# On the cube of the equivariant linking pairing for closed 3-manifolds of rank 1

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*Chern-Simons Gauge Theory: 20 years after*

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In this talk, all the manifolds are **oriented**.

A **homology sphere**
is a closed (connected, compact, without boundary) 3–manifold $N$ such that $H_*(N;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$.





The study of **3–manifold invariants** built from **integrals over configuration spaces** started after the work of Witten on Chern-Simons theory in 1989, with work of Axelrod, Singer, Kontsevich, Bott, Cattaneo, Taubes...

For the knots and links case, many more authors were involved including Bar-Natan, Guadagnini, Martellini, Mintchev, Altschüler, Freidel, Poirier...

In 1999, G. Kuperberg and D. Thurston proved that some of these invariants (the Kontsevich ones)
fit in with the framework of finite type invariants of homology spheres
studied by Ohtsuki, Le, Murakami (2), Goussarov, Habiro, Rozansky, Garoufalidis, Polyak...
and together define a
**universal finite type invariant** for homology 3-spheres.

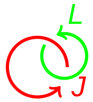




In particular, the **Kuperberg-Thurston** work shows how to write

$$\lambda_{\text{Casson, 1984}}(N) = \frac{1}{6}\int_{(N\setminus\{\infty\})^2\setminus\text{diagonal}} \omega^3$$

for a homology sphere $N$, $\infty \in N$,
and a closed 2-form $\omega$ such that
for any 2-component link $(J, L)$ of $N$, $\int_{J\times L}\omega = lk(J,L)$.




$J \sqcup L : S^1 \sqcup S^1 \to N \setminus \{\infty\}$
induces $J \times L : S^1 \times S^1 \to (N\setminus\{\infty\})^2 \setminus$ diagonal.

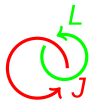
$6\lambda_{\text{Casson}}(N)$ may be viewed as the *cube of the linking form.*

We can also write

$6\lambda_{\text{Casson, 1984}}(N)$ is the algebraic intersection $\langle F_X, F_Y, F_Z\rangle$ of three codimension 2 submanifolds $F_X, F_Y, F_Z$ in $(N \setminus \{\infty\})^2 \setminus$ diagonal,

for a homology sphere $N$, $\infty \in N$,
and 4–dimensional submanifolds $F$ ($F_X$, $F_Y$ and $F_Z$)
Poincaré dual to the previous $\omega$

such that for any 2-component link $(J, L)$ of $N$,




$$\langle J \times L, F\rangle = lk(J, L).$$

See math.GT/0411088 and math.GT/0411431 for details and generalisations.



Background
Statements
The construction

**Theorem**

We can similarly define
an equivariant cube $\mathcal{Q}(M, \mathbb{K})$ of the equivariant linking pairing
for a closed 3–manifold $M$ with $H_1(M; \mathbb{Q}) = \mathbb{Q}$
w.r.t. a framed knot $\mathbb{K}$ such that $H_1(M; \mathbb{Z})/\text{Torsion} = \mathbb{Z}[\mathbb{K}]$
with the following properties.

1. $\mathcal{Q}(M, \mathbb{K}) \in \mathbb{Q}(x, y) = \frac{\mathbb{Q}(x,y,z)}{(xyz=1)}$, $\quad \mathcal{Q}(S^1 \times S^2, S^1 \times u) = 0.$
2. If $N$ is a rational homology sphere,
$$\mathcal{Q}(M \sharp N, \mathbb{K}) = \mathcal{Q}(M, \mathbb{K}) + 6\lambda_{\text{Casson-Walker}}(N).$$
3. Surgery formula
4. Variation under framed knot change
5. And more...

Cube of equivariant linking pairing for $M^3$ with $\beta_1 = 1$

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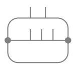
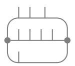
**Relation to the Kontsevich integral**

The invariant $\mathcal{Q}(M, \mathbb{K})$ is equivalent to
the two-loop part of the Kontsevich integral
of the core of the surgery in the surgered rational homology sphere $M(\mathbb{K})$.

In 2004, Garoufalidis and Kricker defined a rational lift of the Kontsevich integral for null-homologous knots in rational homology spheres.
In 2005, Julien Marché proposed a “cubic” definition of an invariant equivalent to the two-loop part.

$x^2y^3 = x^3y^4z$ reads  =  elsewhere.
The Casson invariant is associated with the graph $\theta$.
Our equivariant cube is associated with the above haired $\theta$.

