The goal of the first exercise is to prove the existence of weakly admissible Heegaard diagrams.

Exercise 1. Throughout the exercise Σ is a closed, oriented, connected genus g surface, $z \in \Sigma$ a base point, ω an area form on Σ and λ a primitive of ω on $\Sigma^\circ = \Sigma \setminus \{z\}$.

(a) Given a compactly supported closed one-form θ on Σ°, we define a vector field X by

$$\iota_X \omega = \theta.$$

Let φ_t be the flow of X. Use Cartan's formula $L_X = d\iota_X + \iota_X d$ to prove the following:

1. $\varphi_t^* \omega = \omega$ and $\varphi_t^* \theta = \theta$,
2. $\varphi_t^* \lambda = \lambda + t\theta + dh_t$ where $h_t \colon \Sigma^\circ \to \mathbb{R}$ is a smooth function,
3. If α is an embedded closed curve in Σ°, then

$$\int_{\varphi_t(\alpha)} \lambda = \int_\alpha (\lambda + t\theta).$$

(b) Given pairwise disjoint embedded curves $\alpha_1, \ldots, \alpha_g$ in Σ° such that $\Sigma \setminus (\alpha_1 \cup \ldots \cup \alpha_g)$ is connected, we can find compactly supported 1-forms $\theta_1, \ldots, \theta_g$ with disjoint supports such that

$$\int_{\alpha_i} \theta_j = \delta_{i,j}.$$

(c) Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_g)$ be as in (b). Prove that there exists a compactly supported diffeomorphism $\varphi \colon \Sigma^\circ \to \Sigma^\circ$ such that $\int_{\varphi(\alpha_i)} \lambda = 0$ for all i.

(d) Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a pointed Heegaard diagram. If

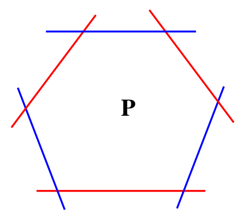
$$\int_{\alpha_i} \lambda = \int_{\beta_i} \lambda = 0$$

for all i, then every periodic domain has both positive and negative coefficients.

Exercise 2. Let $\mathbf{x}, \mathbf{y}, \mathbf{w}$ be Heegaard Floer generators and $u \in \mathcal{M}(\mathbf{x}, \mathbf{y})$, $v \in \mathcal{M}(\mathbf{y}, \mathbf{w})$ holomorphic Heegaard Floer curves. Prove that

$$I(\mathcal{D}(u) + \mathcal{D}(v)) = I(\mathcal{D}(u)) + I(\mathcal{D}(v)).$$

Exercise 3. Let P a region with the shape of a polygon with $2n$ edges alternating on the α- and β-curves and whose vertices are all convex. See the figure for the case $n = 3$.



Prove that $I(P) = 1$ and there is unique holomorphic Heegaard Floer curve u with $\mathcal{D}(u) = P$, up to reparametrisations of the source and translations in the target.

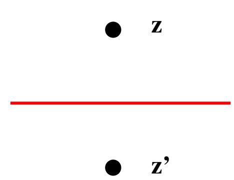
Exercise 4. Let $\mathfrak{s}$ be a Spin^c-structure on M and $c \in H^2(M;\mathbb{Z})$. Prove that

$$c_1(\mathfrak{s} + c) = c_1(\mathfrak{s}) + 2c.$$

Exercise 5. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a pointed Heegaard diagram, $\mathbf{x}$ a Heegaard Floer generator, P a periodic domain and $[P] \in H_2(M;\mathbb{Z})$ the homology class associated to it. Prove that

$$c_1(\mathfrak{s}_z(\mathbf{x})) = e(P) + 2n_{\mathbf{x}}(P).$$

Exercise 6. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a pointed Heegaard diagram and let z' be another basepoint separated by from z by the curve α_i.



Compute $\mathfrak{s}_z(\mathbf{x}) - \mathfrak{s}_{z'}(\mathbf{x})$ for a Heegaard Floer generator $\mathbf{x}$.

Exercise 7. Compute the Heegaard Floer homology groups of $L(p, q)$ using the Heegaard diagrams you drewn in the previous exercise sheet.

Exercise 8. Let M and M' be three-manifolds described by the pointed Heegaard diagrams $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z')$. Define the pointed Heegaard diagram $(\Sigma'', \boldsymbol{\alpha}'', \boldsymbol{\beta}'', z'')$ such that:

- $\Sigma'' = \Sigma \# \Sigma'$, where the connected sum is performed at the base points,
- $\boldsymbol{\alpha}'' = \boldsymbol{\alpha} \cup \boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'' = \boldsymbol{\beta} \cup \boldsymbol{\beta}'$, where $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}'$, $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$ are identified to sets of curves in Σ'' in the obvious way, and

- z'' is a base point in the connected sum region.

Prove that

(a) $(\Sigma'', \boldsymbol{\alpha}'', \boldsymbol{\beta}'', z'')$ is a Heegaard diagram of $M\#N$, and

(b) $\widehat{HF}(M\#N) \cong \widehat{HF}(M) \otimes \widehat{HF}(N)$.

Observe that (b) also implies that $\widehat{HF}$ is invariant under stabilisations.