Graphic surgery formulae for finite type invariants of 3-manifolds

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Setting

- M is a $\mathbb{Q}$-sphere
  (closed, 3-dimensional such that $H_*(M;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$)
- $\infty \in M$.

Theorem 1 (G. Kuperberg, D. Thurston)

$\forall \omega$ closed 2-form on $C_2(M) = (M \setminus \infty)^2 \setminus$ diagonal

- such that
  $\forall J \sqcup K\colon S^1 \sqcup S^1 \to M$ inducing $J \times K : S^1 \times S^1 \to C_2(M)$,

$$\int_{J\times K} \omega = \mathrm{lk}(J,K)$$

- canonically fixed on $\partial \overline{C_2(M)}$, antisymmetric,

$$\lambda_{\text{Casson-Walker}}(M) = \frac{2}{12}\int_{C_2(M)} \omega^3.$$

With $\{i,j\} \subset \{1,2,3,4\}$, associate

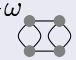
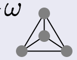
$$\begin{array}{lcll} p_{ij}\colon & C_4(M) = (M\setminus\infty)^4 \setminus \text{all diagonals} & \to & C_2(M) \\ & (x_1,x_2,x_3,x_4) & \mapsto & (x_i,x_j) \end{array}$$

and set $\omega_{ij} = p_{ij}^*(\omega)$.

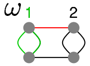
Theorem 2 (M. Kontsevich, G. Kuperberg, D. Thurston)

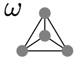
$\forall \omega$ as in Theorem 1,

$$\lambda_2(M) = \int_{C_4(M)} \frac{2}{16}\omega_{\Gamma_1} + \frac{1}{24}\omega_{\Gamma_2}$$

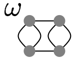
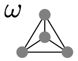
is a non trivial degree 2 invariant of M.

ω  $= \omega_{12} \wedge \omega_{13}^2 \wedge \omega_{24}^2 \wedge \omega_{34}$

ω  $= -\omega_{12} \wedge \omega_{13} \wedge \omega_{14} \wedge \omega_{23} \wedge \omega_{24} \wedge \omega_{34}$

$[\omega] \in H^2(\overline{C_2(M)})$ is Poincaré dual to $F^4 \in H_4(\overline{C_2(M)}, \partial\overline{C_2(M)})$.

$$\int_{X^2 \subset C_2(M)} \omega = \langle F^4, X^2 \rangle_{C_2(M)}.$$

$\int_{C_4(M)} \omega$  and $\int_{C_4(M)} \omega$ 

are algebraic intersections of 6 codimension 2 manifolds F_{ij} in

$$C_4(M) \subset M^4.$$

Thus, $\lambda_2(M)$ is a mixture of

- linking numbers associated to edges
- triple intersections in M associated to vertices

associated to some combination of trivalent diagrams.

For any n, there are (degree n) invariants
that are similar combinations
of such configuration space integrals over $C_{2n}(M)$
associated to trivalent vertices with $2n$ vertices,
and that distinguish $\mathbb{Z}$-spheres that are not distinguished by the invariants of degree less than n.

Notation (surgery)

Let K be a knot in M.

$$M(K;p/q) = M \setminus \mathrm{Int}(T(K)) \cup_{\substack{\partial T(K) \\ (pm(K)+q\ell(K)) \sim S^1 \times *}} D^2 \times S^1$$

If $K = \partial\Sigma$,

$$F^4(M(K;p/q)) = F^4(M) - \frac{q}{2p}(\Sigma \times \Sigma^+ + \Sigma^+ \times \Sigma)$$

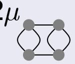
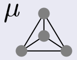
in $C_2(M) \cap C_2(M(K;p/q))$
up to minor corrections.

$$\left(lk_{M(K;p/q)}(U,V) = lk_M(U,V) - \left(\frac{q}{p}\right) lk_M(K,U) lk_M(K,V). \right)$$

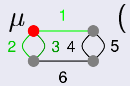
For $i \in \{1,\dots,6\}$, let Σ_i be a surface with boundary a knot $\partial\Sigma_i = K_i$, such that $\Sigma_i \cap K_j = \emptyset$ if $i \neq j$.
Set $\mu_{ijk}(L = (K_1,\dots,K_6)) = -\langle \Sigma_i, \Sigma_j, \Sigma_k \rangle_M$.
For $\sigma \in \mathfrak{S}_6$, $\mu^\sigma_{ijk} = \mu_{\sigma(i)\sigma(j)\sigma(k)}$.
Let $q_1,\dots,q_6 \in \mathbb{Z} \setminus \{0\}$. For $I \subset \{1,\dots,6\}$, $M_I = M(K_i; 1/q_i)_{i \in I}$.

