The goal of the first exercise is to prove the existence of weakly admissible Heegaard diagrams.

**Exercise 1.** Throughout the exercise $\Sigma$ is a closed, oriented, connected genus $g$ surface, $z \in \Sigma$ a base point, $\omega$ an area form on $\Sigma$ and $\lambda$ a primitive of $\omega$ on $\Sigma^\circ = \Sigma \setminus \{z\}$.

(a) Given a compactly supported closed one-form $\theta$ on $\Sigma^\circ$, we define a vector field $X$ by

$$
i_X \omega = \theta.$$

Let $\varphi_t$ be the flow of $X$. Use Cartan’s formula $L_X = d\iota_X + \iota_X d$ to prove the following:

1. $\varphi_t^* \omega = \omega$ and $\varphi_t^* \theta = \theta$,
2. $\varphi_t^* \lambda = \lambda + t\theta + dh_t$ where $h_t : \Sigma^\circ \to \mathbb{R}$ is a smooth function,
3. If $\alpha$ is an embedded closed curve in $\Sigma^\circ$, then

$$
\int_{\varphi_t(\alpha)} \lambda = \int_{\alpha} (\lambda + t \theta).
$$

(b) Given pairwise disjoint embedded curves $\alpha_1, \ldots, \alpha_g$ in $\Sigma^\circ$ such that $\Sigma \setminus (\alpha_1 \cup \ldots \cup \alpha_g)$ is connected, we can find compactly supported 1-forms $\theta_1, \ldots, \theta_g$ with disjoint supports such that

$$
\int_{\alpha_i} \theta_j = \delta_{i,j}.
$$

(c) Let $\alpha = (\alpha_1, \ldots, \alpha_g)$ be as in (b). Prove that there exists a compactly supported diffeomorphism $\varphi : \Sigma^\circ \to \Sigma^\circ$ such that $\int_{\varphi(\alpha)} \lambda = 0$ for all $i$.

(d) Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram. If

$$
\int_{\alpha_i} \lambda = \int_{\beta_i} \lambda = 0
$$

for all $i$, then every periodic domain has both positive and negative coefficients.

**Exercise 2.** Let $x, y, w$ be Heegaard Floer generators and $u \in \mathcal{M}(x, y)$, $v \in \mathcal{M}(y, w)$ holomorphic Heegaard Floer curves. Prove that

$$
I(D(u) + D(v)) = I(D(u)) + I(D(v)).
$$
Exercise 3. Let $P$ a region with the shape of a polygon with $2n$ edges alternating on the $\alpha$- and $\beta$-curves and whose vertices are all convex. See the figure for the case $n=3$.

Prove that $I(P) = 1$ and there is unique holomorphic Heegaard Floer curve $u$ with $D(u) = P$, up to reparametrisations of the source and translations in the target.

Exercise 4. Let $s$ be a Spin$^c$-structure on $M$ and $c \in H^2(M; \mathbb{Z})$. Prove that

$$c_1(s + c) = c_1(s) + 2c.$$  

Exercise 5. Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram, $x$ a Heegaard Floer generator, $P$ a periodic domain and $[P] \in H_2(M; \mathbb{Z})$ the homology class associated to it. Prove that

$$c_1(s_z(x)) = e(P) + 2n_x(P).$$  

Exercise 6. Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram and let $z'$ be another basepoint separated by from $z$ by the curve $\alpha_i$.

Compute $s_z(x) - s_{z'}(x)$ for a Heegaard Floer generator $x$.

Exercise 7. Compute the Heegaard Floer homology groups of $L(p, q)$ using the Heegaard diagrams you drew in the previous exercise sheet.

Exercise 8. Let $M$ and $M'$ be three-manifolds described by the pointed Heegaard diagrams $(\Sigma, \alpha, \beta, z)$ and $(\Sigma', \alpha', \beta', z')$. Define the pointed Heegaard diagram $(\Sigma'', \alpha'', \beta'', z'')$ such that:

- $\Sigma'' = \Sigma \# \Sigma'$, where the connected sum is performed at the base points,
- $\alpha'' = \alpha \cup \alpha'$ and $\beta'' = \beta \cup \beta'$, where $\alpha, \alpha', \beta$ and $\beta'$ are identified to sets of curves in $\Sigma''$ in the obvious way, and
• $z''$ is a base point in the connected sum region.

Prove that

(a) $(\Sigma'', \alpha'', \beta'', z'')$ is a Heegaard diagram of $M \# N$, and

(b) $\widehat{HF}(M \# N) \cong \widehat{HF}(M) \otimes \widehat{HF}(N)$.

Observe that (b) also implies that $\widehat{HF}$ is invariant under stabilisations.