Cube of equivariant linking pairing for $M^3$ with $\beta_1 = 1$

Background
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The infinite cyclic covering
$q: \tilde{M} \to M$

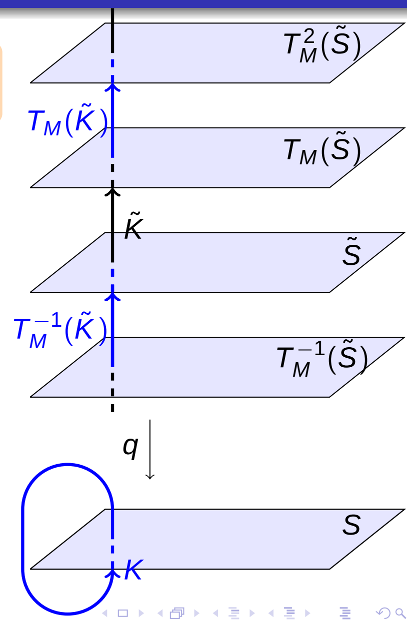
$T_M$ generates covering group

Example : $M = S^1 \times S^2$
$\tilde{M} = \mathbb{R} \times S^2$, $K = S^1 \times u$

$M$ closed 3–manifold,

$K$ knot of $M$ such that
$H_1(M; \mathbb{Z})/\text{Torsion} = \mathbb{Z}[K]$

$S$ closed surface such that
$\langle K, S\rangle = 1$.




Cube of equivariant linking pairing for $M^3$ with $\beta_1 = 1$

$\tilde{M}$ infinite cyclic covering of $M$. $T_M$ generates its covering group.
The action of $T_M$ on $H_1(\tilde{M};\mathbb{Q})$ is denoted as the multiplication by $t_M$.
The $\mathbb{Q}[t_M^{\pm 1}]$-module $H_1(\tilde{M};\mathbb{Q})$ reads

$$H_1(\tilde{M};\mathbb{Q}) = \bigoplus_{i=1}^{k} \frac{\mathbb{Q}[t_M^{\pm 1}]}{\delta_i}$$

where $\delta_i$ divides $\delta_{i+1}$.
$\delta = \delta(M) = \delta_k$ is the annihilator of $H_1(\tilde{M};\mathbb{Q})$ and
$\Delta = \Delta(M) = \prod_{i=1}^{k} \delta_i$ is the Alexander polynomial of $M$.
They are normalised so that

$$\Delta(t_M) = \Delta(t_M^{-1}) \text{ and } \Delta(1) = 1,$$
$$\delta(t_M) = \delta(t_M^{-1}) \text{ and } \delta(1) = 1.$$





**Theorem**

We can construct
an equivariant cube $\mathcal{Q}(M,\mathbb{K})$ of the equivariant linking pairing for a closed 3–manifold $M$ with $H_1(M;\mathbb{Q}) = \mathbb{Q}$
w.r.t. a framed knot $\mathbb{K}$ such that $H_1(M;\mathbb{Z})/\text{Torsion} = \mathbb{Z}[\mathbb{K}]$
with the following properties.

1. $\mathcal{Q}(M,\mathbb{K}) \in \mathbb{Q}(x,y) = \frac{\mathbb{Q}(x,y,z)}{(xyz=1)}$, $\quad \mathcal{Q}(S^1\times S^2, S^1\times u) = 0.$

$$\delta(x)\delta(y)\delta(z)\mathcal{Q}(M,\mathbb{K}) \in \frac{\mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]}{(xyz=1)} = \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$$

   where $\delta$ is the annihilator of $H_1(\tilde{M};\mathbb{Q})$.
2. If $N$ is a rational homology sphere,

$$\mathcal{Q}(M\sharp N,\mathbb{K}) = \mathcal{Q}(M,\mathbb{K}) + 6\lambda_{\text{Casson-Walker}}(N).$$





**The equivariant linking pairing**

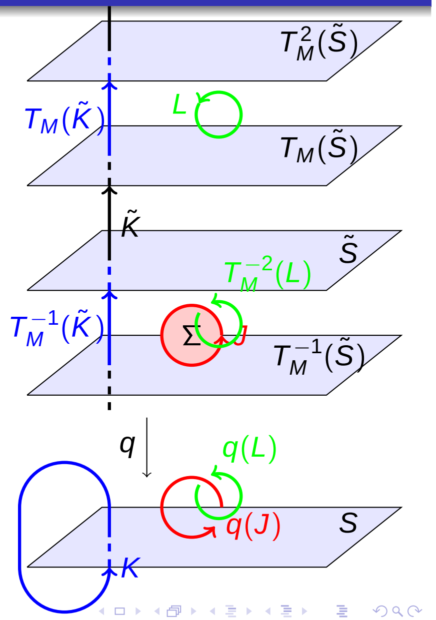
$(J, L)$ 2–component link of $\tilde{M}$,
$q(J) \cap q(L) = \emptyset$
Assume $J = \partial\Sigma$.

$$lk_e(J,L) = \sum_{n\in\mathbb{Z}} t_M^n \langle \Sigma, T_M^n(L)\rangle = \langle \Sigma, L\rangle_e$$

Here, $lk_e(J,L) = t_M^{-2}$

$$lk_e(P(T_M)J, Q(T_M)L) = P(t_M)Q(t_M^{-1})lk_e(J,L)$$

For any knot $J$ of $\tilde{M}$,
$\delta J$ bounds a rational chain.
$lk_e(J,L) = \frac{lk_e(\delta J, L)}{\delta}$.

