$ in a neighborhood of ∂A. Thicken the neighborhood $N(\gamma)$ to $[a-7, b+7] \times \gamma \times [-2, 2]$. Add bands to $\gamma \times [-2, 2]$ so that the disjoint union $\gamma \times [-2, 2]$ is embedded in a connected oriented surface Σ of A with one boundary component. Let $[a-7, b+7] \times \Sigma$ be embedded in A so that this parametrization matches the previous one.

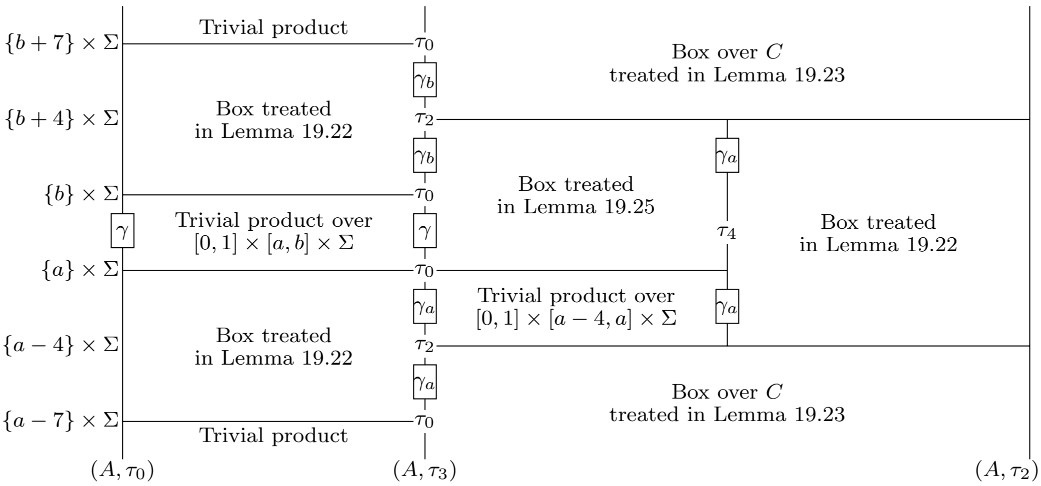


Figure 19.4: Schema of proof for Theorem 19.21

After a possible homotopy of τ_2, there exist annuli $\gamma_a \times [-1, 1]$ and $\gamma_b \times [-1, 1]$ in Σ such that

- τ_0 coincides with τ_2 in a neighborhood of $[a-7, b+7] \times \partial\Sigma$,
- $\tau_{0|\{a-4\}\times\Sigma} = \tau_2 \circ \mathcal{T}_{\gamma_a}$ and $\tau_{0|\{b+4\}\times\Sigma} = \tau_2 \circ \mathcal{T}_{\gamma_b}^{-1}$.

Let

$$\begin{aligned} N(\gamma_3) = N(\gamma) \quad & \sqcup([b+2, b+3] \sqcup [b+5, b+6]) \times \gamma_b \times [-1, 1] \\ & \sqcup([a-6, a-5] \sqcup [a-3, a-2]) \times \gamma_a \times [-1, 1]. \end{aligned}$$

Let $\tau_3 = (N(\gamma_3); \tau_{3,e}, \tau_{3,b})$ be a pseudo-parallelization, which coincides with τ_0 outside $([b+1, b+7]) \times \gamma_b \times [-1, 1] \sqcup ([a-7, a-1]) \times \gamma_a \times [-1, 1]$, and which coincides with τ_2 on $\{a-4\} \times \Sigma$ and on $\{b+4\} \times \Sigma$. According to Lemma 19.22 and to Proposition 19.20,

$$z_n(A; \tau_0, \tau_3) = \frac{p_1(\tau_0, \tau_3)}{4} \beta_n.$$

Set $B = [a-4, b+4] \times \Sigma$ and $C = A \setminus \left( ]a-4, b+4[\times \mathring{\Sigma} \right)$.

$$z_n(A; \tau_2, \tau_3) = z_n(B; \tau_2, \tau_3) + z_n\left(C; \tau_2, \tau_3\right)$$

and p_1 decomposes similarly.

Let us now prove that

$$z_n\left(C; \tau_2, \tau_3\right) = \frac{1}{4} p_1(\tau_{2|C}, \tau_{3|C}) \beta_n.$$

This follows by applying Lemma 19.23, after an isotopy of $[b+4, b+7] \times \Sigma$ which sends $[b+5, b+6] \times \gamma_b \times [-1, 1]$ to itself (at the end) so that $\{b+5\} \times \gamma_b \times [-1, 1]$ is sent to $\{b+6\} \times \gamma_b \times (-[-1, 1])$, and $\{b+6\} \times \gamma_b \times [-1, 1]$ is sent to $\{b+5\} \times \gamma_b \times (-[-1, 1])$.

Since $[\gamma_a] + [\gamma] + [\gamma_b] = 0$ in $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$, Lemma 19.25 and Proposition 19.20 imply that

$$z_n\left(B; \tau_3, \tau_4\right) = \frac{1}{4} p_1(\tau_{3|B}, \tau_4) \beta_n.$$

for a pseudo-parallelization $\tau_4 = (([a-3, a-2] \sqcup [b+2, b+3]) \times \gamma_a \times [-1, 1]; \tau_{4,e}, \tau_{4,b})$ of B that coincides with τ_2 in a neighborhood of ∂B. According to Lemma 19.22,

$$z_n\left(B; \tau_4, \tau_2\right) = \frac{1}{4} p_1(\tau_4, \tau_{2|B}) \beta_n.$$

So $z_n(A; \tau_2, \tau_3) = \frac{1}{4} p_1(\tau_2, \tau_3) \beta_n$ and $z_n(A; \tau_0, \tau_2) = \frac{1}{4} p_1(\tau_0, \tau_2) \beta_n$. For the same reasons, $z_n(A; \tau_1, \tau_2) = \frac{1}{4} p_1(\tau_1, \tau_2) \beta_n$, and the lemma is proved when A is a rational homology ball.

In general, A is assumed to embed into a rational homology ball B, the pseudo-parallelization τ_0 on A extends to a pseudo-parallelization $\tilde{\tau}_0$ of B, and the pseudo-parallelization τ_1 over A extends to a pseudo-parallelization $\tilde{\tau}_1$ of B, which coincides with $\tilde{\tau}_0$ over $B \setminus \mathring{A}$. Therefore

$$z_n(A;\tau_0,\tau_1) = z_n(B;\tilde{\tau}_0,\tilde{\tau}_1) = \frac{p_1(\tilde{\tau}_0,\tilde{\tau}_1)}{4}\beta_n = \frac{p_1(\tau_0,\tau_1)}{4}\beta_n.$$

□

19.5 Proof of Theorem 19.13

Proposition 19.26. *Let $\check{R}$ be a rational homology $\mathbb{R}^3$ equipped with an asymptotically standard pseudo-parallelization τ. Let $\omega(\tau)$ be a homogeneous propagating form of $(C_2(R),\tau)$. Let $n \in \mathbb{N}$. With the notation of Corollary 10.9 and Notation 7.16,*

$$z_n(\check{R},\omega(\tau)) = \mathfrak{z}_n(R) + \frac{1}{4}p_1(\tau)\beta_n$$

and

$$\mathcal{Z}(R) = Z(\check{R},\emptyset,\omega(\tau))\exp\left(-\frac{p_1(\tau)}{4}\beta\right)$$

Proof: Corollary 9.4 implies that for any two pseudo-parallelizations τ_0 and τ_1 of $\check{R}$ that are standard outside B_R,

$$z_n(\check{R},\omega(\tau_1)) - z_n(\check{R},\omega(\tau_0)) = z_n\left([0,1]\times UB_R;\omega\right)$$

for a 2–form ω on $[0,1]\times UB_R$ as in Lemma 9.1. Theorem 19.21 implies that

$$z_n\left([0,1]\times UB_R;\omega\right) = \frac{1}{4}\left(p_1(\tau_1)-p_1(\tau_0)\right)\beta_n.$$

Conclude with Corollary 10.9. □

Proof of Theorem 19.13: Theorem 12.7 implies that

$$Z(\mathcal{C},L,\tau) = \exp(\frac{1}{4}p_1(\tau)\beta)\prod_{j=1}^{k}\left(\exp(I_\theta(K_j,\tau)\alpha)\sharp_j\right)\mathcal{Z}(\mathcal{C},L)$$

for any actual parallelization τ standard on $\partial\mathcal{C}$, and we want to prove that the same equality holds when τ is a pseudo-parallelization that is an actual parallelization over a tubular neighborhood $N(L)$ of L. Proposition 19.26 leaves us with the proof that

$$\check{Z}(\mathcal{C}, L, \tau') = \prod_{j=1}^{k} (\exp(I_\theta(K_j, \tau')\alpha)\sharp_j)\, \check{\mathcal{Z}}(\mathcal{C}, L)$$

for any pseudo-parallelization τ' of $\mathcal{C}$ standard on $\partial\mathcal{C}$ that is an actual parallelization over a tubular neighborhood $N(L)$ of L.

Assume that the restriction τ' to the tubular neighborhood $N(L)$ of L extends to an actual parallelization τ of $\mathcal{C}$.

According to Proposition 14.40,

$$\check{Z}(\mathcal{C}, L, \tau') = \left(\prod_{j=1}^{k} \exp\left(I_j\right) \sharp_j\right) \check{Z}(\mathcal{C}, L, \tau),$$

where I_j is defined for any component K_j of $L = \coprod_{j=1}^{k} K_j$, from a closed 2-form $\tilde{\omega}$ on $[0,1] \times U\mathcal{C}$, which may be expressed as $p^*_{UN(L)} p^*_\tau(\omega_{S^2})$ over $[0,1] \times UN(L)$, where $p_{UN(L)} \colon [0,1] \times UN(L) \to UN(L)$ is the projection to the second factor, according to Proposition 19.19. The factorization via $p_{UN(L)}$ implies that the I_j vanish in this case, so $\check{Z}(\mathcal{C}, L, \tau') = \check{Z}(\mathcal{C}, L, \tau)$. The degree-one part of this equality implies that $I_\theta(K_j, \tau') = I_\theta(K_j, \tau)$, too, and the theorem is proved in this first case.

Let D_r denote the disk of the complex numbers of module less or equal than r. Let τ_s denote the standard parallelization of $\mathbb{R}^3$ and its restriction to $D_4 \times [-2,2]$. Let

$$N(\gamma_2) = (D_3 \setminus \mathring{D}_1) \times [-1,1]$$

be a tubular neighborhood of $(\gamma_2 = \partial D_2 \times \{0\})$. Let $\tau_2 = (N(\gamma_2); \tau_e, \tau_b)$ be a pseudo-parallelization of $D_4 \times [-2,2]$ that coincides with τ_s in a neighborhood of $\partial(D_4 \times [-2,2])$ and that maps e_3 to the vertical direction of $\{0\} \times [-2,2]$ along $\{0\} \times [-2,2]$.

Let $\tau_s^{\mathcal{C}}$ and $\tau_2^{\mathcal{C}}$ satisfy the following set $(*)(\tau_s, \tau_2, \mathcal{C}, L, K_j)$ of assumptions: There is an embedding of $D_4 \times [-2,2]$ in the rational homology cylinder $\mathcal{C}$ equipped with the long tangle representative L so that (the image of) $D_4 \times [-2,2]$ intersects (the image of) L along $\{0\} \times [-2,2]$ and the orientations of $\{0\} \times [-2,2]$ and L match. The component of L that intersects $D_4 \times [-2,2]$ is denoted by K_j. With respect to this embedding, $\tau_s^{\mathcal{C}}$ and $\tau_2^{\mathcal{C}}$ are two pseudo-parallelizations of $\mathcal{C}$ standard on $\partial\mathcal{C}$, which are actual parallelizations over a tubular neighborhood $N(L)$ of L, which coincide outside the image of $D_4 \times [-2,2]$, and which coincide with τ_s and τ_2, respectively, there.

There exists a closed 2-form $\tilde{\omega}$ on $[0,1] \times U(D_4 \times [-2,2])$ that restricts to $\{0\} \times U(D_4 \times [-2,2]) \cup [0,1] \times U(D_4 \times [-2,2])_{|\partial(D_4 \times [-2,2])}$ as $p_{\tau_s}^*(\omega_{S^2})$ and to $\{1\} \times U(D_4 \times [-2,2])$ as a homogeneous propagating form of $C_2(R(\mathcal{C}), \tau_2^{\mathcal{C}})$ does, according to Proposition 19.19. This closed 2-form is actually independent of $(\mathcal{C}, L)$, and so is the induced quantity $I(\tau_2)$ of Proposition 14.40 such that

$$\check{Z}(\mathcal{C}, L, \tau_2^{\mathcal{C}}) = \exp(I(\tau_2))\sharp_j \check{Z}(\mathcal{C}, L, \tau_s^{\mathcal{C}})$$

and $I_\theta(K_j, \tau_2^{\mathcal{C}}) - I_\theta(K_j, \tau_s^{\mathcal{C}})$ is a constant $\ell(\tau_2)$, which can be obtained from the degree one part of $I(\tau_2)$.

Apply this computation when $R(\mathcal{C})$ is $SO(3)$, when $K_j = L = K$ is a knot whose homology class represents the generator of $H_1(SO(3);\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and when $\tau_s^{\mathcal{C}}$ is an actual parallelization. Then $\tau_{2|N(L)}^{\mathcal{C}}$ extends to $\mathcal{C}$ as a parallelization standard near $\partial\mathcal{C}$, as all the parallelizations of $N(K)$ do, since the two homotopy classes of parallelizations $N(K)$ are obtained from one another by composition by the map $\begin{array}{lll} SO(3) \times \mathbb{R}^3 & \to & SO(3) \times \mathbb{R}^3 \\ (\rho, x) & \mapsto & (\rho, \rho(x)) \end{array}$.

Then the first case implies that $I(\tau_2) = \left(I_\theta(K, \tau_2^{\mathcal{C}}) - I_\theta(K, \tau_s^{\mathcal{C}})\right)\alpha = \ell(\tau_2)\alpha$. So

$$\exp(-I_\theta(K_j, \tau_2^{\mathcal{C}})\alpha)\sharp_j \check{Z}(\mathcal{C}, L, \tau_2^{\mathcal{C}}) = \exp(-I_\theta(K_j, \tau_s^{\mathcal{C}})\alpha)\sharp_j \check{Z}(\mathcal{C}, L, \tau_s^{\mathcal{C}})$$

and

$$\prod_{\ell=1}^{k} \left(\exp(-I_\theta(K_\ell, \tau_2^{\mathcal{C}})\alpha)\sharp_\ell\right) \check{Z}(\mathcal{C}, L, \tau_2^{\mathcal{C}}) = \prod_{\ell=1}^{k} \left(\exp(-I_\theta(K_\ell, \tau_s^{\mathcal{C}})\alpha)\sharp_\ell\right) \check{Z}(\mathcal{C}, L, \tau_s^{\mathcal{C}})$$

for any two pseudo-parallelizations $\tau_s^{\mathcal{C}}$ and $\tau_2^{\mathcal{C}}$ that satisfy $(*)(\tau_s, \tau_2, \mathcal{C}, L, K_j)$.

Let τ' be a pseudo-parallelization of $\mathcal{C}$ that is standard on $\partial\mathcal{C}$ and that coincides with an actual parallelization τ_N of $\mathcal{C}$ on $N(L)$, and let τ be a parallelization of $\mathcal{C}$ that is standard on $\partial\mathcal{C}$. The restrictions of τ_N and τ to $\partial N(L)$ are homotopic along the meridians of L and they differ by the action of the generator of $\pi_1(SO(2))$ along parallels on components K_j for K_j in some finite set A. If $A = \emptyset$, the first studied case implies that

$$\prod_{j=1}^{k} \left(\exp(-I_\theta(K_j, \tau')\alpha)\sharp_j\right) \check{Z}(\mathcal{C}, L, \tau') = \check{\mathcal{Z}}(\mathcal{C}, L).$$

Otherwise, perform a homotopy of τ' as in Definition 19.17 to transform τ' to a pseudo-parallelization τ'' equipped with one embedding of $D_4 \times [-2,2]$ per component K_j of A, whose image meets L in K_j along $\{0\} \times [-2,2]$, so that the orientations of $\{0\} \times [-2,2]$ and L match, and τ'' is induced by τ_s

on the image of these embeddings. For $B \subseteq A$, let τ''_B be obtained from τ'' by changing τ_s to τ_2 on the images of the embeddings of $D_4 \times [-2,2]$ that meet an element of B. Then for $B \subseteq A$, and for $K_\ell \in A \setminus B$,

$$\prod_{j=1}^{k} \left(\exp(-I_\theta(K_j, \tau''_{B\cup\{K_\ell\}})\alpha)\sharp_j\right) \check{Z}(\mathcal{C}, L, \tau''_{B\cup\{K_\ell\}})$$

$$= \prod_{j=1}^{k} (\exp(-I_\theta(K_j, \tau''_B)\alpha)\sharp_j) \check{Z}(\mathcal{C}, L, \tau''_B).$$

So

$$\prod_{j=1}^{k} (\exp(-I_\theta(K_j, \tau''_A)\alpha)\sharp_j) \check{Z}(\mathcal{C}, L, \tau''_A) = \prod_{j=1}^{k} (\exp(-I_\theta(K_j, \tau'')\alpha)\sharp_j) \check{Z}(\mathcal{C}, L, \tau'')$$

by induction on $\sharp A$. Since τ''_A and τ are homotopic on $N(L)$, the first proved case implies that

$$\prod_{j=1}^{k} (\exp(-I_\theta(K_j, \tau''_A)\alpha)\sharp_j) \check{Z}(\mathcal{C}, L, \tau''_A) = \check{\mathcal{Z}}(\mathcal{C}, L).$$

So it suffices to prove that

$$\prod_{j=1}^{k} (\exp(-I_\theta(K_j, \tau'')\alpha)\sharp_j) \check{Z}(\mathcal{C}, L, \tau'') = \prod_{j=1}^{k} (\exp(-I_\theta(K_j, \tau')\alpha)\sharp_j) \check{Z}(\mathcal{C}, L, \tau')$$

or that $\prod_{j=1}^{k} (\exp(-I_\theta(K_j, \tau')\alpha)\sharp_j) \check{Z}(\mathcal{C}, L, \tau')$ is invariant by a homotopy of τ', as in Definition 19.17, supported in a ball where τ' is a genuine parallelization (namely, around an image of $D_4 \times [-2,2]$).

Again, the effect on $\check{Z}(\mathcal{C}, L, \tau')$ of such a homotopy depends only on the homotopy inside the ball, according to Proposition 14.40. Since such a ball equipped with the homotopy may be inserted in a tangle equipped with a genuine trivialization τ, we conclude that $\prod_{j=1}^{k} (\exp(-I_\theta(K_j, \tau')\alpha)\sharp_j) \check{Z}(\mathcal{C}, L, \tau')$ is indeed invariant under the above homotopies. This is sufficient to conclude the proof of Theorem 19.13. $\square$

More definitions of Z, involving non-necessarily homogeneous propagating forms and pseudo-parallelizations, are given in Chapter 21.

Chapter 20

Simultaneous normalization of propagating forms

This chapter is devoted to the proof of Propositions 18.32 and 18.35. As shown in Section 18.7, this is sufficient to prove Theorem 18.5. We use real coefficients for homology and cohomology.

20.1 Sketch

First note that the homogeneous boundary form of Definition 19.9 defined on $\partial C_2(R)$ (or the form $p_\tau^*(\omega_{S^2})$ as in Definition 3.9 when pseudo-parallelizations are not involved) is antisymmetric on $\partial C_2(R)$ as in Definition 7.22. So it extends as a closed antisymmetric 2-form $\omega = \omega_\emptyset$ on $C_2(R)$ as in Lemma 3.14.

Also note that if the restriction of ω_I to

$$A_I^{(i)} \times (C_1(R_I) \setminus A_{I,3}^{(i)}) \subset C_2(R_I)$$

equals

$$\sum_{j \in \underline{g_i}} p_1^*(\eta_I(a_j^i)) \wedge p_2^*(\eta_I(z_j^i)) + p_2^*(\omega(p^i))$$

as stated in Proposition 18.35, then the restriction of ω_I to $A_I^{(i)} \times A_I^{(k)}$ equals

$$\sum_{\substack{j = 1, \dots, g_i \\ \ell = 1, \dots, g_k}} lk(z_j^i, z_\ell^k) p^*_{A_I^{(i)}}(\eta(a_j^i)) \wedge p^*_{A_I^{(k)}}(\eta(a_\ell^k)),$$

for $k \neq i$, as wanted in Proposition 18.32.

In order to arrange the propagating forms ω_I as in Propositions 18.32 and 18.35, we will first show how to make ω satisfy the conditions of Proposition 18.35, with respect to the notation before Proposition 18.35.

More precisely, we are going to prove the following proposition, in Subsection 20.2.