Theorem 1

$$\sum_{I \subset \{1,2,\dots,6\}} (-1)^{\sharp I} \lambda_2(M_I) = \left(\prod_{i=1}^{6} q_i\right) \left(2\mu_{\text{(graph)}}(L) + \mu_{\text{(graph)}}(L) \right)$$

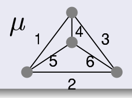



where



$$\mu(L) = \frac{1}{16} \sum_{\sigma \in \mathfrak{S}_6} \mu^\sigma_{123} \mu^\sigma_{145} \mu^\sigma_{263} \mu^\sigma_{465}$$

and

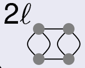
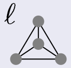


$$\mu(L) = \frac{1}{24} \sum_{\sigma \in \mathfrak{S}_6} \mu^\sigma_{143} \mu^\sigma_{251} \mu^\sigma_{236} \mu^\sigma_{456}.$$

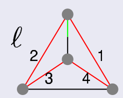
Let K_1, K_2, K_3, K_4 bound Σ_1, Σ_2, Σ_3, Σ_4 in M such that $\Sigma_i \cap K_j = \emptyset$ when $i \neq j$.
Assume $\mu_{ijk}(L_4 = (K_1,\dots,K_4)) = 0 \quad \forall \{i,j,k\} \subset \{1,2,3,4\}$.
For $I \subset \{1,2,3,4\}$, $M_I = M(K_i; 1/q_i)_{i \in I}$.

Theorem 2

$$\sum_{I \subset \{1,2,3,4\}} (-1)^{\sharp I} \lambda_2(M_I) = \left(\prod_{i=1}^{4} q_i\right) \left(2\ell_{\text{(graph)}}(\Sigma) + \ell_{\text{(graph)}}(\Sigma) \right)$$

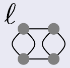



where

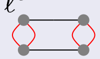
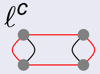


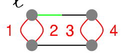
$$\ell(\Sigma) = \frac{3}{24} \sum_{\sigma \in \mathfrak{S}_4} \begin{matrix} lk\left(\Sigma_{\sigma(1)} \cap \Sigma_{\sigma(2)}, \Sigma_{\sigma(3)} \cap \Sigma_{\sigma(4)}\right) \\ lk\left(\Sigma_{\sigma(3)} \cap \Sigma_{\sigma(2)}, \Sigma_{\sigma(1)} \cap \Sigma_{\sigma(4)}\right) \end{matrix}$$

and



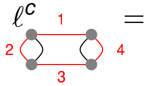
$$\ell(\Sigma) = \ell^d \quad + \ell^c \quad \dots$$

ℓ^d 

$$= \frac{1}{16}\sum_{\sigma\in\mathfrak{S}_4} lk\left(\Sigma_{\sigma(1)}\cap\Sigma_{\sigma(2)}, \Sigma_{\sigma(3)}\cap\Sigma_{\sigma(4)}\right)^2$$

and

ℓ^c 

$$= \frac{4}{16}\sum_{\sigma\in\mathfrak{S}_4} lk^\sigma\left(\Sigma_1\cap\Sigma_2, \Sigma_2\cap\Sigma_3\right) lk^\sigma\left(\Sigma_3\cap\Sigma_4, \Sigma_4\cap\Sigma_1\right)$$

- Theorems 1 and 2 generalize to any degree n invariant (with $3n-$ and $2n-$ component links, respectively).
- They generalize for p/q-surgeries on null-homologous knots in $\mathbb{Q}$-spheres.
- For $\mathbb{Z}$-spheres, together with other results of math.GT/0703347, they generalize Garoufalidis-Goussarov-Polyak comparison of filtrations of the vector space generated by $\mathbb{Z}$-spheres.

Questions

- Study filtrations of the space of $\mathbb{Q}$-spheres.
- Find topological surgery formulae for all configuration space invariants.

Theorem (Le, Murakami, Ohtsuki, Garoufalidis . . .)

- $\lambda_2(M_1\sharp M_2) = \lambda_2(M_1) + \lambda_2(M_2)$ $(\Rightarrow \lambda_2(S^3) = 0)$
- $\lambda_2(M(K;1/q)) - \lambda_2(M) = q^2\lambda_2''(K) + qv_3(K)$.

- $\lambda_2''(K)$ is determined by the Alexander polynomial (Garoufalidis-Habegger).
- Any degree n invariant has a degree n polynomial variation, and an explicit graphic formula for the leading term is in math.GT/0703347.
- E. Auclair explicitly expressed $v_3(K)$ (that is a coefficient of the Jones polynomial for $K\subset S^3$) in terms of invariants of $H_1(\widetilde{M\setminus\Sigma};\mathbb{Q})$.