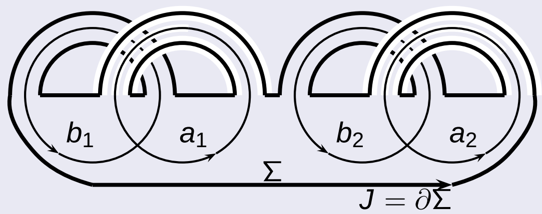







**③ Surgery formula**

Let $J$ be a knot of $M$ that bounds a Seifert surface $\Sigma$ disjoint from $K$, whose $H_1$ goes to 0 in $H_1(M)/\text{Torsion}$.
Let $(a_i, b_i)_{i=1,\dots,g}$ be a symplectic basis of $H_1(\Sigma)$.




then

$$\mathcal{Q}(M(J;p/q),\mathbb{K}) - \mathcal{Q}(M,\mathbb{K}) = 6\frac{q}{p}\lambda_e'(J) + 6\lambda(S^3(\bigcirc;p/q)).$$

### ③ Surgery formula

Let $J$ be a knot of $M$ that bounds a Seifert surface $\Sigma$ disjoint from $K$, whose $H_1$ goes to 0 in $H_1(M)/\text{Torsion}$.
Let $(a_i, b_i)_{i=1,\dots,g}$ be a symplectic basis of $H_1(\Sigma)$. Let

$$\lambda'_e(J) = \frac{1}{12} \sum_{(i,j)\in\{1,\dots,g\}^2} \sum_{\mathfrak{S}_3(x,y,z)} (\alpha_{ij}(x,y) + \alpha_{ij}(x^{-1},y^{-1}) + \beta_{ij}(x,y))$$

where $\alpha_{ij}(x,y)$ is
$lk_e(a_i,a_j^+)(x)lk_e(b_i,b_j^+)(y) - lk_e(a_i,b_j^+)(x)lk_e(b_i,a_j^+)(y)$
and $\beta_{ij}(x,y)$ is
$(lk_e(a_i,b_i^+)(x) - lk_e(b_i^+,a_i)(x))(lk_e(a_j,b_j^+)(y) - lk_e(b_j^+,a_j)(y))$,
then

$$\mathcal{Q}(M(J;p/q),\mathbb{K}) - \mathcal{Q}(M,\mathbb{K}) = 6\frac{q}{p}\lambda'_e(J) + 6\lambda(S^3(\bigcirc;p/q)).$$



$$I_\Delta(t) = \frac{1+t}{1-t} + \frac{t\Delta'(t)}{\Delta(t)}$$

### ④ Framed knot change

Let $\mathbb{K}'$ be another knot of $M$ such that $H_1(M)/\text{Torsion} = \mathbb{Z}[K']$.
Then there exists an antisymmetric polynomial $\mathcal{V}(\mathbb{K},\mathbb{K}')$ in $\mathbb{Q}[t,t^{-1}]$ such that

$$\mathcal{Q}(M,\mathbb{K}') - \mathcal{Q}(M,\mathbb{K}) = \sum_{\mathfrak{S}_3(x,y,z)} \frac{\mathcal{V}(\mathbb{K},\mathbb{K}')(x)}{\delta(x)} I_\Delta(y).$$

Furthermore, for any $k \in \mathbb{Z}$, there exists a pair of framed knots $(\mathbb{K},\mathbb{K}')$ such that $\mathcal{V}(\mathbb{K},\mathbb{K}') = t^k - t^{-k}$.

Example: If $\mathbb{K}'$ is obtained from $\mathbb{K}$ by adding one to the framing, then $\mathcal{V}(\mathbb{K},\mathbb{K}')(t) = -\frac{\delta(t)}{2}\frac{t\Delta'(t)}{\Delta(t)}$.



### Definition of an invariant for 3-manifolds of rank one

For a fixed $(\delta,\Delta)$, define $\overline{\mathcal{Q}}(M)$ in the quotient of $\mathbb{Q}(x,y)$ by the vector space generated by the $(\mathcal{Q}(M,\mathbb{K}') - \mathcal{Q}(M,\mathbb{K}))$ as the class of $\mathcal{Q}(M,\mathbb{K})$.

The invariant $\overline{\mathcal{Q}}(M)$ is equivalent
to a special case of invariants defined by Ohtsuki in 2008,
combinatorially,
for 3-manifolds of rank one.