Proposition 20.1. *Let $\tilde{\omega}$ be a propagating form of $C_2(R)$ as in Definition 3.11. Its restriction to $\partial C_2(R) \setminus UB_R$ may be expressed as $p_\tau^*(\tilde{\omega}_{S^2})$, for some volume-one form $\tilde{\omega}_{S^2}$ of S^2. Let $\tilde{\omega}(p_i)$ (resp. $\tilde{\omega}(p_i)_\iota$) be a degree two form on $\left(C_1(R) \setminus Int(A^{(i)})\right)$ that satisfies the same properties as the form $\omega(p^i)$ (introduced before Proposition 18.35) except[1] that it restricts to $\partial C_1(R) = S^2$ as $\tilde{\omega}_{S^2}$ (resp. as $-\iota_{S^2}^*\tilde{\omega}_{S^2}$) instead of the usual volume form ω_{S^2}. If $\tilde{\omega}$ is antisymmetric, then assume $\tilde{\omega}(p_i)_\iota = \tilde{\omega}(p_i)$.*

There exists a propagating form ω of $C_2(R)$ such that

1. *ω coincides with $\tilde{\omega}$ on $\partial C_2(R)$,*
2. *for every $i \in \underline{x}$, the restriction of ω to*
$$A^{(i)} \times (C_1(R) \setminus A_3^{(i)}) \subset C_2(R)$$
equals
$$\sum_{j=1}^{g_i} p_1^*(\eta(a_j^i)) \wedge p_2^*(\eta(z_j^i)) + p_2^*(\tilde{\omega}(p_i)),$$
where p_1 and p_2 denote the first and the second projection of $A^{(i)} \times (C_1(R) \setminus A_3^{(i)})$ to $C_1(R)$, respectively, and the restriction of ω to
$$(C_1(R) \setminus A_3^{(i)}) \times A^{(i)} \subset C_2(R)$$
equals
$$\sum_{j \in \underline{g_i}} p_1^*(\eta(z_j^i)) \wedge p_2^*(\eta(a_j^i)) - p_1^*(\tilde{\omega}(p_i)_\iota).$$
3. *for every $i \in \underline{x}$, for any $j \in \{1, 2, \dots, g_i\}$,*
$$\int_{\Sigma(a_j^i) \times p^i} \omega = 0 \quad \text{and} \quad \int_{p^i \times \Sigma(a_j^i)} \omega = 0,$$
where $p^i \in \partial A^{(i)}$ and $\partial\Sigma(a_j^i) \subset \{4\} \times \partial A^{(i)}$, for every i, for any $j \in \{1, 2, \dots, g_i\}$,

[1] In our proof of Propositions 18.32 and 18.35, we use Proposition 20.1 only when $\tilde{\omega}_{S^2} = \omega_{S^2}$ and $\tilde{\omega}(p_i) = \tilde{\omega}(p_i)_\iota = \omega(p^i)$, but the general statement is useful in other related work.

4. ω is antisymmetric if $\tilde{\omega}$ is.

Assume that Proposition 20.1 is proved. This is the goal of Subsection 20.2. Recall that we will use Proposition 20.1 only when $\tilde{\omega}_{S^2} = \omega_{S^2}$ and $\tilde{\omega}(p_i) = \tilde{\omega}(p_i)_\iota = \omega(p^i)$. When changing some $A^{(i)}$ into some $A^{(i)\prime}$ with the same Lagrangian, it is easy to change the restrictions of ω inside the parts $(A^{(r)} \times (C_1(R) \setminus A_3^{(r)})$ or $(C_1(R) \setminus A_3^{(r)}) \times A^{(r)} \subset C_2(R))$ mentioned in the statement of Proposition 20.1. Indeed, all the forms $\eta(a_j^i)$, $\eta(z_j^i)$ and $\omega(p^i)$ can be defined on the parts of the R_I where they are required, so that these forms coincide with each other whenever it makes sense, and so that they have the properties that were required for R. (Recall that the $\eta(a_j^i)$ are defined both in $A^{(i)}$ and $A^{(i)\prime}$, and that they are identical near $\partial A^{(i)}$ and $\partial A^{(i)\prime}$, while $\omega(p^i)$ is supported in $\left(R \setminus \left(\cup_{k \in \underline{x}} \mathrm{Int}(A^{(k)})\right)\right)$, and the $\eta(z_j^i)$ restrict to the $A^{(k)}$ as a (fixed by the linking numbers) combination of $\eta(a_\ell^k)$. Define $\omega_0(R_I)$ on
$D(\omega_0(R_I)) =$

$$\left(C_2(R_I) \setminus \left(\cup_{i \in I} p_b^{-1}\left((A_{-1}^{(i)\prime} \times A_3^{(i)\prime}) \cup (A_3^{(i)\prime} \times A_{-1}^{(i)\prime})\right)\right)\right) \cup p_b^{-1}(\mathrm{diag}(\check{R}_I))$$

so that

1. $\omega_0(R_I) = \omega$ on $C_2\left(R \setminus (\cup_{i \in I} A_{-1}^{(i)\prime})\right)$,
2. $$\omega_0(R_I) = \sum_{i=1}^{g_i} p_1^*(\eta(a_j^i)) \wedge p_2^*(\eta(z_j^i)) + p_2^*(\omega(p^i))$$
   on $p_b^{-1}(A^{(i)\prime} \times (\check{R}_I \setminus A_3^{(i)\prime}))$ when $i \in I$,
3. $\omega_0(R_I) = -\iota^*(\omega_0(R_I))$ on $(\check{R}_I \setminus A_3^{(i)\prime}) \times A^{(i)\prime}$ when $i \in I$,
4. On $\partial C_2(R_I)$, $\omega_0(R_I)$ coincides with the homogeneous boundary form $\omega(\tau_I, F)$ of Definition 19.9 for a map F, which is the same for all $I \subseteq \underline{x}$.

Note that this definition is consistent.

Set $R_i = R_{\{i\}}$ and set $D_A(\omega_0(R_i)) = C_2(A_4^{(i)\prime}) \cap D(\omega_0(R_i))$.

Lemma 20.2. *With the above notation, for any $i \in \underline{x}$, the cohomology class of $\omega_0(R_i)$ vanishes on the kernel of the map induced by the inclusion*

$$H_2\left(D_A(\omega_0(R_i))\right) \longrightarrow H_2(C_2(A_4^{(i)\prime})).$$

This lemma was surprisingly difficult to prove for me. It will be proved in Subsection 20.3. Assume it for the moment. Then (the cohomology class of) $\omega_0(R_i)$ is in the image of the natural map

$$H^2(C_2(A_4^{(i)\prime})) \longrightarrow H^2(D_A(\omega_0(R_i))).$$

Therefore $\omega_0(R_i)$ extends to a closed form $\omega_1(R_i)$ on $C_2(A_4^{(i)\prime})$. Change this form to $\omega_{\{i\}} = \frac{\omega_1(R_i)-\iota^*(\omega_1(R_i))}{2}$ to get an antisymmetric homogeneous propagating form of $\left(C_2(R_i), \tau_{\{i\}}\right)$.

Now, for any $I \subseteq \underline{x}$, we may define

$$\omega_I = \begin{cases} \omega_0(R_I) & \text{on } C_2(R_I) \setminus \left(\cup_{i\in I} p_b^{-1}\left((A_{-1}^{(i)\prime} \times A_4^{(i)\prime}) \cup (A_4^{(i)\prime} \times A_{-1}^{(i)\prime})\right)\right) \\ \omega_{\{i\}} & \text{on } C_2(A_4^{(i)\prime}) \text{ for } i \in I \end{cases}$$

since the $C_2(A_4^{(i)\prime})$ do not intersect.

Note that $\int_{\Sigma'(a_j^i)\times p^i} \omega_{\{i\}} = 0$ because $\int_{\Sigma'(a_j^i)\times p^i} \omega_{\{i\}} = \int_{\Sigma'(a_j^i)\times (p^i\times\{4\})} \omega_{\{i\}} + \int_{\partial\Sigma'(a_j^i)\times (p^i\times[0,4])} \omega_{\{i\}}$, $0 = \int_{\Sigma(a_j^i)\times (p^i\times\{4\})} \omega + \int_{\partial\Sigma(a_j^i)\times (p^i\times[0,4]\}} \omega$, where

$$\int_{(\Sigma'(a_j^i)\cap A_0^{(i)\prime})\times (p^i\times\{4\})} \omega_{\{i\}} = \int_{(\Sigma(a_j^i)\cap A_0^{(i)})\times (p^i\times\{4\})} \omega = 0$$

because of the prescribed behaviour of the forms on $A_I^{(i)} \times \left(\check{R}_I \setminus A_{I3}^{(i)}\right)$, and $\int_{\left(-(\partial\Sigma(a_j^i)\times[0,4])\times(p^i\times\{4\})\right)\cup\left(\partial\Sigma(a_j^i)\times(p^i\times[0,4])\right)} (\omega_{\{i\}} - \omega) = 0$ since ω and $\omega_{\{i\}}$ coincide on $C_2\left(A_4^{(i)} \setminus A_{-1}^{(i)}\right)$.

Therefore, the forms ω_I satisfy the conclusions of Propositions 18.32 and 18.35, which will be proved once Proposition 20.1 and Lemma 20.2 are proved. Their proofs will occupy the next two subsections.

20.2 Proof of Proposition 20.1

The homology classes of the $(z_j^i \times (4 \times a_k^i))_{(j,k)\in\{1,\dots,g_i\}^2}$ and $(p^i \times \partial C_1(R))$ form a basis of

$$H_2\left(A^{(i)} \times (C_1(R) \setminus A_3^{(i)})\right) = (H_1(A^{(i)}) \otimes H_1(R \setminus A^{(i)})) \oplus H_2(C_1(R) \setminus A^{(i)}).$$

According to Lemma 3.12, the evaluation of the cohomology class of any propagating form of $C_2(R)$ at these classes is $\ell(z_j^i, (4 \times a_k^i)) = \delta_{kj}$ for the first ones and 1 for the last one. In particular, the form of the statement integrates correctly on this basis.

Let us first prove Proposition 20.1 when $\underline{x} = \{1\}$. Set $A^1 = A$, and forget about the superfluous superscripts 1. Let ω_0 be a propagating two form of $C_2(R)$ that restricts to $\partial C_2(R) \setminus UB_R$ as $p_\tau^*(\tilde{\omega}_{S^2})$, and let ω_b be the closed 2-form defined on $(A_1 \times (C_1(R) \setminus \mathrm{Int}(A_2)))$ by the statement (extended naturally). Since this form ω_b integrates correctly on

$$H_2\left(A_1 \times (C_1(R) \setminus \mathrm{Int}(A_2))\right),$$

there exists a one-form η on $(A_1 \times (C_1(R) \setminus \mathrm{Int}(A_2)))$ such that $\omega_b = \omega_0 + d\eta$.

This form η is closed on $A_1 \times \partial C_1(R)$. Since $H^1\left(A_1 \times (C_1(R) \setminus \mathrm{Int}(A_2))\right)$ maps surjectively to $H^1(A_1 \times \partial C_1(R))$, we may extend η as a closed one-form $\tilde{\eta}$ on $(A_1 \times (C_1(R) \setminus \mathrm{Int}(A_2)))$. Changing η into $(\eta - \tilde{\eta})$, turns η to a primitive of $(\omega_b - \omega_0)$ that vanishes on $A_1 \times \partial C_1(R)$.

Let

$$\chi \colon C_2(R) \to [0,1]$$

be a smooth function supported in $(A_1 \times (C_1(R) \setminus \mathrm{Int}(A_2)))$, and constant with the value 1 on $(A \times (C_1(R) \setminus A_3))$.

Set

$$\omega_a = \omega_0 + d\chi\eta.$$

Then ω_a is a closed form that has the required form on $(A \times (C_1(R) \setminus A_3))$. Furthermore, the restrictions of ω_a and ω_0 agree on $\partial C_2(R)$ since $d\chi\eta$ vanishes there (because η vanishes on $A_1 \times \partial C_1(R)$).

Adding to η a combination η_c of the closed forms $p_2^*(\eta(z_j))$, which vanish on $A_1 \times \partial C_1(R)$, does not change the above properties, but adds

$$\int_{p \times ([2,4] \times a_j)} d(\chi\eta_c) = \int_{p \times (4 \times a_j)} \eta_c$$

to $\int_{p \times \Sigma(a_j)} \omega_a$. Therefore, since the $p_2^*(\eta(z_j))$ generate the dual of $\mathcal{L}_A$, we may choose η_c so that all the $\int_{p \times \Sigma(a_j)} \omega_a$ vanish. After this step, ω_a is a closed form, which takes the prescribed values on

$$PS_a = \partial C_2(R) \cup \left(A \times (C_1(R) \setminus A_3)\right),$$

and such that all the $\int_{p \times \Sigma(a_j)} \omega_a$ vanish. In order to make ω_a take the prescribed values on $\iota(PS_a)$ and integrate as wanted on $\Sigma(a_j) \times p$, we apply similar modifications to ω_a on the symmetric part $(C_1(R) \setminus \mathrm{Int}(A_2)) \times A_1$. The support of these modifications is disjoint from the support of the previous ones. Thus, they do not interfer and transform ω_a into a closed form ω_b with the additional properties:

- ω_b has the prescribed form on $(C_1(R) \setminus A_3) \times A$,
- $\int_{\Sigma(a_j)\times p} \omega_b = 0$, for all $j = 1, \dots g_1$.

Now, the form $\omega = \omega_b$ (resp. $\omega = \frac{\omega_b - \iota^*(\omega_b)}{2}$ if antisymmetry is wanted) has all the required properties, and Proposition 20.1 is proved for $\underline{x} = \{1\}$.

We now proceed by induction on x. We start with a 2-form ω_0 that satisfies all the hypotheses with $\underline{x-1}$ instead of $\underline{x}$, and by the first step, we also assume that we have a 2-form ω_b that satisfies all the hypotheses with $\{x\}$ instead of $\underline{x}$, with the enlarged $A_1^{(x)}$ replacing $A^{(x)}$.

Now, we proceed similarly. There exists a one-form η on $C_2(R)$ such that $\omega_b = \omega_0 + d\eta$. The exact sequence

$$0 = H^1(C_2(R)) \longrightarrow H^1(\partial C_2(R)) \longrightarrow H^2(C_2(R), \partial C_2(R)) \cong H_4(C_2(R)) = 0$$

implies that $H^1(\partial C_2(R))$ is trivial. Therefore, η is exact on $\partial C_2(R)$, we may assume that η vanishes on $\partial C_2(R)$, which we do. Let

$$\chi \colon C_2(R) \to [0,1]$$

be a smooth function supported in $\left(A_1^{(x)} \times (C_1(R) \setminus \mathrm{Int}(A_2^{(x)}))\right)$, and constant with the value 1 on $\left(A^{(x)} \times (C_1(R) \setminus A_3^{(x)})\right)$. Again, we are going to modify η by some closed forms so that

$$\omega_a = \omega_0 + d\chi\eta$$

has the prescribed value on

$$\begin{aligned} PS_a = \partial C_2(R) \quad & \cup \left(\cup_{k=1}^{x} \left(A^{(k)} \times (C_1(R) \setminus A_3^{(k)})\right)\right) \\ & \cup \left(\cup_{k=1}^{x-1} \left((C_1(R) \setminus A_3^{(k)}) \times A^{(k)}\right)\right). \end{aligned}$$

Our form ω_a is as required anywhere except possibly in

$$\left(A_1^{(x)} \times (C_1(R) \setminus \mathrm{Int}(A_2^{(x)}))\right) \setminus \left(A^{(x)} \times (C_1(R) \setminus A_3^{(x)})\right)$$

and in particular in the intersection of this domain with the domains where it was normalized previously, which are included in

$$\left(A_1^{(x)} \times (\partial C_1(R) \cup (\cup_{k=1}^{x-1} A^{(k)}))\right).$$

Recall that η vanishes on $A_1^{(x)} \times \partial C_1(R)$. Our assumptions also imply that η is closed on $A_1^{(x)} \times A^{(k)}$, for any $k < x$. Let us prove that they imply that η is exact on $A_1^{(x)} \times A^{(k)}$, for any $k < x$. To do that, it suffices to check that:

1. For any $j = 1, \dots, g_x$, $\int_{z_j^x \times p^k} \eta = 0$.

2. For any $j = 1, \dots, g_k$, $\int_{p^x \times z_j^k} \eta = 0$.

Let us prove the first assertion. Let $\infty(v) \in \partial C_1(R)$ and let $[p^k, \infty(v)]$ be a path from p^k to $\infty(v)$ in $C_1(R)$ that intersects $\mathcal{C}$ as the path $[p^k, q^k]$ introduced before Proposition 18.35. Since $\int_{z_j^x \times \infty(v)} \eta = 0$,

$$\int_{z_j^x \times p^k} \eta = \int_{\partial(z_j^x \times [p^k, \infty(v)])} \eta = \int_{z_j^x \times [p^k, \infty(v)]} (\omega_b - \omega_0)$$

where $\int_{z_j^x \times [p^k, \infty(v)]} \omega_b = 0$, because the supports of the $\eta(z_\ell^x)$ do not intersect $[p^k, \infty(v)]$. Now,

$$\int_{z_j^x \times [p^k, \infty(v)]} \omega_0 = -\int_{\Sigma(z_j^x) \times \partial[p^k, \infty(v)]} \omega_0 = \int_{\Sigma(z_j^x) \times \{p^k\}} \omega_0.$$

The latter integral vanishes because

1. $\Sigma(z_j^x)$ intersects $A_4^{(k)}$ as copies of $\Sigma(a_\ell^k)$,

2. $\int_{\Sigma(a_\ell^k) \times p^k} \omega_0 = 0$ (this is the third condition of Proposition 20.1), and,

3. the integral of ω_0 also vanishes on the remaining part of $\Sigma(z_j^x) \times p^k$ because ω_0 is determined on $((C_1(R) \setminus A_4^{(k)}) \times A^{(k)})$ and because the support of $\omega(p^k)$ is disjoint from $\Sigma(z_j^x)$.

Let us prove the second assertion, namely that $\int_{p^x \times z_j^k} \eta = 0$ for $j \in \underline{g_k}$. Again, since η vanishes on $\partial C_2(R)$, $\int_{\infty(v) \times z_j^k} \eta = 0$ and therefore

$$\int_{p^x \times z_j^k} \eta = -\int_{[p^x, \infty(v)] \times z_j^k} (\omega_b - \omega_0).$$

$\int_{[p^x, \infty(v)] \times z_j^k} \omega_0 = 0$ because of the behaviour of ω_0 on $(C_1(R) \setminus A_4^{(k)}) \times A^{(k)}$.

$$\int_{[p^x, \infty(v)] \times z_j^k} \omega_b = \int_{(\partial[p^x, \infty(v)]) \times \Sigma(z_j^k)} \omega_b = -\int_{\{p^x\} \times \Sigma(z_j^k)} \omega_b.$$

Again, we know that this integral is zero along the intersection of $\{p^x\} \times \Sigma(z_j^k)$ with $A^{(x)} \times (C_1(R) \setminus A_4^{(x)})$ because $\Sigma(z_j^k)$ does not meet the support of $\omega(p^x)$, and we conclude because $\int_{\{p^x\}\times\Sigma(a^x_\ell)}\omega_b = 0$ and because $\Sigma(z^k_j)$ intersects $A_4^{(x)}$ along copies of $\Sigma(a^x_\ell)$.

Since η is exact on the annoying parts, we can assume that it vanishes identically in these parts.

Thus, ω_a takes the prescribed values on $A^{(x)} \times (C_1(R) \setminus A_4^{(x)})$, ω_a coincides with ω_0 where ω_0 was prescribed, and ω_a integrates correctly along the $\Sigma(a^k_\ell) \times p^k$ and their symmetric with respect to ι, for $k \neq x$. Let us now modify η by adding a linear combination of $p_2^*(\eta(z^x_j))$ that vanishes on the $A_1^{(x)} \times A^{(k)}$, for $k < x$, and thus without changing the above properties, so that the integrals of ω_a along the $\{p^x\} \times \Sigma(a^x_\ell)$ vanish, for $\ell = 1, \dots, g_x$, too. Let $f : H_1(R \setminus \mathrm{Int}(A^{(x)})) \longrightarrow \mathbb{R}$ be the linear map defined by

$$f(a^x_\ell) = \int_{\{p^x\}\times\Sigma(a^x_\ell)} \omega_a.$$

The combination $\eta_c = \sum_{\ell=1}^{g_x} f(a^x_\ell) p_2^*(\eta(z^x_\ell))$ is such that for any $y \in \mathcal{L}_{A^{(x)}}$, $f(y) = \int_{p^x \times y} \eta_c$.

Observe that, for $k < x$, and for $j = 1, \dots, g_k$,

$$\begin{aligned} f(z^k_j) &= \textstyle\sum_{\ell=1}^{g_x} lk(z^k_j, z^x_\ell) f(a^x_\ell) = \int_{\{p^x\}\times\Sigma(z^k_j)} \omega_a \\ &= \textstyle\int_{\{\infty(v)\}\times\Sigma(z^k_j)} \omega_a - \int_{[p^x,\infty(v)]\times z^k_j} \omega_a = 0. \end{aligned}$$

This implies that $f\left(\mathrm{Im}\left(H_1\left(A^{(k)}\right) \longrightarrow H_1\left(R \setminus \mathring{A}^{(x)}\right)\right)\right) = 0$. Thus, η_c vanishes on $A_1^{(x)} \times A_k$ since it may be expressed as

$$\sum_{\ell=1}^{g_x} f(a^x_\ell) \sum_{j=1}^{g_k} lk(z^k_j, z^x_\ell) p_2^*(\eta(a^k_j)) = \sum_{j=1}^{g_k} f(z^k_j) p_2^*(\eta(a^k_j)),$$

there. Changing η into $(\eta - \eta_c)$ does not change ω_a on the prescribed set but removes $\int_{\{p^x\}\times\Sigma(a^x_\ell)} d\chi\eta_c = \int_{\{p^x\}\times(4\times a^x_\ell)} \eta_c = f(a^x_\ell)$ from $\int_{\{p^x\}\times\Sigma(a^x_\ell)} \omega_a$, which becomes 0.

After this step, ω_a is a closed form, which takes the prescribed values on PS_a, such that the integrals of ω_a along the $(\{p^x\} \times \Sigma(a^x_\ell))$ vanish, for $\ell = 1, \dots, g_i$. In order to make ω_a take the prescribed values on $\iota(PS_a)$, we apply similar modifications to ω_a on the symmetric part $(C_1(R) \setminus \mathring{A}_2^{(x)}) \times A_1^{(x)}$. Again, the support of these modifications is disjoint from the support of the previous ones. Thus, they do not interfer and they transform ω_a to a closed form ω_c with the additional properties:

- ω_c has the prescribed form on $(C_1(R) \setminus A_3^{(x)}) \times A^{(x)}$,

- $\int_{\Sigma(a_j^x)\times p^x} \omega_c = 0$, for all $j = 1, \dots g_x$.

