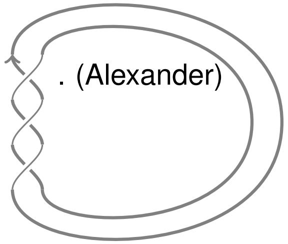
Sur quelques travaux de Pierre Vogel (3)
Sur l'algèbre de diagrammes Λ

Christine Lescop

CNRS, Institut Fourier, Grenoble

IHP 27-28 octobre 2010

Knots in $\mathbb{R}^3$ are closures of braids  . (Alexander)

[*Representation of links by braids: a new algorithm.* Comment. Math. Helv. (1990)]

- 1984, construction of Jones polynomial from representations of braid groups.

Many generalisations involving quantum group representations or representations of Hecke algebras.

[*Représentations et traces des algèbres de Hecke.* Enseign. Math. (1988)]

- 1988, Witten-Reshetikhin-Turaev invariants of 3-manifolds.

Construction simplified by Lickorish
and extended by Blanchet, Habegger, Masbaum and Vogel
to a topological quantum field theory
satisfying the Atiyah axioms of a TQFT.

[Blanchet, Habegger, Masbaum, Vogel, *Topological quantum field theories derived from the Kauffman bracket.* Topology (1995)] Photos : Jean-Yves Le Dimet

- 1984, construction of Jones polynomial
- 1988, Witten-Reshetikhin-Turaev invariants of 3-manifolds
- 1995, Blanchet-Habegger-Masbaum-Vogel TQFT
- ... many more quantum invariants

Problem

Classify knot and 3-manifold invariants.

Partial solution

Theory of finite type invariants
introduced in 1990 by Vassiliev for knots (Birman, Lin, Goussarov, Bar-Natan),
extended to 3-manifold invariants by Ohtsuki in 1995.

The Kontsevich integral

Kontsevich (1991) constructed

$$Z_K : \{\text{ isotopy classes of oriented knots}\} \to \prod_{n\in\mathbb{N}} \mathcal{A}_n(S^1)$$

where

- Z_K is universal:
  ν is a $\mathbb{C}$-valued finite type knot invariant
  $\Leftrightarrow \nu = \psi \circ Z_K$ for some linear $\psi : \mathcal{A}(S^1) \to \mathbb{C}$
  such that $\psi(\prod_{n>N} \mathcal{A}_n(S^1)) = 0$,
- $\mathcal{A}(S^1) = \prod_{n\in\mathbb{N}} \mathcal{A}_n(S^1)$ is identified by Z_K with the graded space of the Vassiliev filtration of the $\mathbb{Q}$-vector space freely generated by knots,
- each $\mathcal{A}_n(S^1)$ is an explicit finite-dimensional $\mathbb{Q}$-vector space of Feynman-Jacobi diagrams.

The LMO invariant

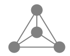
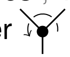
Le, Murakami and Ohtsuki (1996) constructed

$$Z_{LMO} : \frac{\{\text{closed oriented 3-manifolds}\}}{\text{orientation-preserving diffeomorphisms}} \to \prod_{n\in\mathbb{N}} \mathcal{A}_n$$

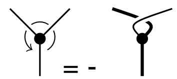
where

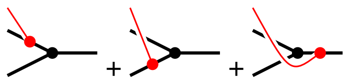
- Z_{LMO} is universal: (Le, 1997)
  ν is a $\mathbb{C}$-valued finite type invariant of homology spheres
  $\Leftrightarrow \nu = \psi \circ Z_{LMO}$ for some linear $\psi : \mathcal{A} \to \mathbb{C}$
  such that $\psi(\prod_{n>N} \mathcal{A}_n) = 0$,
- $\mathcal{A} = \prod_{n\in\mathbb{N}} \mathcal{A}_n$ is identified by Z_{LMO} with the graded space associated to the Ohtsuki filtration of the $\mathbb{Q}$-vector space freely generated by homology spheres,
- each $\mathcal{A}_n$ is an explicit finite-dimensional $\mathbb{Q}$-vector space of Feynman-Jacobi diagrams.