### ⑤ Property

If $\Delta$ has only simple roots,
if $N$ is a rational homology sphere s. t. $\lambda_{Casson-Walker}(N) \neq 0$,
then $\overline{\mathcal{Q}}(M) \neq \overline{\mathcal{Q}}(M\sharp N)$.



Consider the infinite cyclic covering of $M^2$

$$\widetilde{M^2} = \frac{\tilde{M}^2}{(u,v)\sim(T_M(u),T_M(v))} \xrightarrow{q_2} M^2$$

with generating covering transformation $T$.

$$T(\overline{(u,v)}) = \overline{(T_M(u),v)} = \overline{(u,T_M^{-1}(v))}$$

$$q_2^{-1}(\text{diag}(M^2)) = \sqcup_{n\in zz} T^n\left(\overline{\text{diag}(\tilde{M}^2)} = \text{diag}(M^2)\right) = \mathbb{Z}\times\text{diag}(M^2).$$

Our configuration space $\tilde{C}_2(M)$ is obtained from $\widetilde{M^2}$ by replacing $q_2^{-1}(\text{diag}(M^2))$ by its unit normal bundle $\mathbb{Z}\times M\times_\tau S^2$ where $(u,v)$ $\infty$-ly close to the diagonal projects to the fiber $S^2$ as the direction of $\overrightarrow{q(u)q(v)} \in T_{q(u)}M =_\tau \mathbb{R}^3$ for a trivialisation $\tau$ of $TM$.

Topologically, this blow-up amounts to remove an open tubular neighborhood of $q_2^{-1}(\text{diag}(M^2))$.

$$I_\Delta(t) = \frac{1+t}{1-t} + \frac{t\Delta'(t)}{\Delta(t)}$$

**Lemma**

$\forall X \in S^2$, $\exists$ a rational 4–dimensional chain $G$ of $\tilde{C}_2(M)$ such that

$$\partial G = (t-1)\delta\left(M \times_\tau X - I_\Delta K \times_\tau S^2\right).$$

Set $F = \frac{G}{(t-1)\delta}$, then $\langle J \times L, F\rangle_e = lk_e(J, L)$.

**Lemma**

$H_3(\tilde{C}_2(M);\mathbb{Q}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t) = \mathbb{Q}(t)[K \times_\tau S^2]$.





$$A(K) = \overline{\{(K(z), K(z\exp(2i\pi u)); z \in S^1, u \in ]0,1[\}}.$$

Assume that $\tau\colon TM \to M \times \mathbb{R}^3$ maps $TK$ to $\mathbb{R}W$ with $W \in S^2$.

**Lemma**

Let $K_X$ be a parallel of $K$ w.r.t. the parallelisation of $\mathbb{K}$.
$\forall X \in S^2 \setminus \{W, -W\}$,
$\exists$ a rational 4–dimensional chain $G_X$ of $\tilde{C}_2(M)$ such that

$$\partial G_X = (t-1)\delta\left(M \times_\tau X - I_\Delta K_X \times_\tau S^2\right).$$

and $\langle A(K), G_X\rangle_e = 0$.

**Lemma**

$H_4(\tilde{C}_2(M);\mathbb{Q}) \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t) = \mathbb{Q}(t)[S \times_\tau S^2]$.





$$A(K) = \overline{\{(K(z), K(z\exp(2i\pi u)); z \in S^1, u \in ]0,1[\}}.$$

Assume that $\tau\colon TM \to M \times \mathbb{R}^3$ maps $TK$ to $\mathbb{R}W$ with $W \in S^2$.

**Lemma**

Let $K_X$, $K_Y$ and $K_Z$ be disjoint parallels of $K$ w.r.t. its parallelisation. $\forall X, Y, Z \in S^2 \setminus \{W, -W\}$ distinct,
$\exists$ rational 4–dimensional chains $G_X$, $G_Y$ and $G_Z$ of $\tilde{C}_2(M)$ such that

$$\partial G_{X, Y \text{ or } Z} = (t-1)\delta\left(M \times_\tau \{X, Y \text{ or } Z\} - I_\Delta K_{X, Y \text{ or } Z} \times_\tau S^2\right).$$

and $\langle A(K), G_X\rangle_e = \langle A(K), G_Y\rangle_e = \langle A(K), G_Z\rangle_e = 0$.

**Definition of $\mathcal{Q}(M;\mathbb{K})$**

$\mathcal{Q}(M;\mathbb{K}) = \frac{\langle G_X, G_Y, G_Z\rangle_e}{(x-1)\delta(x)(y-1)\delta(y)(z-1)\delta(z)} - \frac{1}{4}p_1(\tau)$.





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$$\langle G_X, G_Y, G_Z\rangle_e = \sum_{(m,n)\in\mathbb{Z}^2} x^m y^n \langle T^{-m}(G_X), T^{-n}(G_Y), G_Z\rangle$$