Now, the form $\omega = \omega_c$ (resp. $\omega = \frac{\omega_c - \iota^*(\omega_c)}{2}$ if antisymmetry is wanted) has all the required properties, and Proposition 20.1 is proved.

□

20.3 Proof of Lemma 20.2

In order to conclude the proofs of Propositions 18.32 and 18.35, we now prove Lemma 20.2.

We first state some homological lemmas.

Lemma 20.3. *Let S be a closed (oriented) surface. Let S and S^+ be two copies of S, let $(c_i)_{i\in\underline{2g}}$ and $(c_i^*)_{i\in\underline{2g}}$ be two dual bases of $H_1(S;\mathbb{Z})$ such that $\langle c_i, c_j^*\rangle = \delta_{ij}$. Let $* \in S$. Let $diag(S \times S^+) = \{(x, x^+) \mid x \in S\}$. We have the following equality in $H_2(S \times S^+)$*

$$[diag(S \times S^+)] = [* \times S^+] + [S \times *^+] + \sum_{i=1}^{2g}[c_i \times c_i^{*+}].$$

PROOF:

$$H_2(S \times S^+) = \mathbb{Z}[* \times S^+] \oplus \mathbb{Z}[S \times *^+] \oplus \bigoplus_{(i,j)\in\underline{2g}^2} \mathbb{Z}[c_i \times c_j^{*+}].$$

The dual basis of the above basis with respect to the intersection form is

$$\left([S \times *^+], [* \times S^+], ([c_i^* \times c_j^+])_{(i,j)\in\underline{2g}^2}\right).$$

To get the coordinates of $[\mathrm{diag}(S\times S^+)]$ in the first decomposition we compute the intersection numbers with the second one. $\langle[\mathrm{diag}(S \times S^+)], [c_i^* \times c_i^+]\rangle = \pm 1$, where the tangent space to $\mathrm{diag}(S \times S^+)$ is parametrized naturally by (u_i, v_i^*, u_i, v_i^*) and the tangent space to $[c_i^* \times c_i^+]$ is parametrized naturally by $(0, w_i^*, x_i, 0)$. So the intersection sign is the sign of the permutation

$$(u, v, w, x) \mapsto (u, w, x, v)$$

which is $+1$.

□

Lemma 20.4. *Let Σ be a connected compact oriented surface with one boundary component $J(S^1)$ equipped with a basepoint $* = J(1)$. Let $(c_i)_{i\in\underline{2g}}$ and $(c_i^*)_{i\in\underline{2g}}$ be two dual bases of $H_1(\Sigma;\mathbb{Z})$ such that $\langle c_i, c_j^*\rangle = \delta_{ij}$. Let Σ and Σ^+ be two copies of Σ and set $J^+ = J^+(S^1) = \partial\Sigma^+$. Define the subspaces $J \times_{*,\leq} J^+$ and $J \times_{*,\geq} J^+$ of $J \times J^+$ by*

$$J \times_{*,\leq} J^+ = \{(J(\exp(2i\pi t)), J(\exp(2i\pi u))) \mid (t,u) \in [0,1]^2, t \leq u\}$$

and

$$J \times_{*,\geq} J^+ = \{(J(\exp(2i\pi t)), J(\exp(2i\pi u))) \mid (t,u) \in [0,1]^2, t \geq u\}.$$

Let $diag(\Sigma\times\Sigma^+)$ be the subspace $\{(x,x) \mid x \in \Sigma\}$ of $\Sigma\times\Sigma^+$. Then the chains

$$C_{*,\leq}(\Sigma,\Sigma^+) = diag(\Sigma\times\Sigma^+) - * \times \Sigma^+ - \Sigma \times *^+ - J \times_{*,\leq} J^+$$

and

$$C_{*,\geq}(\Sigma,\Sigma^+) = diag(\Sigma\times\Sigma^+) - * \times \Sigma^+ - \Sigma \times *^+ + J \times_{*,\geq} J^+$$

are cycles and we have the following equality in $H_2(\Sigma\times\Sigma^+)$

$$[C_{*,\leq}(\Sigma,\Sigma^+)] = [C_{*,\geq}(\Sigma,\Sigma^+)] = \sum_{i=1}^{2g}[c_i \times c_i^{*+}].$$

Proof: Since $\partial(J \times_{*,\leq} J^+) = \mathrm{diag}(J \times J^+) - * \times J^+ - J \times *^+$, $C_{*,\leq}(\Sigma,\Sigma^+)$ is a cycle. Consider the closed surface S obtained from Σ by gluing a disk D along J. According to Lemma 20.3, in $H_2(S\times S^+)$,

$$[\mathrm{diag}(S\times S^+)] = [* \times S^+] + [S \times *^+] + \sum_{i=1}^{2g}[c_i \times c_i^{*+}].$$

This implies that

$$[C_{*,\leq}(\Sigma,\Sigma^+) - C_{*,\leq}(-D,(-D)^+)] = \sum_{i=1}^{2g}[c_i \times c_i^{*+}]$$

in $H_2(S\times S^+)$. Since the cycle $C_{*,\leq}(-D,(-D)^+)$ sits in $D\times D^+$, it is null-homologous there, and since $H_2(\Sigma\times\Sigma^+)$ injects naturally into $H_2(S\times S^+)$, we can conclude that $[C_{*,\leq}(\Sigma,\Sigma^+)] = \sum_{i=1}^{2g}[c_i \times c_i^{*+}]$. The proof for $C_{*,\geq}(\Sigma,\Sigma^+)$ is the same. □

Consider a rational homology handlebody A with a collar $[-4,0]\times\partial A$ of its boundary. Recall that for $s \in [-4,0]$, $A_s = A \setminus (]s,0]\times\partial A)$, $\partial A_s =$

$\{s\} \times \partial A$. Let $(a_i, z_i)_{i=1,\dots g_A}$ be a basis of $H_1(\partial A)$ such that $a_i = \partial(\Sigma(a_i) \subset A)_{i=1,\dots,g}$, where $\Sigma(a_i)$ is a rational chain of A, and $\langle a_i, z_j\rangle = \delta_{ij}$.

Consider a curve a representing an element of $\mathcal{L}_A$ of order k in $H_1(A;\mathbb{Z})$, $k \in \mathbb{N} \setminus \{0\}$. Let $\Sigma = k\Sigma(a)$ be a surface of A immersed in A bounded by ka that intersects $[-1,0] \times \partial A$ as k copies of $[-1,0] \times a$, and that intersects $\mathrm{Int}(A_{-1})$ as an embedded surface and $\mathrm{Int}(A_{-1}) \setminus A_{-2}$ as k disjoint annuli. (See the thick part of Figure 20.1.) For $s \in [-2,0]$, set $\Sigma_s = \Sigma \cap A_s$.

Lemma 20.5. *With the above notation, let $(c_i)_{i=1,\dots 2g}$ and $(c_i^*)_{i=1,\dots,2g}$ be two dual bases of $\frac{H_1(\Sigma_{-2};\mathbb{Z})}{H_1(\partial\Sigma_{-2};\mathbb{Z})}$, $\langle c_i, c_j^*\rangle = \delta_{ij}$. Represent $(c_i)_{i=1,\dots 2g}$ and $(c_i^*)_{i=1,\dots,2g}$ by curves $(c_i)_{i=1,\dots 2g}$ and $(c_i^*)_{i=1,\dots,2g}$ of Σ_{-2}. Let $\Sigma_{-2} \times [-1,1]$ denote a tubular neighborhood of $\Sigma_{-2} = \Sigma_{-2} \times \{0\}$ in A_{-2}. For a curve σ of Σ_{-2}, σ^+ denotes the curve $\sigma \times \{1\}$.*

Then $\sum_{i=1}^{2g} c_i \times c_i^*$ is homologous to $\sum_{(j,\ell)\in \underline{g_A}^2\setminus diag}\langle \Sigma, \Sigma(a_j), \Sigma(a_\ell)\rangle z_j \times z_\ell$ in A^2, where the z_j are pairwise disjoint representatives of the $[z_j]$ on ∂A.

Furthermore, $\sum_{i=1}^{2g} c_i \times c_i^{*+}$ is homologous to

$$\sum_{(j,\ell)\in \underline{g_A}^2\setminus diag}\langle \Sigma, \Sigma(a_j), \Sigma(a_\ell)\rangle z_j \times z_\ell - gUA_{|*}$$

in $C_2(A)$.

Proof: Assume that Σ and the $\Sigma(a_j)$ are transverse to each other. For $(j,\ell) \in \{1,\dots,g_A\}^2$, set $\gamma_{\Sigma j} = \Sigma \cap \Sigma(a_j)$ and $\gamma_{\Sigma\ell} = \Sigma \cap \Sigma(a_\ell)$. If $j \neq \ell$, then set $\gamma_{j\ell} = \Sigma(a_j) \cap \Sigma(a_\ell)$. Then in $H_1(A)$, $c_i = \sum_{j=1}^{g_A}\langle c_i, \Sigma(a_j)\rangle_A z_j = \sum_{j=1}^{g_A}\langle c_i, \gamma_{\Sigma j}\rangle_\Sigma z_j$ and similarly,

$$c_i^* = \sum_{\ell=1}^{g_A}\langle c_i^*, \gamma_{\Sigma\ell}\rangle_\Sigma z_\ell.$$

Thus, in $H_2(A^2)$,

$$c_i \times c_i^* = \sum_{(j,\ell)\in \underline{g_A}^2}\langle c_i, \gamma_{\Sigma j}\rangle_\Sigma \langle c_i^*, \gamma_{\Sigma\ell}\rangle_\Sigma z_j \times z_\ell$$

On the other hand, in $H_1(\Sigma_{-2})/H_1(\partial\Sigma_{-2})$,

$$\gamma_{\Sigma j} = \sum_{i=1}^{2g}\langle c_i, \gamma_{\Sigma j}\rangle_\Sigma c_i^* \text{ and } \gamma_{\Sigma\ell} = -\sum_{i=1}^{2g}\langle c_i^*, \gamma_{\Sigma\ell}\rangle_\Sigma c_i.$$

Then

$$\langle \gamma_{\Sigma j}, \gamma_{\Sigma\ell}\rangle_\Sigma = \sum_{i=1}^{2g}\langle c_i, \gamma_{\Sigma j}\rangle_\Sigma \langle c_i^*, \gamma_{\Sigma\ell}\rangle_\Sigma.$$

In particular, for any $j \in \{1, \dots, g_A\}$,

$$\sum_{i=1}^{2g} \langle c_i, \gamma_{\Sigma j}\rangle_\Sigma \langle c_i^*, \gamma_{\Sigma j}\rangle_\Sigma = 0.$$

If $j \neq \ell$, then $\langle \Sigma, \Sigma(a_j), \Sigma(a_\ell)\rangle = \langle \gamma_{\Sigma j}, \gamma_{\Sigma \ell}\rangle_\Sigma$ and the first assertion is proved. Let us now prove that $\alpha = \sum_{i=1}^{2g} c_i \times c_i^{*+}$ is homologous to

$$\beta = \sum_{(j,\ell)\in \underline{g_A}^2 \setminus \mathrm{diag}} \langle \Sigma, \Sigma(a_j), \Sigma(a_\ell)\rangle z_j \times z_\ell - gUA_{|*}$$

in $C_2(A)$. First note that the homology class of α in $C_2(A)$ is independent of the dual bases (c_i) and (c_i^*). Indeed, since for a curve σ of Σ_{-2}, both $a \times \sigma^+$ and $\sigma \times a^+$ are null-homologous in $C_2(A)$, the class of α in $C_2(A)$ depends only on the class of $\sum_{i=1}^{2g} c_i \otimes c_i^{*+}$ in $H_1(\Sigma_{-2})/H_1(\partial\Sigma_{-2}) \otimes H_1(\Sigma_{-2}^+)/H_1(\partial\Sigma_{-2}^+)$, which is determined by the following property: For any two closed curves e and f of Σ, $\langle e \times f^+, \sum_{i=1}^{2g} c_i \times c_i^{*+}\rangle_{\Sigma\times\Sigma^+} = -\langle e, f\rangle_\Sigma$.

In particular, $[\alpha] = [\sum_{i=1}^{2g} c_i^* \times (-c_i^+)]$. The previous computation tells us that the difference $[\beta - \alpha]$ is a rational multiple of $[UA_{|*}]$. In particular, we can evaluate this multiple by embedding A in a rational homology ball obtained from A by adding thickened disks along neighborhoods of the z_i. Embed this rational homology ball in a rational homology sphere R.

In $H_2(C_2(R); \mathbb{R})$,

$$\begin{aligned}[\beta - \alpha] &= \textstyle\sum_{(j,\ell)\in\{1,\dots,g_A\}^2\setminus \mathrm{diag}} \langle \Sigma, \Sigma(a_j), \Sigma(a_\ell)\rangle lk(z_j, z_\ell)[U\check{R}_{|*}] \\ &\quad -g[U\check{R}_{|*}] - \tfrac{1}{2}\textstyle\sum_{i=1}^{2g} \left(lk(c_i, c_i^{*+}) - lk(c_i^{*-}, c_i)\right)[U\check{R}_{|*}] \\ &= -g[U\check{R}_{|*}] - \tfrac{1}{2}(-2g)[U\check{R}_{|*}] = 0.\end{aligned}$$

Since $[UA_{|*}] \neq 0$ in $H_2(C_2(R); \mathbb{R})$, this equality also holds in $H_2(C_2(A); \mathbb{R})$. $\square$

We now define a cycle $F^2(\Sigma(a))$ of $\partial C_2(A)$, which is associated to the surface $\Sigma = k\Sigma(a)$ introduced before Lemma 20.5. Let $(a \times [-1, 1])$ be a tubular neighborhood of a in ∂A. Let $p(a) \in a$ and view a as the image of a map $a\colon [0, 1] \to a$ such that $a(0) = a(1) = p(a)$. Let $\Sigma^+ = \Sigma_{-1} \cup k\{(t - 1, a(\alpha), t) \mid (t, \alpha) \in [0, 1]^2\}$. So $\partial\Sigma^+ = ka^+$, where $a^+ = a \times \{1\}$.

Let $p(a)^+ = (p(a), 1) = (0, p(a), 1) \in a \times [-1, 1] \subset (\partial A = \{0\} \times \partial A)$.

Recall $a \times_{p(a),\geq} a^+ = \{((a(v), 0), (a(w), +1)) \mid (v, w) \in [0, 1]^2, v \geq w\}$.

Let $T(a)$ be the closure of $\{((a(v), 0), (a(v), t)) \mid (t, v) \in]0, 1] \times [0, 1]\}$ (oriented by (t, v)) in $\partial C_2(A)$.

Let $s_+(\Sigma)$ be the positive normal section of $U(A)_{|\Sigma}$, and let $e(\Sigma(a) = \frac{\Sigma}{k}) = \frac{g+k-1}{k}$, where g is the genus of Σ.

$$e(\Sigma(a)) = \frac{-\chi(\Sigma)}{2k} + \frac{1}{2}.$$

Lemma 20.6. *With the above notation,*

$$\begin{array}{rl} F^2(\Sigma(a)) = & T(a) + a \times_{p(a),\geq} a^+ - p(a) \times \frac{1}{k}\Sigma^+ - \frac{1}{k}\Sigma \times p(a)^+ \\ & +\frac{1}{k}s_+(\Sigma) + e(\Sigma(a))[UA_{|*}] \\ & -\sum_{(j,\ell)\in\{1,\dots,g_A\}^2\setminus diag}\langle \Sigma(a), \Sigma(a_j), \Sigma(a_\ell)\rangle z_j \times z_\ell \end{array}$$

is a cycle, which is null-homologous in $C_2(A)$.

Proof: For $k=1$, (when we are dealing with integral homology handlebodies, for example) it is a direct consequence of Lemma 20.4 and Lemma 20.5 above. Let us now focus on the case $k>1$. Without loss of generality, assume that

$$\Sigma \cap ([-2,-1] \times \partial A) = \{(t-2, a(\alpha), \frac{(j-1)(1-t)}{k}) \mid (t,\alpha) \in [0,1]^2, j \in \underline{k}\}$$

and change the definition of Σ^+ for the proof, without loss of generality, so that

$$\Sigma^+ \cap ([-1,0] \times \partial A) = k([-1,0] \times a \times \{1\})$$

and

$$\Sigma^+ \cap ([-2,-1] \times \partial A) = \{(t-2, a(\alpha), \frac{(j-\frac{1}{2})(1-t)}{k} + t) \mid (t,\alpha) \in [0,1]^2, j \in \underline{k}\}$$

as in Figure 20.1, which represents $\Sigma \cap ([-2,0] \times p(a) \times [-1,1])$ as the thick lines and $\Sigma^+ \cap ([-2,0] \times p(a) \times [-1,1])$ as the thin lines when $k=3$.

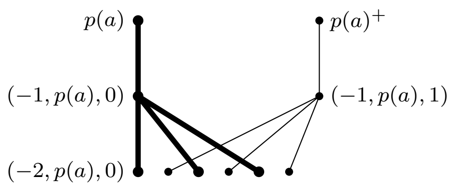


Figure 20.1: $\Sigma \cap ([-2,-1] \times p(a) \times [-1,1])$ and $\Sigma^+ \cap ([-2,-1] \times p(a) \times [-1,1])$

Recall $\Sigma_{-2} = \Sigma \cap A_{-2}$, $\partial\Sigma_{-2} = \cup_{j=1}^k \left(\{-2\} \times a \times \{\frac{j-1}{k}\}\right)$, and let Σ^+_{-2} be a parallel copy of Σ on its positive side with boundary $\partial(-\Sigma^+ \cap ([-2,-1[\times \partial A))$.

Glue abstract disks D_j with respective boundaries $\{-2\} \times (-a) \times \{\frac{j-1}{k}\}$ on $\partial\Sigma_{-2}$ (resp. D_j^+ with boundaries $\{-2\} \times (-a) \times \{\frac{j-\frac{1}{2}}{k}\}$ on $\partial\Sigma^+_{-2}$), and let S (resp. S^+) be the obtained closed surface. For $j=1,\dots,k$, set $p_j = (-2, p(a), \frac{j-1}{k}) \in \partial A_{-2}$ and $p_j^+ = (-2, p(a), \frac{j-\frac{1}{2}}{k})$. Then it follows from Lemma 20.3 that

$$C(S) = \mathrm{diag}(S \times S^+) - p_1 \times S^+ - S \times p_k^+ - \sum_{i=1}^{2g} c_i \times c_i^{*+}$$

is null-homologous in $H_2(S \times S^+)$. Choose closed representatives c_i of the classes c_i of Lemma 20.3 in the interior of Σ_{-2} such that $\left(\Sigma_{-2} \setminus \cup_{i=1}^{2g} c_i\right)$ is connected. Let $[p_1, p_j]$ (resp. $[p_j^+, p_k^+]$) denote a path in $\left(\Sigma_{-2} \setminus \cup_{i=1}^{2g} c_i\right)$ from p_1 to p_j (resp. in $\left(\Sigma_{-2}^+ \setminus \cup_{i=1}^{2g} c_i^+\right)$ from p_j^+ to p_k^+). Adding the null-homologous cycles

$$\partial(-[p_1, p_j] \times D_j^+) = p_1 \times D_j^+ - p_j \times D_j^+ + [p_1, p_j] \times \partial D_j^+,$$

$$\partial(D_j \times [p_j^+, p_k^+]) = D_j \times p_k^+ - D_j \times p_j^+ + \partial D_j \times [p_j^+, p_k^+],$$

for $j = 1, \dots, k$, and the null-homologous cycles of Lemma 20.4

$$(-C_{*,\leq}(D_j, D_j^+))$$

to $C(S)$ transforms it to the still null-homologous cycle

$$\begin{aligned} C(\Sigma_{-2}) = \; & \mathrm{diag}(\Sigma_{-2} \times \Sigma_{-2}^+) - p_1 \times \Sigma_{-2}^+ - \Sigma_{-2} \times p_k^+ - \textstyle\sum_{i=1}^{2g} c_i \times c_i^{*+} \\ & + \textstyle\sum_{j=1}^{k} (\partial D_j \times [p_j^+, p_k^+] + [p_1, p_j] \times \partial D_j^+ + \partial D_j \times_{p(a),\leq} \partial D_j^+). \end{aligned}$$

This cycle $C(\Sigma_{-2})$ can be naturally deformed continuously in $\Sigma \times \Sigma^+$ to the following still null-homologous cycle $C(\Sigma)$, where the level $\{-2\} \times \partial A$ is replaced by the level $\{0\} \times \partial A$. When s tends to (-1), the path $[\{s\} \times p_j^+, \{s\} \times p_k^+]$ becomes a loop $[p_j^+, p_k^+]_{-1}$ on $\Sigma_{-1}^+ = \Sigma^+ \cap A_{-1}$, and the path $[\{s\} \times p_1, \{s\} \times p_j]$ becomes a loop $[p_1, p_j]_{-1}$ on Σ_{-1}.

$$\begin{aligned} C(\Sigma) = \; & \mathrm{diag}(\Sigma \times \Sigma^+) - p(a) \times \Sigma^+ - \Sigma \times p(a)^+ - \textstyle\sum_{i=1}^{2g} c_i \times c_i^{*+} \\ & + \textstyle\sum_{j=1}^{k} \left((-a) \times [p_j^+, p_k^+]_{-1} + [p_1, p_j]_{-1} \times (-a^+)\right) \\ & + k\left(a \times_{p(a),\geq} a^+\right). \end{aligned}$$

Since a bounds $\frac{1}{k}\Sigma$, the cycle $(-a) \times [p_j^+, p_k^+]_{-1}$ is homologous to the cycle $\langle \frac{-1}{k}\Sigma, [p_j^+, p_k^+]_{-1}\rangle_A UA_{|*}$, in $C_2(A)$. Similarly, $[p_1, p_j]_{-1} \times (-a^+)$ is homologous to $\langle \frac{-1}{k}\Sigma^+, [p_1, p_j]_{-1}\rangle_A UA_{|*}$, in $C_2(A)$. Intersections occur where Σ and Σ^+ intersect, in $([-2, -1] \times \partial A)$ as shown in Figure 20.2, where the positive normal to Σ goes from left to right.