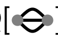
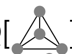
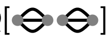
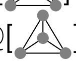
Jacobi diagrams

$\mathcal{D}_n$ is the $\mathbb{Q}$-vector space generated by
trivalent graphs with $2n$ oriented vertices , like  $\in \mathcal{D}_2$,
oriented: equipped with a cyclic order  .

$$\mathcal{A}_n = \frac{\mathcal{D}_n}{AS, Jacobi}$$

(AS or Antisymmetry) 

(IHX or Jacobi) 

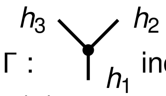
$\mathcal{A}_1 = \mathbb{Q}[$$]$, $\mathcal{A}_2 = \mathbb{Q}[$$] \oplus \mathbb{Q}[$$]$,
$\mathcal{A}_2^c = \mathcal{A}_2^{\text{connected}} = \mathbb{Q}[$$]$.

L finite-dimensional complex Lie algebra
with a non-degenerate bilinear symmetric invariant form

$$b \in (L \otimes L)^* = L^* \otimes L^*.$$

b induces $B\colon L \xrightarrow{\cong} L^*$.
Casimir element: $\Omega = (B^{-1} \otimes B^{-1})(b) \in L \otimes L$.
Γ trivalent graph, E its set of edges, H its set of half-edges.

$$\otimes_{e\in E}\Omega_{(e)} \in \otimes_{h\in H} L_{(h)}.$$

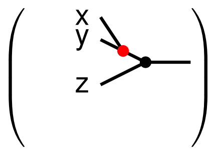
v vertex of Γ :  induces

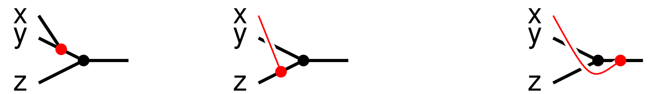
$$\psi_{L,b}(v)\colon \quad L_{(h_1)} \otimes L_{(h_2)} \otimes L_{(h_3)} \to \mathbb{C}, \qquad x \otimes y \otimes z \mapsto b([x,y],z)$$

b invariant $\Leftrightarrow b([x,y],z) = b([z,x],y)$
$\Leftrightarrow \psi_{L,b}(v)$ cyclically invariant tensor of $\otimes^3 L^*$.

Weight map $\psi_{L,b}$: $\psi_{L,b}(\Gamma) = \otimes_v \psi_{L,b}(v)(\otimes_{e\in E}\Omega_{(e)}) \in \mathbb{C}$.

The Jacobi identity for Lie algebras implies that $\psi_{L,b}$ factors through the IHX-Jacobi relation.

$\psi_{L,b}$  : $\otimes^3 L \to L$, $x \otimes y \otimes z \mapsto [[x,y],z]$



Question

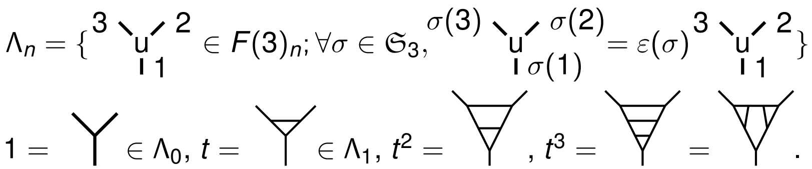
Is any non-trivial element of $\mathcal{A}$ detected by some $\psi_{L,b}$?

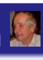
Answer 

No!

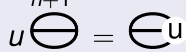
This question was asked for $\mathcal{A}(S^1) = \mathcal{A}^c \oplus \mathcal{B}$
by Bar-Natan in his famous article on Vassiliev invariants.

$F(3)_n$ is the $\mathbb{Q}$-vector space generated by
connected graphs with $(2n+1)$ oriented trivalent vertices
and 3 numbered univalent vertices (or legs)
quotiented by IHX(Jacobi) and AS.



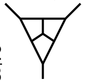
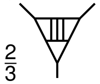
Theorem 

Elements of Λ_n act on spaces of connected Jacobi diagrams by insertion at a vertex.

$\lambda\colon \Lambda_n \to \mathcal{A}^c_{n+1}$, $u \mapsto$  is an isomorphism.

Λ is a commutative graded algebra.