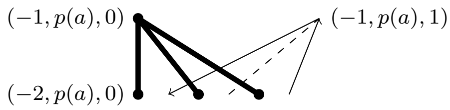


Figure 20.2: The intersection $\Sigma \cap \Sigma^+$ and the loop $[p_j^+, p_k^+]_{-1}$ ($j = 1$, $k = 3$)

$$\langle \frac{-1}{k}\Sigma, [p_j^+, p_k^+]_{-1}\rangle_A = \frac{k-j}{k}$$

and

$$\langle \frac{-1}{k}\Sigma^+, [p_1, p_j]_{-1}\rangle_A = \frac{j-1}{k}.$$

Therefore, Lemma 20.5 implies that the null-homologous cycle $C(\Sigma)$ is homologous to

$$\begin{array}{l}\mathrm{diag}(\Sigma \times \Sigma^+) - p(a) \times \Sigma^+ - \Sigma \times p(a)^+ \\ -\sum_{(j,\ell)\in\{1,\dots,g_A\}^2\setminus\mathrm{diag}} \langle \Sigma, \Sigma(a_j), \Sigma(a_\ell)\rangle z_j \times z_\ell + gUA_{|*} \\ +k\left(a \times_{p(a),\geq} a^+\right) + (k-1)UA_{|*},\end{array}$$

which is naturally homologous to $kF^2(\Sigma(a))$, which is therefore homologous to zero. $\square$

Lemma 20.7. *If A is a rational homology handlebody such that $H_1(A) = \oplus_{j=1}^{g(A)}\mathbb{R}[z_j]$, then*

$$H_3(C_2(A);\mathbb{R}) = \oplus_{j=1}^{g(A)}\mathbb{R}[UA_{|z_j}].$$

Proof: The configuration spaces $C_2(A)$ and $C_2(\mathring{A})$ have the same homotopy type, which is the homotopy type of $\mathring{A}^2 \setminus \mathrm{diag}$, $H_3(\mathring{A}^2) = H_4(\mathring{A}^2) = 0$, $H_4(\mathring{A}^2, \mathring{A}^2 \setminus \mathrm{diag}) \cong H_4(\mathring{A}^2 \times B^3, \mathring{A}^2 \times S^2) = \oplus_{j=1}^{g(A)}\mathbb{R}[z_j \times B^3]$. $\square$

Lemma 20.8. *Let $i \in \underline{x}$. For any $j \in \underline{g_i}$, assume that the chains $\Sigma(a^i_j)$ and $\Sigma'(a^i_j)$ defined before Proposition 18.35 intersect $[-1,4]\times\partial A^{(i)}$ as $[-1,4]\times a^i_j$ and that they may be expressed as $\frac{1}{k'}\Sigma'$ and $\frac{1}{k}\Sigma$, respectively, for immersed surfaces Σ and Σ', which intersect Int$(A^{(i)}_{-1})$ and Int$(A^{(i)\prime}_{-1})$, respectively, as embedded surfaces (as before Lemma 20.5). Fix $p(a^i_j)$ on $\{4\}\times a^i_j$.*

The classes of the cycles $F^2(\Sigma'(a^i_j))$ of $\partial C_2(A^{(i)\prime}_4)$, defined in Lemma 20.6, for $j \in \underline{g_i}$ generate the kernel of the map induced by the inclusion

$$H_2\left(D_A(\omega_0(R_i))\right) \longrightarrow H_2(C_2(A^{(i)\prime}_4))$$

for the domain $D_A(\omega_0(R_i))$ defined before Lemma 20.2.

Proof: First note that $D_A(\omega_0(R_i))$ is homotopically equivalent by retraction to $\partial C_2(A^{(i)\prime}_4)$. The cycles $F^2(\Sigma'(a^i_j))$ sit in $\partial C_2(A^{(i)\prime}_4))$ and bound chains $G^3(a^i_j)$ in $C_2(A^{(i)\prime}_4)$ according to Lemma 20.6. These chains can be assumed to be transverse to the boundary, so that these chains satisfy

$$\langle [G^3(a^i_j)], [UA^{(i)\prime}_{|z^i_k}]\rangle_{C_2(A^{(i)\prime}_4)} = \pm\langle F^2(\Sigma'(a^i_j)), UA^{(i)\prime}_{|\{0\}\times z^i_k}\rangle_{\partial C_2(A^{(i)\prime}_4)} = \pm\delta_{jk}.$$

Therefore, Poincaré duality and Lemma 20.7 imply that

$$H_3(C_2(A_4^{(i)\prime}), \partial C_2(A_4^{(i)\prime})) = \oplus_{j=1}^{g_i}\mathbb{R}[G^3(a_j^i)].$$

The cycle $[G^3(a_j^i)]$ of $\left(C_2(A_4^{(i)\prime}), \partial C_2(A_4^{(i)\prime})\right)$ is mapped to $[F^2(\Sigma'(a_j^i))]$ by the boundary map of the long exact sequence associated to $\left(C_2(A_4^{(i)\prime}), \partial C_2(A_4^{(i)\prime})\right)$.
$\square$

Proof of Lemma 20.2: According to Lemma 20.8, it suffices to prove that $\int_{F^2(\Sigma'(a_j^i))}\omega_0(R_i) = 0$ for any $i \in \underline{x}$, and for any $j \in \underline{g_i}$. Fix $i \in \underline{x}$ and $j \in \underline{g_i}$. Set $a = \{4\} \times a_j^i$. Let F' denote the cycle $F^2(\Sigma'(a))$ of $\partial C_2(A_4^{(i)\prime})$ associated to $\Sigma' = k'\Sigma'(a)$ and to $p(a) = p(a_j^i)$, and let F denote the cycle $F^2(\Sigma(a))$ of $\partial C_2(A_4^{(i)})$ associated similarly to $\Sigma(a)$ and to $p(a)$.

$$\begin{aligned} F' = \; & T(a) + a \times_{p(a),\geq} a^+ - p(a) \times \tfrac{1}{k'}\Sigma'^+ - \tfrac{1}{k'}\Sigma' \times p(a)^+ \\ & + \tfrac{1}{k'}s_+(\Sigma') + e(\Sigma'(a))[UA_{|*}] \\ & - \textstyle\sum_{(p,q)\in\{1,\dots g_i\}^2\setminus \mathrm{diag}} \langle \Sigma'(a), \Sigma'(a_p^i), \Sigma'(a_q^i)\rangle (\{4\} \times z_p^i) \times (\{4\} \times z_q^i) \end{aligned}$$

Set $\Sigma'^+_{-1} = \Sigma'^+ \cap A_{-1}^{(i)\prime}$. The integral of $\omega_0(R_i)$ along

$$\left(-p(a) \times \frac{1}{k'}\Sigma'^+_{-1} - \frac{1}{k'}\Sigma'_{-1} \times p(a)^+\right)$$

is zero because of the prescribed form of $\omega_0(R_i)$ on

$$\left((\check{R} \setminus A_3^{(i)\prime}) \times A^{(i)\prime}\right) \cup \left(A^{(i)\prime} \times (\check{R} \setminus A_3^{(i)\prime})\right).$$

Similarly,

$$\int_{-p(a)\times\Sigma^+_{-1}(a_j^i) - \Sigma_{-1}(a_j^i)\times p(a)^+} \omega = 0,$$

where ω is the form of Proposition 20.1. The part

$$\begin{aligned} C = \; & T(a) + a \times_{p(a),\geq} a^+ - \left(p(a) \times \tfrac{1}{k'}(\Sigma'^+ \setminus \Sigma'^+_{-1})\right) - \left(\tfrac{1}{k'}(\Sigma' \setminus \Sigma'_{-1}) \times p(a)^+\right) \\ & + \tfrac{1}{k'}s_+(\Sigma' \setminus \Sigma'_0) \end{aligned}$$

of F' (or F) sits in the intersection of $C_2(A_4^{(i)})$ and $C_2(A_4^{(i)\prime})$, inside which $\omega = \omega_0(R_i)$. So

$$\int_C \omega = \int_C \omega_0(R_i).$$

Similarly, for any $(p,q) \in \{1,\dots g_i\}^2 \setminus \mathrm{diag}$,

$$\int_{(\{4\}\times z_p^i)\times(\{4\}\times z_q^i)} \omega_0(R_i) = \int_{(\{4\}\times z_p^i)\times(\{4\}\times z_q^i)} \omega.$$

This integral is $lk(z_p^i, z_q^i)$. Since $lk(z_p^i, z_q^i) = lk(z_q^i, z_p^i)$, and since

$$\langle \Sigma'(a), \Sigma'(a_p^i), \Sigma'(a_q^i)\rangle = -\langle \Sigma'(a), \Sigma'(a_q^i), \Sigma'(a_p^i)\rangle,$$

the integral of $\omega_0(R_i)$ along

$$\sum_{(p,q)\in\{1,\dots g_i\}^2\setminus\mathrm{diag}} \langle \Sigma'(a), \Sigma'(a_p^i), \Sigma'(a_q^i)\rangle(\{4\}\times z_p^i)\times(\{4\}\times z_q^i)$$

is equal to zero. The integral of ω along

$$\sum_{(p,q)\in\{1,\dots g_i\}^2\setminus\mathrm{diag}} \langle \Sigma(a), \Sigma(a_p^i), \Sigma(a_q^i)\rangle(\{4\}\times z_p^i)\times(\{4\}\times z_q^i)$$

vanishes similarly.

Since $\int_F \omega = 0$,

$$\begin{aligned}\int_{F'} \omega_0(R_i) &= \int_{F'} \omega_0(R_i) - \int_F \omega \\ &= \int_{\frac{1}{k'}s_+(\Sigma'_0)} \omega_0(R_i) - \int_{s_+(\Sigma_0(a_j^i))} \omega + e(\Sigma'(a)) - e(\Sigma(a)).\end{aligned}$$

When τ, which coincides with τ_i on $[0,4]\times\partial A^{(i)}$, maps e_3 to the positive normal to $\Sigma_0'^{+}$ along $\partial\Sigma_0'^{+}$, according to Proposition 19.18 and to Lemma 19.15,

$$\int_{\frac{1}{k'}s_+(\Sigma'_0)} \omega_0(R_i) = \frac{1}{2}d(\tau(.\times e_2), \{0\}\times a) + \frac{1}{2k'}\chi(\Sigma'_{-2}).$$

So

$$\int_{\frac{1}{k'}s_+(\Sigma'_0)} \omega_0(R_i) + e(\Sigma'(a)) = \frac{1}{2}d(\tau(.\times e_2), \{0\}\times a) + \frac{1}{2} = \int_{s_+(\Sigma_0(a_j^i))} \omega + e(\Sigma(a))$$

and $\int_{F'} \omega_0(R_i) = 0$. When τ does not map e_3 to the positive normal to $\Sigma_0'^{+}$ along $\partial\Sigma_0'^{+}$, perform a simultaneous homotopy on τ and τ_i to make this happen without changing

$$\int_{\frac{1}{k'}s_+(\Sigma'_0)} \omega_0(R_i) - \int_{s_+(\Sigma_0(a_j^i))} \omega.$$

Thus the above proof still implies that $\int_{F'} \omega_0(R_i) = 0$. $\square$

Chapter 21

Much more flexible definitions of $\mathcal{Z}$

21.1 More propagating forms associated to pseudo-parallelizations

In this section, we define non-necessarily homogeneous propagating forms, associated to pseudo-parallelizations, and we give more flexible definitions of Z, which involve these forms. In Section 21.2, we will define propagating chains associated to pseudo-parallelizations, which allow for discrete computations of Z associated to pseudo-parallelizations. Again, as in Chapter 11 and Section 17.1, the corresponding discrete definition of Z will be justified by using non-homogeneous propagating forms ε-dual (as in Definition 11.6) to these propagating chains.

Definition 21.1 (General boundary form associated to a pseudo-parallelization $\tilde{\tau}$)**.** Let A be an oriented 3-manifold with possible boundary, equipped with a pseudo-parallelization $\tilde{\tau} = (N(\gamma); \tau_e, \tau_b)$ as in Definition 19.5. Let ω_s be a 2–form of S^2 invariant under the rotations around the vertical axis, such that $\int_{S^2} \omega_s = 1$. Let ω_i be a 2–form of S^2 such that $\int_{S^2} \omega_i = 1$. Let $\eta_{i,s,1}$ be a 1-form of S^2 such that $\omega_i = \omega_s + d\eta_{i,s,1}$. Let $\varepsilon \in]0, 1/2[$ be the small positive number of Definitions 19.5 and 19.9. Let $\varepsilon_i \in]0, \frac{\varepsilon}{2}[$ and let k be a large integer greater than 3. Let $p(\tau_b)$ denote the projection from $UN(\gamma)$ to S^2 induced by τ_b.

$$p(\tau_b)(\tau_b(t, c, u; X \in S^2)) = X.$$

Let $\eta_{i,s}$ be a 1-form on $U\ (N(\gamma) = [a, b] \times \gamma \times [-1, 1])$ that pulls back through

$p_{[a,b]} \times p_{[-1,1]} \times p(\tau_b)$ and such that $\eta_{i,s} =$

$$\begin{cases} p(\tau_b)^*(\eta_{i,s,1}) & \text{on } [b-\varepsilon_i+\varepsilon_i^k, b] \times \gamma \times [-1,1] \times S^2 \\ & \text{and on } [a,b] \times \gamma \times N(\partial[-1,1]) \times S^2 \\ \frac{p(\tau_b \circ \mathcal{T}_\gamma^{-1})^*(\eta_{i,s,1})}{2} & \\ +\frac{p(\tau_b \circ F(\gamma,\tau_b)^{-1})^*(\eta_{i,s,1})}{2} & \text{on } [a, b-\varepsilon_i-\varepsilon_i^k] \times \gamma \times [-1,1] \times S^2, \end{cases}$$

where $N(\partial[-1,1]) = [-1,1] \setminus [-1+\varepsilon, 1-\varepsilon]$.

Define $\omega = \omega(\tilde{\tau}, \omega_i, k, \varepsilon_i, \eta_{i,s})$ on UA to be

$$\begin{cases} p(\tau_e)^*(\omega_i) & \text{on } U\left(A \setminus \left([a, b-\varepsilon_i+\varepsilon_i^k] \times \gamma \times [-1,1]\right)\right) \\ \quad \frac{p(\tau_b \circ \mathcal{T}_\gamma^{-1})^*(\omega_i)}{2} & \\ + \quad \frac{p(\tau_b \circ F(\gamma,\tau_b)^{-1})^*(\omega_i)}{2} & \text{on } U([a, b-\varepsilon_i-\varepsilon_i^k] \times \gamma \times [-1,1]) \\ p(\tau_b)^*(\omega_s) + d\eta_{i,s} & \text{on } U([b-\varepsilon_i-2\varepsilon_i^k, b-\varepsilon_i+2\varepsilon_i^k] \times \gamma \times]-1,1[), \end{cases}$$

with the notation of Definition 19.9, where τ_e is extended to $U([b-\varepsilon_i+\varepsilon_i^k, b] \times \gamma \times [-1,1])$, so that it coincides with τ_b, there.

Lemma 21.2. *Definition 21.1 of $\omega(\tilde{\tau}, \omega_i, k, \varepsilon_i, \eta_{i,s})$ is consistent.*

Furthermore, we have the following complement, which will not be used in this book, but which is useful for the study of equivariant invariants as in [Les13]. Let c and d be two elements of $]-1,1[$ such that $c < d$. Let v_3 denote the projection on the third coordinate in $\mathbb{R}^3$ and let $S^2_{]c,d[} = \{X \in S^2 \mid v_3(X) \in]c,d[\}$. If ω_s and ω_i are compactly supported in $S^2_{]c,d[}$, then $\eta_{i,s,1}$ can be chosen so that it is compactly supported in $S^2_{]c,d[}$, too, and $\eta_{i,s}$ can be chosen so that it is compactly supported in $[a,b] \times \gamma \times [-1,1] \times S^2_{]c,d[}$.

Proof: The existence of $\eta_{i,s,1}$ compactly supported in $S^2_{]c,d[}$, when ω_s and ω_i are compactly supported in $S^2_{]c,d[}$, comes from the fact that $H^2_c(S^2_{]c,d[}) = \mathbb{R}$, according to Theorem B.3.

Extending $\eta_{i,s}$ to

$$C(\times\gamma) \times S^2 = [b-\varepsilon_i-\varepsilon_i^k, b-\varepsilon_i+\varepsilon_i^k](\times\gamma) \times [-1+\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}] \times S^2$$

is easy: Cover S^2 by three open spaces $S^2_{]c,d[}$, $N(N) = \{X \in S^2 \mid v_3(X) > \frac{c+d}{2}\}$ and $N(S) = \{X \in S^2 \mid v_3(X) < \frac{c+d}{2}\}$. Pick a corresponding partition of unity $(\chi_{S^2_{]c,d[}}, \chi_{N(S)}, \chi_{N(N)})$ of functions compactly supported on $S^2_{]c,d[}$, $N(N)$ and $N(S)$, respectively, whose sum is one. On the product by $C(\times\gamma)$ of a space S, which stands for $S^2_{]c,d[}$, $N(N)$ or $N(S)$, the form $\eta_{i,s}$ is a combination of basic standard one-forms. So it suffices to smoothly extend the $\mathbb{R}$-valued coordinate functions of the forms from $(\partial(C(\times\gamma)))\times S$ to $C(\times\gamma)\times S$, in order to obtain $\eta_{i,s}$ with the wanted properties, so that $\eta_{i,s}$ is compactly supported in $[a,b]\times\gamma\times[-1,1]\times S^2_{]c,d[}$ when $\eta_{i,s,1}$ is compactly supported in $S^2_{]c,d[}$.

Now, Lemma 19.10 ensures that the definitions match (on $[b-\varepsilon_i-2\varepsilon_i^k, b-\varepsilon_i-\varepsilon_i^k]\times\gamma\times]-1,1[$) for any forms that behave as in the definition, since any 2-form ω_s invariant under the rotations around the vertical axis may be expressed as $(\beta\circ v_3)\omega_{S^2}$. $\square$

Definition 21.3. Let A be an oriented 3-manifold equipped with a pseudo-parallelization $\tilde{\tau}$. A *boundary form* of $(A,\tilde{\tau})$ is a 2–form on UA that may be expressed as $\omega(\tilde{\tau},\omega_i,k,\varepsilon_i,\eta_{i,s})$ for some $(\omega_i,k,\varepsilon_i,\eta_{i,s})$ as in Definition 21.1. When $\eta_{i,s,1}$, ω_s and ω_i are compactly supported in $S^2_{]c,d[}$, and when $\eta_{i,s}$ is compactly supported in $[b-\varepsilon_i-2\varepsilon_i^k, b-\varepsilon_i+2\varepsilon_i^k]\times\gamma\times[-1,1]\times S^2_{]c,d[}$, it is said to be *adapted to* $S^2_{]c,d[}$.

Note that Definition 21.1 coincides with Definition 19.9 of a homogeneous boundary form when $\omega_i=\omega_s=\omega_{S^2}$, and when $\eta_{i,s}=0$. Also note that $\omega(\tilde{\tau},\omega_i,k,\varepsilon_i,\eta_{i,s})=\omega(\tilde{\tau},\omega_s,k,\varepsilon_i,0)+d\eta_{i,s}$ on $UN(\gamma)$.

Lemma 21.4. *Under the assumptions of Proposition 19.18, for any boundary form ω of $(M,\tilde{\tau})$ as in Definition 21.3,*

$$\int_{s_+(\Sigma)}\omega=\frac{1}{2}\chi\left(\tilde{\tau}(.\times e_2)_{|\partial\Sigma};\Sigma\right)$$

and

$$\int_{s_-(\Sigma)}\omega=-\frac{1}{2}\chi\left(\tilde{\tau}(.\times e_2)_{|\partial\Sigma};\Sigma\right).$$

Proof: Thanks to Lemma 19.16, Lemma 21.4 is true when Σ does not meet $N(\gamma)$. Therefore, as in the proof of Proposition 19.18, it suffices to prove Lemma 21.4 when Σ is a meridian of the link γ of the pseudo-parallelization $\tilde{\tau}$. Let us treat this case. When $\omega_i=\omega_s$ and $\eta_{i,s}=0$, the proof of Proposition 19.18 applies. In general, let $\tilde{\omega}$ be the form obtained from ω by changing ω_i to ω_s and $\eta_{i,s}$ to 0. The form ω may be written as $\tilde{\omega}+d\eta_{i,s}$ on $UN(\gamma)$, where $\eta_{i,s}$ is expressed as $p(\tau_e)^*(\eta_{i,s,1})$ along $s_+(\Sigma)_{|\partial\Sigma}$. Since $p(\tau_e)$ maps $s_+(\Sigma)_{|\partial\Sigma}$ to a point, $\int_{s_+(\Sigma)}\omega-\int_{s_+(\Sigma)}\tilde{\omega}=\int_{s_+(\Sigma)_{|\partial\Sigma}}p(\tau_e)^*(\eta_{i,s,1})=0$. Similarly, $\int_{s_-(\Sigma)}\omega=\int_{s_-(\Sigma)}\tilde{\omega}$. $\square$

Theorem 19.21 generalizes as follows to these boundary forms.

Theorem 21.5. *Let A be a compact 3-manifold that embeds in a rational homology 3-ball. Assume that A is equipped with two pseudo-parallelizations τ_0 and τ_1 that coincide with a common genuine parallelization along a regular neighborhood of ∂A. For $i \in \underline{3n}$, let $\omega_{0,i}(\tau_0)$ be a boundary form of (A,τ_0) and let $\omega_{1,i}(\tau_1)$ be a boundary form of (A,τ_1) There exists a closed 2-form $\omega(i)$ on $[0,1]\times UA$ that restricts*

- *to $\{0\}\times UA$ as $\omega_{0,i}(\tau_0)$*
- *to $\{1\}\times UA$ as $\omega_{1,i}(\tau_1)$,*
- *to $[0,1]\times UA_{|\partial A}$ as $(Id_{[0,1]}\times p_{\tau_0})^*(\omega_{S,i})$ with respect to a closed 2-form $\omega_{S,i}$ on $[0,1]\times S^2$ and to the projection $Id_{[0,1]}\times p_{\tau_0}\colon [0,1]\times UA_{|\partial A}\to [0,1]\times S^2$.*

Let n be a natural integer. As in Corollary 9.4, set

$$z_n([0,1]\times UA;(\omega(i))_{i\in\underline{3n}}) = \sum_{\Gamma\in\mathcal{D}_n^c}\zeta_\Gamma\int_{[0,1]\times\check{S}_{V(\Gamma)}(TA)}\bigwedge_{e\in E(\Gamma)}p_e^*(\omega(j_E(e))[\Gamma].$$

Then

$$z_n([0,1]\times UA;(\omega(i))_{i\in\underline{3n}}) = \frac{p_1(\tau_0,\tau_1)}{4}\beta_n.$$

Proof: The existence of $\omega_{S,i}$ is proved in Lemma 9.1. The proof of the existence of the form $\omega(i)$ with its prescribed properties is obtained from the proof of Proposition 19.19 by replacing Proposition 19.18 with Lemma 21.4.

When $\tau_0=\tau_1$ and when $\omega_{0,i}(\tau_0)$ is a homogeneous boundary form, $\omega(i)$ can be chosen so that $\omega(i)$ is equal to

$$\begin{cases} p(\tau_e)^*(\omega_{S,i}) & \text{on } [0,1]\times U\left(A\setminus\left([a,b-\varepsilon_i+\varepsilon_i^k]\times\gamma\times[-1,1]\right)\right) \\ \quad\frac{p(\tau_b\circ\mathcal{T}_\gamma^{-1})^*(\omega_{S,i})}{2} & \\ +\ \frac{p(\tau_b\circ F(\gamma,\tau_b)^{-1})^*(\omega_{S,i})}{2} & \text{on } [0,1]\times U([a,b-\varepsilon_i-\varepsilon_i^k]\times\gamma\times[-1,1]) \\ p(\tau_b)^*(\omega_{S^2})+d(t\eta_{i,s}) & \text{on } [0,1]\times U([b-\varepsilon_i-\varepsilon_i^k,b-\varepsilon_i+\varepsilon_i^k]\times\gamma\times]-1,1[), \end{cases}$$

where t stands for the coordinate in $[0,1]$ and $\omega_{S,i}=\omega_{S^2}+d(t\eta_{i,s,1})$, with the notation of Definition 21.1.

In this case, the parts over $(A\setminus([a,b]\times\gamma\times[-1,1]))$ in

$$z_n([0,1]\times UA;(\omega(i))_{i\in\underline{3n}})$$

cancel because the forms do not depend on the factor A (which is factored out via the parallelization τ_e), and the parts over $([a,b]\times\gamma\times[-1,1])$ also cancel because the forms do not depend on the factor γ.

In this special case, we simply find $z_n([0,1] \times UA; (\omega(i))_{i\in\underline{3n}}) = 0$, as announced.

In general, as in the beginning of the proof of Proposition 19.20, and as in Corollary 9.4, $z_n([0,1]\times UA; (\omega(i)))$ depends only on the restriction of the $\omega(i)$ to $\partial\left([0,1] \times UA\right)$.

The above arguments can also be used to prove that

$$z_n([0,1] \times UA; (\omega(i))_{i\in\underline{3n}})$$

is independent of the forms $\omega_{S,i}$. So $z_n([0,1]\times UA; (\omega(i))_{i\in\underline{3n}})$ depends only on the pairs $(\omega_{0,i}(\tau_0), \omega_{1,i}(\tau_1))$. Denote it by $z_n\left((\omega_{0,i}(\tau_0), \omega_{1,i}(\tau_1))_{i\in\underline{3n}}\right)$, observe

$$\begin{aligned} z_n\left((\omega_{0,i}(\tau_0), \omega_{2,i}(\tau_2))_{i\in\underline{3n}}\right) = & \quad z_n\left((\omega_{0,i}(\tau_0), \omega_{1,i}(\tau_1))_{i\in\underline{3n}}\right) \\ & + \; z_n\left((\omega_{1,i}(\tau_1), \omega_{2,i}(\tau_2))_{i\in\underline{3n}}\right) \end{aligned}$$

again, and conclude with the study of the special case, and with Theorem 19.21. □

Definition 21.6. A pseudo-parallelization of a rational homology $\mathbb{R}^3$ is said to be *asymptotically standard* when it coincides with τ_s on $\mathring{B}_{1,\infty}$ (as in Definition 3.6). Let $\check{R}$ be a rational homology $\mathbb{R}^3$ equipped with an asymptotically standard pseudo-parallelization $\tilde{\tau}$. A *propagating form* of $(C_2(R), \tilde{\tau})$ is a propagating form of $C_2(R)$ (as in Definition 3.11) that coincides with a boundary form (of Definition 21.3) of $(\check{R}, \tilde{\tau})$ on $U\check{R}$.

Theorem 7.39 generalizes as follows to pseudo-parallelizations:

Theorem 21.7. *Let $\check{R}$ be a rational homology $\mathbb{R}^3$ equipped with an asymptotically standard pseudo-parallelization $\tau = (N(\gamma); \tau_e, \tau_b)$. Let $L\colon \mathcal{L} \hookrightarrow \check{R}\setminus N(\gamma)$ be a link embedding, which is straight with respect to $\tau_{|\check{R}\setminus N(\gamma)}$. Let $L_{\parallel,\tau}$ denote the parallel of L induced by τ. For any $i \in \underline{3n}$, let $\omega(i)$ be a propagating form of $(C_2(R), \tau)$. Then*

$$Z^s_n(\check{R}, L, (\omega(i))_{i\in\underline{3n}}) = \sum_{\Gamma\in\mathcal{D}^e_n(\mathcal{L})} \zeta_\Gamma I(R, L, \Gamma, (\omega(i))_{i\in\underline{3n}})[\Gamma] \in \mathcal{A}_n(\mathcal{L}).$$

is independent of the chosen $\omega(i)$. Set $Z^s_n(\check{R}, L, \tau) = Z^s_n(\check{R}, L, (\omega(i))_{i\in\underline{3n}})$ and $Z^s(\check{R}, L, \tau) = \left(Z^s_n(\check{R}, L, \tau)\right)_{n\in\mathbb{N}}$. Then

$$Z^s(\check{R}, L, \tau) = Z(\check{R}, L, \tau) = \mathcal{Z}^f(\check{R}, L, L_{\parallel,\tau}) \exp\left(\frac{p_1(\tau)}{4}\beta\right),$$

where $Z(\check{R}, L, \tau)$ is defined in Theorem 19.13 (as in Theorem 7.20), with Definition 19.8 for $p_1(\tau)$, and Definition 7.40 for $\mathcal{Z}^f(\check{R}, L, L_{\parallel,\tau})$.

Theorem 21.7 is a consequence of the following generalization of Theorem 16.7 to pseudo-parallelizations:

Theorem 21.8. *Let $L\colon \mathcal{L} \hookrightarrow R(\mathcal{C})$ denote the long tangle associated with a tangle in a rational homology cylinder equipped with a pseudo-parallelization $\tau = (N(\gamma); \tau_e, \tau_b)$ that coincides with a genuine parallelization in the neighborhood of the image of L. Let $\{K_j\}_{j\in I}$ be the set of components of L. Assume that the bottom (resp. top) configuration of L is represented by a map $y^-\colon B^- \to D_1$ (resp. $y^+\colon B^+ \to D_1$).*

Let $N \in \mathbb{N}$. For $i \in \underline{3N}$, let $\tilde{\omega}(i, S^2) = (\tilde{\omega}(i,t,S^2))_{t\in[0,1]}$ be a closed 2–form on $[0,1]\times S^2$ such that $\int_{S^2}\tilde{\omega}(i,0,S^2) = 1$. There exists a closed 2–form $\tilde{\omega}(i) = (\tilde{\omega}(i,t))_{t\in[0,1]}$ on $[0,1]\times C_2(R(\mathcal{C}))$ such that

$$\tilde{\omega}(i) = (\mathbf{1}_{[0,1]} \times p_{\tau_e})^*(\tilde{\omega}(i,S^2))$$

on $[0,1]\times(\partial C_2(R(\mathcal{C}))\setminus UN(\gamma))$, and the restriction of $\tilde{\omega}(i,t)$ to $U\check{R}(\mathcal{C})$ is a boundary form of $(\check{R}(\mathcal{C}),\tau)$, as in Definition 21.3, for all $t\in[0,1]$.

For such a family $(\tilde{\omega}(i))_{i\in\underline{3N}}$, and for a subset A of $\underline{3N}$ with cardinality $3k$, set

$$Z(\mathcal{C},L,\tau,A,(\tilde{\omega}(i,t))_{i\in A}) = \sum_{\Gamma\in\mathcal{D}^e_{k,A}(\mathcal{L})} \zeta_\Gamma I(\mathcal{C},L,\Gamma,(\tilde{\omega}(i,t))_{i\in A})[\Gamma] \in \mathcal{A}_k(\mathcal{L}).$$

Then

$$Z(\mathcal{C},L,\tau,A)(t) = Z(\mathcal{C},L,\tau,A,(\tilde{\omega}(i,t))_{i\in A})$$

depends only on $(\tilde{\omega}(i,t,S^2))_{i\in A}$ for any t (and on $(\mathcal{C},L,\tau)$).

It will be denoted by $Z(\mathcal{C},L,\tau,A,(\tilde{\omega}(i,t,S^2))_{i\in A})$. When $\tilde{\omega}(i,t,S^2)$ is the standard homogeneous form ω_{S^2} on S^2 for any i, and for all $t\in[0,1]$, $Z(\mathcal{C},L,\tau,.)(t)$ maps any subset of $\underline{3N}$ with cardinality $3k$ to $Z_k(\mathcal{C},L,\tau)$.

Furthermore, with the notation of Definition 16.8, for any orientation of L,

$$Z(\mathcal{C},L,\tau,.)(t) =$$

$$\left(\left(\prod_{j\in I}\widetilde{hol}_{[0,t]}(\eta(.,U_j^+))\sharp_j\right)\widetilde{hol}_{[t,0]\times y^-}(\eta_{B^-,.})Z(\mathcal{C},L,\tau,.)(0)\widetilde{hol}_{[0,t]\times y^+}(\eta_{B^+,.})\right)_{\amalg},$$

where $U_j^+ = p_\tau(U^+K_j)$.

Proof: Fix $i\in\underline{3N}$. Pick a one-form $\tilde{\eta}_{i,s,1}$ on $[0,1]\times S^2$ such that $\tilde{\omega}(i,S^2) = p_{S^2}^*(\omega_{S^2}) + d\tilde{\eta}_{i,s,1}$. Use this form to construct a one-form $\tilde{\eta}_{i,s}(t)$ on $[0,1]\times U([a,b]\times\gamma\times[-1,1])$ that pulls back through $p_{[0,1]}\times p_{[a,b]}\times p_{[-1,1]}\times p(\tau_b)$ and such that its restriction to $\{t\}\times U(N(\gamma))$ satisfies the conditions of Definition 21.1, with respect to $\tilde{\eta}_{i,s,1|\{t\}\times U(N(\gamma))}$.

Next use $\tilde{\eta}_{i,s}$ to construct the restriction of $\tilde{\omega}(i)$ to $[0,1] \times \partial C_2(R(\mathcal{C}))$, such that $\tilde{\omega}(i,t)_{|\{t\}\times U\check{R}(\mathcal{C})} = \omega(\tau, \tilde{\omega}(i,t,S^2), k, \varepsilon_i, \tilde{\eta}_{i,s}(t))$. Now, the existence of $\tilde{\omega}(i)$ follows, as in Lemma 9.1.

To finish the proof of this theorem, first prove the variation formula that expresses $Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i,t))_{i\in A})$ as a function of $Z(\mathcal{C}, L, \tau, A, (\tilde{\omega}(i,0))_{i\in A})$ for forms $\tilde{\omega}(i,t)$ as in the statement. The proof of this variation formula is similar to the proof of Theorem 16.7.

Then note that when $\tilde{\omega}(i,0,S^2)$ is the standard homogeneous form ω_{S^2} on S^2 for any i, $Z(\mathcal{C}, L, \tau, .)(0)$ maps any subset of $\underline{3N}$ with cardinality $3k$ to $Z_k(\mathcal{C}, L, \tau)$, by the definition of Theorem 19.13. Thanks to Lemma 9.1, this proves that $Z(\mathcal{C}, L, \tau, A)(t)$ depends only on the forms $\tilde{\omega}(i,S^2)$ and therefore only on the forms $\tilde{\omega}(i,t,S^2)$. $\square$

Proof of Theorem 21.7: Since the factors $\widetilde{\mathrm{hol}}_{[0,t]}(\eta(., p_\tau(U^+K_j))$ in the formula of Theorem 21.8 vanish when L is a straight tangle with respect to τ, and since the factors $\widetilde{\mathrm{hol}}_{[t,0]\times y^-}(\eta_{B^-,.})$ and $\widetilde{\mathrm{hol}}_{[0,t]\times y^+}(\eta_{B^+,.})$ vanish when L is a link, Theorem 21.7 follows from Theorem 21.8, Theorem 19.13 and Lemma 7.37.

Remark 21.9. When $\tilde{\omega}(i,1,S^2)$ is the standard homogeneous form ω_{S^2} on S^2 for any i, the variation formula of Theorem 21.8 yields alternative expressions of Z.

21.2 Pseudo-sections associated to pseudo-parallelizations

For an asymptotically standard parallelization τ of a punctured rational homology 3-sphere $\check{R}$, a propagating chain of $(C_2(R), \tau)$ is defined in Definition 3.9 to be a 4–chain P of $C_2(R)$ such that $\partial P = p_\tau^{-1}(X)$ for some $X \in S^2$, where $p_\tau^{-1}(X) \subset \partial C_2(R)$, $p_\tau^{-1}(X) \cap \left(\partial C_2(R) \setminus U\check{R}\right)$ is independent of τ, and

$$p_\tau^{-1}(X) \cap U\check{R} = \tau(\check{R} \times \{X\})$$

The restriction of a section $\tau(\check{R} \times X) = \tau(\check{R} \times \{X\})$ to a part A of $\check{R}$ is denoted by $s_\tau(A; X)$.

In this section, we define pseudo-sections $s_{\tilde{\tau}}(\check{R}; X)$ associated to pseudo-parallelizations $\tilde{\tau}$. A *propagating chain* of $(C_2(R), \tilde{\tau})$ is a 4–chain P of $C_2(R)$ such that

$$\partial P = \left(p_\tau^{-1}(X) \cap \left(\partial C_2(R) \setminus U\check{R}\right)\right) \cup s_{\tilde{\tau}}(\check{R}; X)$$

for some $X \in S^2$. Thus the pseudo-sections $s_{\tilde{\tau}}(\check{R}; X)$ will play the same role as the sections $s_\tau(\check{R}; X)$ in the more flexible definition of $\mathcal{Z}$, which will be presented in Section 21.3.

Definition 21.10 (Pseudo-sections $s_{\tilde{\tau}}(.;X)$)**.** Recall the map $F(\gamma,\tau_b)$ of Definition 19.9, and the notation of Definition 19.5.

Let $X \in S^2$ and let $S^1(X)$ be the circle (or point) in S^2 that lies in the plane orthogonal to the axis generated by $(0,0,1)$, that contains X. Let $G_2(X)$ be a 2-dimensional chain $G_2(X)$ in $[-1,1]\times S^1(X)$, whose boundary is $\{\left(u,\rho_{-\alpha(u)}(X)\right), u\in[-1,1]\} + \{\left(u,\rho_{\alpha(u)}(X)\right), u\in[-1,1]\} - 2[-1,1]\times\{X\}$, as in Figure 21.1.

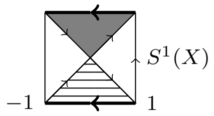


Figure 21.1: The chain $G_2(X)$ in $[-1,1]\times S^1(X)$

Then $s_{\tilde{\tau}}(A;X)$ is the following 3–cycle of $(UA, UA_{|\partial A})$

$$\begin{aligned} s_{\tilde{\tau}}(A;X) &= s_{\tau_e}(A\setminus \mathring{N}(\gamma);X) \\ &+ \ \frac{1}{2}\left(s_{\tau_b\circ\mathcal{T}_\gamma^{-1}}(N(\gamma);X) + s_{\tau_b\circ F(\gamma,\tau_b)^{-1}}(N(\gamma);X) + \{b\}\times\gamma\times G_2(X)\right), \end{aligned}$$

where τ_b and τ_e identify $UA_{|\{b\}\times\gamma\times[-1,1]}$ with $\{b\}\times\gamma\times[-1,1]\times S^2$ in the same way.

We also introduce small deformations of these sections, associated with ε_i such that $0<\varepsilon_i<\varepsilon$ (with respect to the ε of Definition 19.5), as follows. Let $N(\gamma,\varepsilon_i)=[a,b-\varepsilon_i]\times\gamma\times[-1,1]\times S^2$, then $s_{\tilde{\tau}}(A;X,\varepsilon_i)$ is the following 3–cycle of $(UA, UA_{|\partial A})$

$$\begin{aligned} s_{\tilde{\tau}}(A;X,\varepsilon_i) = \ & s_{\tau_e}(A\setminus \mathring{N}(\gamma,\varepsilon_i);X) \\ & +\frac{1}{2}\left(s_{\tau_b\circ\mathcal{T}_\gamma^{-1}}(N(\gamma,\varepsilon_i);X) + s_{\tau_b\circ F(\gamma,\tau_b)^{-1}}(N(\gamma,\varepsilon_i);X)\right) \\ & +\frac{1}{2}\{b-\varepsilon_i\}\times\gamma\times G_2(X), \end{aligned}$$

where τ_e is extended naturally over $[b-\varepsilon,b]\times\gamma\times[-1,1]$, so that it coincides with τ_b, there.

When Σ is a 2–chain that intersects $N(\gamma)$ along sections $N_c(\gamma)=[a,b]\times\{c\}\times[-1,1]$ (which are oriented as meridian disks of $(-\gamma)$), set

$$s_{\tilde{\tau}}(\Sigma;X) = s_{\tilde{\tau}}(A;X)\cap UA_{|\Sigma}.$$

So $s_{\tilde{\tau}}(N_c(\gamma);X)$ equals

$$\frac{s_{\tau_b\circ\mathcal{T}_\gamma^{-1}}(N_c(\gamma);X) + s_{\tau_b\circ F(\gamma,\tau_b)^{-1}}(N_c(\gamma);X) - \{b\}\times\{c\}\times G_2(X)}{2}.$$

Note that $G_2(e_3)$ lies in $[-1,1]\times\{e_3\}$, so $s_\tau(A;\pm e_3)$ is simply given by

$$\begin{array}{rl} s_\tau(A;\pm e_3) = & s_{\tau_e}(A\setminus \mathring{N}(\gamma);\pm e_3) \\ & +\frac{1}{2}\left(s_{\tau_b\circ \mathcal{T}_\gamma^{-1}}(N(\gamma);\pm e_3)+s_{\tau_b\circ F(\gamma,\tau_b)^{-1}}(N(\gamma);\pm e_3)\right). \end{array}$$

Below, we discuss common properties of homology classes of sections and pseudo-sections.

Lemma 21.11. *Let $W\in S^2$. Let Φ be a map from the unit disk D^2 of $\mathbb{C}$ to $SO(3)$ such that $\Phi(\exp(i\beta))$ is the rotation $\rho_{2\beta,W}$ with axis directed by W and with angle 2β, for $\beta\in[0,2\pi]$. Then the map*

$$\begin{array}{rccc} \Phi_W=\Phi(\cdot)(W)\colon & D^2 & \to & S^2 \\ & z & \mapsto & \Phi(z)(W) \end{array}$$

sends ∂D^2 to W, and the degree of the induced map from $D^2/\partial D^2$ to S^2 is (-1).

PROOF: First note that the above degree does not depend on Φ on the interior of D^2, since $\pi_2(SO(3))=0$. View the restriction of Φ_W to ∂D^2, as the path composition of the maps $(\beta\in[0,2\pi]\mapsto\rho_{\beta,W})$ and the inverse of $(\beta\in[0,2\pi]\mapsto\rho_{\beta,-W})$ (that is twice the first map). Consider an arc ξ of a great circle of S^2 from $-W$ to W, then Φ can be regarded as the map from $\xi\times[0,2\pi]$ to $SO(3)$ that maps (X,β) to $\rho_{\beta,X}$, so that the only preimage of $-W$ under Φ_W is (X_0,π), where $X_0\in\xi$ and $X_0\perp W$, and the local degree is easily seen to be (-1). ☐

Lemma 19.16 generalizes as follows to pseudo-parallelizations, with the notation of Definition 19.14.

Proposition 21.12. *Let $e_3=(0,0,1)\in\mathbb{R}^3$. Let Σ be a compact oriented surface immersed in a 3-manifold M equipped with a pseudo-parallelization $\tau=(N(\gamma);\tau_e,\tau_b)$, which is an actual parallelization around the boundary $\partial\Sigma$ of Σ such that $\tau(.\times e_3)$ is a positive normal to Σ along $\partial\Sigma$. Let $s_+(\Sigma)\subset UM$ be the section of $UM_{|\Sigma}$ in UM associated with the positive normal field n to Σ, which coincides with $\tau(\partial\Sigma\times e_3)$ on $\partial\Sigma$. Then*

$$2(s_+(\Sigma)-s_\tau(\Sigma;e_3))-\chi\left(\tau(.\times e_2)_{|\partial\Sigma};\Sigma\right)UM_{|*}$$

and

$$2(s_-(\Sigma)-s_\tau(\Sigma;-e_3))+\chi\left(\tau(.\times e_2)_{|\partial\Sigma};\Sigma\right)UM_{|*}$$

are cycles that are null-homologous in $UM_{|\Sigma}$.