$R = \mathbb{Q}[t, \sigma, \omega]$ is generated by an element t of degree 1, σ of degree 2 and ω of degree 3.
$R_0 = \mathbb{Q}[t] \oplus \omega R$ subalgebra of R. Define $\varphi \colon R_0 \to \Lambda$

$\varphi(t) = t$, $\varphi(\omega) = \frac{2}{3}$  $+ \frac{8}{3} t^3$, $\varphi(\sigma\omega) = \frac{2}{3}$  $+ \frac{17}{3} t^2 \omega - 8t^5$.

$$P_{osp} = 27\omega^2 - 72t\sigma\omega + 40t^3\omega + 4\sigma^3 + 29t^2\sigma^2 - 24t^4\sigma$$

$$P_{gl} = \omega - 2t\sigma + 2t^3, \quad P_{exc} = 27\omega - 45t\sigma + 40t^3$$

Theorem

$\varphi(P_{PV} = t\omega P_{gl} P_{osp} P_{exc}) = 0.$
$\overline{\varphi} \colon \frac{R_0}{RP_{PV}} \to \Lambda$ is injective up to degree 22.

Conjecture

$\overline{\varphi}$ is an isomorphism.

Theorem (Kneissler 1997)

$\overline{\varphi}$ is surjective up to degree 10.

Theorem

Let $\psi_{L,b}$ be the map associated to a simple quadratic Lie superalgebra L. $\exists \chi_{L,b} \colon R \to \mathbb{C}$ algebra morphism such that

$$\psi_{L,b}(\varphi(r \in R_0) \in \Lambda) = \chi_{L,b}(r)\psi_{L,b}(1).$$

$\bigcap_b \mathrm{Ker}(\chi_{sl_2,b}) \cap R_0 = \omega R, \quad \bigcap_b \mathrm{Ker}(\chi_{D(2,1,\alpha),b}) = (t)$
$\bigcap_{\{(L,b);\, L \text{ Lie superalgebra of family } gl\}} \mathrm{Ker}\chi_{L,b} = (P_{gl})$
$\bigcap_{\{(L,b);\, L \text{ Lie superalgebra of family } osp\}} \mathrm{Ker}\chi_{L,b} = (P_{osp})$
$\bigcap_{\{(L,b);\, L \in \{E_6, E_7, E_8, F_4, G_2\}\}} \mathrm{Ker}\chi_{L,b} = (P_{exc}) + (P_{2,exc})$
where $P_{2,exc} = (77t^2 - 36\sigma)(176t^2 - 81\sigma)(494t^2 - 225\sigma)$
$(170t^2 - 81\sigma)(65t^2 - 36\sigma)$.

Theorem (Bertrand Patureau-Mirand, G & T 2002)

There exists $\tilde{\chi} \colon \Lambda \to R_0/(RP_{PV} + Rt\omega P_{gl} P_{osp} P_{2,exc})$ such that $\tilde{\chi} \circ \varphi$ lifts the $\chi_{L,b|R_0}$, and $\tilde{\chi} \circ \overline{\varphi}$ is the natural projection.

P. Vogel's work on the characters $\chi_{L,b}$ lead
Pierre Deligne to state a conjecture (CRAS 1996)
about the existence of a category of representations with a parameter
that lifts the category of representations for the Lie groups A_1, A_2, D_4, G_2, F_4, E_6, E_7 and E_8.

Proposition

Conjecture implies this Deligne conjecture.

Recall $\lambda \colon \Lambda_n \to \mathcal{A}^c_{n+1}$, $u \mapsto u$  $=$  is an isomorphism.

Theorem

Let $C_{PV} = \omega P_{gl} P_{osp} P_{exc} = \frac{P_{PV}}{t}$.

$$\lambda(\varphi(C_{PV})) \in \mathcal{A}_{16} \setminus \{0\}$$

cannot be detected by a weight function associated with a quadratic Lie superalgebra.

Proof :
$\chi_{D(2,1,\alpha),b}(\varphi(C_{PV})) \neq 0 \Rightarrow \varphi(C_{PV}) \neq 0 \Rightarrow \lambda(\varphi(C_{PV})) \neq 0.$
$\chi_{D(2,1,\alpha),b}(\lambda) = 2\chi_{D(2,1,\alpha),b}(t)T = 0 \Rightarrow \psi_{L,b}(\lambda(\varphi(C_{PV}))) = 0$ for any simple quadratic Lie superalgebra.

Latest version of [Algebraic structures on modules of diagrams, 1995, 1997, ...,2010] $\Rightarrow$ simple can be removed !