Proof: Lemma 19.16 gives the result for actual parallelizations. Let us prove it for pseudo-parallelizations. Since $s_\tau(\Sigma; \pm e_3) = s_\tau(M; \pm e_3) \cap UM_{|\Sigma}$ is well defined for any surface Σ, as soon as the above intersection is transverse, there is no loss of generality in assuming that Σ meets $N(\gamma)$ along sections $N_c(\gamma)$, for some $c \in \gamma$.

Since the cycles of the statement behave well under gluing (or cutting) surfaces along curves that satisfy the boundary assumptions, and since these assumptions are easily satisfied after a homotopy as in Definition 19.17 (when the cutting process is concerned), it suffices to prove the lemma for a meridian disk Σ of γ. For such a disk $s_\tau(\Sigma; e_3)$ is the average of two genuine sections corresponding to trivializations τ_1 and τ_2. So, using the notation of Lemma 19.15,

$$2(s_+(\Sigma) - s_\tau(\Sigma; e_3)) - \left(\frac{d(\tau_1(. \times e_2), \partial\Sigma) + d(\tau_2(. \times e_2), \partial\Sigma)}{2} + \chi(\Sigma)\right) UM_{|*}$$

is a null-homologous cycle. Since the exterior trivialization τ_e is such that

$$d(\tau_e(. \times e_2), \partial\Sigma) = \frac{d(\tau_1(. \times e_2), \partial\Sigma) + d(\tau_2(. \times e_2), \partial\Sigma)}{2},$$

we are done for the first cycle. The second cycle is treated similarly. $\square$

The obvious property that *genuine sections $s_\tau(\Sigma; X)$ and $s_\tau(\Sigma; Y)$ corresponding to distinct X and Y of S^2 are disjoint* generalizes as follows for pseudo-sections.

Lemma 21.13. *Let Σ be an oriented surface embedded in a 3-manifold A, equipped with a pseudo-parallelization $\tilde{\tau} = (N(\gamma); \tau_e, \tau_b)$, such that Σ intersects $N(\gamma)$ only along sections $N_{c_i}(\gamma) = [a, b] \times \{c_i\} \times [-1, 1]$, in the interior of Σ. Then if $Y \in S^2$ and if $Z \in S^2 \setminus S^1(Y)$, the algebraic intersection $\langle s_{\tilde{\tau}}(\Sigma; Y), s_{\tilde{\tau}}(\Sigma; Z)\rangle_{UA_{|\Sigma}}$ of $s_{\tilde{\tau}}(\Sigma; Y)$ and $s_{\tilde{\tau}}(\Sigma; Z)$ in $UA_{|\Sigma}$ is zero.*

Proof: Recall that $S^1(X)$ denotes the circle in S^2 that lies in a plane orthogonal to the axis generated by $e_3 = (0, 0, 1)$, and that contains X. Let us consider the contribution to $\langle s_{\tilde{\tau}}(\Sigma; Y), s_{\tilde{\tau}}(\Sigma; Z)\rangle_{UA_{|\Sigma}}$ of an intersection point c of γ with Σ. Such a contribution may be expressed as

$$\pm\frac{1}{4}\left(\begin{array}{cc} & \langle s_{\tau_b \circ \mathcal{T}_\gamma^{-1}}(N_c(\gamma); Y), s_{\tau_b \circ F(\gamma,\tau_b)^{-1}}(N_c(\gamma); Z)\rangle_{N_c(\gamma)\times S^2} \\ + & \langle s_{\tau_b \circ F(\gamma,\tau_b)^{-1}}(N_c(\gamma); Y), s_{\tau_b \circ \mathcal{T}_\gamma^{-1}}(N_c(\gamma); Z)\rangle_{N_c(\gamma)\times S^2} \end{array}\right).$$

We prove that both intersection numbers are ± 1 and that their signs are opposite. Consider an arc ξ of great circle from e_3 to $-e_3$. Identify ξ with $[a, b]$ via an orientation-preserving diffeomorphism, so that the map F of

Definition 19.9 can be regarded as the map that maps $(V \in \xi, u \in [-1,1])$ to the rotation $\rho_{\alpha(u),V}$ with axis V and angle $\alpha(u)$. Then the intersection

$$\langle s_{\tau_b \circ F(\gamma,\tau_b)^{-1}}(N_c(\gamma);Z), s_{\tau_b \circ \mathcal{T}_\gamma^{-1}}(N_c(\gamma);Y)\rangle$$

in

$$UA_{|N_c(\gamma)} =_{\tau_b \circ F(\gamma,\tau_b)^{-1}} N_c(\gamma) \times S^2$$

is the algebraic intersection

$$\langle N_c(\gamma) \times \{Z\}, F(\gamma,\tau_b) \circ \mathcal{T}_\gamma^{-1}(N_c(\gamma) \times \{Y\})\rangle_{N_c(\gamma)\times S^2},$$

which is the degree of the map

$$\begin{array}{llll} f_Y\colon & \xi \times [-1,1] & \to & S^2 \\ & (V,u) & \mapsto & \rho_{\alpha(u),V} \circ \rho_{\alpha(-u),e_3}(Y) \end{array}$$

at Z, while the other intersection is the degree of the map f_Z at Y. The boundary of the image of f_Y is $-2S^1(Y)$, where $S^1(Y)$ is oriented as the boundary of the closure of the connected component of $S^2 \setminus S^1(Y)$ that contains e_3. Therefore, the degree increases by 2 from the component of $S^2 \setminus S^1(Y)$ that contains e_3 to the component of $(-e_3)$. On the other hand, the degree of f_Y on the component of e_3 is independent of $Y \neq e_3$ and, with the notation of Lemma 21.11, $\deg(f_{-e_3}) = \deg(\Phi_{-e_3}) = -1$. So the degree of f_Y on the component of $-e_3$, which is independent of $Y \neq (-e_3)$, is 1. Thus the degree of f_Y at Z and the degree of f_Z at Y are opposite to each other. □

Lemma 21.14. *Let A be a rational homology handlebody equipped with two pseudo-parallelizations $\tilde{\tau}_0$ and $\tilde{\tau}_1$ that coincide with the same genuine parallelization near the boundary of A. Let $X \in S^2$. Let $\eta \in]0,\varepsilon[$. There exists a rational 4-chain $H(\tilde{\tau}_0,\tilde{\tau}_1,X,\eta)$ in UA such that*

$$\partial H(\tilde{\tau}_0,\tilde{\tau}_1,X,\eta) = s_{\tilde{\tau}_1}(A;X,\eta) - s_{\tilde{\tau}_0}(A;X,\eta).$$

Proof: Let us show that $C = s_{\tilde{\tau}_1}(A;X,\eta) - s_{\tilde{\tau}_0}(A;X,\eta)$ vanishes in

$$H_3(UA;\mathbb{Q}) = H_1(A;\mathbb{Q}) \otimes H_2(S^2;\mathbb{Q}).$$

In order to do so, we prove that the algebraic intersections of C with $s_{\tilde{\tau}_0}(S;Y)$ vanish for 2–cycles S of $(A,\partial A)$ that generate $H_2(A,\partial A)$, for $Y \in S^2 \setminus S^1(X)$. Of course, $s_{\tilde{\tau}_0}(\partial A;X)$ does not intersect $s_{\tilde{\tau}_0}(S;Y)$. According to Lemma 21.13, $s_{\tilde{\tau}_0}(S;X)$ does not intersect $s_{\tilde{\tau}_0}(S;Y)$ in $UA_{|S}$ algebraically.

So $s_{\tilde{\tau}_0}(A;X)$ does not intersect $s_{\tilde{\tau}_0}(S;Y)$ algebraically, either. Since Proposition 21.12 guarantees that $(s_{\tilde{\tau}_0}(S;X)-s_{\tilde{\tau}_1}(S;X))$ bounds in $UA_{|S}$, $s_{\tilde{\tau}_1}(S;X)$ does not intersect $s_{\tilde{\tau}_0}(S;Y)$ algebraically in $UA_{|S}$. So $s_{\tilde{\tau}_1}(A;X)$ does not intersect $s_{\tilde{\tau}_0}(S;Y)$ algebraically. This proves the existence of the wanted 4–chain $H(\tilde{\tau}_0,\tilde{\tau}_1,X,\eta)$. □

21.3 Definition of Z with respect to pseudo-sections

Definition 21.15. Let $\check{R}$ be a rational homology $\mathbb{R}^3$ equipped with an asymptotically standard pseudo-parallelization $\tilde{\tau}$. A *propagating chain* of $(C_2(R),\tilde{\tau})$ is a propagating chain of $C_2(R)$ (as in Definition 3.11) whose boundary intersects $U\check{R}$ as a chain $s_{\tilde{\tau}}(\check{R};X,\varepsilon_i)$ as in Definition 21.10.

Again, for a given $s_{\tilde{\tau}}(\check{R};X,\varepsilon_i)$, a propagating chain whose boundary intersects $U\check{R}$ as $s_{\tilde{\tau}}(\check{R};X,\varepsilon_i)$ exists because $H_3(C_2(R);\mathbb{Q})=0$.

Lemma 21.16. *For any rational homology $\mathbb{R}^3$ $\check{R}$ equipped with an asymptotically standard pseudo-parallelization $\tilde{\tau}$, for any positive number α, and for any propagating chain P of $(C_2(R),\tilde{\tau})$, transverse to $\partial C_2(R)$, there exists a propagating form of $(C_2(R),\tilde{\tau})$ (as in Definition 21.6) that is α-dual to P (as in Definition 11.6).*

PROOF: There is no loss of generality in assuming that P intersects a collar $[-1,0]\times\partial C_2(R)$ as $[-1,0]\times\partial P$.

Assume that the boundary of P intersects $U\check{R}$ as a chain $s_{\tilde{\tau}}(\check{R};X,\varepsilon_i)$. Let ω_i be a 2–form such that $\int_{S^2}\omega_i=1$, which is α_X-dual to X for some small $\alpha_X\in]0,\alpha[$.

Set $B(\varepsilon_i)=[b-\varepsilon_i-\varepsilon_i^k,b-\varepsilon_i+\varepsilon_i^k]$.

Forms α-dual to a given chain can be constructed as in Lemma B.4. In particular, this process can be used to construct a form ω_1 α-dual to P, which pulls back through $\partial C_2(R)$ on $[-1,0]\times\partial C_2(R)$, which may be expressed as in Definition 21.1 outside $[-1,0]\times U\left(B(\varepsilon_i)\times\gamma\times[-1,1]\right)$, and which factors through $B(\varepsilon_i)\times[-1,1]\times S^2$ on

$$[-1,0]\times U\left(B(\varepsilon_i)\times\gamma\times[-1,1]\right).$$

It remains to see that ω_1 can be assumed to have the form prescribed by Definition 21.1 on $[-1,0]\times U\left(B(\varepsilon_i)\times\gamma\times[-1,1]\right)$.

We also have a form ω_2 as in Definition 21.6, which also pulls back through $\partial C_2(R)$ on $[-1,0]\times\partial C_2(R)$ and which coincides with ω_1 on $[-1,0]\times U\left(\check{R}\setminus(B(\varepsilon_i)\times\gamma\times[-1+\varepsilon,1-\varepsilon])\right)$.

The two forms factor through $[b-\varepsilon_i-2\varepsilon_i^k, b-\varepsilon_i+2\varepsilon_i^k]\times[-1,1]\times S^2$, and they are cohomologous there. So there exists a one-form η on $[b-\varepsilon_i-2\varepsilon_i^k, b-\varepsilon_i+2\varepsilon_i^k]\times[-1,1]\times S^2$ such that $\omega_2-\omega_1=p^*(d\eta)$, where p is the natural projection that forgets the $[-1,0]$ factor and the γ factor, and $d\eta$ vanishes outside $B(\varepsilon_i)\times[-1+\varepsilon,1-\varepsilon]\times S^2$

In order to conclude the proof, it suffices to prove that such η can be chosen so that η is zero outside $B(\varepsilon_i)\times[-1+\varepsilon,1-\varepsilon]\times S^2$, too. Therefore, it suffices to prove that the cohomology class of η is zero on

$$\left(\left([b-\varepsilon_i-2\varepsilon_i^k, b-\varepsilon_i+2\varepsilon_i^k]\times[-1,1]\right)\setminus\left(B(\varepsilon_i)\times[-1+\varepsilon,1-\varepsilon]\right)\right)\times S^2.$$

Thus, it suffices to prove that the integral of η along $\partial([b-\varepsilon_i-2\varepsilon_i^k, b-\varepsilon_i+2\varepsilon_i^k]\times[-1,1]\times\{Y\})$ is zero for some $Y\in S^2$.

This integral is also the integral of $(\omega_2-\omega_1)$ along $[b-\varepsilon_i-2\varepsilon_i^k, b-\varepsilon_i+2\varepsilon_i^k]\times[-1,1]\times\{Y\}$ or along $(N_c(\gamma)=[a,b]\times\{c\}\times[-1,1])\times\{Y\}$, where the integral of ω_1 is the algebraic intersection of $s_{\tau_b}([a,b]\times\{c\}\times[-1,1];Y)$ with P.

Note that nothing depends on the trivialization τ_b, which identifies

$$U\left([a,b]\times\gamma\times[-1,1]\right)$$

with $([a,b]\times\gamma\times[-1,1])\times S^2$. Therefore, there is no loss of generality in assuming that τ_b maps e_3 to the direction of $(-\gamma)$, which is the positive normal to the meridian disks, so that $(N_c(\gamma)\times\{e_3\})=s_+(N_c(\gamma))$.

Lemma 21.4 gives the following expression for the integral of ω_2 along $s_+(N_c(\gamma))$.

$$\int_{s_+(N_c(\gamma))}\omega_2=\frac{1}{2}\chi\left(\tau_e(.\times e_2)_{|\partial N_c(\gamma)};N_c(\gamma)\right),$$

while $\int_{s_+(N_c(\gamma))}\omega_1=\langle s_+(N_c(\gamma)),P\rangle$. On the other hand, according to Proposition 21.12,

$$2(s_+(N_c(\gamma))-s_{\tilde{\tau}}(N_c(\gamma);e_3))-\chi\left(\tau_e(.\times e_2)_{|\partial N_c(\gamma)};N_c(\gamma)\right)U\check{R}_{|*}$$

is null-homologous in $U\check{R}_{|*}$. So its algebraic intersection with P is zero if $X\neq e_3$, and the algebraic intersection of $s_{\tilde{\tau}}(N_c(\gamma);e_3)$ with P is also zero in this case according to Lemma 21.13. Therefore $\int_{s_+(N_c(\gamma))}\omega_1=\int_{s_+(N_c(\gamma))}\omega_2$ and the lemma is proved, when $e_3\neq X$. If $e_3=X$, choose $Y=-e_3$ and conclude by computing the integrals along $s_-(N_c(\gamma))$, similarly. $\square$

For a family of propagating chains P_i of $(C_2(R),\tilde{\tau})$ in general $3n$-position with respect to L as in Definition 11.3, and for propagating forms $\omega(i)$, which are α-dual to them for a sufficiently small α as in Lemma 21.16, the integrals involved in $Z_n^s(\check{R}, L, \tilde{\tau})$ in Theorem 21.7 can be computed as algebraic intersections of preimages of transverse propagating chains of $(C_2(R), \tilde{\tau})$ as in Sections 11.1 and 17.1.

This yields the announced discrete definition of $Z_n^s(\check{R}, L, \tilde{\tau})$ with respect to pseudo-parallelizations.

Appendix A

Some basic algebraic topology

This appendix reviews the main well-known results in algebraic topology that are used thoughout the book. The proofs of these results can be found in several books about basic algebraic topology such as [Gre67, Spa81]. Here we state only the weak versions that are used in the book, together with some sketches of proofs and some hints to provide feelings of how it works.

A.1 Homology

We first review some properties of homology.

A *topological pair* (X, Y) consists of a topological space X and a subspace Y of X. A *map* $f\colon (X, Y) \to (A, B)$ between such pairs is a continuous map from X to A such that $f(Y) \subseteq B$. A topological space X is identified with the pair $(X, \emptyset)$.

The coefficient ring Λ of the homology theories $H(.) = H(.; \Lambda)$ that we consider in this book is $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$.

A *covariant functor* H from the category of topological pairs (X, Y) to the category of graded Λ-modules and homomorphisms of degree 0 maps any topological pair (X, Y) to a sequence $H(X, Y; \Lambda) = H(X, Y) = (H_k(X, Y))_{k\in\mathbb{Z}}$ of Λ-modules, and it maps any map $f\colon (X, Y) \to (A, B)$ between pairs to a sequence of Λ-linear morphisms $H(f) = (H_k(f)\colon H_k(X, Y) \to H_k(A, B))_{k\in\mathbb{Z}}$ so that if $g\colon (A, B) \to (V, W)$ is another map between pairs $H_k(g \circ f) = H_k(g) \circ H_k(f)$ and H_k maps the identity map of (X, Y) to the identity map of $H_k(X, Y)$.

The homology theories (H, ∂) that we consider consist of

- a *covariant functor* H from the category of topological pairs (X, Y) to the category of graded Λ-modules and homomorphisms of degree 0, and

- a *natural transformation* ∂ from $H(X,Y)$ to $H(Y) = H(Y,\emptyset)$, which is a sequence of linear maps $\partial_k(X,Y)\colon H_k(X,Y) \to H_{k-1}(Y)$ such that for any map $f\colon (X,Y) \to (A,B)$, $\partial_k(A,B) \circ H_k(f) = H_{k-1}(f_{|Y}) \circ \partial_k(X,Y)$.

They satisfy the following Eilenberg and Steenrod axioms [Spa81, Chap. 4, Sec. 8, p.199], which characterize them for the spaces that are considered in this book, which are all homotopy equivalent to finite CW-complexes:

- **Homotopy axiom:** For two topological pairs (X,Y) and (A,B), if $f\colon [0,1] \times X \to A$ is a continuous map such that $f([0,1] \times Y) \subseteq B$, and if $f_t\colon X \to A$ maps $x \in X$ to $f(t,x)$, then $H(f_0) = H(f_1)$.

- **Exactness axiom:** For any topological pair (X,Y) with inclusion maps $i\colon Y \hookrightarrow X$ and $j\colon (X,\emptyset) \hookrightarrow (X,Y)$, there is a long exact[1] sequence:

$$\dots \xrightarrow{\partial_{k+1}(X,Y)} H_k(Y) \xrightarrow{H_k(i)} H_k(X) \xrightarrow{H_k(j)} H_k(X,Y) \xrightarrow{\partial_k(X,Y)} H_{k-1}(Y) \xrightarrow{H_{k-1}(i)} \dots$$

- **Excision axiom:** For any topological pair (X,Y), if U is an open subset of X whose closure lies in the interior of Y, then the inclusion map $e\colon (X \setminus U, Y \setminus U) \to (X,Y)$ induces isomorphisms $H_k(e)\colon H_k(X \setminus U, Y \setminus U) \to H_k(X,Y)$ for any $k \in \mathbb{Z}$.

- **Dimension axiom:** If X has one element, then $H_0(X) \cong \Lambda$ and $H_k(X) \cong \{0\}$ if $k \neq 0$.

An example of such a homology theory is the *singular homology* described in [Gre67, Chap. 10 and 13].

Let us show a few examples of properties of homology and computations, from the given axioms. First note that the functoriality implies that if a map $f\colon (X,Y) \to (A,B)$ between topological pairs is a homeomorphism, then $H(f)$ is an isomorphism.

Also note that the homotopy axiom implies that the homologies of $\mathbb{R}^n$ and of its unit ball B^n are isomorphic to the homology of a point, which is determined by the dimension axiom.

We fix a generator $[x]$ of $H_0(\{x\}) = \Lambda[x]$.

Here are other easy consequences of the axioms.

Proposition A.1. *Let X and Y be two topological spaces and let i_X and i_Y denote their respective inclusion maps into their disjoint union $X \sqcup Y$.*

[1]Such a sequence of homomorphisms is *exact* if and only if the image of a homomorphism is equal to the kernel of the following one.

Then for any $k \in \mathbb{Z}$, $H_k(i_X)\colon H_k(X) \to H_k(X \sqcup Y)$ and $H_k(i_Y)\colon H_k(Y) \to H_k(X \sqcup Y)$ are injective and

$$H_k(X \sqcup Y) = H_k(i_X)(H_k(X)) \oplus H_k(i_Y)(H_k(Y)).$$

Proof: Since the excision isomorphism $H_k(e_X)\colon H_k(X) = H_k(X, \emptyset) \to H_k(X \sqcup Y, Y)$ may be expressed as

$$H_k(e_X) = H_k(j\colon (X \sqcup Y, \emptyset) \hookrightarrow (X \sqcup Y, Y)) \circ H_k(i_X\colon X \hookrightarrow X \sqcup Y)$$

$H_k(j)$ is surjective and $H_k(i_X)$ is injective. Similarly, $H_k(i_Y)$ is injective. So the long exact sequence of $(X \sqcup Y, Y)$ yields short exact sequences

$$0 \to H_k(Y) \xrightarrow{H_k(i_Y)} H_k(X \sqcup Y) \xrightarrow{H_k(e_X)^{-1} \circ H_k(j)} H_k(X) \to 0$$

for any integer k.

Since $(H_k(e_X)^{-1} \circ H_k(j)) \circ H_k(i_X)$ is the identity map, the exact sequence splits and we get the result. $\square$

Proposition A.2. *Let X be a topological space and let Y and Z be two subspaces of X such that $Z \subseteq Y$. Then the long sequence*

$$\begin{tikzcd}
H_{k+1}(X,Y) \arrow[r, "\partial_{k+1}"] & H_k(Y,Z) \arrow[r, "H_k(i)"] & H_k(X,Z) \arrow[r, "H_k(j)"] & H_k(X,Y) \arrow[dll, "\partial_k"'] & \\
 & H_{k-1}(Y,Z) \arrow[r, "H_{k-1}(i)"'] & H_{k-1}(X,Z) \arrow[r, "H_{k-1}(j)"] & H_{k-1}(X,Y) \arrow[r, "\partial_{k-1}"] & H_{k-2}()
\end{tikzcd},$$

where the $H_k(i)$ and the $H_k(j)$ are induced by inclusion, and

$$(\partial_k = \partial_k(X,Y,Z))\colon H_k(X,Y) \to H_{k-1}(Y,Z),$$

is the composition of $\partial_k(X,Y)\colon H_k(X,Y) \to H_{k-1}(Y)$ and the map induced by the inclusion from $H_{k-1}(Y)$ to $H_{k-1}(Y,Z)$, is exact. It is called the long exact sequence of homology of the triple (X,Y,Z).

Sketch of proof: In order to prove that $H_k(j) \circ H_k(i) = 0$, note that the inclusion map from (Y,Z) to (X,Y) factors through (Y,Y). Next chase in commutative diagrams using functoriality and the exactness axiom, which is this exact sequence when $Z = \emptyset$. $\square$

For any topological space X with k connected components. Pick a basepoint x_i in each connected component. Let X_0 be the set of these basepoints. Let $i\colon X_0 \hookrightarrow X$ be the inclusion map, and let p be the map that maps each connected component to its basepoint. By functoriality $H(p) \circ H(i)$ is the identity map. Therefore $H(i)$ is injective. In particular $H_0(i)$ injects $\oplus_{x_i \in X_0} \Lambda[x_i]$ into $H_0(X)$, $[x_i]$ will also denote $H_0(i)([x_i])$. Note that the homotopy axiom implies that the element $[x_i]$ of $H_0(Y_i)$ is independent of the basepoint x_i of a connected component Y_i if this component is path-connected.

Lemma A.3.

$$H_k(\mathbb{R}, \mathbb{R} \setminus \{0\}) = \begin{cases} \Lambda\partial_1^{-1}(\mathbb{R}, \mathbb{R} \setminus \{0\})([1] - [-1]) \cong \Lambda & \text{if } k = 1 \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof: Proposition A.1 and the observations before imply that

$$H_0(\mathbb{R} \setminus \{0\}) = \Lambda[-1] \oplus \Lambda[+1],$$

where the morphism $H_0(i)$, induced by inclusion from $H_0(\mathbb{R}\setminus\{0\})$ to $H_0(\mathbb{R}) = \Lambda[1]$, maps $[-1]$ and $[+1]$ to the preferred generator $[-1] = [0] = [1]$ of $H_0(\mathbb{R})$. From the long exact sequence associated to $(\mathbb{R}, \mathbb{R} \setminus \{0\}, \emptyset)$, we deduce that $\partial_1(\mathbb{R}, \mathbb{R} \setminus \{0\})\colon H_1(\mathbb{R}, \mathbb{R} \setminus \{0\}) \longrightarrow H_0(\mathbb{R} \setminus \{0\})$ is an isomorphism onto its image, which is the kernel $\Lambda([+1] - [-1])$ of $H_0(i)$. The long exact sequence also implies that $H_q(\mathbb{R}, \mathbb{R} \setminus \{0\}) = \{0\}$ if $q \neq 1$. $\square$

The following proposition can also be proved by applying the axioms and the above observations.

Proposition A.4. *Let $n \in \mathbb{N}$ and let $k \in \mathbb{Z}$. Recall*

$$B^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid \parallel x \parallel^2 = \sum_{i=1}^n x_i^2 \leq 1\},$$

$S^{n-1} = \partial B^n$, and set $B_+^{n-1} = \{x \in S^{n-1} \mid x_1 \geq 0\}$, $B_-^{n-1} = \{x \in S^{n-1} \mid x_1 \leq 0\}$, $S_+^{n-2} = \{x \in S^{n-1} \mid x_1 = 0\}$. The morphisms of the sequence

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \overset{H_k(i)}{\leftarrow} H_k(B^n, S^{n-1}) \overset{\partial}{\rightarrow} H_{k-1}(S^{n-1}, B_-^{n-1})$$
$$\leftarrow H_{k-1}(B_+^{n-1}, S_+^{n-2}) \overset{H_{k-1}(p)}{\longrightarrow} H_{k-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \setminus \{0\})$$

where

- *the unlabeled morphism and $H_k(i)$ are induced by inclusions,*
- *∂ is the morphism $\partial(B^n, S^{n-1}, B_-^{n-1})$ of Proposition A.2,*
- *$p : B_+^{n-1} \longrightarrow \mathbb{R}^{n-1}$ forgets the first coordinate and shifts the numbering of the remaining ones by (-1),*

are isomorphisms. In particular, when $k = n$, their composition

$$D_n\colon H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to H_{n-1}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \setminus \{0\})$$

is an isomorphism. Let $[\mathbb{R}, \mathbb{R} \setminus \{0\}] = \partial_1^{-1}(\mathbb{R}, \mathbb{R} \setminus \{0\})([1] - [-1])$ be the preferred generator of $H_1(\mathbb{R}, \mathbb{R} \setminus \{0\})$ of Lemma A.3. Define $[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}] = D_n^{-1}[\mathbb{R}^{n-1}, \mathbb{R}^{n-1} \setminus \{0\}]$, inductively. Then

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = \begin{cases} \Lambda[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}] \cong \Lambda & \text{if } k = n \\ \{0\} & \text{otherwise.} \end{cases}$$

Let $[B^n, S^{n-1}] = H_n(i)^{-1}([\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}])$. Then

$$H_k(B^n, S^{n-1}) = \begin{cases} \Lambda[B^n, S^{n-1}] \cong \Lambda & \text{if } k = n \\ \{0\} & \text{otherwise.} \end{cases}$$

□

As a corollary, we get the following proposition:

Proposition A.5. *Let $n \in \mathbb{N}$. Assume $n \geq 1$.*

$$H_k(S^n) = \begin{cases} \Lambda[S^n] \cong \Lambda & \text{if } k = n \\ \Lambda[(1, 0, \ldots, 0)] \cong \Lambda & \text{if } k = 0 \\ \{0\} & \text{otherwise,} \end{cases}$$

where $[S^n] = \partial_{n+1}(B^{n+1}, S^n)([B^{n+1}, S^n])$

Proof: Use the exact sequence associated to (B^{n+1}, S^n) and the previous proposition. □

Proposition A.6. *Let $n \in \mathbb{N}$. Let ϕ be a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $\phi(0) = 0$. If ϕ preserves the orientation, then ϕ induces the identity map on $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. Otherwise, $H_n(\phi)([\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]) = -[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]$.*

Proof: This is easy to see, when $n = 1$, with the generator $[\mathbb{R}, \mathbb{R} \setminus \{0\}] = \partial_1(\mathbb{R}, \mathbb{R} \setminus \{0\})^{-1}([+1] - [-1])$. In general, let $\iota_n\colon \mathbb{R}^n \to \mathbb{R}^n$ map $(x_1, x_2, \ldots, x_{n-1}, x_n)$ to $(x_1, x_2, \ldots, x_{n-1}, -x_n)$. With the notation of Proposition A.4, $H_{n-1}(\iota_{n-1}) \circ D_n = D_n \circ H_n(\iota_n)$. So we can inductively see that $H_n(\iota_n)([\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]) = -[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]$. Finally, since any linear isomorphism that reverses the orientation, is homotopic to ι_n through linear isomorphisms, which map $\mathbb{R}^n \setminus \{0\}$ to itself, $H_n(\phi)([\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]) = -[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]$ for any such isomorphisms. The result follows for orientation-reversing diffeomorphisms, which are homotopic to linear isomorphisms through maps that preserve $\mathbb{R}^n \setminus \{0\}$ near 0 (this is enough thanks to the excision morphism). $\square$

For a topological space X equipped with a triangulation T as in Subsection 2.1.2, it also follows from the axioms (though it is slightly longer to prove) that the homology $H(X)$ of X can be computed as follows. Equip each simplex of the triangulation with an arbitrary orientation, and let $C_k(T) = C_k(T; \Lambda)$ denote the Λ-module freely generated by the simplices of dimension k of T. Define the *boundary map* $\partial_k \colon C_k(T) \to C_{k-1}(T)$, which maps a k-dimensional simplex Δ to its algebraic boundary

$$\partial_k(\Delta) = \sum_{\delta} \varepsilon(\Delta, \delta)\delta,$$

where the sum runs over all the $(k-1)$-dimensional simplices in the boundary of Δ and $\varepsilon(\Delta, \delta)$ is 1 if δ is oriented as (part of) the boundary of Δ (with the outward normal first convention as usual) and (-1) otherwise. Then $\partial_k \circ \partial_{k+1} = 0$ so that $(C(T), \partial) = (C_k(T), \partial_k)_{k \in \mathbb{Z}}$ is a *chain complex* and its homology

$$H(C(T), \partial) = \left(H_k(C(T), \partial) = \frac{\mathrm{Ker}\partial_k}{\mathrm{Im}\partial_{k+1}} \right)_{k \in \mathbb{Z}}$$

is canonically isomorphic to the homology of X.

The elements of $\mathrm{Ker}\partial_k$ are the *k-dimensional cycles* of $C(T)$, and the elements of $\mathrm{Im}\partial_{k+1}$ are the *k-dimensional boundaries* of $C(T)$. The elements of $C_k(T) = C_k(T; \Lambda)$ are called the *simplicial chains* of T of dimension k with coefficients in Λ

This proves that the homology of an n-dimensional manifold M that can be equipped with a triangulation T vanishes in degrees higher than n and in negative degrees. (The existence of a homology theory that satisfies the axioms and Proposition A.4 imply that the notion of dimension is well defined for topological manifolds.) If M is connected, then the coefficients of two simplices that share an $(n-1)$-simplex in an n-dimensional cycle must coincide when the orientations of the simplices are consistent along the face, and they must be opposite to each other otherwise. In particular, when $\Lambda = \frac{\mathbb{Z}}{2\mathbb{Z}}$, if M is connected, then the existence of a nonzero n-dimensional cycle implies that the boundary of M is empty and that M is compact. When $\Lambda = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, if M is connected, then the existence of a nonzero n-dimensional cycle furthermore implies that M is orientable.

Assume that M is a compact, n-dimensional, connected, triangulable, oriented manifold, with empty boundary, and assume that the n-simplices of T are equipped with the induced orientation. Then the homology class of the cycle that is the sum of all these simplices is called the *fundamental class* of M. It is denoted by $[M]$. (When $\Lambda = \frac{\mathbb{Z}}{2\mathbb{Z}}$, this definition does not require an orientation of M.) Let B^n be an n-dimensional closed ball embedded in M by an orientation-preserving embedding, and let 0 be the center of this ball. Thanks to the excision axion, the inclusions induce isomorphisms from $H_n(B^n, B^n \setminus \{0\})$ to $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ and from $H_n(B^n, B^n \setminus \{0\})$ to $H_n(M, M \setminus \{0\})$. Let $[B^n, B^n \setminus \{0\}]$ denote the generator of $H_n(B^n, B^n \setminus \{0\})$ that maps to $[\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}]$ by the first isomorphism, and let $[M, M \setminus \{0\}]$ denote the image of $[B^n, B^n \setminus \{0\}]$ by the second isomorphism. Then the inclusion induces an isomorphism from $H_n(M)$ to $H_n(M, M \setminus \{0\})$, which maps the generator $[M]$ of $H_n(M)$ to the generator $[M, M \setminus \{0\}]$ of $H_n(M, M \setminus \{0\})$. This provides a definition of the generator $[M]$ of $H_n(M)$, which is independent of the triangulation.

Let X be a topological space equipped with a triangulation T as above, and let Y be a closed subspace of X that is a union of simplices of T. Let T_Y be the corresponding triangulation of Y. Then the homology $H(X,Y)$ can be computed as the homology of the chain complex $(C(T,T_Y), \partial(T,T_Y))$, where $C_k(T,T_Y) = \frac{C_k(T)}{C_k(T_Y)}$ and $\partial_k(T,T_Y)\colon C_k(T,T_Y) \to C_{k-1}(T,T_Y)$ is the map induced by the previous boundary map ∂_k. When M is a connected, compact, oriented n-dimensional manifold with boundary, equipped with a triangulation T whose n-dimensional simplices are oriented by the orientation of M, the homology class of the cycle that is the sum of all the n-dimensional simplices of T is called the *fundamental class* of $(M, \partial M)$. It is denoted by $[M, \partial M]$. Again, if B^n is an n-dimensional closed ball embedded in M by an orientation-preserving embedding, the inclusions induce isomorphisms from $H_n(B^n, B^n \setminus \{0\})$ to $H_n(M, M \setminus \{0\})$ and from $H_n(M, \partial M)$ to $H_n(M, M \setminus \{0\})$, and the image of $[B^n, B^n \setminus \{0\}]$ by the first isomorphism coincides with the image of $[M, \partial M]$. Is is denoted by $[M, M \setminus \{0\}]$.

These considerations allow us to talk about the homology class of a compact oriented p-dimensional submanifold P of a manifold M. It is the image of $[P] \in H_p(P)$ in $H_p(M)$ under the map induced by the inclusion, and it is often still denoted by $[P]$. When P is a compact oriented p-dimensional manifold with boundary embedded in a topological space X so that ∂P is embedded in a subspace Y of X, we define the class $[P, \partial P]$ of $(P, \partial P)$ in $H_p(X,Y)$, similarly.

With these conventions, when M is a connected, compact, oriented, n-dimensional manifold with boundary, such that M can be equipped with a triangulation, the boundary map ∂_n in the homology sequence of the pair $(M, \partial M)$ maps $[M, \partial M]$ to the class $[\partial M]$, which is the sum of the classes of the connected components of ∂M equipped with the orientation induced by the orientation of M with respect to the outward normal first convention.

All the manifolds considered in this book can be equipped with triangulations. When M is a manifold equipped with a triangulation T, and when P is a p-dimensional compact oriented manifold embedded in M that is a union of simplices of T, then the homology class of P vanishes if and only if the chain that is the sum of the simplices of dimension p of P (equipped with the orientation of P) is the (algebraic) boundary of a simplicial chain of T of dimension $p+1$.

Homology with various coefficients are related by the universal coefficient theorem. See [Gre67, 29.12] for example. Here is an excerpt of this theorem.

Theorem A.7. *When $\Lambda = \mathbb{Q}$ or $\mathbb{R}$, for any topological space X,*

$$H_k(X;\Lambda) = H_k(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda.$$

In this book, we mostly use cohomology with coefficients in $\mathbb{Q}$, $\frac{\mathbb{Z}}{2\mathbb{Z}}$ or $\mathbb{R}$. In this case, it can be defined by the following excerpt of the universal coefficient theorem for cohomology, which can be found in [Spa81, Chap.5, Sec.5, Thm. 3, page 243], for example.

Theorem A.8. *When Λ is a field, for any $k \in \mathbb{Z}$,*

$$H^k(X,Y;\Lambda) = Hom_\Lambda(H_k(X,Y;\Lambda),\Lambda).$$

Note the sign $=$ in the above theorems, which means that the identifications are canonical. For a continuous map $f\colon (X,Y) \to (A,B)$ and an integer $k \in \mathbb{Z}$,

$$H^k(f;\Lambda)\colon H^k(A,B;\Lambda) \to H^k(X,Y;\Lambda)$$

maps a linear map g of $H^k(A,B;\Lambda)$ to $g \circ H_k(f;\Lambda)$.

For a general Λ, and for a pair (X,Y) of topological spaces such that X is equipped with a triangulation T as above, and Y is equipped with a subtriangulation T_Y of T as before, the cohomology $H^*(X,Y;\Lambda)$ of (X,Y) is the cohomology of the complex $(C^*(T,T_Y;\Lambda), \partial^*(T,T_Y;\Lambda))$, where $C^k(T,T_Y;\Lambda) = \mathrm{Hom}(C_k(T,T_Y;\Lambda);\Lambda)$ and $\partial^k\colon C^k(T,T_Y;\Lambda) \to C^{k+1}(T,T_Y;\Lambda)$ maps a linear form g to $g \circ \partial_{k+1}$.

$$H^k(X,Y;\Lambda) = \frac{\mathrm{Ker}(\partial^k)}{\mathrm{Im}(\partial^{k-1})}.$$

The elements of $\mathrm{Ker}(\partial^k)$ are the *k–dimensional cocycles* and the elements of $\mathrm{Im}(\partial^{k-1})$ are the *k–dimensional coboundaries.*

Here is a weak version of the Poincaré duality theorem. See [Gre67, Chap. 26 to 28, in particular (28.18)].

Theorem A.9. *Let M be a compact, n-dimensional manifold with possible boundary. Let H denote the singular homology. Then there are canonical Poincaré duality isomorphisms from $H^k(M, \partial M; \frac{\mathbb{Z}}{2\mathbb{Z}})$ to $H_{n-k}(M; \frac{\mathbb{Z}}{2\mathbb{Z}})$ and from $H^k(M; \frac{\mathbb{Z}}{2\mathbb{Z}})$ to $H_{n-k}(M, \partial M; \frac{\mathbb{Z}}{2\mathbb{Z}})$ for any $k \in \mathbb{Z}$.*

If M is oriented, then for any $\Lambda \in \frac{\mathbb{Z}}{2\mathbb{Z}}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, there are canonical Poincaré duality isomorphisms from $H^k(M, \partial M; \Lambda)$ to $H_{n-k}(M; \Lambda)$ and from $H^k(M; \Lambda)$ to $H_{n-k}(M, \partial M; \Lambda)$. The evaluation $P^{-1}(A)(B)$ of the image $P^{-1}(A) \in H^k(.; \Lambda)$ of a smooth submanifold A under the inverse of such a Poincaré duality isomorphism, at the class of a submanifold or a simplicial cycle B, of dimension $n-k$, transverse to A, is the algebraic intersection $\langle A, B\rangle_M$.

Here is a weak version of the Künneth theorem [Spa81, Chap.5, Sec.3, Thm. 10 (and Chap.5, Sec.2, Lem. 5)].

Theorem A.10. *Let H denote the singular homology with coefficients in a commutative field Λ. Then for any two topological spaces X and Y, for any $k \in \mathbb{N}$,*

$$H_k(X \times Y) = \oplus_{i=0}^k H_i(X) \otimes_\Lambda H_{k-i}(Y)$$

Again, the sign $=$ means that the identification is canonical. The tensor product of homology classes, represented by embeddings of oriented compact manifolds without boundary P into X, and Q into Y, represents the homology class of $P \times Q$.

We end this section by stating a weak version of the following Mayer-Vietoris exact sequence, which can be recovered from the Eilenberg-Steenrod axioms. See [Gre67, Chap. 17, Thm. 17.6], for example.

Theorem A.11. *Let X be a topological space. Let A and B be subspaces of X such that $X = A \cup B$ and the inclusion maps induce isomorphisms from $H(A, A\cap B)$ to $H(X, B)$ and from $H(B, A\cap B)$ to $H(X, A)$. Let $i_A \colon A\cap B \hookrightarrow A$, $i_B \colon A \cap B \hookrightarrow B$, $j_A \colon A \hookrightarrow X$, $j_B \colon B \hookrightarrow X$ denote the inclusion maps.*

Then there is a long exact sequence

$$\cdots \to H_{k+1}(X) \xrightarrow{\partial_{MV,k+1}} H_k(A \cap B) \xrightarrow{i_{MV,k}} H_k(A) \oplus H_k(B) \xrightarrow{j_{MV,k}} H_k(X) \xrightarrow{\partial_{MV,k}} \dots$$

such that $j_{MV,k}(\alpha \in H_k(A), \beta \in H_k(B)) = H_k(j_A)(\alpha) + H_k(j_B)(\beta)$, $i_{MV,k}(\gamma) = (H_k(i_A)(\gamma), -H_k(i_B)(\gamma))$ and $\partial_{MV,k+1}$ is the composition of the map induced by inclusion from $H_{k+1}(X)$ to $H_{k+1}(X, B)$, the inverse of the isomorphism from $H_{k+1}(A, A \cap B)$ to $H_{k+1}(X, B)$ and the boundary map $\partial_{k+1}(A, A \cap B)$ of the long exact sequence of $(A, A \cap B)$ from $H_{k+1}(A, A \cap B)$ to $H_k(A \cap B)$.

A.2 Homotopy groups

Let X be a topological space equipped with a basepoint x. For $n \geq 1$, the homotopy group $\pi_n(X, x)$ is the set of homotopy classes of maps from $[0,1]^n$ to X which map $\partial([0,1]^n)$ to x, equipped with the product that maps a pair $([f],[g])$ of homotopy classes of maps f and g such that f maps $[1/2,1] \times [0,1]^{n-1}$ to x and g maps $[0,1/2] \times [0,1]^{n-1}$ to x to the class $[f][g]$ of the map that coincides with f on $[0,1/2] \times [0,1]^{n-1}$ and with g on $[1/2,1] \times [0,1]^{n-1}$. The set of path-connected components of X is denoted by $\pi_0(X)$.

This product is commutative when $n \geq 2$.

Remark A.12. Classically, the set $\pi_n(X, x)$ is defined as the set of homotopy classes of maps from $(S^n, (1, 0, \ldots, 0))$ to (X, x). For $n \geq 1$, the two definitions coincide since S^n is homeomorphic to the quotient $\frac{[0,1]^n}{\partial[0,1]^n}$.

Examples A.13. Let k and n be positive integers, such that $1 \leq k \leq n$. A standard approximation theorem [Hir94, Chapter 2, Theorem 2.6, p. 49] implies that any continuous map from S^k to S^n is homotopic to a smooth map. The Morse-Sard theorem ensures that if $k < n$, any smooth map is valued in the complement of a point in S^n, which is contractible. Therefore, $\pi_k(S^n, *) = \{1\}$ if $1 \leq k < n$. The reader can develop those arguments to prove that there is a canonical isomorphism from $\pi_n(S^n, *)$ to $\mathbb{Z}$, which maps the homotopy class of a smooth map from $(S^n, *)$ to itself to its degree.

A weak version of the *Hurewicz theorem*, which relates homotopy groups to homology groups, ensures that for any path connected topological space X equipped with a basepoint x, $H_1(X;\mathbb{Z})$ is the abelianization of $\pi_1(X, x)$. See [Gre67, (12.1)], for example.

A map $p \colon E \to B$ is called a *weak fibration* if it satisfies the following *homotopy lifting property with respect to cubes*:

For any integer $n \in \mathbb{N}$, for any pair $(h_0 \colon [0,1]^n \times \{0\} \to E, H \colon [0,1]^{n+1} \to B)$ of continuous maps such that $H_{|[0,1]^n \times \{0\}} = p \circ h_0$, there exists a continuous extension h of h_0 to $[0,1]^{n+1}$ such that $H = p \circ h$.

To such a weak fibration, we associate the following long exact sequence in homotopy [Spa81, Chap. 7, Sec. 2, Thm. 10].

Theorem A.14. *Let $p \colon E \to B$ be a weak fibration. Let $e \in E$ be a basepoint of E. Let $b = p(e)$ denote its image under p and let $F = p^{-1}(b)$ denote the fiber over b. Then we have the following long exact sequence:*

$$\ldots \pi_{n+1}(B, b) \to \pi_n(F, e) \to \pi_n(E, e) \to \pi_n(B, b) \to \pi_{n-1}(F, e) \ldots$$

$$\cdots \to \pi_1(B, b) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B),$$

where the maps between the π_n *are induced by the inclusion* $F \hookrightarrow E$ *and by* p*, respectively, and the map from* $\pi_n(B, b) \to \pi_{n-1}(F, e)$ *is constructed as follows. An element of* $\pi_n(B, b)$ *is represented by a map* $H\colon [0,1]^n \to B$*, which has a lift* $h\colon [0,1]^n \to E$ *that maps* $[0,1]^{n-1} \times \{0\} \cup (\partial[0,1]^{n-1} \times [0,1])$ *to* e*. Such an element is mapped to the homotopy class of the restriction of* h *to* $[0,1]^{n-1} \times \{1\}$*. The last three maps of the exact sequence are just maps between sets. Exactness means that the preimage of the component of the basepoint is the image of the previous map.*

Remark A.15. In order to define the map from $\pi_n(B, b)$ to $\pi_{n-1}(F, e)$ in the above theorem, we use the fact that the pair

$$\big([0,1]^n, [0,1]^{n-1} \times \{0\} \cup \big(\partial[0,1]^{n-1} \times [0,1]\big)\big)$$

is homeomorphic to $([0,1]^n, [0,1]^{n-1} \times \{0\})$ for all $n \in \mathbb{N} \setminus \{0\}$.

A path p from a point x to another point y of X induces an isomorphism from $\pi_n(X, x)$ to $\pi_n(X, y)$ for any integer n. So the basepoint is frequently omitted from the notation $\pi_n(X, x)$ when X is path connected.

A map $p\colon E \to B$ is called a *covering map* if every $b \in B$ has a neighborhood U such that $p^{-1}(U)$ is a disjoint union of subsets of E, each of which is mapped homeomorphically onto U by p. Such a covering map is an example of a weak fibration, for which p induces isomorphisms from $\pi_n(E, e)$ to $\pi_n(B, p(e))$, for any $e \in E$, and for any $n \geq 2$.

Appendix B

Differential forms and De Rham cohomology

Here are a few well-known results about differential forms that are used thoughout the book. A more complete account can be found in [BT82].

B.1 Differential forms

Let M be a smooth manifold with possible boundary and corners. Degree 0 forms of M are smooth functions on M. The differential $df = Tf \colon TM \to \mathbb{R}$ of a smooth map f from M to $\mathbb{R}$ is an example of a degree 1 form of M. In general, a *degree k differential form* on M is a smooth section of the bundle $\bigwedge^k (TM)^* = \mathrm{Hom}(\bigwedge^k TM; \mathbb{R})$. So, such a form ω maps any $m \in M$ to an element $\omega(m)$ of $\mathrm{Hom}(\bigwedge^k T_m M; \mathbb{R})$, smoothly, in the sense below. On an open part U of a manifold, which is identified with an open subspace of $\mathbb{R}^n$ via a chart $\phi \colon U \to \mathbb{R}^n$, which maps $u \in U$ to $\phi(u) = (\phi_1(u), \ldots, \phi_n(u))$, the degree k forms are uniquely expressed as

$$\sum_{(i_1,\ldots,i_k) \in \mathbb{N}^k \mid 1 \le i_1 < \cdots < i_k \le n} f_{i_1 \ldots i_k} d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}.$$

for smooth maps $f_{i_1 \ldots i_k} \colon U \to \mathbb{R}$. The vector space of degree k differential forms on M is denoted by $\Omega^k(M)$. The differential $T\psi \colon TM \to TN$ of a smooth map ψ from a manifold M to a manifold N induces the pull-back of differential forms

$$\begin{array}{rlll} \psi^* \colon & \Omega^k(N) & \to & \Omega^k(M) \\ & \omega & \mapsto & (x \mapsto \omega_{\psi(x)} \circ \bigwedge^k T_x \psi). \end{array}$$

The antisymmetric bilinear exterior product $\wedge$ equips $\oplus_{k \in \mathbb{N}} \Omega^k(M)$ with a structure of graded algebra, such that $\psi^*(\omega \wedge \omega') = \psi^*(\omega) \wedge \psi^*(\omega')$ for any two forms ω and ω' on N. This graded algebra is equipped with a unique operator $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ such that:

- for any $f \in \Omega^0(M)$, $df = Tf$ is the differential of f, and $d \circ d(f) = 0$,
- for any $\alpha \in \Omega^{|\alpha|}(M)$ and $\beta \in \Omega^{|\beta|}(M)$, $d(\alpha\wedge\beta) = d\alpha\wedge\beta+(-1)^{|\alpha|}\alpha\wedge d\beta$,
- the derivation operator d commutes with the above pull-backs, $d\psi^*(\omega) = \psi^*(d\omega)$.

Then $d \circ d = 0$.

The *support* of a differentiable form is the closure of the set where it does not vanish.

Let $x_i\colon \mathbb{R}^k \to \mathbb{R}$ be the usual coordinate functions. The *integral* of a degree k differential form $\omega = f dx_1 \wedge \cdots \wedge dx_k$ over a k-dimensional compact submanifold C of $\mathbb{R}^k$ with boundary and corners (like $[0,1]^k$) is $\int_C \omega = \int_C f dx_1 \dots dx_k$. Then for any smooth map ψ of $\mathbb{R}^k$ that restricts to an orientation-preserving diffeomorphism from C to its image,

$$\int_C \psi^*(\omega) = \int_{\psi(C)} \omega.$$

This property allows us to define the integral of a k-form ω over any k-dimensional compact submanifold C of a manifold M, identified to a subspace of $\mathbb{R}^k$ by a diffeomorphism $\phi\colon C \to \mathbb{R}^k$ onto its image, as

$$\int_C \omega = \int_{\phi(C)} \phi^{-1*}(\omega),$$

unambiguously. This definition extends naturally to general compact manifolds with boundaries and corners.

One of the most useful theorems in this book is the following Stokes theorem. See [BT82, Theorem 3.5, Page 31].

Theorem B.1. *Let ω be a degree d form on an oriented $d+1$ smooth compact manifold M with boundary ∂M, then*

$$\int_{\partial M} \omega = \int_M d\omega$$

This theorem applies to manifolds with ridges, and $\int_{\partial M} \omega$ is the sum of the $\int_C \omega$, over the codimension zero faces C of ∂M, which are d manifolds with boundaries.

B.2 De Rham cohomology

A degree k differential form ω on M is *closed* if $d\omega = 0$. It is *exact* if $\omega \in d\Omega^{k-1}(M)$. Define the degree k De Rham cohomology module of M to be

$$H^k_{dR}(M) = \frac{\operatorname{Ker}\left(d\colon \Omega^k(M) \to \Omega^{k+1}(M)\right)}{d(\Omega^{k-1}(M))}.$$

For a compact submanifold N of M, or for $N = \partial M$, as in [BT82, pages 78-79], set

$$\tilde{\Omega}^k(M,N) = \Omega^k(M) \oplus \Omega^{k-1}(N)$$

and define $d\left((\omega,\eta) \in \tilde{\Omega}^k(M,N)\right) = (d\omega, i^*(\omega) - d\eta)$, where $i\colon N \hookrightarrow M$ is the inclusion. The relative degree k De Rham cohomology module of (M,N) is

$$H^k_{dR}(M,N) = \frac{\operatorname{Ker}\left(d\colon \tilde{\Omega}^k(M,N) \to \tilde{\Omega}^{k+1}(M,N)\right)}{d(\tilde{\Omega}^{k-1}(M,N))}.$$

We have a natural short exact sequence of chain complexes

$$0 \to \Omega^{k-1}(N) \to \tilde{\Omega}^k(M,N) \to \Omega^k(M) \to 0,$$

where $\eta \in \Omega^{k-1}(N)$ is mapped to $(0, (-1)^{|\eta|}\eta)$ and $(\omega,\eta) \in \tilde{\Omega}^k(M,N)$ is mapped to $(\omega, 0)$. This sequence induces a natural long exact cohomology sequence

$$\to H^{k-1}_{dR}(N) \to H^k_{dR}(M,N) \to H^k_{dR}(M) \to H^k_{dR}(N) \to$$

where the map from $H^k_{dR}(M)$ to $H^k_{dR}(N)$ is induced by the restriction.

The degree k forms with compact support in M form a subspace $\Omega^k_c(M)$ of $\Omega^k(M)$, and the degree k De Rham cohomology module with compact support of M is

$$H^k_{dR,c}(M) = \frac{\operatorname{Ker}\left(d\colon \Omega^k_c(M) \to \Omega^{k+1}_c(M)\right)}{d(\Omega^{k-1}_c(M))}.$$

For any smooth map ψ from M to another manifold M' that maps N to a submanifold N', the pull-back ψ^* induces maps still denoted by ψ^* from $H^k_{dR}(M')$ to $H^k_{dR}(M)$, and from $H^k_{dR}(M',N')$ to $H^k_{dR}(M,N)$. If ϕ is another such smooth map from the pair (M',N') to another such (M'',N''), then

$$\psi^* \circ \phi^* = (\phi \circ \psi)^*.$$

When such a map ψ is proper, (i.e. when the preimage of a compact is compact), ψ also induces $\psi^*\colon H^k_{dR,c}(M') \to H^k_{dR,c}(M)$.

The following standard lemma implies that $\psi^*\colon H^k_{dR}(M') \to H^k_{dR}(M)$ depends only on the homotopy class of $\psi\colon M \to M'$.

Lemma B.2. *Let V and W be two smooth manifolds and let $h\colon [0,1]\times V \to W$ be a smooth homotopy. Let ω be a degree d closed form on W. Then*

$$h_t^*(\omega) - h_0^*(\omega) = d\eta_t(h,\omega)$$

for any $t \in [0,1]$, where $\eta_t(h,\omega)$ is the following degree $(d-1)$ form on V. For $u \in [0,1]$, let $i_u\colon V \to [0,1]\times V$ map $v \in V$ to $i_u(v) = (u,v)$. Let $h^*(\omega)_{(u,v)}(\frac{\partial}{\partial t}\wedge .)$ be obtained from $h^*(\omega)$ by evaluating it at the tangent vector $\frac{\partial}{\partial t}$ to $[0,1]\times\{v\}$ at (u,v). (Thus $h^*(\omega)_{(u,v)}(\frac{\partial}{\partial t}\wedge .)$ is the value at $(u,v)\in[0,1]\times V$ of a degree $(d-1)$ form of $[0,1]\times V$.) Then

$$\eta_t(h,\omega)_v = \int_0^t i_u^*\left(h^*(\omega)_{(u,v)}\left(\frac{\partial}{\partial t}\wedge .\right)\right)du$$

Proof: Note $h_u^*(\omega) = i_u^*(h^*(\omega))$ and write

$$h^*(\omega) = \omega_1 + dt\wedge h^*(\omega)\left(\frac{\partial}{\partial t}\wedge .\right),$$

where $h_u^*(\omega) = i_u^*(\omega_1)$. Since ω is closed, $dh^*(\omega)=0$, therefore

$$0 = d\omega_1 - dt\wedge d(h^*(\omega)(\frac{\partial}{\partial t}\wedge .)).$$

On the other hand, for the natural projection $p_V\colon [0,1]\times V\to V$,

$$(d\omega_1)_{(u,v)} = dt\wedge\frac{\partial}{\partial t}(\omega_1) + p_V^*(dh_u^*(\omega)).$$

Hence

$$i_u^*(\frac{\partial}{\partial t}(\omega_1)) = i_u^*(d(h^*(\omega)(\frac{\partial}{\partial t}\wedge .)))$$

and $\frac{\partial}{\partial u}h_u^*(\omega) = \frac{\partial}{\partial u}i_u^*(\omega_1) = i_u^*(\frac{\partial}{\partial t}(\omega_1)) = i_u^*(d(h^*(\omega)(\frac{\partial}{\partial t}\wedge .)))$. Since $h_t^*(\omega)_v - h_0^*(\omega)_v = \int_0^t\frac{\partial}{\partial u}h_u^*(\omega)_v du$, the lemma follows. □

In particular, if there exist smooth maps $f\colon M\to N$ and $g\colon N\to M$ such that $f\circ g$ is smoothly homotopic to the identity map of N and $g\circ f$ is smoothly homotopic to the identity map of M, then f^* is an isomorphism from $H^k_{dR}(N)$ to $H^k_{dR}(M)$, for any k, and all the smoothly contractible manifolds such as $\mathbb{R}^n$ have the same de Rham cohomology as the point. So for such a manifold C, $H^k_{dR}(C) = \{0\}$ for any $k\neq 0$, and $H^0_{dR}(C)=\mathbb{R}$.

More generally, in 1931, Georges De Rham identified the De Rham cohomology of a smooth n-dimensional manifold M to its singular cohomology with coefficients in $\mathbb{R}$. See [War83, Pages 205-207], or [BT82, Theorems 8.9 Page 98, 15.8 page 191].

The De Rham isomorphism maps the cohomology class $[\omega]$ of a closed degree k form to the linear form of $\mathrm{Hom}(H_k(M;\mathbb{R});\mathbb{R}) = H^k(M;\mathbb{R})$ that maps the homology class of a closed oriented k-dimensional submanifold N without boundary of M to $\int_N \omega$. (If M is smoothly triangulated, any form ω defines a simplicial cochain that is a map from the set $\Delta_k(M)$ of k-simplices to $\mathbb{R}$, and Stokes's theorem guarantees that the induced map from $\Omega^k(M)$ to $\mathbb{R}^{\Delta_k(M)}$ commutes with differentials, and induces a morphism from $H^k_{dR}(M)$ to $H^k(M;\mathbb{R}) = \mathrm{Hom}(H_k(M;\mathbb{R});\mathbb{R})$ that maps $[\omega]$ as stated above.)

We also have the following theorem [GHV72, page 197] or [BT82, (5.4) and Remark 5.7].

Theorem B.3. *For any oriented manifold M without boundary of dimension n (whose cohomology is not necessarily finite dimensional), the morphism $\left(\omega \mapsto (\omega_2 \mapsto \int_M \omega \wedge \omega_2)\right)$ induces an isomorphism from $H^k(M;\mathbb{R})$ to $Hom(H_c^{n-k}(M);\mathbb{R})$, for any integer k.*

In particular, when the real cohomology of M is finite dimensional, so is its homology, and $H_c^{n-k}(M)$ is isomorphic to $H_k(M;\mathbb{R})$. Below, we exhibit the image of the homology class of an oriented compact submanifold A in $H_k(M;\mathbb{R}) = \mathrm{Hom}(H^k(M);\mathbb{R})$ under a canonical isomorphism from $H_k(M;\mathbb{R})$ to $H_c^{n-k}(M)$.

Let A be a compact oriented k-dimensional submanifold without boundary of the manifold M, and let $N(A)$ be a tubular neighborhood of M. For trivializing open parts U of A, $N(A)$ restricts as an embedding $U \times \mathbb{R}^{n-k} \hookrightarrow M$ and we may assume that these local embeddings are compatible, in the sense that, for overlapping U and V, the corresponding embeddings are obtained from one another by precomposition by some $(u,x) \mapsto (u, \psi(u)(x))$, for a map $\psi\colon U \cap V \to SO(n-k)$. Let ω be a compactly supported $(n-k)$-form on $\mathbb{R}^{n-k}$ invariant under the $SO(n-k)$-action (for example, the product of $dx_1 \wedge \cdots \wedge dx_{n-k}$ by a function of the distance d to the origin, which vanishes for d big enough, and whose derivatives vanish for d small enough) such that $\int_{\mathbb{R}^{n-k}} \omega = 1$. Pull back ω on $N(A)$ via the local projections from $U \times \mathbb{R}^{n-k}$ to $\mathbb{R}^{n-k}$. Our conditions ensure that this closed pull-back ω_A is defined consistently on $N(A)$. Extend it by 0 outside $N(A)$. The integral of this closed form ω_A over a compact $(n-k)$ submanifold B of M transverse to A is the algebraic intersection of A and B when the support of ω_A is close enough to A.

Note that the support of ω_A may be chosen in an arbitrarily small neighborhood of A. In the words of Definition 11.6, the form ω_A is α-dual to A for an arbitrarily small positive number α.

Lemma B.4. *Let M be a compact manifold of dimension n with possible boundary. Assume that M is equipped with a smooth triangulation. The above process can be extended to produce canonical* Poincaré duality isomorphisms *from $H_a(M, \partial M; \mathbb{R})$ to $H^{n-a}_{dR}(M)$, where such a Poincaré duality isomorphism maps the class of an a-dimensional cycle A of $(M, \partial M)$ to the class of a closed $(n-a)$-form ω_A α-dual to A for an arbitrarily small positive number α, as follows.*

PROOF: Let A be a simplicial a-cycle of $(M, \partial M)$. The cycle A is a linear combination of a-dimensional simplices of M. Let $\overline{A}$ denote the support of A, which is the union in M of the closed simplices with non-zero coefficient of A. We construct a form ω_A α-dual to A as follows. Let $A^{(k)}$ be the intersection of $\overline{A}$ with the k-skeleton of $(M, \partial M)$. (If $k \leq 0$, $A^{(k)} = \emptyset$.) First construct a closed form ω_A α-dual to A, outside a small neighborhood $N(A^{(a-1)})$ of $A^{(a-1)}$, so that its support is in an arbitrarily small neighborhood of A, as explained above in the case of manifolds. Then extend ω_A around each $(a-1)$-simplex Δ of A, outside a small neighborhood $N(A^{(a-2)})$ of $A^{(a-2)}$ such that the intersection of a neighborhood of Δ with the complement of $N(A^{(a-2)})$ is diffeomorphic to $\Delta' \times D^{n-a+1}$, for some truncation Δ' of Δ. See Figure B.1. The form ω_A is defined and closed on a neighborhood of $\Delta' \times \partial D^{n-a+1}$. Furthermore, its integral along $\{x\} \times \partial D^{n-a+1}$ is the algebraic intersection of ∂D^{n-a+1} and A (up to sign), which is the coefficient of Δ in ∂A (up to sign, again), which is zero so that ω_A may be written as $d\eta$ on a neighborhood of $\Delta' \times \partial D^{n-a+1}$. Let χ be a map on $\mathbb{R}^{n-a+1}$ that takes the value 1 outside a small neighborhood of 0 and that vanishes in a smaller neighborhood of 0 so that $d\chi\eta$ makes sense on $\Delta' \times D^{n-a+1}$ and extends ω_A as a closed form.

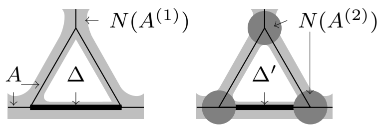


Figure B.1: The neighborhoods $N(A^{(1)})$ and $N(A^{(2)})$, a simplex Δ and its truncation Δ', when A is one-dimensional

This process allows us to define a closed form ω_A α-dual to A outside a small neighborhood $N(A^{(a-2)})$ of $A^{(a-2)}$. Iterate the process in order to obtain such a form outside $N(A^{(a-3)})$, outside $N(A^{(a-4)})$, and on the whole M. Note that the forms will be automatically exact around the truncated smaller cells since $H^{n-a}(\partial D^{n-a+k}) = 0$ when $k > 1$. $\square$

In this setting, the correspondence between chain boundaries and the differentiation operator d can be roughly seen as follows. Let A be a compact oriented submanifold of dimension a with boundary of a manifold M, and let $[-1,1] \times \partial A$ be embedded in the interior of M so that $\{0\} \times \partial A = \partial A$ and $[-1,0] \times \partial A$ is a neighborhood of ∂A in A. Let $N(A^+)$ be the total space of the normal bundle to $A^+ = A \cup_{\partial A} ([0,1] \times \partial A)$ embedded in M. Let $p_{A^+} \colon N(A^+) \to A^+$ denote the corresponding projection. Let χ be a function on A^+, which maps A to 1 and a neighborhood of ∂A^+ to 0, and which factors through the projection onto $[0,1]$ on $[0,1] \times \partial A$. Let ω_{A^+} be a closed form compactly supported in $N(A^+)$, which is dual to A^+ there. Then

$$\omega_A = (\chi \circ p_{A^+})\omega_{A^+}$$

is a form that vanishes near the boundary of $\partial \overline{N(A^+)}$ and extends as 0 outside $N(A^+)$. The form ω_A is closed everywhere except over $[0,1] \times \partial A$, where $d\omega_A = d(\chi \circ p_{A^+}) \wedge \omega_{A^+}$, so that $d\omega_A$ is dual to $\partial A \times \{\frac{1}{2}\}$, up to sign.

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Terminology